The phenomenon of quenching in the presence of convection

Sang Ro Park
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The phenomenon of quenching in the presence of convection

Park, Sang Ro, Ph.D.
Iowa State University, 1989
The phenomenon of quenching
in the presence of convection

by

Sang Ro Park

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INTRODUCTION

In this paper, we consider four nonstandard initial-boundary value problems for Burger's equation, namely

$$(A_1) \quad u_t = u_{xx} + \varepsilon uux \quad \text{on} \quad (0,1) \times (0,T)$$

$u_x(1,t) = a(1 - u(1,t))^{-p} \quad \text{on} \quad (0,T)$

$u(0,t) = 0 \quad \text{on} \quad (0,T)$

$u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1].$

$$(B_1) \quad u_t = u_{xx} + \varepsilon uux \quad \text{on} \quad (0,1) \times (0,T)$$

$u(1,t) = 0 \quad \text{on} \quad (0,T)$

$-u_x(0,t) = a(1 - u(0,t))^{-p} \quad \text{on} \quad (0,T)$

$u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1].$

$$(C_1) \quad u_t = u_{xx} + \varepsilon uux \quad \text{on} \quad (0,1) \times (0,T)$$

$u(0,t) = 0 \quad \text{on} \quad (0,T)$

$u_x(1,t) = \frac{a}{u^p(1,t)} \quad \text{on} \quad (0,T)$

$u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1]$. 
and

\[(D_1) \quad u_t = u_{xx} + \varepsilon uu_x \quad \text{on} \quad (0,1) \times (0,T)\]

\[-u_x(0,t) = \frac{a}{u^p(0,t)} \quad \text{on} \quad (0,T)\]

\[u(1,t) = 0 \quad \text{on} \quad (0,T)\]

\[u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1]\]

Here \(p > 0, \varepsilon, a > 0\) and \(T \leq \infty\). We consider only positive stationary solutions of \((A_1), (B_1), (C_1)\) and \((D_1)\) for \(\varepsilon > 0\) because

\[u(x,t) = v(1-x,t)\]

defines a one-to-one, onto correspondence between the solutions of \((A_1)\) and those of \((B_1)\), and the solutions of \((C_1)\) and those of \((D_1)\), respectively. This observation permits us to construct all the stationary solutions of \((A_1), (B_1), (C_1)\) or \((D_1)\) for all real \(\varepsilon\), if we know only the positive stationary solutions of \((A_1), (B_1), (C_1)\) and \((D_1)\) for \(\varepsilon \geq 0\). (Nontrivial stationary solutions of \((A_1), (B_1), (C_1), (D_1)\) are necessarily of one sign.)

In a recent paper [11], Levine studied two nonstandard initial-boundary value problems for Burger's equation, namely

\[(E) \quad u_t = u_{xx} + \varepsilon uu_x \quad \text{on} \quad (0,1) \times (0,T)\]

\[u_x(1,t) = a u^p(1,t) \quad \text{on} \quad (0,T)\]

\[u(0,t) = 0 \quad \text{on} \quad (0,T)\]

\[u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1].\]
and

\[(F) \quad u_t = u_{xx} + \varepsilon u_x u \quad \text{on} \quad (0,1) \times (0,T)\]
\[u(1,t) = 0 \quad \text{on} \quad (0,T)\]
\[-u_x(0,t) = au^p(0,t) \quad \text{on} \quad (0,T)\]
\[u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1].\]

Here \(p > 0\), \(\varepsilon\), \(a > 0\), while \(u^p\) is defined as \(|u|^{p-1}u\). When \(\varepsilon = 0\), (E) and (F) are essentially the same problem. They have been studied from the point of view of potential well-theory (in several space dimensions) in a recent series of papers [15,16]. In [11], a complete stability-instability analysis was given for (E) and (F). In this paper we completely analyze stationary solutions in the case of \(p < 0\), and for dynamical behavior we get partial results for (E) and complete results for (F).

In [10], Kawarada defined quenching and blows up at quenching point for the solutions of \(u_t = u_{xx} + \frac{1}{1-u}\) for \(0 < x < L, t > 0\) with \(u(0,t) = u(1,t) = 0\) and \(u(x,0) = 0\). Let us write down two problems.

\[(G) \quad u_t = u_{xx} + \varepsilon \phi(u(x,t)) \quad 0 < x < 1, \ t > 0,\]
\[u(x,0) = u(0,t) = u(1,t) = 0 \quad 0 \leq x \leq 1, \ t \geq 0,\]

\[(H) \quad u_t = u_{xx} \quad 0 < x < 1, \ t > 0,\]
\[u(x,0) = u(0,t) = 0 \quad 0 \leq x \leq 1, \ t \geq 0\]
\[u_x(1,t) = \varepsilon \phi(u(1,t)) \quad t > 0\]

where \(\phi : (-\infty, 1) \to (0, \infty)\) is an increasing \(C^1\) function with \(\phi(0) = 1\) and
\[ \lim_{x \to 1^-} \phi(x) = -\infty. \] Acker and Walter \cite{2,3} and, independently, Levine and Montgomery \cite{14}, proved that there exists \( \varepsilon_0 > 0 \) such that

i) If \( \varepsilon \leq \varepsilon_0 \), then the solution \( u \) of (G) is global, and its limit is a stationary solution of (G).

ii) If \( \varepsilon > \varepsilon_0 \), then there is \( T = T(\varepsilon) < \infty \) such that the solution \( u \) of (G) quenches.

Kawarada \cite{10} showed that if \( \phi(u) = \frac{1}{1-u} \), then \( \varepsilon_0 \leq 2\sqrt{2} \). In fact, the above four authors showed for this \( \phi \), \( \varepsilon_0 \approx 1.5307 \ldots \). Acker and Walter \cite{2}, considered quenching problems in higher dimensions. They considered, for example

\[
\begin{align*}
(K) & \quad u_t = \Delta u + \phi(u) \quad (x,t) \in D_\alpha \times [0,T) \\
& \quad u(x,0) = 0 \quad x \in D_\alpha \\
& \quad u(x,t) = 0 \quad (x,t) \in \partial D_\alpha \times (0,T)
\end{align*}
\]

where \( D_\alpha \), for \( \alpha > 0 \), is a dilation of a reference domain \( D = D_1 \). They showed the existence of a number \( a_0 > 0 \) such that if \( a < a_0 \), then no quenching occurred, while if \( a > a_0 \), then quenching occurred in finite time. Unlike the one dimensional case, nothing was asserted for the critical case \( a = a_0 \).

In \cite{1} A. Acker and B. Kawohl showed that under the assumptions \( u = u(r,t) \) is a radial solution of (K), \( u_0 = u_0(r) \geq 0 \), \( \frac{\partial u_0}{\partial r} \leq 0 \), and \( \Delta u_0 + \phi(u_0) \geq 0 \) with some restrictions on the nonlinearity \( \phi(u) \), if \( u \) quenches, then the only quenching point is the origin and \( u_t(0,t) \) is unbounded. Recently, Deng and Levine \cite{6} generalized this result to the problem

\[
\begin{align*}
(L) & \quad u_t = \Delta u + \phi(u) \quad x \in \Omega, \quad 0 < t < T \\
& \quad u = 0 \quad x \in \partial \Omega, \quad 0 < t < T \\
& \quad u(x,0) = u_0(x) \quad x \in \Omega,
\end{align*}
\]
where $\Omega$ is a bounded convex domain in $\mathbb{R}^n$ with smooth boundary, $\phi$ satisfies $\phi'(u) \geq 0$ and $\phi''(u) \geq 0$, while the initial datum satisfies $0 \leq u_0 < 1$ and $\Delta u_0 + \phi(u_0) \geq 0$. They proved that if $u$ quenches in finite or infinite time, then the quenching points are in compact set, and $u_t$ blows up at finite quenching time. In [11], Levine proved that for (H), there exists $\varepsilon_0 > 0$ such that

i) If $\varepsilon \leq \varepsilon_0$, the solution $u$ of (H) is global,

ii) If $\varepsilon > \varepsilon_0$, then there exists time $T(\varepsilon)$ such that $u$ quenches at $T$ and

$$\lim_{t \to T^+} u(x, t) = \lim_{t \to T^-} u_t(x, t) = \infty.$$ 

So, we automatically have a question about the quenching problem for $(A_1)$ and $(C_1)$. We examine the quenching of $u$, and the blow up of $u_t$.

Our results are in the spirit of the framework considered by Hirsch [8] and Matano [18,19]. However, application of their results to our problem is complicated by the presence of the nonlinear term in the boundary condition. Also we make very strong use of the qualitative dependence of the stationary solution upon $\varepsilon$, which is probably special to the one space dimensional character of our problem.

The plan of the thesis is as follows. In Section 1, we characterize the set of non-negative stationary solutions for generalizations of $(A_1), (B_1), (C_1)$ and $(D_1)$ and get general results for generalizations of the problems $(A_1), (B_1), (C_1)$ and $(D_1)$. We then briefly discuss the question of local existence and continuation. The results of this section are modification of those of Levine [11]. The proofs are expanded versions of his and are included for the convenience of the reader. In Section 2, we obtain the set of stationary solutions of $(A_1)$. We then study the quenching and nonquenching point. We examine the question of blowing up of $u_t$ at the quenching point. We also examine the questions of stability and instability of the set of stationary solutions of $(A_1)$. In Section 3, we characterize the set of nonnegative stationary solutions for $(B_1)$. We then examine the questions of stability and instability of the set of stationary solutions of $(B_1)$. We also discuss the quenching problems.
In Section 4, we characterize the set of nonnegative stationary solutions for \((C_1)\). We examine the questions of stability and instability of the set of stationary solutions of \((C_1)\). We also discuss that some solutions of \((C_1)\) blow up in infinite time, and we prove a result using perturbation theory. In Section 5, we characterize the set of nonnegative stationary solutions for \((D_1)\). Next, we examine the questions of stability and instability of the set of stationary solutions of \((D_1)\). In Section 6, we characterize the set of nonnegative stationary solutions for

\[
(N_1) \quad u_t = u_{xx} + \varepsilon uu_x \quad \text{on} \quad (0,1) \times (0,T)
\]
\[
u(0,t) = 0 \quad \text{on} \quad (0,T)
\]
\[
u_x(1,t) + \frac{\varepsilon}{2} u_x^2(1,t) = au^p(1,t) \quad \text{on} \quad (0,T)
\]
\[
u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1]
\]

and

\[
(N_2) \quad u_t = u_{xx} + \varepsilon uu_x \quad \text{on} \quad (0,1) \times (0,T)
\]
\[
u_x(0,t) + \frac{\varepsilon}{2} u_x^2(0,t) = -au^p(0,t) \quad \text{on} \quad (0,T)
\]
\[
u(1,t) = 0 \quad \text{on} \quad (0,T)
\]
\[
u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1]
\]

Finally we get results numerically in Appendix I, Appendix II and Appendix III.
1. GENERAL PROPERTIES AND RESULTS FOR (A), (B)

1.1. Stationary Solutions

Here we consider stationary solutions for

(A) \[ u_t = u_{xx} + \varepsilon (f(u))_x \quad \text{on} \quad (0,1) \times (0,T) \]
\[ u(0,t) = 0 \quad \text{on} \quad (0,T) \]
\[ u_x(1,t) = ag(u(1,t)) \quad \text{on} \quad (0,T) \]
\[ u(x,0) = u_0(x) \quad \text{on} \quad [0,1] \]

(B) \[ u_t = u_{xx} + \varepsilon (f(u))_x \quad \text{on} \quad (0,1) \times (0,T) \]
\[ u(1,t) = 0 \quad \text{on} \quad (0,T) \]
\[ -u_x(0,t) = ag(u(0,t)) \quad \text{on} \quad (0,T) \]
\[ u(x,0) = u_0(x) \quad \text{on} \quad [0,1] \]

where \( \varepsilon, a > 0, T \leq \infty, f, g \) are real valued, continuously differentiable functions. For example, if \( f(u) = \frac{u^2}{2}, \quad g(u) = \frac{1}{(1-u)^\beta}, \quad \beta > 0, \) then (A), (B) become \((A_1), (B_1)\), respectively and if \( f(u) = \frac{u^2}{2}, g(u) = au^{-p}, \) \((A), (B)\) become \((C_1), (D_1)\) respectively.

The following lemma is a simple consequence of the first and second maximum principles for elliptic equations.

**Lemma 1.1.** Let \( f \) be twice continuously differentiable and \( g \neq 0 \). Then nonzero stationary solutions of (A) cannot change sign, and positive stationary solutions \( w(x) \) of (A) satisfy \( w'(x) > 0 \) on \([0,1]\).

**Proof:** For the first statement, we assume that \( w(x_0) = 0 \) for some \( 0 < x_0 < 1 \) and \( w(x) > 0 \) for \( 0 < x < x_0 \). Then there exists a point \( x_1 \) between 0 and \( x_0 \) such that \( w(x) \) has maximum at \( x_1 \). Then \( 0 = w_{xx}(x_1) < 0 \). This is not possible.
For the second statement, we must have \( w'(0) \geq 0 \). This inequality is strict unless \( w \equiv 0 \) (i.e., otherwise if \( w(0) = 0 \), \( w'(0) = 0 \), then \( w = 0 \) by uniqueness.) If \( w' \) changed sign on \([0,1)\), \( w \) would have an interior maximum which cannot happen unless \( w \equiv \text{constant} = 0 \). If \( w'(1) = 0 \), then \( a = 0 \) because \( g(w(1)) \neq 0 \), but \( a > 0 \) by definition. If, for some \( x_0 \in (0,1) \), \( w'(x_0) = 0 \) and \( w'(x) \geq 0 \) for \( x \neq x_0 \), then \( w''(x_0) = 0 \). Let \( v = w' \). Then

\[
\begin{align*}
v'' + \varepsilon f'(w)v' + \varepsilon f''(w)v^2 &= 0, \\
v(x_0) = v'(x_0) &= 0.
\end{align*}
\]

Hence \( v \equiv 0 \), that is \( w(x) \equiv w(1) \equiv 0 \). This is a contradiction!

The proof of Lemma 1.2 is similar to that of Lemma 1.1. So we omit it here.

**Lemma 1.2.** Let \( f \) be twice continuously differentiable and \( g \neq 0 \). Then nonzero stationary solutions of \((B)\) cannot change sign, and positive solutions \( w(x) \) of \((B)\) satisfy \( w'(x) < 0 \) on \([0,1]\).

**Theorem 1.1.** Let \( f'(u) > 0 \) and be monotonically increasing for \( u > 0 \) and \( g > 0 \). Let \( w(x) \) be a positive stationary solution of \((A)\), in \( C^2 \) on \((0,1)\), and in \( C^1 \) on \([0,1]\). Let \( w_1 \equiv w(1) \). Then

\[
(1.1) \quad \int_0^x \frac{d\sigma}{ag(w_1) + \varepsilon (f(w_1) - f(\sigma))} = x, \quad w(1) < 1
\]

for \( 0 \leq x \leq 1 \). Conversely, if \( 0 < w_1 < 1 \) solves

\[
(1.2) \quad \int_0^{w_1} \frac{d\sigma}{ag(w_1) + \varepsilon (f(w_1) - f(\sigma))} = 1,
\]

and \( w \) solves \((1.1)\) with this degree of smoothness, with \( w(1) = w_1 \), then \( w \) is a positive stationary solution of \((A)\).
PROOF: To prove (1.1), we know that

\[ w''(x) + \varepsilon [f(w(x))]' = 0, \quad 0 < x < 1, \]
\[ w(0) = 0, \quad w'(1) = ag(w(1)), \]

so

\[ \int_1^x \frac{d}{d\tau} [w'(\tau) + \varepsilon f(w(\tau))] d\tau = 0, \]
\[ w'(x) + \varepsilon f(w(x)) - [w'(1) + \varepsilon f(w(1))] = 0, \]

which implies that

\[ w'(x) = -\varepsilon f(w(x)) + ag(w(1)) + \varepsilon f(w(1)) \]
\[ = ag(w(1)) + \varepsilon [f(w_1) - f(w(x))] > 0. \]

Therefore,

\[ w(x) \int_0^{w_1} \frac{dw}{ag(w_1) + \varepsilon [f(w_1) - f(w)]} = x. \]

For the converse, we observe that if \( w(\cdot) \) satisfies (1.1), then \( w(x) < w_1 \) for \( x < 1 \). To see this, suppose that \( w(\bar{x}) \geq w_1 \) for some \( \bar{x} \in [0, 1) \). If \( h(\sigma) = ag(w_1) + \varepsilon [f(w_1) - f(\sigma)] \) has no roots on \( [0, w(\bar{x})] \), then \( h(\sigma) > 0 \) and

\[ 1 = \int_0^{w_1} \frac{d\sigma}{h(\sigma)} \leq \int_0^{w(\bar{x})} \frac{d\sigma}{h(\sigma)} = \bar{x}. \]

Hence \( \bar{x} = 1 \). If \( h(\sigma) \) has a root, say \( \bar{\sigma} \), then this root is unique because \( f' > 0 \) on \([0, w(\bar{x})] \). Moreover, \( \bar{\sigma} > \sup\{w(x)|0 \leq x \leq 1\} \); otherwise there would be an
$x^0 \in [0, 1]$ with $\bar{\sigma} = w(x^0)$. But then

$$h(\sigma) = \varepsilon [f(\bar{\sigma}) - f(\sigma)]$$

$$= \varepsilon f'(\sigma_0) (\bar{\sigma} - \sigma), \quad \sigma < \sigma_0 < \bar{\sigma}$$

$$\leq \varepsilon f'(\bar{\sigma}) (\bar{\sigma} - \sigma),$$

so that

$$x^0 = \int_{0}^{w(x^0)} \frac{d\sigma}{h(\sigma)} = \int_{0}^{\bar{\sigma}} \frac{d\sigma}{\varepsilon [f(\bar{\sigma}) - f(\sigma)]}$$

$$= \int_{0}^{\bar{\sigma}} \frac{d\sigma}{\varepsilon f'(\sigma_0) (\bar{\sigma} - \sigma)}$$

$$\geq \int_{0}^{\bar{\sigma}} \frac{d\sigma}{\varepsilon f'(\bar{\sigma}) (\bar{\sigma} - \sigma)}$$

$$= \frac{1}{\varepsilon f'(\bar{\sigma})} \int_{0}^{\bar{\sigma}} \frac{d\sigma}{(\bar{\sigma} - \sigma)} = \infty$$

for $x^0 \neq 0$. If $x^0 = 0$ and $w(0) > 0$, then, by the above argument, we have a contradiction. So $w(0) = 0$, and thus $w(0) = \bar{\sigma}$ is impossible. Thus $w(x) \leq \bar{\sigma} - \delta$ for some $\delta > 0$ and all $x \in [0, 1]$, and it follows that $h(\sigma)$ has no zero on $[0, w(\bar{x})]$. Hence $w(x) \leq w_1$ by the above argument.

Therefore, we may differentiate (1.1) to find that

$$w'(x) + \varepsilon f(w(x)) = a_1(w_1) + \varepsilon f(w_1),$$

and

$$w''(x) + \varepsilon [f(w(x))]' = 0.$$
Also 
\[ w'(1) = a(w_1), \]
\[ w'(x) \geq \varepsilon [f(w_1) - f(w(x))] \geq 0. \]

Hence \( w(0) = \lim_{x \to 0^+} w(x) \) exists, and by (1.1) \( w(0) = 0. \)

**Theorem 1.2.** Let \( f(u) \geq 0 \) for \( u \geq 0. \) Let \( w(x) \) be a positive stationary solution of (B), in \( C^2 \) on \((0,1)\) and \( C^1 \) on \([0,1] \). Let \( w_0 = w(0). \) Then

\[ (1.3) \quad \int_0^x \frac{d\sigma}{a(w_0) - \varepsilon f(w_0) + \varepsilon f(\sigma)} = 1 - x. \]

for \( 0 \leq x \leq 1 \) and \( a(w_0) - \varepsilon f(w_0) > 0. \) Conversely, if \( w(0) = w_0 > 0, a(w_0) - \varepsilon f(w_0) > 0, \) \( w_0 \) solves

\[ (1.4) \quad \int_0^x \frac{d\sigma}{a(w_0) - \varepsilon f(w_0) + \varepsilon f(\sigma)} = 1, \]

\( w \) solve (1.3) with this degree of smoothness, and \( w(0) = w_0, \) then \( w \) is a positive stationary solution of (B).

**Proof:** To prove (1.3), we know that

\[ w''(x) + \varepsilon [f'(w(x))] = 0, \quad 0 < x < 1, \]
\[ w(1) = 0, w'(0) = -a(g(w(0))), \]

so we have

\[ \int_0^x \frac{d}{d\tau} \{w'(\tau) + \varepsilon [f(w(\tau))]\} = 0. \]

Hence

\[ w'(x) + \varepsilon [f(w(x))] - w'(0) - \varepsilon f(w(0)) = 0, \]
Thus
\[ w(x) = -\int_0^\infty \frac{d\sigma}{ag(\sigma) - \varepsilon f(\sigma)} = -\int_1^x d\sigma = 1 - x, \]
and
\[ w'(1) = -[ag(\sigma) - \varepsilon f(\sigma) - \varepsilon f(w(1))] < 0. \]
Since \( f(0) = 0 \),
\[ ag(\sigma) - \varepsilon f(\sigma) > 0. \]
For the converse, we observe that if \( w(\cdot) \) satisfies (1.3), then \( w(x) < w_0 \) for \( 0 < x < 1 \).
To see this, suppose that \( w(x) > w_0 \) for some \( x \in (0,1] \). Since \( h(\sigma) = ag(\sigma) - \varepsilon f(\sigma) + \varepsilon f(w_0) \) has no roots on \( (0,x] \), \( h(\sigma) > 0 \) and
\[ \frac{1}{\int_0^x d\sigma} \frac{w(x)}{h(\sigma)} < \int_0^\infty \frac{d\sigma}{h(\sigma)} = 1 - \bar{x}, \]
which implies \( \bar{x} = 0 \). Hence \( w(x) < w_0 \) if \( 0 < x < 1 \). Therefore we may differentiate (1.3) to find that
\[ w'(x) + \varepsilon f(w(x)) = -[ag(\sigma) - \varepsilon f(\sigma)] \]
\[ \text{and} \]
\[ w''(x) + \varepsilon f'(w_0) = 0. \]
Also, \( w'(0) = -ag(w_0) \). We know that
\[ w'(x) = -[ag(\sigma) - \varepsilon f(\sigma) - \varepsilon f(w(x))] < 0, \]
so, \( w(1) \equiv \lim_{x \to 1^-} w(x) \) exists, and, by (1.3), \( w(1) = 0 \).

With somewhat further restrictions on \( f \), we can prove that these stationary solutions are ordered.

**Theorem 1.3.** Let \( f' \) be locally Lipschitz continuous on \( [0, \infty) \). Let \( u(x), v(x) \) be the stationary solutions of (A) with \( u(1) = \mu_1, v(1) = v_1 \) and \( 1 > \mu_1 > v_1 > 0 \). Then \( u(x) > v(x) \) for all \( x \in (0,1] \).
PROOF: Suppose that $u(x) \leq v(x)$ for some $x \in (0, 1]$. Take $x_0$ such that $x_0 = \max\{x \mid u(x) = v(x)\}$. Then $u(x) > v(x)$ for $x_0 < x < 1$. We can now find $x_1$ such that $v(x_1) = u(x_1), 0 \leq x_1 < x_0$, and $u(x) > v(x)$ for $x_1 < x < x_0$. So $u'(x_1) < v'(x_1)$ and $u'(x_0) > v'(x_0)$. But due to the autonomous nature of the problem, solutions may be identified with orbits in the $(u, u')$ phase space, and the above conditions imply that the orbits of $u$ and $v$ have a point of intersection. By uniqueness for the initial value problem, the orbits of $u$ and $v$ must coincide and so $u = v$. Hence $u_1 = v_1$, which cannot happen.

THEOREM 1.4. Let $u_0, v_0$ be solutions of (1.4) with $u_0 > v_0 > 0, ag(u_0) - \varepsilon f(u_0) > 0, ag(v_0) - \varepsilon f(u_0) > 0$ and let $u(x), v(x)$ denote the corresponding solutions of (1.3). If $f'$ is strictly increasing, then $u(x) > v(x)$ on $(0, 1]$.

One can find the proof of Theorem 1.4 in Levine [11]. It is therefore omitted.

We shall assume all solutions are $C^2$ in $x$ and $C^1$ in $t$ on $D_T \equiv (0, 1) \times (0, T)$ and continuous in the parabolic cylinder $[0, 1] \times [0, T) \equiv D_T \cup \Gamma_T$. For convenience we shall assume $f$ is $C^2$.

LEMMA 1.3. Let $f$ be $C^2$ on $R^1$ and $g$ be $C^1$ on $(0, 1)$. Let $u$ solve (A) in $D_T$, be $C^2$ in $x$, $C^1$ in $t$ in $D_T$, and continuous in $D_T \cup \Gamma_T$. If $u(x, t) > u(x, 0) > 0 (0 < u(x, t) < u(x, 0))$ in $D_T \cup \Gamma_T$ except $x = 0$, then $u_t(x, t) > 0 (< 0)$ in $D_T$ and on $x = 1, 0 < t < T$.

PROOF: To prove this we work in $D_T - \delta$ and let $0 < h < \delta/2$ for some $\delta > 0$. We let

$$v(x, t) = u(x, t + h) - u(x, t).$$

Then

$$v(0, t) \geq 0, \quad \text{if } \quad 0 < t < T,$$
$$v(x, 0) > 0, \quad \text{on } \quad [0, 1].$$
and \( v \) satisfies

\[
v_t = v_{xx} + \varepsilon f'(u(x, t + h)) v_x + \varepsilon f''(\zeta(x, t, h)) u_x v,
\]

where \( \zeta \) is between \( u(x, t + h) \) and \( u(x, t) \). Furthermore, for \( 0 < t \leq T - \delta \) we have

\[
v_x(1, t) = a g'(\eta(1, t, h)) v(1, t),
\]

where \( \eta \) is between \( u(1, t+h) \) and \( u(1, t) \). The hypotheses are such that the coefficients of \( v, v_x \) are bounded in and therefore, by the first and second maximum principle, \( v \geq 0 \) in \( \bar{D}_{T-\delta} \). It follows that \( u_t \geq 0 \) whenever it exists.

By interior regularity, \( u_t \) exists in \( D_T \), and, by boundary regularity arguments, \( u_t \) exists on \( x = 1, 0 < t < T \). Now \( \psi = u_t \) satisfies \( \psi \geq 0 \), where it exists, and

\[
\psi_t = \psi_{xx} + \varepsilon f'(u)\psi_x + \varepsilon f''(u)u_x\psi
\]

in \( D_{T-\delta} \). Also for \( x = 1, 0 < t \leq T - \delta \), we have

\[
\psi_x = a g'(u)\psi
\]

while for \( 0 < t \leq T - \delta \),

\[
\psi(0, t) = 0,
\]

and

\[
\psi(x, 0) \geq 0
\]

on \( (0, 1) \). Therefore \( \psi > 0 \) in \( D_{T-\delta} \), unless \( \psi \equiv 0 \), by the strong maximum principle. But \( \psi \equiv 0 \) implies \( u(x, 0) \equiv u(x, t) \).

**Lemma 1.4.** Suppose that \( f' \) is increasing and that \( u > 0 \) on \( \{1\} \times [0, T) \).

Let \( u(x, t) \) solve (A). If \( u(x, 0) > 0 \) on \([0, 1]\), then \( u > 0 \) on \( D_T \cup \Gamma_T \), except at \( x = 0 \). Suppose also that \( \frac{g(u)}{w} \) is increasing on \([w_0, 1)\) for some \( w_0 > 0 \). If \( w(x) \) is a positive stationary solution of (A), and if \( w(1) > w_0 \), then there exists \( \sigma_0(0 < \sigma_0 < 1) \)
such that if $\sigma \in [0, \sigma_0)$ and $u(x, 0) \leq (1 - \sigma)w(x)$, then $u(x, t) \leq (1 - \sigma)w(x)$. If $u(x, 0) \geq (1 + \sigma)w(x)$ for some $\sigma > 0$, then either $u(x, t) \geq (1 + \sigma)w(x)$ on $D_T \cup \Gamma_T$ or $u$ quenches.

**Proof:** If $u$ has a negative minimum in $\hat{D}_{T-\delta}$ for some $\delta > 0$, then for any $\mu > 0$, $v = e^{-\mu t}u$ also would have a negative minimum in $\hat{D}_{T-\delta}$. We have

$$v_t = -\mu e^{-\mu t}u + e^{-\mu t}u_t$$

$$= -\mu e^{-\mu t}u + e^{-\mu t}[u_{xx} + \varepsilon f'(u)u_x]$$

$$= u_{xx} + \varepsilon f'(u)u_x - \mu v.$$  

From this, a negative minimum cannot occur in the interior. So $u(x, t) > 0$ in $\hat{D}_{T-\delta}$ except at $x = 0$. To prove the second statement, we can find $\sigma_0 (0 < \sigma_0 < 1)$ such that $(1 - \sigma_0)w(1) > w_0$. Let $\sigma \in [0, \sigma_0)$, let $v(x) = (1 - \sigma)w(x)$, and note that

$$v_{xx} + f'(v)v_x = (1 - \sigma)w_{xx} + f'((1 - \sigma)w(x))(1 - \sigma)w_x(x)$$

$$= (1 - \sigma)[w_{xx} + f'((1 - \sigma)w(x))w_x(x)]$$

$$\leq (1 - \sigma)[w_{xx} + f'(w(x))w_x(x)]$$

$$\leq 0,$$

since $w_x > 0$ on $[0, 1]$ and $f'$ is assumed increasing. Moreover on $x = 1$,

$$v_x - g(v) = (1 - \sigma)g(w) - g((1 - \sigma)w)$$

$$= (1 - \sigma)w \left[ \frac{g(w)}{w} - \frac{g((1 - \sigma)w)}{(1 - \sigma)w} \right]$$

$$\geq 0,$$

since $(1 - \sigma)w \geq (1 - \sigma_0)w > w_0$, and $\frac{g(w)}{w}$ is increasing on $[w_0, 1)$.

We now set

$$\psi(x, t) = e^{(\lambda x + \mu t)}(v(x) - u(x, t))$$
and find that in $D_{T-\delta}$

$$
\psi_t \geq \psi xx + (-2\lambda + f'(v)) \psi x + \left( \lambda^2 + \mu - \lambda f'(v) + f''(\zeta)(u x) \right) \psi.
$$

Also at $x = 1$, $0 < t \leq T - \delta$, we have

$$
\psi x \geq (g'(\nu) + \lambda)\psi.
$$

Let $\lambda$ and $\mu$ be chosen to make the coefficients of $\psi$ in these last two inequalities negative. Then, if $\psi$ has a negative minimum in $D_{T-\delta}$, it must occur at $x = 0$ or at $(1,0)$. At $(1,0)$, however, $\psi(1,0) \geq 0$. Therefore, $\psi \geq 0$, and the second statement is proved. An argument similar to the above shows us that if $u(x,0) \geq (1 + \sigma)w(x)$, then $u(x,t) \geq (1 + \sigma)w(x)$ on $D_T \cup \Gamma_T$.

**Theorem 1.5.** Let $f'$ be strictly increasing on $(0,\infty)$, and let $g$ be strictly increasing on $(0,1)$. Suppose that $g(w)/w$ is increasing on $[w_0,1)$ and that the roots, $w$, of (1.2) are isolated with $w > w_0$. Then there is at most one positive stationary solution of (A), call it $w(x)$, which satisfies $w(1) > w_0$. Moreover there exists $\sigma_0$ such that if $u(x,t)$ solves (A) on $D_T \cup \Gamma_T$, and $0 \leq u(x,0) \leq (1 - \sigma)w(x)$ on $[0,1]$, for $\sigma \in [0,\sigma_0)$, then we may take $T = +\infty$, and $0 \leq u(x,t) \leq (1 - \sigma)w(x)$ for all $x \in (0,1], t \in [0,\infty)$. Therefore, $w(x)$ is unstable from below (when it exists and $w(1) > w_0$).

**Proof:** Let $w_1, w_2$ be two stationary solutions of (A) with $w_0 < w_1(1) < w_2(1)$, and assume that there are no solutions of (1.2) in $(w_1(1), w_2(1))$. By Theorem 1.3, we have $w_1(x) < w_2(x)$ on $[0,1]$. Moreover, $w_i'(x) > 0$ for $i = 1,2$ on $(0,1]$ by Lemma 1.1. With $q(x) = \frac{w_1'(x)}{w_2'(x)}$, we have $q'(x) = \varepsilon \left( f'(w_2) - f'(w_1) \right) q > 0$ on $[0,1]$. From this, it follows that $w_1'(0) \leq w_2'(0)$ which, by uniqueness, must be strict. Moreover, $0 < \frac{w_i'(1)}{w_2'(1)} < 1$ because $g$ is strictly increasing. Set $r_i = 1 - \frac{w_i'(1)}{w_2'(1)}$, $i = 0,1$. Then $r_i \in (0,1)$, and on $(0,1], (1 - r_0)w_2(x) < w_1(x) < (1 - r_1)w_2(x)$. To prove this, let $k(x) = w_1(x) - (1 - r_0)w_2(x)$. Then $k'(x) = w_1'(x) - (1 - r_0)w_2'(x) > 0$ since $q(x)$ is
increasing, and we know that $k(0) = 0$. Hence $(1 - r_0)w_2(x) < w_1(x)$. Similarly, we can prove the second inequality.

Since $w_1(1) > w_0$, there exists $\sigma_0 > 0$ such that $(1 - \sigma_0)w_1(1) > w_0$. Choose $s$ ($0 < s < 1$) such that $1 - (1 - r_1)^s < \sigma_0$. Let $u(x, t)$ solve (A) with $u(x, 0) = (1 - r_1)^sw_2(x)$. Then $(1 - r_1)^{s-1}w_1(x) < u(x, 0) \leq (1 - r_1)^sw_2(x)$. By Lemma 1.4 and this inequality,

$$(1 - r_1)^{s-1}w_1(x) \leq u(x, t) \leq (1 - r_1)^sw_2(x) \text{ on } D_T \cup \Gamma_T.$$ 

From this a priori bound and the continuation theorem below, found $T = +\infty$. Since $u_t \leq 0$ on $[0, 1] \times (0, \infty)$ by Lemma 1.3, 

$$\lim_{t \to \infty} u(x, t) = \phi(x)$$

exists and $w_1(x) < (1 - r_1)^{s-1}w_1(x) \leq \phi(x) \leq (1 - r_1)^sw_2(x) < w_2(x)$ on $[0, 1]$.

Let 

$$F(x, t) = \int_0^1 G(x, y)u(y, t)dy,$$

where 

$$G(x, y) = \begin{cases} x & \text{if } 0 \leq x \leq y \leq 1 \\ y & \text{if } 0 \leq y \leq x \leq 1 \end{cases}$$

then 

$$\lim_{t \to \infty} F(x, t) = \int_0^1 G(x, y)\phi(y)dy \text{ and is finite.}$$

Calculating $F_t$, we see that 

$$F_t = \int_0^1 G(x, y)u_t(y, t)dy$$

$$= -u(x, t) - \int_0^x \varepsilon f(u(y, t))dy + x[a g(u(1, t)) + \varepsilon f(u(1, t))]$$

$$\to -[\phi(x) + \int_0^x \varepsilon f(\phi(y))dy] + x[a g(\phi(1)) + \varepsilon f(\phi(1))]$$
as \( t \to \infty \).

This limit, which is nonpositive, is in fact zero for \( x \in [0,1] \); otherwise, \( F \) would not have a finite limit as \( t \to \infty \). Therefore,

\[
\phi(x) + \int_0^x \varepsilon f(\phi(y)) \, dy = x[a_g(\phi(1)) + \varepsilon f(\phi(1))],
\]

and hence \( \phi \) is a stationary solution of (A) with \( \phi(1) \in (w_1(1), w_2(1)) \), which is the desired contradiction.

The second statement of the theorem follows from the lemma and the continuation theorems. Thus \( w(x) \) is unstable from above and below in the class of continuous functions on \([0,1]\) vanishing at \( x = 0 \).

**EXAMPLE 1.1:** Suppose \( f(u) = u^2, g(u) = \frac{1}{(1-u)^\beta} \). Then we know that \( w_0 = \frac{1}{\beta+1} \).

So, the theorem holds for \( w(1) > \frac{1}{\beta+1} \).

**THEOREM 1.6.** Let \( \varepsilon \geq 0 \). Let \( f \) be a \( C^1 \) increasing function on \([0,\infty)\) with \( f'(u) > 0 \) for \( u > 0 \). \( a_g \) an increasing function on \([0,1]\) and \( a_g > 0 \) on \([0,1]\). Let \( w(x,\varepsilon) \) be in a \( C^1 \) (in \( \varepsilon \)) branch of positive solutions on some \( \varepsilon \) interval and let \( w_1(\varepsilon) = w(1,\varepsilon) \). If \( w_1'(\varepsilon) > 0 \) on this branch, the solutions, are stable while if \( w_1'(\varepsilon) < 0 \) on this branch the solutions are unstable. (Here \( w_1'(\varepsilon) = \frac{\partial w_1(1,\varepsilon)}{\partial \varepsilon} \).)

The proof of Theorem 1.6 is similar to that of Theorem 3.6A in Levine [11]. So, we omit it here.

We now examine the questions of stability and instability for the time dependent problems (B). We shall again assume that all solutions are \( C^2 \) in \( x \) and \( C^1 \) in \( t \) on \( D_T \) and continuous in the parabolic cylinder \( D_T \cup \Gamma_T \), and for convenience, that \( f \) is \( C^2 \). As noted earlier the structure conditions on \( f, g \) are somewhat different in this problem.

There are parallel results however.
LEMMA 1.5. Let \( f'(u) \) be increasing and \( u > 0 \) on \( \{0\} \times [0,T) \). Let \( u \) be a solution of (B) on \( D_T \cup \Gamma_T \). If \( u(x,0) > 0 \) on \([0,1]\), then \( u > 0 \) on \( D_T \cup \Gamma_T \) except \( x = 1 \). Suppose that \( g(w)/w \) is decreasing on \([0,w_0]\). If \( w(x) \) is a positive stationary solution of (B) and \( w(0) < w_0 \), then there is a \( \sigma_0 \) such that for \( \sigma \in [0,\sigma_0) \), if \( u(x,0) \leq (1 + \sigma)w(x) \) on \([0,1]\), then \( u(x,t) \leq (1 + \sigma)w(x) \) on \( D_T \cup \Gamma_T \). If \( u(x,0) \geq (1 - \sigma)w(x) \) on \([0,1]\) for some \( \sigma \in (0,1) \), then \( u(x,t) \geq (1 - \sigma)w(x) \).

PROOF: If \( u \) has a negative minimum in \( \bar{D}_{T-\delta} \) for some \( \delta > 0 \), then for any \( \mu > 0 \),

\[
v = e^{-\mu t} u
\]

also would have a negative minimum in \( \bar{D}_{T-\delta} \). We know that

\[
v_t = v_{xx} + \varepsilon f'(u)v_x - \mu v
\]

so, from this, a negative minimum cannot occur in \( D_{T-\delta} \) so, \( u(x,t) > 0 \) in \( \bar{D}_{T-\delta} \) except at \( x = 1 \). To prove the second statement, we can choose \( \sigma_0 \) such that \((1 + \sigma_0)w(0) < w_0 \), and we let

\[
v(x) = (1 + \sigma)w(x).
\]

Then

\[
v_{xx} + \varepsilon f'(v)v_x = (1 + \sigma) \left\{ w_{xx} + \varepsilon f'((1 + \sigma)w) w_x \right\}
= (1 + \sigma) \varepsilon \left\{ f'((1 + \sigma)w) - f(w) \right\} w_x \leq 0.
\]

At \( x = 0 \), we have

\[
v_x + ag(v) = (1 + \sigma)w_x + ag(v)
= ag(v) - (1 + \sigma)ag(w)
= a(1 + \sigma)w \left\{ \frac{g((1 + \sigma)w)}{(1 + \sigma)w} - \frac{g(w)}{w} \right\}
\leq 0.
\]
Define

\[ \psi(x, t) = e^{(\lambda x + \mu t)}(v(x) - u(x, t)). \]

Then

\[
\psi_t = \mu \psi + e^{(\lambda x + \mu t)}[-u_{xx} - \varepsilon f'(u)u_x]
\]

\[
\geq \mu \psi + e^{(\lambda x + \mu t)}[v_{xx} - u_{xx} + \varepsilon f'(v)v_x - u_x \varepsilon f'(u)]
\]

\[
= \psi_{xx} + [\varepsilon f'(u) - 2\lambda] \psi_x + [\lambda^2 + \mu - \varepsilon \lambda f'(u) + \varepsilon f''(\zeta)\psi_x] \psi,
\]

where \( \zeta \) is between \( v(x, t) \) and \( u(x, t) \). At \( x = 0 \), we have

\[
\psi_x = \lambda \psi + e^{(\lambda x + \mu t)}(v_x - u_x)
\]

\[
\leq \lambda \psi + e^{(\lambda x + \mu t)}[-ag(v) + ag(u)]
\]

\[
= \lambda \psi - a g'(\eta) \psi
\]

\[
= (\lambda - a g'(\eta)) \psi,
\]

where \( \eta \) is between \( v \) and \( u \). If we choose \( \lambda \) such that in the second inequality the coefficient of \( \psi \) is positive and \( \mu \) such that in the first inequality the coefficient of \( \psi \) is negative, then the desired result follows.

For the last statement, we let

\[ v_1(x) = (1 - \sigma)w(x) \]

Then

\[
v_{1xx} + \varepsilon f'(v_1)v_{1x} = (1 - \sigma)[w_{xx} + \varepsilon f'((1 - \sigma)w)w_x]
\]

\[
= (1 - \sigma)w_x \varepsilon[f'((1 - \sigma)w) - f'(w)]
\]

\[
\geq 0
\]

because \( w_x \leq 0 \). At \( x = 0 \), we have

\[
v_{1x} + ag(v_1) = ag((1 - \sigma)w) - a(1 - \sigma)g(w)
\]

\[
= a(1 - \sigma)w \left[ \frac{g((1 - \sigma)w)}{(1 - \sigma)w} - \frac{g(w)}{w} \right]
\]

\[
\geq 0
\]
because \( w \leq w(0) < w_0 \). Define

\[
\psi_1(x, t) = e^{\lambda x + \mu t} (v_1(x) - u(x, t)).
\]

Then

\[
\psi_t = \mu \psi + e(\lambda x + \mu t)[-u_{xx} - \varepsilon f'(u)u_x]
\leq \mu \psi + e(\lambda x + \mu t)[v_{1xx} - u_{xx} + \varepsilon f'(v_1)v_{1x} - \varepsilon f'(u)u_x]
= \psi_{xx} + [\varepsilon f'(u) - 2\lambda] \psi_x + [\lambda^2 + \mu + \varepsilon f''(\xi_1)v_{1x} - \lambda \varepsilon f'(u)] \psi,
\]

where \( \xi_1 \) is between \( v_1 \) and \( u \). At \( x = 0 \), we have

\[
\psi_x \geq \lambda + e^{\lambda x + \mu t}[-ag(v_1) + ag(u)]
= \lambda \psi - ag'(u_1) \psi
= [\lambda - ag'(u_1)] \psi,
\]

where \( u_1 \) is between \( v_1 \) and \( u \). If we choose \( \lambda \) such that in the second inequality the coefficient of \( \psi \) is positive and \( \mu \) such that in the first inequality the coefficient of \( \psi \) is negative, then the desired result follows.

**Theorem 1.7.** Let \( f' \) be strictly increasing on \([0, \infty)\), \( -\frac{g(u)}{u} \) be increasing on \([0, w_0]\) and \( g \) strictly increasing on \([0, 1]\). Suppose that the roots of (1.4) which are less than \( w_0 \) are isolated, and satisfy the conditions of Theorem 1.2. Then there is at most one positive stationary solution \( w(x) \) of (B) which is less than \( w_0 \) and it is stable.

**Proof:** The stability follows from Lemma 1.5 and the continuation theorems below. Let \( w_1, w_2 \) be positive stationary solutions such that \( w_1(x) < w_2(x) \) on \([0, 1]\), \( w_2(0) < w_0 \), and there are no roots of (1.4) in \((w_1(0), w_2(0))\). It follows from Lemma 1.2 that \( w'_i(x) < 0 \) on \([0, 1]\) for \( i = 1, 2 \). With \( q(x) = \frac{w'_1(x)}{w'_2(x)} \), we have

\[
q'(x) = \varepsilon [f'(w_2(x)) - f'(w_1(x))]q(x) > 0.
\]
From this, \( q(x) \) is increasing. Since \( w_i'(x) < 0 \) \((i = 1, 2)\) and \( g \) is strictly increasing,

\[
0 < \frac{w_1'(0)}{w_2'(0)} = \frac{g(w_1(0))}{g(w_2(0))} < 1
\]

Set \( r_0 = 1 - \frac{w_1'(0)}{w_2'(0)} \). Then \( 0 < r_0 < 1 \), and we claim that \( w_1(x) < (1 - r_0)w_2(x) \). To prove this, let \( k(x) = (1 - r_0)w_2(x) - w_1(x) \). Then \( k'(x) = (1 - r_0)w_2'(x) - w_1'(x) \). Since \( q(x) \) is increasing,

\[
1 - r_0 = \frac{w_1'(0)}{w_2'(0)} < \frac{w_1'(x)}{w_2'(x)},
\]

and since \( w_2'(x) < 0 \), we have \((1 - r_0)w_2'(x) > w_1'(x)\). So \( k'(x) > 0 \). Furthermore, we know that

\[
k(0) = (1 - r_0)w_2(0) - w_1(0)
= \frac{w_1'(0)}{w_2'(0)} w_2(0) - w_1(0) > 0
\]

because \( \frac{-g(w)}{u} \) is increasing on \([0, w_0]\) and \( w_2(0) < w_0 \). So, \( k(x) > 0 \), and hence the claim is established.

Let \( \delta > 0 \) such that \((1 - r_0)w_2(0) < (1 + \delta)w_1(0) < w_2(0)\). If we set

\[
v(x) = \left(\frac{1 + \delta}{1 - r_0}\right) w_1(x) - w_2(x)
= (1 + \delta)w_1(x) - w_2(x),
\]

then

\[
v(0) = \left(\frac{1 + \delta}{1 - r_0}\right) w_1(0) - w_2(0) > 0, v(1) = 0
\]

and

\[
v'(x) = (1 + \delta)w_1'(x) - w_2'(x).
\]
Also we have

\[ v''(x) = (1 + \delta)w''_1(x) - w''_2(x) \]

\[ = -\varepsilon(1 + \delta)f'(w_1)w_1' + \varepsilon f'(w_2)w_2' \]

\[ < -\varepsilon(1 + \delta)f'(w_2)w_1' + \varepsilon f'(w_2)w_2' \]

\[ = -\varepsilon f'(w_2)[(1 + \delta)w_1' - w_2'] \]

\[ = -\varepsilon f'(w_2)v'(x). \]

Consequently, \( v \) cannot have a minimum on \((0,1)\), and so \( v(x) > 0 \) on \((0,1)\). Therefore \((1 + \delta)w_1(x) > (1 - \tau_0)w_2(x)\).

If \( u \) solves (B) with \( u(x,0) = (1 + \delta)w_1(x) \), then \( u(x,t) > 0 \) except \( x = 1 \) on \( D_T \cup \Gamma_T \). To prove this, first observe that from (B) \( u(x,t) \) cannot have a negative minimum in \( D_T \). Furthermore if \( u(x,t) \) has a negative minimum at \( x = 0, t = t_0 \), then \( u_x(0,t_0) > 0 \). But then \( 0 > -u_x(0,t_0) = g(u(0,t_0)) > 0 \) which is a contradiction. Hence \( u(x,t) > 0 \) except \( x = 1 \).

By the Lemma 1.5, \((1 - \tau_0)w_2(x) < u(x,t)\) on \( D_T \cup \Gamma_T \) except \( x = 1 \). Since \((1 + \delta)w_1(0) < w_2(0) < w_0\), it also follows from Lemma 1.5 that \( u(x,t) \leq (1 + \delta)w_1(0) \). Therefore, by the continuation theorems, we may take \( T = +\infty \). However, by the above \( u(x,t) \leq u(x,0) \) on \([0,1]\) so that \( u_t \leq 0 \) for all \( t \) (the proof is the same as that of Lemma 1.3) and hence \( \lim_{t \to \infty} u(x,t) = \phi(x) \) exists for all \( x \in [0,1] \). Exactly as in Theorem 1.5, we easily establish that \( \phi \) is a stationary solution, and hence \( \phi(0) \) is a root of (1.4). Since

\[ w_1(0) < (1 - \tau_0)w_2(0) < \phi(0) < (1 + \delta)w_1(0) < w_2(0), \]

we have reached the desired contradiction.

EXAMPLE 1.2: If \( g(u) = \frac{1}{(1-u)\beta} \), then \( w_0 \leq \frac{1}{\beta+1} \).

The proof of Theorem 1.8 (below) is similar to that of Theorem 3.6B in Levine [11] and is omitted.
THEOREM 1.8. Let $f' > 0$ on $(0, \infty)$ and suppose that $w(x, \varepsilon)$ is a $C^1$ branch of stationary solutions of $(B)$ along which $ag(w_0) - \varepsilon f(w_0) > 0$. ($w_0(\varepsilon) \equiv w(0, \varepsilon)$). If $w'_0(\varepsilon) = \frac{\partial w(0, \varepsilon)}{\partial \varepsilon} > 0$, this is a branch of unstable stationary solutions. If $f'' > 0$ and $w'_0(\varepsilon) < 0$, this is a branch of stable stationary solutions.

1.2. Local Existence and Continuation

In this section we shall establish the existence of solutions of $(A)$, and $(B)$ on $D_T \cup \Gamma_T$ for sufficiently small $T$ and certain initial values. This result follows from results in [7]. However, we include an elementary proof here for completeness (see also [11]).

We assume that $f, g$ are defined on $R^1$ and $[0,1)$ respectively and $f(0) = 0$. We shall also assume that $f$ is uniformly Lipschitz in compact subsets of $R^1$, that $g$ is continuous and is uniformly Lipschitz on compact subsets of $[0,1)$. We shall also define, for $0 < M < 1$,

\begin{equation}
(1.5) \quad f_M \equiv \sup_{|u| \leq M} |f(u)|
\end{equation}

and

\begin{equation}
(1.6) \quad g_M \equiv \sup_{|u| \leq M} |g(u)|
\end{equation}

We shall discuss problem $(A)$. The arguments for $(B)$ are similar and are omitted.

Let $G(x, y; t)$ denote the Green's function for

\[ Lu = u_t - u_{xx} \quad 0 < x < 1, t > 0 \]

with boundary conditions

\[ u(0, t) = u_x(1, t) = 0, \quad t > 0. \]

i.e.

\[ G(x, y; t) = 2 \sum_{n=1}^{\infty} \sin(\lambda_n x)\sin(\lambda_n y)e^{\lambda_n^2 t} \]
where $\lambda_n = \frac{1}{2}(2n - 1)\pi$. Then $G_x(1, y; t) = G_y(x, 1; t) = G(0, y; t) = G(x, 0; t) = 0$ and $u$ is a solution of (A) on $[0, 1] \times [0, T)$ iff for $(x, t) \in [0, 1] \times [0, T)$,

\begin{equation}
(1.7)
\begin{aligned}
u(x, t) &= \frac{1}{1} \int_{0}^{1} G(x, y; t)u_0(y)dy - \int_{0}^{1} \int_{0}^{t} \epsilon G_y(x, y; t - \eta)f(u(y, \eta)) d\eta dy
+ \int_{0}^{t} G(x, 1; t - \eta)[\epsilon f(u(1, \eta)) + \alpha g(u(1, \eta))]d\eta
\equiv T(u(x, t)).
\end{aligned}
\end{equation}

In order to show that (1.7) is solvable for sufficiently small $T$, we use a contraction mapping argument. We define

\begin{equation}
(1.81)
u_1(x, t) = 0
\end{equation}

and then, recursively define

\begin{equation}
(1.82)
u_{n+1}(x, t) = T\nu_n(x, t).
\end{equation}

**Theorem 1.9.** Let the initial datum for problem (A) be continuous, 
\[\sup_{0 \leq x \leq 1} |u_0(x)| < 1\] and satisfy

\begin{equation}
(1.9)
0 < d_1 < \int_{0}^{1} G(1, y; t)u_0(y)dy
\end{equation}

for $0 \leq t \leq 1$, say. Then for sufficiently small $T$, (A) has a unique solution which satisfies

\begin{equation}
(1.10)
u(1, t) \geq \frac{d_1}{2}
\end{equation}
The solution is \( C^1 \) in \( t \) and \( C^2 \) in \( x \) on \( (0,1) \times (0,T) \) and continuous on \( \bar{D}_T \).

**Proof:** The proof is fairly standard. We shall only sketch the arguments. First, define

\[
d_2 = \sup_{0 \leq x \leq 1} |u_0(x)|
\]

\[
\mu(t) = \sup_{0 \leq x \leq 1} \sup_{0 \leq t' \leq t} \int_0^1 |G_y(x,1; t' - \eta)| d\eta
\]

\[
\nu(t) = \sup_{0 \leq x \leq 1} \sup_{0 \leq t' \leq t} \int_0^1 \int_0^1 |G_y(x,y; t' - \eta)| dy d\eta
\]

Clearly \( \mu(t) \to 0 \) monotonically as \( t \to 0^+ \). Inspection of the principal part of \( G \) shows us that the same is true for \( \nu(t) \). For fixed, \( 1 > M > d_2 \), choose \( T \) so small that

\[
(1.11) \quad \nu(T)f_M + \mu(T)(f_M + g_M) < \min(M - d_2, \frac{1}{d_1})
\]

\[
(1.12) \quad \beta_1 = \nu(T) \cdot \sup_{|\xi| \leq M} \varepsilon |f'(|\xi|)| < \frac{1}{2}
\]

\[
(1.13) \quad \beta_2 = \mu(T) \left( \varepsilon \sup_{|\xi| \leq M} |f'(\xi)| + \sup_{|\xi| \leq M} a|g'(\xi)| \right) < \frac{1}{2}
\]

Then \( \beta = \beta_1 + \beta_2 < 1 \).

It follows from (1.8)\(_1\), (1.8)\(_2\), (1.11) and induction on \( n \) that, on \( \bar{D}_T \)

\[
(1.14) \quad \|u_n\|_{\bar{D}_T} \equiv \sup_{\bar{D}_T} |u_n(x,t)| \leq M.
\]
for all \( n = 1, 2, \ldots \). Moreover, on \( x = 1 \), we have from (1.8)_2 and (1.11),

\[
\tag{1.15} u_n(1, t) \geq \frac{1}{2} d_1
\]

for \( n = 2, 3, \ldots \). A standard argument shows that if

\[
\tag{1.16} \gamma_n \equiv \|u_{n+1} - u_n\|_{\bar{D}_T},
\]

then

\[
\tag{1.17} \gamma_{n+1} \leq \beta^{n-1} \gamma.
\]

Therefore \( \{u_n\} \) is uniformly convergent on \( \bar{D}_T \) and

\[
\tag{1.18} u(x, t) = \lim_{n \to \infty} u_n(x, t)
\]

solves (1.7) with \( u(1, t) \geq \frac{1}{2} d_1 \).

The asserted interior regularity follows from the properties of \( G \) and the continuity of \( u \) in \( \bar{D}_T \). We omit the standard arguments.

A similar statement and argument holds for (B). This result allows us to establish a precise version of the statement. "If \( |u| < 1 \) on \( \bar{D}_T \) and \( u(1, T) > 0 \) and \( u \) is a classical solution of (A) on \( D_T \cup \Gamma_T \), then \( u \) may be continued as a classical solution on \( D_{T+\delta} \cup \Gamma_{T+\delta} \) for some \( \delta > 0 \), with \( u(1, t) > 0 \) on \( [T, T + \delta) \)."
2. STATIONARY SOLUTIONS AND DYNAMICAL RESULTS FOR \((A_1)\)

2.1. Stationary Solutions

We now consider stationary solutions for

\[
(A_1) \quad u_t = u_{xx} + \varepsilon \left( \frac{u^2}{2} \right) \quad \text{on} \quad (0,1) \times (0,T)
\]

\[
u(0,t) = 0 \quad \text{on} \quad (0,T)
\]

\[
u_x(1,t) = \frac{1}{(1 - u(1,t))^{\beta}} \quad \text{on} \quad (0,T)
\]

\[
u(x,0) = \nu_0(x) \quad \text{prescribed on} \quad [0,1]
\]

For problem \((A_1)\) solution of (1.2) is equivalent to solving

\[
L(w_1) = \int_0^{w_1} \frac{d\sigma}{2a \varepsilon (1 - w_1)^{-\beta} + w_1^2 - \sigma^2} = \frac{\varepsilon}{2}
\]

for \(w_1\). Let \(u = w_1, \lambda = \frac{2a}{\varepsilon}\) and \(D(u,\sigma) = \lambda (1-u)^{-\beta} + u^2 - \sigma^2\). Then \(D(u,0) \geq D(u,u) = \lambda (1-u)^{-\beta}\), and

\[
L(u) = \int_0^{u} \frac{d\sigma}{\lambda (1-u)^{-\beta} + u^2 - \sigma^2} = \frac{u}{\int_0^{u} \frac{d\sigma}{D(u,\sigma)}}.
\]

Also, we have

\[
L'(u) = \frac{1}{D(u,u)} - D'(u) \int_0^{u} \frac{d\sigma}{D^2(u,\sigma)}
\]

where \(D'(u) = \lambda \beta (1-u)^{-(\beta+1)} + 2u\). We assume that \(L'(u) = 0\) at \(u\). Then

\[
\frac{1}{D(u,u)} = D'(u) \int_0^{u} \frac{d\sigma}{D^2(u,\sigma)}.
\]
and

$$u''(u) = \frac{-1}{D^2(u,u)}[D_u + D_{\sigma}](u,u) - \frac{D'(u)}{D^2(u,u)} - D''(u) \int_0^u \frac{d\sigma}{D^2(u,\sigma)}$$

$$+ 2(D'(u))^2 \int_0^u \frac{d\sigma}{D^3(u,\sigma)}$$

$$\leq \frac{-1}{D^2(u,u)}[2D'(u) - 2u] - \frac{D''(u)}{D(u,u)D'(u)} + \frac{2D'(u)}{D^2(u,u)}$$

$$= \frac{2u}{D^2(u,u)} \frac{D''(u)}{D(u,u)D'(u)}.$$

Hence, we have

$$(2.4) \quad D(u,u)D^2(u,u)D'(u)L''(u) \leq 2D^2(u,0) - D''(u)D^2(u,u)$$

from $\frac{D^2(u,0)}{D(u,u)} \geq uD'(u)$.

$$(2.5) \quad 2D^2(u,0) - D''(u)D^2(u,u) = 2(\lambda(1-u)^{-\beta} + u^2)^2$$

$$- (\lambda(\beta+1)(1-u)^{-(\beta+2)} + 2)(\lambda(1-u)^{-\beta})^2.$$
then

\[
\int_0^1 \frac{w_0 \delta \sigma}{2a(1 - w_0)^{-\beta} + \varepsilon w_0^2 (1 - \sigma^2)} = \frac{1}{2}.
\]

We plot \( a(\varepsilon) \) below.

Figure 2.1. \( a(\varepsilon) \)

Hence we get Theorem 2.1.

**THEOREM 2.1.** For given \( \beta > 0 \), there exist \( a_1, \varepsilon(a, \beta) \) such that

(i) If \( \varepsilon > \varepsilon(a, \beta) \), there is no stationary solution.

(ii) If \( \varepsilon = \varepsilon(a, \beta) \), there is only one stationary solution.

(iii) If \( 0 < \varepsilon < \varepsilon(a, \beta) \), there are two stationary solutions for any \( a, 0 < a < a_1 \) and if \( a > a_1 \), there is no stationary solution for arbitrary \( \varepsilon > 0 \).
We plot $w(\varepsilon)$ below.

![Graph of $w(\varepsilon)$](image)

**Figure 2.2.** $w(\varepsilon)$

**Remark 2.1:** The numerical method indicates that $L'(u) = 0$ has exactly one zero.

### 2.2. Dynamical Results

We know that $w(x)$ is a stationary solution of (A) iff

$$w(x) = G(x, 1) \frac{a}{(1 - w(1))^\beta} + \varepsilon \int_0^1 G(x, y) \frac{d}{dy} [f(w(y))] dy$$

where

$$G(x, y) = \begin{cases} y, & 0 \leq y \leq x \leq 1 \\ x, & 0 \leq x \leq y \leq 1. \end{cases}$$

Suppose that $u(x, t)$ is a solution of (A) which does not quench. Then

$$F(x, t) = \int_0^1 G(x, y) u(y, t) dy$$

is bounded in $[0, 1] \times [0, \infty)$. And

$$F_t = \int_0^1 G(x, y) u_t(y, t) dy$$

$$= -u(x, t) + G(x, 1) \frac{a}{(1 - u(1, t))^\beta} + \varepsilon \int_0^1 G(x, y) [f(u(y, t))]' dy$$
If $u_t \geq 0$, then $F_t \geq 0$ and $u$ increases to a solution of (2.6).

**Lemma 2.1.** Let $u$ be a solution of (A) which does not quench and let $u(x,0) = w(x,\varepsilon_1,a_1)$ for $0 < \varepsilon < \varepsilon_0, 0 < a_1 < a$. (i.e. $w(x,\varepsilon_1,a_1)$ is a stationary solution of (A) with $\varepsilon = \varepsilon_1, a = a_1$). Then $u_t \geq 0$ for all $t$.

**Proof:** Let $v(x,t) = u_t(x,t)$ and $\psi(x,t) = e^{(\lambda x + \mu t)}v(x,t)$. Then

$$
\psi_t = \psi_{xx} + [-2\lambda + \varepsilon f'(u)]\psi_x + \lambda^2 + \mu - \epsilon f''(u)u_x - \lambda \epsilon f'(u)\psi,
$$

and at $x = 1$

$$
\psi_x(1,t) = \left(\lambda + \frac{a\beta}{(1 - u(1,t))\beta + 1}\right)\psi.
$$

We can choose $\lambda, \mu$ such that both coefficients are negative. Also we calculate that

$$
u_t(x,0) = w_{xx}(x,\varepsilon_1,a_1) + \varepsilon f'(w(x,\varepsilon_1,a_1))w_x(x,\varepsilon_1,a_1)
\geq w_{xx}(x,\varepsilon_1,a_1) + \varepsilon_1 f'(w(x,\varepsilon_1,a_1))w_x(x,\varepsilon_1,a_1) = 0.
$$

Hence $\psi$ has no negative minimum. So $u_t \geq 0$.

**Theorem 2.2.** Suppose that $\beta > 0$

i) If $\varepsilon > \varepsilon(a,\beta)$ for $0 \leq a < a_1$, or if $a > a_1$, then every solution of (A) with $u_0 \in C^1[0,1], u'_0(0) > 0$ and $0 < u_0 < 1$ quenches.

ii) If $\varepsilon = \varepsilon(a,\beta)$ for $0 < a < a_1$, we have the following:

(a) If $u_0(x) \leq w(x,\varepsilon(a,\beta))$, then $u$ is a global solution and $\lim_{t \to \infty} u(x,t) = w(x,\varepsilon(a,\beta))$.

(b) For every $\delta > 0$, there exists $u_0 \geq 0$ with $u_0 < 1$ and $\|u_0(\cdot) - w(\cdot,\varepsilon(a,\beta))\|_{L^\infty} < \delta$ such that $u$ quenches.

iii) If $0 < \varepsilon < \varepsilon(a,\beta)$ for $0 \leq a < a_1$, we have the following:
(a) If \( u_0(x) < w_+(x, \varepsilon) \) and \( u_0 \in C^1[0, 1] \), then \( u \) is global and \( \lim_{t \to \infty} u(x, t) = w_-(x, \varepsilon) \).

(b) For every \( \delta > 0 \), there is a \( u_0(<1) \) with \( \| u_0(\cdot) - w_+(\cdot, \varepsilon) \|_{L^\infty} < \delta \) such that \( u \) quenches.

PROOF: Since \( u_0(x) > 0, u'_0(0) > 0 \). So, \( u'_0(x) > s_1 > 0 \) for some \( s_1 > 0 \) and on \([0, \delta_1]\) for some \( \delta_1 > 0 \). Let \( s_2 \) be the minimum of \( u(x) \) on \([\delta_1, 1]\). Next, define

\[
\| w \| = \max\{ \| w \|_{L^\infty}, \| w_x \|_{L^\infty} \}.
\]

Then we know that \( \| w(\varepsilon, a) \| \to 0 \) as \( \varepsilon \to 0, a \to 0 \), and hence can find \( w(\varepsilon_1, a_1) \) such that

\[
\| w(\varepsilon_1, a_1) \| < \min \{ s_1, s_2 \}, \varepsilon_1 < \varepsilon, a_1 < a.
\]

Then \( u_0(x) > w(\varepsilon_1, a_1) \). To prove (i), suppose that \( u \) does not quench. Then \( T = +\infty \). Let \( v \) be a solution of \((A)\) with \( v(x,0) = w(\varepsilon_1, a_1) \). By comparison, \( u(x, t) \geq v(x, t) \), and so by Lemma 2.1, \( v_t \geq 0 \). Hence in \((2.7)\), \( v(x, t) \to w(x) \) as \( t \to \infty \), and \( w(x) \) is a stationary solution of \((A)\). But there are no such solutions.

Part (ii) is clear!

To prove (iii) (a), we observe that since \( 0 \leq u_0(x) < w_+(x, \varepsilon(a, \beta)) \), there is a number \( \sigma > 0 \) such that \( 0 \leq u_0(x) < w_+(x, \varepsilon(a, \beta) + \sigma) \). We let

\[
v_0(x) = w_+(x, \varepsilon(a, \beta) + \sigma).
\]

Then, for as long as both solutions exist,

\[
u(x, t; u_0) \leq u(x, t, v_0)
\]

Also we know that

\[
 u_t(x, 0, v_0) = v_{0xx}(x) + \varepsilon(a, \beta)[f(v_0)]' \\
 < v_{0xx} + (\varepsilon(a, \beta) + \sigma)[f(v_0)]' = 0.
\]
Hence \( u_t(x, 0, v_0) < 0 \) on \((0,1)\). Therefore, by standard arguments \( u_t(x, t; v_0) \leq 0 \), and, consequently, \( u(x, t; v_0) \leq v_0 \) and \( u(x, t; v_0) \) is global. Therefore so is \( u(x, t; u_0) \).

Also \( \lim_{t \to \infty} u(x, t; v_0) = \psi(x) \) exists. By an argument similar to that used to prove part (i), \( \psi(x) \) is a stationary solution. But then \( \psi(x) = w_-(x, \varepsilon) \). Also, by the argument in part (i), we can find \( \varepsilon_1 < \varepsilon, a_1 < a \) such that \( u(x, t; w(x, \varepsilon_1, a_1)) \) exists globally and

\[
\lim_{t \to \infty} u(x, t; w(x, \varepsilon_1, a_1)) = f'(x),
\]

which must also be a stationary solution. Since

\[
u(x, t; w(x, \varepsilon_1, a_1)) \leq u(x, t; u_0) \leq u(x, t, v_0),
\]

it follows that \( F(x) = w_-(x, \varepsilon) \) and

\[
\lim_{t \to \infty} u(x, t; u_0) = w_-(x, \varepsilon)
\]

pointwise as desired.

To prove (iii), (b), given \( \delta > 0 \), we choose \( \sigma > 0 \) so small that

\[
\|w_+(\cdot, \varepsilon - \sigma) - w_+(\cdot, \varepsilon)\| < \delta
\]

and \( w_+(1, \varepsilon - \sigma) > w_+(1, \varepsilon) \). We now set \( u_0(x) = w_+(x, \varepsilon - \sigma) \) and observe that for \( u(x, t; u_0) \), we have \( u_t(x, 0; u_0) > 0 \) on \((0,1)\). Again we find that \( u_t(x, t; u_0) > 0 \), whenever \( u_t \) exists. Thus if \( u \) does not quench, then \( T = +\infty \) and \( u(x, t) \to w(x) \), where \( w(x) \) is a solution of (2.6) which must be a stationary solution of \( A \) with \( w(1) > w_+(1, \varepsilon) \). However, this is not possible. So \( u \) must quench.

**THEOREM 2.3.** Let \( u_0'(x) \geq 0 \) on \([0,1]\) and \( u \in C^1([0,1] \times [0, T)) \) solve (A) with \( u(x, 0) = u_0(x) \), and suppose \( u \) quenches at time \( T, u \geq 0 \). Then quenching occurs at \( x = 1, \) and \( u_t(1, t) \) blows up as \( t \) approaches \( T \) from below if \( u \in C([0,1] \times [0, T]) \) (or \( u(x, t) \geq u_0(x) \)).

**PROOF:** First, we want to prove \( u_x \geq 0 \). Let

\[
v(x, t) = u_x(x, t).
\]

Then

\[
v_t = v_{xx} + \varepsilon f''(u)u_x v + \varepsilon f'(u)v_x.
\]

For

\[
w = e^{\lambda t} v(x, t)
\]
we calculate that

\[ w_t = \lambda e^{\lambda t} v(x, t) + e^{\lambda t} v_0(x, t) \]
\[ w_x = e^{\lambda t} v_x, w_{xx} = e^{\lambda t} v_{xx}. \]

Hence

\[ w_t = w_{xx} + [\lambda + \varepsilon f''(u)v]w + \varepsilon f'(u)w_x. \]

We can choose \( \lambda \) such that

\[ \lambda + \varepsilon f''(u)v < 0. \]

Suppose \( v \) has a negative minimum in the domain. Then \( w \) has also a negative minimum in the domain, and

\[ w_t = 0, w_x = 0, w_{xx} \geq 0, w < 0. \]

Hence

\[ 0 = w_{xx} + [\lambda + \varepsilon f''(u)v]w > 0 \]

at that point which cannot happen. Hence \( w \geq 0 \), and so \( v \geq 0 \). Therefore \( u_x \geq 0 \), and hence \( u \) quenches at \( x = 1 \). Next, we can find \( u \) from the representation formula
as follows; For $t < T$,

(2.8) \[
    u(x,t) = \int \int G(x,\xi,t-\tau)f'(u(\xi,\tau))u_\xi(\xi,\tau)d\tau d\xi \\
    + \int G(x,1,t-\tau)\varphi(u(1,\tau))d\tau \\
    + \int u_0(\xi)G(x,\xi,t)d\xi
\]

where $G(x,\xi,t) = 2 \sum_{n=0}^{\infty} \sin \lambda_n x \sin \lambda_n \xi \exp(-\lambda_n t), \lambda_n = \frac{1}{2}(2n + 1)\pi$

and $\varphi(u) = \frac{a}{(1-u)^\beta}$.

After differentiating and integrating by parts we find that for $0 < t < T$,

(2.9) \[
    u_t(x,t) = \int \int G_t(x,\xi,t-\tau)f'(u(\xi,\tau))u_\xi(\xi,\tau)d\tau d\xi \\
    + \int u_0(\xi)G_t(x,\xi,t)d\xi \\
    + \int \frac{\partial}{\partial \tau}[G(x,1 : t-\tau) \cdot \varphi(u(1,\tau))] d\tau,
\]
and so

\[
u_t(x,t) = \int \int G_t(x,\xi,t-\tau)f'(u(\xi,\tau))u_\xi(\xi,\tau)d\tau d\xi
\]

\[
+ \int u_0(\xi)G_t(x,\xi,t)d\xi
\]

\[
- \int \frac{\partial}{\partial r}[G(x,1;t-\tau)]\varphi(u(1,0))d\tau
\]

\[
= \int \int G_t(x,\xi,t-\tau)f'(u(\xi,\tau))u_\xi(\xi,\tau)d\tau d\xi
\]

\[
+ \int u_0(\xi)G_t(x,\xi,t)d\xi
\]

\[
+ G(x,1;t)\varphi(u(1,0)) + \int G(x,1,t-\tau)\varphi'(u(1,\tau))u_\tau(1,\tau)d\tau.
\]

Since \( u \in (([0,1] \times [0,T])(or \ u(x,t) \geq u_0(x)) \), there exists a \( \delta > 0 \) such that

\[
u(1,t) \geq u(1,0) \text{ for } T - \delta \leq t \leq T.
\]

Hence \( u_t(1,t) \geq c_1 + c_2[\varphi(u(1,t)) - \varphi(u(1,0))] \).
where

\[ c_1 = \inf_{T-\delta \leq t \leq T} \left\{ \int_0^1 \int_0^t \frac{1}{T-\delta \leq t \leq T} \left( \int_0^1 \int_0^t G_t(1, \xi, t-\tau) f'(u(\xi, \tau)) u_\xi(\xi, \tau) d\tau d\xi \right. \right. \\
\left. + \int_0^1 u_0(\xi) G_t(1, \xi, t) d\xi \right. \right. \\
+ G(x, 1 : t) \varphi(u(1, 0)) \right\} \]

and

\[ c_2 = \inf_{0 \leq \tau \leq t} \{ G(1, 1, t - \tau) \} \]

\[ = 2 \sum_{n=0}^{\infty} \sin \lambda_n \sin \lambda_n \exp(-\lambda_n(t - \tau)) \]

\[ = 2 \sum_{n=0}^{\infty} \exp(-\lambda_n(t - \tau)) > 0. \]

Therefore it follows that

\[ \lim_{t \to T} u_t(1, t) = \infty. \]
3. STATIONARY SOLUTIONS AND DYNAMICAL RESULTS FOR \((B_1)\)

3.1. Stationary Solutions

Here we consider stationary solutions for

\[(B_1) \quad u_t = u_{xx} + \varepsilon \left(\frac{u^2}{2}\right)_x \quad \text{on} \quad (0,1) \times (0,T)\]

\[u(1,t) = 0 \quad \text{on} \quad (0,T)\]

\[-u_x(0,t) = \frac{a}{(1 - u(0,t))^\beta} \quad \text{on} \quad (0,T)\]

\[u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1].\]

For problem \((B_1)\) solutions of (1.4) is equivalent to solving

\[F(w_0) = \int_0^w \frac{d\sigma}{2a \varepsilon (1-w_0)^{-\beta} - w_0^2 + \sigma^2} = \frac{\varepsilon}{2}\]

for \(w_0\). Let \(D(w_0, \sigma) = \lambda (1-w_0)^{-\beta} - w_0^2 + \sigma^2\) where \(\lambda = \frac{2a}{\varepsilon}\).

Then \(D(w_0, 0) \leq D(w_0, \sigma) \leq D(w_0, w_0)\). Let \(L(u) = \int_0^u \frac{d\sigma}{D(u, \sigma)}\). Then we have

\[L'(u) = \frac{1}{D(u,u)} - \int_0^u \frac{D_u(u, \sigma)}{D^2(u, \sigma)} d\sigma\]

\[= \frac{1}{D(u,u)} - D'(u) \int_0^u \frac{d\sigma}{D^2(u, \sigma)},\]

where \(D'(u) = \lambda \beta (1-u)^{-(\beta+1)} - 2u\). Assume that \(L'(u) = 0\). Then

\[\frac{1}{D(u,u)} = D'(u) \int_0^u \frac{d\sigma}{D^2(u, \sigma)}.\]
Next, we calculate $L''(u)$.

\[
L''(u) = \frac{-1}{D^2(u, u)} [D_u + D_\sigma](u, u) - \frac{D'(u)}{D(u, u)} - D''(u) \int_0^u \frac{d\sigma}{D^2(u, \sigma)} \\
+ 2 \left( \frac{D'(u)}{D(u, u)} \right)^2 \int_0^u \frac{d\sigma}{D^3(u, \sigma)}
\]

\[
= \frac{-1}{D^2(u, u)} [2D'(u) + 2u] - \frac{D''(u)}{D(u, u)D'(u)} + 2 \left( \frac{D'(u)}{D(u, u)} \right)^2 \int_0^u \frac{d\sigma}{D^3(u, \sigma)}
\]

\[
\leq \frac{-1}{D^2(u, u)} [2D'(u) + 2u] - \frac{D''(u)}{D(u, u)D'(u)} + \frac{2}{D(u, 0)} \cdot \frac{D'(u)}{D(u, u)}
\]

\[
= \frac{-1}{D^2(u, u)} \left[ 2D'(u) + 2u + \frac{D''(u)D(u, u)}{D'(u)} - \frac{2D'(u)D(u, u)}{D(u, 0)} \right].
\]

Hence

\[
L''(u)D^2(u, u)(D'(u))^2D(u, 0)
\]

\[
\leq -D'(u) \left[ (2D'(u) + 2u)D'(u)D(u, 0) + D''(u)D(u, u)D(u, 0) \right]
\]

\[
-2 \left\{ D'(u) \right\}^2 D(u, u)]
\]

\[
= D'(u) \left\{ 2D'(u)^2[\lambda(1 - u)^{-\beta} - \lambda(1 - u)^{-\beta} + u^2] - 2uD'(u)D(u, 0) \right. \\
- D''(u)D(u, u)D(u, 0) \right\}
\]

\[
= D'(u) \left\{ 2u^2 \left( D'(u) \right)^2 - 2u[\lambda \beta(1 - u)^{-(\beta + 1)} - 2u]\left( \lambda(1 - u)^{-\beta} - u^2 \right) \\
- \left[ \lambda \beta(1 - u)^{-(\beta + 2)} - 2\cdot \lambda(1 - u)^{-\beta}\left( \lambda(1 - u)^{-\beta} - u^2 \right) \right] \right\}
\]

\[
= \{ \lambda \beta(1 - u)^{-(\beta + 1)} - 2u \} \left\{ 2u^2(\lambda \beta(1 - u)^{-(\beta + 1)} - 2u) \right\}
\]
\[-2u \cdot [\lambda \beta (1 - u)^{(\beta + 1)} - 2u] \\
\cdot [\lambda (1 - u)^{-\beta} - u^2] - [\lambda \beta (\beta + 1)(1 - u)^{-(\beta + 2)} - 2] \\
\cdot \lambda (1 - u)^{-\beta} [\lambda (1 - u)^{-\beta} - u^2] \}\]

\[\to -\infty \text{ as } \lambda \to \infty \text{ on } [0, \frac{2}{\beta + 2}]\]

because the coefficient of \(\lambda^3\) is negative. So, we can find \(\lambda_1(\beta)\) such that \(L''(u) < 0\) if \(\lambda > \lambda_1(\beta)\). We need another condition from \(ag(u) - \epsilon f(u) > 0\),

(3.2) \(\lambda (1 - u)^{-\beta} - u^2 > 0\)

which is equivalent to

(3.3) \(\lambda > (1 - u)^{\beta} u^2 \equiv G_1(u)\).

Since \(G_1'(u) = (1 - u)^{\beta - 1} u [2 - (\beta + 2)u]\), \(G_1\) has maximum at \(u = \frac{2}{\beta + 2}\). Let

\(\lambda_c \equiv G_1 \left( \frac{2}{\beta + 2} \right)\).

CASE 1. \(\lambda_1 \geq \lambda_c\):

For \(\lambda > \lambda_1\), \(L'(u) = 0\) has exactly one zero. i.e. there is only one \(u_0\) such that

\[L(u_0) = \int_0^1 \frac{u_0 d\sigma}{\lambda (1 - u_0)^{-\beta} - u_0^2 (1 - \sigma^2)} = \frac{1}{2} \epsilon\]

for some \(\epsilon\). For such \(u_0\) we have

\[\int_0^1 \frac{u_0 d\sigma}{2a(1 - u_0)^{-\beta} - \epsilon u_0^2 (1 - \sigma^2)} = \frac{1}{2} \cdot\]

We plot \(a(\epsilon)\) below.
Thus we get Theorem 3.1.

**Theorem 3.1.** For \( \lambda > \lambda_1 \) and for given \( \beta > 0 \), there exist \( a_0(\beta), a_1(\beta), \varepsilon(a, \beta) \) such that

(i) If \( \varepsilon > \varepsilon(a, \beta) \), there is no stationary solution.

(ii) If \( \varepsilon = \varepsilon(a, \beta) \), there is only one stationary solution.

(iii) If \( 0 < \varepsilon < \varepsilon(a, \beta) \), there are two stationary solutions for any \( a, a_0 \leq a < a_1 \) and if \( a \geq a_1 \), there is no stationary solution for arbitrary \( \varepsilon > 0 \).

We plot \( w_0(\varepsilon) \) below.
CASE 2. $\lambda_c > \lambda_1$.

If $\lambda > \lambda_c$, then this is the same as case 1. So, we can consider $\lambda_1 < \lambda \leq \lambda_c$. Then there exist $\alpha_1, \alpha_2$ such that $0 < \alpha_1 < \frac{2}{\beta+2} < \alpha_2 < 1$ and $G_1(\alpha_1) = \pi_1(\alpha_2) = \lambda$. We plot $G_1(u)$ below.

![Graph of $G_1(u)$](image)

**Figure 3.3.** $G_1(u)$

First we consider the case $L(\alpha_2) > L(\alpha_1)$.

**Theorem 3.2.**

(i) If $0 < \varepsilon < 2L(\alpha_1)$, then there are two stationary solutions.

(ii) If $2L(\alpha_1) < \varepsilon < 2L(\alpha_2)$, then there is only one stationary solution.

(iii) If $\varepsilon > 2L(\alpha_2)$, then there are no stationary solutions.

We plot $w_0(u)$ below.

![Graph of $w_0(u)$](image)

**Figure 3.4.** $w_0(\varepsilon)$
Next, we consider the case when $L(\alpha_1) > L(\alpha_2)$ for which we get similar results. We plot $w_0(\varepsilon)$ below.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.5.png}
\caption{$w_0(\varepsilon)$}
\end{figure}

REMARK 3.1: The numerical method shows $\lambda_1 \equiv 0$. (See Appendix I.)

3.2. Dynamical Results

We know that $w(x)$ is a stationary solution of $(B_1)$ iff

\begin{equation}
(3.4) \quad w(x) = -G(0, x) \frac{a}{(1 - w(0))^{\beta}} + \epsilon \int_0^1 G(\xi, x) \frac{d}{d\xi} [f(w(\xi))] d\xi,
\end{equation}

where

\[
G(\xi, x) = \begin{cases} 
1 - x & 0 \leq \xi < x \leq 1 \\
1 - \xi & 0 \leq x < \xi \leq 1
\end{cases}
\]

Suppose that $u(x, t)$ is a solution of $(B_1)$ which does not quench. Then

\[
F(x, t) = \int_0^t G(\xi, x) u(\xi, t) d\xi
\]
is bounded in \([0, 1] \times [0, \infty)\), and

\[
F_t(x,t) = \int_0^t G(\xi, x) u_t(\xi, t) d\xi \\
= \int_0^1 G(\xi, x) [u_{\xi\xi} + \varepsilon f(u(\xi, t))] d\xi \\
= \int_0^1 G(\xi, x) u_{\xi\xi} + \varepsilon \int_0^1 G(\xi, x) [f(u(\xi, t))] d\xi \\
= -G(0, x) \frac{a}{(1-u(0,t))\beta} + \varepsilon \int_0^1 G(\xi, x) [f(u(\xi, t))] d\xi \\
- u(x, t).
\]

If \(u_t \geq 0\), then \(F_t \geq 0\) and \(u\) increases to a solution of (3.4).

**Lemma 3.1.** Let \(u\) be a solution of (\(B_1\)) which does not quench, and let \(u(x, 0) = w(x, \varepsilon_1)\) for \(\varepsilon < \varepsilon_1\). Then \(u_t \leq 0\) for all \(t\).

**Proof:** We let \(v(x, t) = u_t(x, t)\) and \(\psi(x, t) = e^{(\lambda x + \mu t)} v(x, t)\). Then we find that

\[
\psi = \psi_{xx} + [-2\lambda + \varepsilon f'(u)] \psi_x + \\
\left[\lambda^2 + \mu - \varepsilon f''(u) u_x - \lambda \varepsilon f'(u)\right] \psi,
\]

and at \(x = 0\)

\[
\psi_x(0, t) = (\lambda - \frac{a\beta}{(1-u(0,t))\beta+1}) \psi.
\]

We now choose \(\lambda\) such that the coefficient of the second equation is positive and \(\mu\) such that the coefficient of \(\psi\) of the first equation is negative. Also we calculate

\[
\begin{align*}
\psi_x(0, t) &= w_{xx}(x, \varepsilon_1) + \varepsilon f'(w(x, \varepsilon_1)) w_x(x, \varepsilon_1) \\
&\geq w_{xx}(x, \varepsilon_1) + \varepsilon_1 f'(w(x, \varepsilon_1)) w_x(x, \varepsilon_1) = 0.
\end{align*}
\]
Hence \( \psi \) has no negative minimum, and \( u_t \geq 0 \). The proof of Lemma 3.2 is similar to that of Lemma 3.1; we omit it here.

**Lemma 3.2.** Let \( u \) be a solution of (B1) which does not quench, and let \( u(x,0) = w(x,\varepsilon_1) \) for \( 0 < \varepsilon_1 < \varepsilon \). Then \( u_t \leq 0 \) for all \( t \).

**Theorem 3.3.** For \( \lambda > \lambda_1, \beta > 0 \),

i) If \( \varepsilon > \varepsilon(a,\beta) \), then for every solution of (B1) with \( 0 \leq u_0 < 1, u(0,t) > 0 \) quenches.

ii) If \( \varepsilon = \varepsilon(a,\beta) \), we have the following:
   
   (a) If \( u_0(x) > w(x,\varepsilon(a,\beta)) \), then \( u \) is a global solution and \( \lim_{t \to \infty} u(x,t) = w(x,\varepsilon(a,\beta)) \).

   (b) For every \( \delta > 0 \), there exists \( u_0 \geq 0 \) with \( u_0 < 1 \) and
   
   \[ \|u_0(\cdot) - w(\cdot,\varepsilon(a,\beta))\|_{L^\infty} < \delta \]
   
   such that \( u \) quenches.

iii) If \( 0 < \varepsilon < \varepsilon(a,\beta)a_0 \leq a < a_1 \), we have the following:

   (a) If \( w(x) < u_0(x) < w_+(x,\varepsilon) \) and \( u_0 \in C^1[0,1] \), then \( u \) is global and
   
   \[ \lim_{t \to +\infty} u(x,t) = w_+(x) \]. Furthermore, if \( 0 < u_0(x) < w_-(x,\varepsilon) \) and
   
   \[ u_0''(x) + \varepsilon f'(u_0)u_0' \geq 0 \], then \( u \) is global and \( \lim_{t \to \infty} = w_-(x,\varepsilon) \).

   (b) If \( a > a_1 \), then every solution of (B1) with \( 0 < u_0(x) < 1 \) quenches.

**Proof:**

(i) We suppose that \( u(x,t) \) does not quench. Then we can find \( w_+(x,\varepsilon_1) \) such that \( 1 > w_+(x,\varepsilon_1) > u_0(x) \) and \( \varepsilon_1 < \varepsilon \). Let \( v(x,t) \) be a solution of (B1) with \( v(x,0) = w_+(x,\varepsilon_1) \). By the comparison theorem, \( v(x,t) \geq u(x,t) \), and by Lemma 3.2, \( v_t \leq 0 \). Hence \( 0 < \lim_{t \to \infty} u(x,t) \leq \lim_{t \to \infty} v(x,t) = 0 \), because \( v \) has no stationary solution.

But this implies \( u(x,t) \to 0 \), and so \( u_x(0,t) = \frac{-a}{(1-u(0,t))^\beta} \to 0 \) which cannot happen. Thus, it must be that \( u(x,t) \) quenches.
(ii) Clear!

(iii) (a) We can choose \( w_-(x, \varepsilon) \) such that \( w_-(x, \varepsilon) < w_0(x) < w_+(x, \varepsilon) \), for \( \varepsilon < \varepsilon_1 \).

Let \( v(x, t) \) be a solution of \((B_1)\) with \( v(x, 0) = w_-(x, \varepsilon_1) \). Then, by Lemma 3.1, \( v_t \geq 0 \), and by the comparison theorem, \( v(x, t) \leq u(x, t) \leq w_+(x, \varepsilon) \). Moreover, \( v(x, t) \to \phi(x) \), and \( \phi(x) \) is a stationary solution. So \( \phi(x) = w_+(x, \varepsilon) \), and thus \( u(x; t) \to w_+(x, \varepsilon) \) as \( t \to \infty \).

Since \( u(x, t) \leq w_-(x, \varepsilon) \), \( u \) is global. We know that \( u_t \geq 0 \). Thus \( u(x, t) \to w_-(x, \varepsilon) \).

(b) is similar to (i), so we omit it here.

The proofs of the following theorems are similar to that of Theorem 3.1 and so are omitted. Here we assume \( \lambda_c > \lambda_1 \).

**Theorem 3.4.** Assume \( L(\alpha_2) > L(\alpha_1) \).

(i) For \( 0 < \varepsilon < 2L(\alpha_1) \), we have the following:

(a) If \( w_-(x, \varepsilon) < w_0(x) < w_+(x, \varepsilon) \) and \( u_0 \in C^1[0,1] \), then \( u \) is global and \( \lim_{t \to \infty} u(x, t) = w_+(x, \varepsilon) \).

(b) If \( 0 < u_0(x) < w_-(x, \varepsilon) \), \( u_0''(x) + \varepsilon f'(u_0)u_0'(x) \geq 0 \), then \( u \) is global and \( \lim_{t \to \infty} u(x, t) = w_-(x, \varepsilon) \).

(ii) For \( 2L(\alpha_1) \leq \varepsilon < 2L(\alpha_2) \), we have the following: If \( 0 < u_0(x) < w_+(x, \varepsilon) \) and \( u_0 \in C^1[0,1] \), then \( u \) is global and \( \lim_{t \to \infty} u(x, t) = w_+(x, \varepsilon) \).

(iii) If \( \varepsilon > 2L(\alpha_2) \), then every solution of \((B_1)\) with \( 0 < u_0 < 1 \) and \( u(0,t) > 0 \) quenches.

**Theorem 3.5.** Assume \( L(\alpha_1) > L(\alpha_2) \).

(i) For \( 0 < \varepsilon < 2L(\alpha_2) \), we have the following:

(a) If \( w_-(x, \varepsilon) < w_0(x) < w_+(x, \varepsilon) \) and \( u_0 \in C^1[0,1] \), then \( u \) is global and \( \lim_{t \to \infty} u(x, t) = w_+(x, \varepsilon) \).
(b) If $0 < u_0(x) < \omega(x, \varepsilon)$, $u_0''(x) + \varepsilon f'(u_0)u_0'(x) \geq 0$, then $u$ is global and
\[ \lim_{t \to \infty} u(x, t) = \omega(x, \varepsilon). \]

(ii) For $wL(\alpha_2) \leq \varepsilon < 2L(\alpha_1)$, we have the following:

(a) If $\omega(x) < u_0(x)$, then every solution of $(B_1)$ quenches.

(b) If $0 < u_0(x) < \omega_2(x)$, $u_0''(x) + \varepsilon f'(u_0)u_0'(x) \geq 0$, then $u$ is global and
\[ \lim_{t \to \infty} u(x, t) = \omega(x, \varepsilon). \]
4. STATIONARY SOLUTIONS AND DYNAMICAL RESULTS FOR \((C_1)\)

4.1. Stationary Solutions

Here we consider stationary solutions for

\[
\begin{align*}
(C_1) & \quad u_t = u_{xx} + \varepsilon \left[ \frac{u^2}{2} \right]_x \quad \text{on} \quad (0, 1) \times (0, T) \\
& \quad u(0, t) = 0 \quad \text{on} \quad (0, T) \\
& \quad u_x(1, t) = \frac{a}{w^p(1, t)} \quad \text{on} \quad (0, T) \\
& \quad u(x, 0) = u_0(x) \quad \text{prescribed on} \quad [0, 1].
\end{align*}
\]

In this case, (1.2) is equivalent to

\[
(4.1) \quad \int_0^1 \frac{dy}{(2a/\varepsilon)w_1^{-(p+2)} + 1 - y^2} = w_1\varepsilon/2.
\]

Let \(v = (2a/\varepsilon)w_1^{-(p+2)}\), \(\delta = \varepsilon/2(\varepsilon/2a)^{-1/(p+2)}\). Then

\[
(4.2) \quad \int_0^1 \frac{dy}{v + 1 - y^2} = \delta v^{1/(p+2)},
\]

and so

\[
(4.3) \quad v^{1/(p+2)} \int_0^1 \frac{dy}{v + 1 - y^2} = \delta.
\]

Now define \(\alpha = (v + 1)^{\frac{1}{2}}\), and \(q = 1/(p + 2)(0 < q < \frac{1}{2})\), and let

\[
(4.4) \quad \phi(\alpha) = \frac{1}{2} (\alpha^2 - 1)^q \alpha^{-1} \ln((\alpha + 1)/(\alpha - 1)) = \delta
\]
where \( \phi(\alpha) \) is defined on \((1, \infty)\). We want to find the number of positive solutions of \((4.4)\). It is easily seen that

\[
\phi'(\alpha) = \frac{1}{2} (\alpha^2 - 1)^q - 1 \alpha^2 k(\alpha)
\]

where

\[
k(\alpha) = \left[ (2q - 1) \alpha^2 + 1 \right] \ln((\alpha + 1)/(\alpha - 1)) - 2\alpha.
\]

We have

\[
\frac{1}{2} k'(\alpha) = (2q - 1) \alpha \ln((\alpha + 1)/(\alpha - 1)) - 2q \alpha^2 (\alpha^2 - 1)^{-1},
\]

since \(0 < q < \frac{1}{2}, 2q - 1 < 0\), and hence \(k'(\alpha) < 0\). However

\[
\lim_{\alpha \to 1^+} k(\alpha) = +\infty,
\]

while \(k(\alpha) \approx 4(q - 1)\alpha(\alpha < 0)\) as \(\alpha \to \infty\). Therefore \(k\) has exactly one sign change and first increases and then decreases on \((1, \infty)\). We note also that

\[
\lim_{\alpha \to 1^+} \phi(\alpha) = 0
\]

and

\[
\phi(\alpha) = \frac{1}{2} \alpha^{2q} / \alpha \ln((\alpha + 1)/(\alpha - 1))
\]

as \(\alpha \to \infty\), so

\[
\lim_{\alpha \to \infty} \phi(\alpha) = 0.
\]

Thus the equation \((4.4)\) has zero, one or two solutions accordingly as

\[
\epsilon > [2\phi_0]^{(p+2)}/(p+1)(2a)^{-1}/(p+1),
\]

\[
\epsilon = [2\phi_0]^{(p+2)}/(p+1)(2a)^{-1}/(p+1),
\]

or

\[
\epsilon < [2\phi_0]^{(p+2)}/(p+1)(2a)^{-1}/(p+1),
\]
where
\[ \phi_0 = \max_{1<\alpha<\infty} \phi(\alpha). \]

Therefore, we get the following:

**Theorem 4.1.** Given \( a, p > 0 \), there is \( \varepsilon(a, p) > 0 \) such that

(i) If \( \varepsilon > \varepsilon(a, p) \), then there is no stationary solution,

(ii) If \( \varepsilon = \varepsilon(a, p) \), then there is only one stationary solution,

(iii) If \( 0 < \varepsilon < \varepsilon(a, p) \), then there are two stationary solutions, say, \( w_+(x, \varepsilon) \) and \( w_-(x, \varepsilon) \).

**Remark 4.1:** In above example, there is at most one \( w(x) \) stationary solution of \( (C_1) \) such that \( w(1) \in (0, (ap/\varepsilon)^{1/(p+2)}) \). To prove this, suppose that there are two stationary solutions of \( (C_1) \), which we will call \( w_1(x) \) and \( w_2(x) \). Then \( w_1(1) < w_2(1) \) and \( w_1(1), w_2(1) \in (0, (ap/\varepsilon)^{1/(p+2)}) \). We know that \( f(u) + g(u) \) is decreasing in this interval because

\[ [f(u) + g(u)]' = 1/u^{p+1} [\varepsilon u^{p+2} - ap]. \]

From the conservation laws, we see that for any \( x \) in \((0,1)\)

\[ w_1(x)(0) = g(w_1(1)) + f(w_1(1)) = w_1(x) + f(w_1(x)) \]

and

\[ w_2(x)(0) = g(w_2(1)) + f(w_2(1)) = w_2(x) + f(w_2(x)). \]

Since \( g + f \) is decreasing, we find

\[ w_1(x)(0) > w_2(x)(0), \]

and, since \( w_1(0) = w_2(0) = 0 \),

\[ w_1(x) > w_2(x) \]
in a neighborhood of $x = 0$. Because $w_2(1) > w_1(1)$, we can find $x_-$ in $(0, 1)$ which is the first point in $(0, 1)$ such that $w_1(x_-) = w_2(x_-)$. We see from the above that

$$w_{1x}(x_-) > w_{2x}(x_-),$$

and this inequality holds in a left open neighborhood of $x_-$, say $[x_- - \delta, x_-]$. But then

$$0 = w_1(x_-) - w_2(x_-) = \int_{x_- - \delta}^{x_-} [w_{1x}(x) - w_{2x}(x)] dx + [w_1(x_- - \delta) - w_2(x_-)] > 0,$$

which is a contradiction.

4.2. Dynamical Results

We know that $w(x)$ is a stationary solution of $(C_1)$ iff

$$w(x) = G(x, 1)a/[w(1)]^P + \int_0^1 G(x, y)d/dy[f(w(y))]dy,$$

where

$$G(x, y) = \begin{cases} y, & 0 \leq y \leq x \leq 1 \\ x, & 0 \leq x \leq y \leq 1. \end{cases}$$

Suppose that $u(x, t)$ is a solution of $(C_1)$ which does not blow up (i.e., which is bounded). Then

$$F(x, t) = \int_0^1 G(x, y)u(y, t)dy$$

is bounded in $[0, 1] \times [0, \infty)$ and

$$F_t(x, t) = \int_0^1 G(x, y)u_t(y, t)dy$$

$$= -u(x, t) + G(x, 1)a/[u(1, t)]^P + \int_0^1 G(x, y)[f(y, t)]'dy.$$
If $u_t \geq 0$, then $F_t \geq 0$ and $u$ increases to a solution of (4.6).

**Lemma 4.1.** Let $u$ be a solution of $(C_1)$ with $u(x,0) = w(x, \epsilon_1, a_1)$ for $0 < \epsilon_1 < \epsilon$ and $0 < a_1 < a$. Then $u_t \geq 0$ for all $t$.

**Proof:** Let $u_t(x,t) = v$. Then $v(0,t) = 0$, and

$$u_t(x,0) = w_{xx}(x, \epsilon_1, a_1) + \epsilon f'(w(x, \epsilon_1, a_1))w_x(x \epsilon_1, a_1)$$

$$\geq w_{xx}(x, \epsilon_1, a_1) + \epsilon_1 f'(w(x, \epsilon_1, a_1))w_x(x \epsilon_1, a_1)$$

$$= 0.$$

Next define

$$\Phi(x,t) = \exp(\tau x + \mu t)v(x,t).$$

We then have that

$$\Phi_t = \Phi_{xx} + [-2\tau + \epsilon f'(u)]\Phi_x + [\tau^2 + \mu - \epsilon''(u)u_x - \tau \mu f'(u)]\Phi,$$

and at $x = 1$,

$$\Phi_x(1,t) = \{\tau - ap/\{u(1,t)^{p+1}\}\} \Phi.$$

Let $\tau$ and $\mu$ be chosen such that each coefficient of $\Phi$ are negative. Then it follows that $\Phi$ has no negative minimum. Therefore, $u_t \geq 0$.

**Lemma 4.2.** Let $u$ be a solution of $(C_1)$ with $u(x,0) = u_0(x)$. We assume that $u_0''(x) + \epsilon u_0(x)u_0'(x) \geq 0$ on $[0,1]$, and we assume the corner compatibility conditions. Then $u_t \geq 0$ for all $t$.

**Proof:** Let $u_t(x,t) = v$. Then $v(0,t) = 0$ for all $t > 0$, and

$$u_t(x,0) = u_0''(x) + \epsilon u_0(x)u_0'(x) \geq 0 \text{ on } [0,1].$$

By the maximum principle, we have that $u_t \geq 0$. 
The proof of Lemma 4.3 is similar to that of Lemma 4.2, and so we omit it.

**Lemma 4.3.** Let $u$ be a solution of $(C_1)$ with $u(x,0) = u_0(x)$. Assume that $u_{xx}(1,t) \leq 0$ for all $t > 0$ and $u'_0(x) \leq 0$ on $[0,1]$. Then $u_{xx} \leq 0$ on $[0,1] \times [0,T)$.

**Theorem 4.2.** If $\varepsilon > \varepsilon(a,p)$, then every solution of $(C_1)$ with $u'_0(0) > 0$ and $u_0$ in $C^1[0,1]$ blows up in finite or infinite time.

**Proof:** We suppose that a solution $u(x,t)$ of $(C_1)$ with above conditions does not blow up. Then it is bounded. Since $u'_0(0) > 0$, it follows that $u'_0(x) > s_1 > 0$ for some $s_1$ and on $[0,\delta_1)$ for some $\delta_1 > 0$. Let $s_2$ be the minimum of $u_0(x)$ on $[\delta_1,1]$. We define

$$|w| = \max \{\|w\|^\infty_L, \|w_x\|^\infty_L\}.$$  

Then

$$\|w(\varepsilon,a)\| \to 0 \quad \text{as} \quad \varepsilon \to 0, a \to 0,$$

and so we can find $w(\varepsilon_1,a_1)$ such that

$$|w(\varepsilon_1,a_1)| < \min\{s_1,s_2\}, \varepsilon_1 < \varepsilon, a_1 < a.$$

Then $u_0(x) > w(x,\varepsilon_1,a_1)$. Let $v$ be a solution of $(C_1)$ with $v(x,0) = w(x,\varepsilon_1,a_1)$. Then the comparison theorem shows that $u(x,t) \geq v(x,t)$, and, by Lemma 4.1., $v_t \geq 0$. Hence in (4.7), $v(x,t) \to w(x)$ as $t \to \infty$. But there is no stationary solution of $(C_1)$, and hence $v$ blows up. Therefore $u$ blows up.

**Theorem 4.3.** For $0 < \varepsilon < \varepsilon(a,p)$, we have the following:

(a) If $u_0(x) < w_+(x,\varepsilon), u_0 \in C^1([0,1])$, and $u'_0(0) > 0$, then $u$ is global and $\lim_{t \to \infty} u(x,t) = w_-(x,\varepsilon)$.

(b) For any $\delta > 0$, there exists $u_0(x)$ with $\|u_0(\cdot) - w_+(\cdot,\varepsilon)\| \leq \delta$ such that $u$ blows up.
PROOF: Since $0 \leq u_0(x) < w_+(x, \varepsilon)$, there is a $\sigma > 0$ such that $0 \leq u_0(x) < w_+(x, \varepsilon + \sigma)$. We put $v_0(x) = w_+(x, \varepsilon + \sigma)$.

Then, for as long as both solutions exist, $u(x, t; u_0) \leq u(x, t; v_0)$. Also,

$$u_t(x, 0, v_0) = v_{0xx}(x) + \varepsilon[f(v_0)]'$$
$$< v_{0xx}(x) + (\varepsilon + \sigma)[f(v_0)]'$$
$$= 0,$$

and hence $u_t(x, 0; v_0) < 0$ on $[0, 1)$. Therefore, by standard arguments, $u_t(x, t; v_0) \leq 0$, and, consequently, $u(x, t; v_0) \leq v_0$. So $u(x, t; v_0)$ is global. We can also find $w(x, \varepsilon_1, a_1)$ as in Theorem 4.2. Hence

$$u(x, t; w(x, \varepsilon_1, a_1)) \leq u(x, t; u_0) \leq u(x, t; v_0),$$

and so

$$\lim_{t \to \infty} u(x, t; u_0) = \phi(x),$$

a stationary solution. We have $\phi(1) \leq v_0(1) = w_+(1, \varepsilon) < w_+(1, \varepsilon)$, and so with $\phi(x)$ being

$$\phi(x) = w_-(x, \varepsilon).$$

To prove (b), given $\delta > 0$, we choose $\sigma > 0$ so that

$$\|w_+(\cdot, \varepsilon - \sigma) - w_+(\cdot, \varepsilon)\| < \delta$$

and

$$w_+(1, \varepsilon - \sigma) > w_+(1, \varepsilon).$$

Now set

$$u_0(x) = w_+(x, \varepsilon - \sigma),$$

and observe that for $u(x, t, u_0)$, we have $u_t(x, 0; u_0) \geq 0$. If $u$ does not blow up, then $T = \infty$ and $u(x, t) \to w(x)$ with $w(1) > w_+(1, \varepsilon)$. This is impossible. Therefore, $u$ must blow up.
REMARK 4.2. If we assume that the continuity of $u_t, u_{xx}$ at $x = 1$, and $u_{xx} \leq 0$, then

$$u_t(1,t) \leq \varepsilon a u^{1-p}(1,t).$$

If we further assume that $u'_0(x) \geq 0$, then $u_x \geq 0$. We find after a quadrature that

$$u(x,t) \leq u(1,t) \leq (\varepsilon a t + u^p(1,0))^\frac{1}{p} \equiv \Theta(t).$$

The numerical calculations indicate that $u_{xx} \leq 0$. (See Appendix II.) We can summarize this as follows:

THEOREM 4.4. Let $u$ be a solution of $(C_1)$ which satisfies the conditions of Lemma 4.2 and Lemma 4.3. Also assume that $u$ is a solution of $(C_1)$ as in Theorem 4.2. Then $u$ is global, and so $u$ blows up in infinite time.

We also have the following asymptotic results for $u_{xx}$ at $x = 1$.

THEOREM 4.5. Let $u$ be a solution of $(C_1)$. Then

$$u_{xx}(1,t) < 0 \quad \text{as} \quad a \to \infty.$$ 

PROOF: Let $\xi = \frac{1-x}{\sigma}, \tau = \frac{t}{\sigma^2}$ and $u(x,t) = a^{\frac{1}{p+1}} v(\xi, \tau)$, where $\sigma^{-2} = a^{\frac{1}{p+1}}$. Then we have

\begin{align*}
(4.8) \quad & v_T = v_{\xi \xi} - \varepsilon \cdot \frac{1}{\sigma} v_{\xi} \quad \text{for} \quad \xi > 0, \\
(4.9) \quad & v_T = v_{\xi \xi} + \varepsilon v^{1-p} \quad \text{for} \quad \xi = 0.
\end{align*}

We write

\begin{align*}
(4.10) \quad v = v_0 + \sigma v_1 + \sigma^2 v_2 + \ldots,
\end{align*}
and from (4.8) we have

\[(4.11) \quad v_{0,\tau} + \sigma v_{1,\tau} + \sigma^2 v_{2,\tau} + \ldots = v_{0,\xi\xi} + \sigma v_{1,\xi\xi} + \sigma^2 v_{2,\xi\xi} + \ldots - \varepsilon \frac{1}{\sigma} [v_0 + \sigma v_1 + \sigma^2 v_2 + \ldots]\]

\[\left[ v_{0,\xi} + \sigma v_{1,\xi} + \sigma^2 v_{2,\xi} + \ldots \right].\]

Comparing the coefficients of powers of $\sigma$, we have the following:

\[(4.12) \quad \sigma^{-1} : v_0 = 0.\]

\[\sigma^0 : v_{0,\tau} = v_{0,\xi\xi} - \varepsilon [v_0 v_{1,\xi} + v_1 v_{0,\xi}].\]

\[\sigma^1 : v_{1,\tau} = v_{1,\xi\xi} - \varepsilon [v_0 v_{2,\xi} + v_1 v_{1,\xi} + v_2 v_0,\xi].\]

Therefore,

\[(4.13_1) \quad v_0 = v_0(\tau),\]

\[(4.13_2) \quad v_1(\xi, \tau) = -\frac{v_0'(\tau)\xi}{\varepsilon v_0(\tau)} + \phi_1(\tau),\]

and

\[(4.13_3) \quad \phi_1'(\tau) - \xi \frac{1}{\varepsilon} \left[ \frac{v_0'(\tau)}{v_1(\tau)} \right]' = -\varepsilon v_0 v_{2,\xi}\]

\[- \varepsilon \left[ \frac{v_0'(\tau)}{\varepsilon v_0(\tau)} \xi + \phi_1(\tau) \right] \left[ \frac{-v_0'(\tau)}{\varepsilon v_0(\tau)} \right],\]

where $\phi_1$ is any function of $\tau$. Differentiating (4.13)_3 with respect to $\xi$ yields

\[(4.14) \quad v_{2,\xi\xi} = \frac{1}{v_0(\tau)} \left( \frac{1}{\varepsilon} \right)^2 \left[ \frac{v_0'(\tau)}{v_0(\tau)} \right]' - \left( \frac{v_0'(\tau)}{v_0(\tau)} \right)^2].\]
To get \( v_0(\tau) \), we use the boundary condition (4.9) to obtain

\[
(4.15) \quad v'_0(\tau) = \varepsilon v_0^{1-p}.
\]

So

\[
(4.16) \quad \frac{v'_0(\tau)}{v_0(\tau)} = \varepsilon v_0^{-p} = \frac{\varepsilon}{v_0^p},
\]

and

\[
(4.17) \quad \left[ \frac{v'_0(\tau)}{v_0(\tau)} \right]' = -\varepsilon pv_0'(\tau) = \frac{-\varepsilon p v_0(\tau)}{v_0^{p+1}}.
\]

Thus we have

\[
(4.18) \quad v_{2,\xi\xi} = \frac{1}{v_0(\tau)} (\varepsilon)^2 \left[ \frac{-(\varepsilon)^2 p}{v_0^{2p}(\tau)} - \frac{(\varepsilon)^2}{v_0^{2p}} \right]
\]

\[
= \frac{-(p+1)}{v_0^{2p+1}(\tau)},
\]

and (4.16) yields

\[
(4.19) \quad v_0(\tau) = [\varepsilon \rho + v_0^p(0)]^{\frac{1}{p}}.
\]

Since \( p > 0 \), it follows from (4.18) that \( v_{2,\xi\xi} < 0 \) as \( \sigma \to 0 \). Finally, \( u_{xx}(1,t) < 0 \) as \( a \to \infty \), since \( u_{xx}(1,t) = \frac{1}{\sigma} + \frac{a+1}{\sigma^2} = v_{\xi\xi} \).

**Theorem 4.6.** Let \( u \) be a solution of (C1) with \( u_0''(x) \leq 0 \) and \( u \in C^2 \).

Suppose that \(-1 \leq p\). Then \( u_{xx}(1,t) \leq 0 \).

**Proof:** Put \( v = u_{xx} \). Then we have

\[
v_t = v_{xx} + [2\varepsilon u_x + \varepsilon u_x]v + \varepsilon uv_x
\]

and \( v(0,t) = 0 \) because \( v(0,t) = -\varepsilon u_x + u_t|_{x=0} \).
At \( x = 1, \)
\[
v_x = (u_t - eux)x = (u_x)_t - euv - eux^2
\]
\[
= g'(u)u_t - euv - eux^2
\]
\[
= g'(u)[v + eux_x] - euv - eux^2
\]
Hence we have, at \( x = 1, \)
\[
v_x + [e - g'(u)]v = +eux_xg'(u) - eux^2
\]
\[
= -\varepsilon[-ug(u)g'(u) + g^2(u)]
\]
Since \( g(u) = au^{-p}, -ug'(u) + g(u) = +apu^{-p} + au^{-p} = au^{-p}[+p + 1] \geq 0. \) We put
\[
h_1(x,t) = 2eux(x,t) + eux_x(x,t), h_2(x,t) = eu(x,t), h_3(1,t) = eu(1,t) - g'(u(1,t))
\]
and \( h_4(1,t) = -\varepsilon g(1,t)[-u(1,t)g'(u(1,t)) + g(u(1,t))]. \) Then we have
\[
\begin{cases}
v_t = v_{xx} + h_1(x,t)v + h_2(x,t)v_x \\
v(0,t) = 0 \\
v(x,0) \leq 0 \\
v(x,1) + h_3(1,t)v(1,t) = h_4(1,t)
\end{cases}
\]
(4.20) Let \( 0 < t < T. \) Take \( \delta > 0 \) such that \( 0 < t < T - \delta. \) Let \( \Omega_\delta = [0,1] \times [0,T - \delta]. \) Then \( h_i \) are bounded on \( \Omega_\delta, i = 1,2,3,4. \) We let \( \psi(x,t) = e^{(\lambda x + \mu t)v(x,t)}. \)

Then
\[
\psi_t = \psi_{xx} + (h_2 - 2x)\psi_x + (\mu + \lambda^2 + h_1 - \lambda h_2)\psi
\]
and at \( x = 1, \)
\[
\psi_x + (h_3 - \lambda)\psi = e^{\lambda + \mu t})h_4(1,t)
\]
(4.22) We choose \( \lambda \) such that the coefficient of \( \psi \) in (4.22) is positive and \( \mu \) such that the coefficient of \( \psi \) in (4.21) is negative. Hence we have \( \psi \leq 0. \) Therefore \( v \leq 0 \) on \( \Omega_\delta. \) We are done.
5. STATIONARY SOLUTIONS FOR (D_1)

5.1. Stationary Solutions

Here we consider stationary solutions for

(D_1) \quad u_t = u_{xx} + \varepsilon \left( \frac{u^2}{2} \right)_x \quad \text{on} \quad (0,1) \times (0,T)

\quad u(1,t) = 0 \quad \text{on} \quad (0,T)

\quad -u_x(0,t) = \frac{a}{u^p(0,t)} \quad \text{on} \quad (0,T)

\quad u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1]

In this case, solution of (1.4) is equivalent to solving

\begin{equation}
F(w_0) \equiv \int_0^1 \frac{1}{(\frac{2a}{\varepsilon}) w_0^{p-2} - 1 + \sigma^2} \, d\sigma = \frac{1}{2} \varepsilon w_0,
\end{equation}

for \( w_0 \) with the additional condition that

\( \left( \frac{2a}{\varepsilon} \right) w_0^{p-2} > 1. \)

In this case, it is convenient to define

\( \beta = \left( \frac{2a}{\varepsilon} \right) w_0^{p-2} - 1 \)

and seek positive solutions of

\begin{equation}
G(\beta) \equiv \int_0^1 \frac{d\sigma}{\beta + \sigma^2} = \delta (\beta + 1)^{\frac{-1}{p+2}} \equiv H(\beta)
\end{equation}

where \( \delta = \frac{\varepsilon}{2} \left( \frac{2a}{\varepsilon} \right)^{\frac{1}{p+2}}. \) Next, with \( \alpha^2 = \beta, \alpha > 0, \) we have that

\( I(\alpha) \equiv \frac{\alpha^2 + 1}{\alpha^{p+2}} \tan^{-1} \left( \frac{1}{\alpha} \right) = \delta, \)
and, with \( q = \frac{1}{p+2} \), that

\[
I'(\alpha) = 2q(\alpha^2 + 1)q^{-1} \tan^{-1} \left( \frac{1}{\alpha} \right) + (-1)\alpha^{-2}(\alpha^2 + 1)^q \tan \left( \frac{1}{\alpha} \right)
\]

\[
+ (\alpha^2 + 1)^q \alpha^{-1} \frac{(-1)}{\alpha^2 + 1}
\]

\[
= (\alpha^2 + 1)^q - 1 \alpha^{-2}[2q\alpha^2 \tan^{-1} \left( \frac{1}{\alpha} \right) - (\alpha^2 + 1) \tan \left( \frac{1}{\alpha} \right) - \alpha]
\]

\[
= (\alpha^2 + 1)^q - 1(\{(2q-1)\alpha^2 - 1\} \tan^{-1} \left( \frac{1}{\alpha} \right) - \alpha).
\]

Now \( p > 0 \), and so \( 0 < q < \frac{1}{2} \), i.e., \( 2q - 1 < 0 \). Hence \( I'(\alpha) < 0 \), and thus \( I(\alpha) \) is strictly decreasing. Also, we know that

\[
\lim_{\alpha \to 0^+} I(\alpha) = \lim_{\alpha \to 0^+} \frac{(\alpha^2 + 1)^{p+2} \tan^{-1} \left( \frac{1}{\alpha} \right)}{\alpha} = \infty,
\]

and

\[
\lim_{\alpha \to +\infty} I(\alpha) = \lim_{\alpha \to +\infty} \left[ (\alpha^2 + 1)\alpha^{-p-2} \right] \frac{1}{p+2} \tan^{-1} \left( \frac{1}{\alpha} \right)
\]

\[
= \lim_{\alpha \to +\infty} \left[ \alpha^{-p} + \alpha^{-p-2} \right] \frac{1}{p+2} \tan^{-1} \left( \frac{1}{\alpha} \right)
\]

\[
= 0.
\]

So \( I(\alpha) = 0 \) has exactly one solution.

**Theorem 5.1.** Given \( \epsilon, \alpha, p > 0 \), \( (D_1) \) has exactly one stationary solution.

### 5.2. Dynamical Results

The proof of Theorem 5.2 is similar to that of Lemma 1.5; so we omit it here.

**Theorem 5.2.** Let \( f'(u) \) be an increasing function. Let \( u > 0 \) on \( \{0\} \times [0, T) \). Let \( u \) be a solution of (B) on \( D_T \cup \Gamma_T \). If \( u(x, 0) > 0 \) on \( (0, 1) \), then \( u > 0 \) on \( D_T \cup \Gamma_T \) except at \( x = 1 \).
Suppose that \( g(u)/u \) is decreasing on \((0, \infty)\), that \( w(x) \) is a positive stationary solution of \((B)\), and that \( u \) is a solution of \((B)\). Let \( \sigma > 0 \). If \( u(x, 0) \leq (1 + \sigma)w(x) \) on \([0, 1]\), then \( u(x, t) \leq (1 + \sigma)w(x) \) on \( D_T \cup \Gamma_T \), while if \( u(x, 0) \geq (1 - \sigma)w(x) \) on \([0, 1]\), then \( u(x, t) \geq (1 - \sigma)w(x) \) on \( D_T \cup \Gamma_T \).

**Remark 5.1:** In \((D_1)\), \( \frac{g(u)}{u} = \frac{a}{u^p u} = \frac{a}{u^{p+1}} \). Hence \( \frac{g(u)}{u} \) is decreasing on \((0, \infty)\), and so the stationary solution is stable by Theorem 5.2.
6. STATIONARY SOLUTIONS FOR \((N_1)\) AND \((N_2)\)

6.1. Stationary Solutions for \((N_1)\)

Here we consider stationary solutions for

\[
\begin{align*}
(N_1) \quad u_t &= u_{xx} + \frac{\varepsilon}{2}(u^2)_x \quad \text{on} \quad (0,1) \times (0,T) \\
u(0, t) &= 0 \quad \text{on} \quad (0,T) \\
u_x(1, t) + \frac{\varepsilon}{2}u^2(1, t) &= u^p(1, t) \quad \text{on} \quad (0,T) \\
u(x, 0) &= u_0(x) \quad \text{prescribed on} \quad [0,1].
\end{align*}
\]

In this case (1.2) is equivalent to

\[
F(w_1) = \int_0^1 \frac{d\sigma}{(\frac{2\alpha}{\varepsilon}w_1^{p-2} - \sigma^2)} = \frac{\varepsilon}{2}w_1.
\]

If \(p \geq 2\), then

\[
F'(w_1) = \int_0^1 \frac{-2\varepsilon(p-2)w_1^{p-3}d\sigma}{(\frac{2\alpha}{\varepsilon}w_1^{p-2} - \sigma^2)^2} < 0,
\]

so there is only one solution for \(\varepsilon > 0\). For \(p \leq 1\), we set \(v_1 = \frac{2\alpha}{\varepsilon}w_1^{p-2}\),

\[
\delta = \left(\frac{\varepsilon}{2}\right)^{p-2} \left(\frac{1}{\alpha}\right)^{p-2}
\]

and seek the number of positive solutions of

\[
Q(v_1) = \frac{v_1^{p-1}}{\int_0^1 \frac{d\sigma}{v_1 - \sigma^2}} = \delta v_1^{p-2} \equiv R(v_1).
\]

We know that \(\lim_{v_1 \to 1^+} Q(v_1) = \infty\).

\[
Q'(v_1) = \int_0^1 \frac{-\sigma^2d\sigma}{(v_1 - \sigma^2)^2} < 0,
\]
and \( R'(v_1) = \delta \left( \frac{p-1}{p-2} \right) v_1^{\frac{p-1}{p-2}-1} > 0 \), so \( Q \) is decreasing and \( R \) is increasing. Hence there is only one solution of (6.2). Thus we have one solution of (6.1).

For \( 1 < p < 2 \), we know that \( v_1 > 1 \), so only the case where \( v_1 > 1 \) need be analyzed. Put \( v^2 = v_1 \), then (6.1) is equivalent to

\[
F(v) = v^{\frac{-p}{p-2}} \ln \left( \frac{v+1}{v-1} \right) = \delta,
\]

where \( \delta = 2(\frac{p}{2})^{\frac{p-1}{p-2}} \left( \frac{1}{a} \right)^{\frac{1}{p-2}} \). Let \( q = \frac{p}{2-p} \) \( (q > 1) \), then \( F(v) = v^q \ln \left( \frac{v+1}{v-1} \right) \) and

\[
F'(v) = q v^{q-1} \ln \left( \frac{v+1}{v-1} \right) + q \frac{(-2)}{v^2-1} = v^{q-1} \left[ q \ln \left( \frac{v+1}{v-1} \right) - \frac{2v}{v^2-1} \right] = v^{q-1} k(v),
\]

where \( k(v) = q \ln \left( \frac{v+1}{v-1} \right) - \frac{2v}{v^2-1} \). We have

\[
k'(v) = q \frac{-2}{v^2-1} + \frac{-2(v^2-1) + 2v(2v)}{(v^2-1)^2}
\]

\[
= \frac{1}{v^2-1} \left[ -2q + \frac{2v^2 + 2}{v^2-1} \right]
\]

\[
= \frac{1}{v^2-1} \left[ -2q + \frac{2(v^2-1) + 4}{v^2-1} \right]
\]

\[
= \frac{1}{v^2-1} \left[ 2 - 2q + \frac{4}{v^2-1} \right].
\]

There exists \( v_0 \) such that \( k'(v) > 0 \) on \((1,v_0)\) and \( k'(v) < 0 \) on \((v_0,\infty)\). So \( k \) is increasing on \((1,v_0)\) and decreasing on \((v_0,\infty)\). We plot \( k(v) \) below.
We know that

\[ \lim_{v \to 1^+} k(v) = -\infty \quad \text{and} \quad \lim_{v \to \infty} k(v) = 0. \]

Hence there exists \( 1 < v_0^* < v_0 \) such that

\[
\begin{align*}
    k(v) &< 0 \quad \text{on} \quad (1, v_0^*), \\
    k(v) &> 0 \quad \text{on} \quad (v_0^*, \infty).
\end{align*}
\]

Thus,

\[
\begin{align*}
    F'(v) &< 0 \quad \text{on} \quad (1, v_0^*), \\
    F'(v) &> 0 \quad \text{on} \quad (v_0^*, \infty).
\end{align*}
\]

Therefore, \( F \) is decreasing on \( (1, v_0^*) \) and \( F \) is increasing on \( (v_0^*, \infty) \).

We plot \( F(v) \) below.
Case 1. If $F(v_0^*) > \delta$, then there are two stationary solutions.

Case 2. If $F(v_0^*) = \delta$, then there is only one stationary solution.

Case 3. If $F(v_0^*) < \delta$, then there is no stationary solution.

Next, we want to find the bifurcation diagrams for $p \geq 2$. Note that (6.1) is equivalent to

\[
\int_0^1 \frac{d\sigma}{2aw_1^p - \varepsilon \sigma^2} = \frac{1}{2} w_1.
\]

Suppose that $w_1(\varepsilon)$ is decreasing on some interval $(\varepsilon_1, \varepsilon_2)(\varepsilon_2 > \varepsilon_1)$. Then $(2aw_1^p - \varepsilon \sigma^2)$ is decreasing on $(\varepsilon_1, \varepsilon_2)$, and hence

\[
\int_0^1 \frac{d\sigma}{2aw_1^p - \varepsilon \sigma^2}
\]

is increasing. But $\frac{1}{2} w_2(\varepsilon)$ is decreasing. This is a contradiction. Therefore it must be that $w_1(\varepsilon)$ is increasing. We also know that $w_1(\varepsilon) \to \left(\frac{\varepsilon}{2a}\right)^{p-2}$ as $\varepsilon \to \infty$.

Put $\alpha^2 = \frac{2a}{\varepsilon} w_1^{p-2}$. Then (6.1) is equivalent to

\[
F(\alpha) \equiv \frac{1}{2\alpha} \ln \left(\frac{\alpha + 1}{\alpha - 1}\right) = \left(\frac{\varepsilon}{2}\right)^{p-2} \left(\frac{1}{a}\right)^{p-2} \alpha^{p-2}.
\]
We know that \( \alpha \to \infty \) as \( \varepsilon \to 0^+ \), and, furthermore, that

\[
\ln\left(\frac{\alpha + 1}{\alpha - 1}\right) = \ln(1 + \frac{2}{\alpha - 1}) \simeq \frac{2}{\alpha} \quad \text{as} \quad \alpha \to \infty.
\]

So

\[
F(\alpha) \approx \frac{1}{2\alpha} \cdot \frac{2}{\alpha} = \frac{1}{\alpha^2} \quad \text{as} \quad \alpha \to \infty.
\]

Thus, as \( \alpha \to \infty \), we have

\[
\frac{1}{\alpha^2} = \left(\frac{\varepsilon}{2}\right)^{p-2} \left(\frac{1}{a}\right)^{p-2} \alpha^{p-2},
\]

that is,

\[
(6.6) \quad \alpha^2 = \frac{2}{\varepsilon} a^{p-1}.
\]

From (6.6) we have

\[
w_1 = \left(\frac{\varepsilon}{2}\right)^{p-2} \left(\frac{1}{a}\right)^{p-2} \cdot \left(\frac{1}{\alpha^2}\right)^{p-2}
\]

\[
= \left(\frac{\varepsilon}{2}\right)^{p-2} \left(\frac{1}{\varepsilon}\right)^{p-2} \cdot \left(\frac{1}{a}\right)^{p-2} \cdot \left(\frac{1}{\alpha^2}\right)^{p-2}
\]

\[
= a^{p-2} + \frac{1}{(p-1)(p-2)} = a(\frac{2-p}{(p-1)(p-2)}).
\]

Hence, \( \lim_{\varepsilon \to 0^+} w_1(\varepsilon) \) exists and it is finite. Therefore, we have the bifurcation diagram, Figure 21, in Appendix III.

For \( p < 1 \), we have the bifurcation diagram, Figure 22, in Appendix III. For \( 1 < p < 2 \), we have the bifurcation diagram, Figure 23, in Appendix III. For \( p = 1 \), we have the bifurcation diagram, Figure 24, in Appendix III.
REMARK 6.1: If we introduce a new parameter $\beta(0 \leq \beta \leq 1)$, then we can write the general form

\[(N_R) \quad u_t = u_{xx} + \frac{\varepsilon}{2}[u^2]_x \quad \text{on} \quad (0,1) \times (0,T)\]

\[u_x(1,t) + (1 - \beta)\frac{\varepsilon}{2}u^2(1,t) = au^p(1,t) \quad \text{on} \quad (0,T)\]

\[u(0,t) = 0 \quad \text{on} \quad (0,T)\]

\[u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1]\]

of $(N_1), (C_1)$ and $(E)$. If $\beta = 0$, then $(N_R)$ and $(N_1)$ one the same problem. If $\beta = 1$, then $(N_R)$ and $(C_1), (E)$ are the same according to $p < 0$ or $p > 0$ respectively. We know that if $\beta = 0$, $p \leq 1$ or $p \geq 2$, then there is at most one stationary solution, and if $\beta = 0$, $1 < p < 2$, we have at most two stationary solutions. Also, if $\beta = 1$, $p \geq 1$, there is at most one stationary solution, and if $\beta = 1$, $p < 1$, we have at most two stationary solutions. So, we automatically have questions for $0 < \beta < 1$. If $p > 2$, we have at most one stationary solution.

\[(1.2) \quad \text{is equivalent to}\]

\[\int_0^1 \frac{d\sigma}{v_1^2 - \sigma^2} = \delta (v_1^2 - \beta) \frac{1}{p-2}\]

where \[v_1^2 = \beta + \frac{2a}{\varepsilon}w_1^{p-2}, (v_1 \geq 1), \delta = \left(\frac{\varepsilon}{2}\right)^{p-1} \frac{1}{a} \frac{1}{p-2},\]

so we have

\[F(v_1) = \frac{1}{2v_1}(v_1^2 - \beta) \frac{1}{2-P} \ln\left(\frac{v_1 + 1}{v_1 - 1}\right) = \left(\frac{1}{a}\right) \frac{1}{p-2} \left(\frac{\varepsilon}{2}\right)^{p-1} \equiv \delta.\]

We want to analyze $F(v_1)$, but it is very hard. Hence we will analyze it and its bifurcation diagrams numerically. Our computations show the following: If $1 \leq p < 2$, then we can have no more than two and, for some $\varepsilon$, exactly two solutions. (See Figure 30(a), 30(b), 31(a), 31(b) and 31(c) in Appendix III). For $p < 1$, we can have no more than three, and for some $\varepsilon$, exactly three solutions. (See Figure 32(a) and 32(b)).
6.2 Stationary Solutions for \((N_2)\)

Here we consider stationary solutions for

\[
(N_2) \quad u_t = u_{xx} + \frac{\varepsilon}{2}(u^2)x \quad \text{on} \quad (0, 1) \times (0, T)
\]

\[
u_x(0, t) + \frac{\varepsilon}{2}u_x^2(0, t) = -au^p(0, t) \quad \text{on} \quad (0, T)
\]

\[
u(1, t) = 0 \quad \text{on} \quad (0, T)
\]

\[
u(x, 0) = u_0(x) \quad \text{prescribed on} \quad [0, 1]
\]

In this case (1.4) is equivalent to

\[
F(w_0) = \int_0^1 \frac{d\sigma}{(\frac{2\alpha}{\varepsilon})w_0^{p-2} + \sigma^2} = \frac{\varepsilon}{2}w_0
\]

If \(p \geq 2\),

\[
F^1(w_0) = \int_0^1 \frac{-(\frac{2\alpha}{\varepsilon})(p-2)w_0^{p-3}d\sigma}{(\frac{2\alpha}{\varepsilon})w_0^{p-2} + \sigma^2} < 0,
\]

so there is only one solution of (6.7) for \(\varepsilon > 0\). For \(1 \leq p < 2\), put \(\frac{2\alpha}{\varepsilon}w_0^{p-2} = \beta\) and \(\delta = (\frac{\varepsilon}{2})^{p-2} (\frac{1}{\alpha})^{p-2}\). We then seek the number of positive solutions of

\[
Q(\beta) \equiv \beta \int_0^1 \frac{d\sigma}{\beta + \sigma^2} = \delta \beta^{\frac{p}{p-2}} \equiv R(\beta).
\]

\(R(\beta)\) is decreasing on \((0, \infty)\), but

\[
Q'(\beta) = \int_0^1 \frac{\sigma^2d\sigma}{(\beta + \sigma^2)^2} > 0.,
\]

and so there is only one solution of (6.8). For \(p < 1\), define

\[
I(\beta) \equiv \beta^{2-p} \int_0^1 \frac{d\sigma}{\beta + \sigma^2} = \delta,
\]
and put \( q = \frac{1}{2-p} \). Then \( 0 < q < 1 \), and

\[
(6.10) \quad I(\beta) = \beta^q \int_0^1 \frac{d\sigma}{\beta + \sigma^2},
\]

where \( \beta > 0 \). Let \( \beta = \alpha^2 (\alpha > 0) \). Then we have

\[
(6.11) \quad I(\alpha) = \alpha^{2q} \int_0^1 \frac{d\sigma}{\alpha^2 + \sigma^2} = \alpha^{2q-1} \tan^{-1} \left( \frac{1}{\alpha} \right).
\]

If \( 0 < q \leq \frac{1}{2} \), then \( I(\alpha) \) is decreasing on \((0, \infty)\). So we have only one solution of (6.9). If \( \frac{1}{2} < q < 1 \), we have

\[
(6.12) \quad I'(\alpha) = \alpha^{2q-2} k(\alpha),
\]

where \( k(\alpha) = (2q-1) \tan^{-1} \left( \frac{1}{\alpha} \right) + \frac{(-1)\alpha}{\alpha^2 + 1} \). We have

\[
(6.13) \quad k'(\alpha) = \frac{2}{(\alpha^2 + 1)^2} [(1-q)\alpha^2 - q],
\]

so if we set \( \alpha^+ = \sqrt{\frac{q}{1-q}} \), then \( k'(\alpha) < 0 \) on \([0, \alpha^+]\) and \( k'(\alpha) > 0 \) on \((\alpha^+, \infty)\). Also, we know that \( k(\alpha^+) \leq 0 \). So, we get the graph of \( k \) as follows:
because \( \lim_{\alpha \to \infty} k(\alpha) = 0 \). Hence there exists \( \alpha_0 \) such that \( k(\alpha_0) = 0 \), and thus we have \( I'(\alpha) > 0 \) on \((0, \alpha_0)\) and \( I'(\alpha) < 0 \) on \((\alpha_0, \infty)\). So \( I(\alpha) \) has a maximum at \( \alpha = \alpha_0 \), and, since \( \lim_{\alpha \to \infty} I(\alpha) = 0 \), we get the graph of \( I(\alpha) \) as follows:

Thus \( I(\alpha) = \delta \) has none, one or two solutions according to \( \delta > I(\alpha_0) \), \( \delta = I(\alpha_0) \) or \( 0 < \delta < I(\alpha_0) \). Next, we want to find the bifurcation diagrams. For \( p \geq 2 \), (6.7) is
equivalent to

\begin{equation}
\frac{1}{2} \int_0^1 \frac{d\sigma}{2aw_0^{p-2} + \varepsilon^2} = \frac{1}{2} w_0.
\end{equation}

Suppose that \( w'_0(\varepsilon) \geq 0 \) on some interval \((\varepsilon_1, \varepsilon_2)(\varepsilon_2 > \varepsilon_1)\). Then from (6.14), we would have

\begin{equation}
\frac{1}{2} \int_0^1 \frac{-[2a(p-2)w_0^{p-2} w'_0(\varepsilon) + \sigma^2]d\sigma}{[2aw_1^{p-2} + \varepsilon^2]} = \frac{1}{2} w'_0(\varepsilon).
\end{equation}

But then, in (6.15), the lefthand side is negative, and the righthand side is nonnegative on \((\varepsilon_1, \varepsilon_2)\). This is a contradiction. Hence we have that \( w_0(\varepsilon) \) is decreasing \((0, \infty)\).

Also, we know that \( \lim_{\varepsilon \to 0^+} w_0(\varepsilon) = (\frac{1}{a})^{\frac{1}{p-1}} \) and \( \lim_{\varepsilon \to \infty} w_0(\varepsilon) = 0 \). So, we have

the bifurcation diagram, Figure 25, in Appendix III. For \( 1 < p < 2 \), we have the bifurcation diagram, Figure 26, in Appendix III. For \( p = 1 \), we have the bifurcation diagram, Figure 27, in Appendix III.

For \( 0 < p < 1 \), we have the bifurcation diagram, Figure 28, in Appendix III.

For \( p \leq 0 \), we have the bifurcation diagram, Figure 29, in Appendix III.

**REMARK 6.2:** If we introduce a new parameter \( \beta(0 \leq \beta \leq 1) \), then we can write the general form

\begin{align*}
(N_L) & \quad u_t = u_{xx} + \frac{\varepsilon}{2} u^2_{xx} \\
& \quad \text{on} \quad (0,1) \times (0,T) \\
& \quad u_x(0,t) + (1 - \beta) \frac{\varepsilon}{2} u^2(0,t) = -au^p(0,t) \quad \text{on} \quad (0,T) \\
& \quad u(1,t) = 0 \quad \text{on} \quad (0,T) \\
& \quad u(x,0) = u_0(x) \quad \text{prescribed on} \quad [0,1]
\end{align*}

of \((N_2), (D_1)\) and \((F)\). If \( \beta = 0 \), then \((N_L)\) and \((N_2)\) are the same problem; if \( \beta = 1 \), \((N_L)\) and \((D_1)\), \((F)\) are the same according to \( p < 0 \) or \( p > 0 \), respectively. We know
that if \( \beta = 0, \ p \leq 0 \) or \( p \geq 1 \), then we have at most one stationary solution, and if \( \beta = 0, \ 0 < p < 1 \), we have at most two stationary solutions. Also, if \( \beta = 1, \ p \geq 2 \) or \( p \leq 1 \), then we have at most one stationary solution, and if \( \beta = 1, \ 1 < p < 2 \), we have at most two stationary solutions. Thus we automatically have questions for \( 0 < \beta < 1 \). If \( p \geq 2 \), we have trivially one stationary solution.

(1.4) is equivalent to

\[
\int_0^1 \frac{d\sigma}{\alpha^2 + \sigma^2} = \delta \cdot (\alpha^2 + \beta)^{-\frac{1}{p-2}}
\]

where \( \alpha^2 = \frac{2\varepsilon w_0^p - \beta}{\alpha > 0}, \delta = \left( \frac{\varepsilon}{2} \right)^{\frac{p-1}{p-2}} \left( \frac{1}{\alpha} \right)^{\frac{1}{p-2}} \), so we have

\[
I(\alpha) \equiv \frac{(\alpha^2 + \beta)^{\frac{2-p}{p}}}{\alpha} \tan^{-1} \left( \frac{1}{\alpha} \right) = \delta.
\]

We want to analyze \( I(\alpha) \), but it is very hard. Hence we will analyze it and its bifurcation diagrams numerically. If \( 1 < p < 2 \), we have at most two stationary solutions (see Figure 33(a) and 33(b) in Appendix III). If \( p = 1 \), we have at most one stationary solution (see Figure 34(a) and 34(b) in Appendix III). If \( p < 1 \), we have one stationary solution (see Figure 35(a) and 35(b) in Appendix III).

6.3. Dynamical Results for (\( N_1 \)) and (\( N_2 \))

We know that \( w(x) \) is a stationary solution of (\( N_1 \)) iff

\[
(6.16) \quad w(x) = G(x,1)(aw(1)^p - \frac{\varepsilon}{2} w^2(1)) + \int_0^1 G(x,y) \frac{d}{dy}[f(w(y))]dy
\]

where

\[
G(x,y) = \begin{cases} 
    y, & 0 \leq y \leq x \leq 1 \\
    x, & 0 \leq x \leq y \leq 1.
\end{cases}
\]
suppose that $u(x,t)$ is a solution of $(N_1)$ which does not blow up (i.e., which is bounded). Then

$$F(x,t) = \int_0^1 G(x,y)u(y,t)dy$$

is bounded in $[0,1] \times [0,\infty)$ and

$$F_t(x,t) = \int_0^1 G(x,y)u_t(y,t)dy$$

(6.17)

$$= -u(x,t) + G(x,1)[au^p(1,t) - \frac{\varepsilon}{2}u^2(1,t)]$$

$$+ \int_0^1 G(x,y)[f(u(y,t))]'dy.$$

If $u_t \geq 0$, then $F_t \geq 0$ and $u$ increasing to a solution of (6.16).

**Lemma 6.1.** Let $u$ be a solution of $(N_1)$ with $u(x,0) = w(x,\varepsilon_1,a)$ for $0 < \varepsilon_1 < \varepsilon (\varepsilon_1 > \varepsilon)$ then $u_t > 0 (< 0)$ for all $t$.

The proof of Lemma 6.1 is similar to that of Lemma 4.1, and so we omit it.

**Theorem 6.1.** For $p \geq 2$, every stationary solution of $(N_1)$ is stable, and for $0 < p < 1$, every stationary solution of $(N_1)$ is unstable.

The proof of Theorem 6.1 find in Theorem 3.6A in Levine [11].

**Theorem 6.2.** Let $u(x,t)$ be a solution of $(N_1)$. For $1 < p < 2$, and for $a > 0$, there exists $\varepsilon_0(a,p)$ such that for $0 < \varepsilon < \varepsilon(a,\beta)$, we have

(a) If $u_0(x) < w_+(x,\varepsilon), u_0 \in C^1([0,1])$, and $u_0'(0) > 0$, then $u$ is global and

$$\lim_{t \to \infty} u(x,t) = w_-(x,\varepsilon)$$

(b) For any $\delta > 0$, there exists $u_0(x)$ with $\|u_0(\cdot) - w_+(\cdot,\varepsilon)\|_{L^\infty} < \delta$ such that $u$ blows up.
The proof of Theorem 6.2 is similar to that of Theorem 4.3, so we omit it.

**Theorem 6.3.** Let $u(x,t)$ be a solution of $(N_1)$ and $1 < p < 2, a > 0$ with $u_0'(0) > 0$ and $u_0$ in $C^1[0,1]$. Let $\varepsilon_0(a,p)$ be the same as that in Theorem 6.2. If $\varepsilon > \varepsilon_0(a,p)$, then $u(x,t)$ blows up in finite or infinite time.

The proof of Theorem 6.3 is similar to that of Theorem 4.2, so we omit it.

**Theorem 6.4.** Let $u$ be a positive solution of $(N_1)$ with $u''_0 + \varepsilon u_0 u'_0 \geq 0$ and $u \in C^2$. Suppose that $1 < p < 2, \varepsilon > \varepsilon(a,p)$. Then $u(x,t)$ blows up in finite time.

**Proof:** Suppose that $u(x,t)$ is bounded. Since $u_t \geq 0$, there exists a stationary solution. This cannot happen. Thus $u(x,t)$ blows up. Assume that $u(x,t)$ blows up in infinite time. Then there exists time $T_0$ such that $u_x(1,t) = au^p(1,t) - \frac{a}{2} u^2(1,t) < 0$ for $t \geq T_0$. Since $u(0,T_0) = 0$, and $u_x(1,T_0) < 0$, there exist $x_0 \in (0,1)$ such that $u(x_0,T_0)$ is the maximum of $u(x,T_0)$ on $[0,1]$. Thus $u_x(x_0,T_0) = 0$, and hence $0 \leq u_t(x_0,T_0) = u_{xx}(x_0,T_0) < 0$. This is a contradiction. Therefore, $u(x,t)$ blows up in finite time.

**Remark 6.3:** For $(N_2)$, see Lemma 3.4B and Theorem 3.5B in Levine [11] for nonexistence problems.
7. ACKNOWLEDGMENTS

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8. REFERENCES


Here we solve the equation

\[(9.1) \quad \int_0^{w_0} \frac{d\sigma}{\lambda(1-w_0)^{-\beta} - w_0^2 + \sigma^2} = \varepsilon/2\]

numerically, where \(\lambda = 2a/\varepsilon\). We can change (9.1) to

\[(9.2) \quad f(w_0) \equiv \frac{1}{w_0} \int_0^1 \frac{d\sigma}{\lambda(1-w_0)^{-\beta}w_0^{-2} - 1 + \sigma^2} = \varepsilon/2.\]

Define

\(\alpha(w_0) = \lambda(1-w_0)^{-\beta}w_0^{-2} - 1.\)

We know that \(\alpha > 0\), and \(\alpha\) has minimum at \(w_0 = \frac{2}{\beta+2}\). So we have two cases:

(Case i) \(\alpha(w_0^1) > 0.\)

(Case ii) \(\alpha(w_0^2) \leq 0.\)

In case i), we can graph on \((0,1)\) and in case ii), there are computable numbers, \(w_0^1(\lambda), w_0^2(\lambda)\), such that we can graph \(\alpha\) on \((0,w_0^1(\lambda))\) and \((w_0^2(\lambda),1)\). (We know that \(\alpha(w_0^1(\lambda)) = \alpha(w_0^2(\lambda)) = 0\).) We graph various values of \(\lambda\) and \(\beta\).

(9.2) is equivalent to

\[(9.3) \quad f(w_0) \equiv \frac{1}{w_0\sqrt{\alpha(w_0)}} \tan^{-1} \left( \frac{1}{\sqrt{\alpha(w_0)}} \right) = \frac{\varepsilon}{2}\]

where \(\alpha(w_0) = \lambda(1-w_0)^{-\beta}w_0^{-2} - 1 > 0\). We suppose that \(\lambda \geq 1\). Then it follows \(\alpha(w_0^0) > 0\) on \((0,1)\) because \(\alpha(w_0^0) = \lambda \left( \frac{\beta}{\beta+2} \right)^{-\beta} \left( \frac{2}{\beta+2} \right)^{-2} - 1, 0 < \frac{\beta}{\beta+2} < 1\), and \(0 < \frac{2}{\beta+2} < 1\). So this case belongs to case (i). We graph for the following:

i) \(\lambda = 1, \beta = 2: \) Figure 1.

ii) \(\lambda = 100, \beta = 2: \) Figure 2.

iii) \(\lambda = 1, \beta = 0.1: \) Figure 3.

iv) \(\lambda = 100, \beta = 0.001: \) Figure 4.
Next, we know that

\[ \alpha(w_0^0) = \lambda \left( \frac{\beta}{\beta + 2} \right)^{-\beta} \left( \frac{2}{\beta + 2} \right)^{-2} - 1. \]

So when \( \beta = 2 \), then \( \alpha(w_0^0) = \lambda \left( \frac{1}{2} \right)^{-2} \left( \frac{1}{2} \right)^{-2} - 1. \) Thus if \( \lambda > \frac{1}{16} \), then (case i) occurs; if \( \lambda \leq \frac{1}{16} \), then (case ii) occurs. In the last case, we should solve the equation

\[ \lambda(1 - w_0)^{-2} w_0^{-2} - 1 = 0, \]

that is

\[ w_0^4 - 2w_0^3 + w_0^2 - \lambda = 0. \]

Setting \( w_0 = y + \frac{1}{2} \), we get

\[ y^4 - \frac{1}{2} y^2 + \frac{1}{16} - \lambda = 0. \]

Put \( x = y^2 \). Then we have

\[ x^2 - \frac{1}{2} x + \frac{1}{16} - \lambda = 0, \]

and hence

\[ x = \frac{1}{4} \pm \sqrt{\lambda}. \]

Since \( 0 < \lambda \leq \frac{1}{16} \), we have \( x \geq 0 \) in both cases, and so

\[ y = \pm \sqrt{x} = \pm \sqrt[+]{\frac{1}{4} \pm \sqrt{\lambda}}. \]

Therefore,

\[ w_0 = \pm \sqrt[+]{\frac{1}{4} \pm \sqrt{x} + \frac{1}{2}} \]

\[ = \begin{cases} 
\sqrt[+]{\frac{1}{4} + \sqrt{\lambda} + \frac{1}{2}} > 1 \\
\sqrt[+]{\frac{1}{4} - \sqrt{\lambda} + \frac{1}{2}} > 0 \\
-\sqrt[+]{\frac{1}{4} - \sqrt{\lambda} + \frac{1}{2}} > 0 \\
-\sqrt[+]{\frac{1}{4} + \sqrt{\lambda} + \frac{1}{2}} < 0
\end{cases} \]
and so, 
\[ w_0^1(\lambda) = -\sqrt{\frac{1}{4} - \sqrt{\lambda}} + \frac{1}{2}, \quad w_0^2(\lambda) = \sqrt{\frac{1}{4} - \sqrt{\lambda}} + \frac{1}{2}. \]

We will display the graphs of \( f \) in each of the following cases:

i) \( \beta = 2, \lambda = \frac{1}{32} \): Figure 5.

ii) \( \beta = 2, \lambda = \frac{1}{81} \): Figure 6.

iii) \( \beta = 2, \lambda = 0.1 \): Figure 7.

If \( \beta = 1 \), then \( \alpha(w_0^0) = \lambda(\frac{1}{3})^{-1}(\frac{2}{3})^{-2} - 1 \). So, when \( \lambda > \frac{4}{27} \), (case i) occurs, and if \( \lambda \leq \frac{4}{27} \), then (case ii) occurs. In the last case, we should solve the equation \( \lambda(1 - w_0)^{-2} w_0^{-2} - 1 = 0 \) for each \( \lambda \), that is

\[
\begin{align*}
(1 - w_0)w_0^2 - \lambda &= 0, \\
w_0^3 - w_0^2 + \lambda &= 0.
\end{align*}
\]

The following cases are displayed:

(i) \( \beta = 1, \lambda = \frac{1}{2} \) : Figure 8.

(ii) \( \beta = 1, \lambda = \frac{1}{27} \) : Figure 9 \( (\alpha(w_0^0) \approx 0.2175, \alpha(w_0^1) \approx 0.9598) \).

(iii) \( \beta = 1, \lambda = \frac{1}{81} \) : Figure 10 \( (\alpha(w_0^0) \approx 0.118332, \alpha(w_0^1) \approx 0.987336) \).
Figure 1. $f(\omega_0)$ for $\lambda = 1, \beta = 2$. 
Figure 2. $f(\omega_0)$ for $\lambda = 100, \beta = 2$. 
Figure 3. $f(w_0)$ for $\lambda = 1$, $\beta = 0.1$. 
Figure 4. $f(\omega_0)$ for $\lambda = 100, \beta = 0.001$. 
Figure 5. $f(w_0)$ for $\lambda = 1/32$, $\beta = 2$. 
Figure 6. $f(\omega_0)$ for $\lambda = 1/81, \beta = 2.$
Figure 7. $f(\omega_0)$ for $\lambda = 0.1$, $\beta = 2$. 
Figure 8. $f(\omega_0)$ for $\lambda = 1/2$, $\beta = 1$. 
Figure 9. $f(\omega_0)$ for $\lambda = 1/27, \beta = 1$. 
Figure 10. $f(w_0)$ for $\lambda = 1/81$, $\beta = 1$. 
10. APPENDIX II

We want to solve initial-boundary value problem for

\[
\frac{\partial u}{\partial t} + \frac{\partial G(u)}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0
\]

by numerical method.

The limitation on the time step induced by the stability conditions associated with explicit schemes if often too restrictive in applications, and consequently an implicit scheme is used.

In here, we used the leapfrog Crank-Nicolson scheme, which is

\[
\frac{1}{2\Delta t}(U_i^{n+1} - U_i^{n-1}) + \Delta x G_i^n - \frac{1}{2} \Delta xx(U_i^{n+1} + U_i^{n-1}) = 0
\]

The accuracy is of second order in time as well as in space, and the stability is the CFL condition \(|T| = \frac{A(f)}{\Delta x} \Delta t \leq 1\) where \(A = dG/df\). But this scheme introduces oscillations in the numerical solution. The appearance of above oscillations is avoided by using the following averaging technique at time \(N\Delta t\), which is introduced by Orszag and Tang (1979).

1. Compute a provisional value \(f_i^{N+1}\), at \(N + 1\) using Eq. (10.2).
2. Define a new value \(f_i^N = \frac{1}{4}(f_i^{N+1} + 2f_i^N + f_i^{N-1})\).
3. Compute the new value \(f_i^{N+1}\) using eq. (A.II.2) with \(f_i^N\) replaced by \(f_i^N\).

Such a procedure conserves the second-order accuracy.

The scheme (9.2) is conditionally stable. Unconditional stability can be obtained if the nonlinear term \(\partial G/\partial x\) is evaluated at time \((n + 1)\Delta t\) and then linearized in a convenient manner. Since \(A(u) = cu\), we have the linearization as follows:

\[
\frac{\partial G}{\partial x} i^{n+1} = A_i^n(\frac{\partial U}{\partial x})^{n+1} + A_i^{n+1}(\frac{\partial U}{\partial x})^{n} - A_i^n(\frac{\partial U}{\partial x})^{n} + 0(\Delta x)^2
\]

\[
\approx A_i^n \Delta x U_i^{n+1} + A_i^{n+1} \Delta x f_i^n - A_i^n \Delta x f_i^n,
\]
where \( \Delta x^0 U_i = \frac{U_{i+1} - U_{i+1}}{2\Delta x} \)
\[ \Delta x^2 U_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{(\Delta x)^2}, \]
which leads to second-order accuracy.

Here, we use a three step-method because we don’t know the value at time \( t = \Delta t \). For this, we used explicit method which is a predictor-corrector method. We can write (9.1) as
\[ \frac{\partial^2 u}{\partial x^2} = \psi(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}). \]
The predictor is
\[ (10.3) \frac{1}{(h)^2} \delta_x^2 U_{i,j+\frac{1}{2}} = \psi[ih, (j + \frac{1}{2})k, U_{i,j}, \frac{1}{2h} \mu \delta_x U_{i,j}, \frac{2}{k} (U_{i,j+\frac{1}{2}} - U_{i,j})], \]
and the corrector is
\[ (10.4) \frac{1}{2h^2} \delta_x^2 [U_{i,j+1} + U_{i,j}] = \psi[ih, (j + \frac{1}{2})k, U_{i,j+\frac{1}{2}}, \frac{1}{4h} \mu \delta_x (U_{i,j+1} + U_{i,j}), \frac{1}{k} (U_{i,j+1} - U_{i,j})], \]
where \( \delta y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}, \)
\[ y_n = \frac{1}{2} [y_{n+\frac{1}{2}} + y_{n-\frac{1}{2}}], \]
\[ h = \Delta x, k = \Delta t. \]

We solve the following I.B.V.P. in the following special case:

(I.B.V.P.) \[ \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left[ \frac{\varepsilon}{2} u^2 \right] - \frac{\partial^2 u}{\partial x^2} = 0 \] on \( (0, 1) \times (0, T) \)
\[ u_x(1, t) = \frac{a}{u^p(1, t)} \] on \( (0, T) \)
\[ u(0, t) = 0 \] on \( (0, T) \)
\[ u(x, 0) = u_0(x) \] prescribed on \([0, 1]\)
Let \( u_0(x) = (a)^{\frac{1}{p+1}} x \)

i) \( \varepsilon = 1, p = 2, a = 100 \): Figure 11

ii) \( \varepsilon = 1, p = 1, a = 100 \): Figure 12

iii) \( \varepsilon = 1, p = -2, a = 1 \): Figure 13


iv) \( \varepsilon = 1, p = 1, a = 10 \): Figure 14

v) \( \varepsilon = 1, p = 0, a = 1 \): Figure 15

vi) \( \varepsilon = 1, p = 1, a = 1 \): Figure 16

vii) \( \varepsilon = 1, p = 2, a = 1 \): Figure 17

viii) \( \varepsilon = 1, p = 2, a = 10 \): Figure 18

ix) \( \varepsilon = 0, p = 2, a = 1 \): Figure 19

Let \( u_0(x) = (2a)^{\frac{1}{p+1}} x \sqrt{x} \)

x) \( \varepsilon = 1, p = 2, a = 100 \): Figure 20(a).

Let \( u_0(x) = 2a^{\frac{1}{p+1}} x^2 \)

xi) \( \varepsilon = 1, p = 2, a = 100 \): Figure 21(b).
Figure 11. $u(x, t)$ for $\epsilon = 1$, $p = 2$, $a = 100$. 

View Point is (-7.5, -7.5, 12.0)
Figure 12. $u(x,t)$ for $\epsilon = 1$, $p = 1$, $a = 100$. 

View Point is (-7.5,-7.5,12.0)
View Point is (-7.5, -7.5, 12.0)

Figure 13. $u(x, t)$ for $\epsilon = 1$, $p = -2$, $a = 1$. 
Figure 14. \( u(x,t) \) for \( \epsilon = 1, \ p = 1, \ a = 10 \).
Figure 15. $u(x,t)$ for $c = 1, p = 0, a = 1$.
View Point is (-7.5, -7.5, 12.0)

Figure 16. $u(x,t)$ for $\epsilon = 1$, $p = 1$, $a = 1$. 
View Point is (-7.5, -7.5, 12.0)

Figure 17. $u(x, t)$ for $\epsilon = 1, p = 2, a = 1$. 
View Point is (-7.5, -7.5, 12.0)

Figure 18. $u(x, t)$ for $\epsilon = 1$, $p = 2$, $a = 10$. 
Figure 19. $u(x,t)$ for $\epsilon = 0$, $p = 2$, $\alpha = 1$. View Point is (-7.5, -7.5, 12.0).
Figure 20(a). $u(x,t)$ for $\epsilon = 1$, $p = 2$, $a = 100$, $u_0(x) = (2a)^{\frac{1}{p+1}} x$. 

View Point is (-7.5,-7.5,12.0)
Figure 20(b). $u(x, t)$ for $\epsilon = 1, \; p = 2, \; a = 100, \; u_0(x) = (2a)^{\frac{1}{p+1}}x^2$. 

View Point is (-7.5, -7.5, 12.0)
11. APPENDIX III

We graph the bifurcation diagrams from Figure 21 to Figure 29. From Remark 6.1, we get

\[ F(v_1) = \frac{1}{2v_1}(v_1^2 - \beta)^{\frac{1}{2-p}} \ln\left(\frac{v_1 + 1}{v_1 - 1}\right). \]

We will graph \( F \) as follows;

i) \( \beta = \frac{1}{2}, \ p = 1.5, \ \text{Figure 30} \)

ii) \( \beta = \frac{1}{2}, \ p = 1, \ \text{Figure 31} \)

iii) \( \beta = \frac{9}{10}, \ p = 0.5, \ \text{Figure 32}. \)

From Remark 6.2, we get

\[ I(x) = \frac{(\alpha^2 + \beta)^{\frac{1}{2-p}}}{\alpha} \tan^{-1}\left(\frac{1}{\alpha}\right). \]

We will graph \( I \) as follows;

iv) \( \beta = \frac{1}{2}, \ p = 1.5, \ \text{Figure 33} \)

v) \( \beta = \frac{1}{2}, \ p = 1, \ \text{Figure 34} \)

vi) \( \beta = \frac{1}{2}, \ p = 0.5, \ \text{Figure 35}. \)
Figure 21. $w_1(\varepsilon)$ for $p \geq 2$.

Figure 22. $w_1(\varepsilon)$ for $p < 1$. 
Figure 23. $w_1(\varepsilon)$ for $1 < p < 2$.

Figure 24. $w_1(\varepsilon)$ for $p = 1$. 
Figure 25. $w_0(\epsilon)$ for $p \geq 2$.

Figure 26. $w_0(\epsilon)$ for $1 < p < 2$. 
Figure 27. $w_0(\varepsilon)$ for $p = 1$.

Figure 28. $w_0(\varepsilon)$ for $0 < p < 1$.

Figure 29. $w_0(\varepsilon)$ for $p \leq 0$. 
Figure 30(a). $F(v_1)$ for $\beta = \frac{1}{2}$, $p = 1.5$
Figure 30(b). $w_1(\epsilon)$ for $1 < p < 2$. 
Figure 31(a). $F(v_1)$ for $\beta = \frac{1}{2}, p = 1$
Figure 31(b). \( w_1(\varepsilon) \) for \( p = 1, a > 1 \).

Figure 31(c). \( w_1(\varepsilon) \) for \( p = 1, a < 1 \).
Figure 32(a). $F(v_1)$ for $\beta = \frac{9}{10}$, $p = 0.5$
Figure 32(b). $w_1(e)$ for $p < 1$. 
Figure 33(a). $I(\alpha)$ for $\beta = \frac{1}{2}$, $p = 1.5$
Figure 33(b). $w_0(\epsilon)$ for $1 < p < 2$. 
Figure 34(a). $I(\alpha)$ for $\beta = \frac{1}{2}, \ p = 1$
Figure 34(b). $w_0(\epsilon)$ for $p = 1$. 

$\left( \frac{\beta \epsilon}{2a} \right)^{\frac{1}{p-2}} = \omega_0$
Figure 35(a). $I(\alpha)$ for $\beta = \frac{1}{2}$, $p = 0.5$
Figure 35(b). $w_0(\epsilon)$ for $p < 1$. 