Boundary element solution of Poisson's equations in axisymmetric laminar flows

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Boundary element solution of Poisson's equations in axisymmetric laminar flows

Tevis, Joe W., Ph.D.
Iowa State University, 1989

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Boundary element solution of Poisson's equations in axisymmetric laminar flows

by

Joe W. Tevis

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TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER I. STATEMENT OF PROBLEM</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Motivation of Research</td>
<td>4</td>
</tr>
<tr>
<td>Research Objectives</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER II. BACKGROUND INFORMATION WITH LITERATURE REVIEW</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution of the Primitive Variable Equations</td>
<td>31</td>
</tr>
<tr>
<td>Solution of the Vorticity Transport Equation</td>
<td>34</td>
</tr>
<tr>
<td>Solution of Poisson's Equation</td>
<td>51</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER III. BOUNDARY ELEMENT SOLUTION OF POISSON'S VELOCITY EQUATION</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Biot-Savart Integral Solution</td>
<td>63</td>
</tr>
<tr>
<td>Direct BEM Formulation</td>
<td>73</td>
</tr>
<tr>
<td>Determination of Unknown Boundary Quantities</td>
<td>84</td>
</tr>
<tr>
<td>Testing the Solutions of Poisson's Velocity Equation</td>
<td>95</td>
</tr>
<tr>
<td>Test Results and Conclusions</td>
<td>103</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER IV. NUMERICAL EVALUATION OF BOUNDARY ELEMENT INTEGRALS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evaluation of Elliptic Integrals</td>
<td>106</td>
</tr>
<tr>
<td>Numerical Integration</td>
<td>108</td>
</tr>
<tr>
<td>Testing and Refinement of the Integration Algorithms</td>
<td>136</td>
</tr>
<tr>
<td>Comments and Conclusions</td>
<td>142</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER V. SOLUTION OF ILL-POSED POISSON'S VELOCITY EQUATION BY REGULARIZATION</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Integral Equation</td>
<td>146</td>
</tr>
</tbody>
</table>
APPENDIX C. FINITE DIFFERENCE SOLUTION OF THE VORTICITY TRANSPORT EQUATION

Time-Split MacCormack

Alternate Direction Implicit

ACKNOWLEDGMENTS
## LIST OF FIGURES

| Figure 1.1 | Spool valve with cross-sectioned body | 8   |
| Figure 1.2 | Orifice model of valve metering flow  | 9   |
| Figure 1.3 | Pressure distribution in a spool valve cavity during flow | 16  |
| Figure 1.4 | Control volume analysis of poppet force balance | 17  |
| Figure 1.5 | Illustration of axisymmetric spool geometry in cylindrical polar coordinates | 24  |
| Figure 2.1 | First order upwind and central differencing | 41  |
| Figure 3.1 | Transformation of a position vector | 65  |
| Figure 3.2 | Transformation of a velocity vector | 66  |
| Figure 3.3 | Arbitrary line segment on integration surface | 68  |
| Figure 3.4 | Concentration of vorticity in a vortex sheet | 93  |
| Figure 3.5 | Impulsively started pipe flow | 98  |
| Figure 3.6 | Fully developed of Poiseuille pipe flow | 98  |
| Figure 4.1 | Transformation from global to local coordinates | 112 |
| Figure 4.2 | Rectangular numerical grid | 119 |
| Figure 4.3 | Integration of triangular elements | 124 |
| Figure 4.4 | Second order interpolation functions | 133 |
| Figure 4.5 | Plot of surface integrand | 138 |
| Figure 4.6 | Typical convergence in the numerical integration of interior integrals | 140 |
| Figure 4.7 | The effect of angular divisions, $N_\phi$, on the % mass balance error for two test flow problems | 141 |
Figure 5.1 Output $v_z$ profile for the impulsively started flow problem using the approximate solution regularization constraint 167

Figure 5.2 Output $v_z$ profile for the impulsively started pipe flow problem using the 2nd order smoother regularization constraint 168

Figure 5.3 Output $v_z$ profile for the fully developed pipe flow problem using the 2nd order smoother regularization constraint 171

Figure 5.4 Output $v_z$ profile for the fully developed pipe flow problem using the approximate solution regularization constraint 172

Figure 5.5 $v_z$ solution error, $e_{v_z}$, vs the log of the regularization coefficient, $\alpha$ 176

Figure 5.6 % mass balance error, $e_{mb}$, vs the log of the regularization coefficient, $\alpha$ 176

Figure 5.7 Continuity error, $e_c$, vs the log of the regularization coefficient, $\alpha$ 177

Figure 5.8 Expanded plot of continuity error vs $\alpha$ 178

Figure 6.1 Stability test results using the Biot-Savart velocity formulation 190

Figure 6.2 Stability test results using the direct BEM velocity formulation 191

Figure 6.3 Steady state $v_z$ profile plots for entrance flow using four different algorithms 193

Figure 6.4 Streamlines for steady state entrance flow using Biot-Savart velocity and time-split MacCormack vorticity transport solutions 195

Figure 6.5 Streamlines of the steady state solution for the rotary coupler flow problem 197

Figure 6.6 Streamlines of the steady state modeled valve flow 198

Figure 6.7 Record of continuity error during iteration of the model valve problem 200
Figure 6.8  Profile plots at four intervals for time dependent developing flow at a pipe entrance 205

Figure 7.1. Plot of constant pressure lines for the pipe entrance flow problem 219

Figure 7.2. Plot of constant pressure lines for the rotary coupler flow problem 220

Figure 7.3. Plot of constant pressure lines for the model valve problem 220
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 3.1</td>
<td>Calculated errors using the Biot-Savart and direct BEM formulations for solution of Poisson's velocity equation</td>
<td>103</td>
</tr>
<tr>
<td>Table 5.1</td>
<td>Condition numbers of ill-posed and well-posed impulsively started flow problems</td>
<td>149</td>
</tr>
<tr>
<td>Table 5.2</td>
<td>Non-regularized and regularized solutions for the output velocity in fully developed pipe flow</td>
<td>166</td>
</tr>
<tr>
<td>Table 5.3</td>
<td>Computed errors for several regularized solutions of the impulsively started pipe flow problem</td>
<td>169</td>
</tr>
<tr>
<td>Table 5.4</td>
<td>Computed errors for several regularized solutions of the fully developed or parabolic flow problem</td>
<td>170</td>
</tr>
</tbody>
</table>
CHAPTER I. STATEMENT OF PROBLEM

Introduction

A successful engineering design typically incorporates three different types of information derived from three different sources. These three sources are: 1. technical analysis or mathematical modeling specific to the problem 2. prototype testing and 3. basic engineering principles and previous design experience. The design engineer's job is to obtain this information from calculations, testing, or consultation and then use it creatively to solve the design problem at hand. It may be said that the process of obtaining information is the "engineering" and the interpretation and application of the information is the "art" in the "art of engineering".

The relative importance of each type of information used to complete a particular design will of course depend on the nature and complexity of the phenomenon involved but also on the practical considerations of time and money. For instance, it may be possible to perform a very detailed and exact mathematical analysis for a new design but the benefits may be too costly in terms of the computing time and/or labor. It may be more expedient to do some rough calculations, then build and test a prototype or several prototypes. Again it is the design engineer's job to decide
how much if any mathematical modeling should be performed preliminary to prototype construction and testing.

In general it is advisable to perform as much appropriate mathematical modeling as the time and cost restraints will permit, because it is usually more efficient to correct mistakes and make improvements on paper than in the actual product material. The key word when deciding upon the extent of mathematical modeling is "appropriate". Extensive mathematical modeling, which in this discussion implies computer analysis, is not always an appropriate part of the design process. In fact it can even become dangerous if too much faith is placed in the output of "canned" programs. It has become a very big temptation in the last few years to substitute a "bunch of numbers" for sound qualitative engineering analysis.

It must be remembered that computer analysis cannot take the place of a good understanding of engineering principles and the physics of the problem. Rather it should supplement the design process, with the basic engineering principles supplying the direction and interpretation and the computer analysis supplying the details.

However there is one general class of design projects which often is a good candidate for an intense mathematical modeling effort. This is projects which are refinements or improvements of a design that has been in use for several
years. As a design evolves by long in service use, each small improvement becomes more costly in terms of both time and money. One possible reason for this is that the design may become more physically complex making prototype construction more expensive and thus a less desirable design tool. Also as a design evolves, it may also become more conceptually complex by employing principles and/or procedures which go beyond the current industry conventions. These new elements of the design may be original contributions to the science of engineering but more often they are a new application of an old idea from another engineering or scientific discipline. In either case the large money, time, and labor investments in mathematical modeling are usually justified since it is being applied to a design which has already proven itself functionally and economically. One such design problem has served as the motivation for the research reported in this dissertation. In the remaining sections of this chapter this design problem is described followed by an explanation of how the final research project evolved from this problem. Finally, the research topic is outlined including the inherent limitations and potential applications.
Motivation of Research

In fluid power or hydraulic systems the output actuators, cylinders and motors, are often controlled by spool control valves and poppet relief valves. The design of these valves has been developed over years of testing and redesign and in most cases adequately satisfy all functional requirements. For the spool control valve the function is to provide the cylinder or motor with the fluid flow rate which gives the desired actuator displacement-time relationship. The function of relief valves is to limit the system pressure to levels which are safe for all of the system components and fluid conduits.

Still, over the years engineers and operators have had to live with some annoying and sometimes dangerous shortcomings of the valve designs. The most noteworthy of these problems is spool or poppet dynamic instability. This problem occurs only in servo or automatic valves, not in manually operated valves. Characteristic of this dynamic instability is high frequency pressure oscillations which can result in damaging impact loading of the valves and actuators or in the worst case loss of load control.

Attempts to study and hopefully eliminate dynamic instabilities in valves necessarily start with a motion study of the spool or poppet. This type of dynamic analysis may be attempted either by frequency analysis with automatic control
theory or real time modeling on a digital computer. A study of poppet type relief valve stability using the first method was done by Wandling and Johnson (1972). The stated objective in that paper was to develop and test a method to predict instabilities in a poppet relief but not necessarily to design a stable valve. However the paper does contain an informative discussion of the factors that affect stability and the many problem areas in performing such an analysis.

In both methods of dynamic analysis, frequency analysis or real time modeling, the first step is to develop the model equation for the motion of the spool or poppet. The common starting point for developing this model equation is the classical spring-mass-damper system, Eq. 1.1.

\[ m_v \ddot{x}_v + c \dot{x}_v + k_s x_v = \Sigma F \] (1.1)

where:
- \( m_v \) = spool or poppet mass
- \( c \) = damping coefficient
- \( k_s \) = spring constant
- \( x_v \) = spool or poppet displacement
- \( \Sigma F \) = summation of forces on the spool or poppet

Before this ordinary differential equation can be solved all of the coefficients and \( \Sigma F \) must be determined for the valve being modeled. The spool or poppet mass, \( m_v \), is of course constant for a given valve. The spring constant, \( k_s \), may actually be a function of \( x_v \) but this function remains constant during the operation of the valve. However the
damping coefficient, \( c_v \), and the summation of spool forces, \( \Sigma F \), are not constant as they depend on system parameters and fluid properties which are constantly changing. The system parameters involved are pressure and flow rate supplied by the pump, \( P_S \) and \( Q_S \), and the pressure and flow rate required by the load, \( P_L \) and \( Q_L \).

Determination of the time dependent system parameters, \( P_S, P_L, \) and \( Q_L \), even in the simplest case is quite involved. Typically it is necessary to consider such things as pump performance characteristics, prime mover performance characteristics, actuator size and characteristics, load cycle, and valve characteristics. Though each of these areas may be important in the analysis, this research will investigate only the last, valve performance characteristics, and more specifically the design equations used to model valve performance.

The subject valve modeling equations describe the relationship between the spool or poppet position, \( x_v \), and the previously mentioned system parameters. In their most useful and common form \( Q_L \) is calculated as a function of \( x_v \), \( P_S \), and \( Q_S \).

The summation of spool forces, \( \Sigma F \), may include many terms but the most important by far are the static pressure force and the fluid or Bernoulli force. The equation used to calculate the static pressure force is quite simple, \( F = P \cdot A \).
The major difficulty in applying this equation lies in determining the pressure values, $P$, which may be inside and/or outside the valve.

The fluid or Bernoulli force on a valve spool or poppet is the direct result of pressure variation within the valve or more specifically within a valve cavity. The pressure variation is caused by the acceleration of the hydraulic fluid as it flows through a large opening and out of a smaller opening. The pressure distribution will of course depend on the flow rate, but also on valve geometry and physical properties of the fluid. For "large" flow rates this component force will have a major impact on the character of the valve modeling equation, Eq. 1.1.

At this point the scope of the research has been narrowed from the initial design problem of controlling or eliminating dynamic valve instabilities to developing an accurate valve modeling equation of a specific form. In this process of selecting a topic for study, several possible problem areas have been ignored. Among these are line dynamics, compressibility variations due to trapped air, and load dynamics to name a few. This is unfortunate since a more complete understanding of these phenomena would be valuable. However, as shall later be explained, the goal of this research is not to solve the problem of valve stability
but only to develop a design tool to be used on such problems.

The desire to develop an accurate spool dynamics model has isolated two design calculations to be studied. They are the calculation of $Q_L$ as a function of $x_v$, $P_S$, $P_L$, and $Q_S$, and the calculation of pressures within and immediately adjacent to the valve. The investigation will start with a description and development of the conventional methods used to model a valve followed by a discussion of their strengths and weaknesses. Finally an alternative approach to each design equation will be explained including a discussion of their strengths and weaknesses.

Cessna model 33100 directional control valve

Figure 1.1. Spool valve with cross-sectioned body
Valve model equations

As a valve spool or poppet is displaced, the flow area or metering areas within the valve change. This in turn changes the flow rate of fluid through the valve. The equations describing this change are the valve model equations. Figure 1.1 has been inserted to help illustrate what is happening inside such a valve during operation. Figure 1.1 is a cross-sectional drawing of a spool type control valve.

The conventional valve model equation is developed by first modeling the metering flow through the valve openings as sharp edged orifice flow. The assumed geometry of this type of flow is illustrated in Fig. 1.2. By making the orifice assumption, the flow through each metering opening may be described by the common orifice equation.

Figure 1.2. Orifice model of valve metering flow
The assumptions necessary in the development of the orifice equation are as follows:

1. potential or streamline flow from point 1 to point 2
   (i.e., Bernoulli's equation is applicable)

2. incompressible flow

3. all kinetic energy of the jet is not recovered
   (i.e., $P_2 = P_3$)

4. steady flow

$A_0$ is the valve metering area and $C_d$ is the discharge coefficient. $C_d$ is an empirically determined coefficient to account for the existence of a vena contracta at point 2 and the presence of viscous friction. Application of Bernoulli's equation and the continuity equation between points 1 and 2 yields the orifice equation Eq. 1.2.

$$Q = C_d A_0 \frac{2(P_1-P_2)}{\rho}^{1/2}$$  \hspace{1cm} (1.2)

$Q =$ flow rate (vol/time)

$\rho =$ fluid mass density

$P_1-P_2 =$ pressure drop across metering opening

The biggest advantage of using the orifice model is its simplicity, the equation can be evaluated quickly for each metering opening. A complete valve model is constructed by correctly combining these individual model equations. The derivation of the complete model may become tedious for valves with many metering openings but even then the calculation of the final equation is trivial.
If desired, these orifice flow equations can be used to develop an even simpler valve model, the linearized valve equation, Eq. 1.3. This equation is based on the assumption that the flow through the valve is a function only of spool position and pressure gradients, which is expressed mathematically by \( Q_f = f(x_v, \Delta P) \). The linearized valve equation is obtained by taking the partial differential expansion of this equation and ignoring all higher order terms.

\[
\Delta Q = K_q \Delta x_v + K_p \Delta P \tag{1.3}
\]

In Eq. 1.3 \( K_q \) and \( K_p \) are the flow gain and flow-pressure coefficient, respectively, for the valve. They are defined as follows:

\[
K_q = \frac{\partial Q}{\partial x_v}, \quad K_p = -\frac{\partial Q}{\partial P}
\]

Approximate values for \( K_q \) and \( K_p \) can be obtained by differentiation of the orifice based valve model equation for a specified range of \( P_1-P_2 \) and \( x_v \). This of course means that each set of \( K_q \) and \( K_p \) can be used for only part of the valve's operating range. It is left to the engineer to decide upon an acceptable operating window for each set of valve coefficients.

Another characteristic of the orifice based valve model equation is its general nature. The assumptions made in its development allow it to be used with most all flow conditions regardless of changes in flow regime (laminar or turbulent).
and fluid viscosity. This is not to say that the equations are not better suited to some types of flow, only that it is not reflected in the model equations.

The general model equation does eliminate the need for the designer to consider flow regimes in the analysis but this simplicity is not accomplished without penalty. In making the necessary assumptions, critical information about the particular valve and operating condition is lost. First of all the metering area is assumed to be planar when actually it is annular (the area between two concentric circles). Also, in their given form, the equations cannot be used to predict the effect of changes in flow cavity geometry and dimensions.

Another shortcoming of the orifice based model equations is the absence of viscosity, a characteristic which was previously labeled an advantage. This is particularly disturbing since a change in viscosity is the easiest and best way to input temperature effects into the model. Anyone who has operated a hydraulic system in sub-zero weather has experienced the effect of temperature changes on system performance. The changes can be dramatic but the conventional valve model has no way to predict what these changes will be.
Alternative valve model

As was already explained the function of the valve model equation is to predict the flow rate through the valve and to the load, \( Q_L \), for different spool position, \( x_V \), and system parameters. Another way to accomplish this is to first determine the fluid velocity field in the valve cavity for the existing flow conditions and cavity geometry. The flow rate either in or out of the valve can then be calculated by integrating the normal velocity across the appropriate boundary, as shown below.

\[
Q_L = \int_{A} (\vec{V} \cdot \vec{n}) \, dA \quad (1.4)
\]

The main advantage of using the velocity field approach rather than the orifice flow model approach is that it is possible to incorporate much more information about the particular valve and flow conditions being modeled. The geometry of the flow cavity can be modeled exactly or approximately, whatever is deemed sufficient. Also, it is possible to include viscosity in the model, as in boundary layer flow, or exclude it, as in potential flow. Neither of these options is available with the orifice based model equations.

The complete explanation of the proposed method of determining the velocity field in the valve cavity is the subject of a forthcoming complete chapter. Briefly stated,
it is the numerical solution of the governing partial
differential equations for specified boundary conditions.
The governing equation will be determined in part by the
specified flow regime, dimension of the problem, and the
selected coordinate system. There also are other more subtle
factors which influence the form of the governing equation,
all of which will be discussed in detail later.

**Conventional spool pressure force calculation**

Earlier it was stated that the two most important forces
on a spool are the static pressure force and the Bernoulli
force, and indeed these are the two usually emphasized in
fluid power texts. The same textbooks also give equations to
be used for estimating the values of the forces. However,
the derivation of these equation, in most cases, do not
accurately reflect the "physics of the problem" or the source
of these terms. Before an improved spool force analysis can
be developed this void must be filled. To this end a short
qualitative analysis of the pressure distribution in a poppet
valve during operation will be made. A similar analysis
could be made for a spool valve but the effect of the
Bernoulli force on the spool is very subtle and much harder
to visualize.

Consider the flow cavity in Fig. 1.3 with the ports
marked inlet and outlet. As is often the case one of the
ports, the inlet in this case, is essentially fully open at
all times. All flow control or metering is done with the other port, the outlet in this case.

The static pressure is simply the summation of all $P \times A$ forces acting on the spool. Calculation of these components of course requires that all of the areas and pressures acting on these areas be known. The areas are easily calculated by geometry and may be balanced or unbalanced for a given pressure field. The pressures may be determined by the pump characteristics, load dynamics or pressure drops across valve metering ports depending on the location of the pressure relative to the other components in the system. In the case of pressure drops in valves, the analysis will usually employ some form of the orifice flow equation and thus will inherit all of the assumptions and potential errors that were discussed in the valve model development.

There is one important and frequently unmentioned assumption in the static pressure force analysis; namely, it is assumed that in any valve cavity the pressure distribution is uniform. This means that the resultant force on the spool from the pressure in any valve cavity will be zero unless the areas involved are unbalanced, such as when one end of the spool exits the valve body.

It is known that when fluid enters a cavity through a large opening and exits through a relatively smaller metering opening, the pressure distribution will closely resemble the
one in Fig. 1.3. As a quick explanation, make the gross assumption of potential flow and consider a streamline from the cavity inlet to the outlet. Bernoulli's equation then dictates that the pressure must decrease as the fluid accelerates along the streamline toward the outlet. Although this is acknowledged, the uniform pressure distribution assumption must be made since the simple conventional spool force methods cannot predict the actual distribution.

To compensate for the inability to predict actual pressure distributions within a valve, the effects of the non-uniformity are estimated using a control volume analysis.

![Figure 1.3. Pressure distribution in a spool valve cavity during flow](image_url)
Consider a control volume of fluid whose shape is defined by the flow cavity geometry, Fig. 1.4. The same analysis can be performed for a spool valve but it is much more subtle.

Note that the control volume is actually a 2 dimensional planar model of the 3 dimensional flow cavity. Fortunately, given the established tone of the analysis, this simplifying assumption is not important. This will not be allowed in the more detailed velocity field analysis which will be offered as an improved alternative.

Figure 1.4. Control volume analysis of poppet force balance
Performing a force analysis in the axial direction on the control volume in Fig. 1.4 yields Eq. 1.5. In this
\[ \int_A P \cdot dA - (\dot{m}V)_{\text{out}} \cos \theta = 0 \] 
(1.5)
equation the first term represents the force on the spool due to the non-uniform pressure distribution within a specified cavity and shall be called \( F_B \). Before \( F_B \) can be calculated, estimates must be made for \( \dot{m} \) and \( V \). Again the orifice equation is pressed into service and along with the given expressions for \( \dot{m} \) and \( V \) the Bernoulli force, \( F_B \), can be written as Eq. 1.6.
\[ \dot{m} = \rho Q_L \]
\[ V = Q_L / A_0 \]
\[ F_B = 2C_d C_C A_0 (P_1 - P_2) \cos \theta \]
(1.6)

Still there remains one more parameter to be estimated before \( F_B \) can be calculated, that is the angle \( \theta \). As illustrated in Fig. 1.4, \( \theta \) is the discharge angle of the fluid jet out of the metering opening. The actual discharge angle is a function of several factors including radial clearance between the spool and the valve body, wear on the spool, and spool position. Needless to say the range of \( \theta \) values which a spool valve may experience during operation is quite large. Von Mises (Merritt, 1967, p. 103) reported that \( \theta \) can vary from 21 degrees to 69 degrees due to changes in spool position alone, which translates to a 260% change in
F_B. This information is from an analytic analysis which assumed 2 dimensional potential flow.

Though the work of Von Mises has provided a way to represent $\theta$ as a function of $\kappa_\gamma$, it seems that this is rarely done. As in the work of Wandling and Johnson (1972) it is common practice to select a constant value for $\theta$, and that value is usually close to 69 degrees.

The conventional spool pressure analysis is seen to have some significant shortcomings. Just as in the conventional valve model equations, most of the guilty simplifying assumptions are related to using the sharp edged orifice equation to model valve flow. In addition it is necessary to select a value for the discharge angle, $\theta$, from a large range possible of values.

**Alternative spool pressure force analysis**

In the spool force analysis just presented many of the simplifying assumptions were necessary because the actual pressure distribution inside the valve was not known. Any technique that could determine these distributions as a function of the cavity geometry, fluid properties, and flow regime should be an improvement over the conventional method. The proposed alternative spool pressure force calculation method employs such a technique.

Determining the pressure field in a valve cavity, as will be demonstrated, is quite tedious. But once obtained,
the spool pressure force calculation is very straightforward. The resultant force on the spool due to fluid pressure is obtained by integrating the known pressure over the spool surface area. It is not necessary to calculate separately the static pressure forces and Bernoulli forces.

Pressure field determination requires the solution of the appropriate partial differential equation(s) for a specified set of boundary conditions. This procedure is similar to the velocity field analysis, so much so in fact that they are almost the same problem. As will be shown, a given velocity field also defines an associated unique pressure field. This is quite convenient, as it means that the same analysis procedure could produce both an improved valve flow model and an improved spool pressure force estimation.

Research Objectives

Thus far this chapter has been devoted to the discussion of a particular engineering design problem, valve dynamic stability, and the common methods employed to analyze the problem. Also there was a discussion of the advantages and disadvantages of these methods and finally a proposal of an improved analysis approach.

Given the effort spent on development of the valve stability problem, it may seem contradictory that the
objective of the research is not to solve this problem. The
design problem was used only as a vehicle to motivate the
research whose broad objective is to develop an analysis or
design tool to solve such design problems. The tool
developed is a computer implemented numerical method and thus
the research is in the area of applied mathematics and of a
basic nature. This being true and since the research was
officially administered by the College of Engineering, there
was a need, perhaps unjustified, to established a strong tie
between the numerical work and a real engineering problem.
Even though the stability problem will not be addressed after
this chapter, the numerical tool was developed with a
constant eye towards its application to this problem and
other design problems requiring similar information.

The alternative methods for the valve model equation and
spool pressure force analysis have been described only in
general terms. Namely, by determining the velocity and
pressure fields for a specified valve cavity and boundary
conditions. These methods will now be described in
sufficient detail to establish the governing partial
differential equations which are to be solved. This will
require consideration of flow regime, physical properties of
the fluid, and the necessary dimension of the problem.
Assumptions and restrictions

The first step in establishing the correct governing equations in a fluid flow problem is to define the fluid being modeled. The most important fluid properties in most cases are fluid density and viscosity and shall be the only ones discussed.

The fluid is first of all assumed to have a constant density. Physically, this implies that the fluid is incompressible with no entrained air. This is done even though it is known that air will at sometime enter every hydraulic system. However, the phenomenon of entrained air is so complicated it could only be modeled, if at all, by a statistical or stochastic technique. For this reason the possibility of entrained air will not be considered.

Constant density also implies no temperature dependence, which is a necessary condition to uncouple the momentum and energy equations. Physically, this assumption is unacceptable only in cases where buoyancy is the primary driving force of the fluid flow.

From a mathematical standpoint, the assumption of constant density is critical. The conservation of mass equation, which must be satisfied everywhere in the flow field, is $\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$. If $\rho$ is assumed constant the conservation of mass or continuity equation is reduced to $\nabla \cdot \mathbf{v} = 0$. This identity is used numerous times in the
derivation of the governing partial differential equations as well as their numerical approximations.

The fluid is also assumed to be Newtonian. This permits the fluid shear stress to be expressed as a linear function of velocity gradients or shear rate. This assumption, along with \( v \cdot \hat{v} = 0 \), permit the fluid stress to be expressed in the form necessary to reduce the conservation of momentum equations to the common Navier-Stokes equations. For example, the normal stress on a differential cube of fluid in the \( x \) direction is \( \tau_{xx} = -p + 2\mu \varepsilon_{xx} + \lambda(v \cdot \hat{v}) \) where:

- \( \tau_{xx} \) = normal stress
- \( \varepsilon_{xx} \) = strain rate on the \( x \) face in the \( x \) direction
- \( \mu \) = absolute viscosity
- \( \lambda \) = bulk viscosity.

For constant \( \rho \) this is reduced to \( \tau_{xx} = -p + 2\mu \varepsilon_{xx} \).

Another necessary condition of the mathematical derivation is that viscosity be constant throughout the flow region. This does not mean that viscosity cannot change with time only that at any given time the viscosity is the same everywhere in the flow region.

Fluid flow in hydraulic valves is a true 3 dimensional phenomenon. Still, with judicious coordinate system selection it can often be modeled by a 2 dimensional system of equations. Consider Fig. 1.5 which is an attempt to illustrate a control valve spool in 3 dimensions. At the end
of the spool is the axis for the cylindrical polar coordinate system \((r, \theta, z)\). The geometry of the spool is axisymmetric since it has no \(\theta\) dependence and therefore can be described by 2 dimensions in cylindrical polar coordinates. The same principle holds for the valve body though this is not shown in the drawing.

Since the character of the flow in the valve is greatly influenced by the flow cavity geometry it is reasoned that if both the spool and body are axisymmetric so will be the flow.

![Illustration of axisymmetric spool geometry in cylindrical polar coordinates](image)

**Figure 1.5.** Illustration of axisymmetric spool geometry in cylindrical polar coordinates
This means that in cylindrical polar coordinates the governing equations of the flow will be 2 dimensional. Note that if rectangular were used the governing equations would necessarily be three dimensional.

Even though there are many valves with 3 dimensional geometries, the reduction in numerical complexity afforded by the axisymmetric assumption is hard to resist, especially as a first approximation. For this reason all numerical techniques developed in this research are for axisymmetric geometries. Also all component equations are written in cylindrical polar coordinates.

The final consideration in the specification of the governing equations is flow regime, which essential means either laminar or turbulent. As has been hinted though not specifically stated, the immediate goal is to define a fluid flow which is governed by the vector form of the Navier-Stokes equations. Theoretically all flows, laminar, transitional, and turbulent, are governed by the Navier-Stokes. Realistically though, numerical solutions can only be obtained for laminar flows and analytic solutions only for the simplest of these.

It is generally accepted that the direct solution of Navier-Stokes equations for turbulent flow would require such small spacial and temporal scales that it would be impractical if not impossible. There are, however,
alternatives to the single-valued determination of field variables which is a workable approach in laminar flows. A alternative approach for turbulent flows is to decompose each variable into a stable mean value component, denoted by a overbar, and a fluctuating component, denoted by a prime. For example the pressure variable, p, is replaced by $\bar{p} + p'$. Each new variable is then subjected to some form of filtering or averaging process. The simplest and most familiar is the Reynold's time-averaging proposed in 1895 by Osbourne Reynolds (White, 1974, p. 453). When the filtered values are substituted into the Navier-Stokes equations several new terms appear. The most notable of these is the turbulent stress tensor usually called the turbulent stress. Better estimation of this stress has received most of the attention in research of turbulent flows using this approach. Recent examples include Srinivas and Fletcher (1984) and McDonough and Bywater (1986).

Another approach to turbulence modeling is usually referred to as statistical fluid mechanics as it employs the statistical theories of probability. In the words of Monin and Yaglom "The basic feature of probability-theory approach (or, more commonly, the statistical approach) to the theory of turbulence is the transition from the consideration of a single turbulent flow to the consideration of the statistical ensemble of all similar flows, created by some set of fixed
external conditions" (Monin and Yaglom, 1965, p. 209). In this approach each deterministic field variable is replaced by a stochastic variable defined by a selected set of statistical parameters.

All of the details and variations of the modified governing equations for turbulent flow will be left to other more qualified sources. For current purposes it is sufficient to note that both of the variable substitutions result in a set of governing equations which are drastically different from the original Navier-Stokes and continuity equations. This of coarse means that laminar and turbulent flows require completely different modeling algorithms. The numerical techniques developed in this research will produce valid solutions only for laminar flows.

Since it is probable that most interesting engineering fluid flows are turbulent, the value of a technique valid for only laminar flows may be questioned. At the risk of appearing to plead the case two arguments are presented in support of the research. First of all, consider power hydraulic control valves, though most of the troublesome flows are turbulent, not all of them are. An interesting example of this occurs when the valve is being used to move loads very slowly. In many cases, particularly in closed-centered hydraulic systems, the fluid velocities are such that laminar flow will exist within the valve.
The strongest justification for doing the research, however, has nothing to do with fluid flow problems. As was mentioned earlier the research objective was not to solve the motivating valve stability problem but rather to develop a design tool to help with this and similar problems. Given the fundamental nature of the governing equations a "similar problem" could be selected from engineering disciplines other than fluid mechanics. Any problem which is governed by a partial differential equation similar to those of unsteady laminar flow will benefit from this research. This is perhaps the advantage of basic research over applied research.

Summary

Objective

Develop a computer implemented numerical technique to determine velocity and pressure fields for initial and boundary conditions common to fluid flow in hydraulic flow valves. Also test the computer program on flow problems which have analytic solutions or other numerical solutions for comparison.

Restrictions

The flows to be modeled in this research are subject to the following restrictions. The fluid properties of importance are: 1. constant density (incompressible), 2. uniform viscosity, and 3. Newtonian velocity gradient-shear stress relationship. The problem dimension is assumed to be 2 dimensional with all scalar
component equations written in cylindrical polar coordinates. Also the model is valid only for laminar flow.

The classical partial differential equations which govern the described fluid flow are the Navier-Stokes and continuity equations. The forms of these equations and a review of numerical solution procedures is the subject of Chapter II.
CHAPTER II. BACKGROUND
INFORMATION WITH LITERATURE REVIEW

As was stated in Chapter I, a conscience effort has been made to limit the research to flows which are governed by the Navier-Stokes (momentum) equation, which are given in their vector form, Eq. 2.1. Since assumptions have been made necessary to uncouple the momentum equation from the energy equation, the only other governing equation is the continuity equation or conservation of mass, Eq. 2.2, also given in its vector form.

\[ \rho \frac{D\vec{v}}{Dt} = -\nabla p + \mu \nabla^2 \vec{v} \]  
(2.1)

\[ \nabla \cdot \vec{v} = 0 \]  
(2.2)

The vector forms of these equations are valid for the standard engineering coordinate systems: rectangular, cylindrical polar, and spherical polar. Before such a vector equation can be solved numerically, it must first be converted to scalar component form. This is done by performing all of the vector operations as defined for the chosen coordinate system. As was explained earlier, for axisymmetric flows the most efficient coordinate system is cylindrical polar. The cylindrical polar component forms of the Navier-Stokes (N-S) equations are shown in Eq. 2.3.
\[
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[ v^2 v_r - \frac{v_r}{r^2} \right]
\]

\[
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu v^2 v_z
\]  

(2.3)

Solution of the Primitive Variable Equations

The equation set of Eq. 2.3 contains three field variables: \( v_r, v_z, \) and \( p. \) These are called the primitive variables and the equations containing them are called the primitive variable form of the Navier-Stokes equation. Since all important variables are used, it is possible, though not always easy, to specify any of them as boundary conditions which is sometimes desirable. However the resulting complexity from using three variables and problems associated with the pressure field sometimes make them unattractive, especially in problems where the pressure field is not of interest. This point will be expanded later as well as presenting a possible alternative.

Finite difference methods

Numerical problems caused by the pressure variable are related to the elliptic nature of the pressure field. Since the primitive variable Navier-Stokes equations are parabolic, it is difficult to maintain the correct physical characteristics of the pressure field with direct solution. An early attempt to resolve this inconsistency was presented
by Chorin (1968). He proposed including an artificial compressibility term in the compressible continuity equation. This changes the governing equations to a mixed set of hyperbolic-parabolic partial differential equations. Iterative procedures are employed which gradually decrease the compressibility to zero which preserves the elliptic character of the pressure field. These equations can be solved with any of the many explicit techniques developed for the standard linearized Burger's equation (Anderson, Tannehill, and Pletcher, 1984) or the implicit methods developed by Steger and Kutler (1976). A disadvantage of the later is that it is not time accurate and therefore can only be used as a means to converge to a steady state solution.

Probably the more common approach to solving the primitive variable Navier-Stokes equations is to solve for the pressure field separately with the Poisson equation for pressure. Examples of this multi-step approach have been presented by Ghia et al. (1979), Patankar (1981), and Kwak et al. (1986). An additional feature of the method contained in the last reference is that it can also be used on turbulent flow problems.

**Finite element methods**

It is also possible to use finite element techniques to solve the primitive variable equations. The major differences between the prominent finite element methods are:
1. method of time step integration, 2. handling of the pressure variable, and 3. method used to formulate the finite element integrals.

Mizukami and Tsuchiya (1984) developed a 3 dimensional time dependent solution of the Navier-Stokes equations using an explicit Euler's time marching method based on Helmholtz's decomposition theorem. This results in a two step system of governing equations that do not contain pressure as a primary variable which in most cases is an advantage. These are solved for the velocity field using the standard Galerkin finite element formulation and pressure is obtained from a secondary equation. This method was tested on the problem of entrance flow in a square duct.

Another 3 dimensional time dependent finite element method was developed by Gresho et al. (1984). A modified Galerkin formulation was used which is reported to be more time efficient than the standard Galerkin method. Test results are reported for two problems: 1. steady flow in a lid-driven cavity, and 2. flow past a circular cylinder.

Van De Vosse and Segal (1986) developed a 2 dimensional unsteady technique which uses the penalty function approach to eliminate pressure from the component momentum equations. Two time integration methods were tested, Euler implicit and Crank-Nicolson, on two model problems: 1. oscillating channel flow, and 2. flow over a circular cylinder.
Segal (1985) reviewed three common finite element formulations for the steady state Navier-Stokes equations. They were: 1. standard Galerkin, 2. Galerkin with pressure penalty function, which eliminates pressure from the equations, and 3. divergence-free Galerkin. Another comparative discussion is given by Gresho and Lee (1981) with emphasis on the consequences of using non-Galerkin formulations.

**Boundary element methods**

Boundary element methods may also be used to solve the primitive variable momentum equations. This method is similar to finite elements since they both involve integral equations but there are distinct differences. For some types of problems, though not for the current one, only boundary integrals are required and for all problems the final formulation is explicit rather than implicit as in finite elements. These differences can result in significant savings in computing time and/or storage. Such a technique for steady state Navier-Stokes flow is presented by Bush and Tanner (1983).

**Solution of the Vorticity Transport Equation**

The primitive variable approach to solution of Navier-Stokes flow is very general as it can be applied with any coordinate system and any dimension problem. In fact it
is the only choice for 3 dimensional problems. However even for 2 dimensional problems the complexity of the numerical formulation and problems with the pressure variable have led numerical analysis in search of alternatives. In some special cases it is acceptable to solve a reduced set of governing equations such as the Parabolized Navier-Stokes equations (Anderson et al., 1984, p. 424). Another very important approach, which requires no additional simplifying assumptions, replaces the primitive variables of component velocities, \( v_r \) and \( v_z \), with the derived variable of vorticity, \( \omega \). Mathematically, vorticity is defined as the curl of the velocity vector, i.e., \( \vec{\omega} = \nabla \times \vec{V} \). Physically, it can roughly be described as a measure of the angular velocity at a given point in a flow field.

With the new variable set, \( \omega \) and \( p \), the primitive variable equations are replaced by three equations, which are slightly different for the two most common methods. For both methods the first is always the vorticity transport equation but there are two possible choices for the second. In one method the stream function field is determined with Poisson's stream function equation, \( \nabla^2 \psi = -\omega \), where \( \psi \) is the stream function. The velocity is then determined point by point with finite difference representations of Eq. 2.4.

\[
v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \quad v_z = \frac{1}{r} \frac{\partial \psi}{\partial r}
\]  

(2.4)
In the other method, the velocity field is calculated directly by using the most recent vorticity field in Poisson's equation for velocity, \( \nabla^2 \vec{v} = -\nabla \times \vec{\omega} \). The third equation for both methods is Poisson's equation for pressure. Its solution is optional and need only be solved for problems where pressure is of interest. The appropriate solution methods for all of the Poisson PDEs are very similar and will be discussed later in this chapter.

The vorticity transport equation is derived by taking the curl of the vector Navier-Stokes equation, Eq. 2.1. The result of this vector operation is Eq. 2.5 (Appendix B).

\[
\frac{\partial \vec{\omega}}{\partial t} = -(\vec{\nabla} \cdot \vec{v}) \vec{\omega} + (\vec{\omega} \cdot \vec{v}) \vec{v} + \nu \nabla^2 \vec{\omega} \tag{2.5}
\]

Equation 2.5 describes the convection and diffusion of vorticity in a flow field as a function of time, velocity field, and boundary conditions.

For 3 dimensional problems, the advantages of the vorticity variable approach over the primitive variable is negligible, however for 2 dimensional problems the reduction in numerical complexity is dramatic. This is illustrated by comparing Eq. 2.6 which is the component equation for axisymmetric flow expressed in cylindrical polar coordinates to Eq. 2.3 which are the primitive variable component forms.

\[
\frac{\partial \omega}{\partial t} = -\frac{\partial (v_z \omega)}{\partial z} - \frac{\partial (v_r \omega)}{\partial r} + \nu \left[ \nabla^2 \omega - \frac{\omega}{r^2} \right] \tag{2.6}
\]
The attraction of the vorticity transport approach as an alternative to the primitive variable approach is obvious. First of all there is only one component equation as compared to two for the primitive variable method. Second, pressure is not a primary variable thus eliminating all of the numerical complications associated with it. It is mainly for these reasons that this approach has been selected to model the axisymmetric flows being studied in this research.

As with the primitive variable equations there have been a number of proposed solution techniques for the vorticity transport equation. Thus far most of the work has been done with finite difference formulations. Only recently have integral and integro-differential solutions been developed, i.e., finite element and boundary element methods. Because of its relative simplicity when applied to regular grids and the wealth of available literature it was decided to use finite difference in this research. A detailed review of literature on finite difference solution methods is presented first followed by a brief review of the integral and integro-differential formulations, which are included for the sake of comparison.

**Finite difference methods**

The variations found in the published finite difference solutions to the vorticity transport equation usually are in one of three general areas. They are: 1. differencing
formula and order (of truncation error), 2. stability criterion, and 3. boundary conditions. The differencing formula is the discrete approximation of each the partial derivative in the equation and should reflect the physics of the problem. The order of the truncation error will depend both on the desired accuracy and on the method used to derive the representation. In general higher order representations result in smaller truncation errors. The last group, boundary conditions, is often considered the most important aspect of numerical modeling as they are an essential link between the physical problem and the numerical problem.

**Differencing formula** In the vorticity transport equation there are three types of partial differentials to be approximated by finite difference representations. They are: 1. first order in time, 2. first order non-linear spatial, and 3. second order spatial. Considering the number of possible combinations of differencing type and order, it would be unrealistic to test and review each one. Fortunately this is not necessary as there are only a few that have proved useful.

The first order time derivative is usually represented by either a first order explicit or first order implicit formula. As noted in an early work on the solutions of axisymmetric flows by Strawbridge and Hooper (1968), the major difference between the two is the allowable time step
size, $\Delta t$. He concluded that, in general, implicit methods permit larger time step, but also require more computation time owing to the necessary iterative solution of a large set of simultaneous equations. Torrance (1968) conducted a comparison of five finite difference methods to solve natural convection in a cylinder. Evaluations of two implicit, one first order and one second order, and one explicit led to the same conclusions as Strawbridge and Hooper. Roach and Mueller (1969) used only explicit time marching as did Mei and Plotkin (1985). It should be noted, however, that the Mei and Plotkin method used time marching only to reach an asymptotic steady state solution and was not time accurate.

Arguably the most troublesome terms in the vorticity transport equation are the first order non-linear terms which describe the convection of vorticity. In addition to differencing variations, there is a fundamental difference in the form of these partial derivatives which separates all methods into two mutually exclusive groups: conservative or non-conservative. A partial differential equation (PDE) is said to be conservative if it can be expressed as the divergence of a physical property. Numerical methods which are based on such PDEs are also said to be conservative. In the vorticity transport equation the physical property is vorticity, therefore conservative methods will more closely satisfy the conservation of vorticity. (It is interesting to
note that in the primitive variable equations this translates to the conservation of mass.) Torrance (1968) verified the advantage of conservative formulations in his study of five finite difference methods. Similar results were reported by Atias, Wolfshtein, and Israeli (1977) in a study of steady solutions to lid-driven flow in a square cavity. Here a second order non-conservative method was shown not to be conservation accurate whereas two types on conservative methods were accurate. Other researchers who used conservative methods exclusively were Mei and Plotkin (1985) and Roach and Mueller (1969).

The majority of researchers in computational fluid dynamics recommend using "upwind" differencing of the convection terms rather than central differencing. The argument used is that it is the more physically correct model of the mechanics of vorticity transport since it attempts to use only upwind or upstream information to predict new downstream values. Upwinding assures that the effect of a perturbation is advected only in the direction of velocity. Such a numerical method is said to possess the transport property.

The difference between upwind and central first order finite difference representation at the node \((i,j)\) is illustrated in Fig. 2.1. The nodal values used in each formula are indicated by the symbol \(\times\). The upwind node
pattern corresponds only to the convective velocity direction shown by the arrows (→), whereas the central difference formula is insensitive to velocity direction.

Torrance (1968) noted that first order representations of the convection terms will contain large truncation errors and should be avoided. Also, both first and second order representations contain a term having the same form as the diffusion or viscous term, i.e., a false diffusivity or viscosity. This is exposed by developing the modified PDE, which is the actual PDE solved by a finite difference method (Anderson et al., 1984, p. 90). The false diffusivity makes the flow appear more viscous than it actually is thus damping out numerical instabilities, which is desirable, and real oscillations, which is not desirable. The coefficient of

\[
\begin{array}{ccccccc}
 i-2 & i-1 & i & i+1 & i+2 & j+2 & i-2 & i-1 & i & i+1 & i+2 \\
 \rightarrow | | | | | | | | | | | \rightarrow j+2 \\
 \rightarrow | | | | | | | | | | | \rightarrow j+1 \\
 \rightarrow | | | | | | | | | | | \rightarrow j \\
 \rightarrow | | | | | | | | | | | \rightarrow j-1 \\
 \end{array}
\]

upwind differencing          central differencing

Figure 2.1. First order upwind and central differencing
the false diffusivity term is proportional to the grid spacing. Therefore in order to keep this term small, it is necessary to use very small grid spacings when standard upwinding is used. Mei and Plotkin (1985) also recognized this problem and recommended a modified second order representation which is artificial diffusion free. Gresho and Lee (1981) expounded on the dangers and false security found in upwinding of the convection terms. They strongly recommend using central difference with variable grid spacing to suppress numerical oscillations rather than relying on the false viscosity of the upwinding method. Leonard (1979) on the other hand submits that this approach is computationally inefficient and has developed a artificial diffusion free third order method as a remedy.

The finite difference representation of the diffusion terms is much more consistent throughout the literature. Nearly all references used some form of second order \(O(\Delta x^2)\) central differencing formula, with some being explicit and some implicit. Torrence (1968) tested both but did not report any differences.

**Stability analysis** In order to ensure accurate modeling of the subject flow it is necessary to place practical limitations on spacial grid spacing and on time step size, when time accurate solutions are desired. In the interest of computing efficiency, the largest step size
should be used which does not distort the model. As a review of literature reveals, this is a grey area in computational fluid dynamics. Although there are some useful and mathematically eloquent stability analysis methods, each new problem will usually require some trial and error to find a workable combination of $\Delta t$, $\Delta r$, and $\Delta z$ (for axisymmetric flows in cylindrical polar coordinates).

For the problem of flow in a rotating cylinder, Strawbridge and Hooper (1968) used the Von Neumann method (Anderson et al., 1984, p. 71) to derive the following time step size criterion:

$$\Delta t \left[ \frac{8}{3\Delta r^2} + \frac{2}{\Delta z^2} + \frac{|v_r|}{\Delta r} + \frac{|v_z|}{\Delta z} \right] \leq 1.$$

An analysis of the truncation error and modified PDE led to practical spatial size limits. Roach and Mueller (1969) recommend the following maximum time step for two dimensional planar flow:

$$\Delta t = \left( \frac{v_r}{\Delta r} + \frac{v_z}{\Delta z} + 2\left( R_0 \right) \left( \frac{1}{\Delta r^2} + \frac{1}{\Delta z^2} \right) \right)^{-1}$$

where $R_0$ is the mesh Reynolds number.

Another approach to the problem of numerical stability of the vorticity transport equation is to use the complete non-linear Burger's equation as the model equation. This opens the door to a host of tested methods for establishing critical time step size and grid spacing based on $R_0$ (Anderson et al., 1984, p. 154). An example which was used
in this research is the empirical formula developed by Tannehill for the MacCormack Method:

\[
\Delta t \leq \frac{(\Delta x)^2}{|A|\Delta x + 2\mu}
\]

**Boundary conditions**

The importance of realistic and accurate boundary conditions in numerical analysis cannot be overstated. In the solution of the vorticity form of the Navier-Stokes equations there are three general types of boundary conditions which are most common. They are: 1. constant value or a function of conditions external to the computational region, 2. no-slip and non-porous boundaries, and 3. non-restrictive outflow boundaries. The first group defines how the flow field is interfaced with its immediate environment and should present no numerical difficulties.

The second boundary condition group represents the commonly assumed conditions of a viscous fluid contacting a impermeable solid container or wall. The velocity conditions are simple, both the normal and tangential velocities are constant and equal to zero at the boundary. The no-slip boundary vorticity is much less straightforward and is quite troublesome. First of all, for unsteady flows its value is not constant and secondly, the growth of vorticity on a no-slip boundary is not a transport phenomenon. This means that it cannot correctly be determined by the vorticity transport equation. Instead most analysts first equate
vorticity to the velocity gradients at the boundary, Eq. 2.7, and then approximate them with either first or second order finite difference representations of Eq. 2.4. Roach and

$$\omega = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$$

(2.7)

Mueller (1969) expanded the stream function, $\psi$, out from the wall and truncated to the result $\omega_w = -2(\psi_{w+1} - \psi_w)/\Delta n^2$ where $w$ refers to the node at the boundary and $\Delta n$ is the normal distance from the boundary to the nearest node. Similar formulas were used by Mei and Plotkin (1985), Strawbridge and Hooper (1968) and Torrance (1968).

Gupta and Manohar (1979) discussed the approximation of the no-slip vorticity and warned that second order formulas, as used by Roach, can often lead to inaccurate and/or unstable solutions. Wu (1976) concurs with this assessment and also explains some serious shortcomings of the first order formulas as well. Briefly, the problem is that the first order formula implies a zero pressure gradient tangential to the boundary. The result is a solution valid only for that special case which may be inconsistent with other boundary conditions.

To overcome these problems Wu (1976) developed an alternative technique which does not involve finite difference approximations of the velocity gradients. Instead
it is based on finding the no-slip vorticities which are
kinematically compatible to all constant boundary conditions
and the interior vorticity field as determined by the
vorticity transport equation. This technique is actually an
extension of the normal collocation procedure necessary to
find the unspecified boundary values in all boundary element
solutions of elliptic partial differential equations. This
topic is covered in a later section.

The last type of boundary condition of interest is the
outflow boundary condition. Proper specification of this
condition is especially difficult for problems which are
influenced by downstream flow behavior. These boundary
conditions must somehow communicate to the computational flow
domain what is happening, or should happen, downstream.
Depending on the problem, this information may be in the form
of a pressure, pressure gradient, velocity gradients, etc. A
common example of this type of boundary appears in the study
of the entrance flow in a circular pipe. Usually, the
outflow boundary of the numerical flow domain will be
positioned just beyond the predicted point of fully developed
velocity profile. The general philosophy for this and
similar problems is to specify the least restrictive boundary
condition which produces a stable solution. This is
appropriate for flows which are known to be changing very
slowly downstream from the defined outflow boundary.
For outflow boundaries in unsteady problems Roach and Mueller (1970) used two different boundary conditions. For \( R < 10 \) the condition used was \( \frac{\partial^2 \omega}{\partial z^2} \bigg|_0 = \frac{\partial^2 \omega}{\partial z^2} \bigg|_{0-1} \) where \( 0 \) is the outflow boundary node and \( z \) is assumed to be normal to the boundary. The more restrictive condition of \( \omega_0 = \omega_{0-1} \) was used for \( R > 10 \).

Often times the set of known outflow boundary conditions may not be appropriate, resulting in an ill-posed problem. If the above approach does not appear sufficient it may be necessary to use an iterative procedure or solve a redefined well-posed problem.

Finite element methods

An often cited disadvantage of the vorticity form of the Navier-Stokes equations is the unfortunate boundary condition situation. As was mentioned earlier two separate equations must be solved in order to obtain the desired velocity field. In the early days of finite element, it was generally accepted that the two would be solved separately; first, the vorticity transport equation and second, the Poisson's equation for velocity. The problem with this approach is that the no-slip vorticity is needed to solve the vorticity transport equation but unfortunately these are not known \textit{a priori} thus requiring a iterative procedure.

The common approach to overcoming this problem was to estimate the no-slip vorticity by normal velocity gradients
at the boundary, as described earlier in the finite difference section. These gradients were approximated using nodal velocities of the previous time step (or iteration level). The new vorticity field is then used to solve Poisson's stream function equation for the stream function field which in turn is used to define the new velocity field using Eq. 2.7. These new velocities are then used to make a better estimate for the no-slip vorticities and the process is repeated until the solution converges. An example of this iterative technique is given by Taylor and Hood (1973).

Because of the perceived need to use iterative procedures many analysts have recommended using the primitive variable rather than vorticity formulations (Taylor and Hood, 1973). However, there have recently been techniques developed that permit the vorticity transport equation and Poisson's stream function equation to be solved simultaneously without knowing the no-slip vorticities beforehand. This naturally requires the solution of a larger set of simultaneous equations but eliminates the need for iteration. One of the first to propose such a technique was Campion-Renson and Crochet (1978). Additional examples are presented by Stevens (1982), for natural convection, and Mizukami (1983), for obstructed flow in a cavity.

The non-iterative technique has removed what was considered to be a serious deficiency of the vorticity
formulation of the Navier-Stokes equations. The only possible drawback is the increased computer memory and time required to solve the larger set of equations. The choice is then determined by available computer hardware and not available numerical techniques.

Another point of discussion in the formulation of finite element solutions is the way the convection terms are handled; specifically, the form of the weights used in the Galerkin formulation of the weighted residual method (Zienkiewicz and Heinrich, 1978, p. 1). The issue is the same as was discussed in the section of finite difference methods; should the non-linear convection terms be modeled with upwinding methods or linear methods. The concept of upwinding is easier to understand in the context of finite difference via the modified equation, but the effect on the model are the same when finite element is used. The artificial diffusivity or viscosity tends to damp out disturbing oscillations in the solution. Many analysts consider this a positive characteristic, while others (Gresho and Lee, 1981) see it as an undesirable masking of important modeling information.

**Boundary element methods**

Finite element methods (FEM) are implicit and thus always require the solution of a set of simultaneous equations. Boundary element methods (BEM), on the other
hand, always produce explicit integral equations thus giving them at least one advantage over FEM. This is illustrated by the BEM solution of the Navier-Stokes and continuity equations presented by Wang and Wu (1986). Their three step solution of Navier-Stokes type flow used an explicit integral equation as the first of two steps to determine all vorticity values. The second step is the determination of the difficult no-slip vorticities. Since the procedure is based on the boundary element solution of Poisson's equation, it will be discussed in that section.

In addition to being explicit, the BEM integral equations are selective in that they need only be evaluated on the boundaries with non-zero velocity and interior areas with non-zero. Thus in the separated flow problem with considerable regions of non-viscous or potential flow it would not be necessary to integrate over the entire flow region as in the FEM solution of the same problem.

Before moving on, there is one observation worth noting. In the review of both the FEM and BEM solutions of the vorticity transport equation there was a conspicuous lack of time dependent or unsteady solution techniques. Brebbia (1985, p. 214) does present a time dependent direct BEM solution with a few examples but there were no applications of these techniques in the literature. At this point it is not known if this is a reflection of the interests and
priorities of the contributing researchers or a fundamental inadequacy of the integral formulations.

Solution of Poisson's Equation

Thus far several combinations of equation form and numerical technique have been outlined for the complete solution of Navier-Stokes flow. All but one of these methods require the solution of at least one Poisson's equation which have the general form $v^2u = g$. In this equation $u$ is an unknown scalar variable distributed over the domain, $g$ is a scalar function which is known over the entire domain and $v^2$ is the Laplacian vector operator.

Poisson's equations are elliptic PDEs and are the governing equations for a class of problems known as boundary value problems. An important characteristic of elliptic PDEs is that the solution at any point is dependent on all boundary conditions. Any numerical technique used to solve boundary value problems must reflect this characteristic if it is to be successful.

In the solution methods presented for Navier-Stokes flow three different Poisson's equations were used. They are: 1. pressure equation (Eq. 2.8), 2. vector potential equation (Eq. 2.9) and 3. velocity equation (Eq. 2.10).

\[ v^2p = -\rho v \cdot (\vec{V} \cdot \vec{V}) \]  
\[ v^2\vec{B} = -\vec{\omega} \]
\[ \nabla^2 \hat{\psi} = - (\nabla \times \hat{\omega}) \] (2.10)

The pressure equation is scalar and is given in its most general form using vector notation. The rectangular coordinate component form of the pressure equation is derived in most fluids texts but the axisymmetric form is much harder to find and is therefore derived in Chapter VII.

The vector potential equation is actually a vector equation but for axisymmetric flow expressed in cylindrical polar coordinates there is only one non-zero component, \( B_\theta \). This can be related to the more familiar stream function, \( \psi \), by the equation \( \psi = -B_\theta r \) (Milne-Thomson, 1960, p. 552).

The velocity equation is also a vector Poisson's equation but it transforms to two scalar component equations rather than one. For finite difference and finite element methods this will require the simultaneous solution of two scalar Poisson's equations. This is not the case for boundary element methods but it does complicate the derivation.

Boundary value problems governed by Poisson's equations are one of the most common and important class of problems found in engineering. Besides the ones already mentioned some of the more common are potential flow, steady state electromagnetic, and steady state temperature distribution in a solid. Given the importance of these problems and the
relative simplicity of the governing equation it is not surprising that boundary value problems were one of the first to be attacked by the first numerical analysts. From the earliest attempts, methods have evolved which are capable of solution accuracy sufficient for most engineering and scientific applications. Most recent and ongoing research of finite element and finite difference solution of Poisson's equation are in the areas of improved computing efficiency and incorporation of boundary conditions. For this reason the review of these methods is brief. However boundary element solutions are fairly recent developments and there are still many interesting applications to be studied and techniques to be refined. In fact a major portion of the present research is the development of the BEM solution of Poisson's velocity equation for axisymmetric geometries. For this reason the BEM review is more detailed. Also some background information will be presented, with this discussion continuing in a later chapter.

**Finite difference methods**

In order to emulate the boundary dependent nature of boundary value problems, it is necessary to use centered finite difference formula to solve Poisson's equation. This will always result in an implicit technique requiring the solution of simultaneous equations. Though the differencing formulas may vary slightly, the major differences in
available procedures are in the techniques used to solve the resulting equation system.

As is characteristic of implicit finite difference formulations, the coefficient matrix of the equation system is sparsely populated and often diagonally dominate which can be solved more efficiently than densely populated matrix equations. This increases the number of practical and efficient solution techniques to choose from. Several of the current favorites are described by Anderson et al. (1984).

**Finite element methods**

As with finite difference methods there is not much new in finite element solutions of Poisson's equation. Most publications on Navier-Stokes solutions give little attention, if any, to any of the three common Poisson's equations. A possible exception is the work by Campion-Renson and Crochet (1978) where the stream function and vorticity transport equation were solved simultaneously. Standard finite element formulations for Poisson's equation can be found in most elementary finite element texts, for example Desai (1979, p. 299).

**Boundary element methods**

The theory of BEM solutions for Poisson's equation, $\nabla^2 u(\mathbf{R}) = g(\mathbf{R}_0)$, is well understood and well documented (Brebbia and Walker, 1980). The development of a BEM solution for any PDE depends on the existence of a fundamental solution,
P(\vec{R}, \vec{R}_0), for the type of PDE being solved. This fundamental solution may also be referred to as a unit solution, principal solution, or influence function. The later is perhaps the most descriptive, since they (fundamental solutions) describe quantitatively the "influence" of a point load or source, located by the position vector \vec{R}_0, on the solution u, located by the position vector \vec{R}.

In a computational region containing only one point source, e(\vec{R}_0), the solution is simply \( u(\vec{R}) = e(\vec{R}_0) P(\vec{R}, \vec{R}_0) \). The complete solution for u in an infinite region is the integral obtained by the superposition of all the point source solutions in the region. For problems with finite regions an integral over the boundary must be included to accommodate the boundary conditions. There are two common methods of incorporating the boundary values, one leads to the direct formulation the other to the indirect formulation (Brebbia and Walker, 1980). It can be shown that these two formulations are equivalent (Banerjee and Butterfield, 1981, p. 57).

In this research only the direct Boundary Element solution for Poisson's equation is used, Eq. 2.11, expressed in terms of the fundamental solution, P, body force or source function, g, and the boundary conditions of the unknown function, u. \( \vec{R} \) is the position vector of the field point and the subscript, \( \circ \), indicates quantities or operations
associated with the load point. This equation can be used to solve all of the component equations derived from Eqs. 2.8-2.9.

\[ u(\mathbf{R}) = \int_{\Omega} g \phi \, dV_0 + \int_{\partial \Omega} \left[ \frac{\partial u_0}{\partial n_0} - P \frac{\partial u_0}{\partial n_0} \right] \, dS_0 \quad (2.11) \]

Equation 2.11 contains one integral over the surface \( S_0 \) and one integral over the interior \( \Omega \). The interior integral vanishes for \( g=0 \) which corresponds to Laplace's equation, \( \nabla^2 u = 0 \). This conveniently leaves only the surface integral making BEM a very attractive solution method for Laplace's equation.

Direct solution of the integral equation for \( u(\mathbf{R}) \) appears to require knowledge of both \( u_0 \) and its normal derivative, \( \partial u_0 / \partial n_0 \), on the boundary. Actually it is necessary to specify only one of the two, at each point on the boundary. This requirement is consistent with the boundary conditions necessary to yield a unique solution to all boundary value problems. In digital solutions this translates to specifying either \( u_0 \) or \( \partial u_0 / \partial n_0 \) at every boundary node. When this is done it is possible to rearrange the discrete version of Eq. 2.11, which is developed later, such that it can be used to solve for the remaining unknown boundary values. These values may then be substituted into
the original equation to obtain explicit solutions of \( u \) at any desired field point on the boundary or in the interior.

An interesting alternative to the direct solution of Poisson's velocity equation (Eq. 2.10) is Eq. 2.12. This 3-D vector integral equation was presented by Wu and Thompson (1973). The complete derivation is found in Banerjee

\[
\mathbf{V}(\mathbf{R}) = \frac{1}{D_k} \left[ \mathbf{\omega} \times (\mathbf{R}_0 - \mathbf{R}) \right] + \frac{1}{D_k} \left[ \mathbf{V}_0 \cdot \mathbf{n}_0 \right] \frac{(\mathbf{R}_0 - \mathbf{R}) - (\mathbf{V}_0 \times \mathbf{n}_0) \times (\mathbf{R}_0 - \mathbf{R})}{|\mathbf{R}_0 - \mathbf{R}|^3} \]  

(2.12)

Butterfield (1981, p. 333). Equation 2.12 was developed using the 3 dimensional vector fundamental solution and is therefore applicable to all standard orthonogonal coordinate systems. The same expression for \( \mathbf{V} \) can be obtained more directly by using the integral expression for vector potential \( \mathbf{B} \) (Milne-Thomson, 1960, p. 547) and the definition \( \mathbf{V} = \nabla \times \mathbf{B} \). This result is analogous to the Biot and Savart formula for the magnetic effect of an electrical current.

Both the direct BEM and Biot-Savart formulations are valid solutions of Poisson's velocity equation but there are major differences in the boundary values used. As in all direct BEM integrals the ones for each component velocity, \( v_R \) and \( v_Z \) in axisymmetric flows, will contain that velocity and its normal derivative. The Biot-Savart integrals, however, will contain both component velocities but no normal derivatives thus allowing the analyst the flexibility to
select the formulation which best utilizes the available boundary information.

Before applying the direct BEM formulation to the differential equations Eqs. 2.8, 2.9, and 2.10, it is necessary derive the scalar component equations and define \(dV_0\) and \(dS_0\) consistent with the selected coordinate system. In cylindrical polar coordinates \(dV_0\) and \(dS_0\) are defined as \(rd\theta drdz\) and \(rd\theta ds\) respectively. Since by definition axisymmetric problems are not \(\theta\) dependent, it is possible to integrate over \(\theta\) thus reducing the original 3 dimensional problem to 2 dimensions. Similarly, when all vector operations, \(dV_0\), and \(dS_0\) in Eq. 2.12 are defined consistent with the cylindrical polar coordinates it also can be reduced to 2 scalar equations, \(V_R\) and \(V_Z\), instead of 3.

The reduction from 3 to 2 dimensions does not come without penalty. All integrands will contain elliptic integrals which cannot evaluated in closed form but instead must be evaluated numerically. This is done by first writing each integrand in terms of the standard elliptic integrals of the first and second kind, \(E(k)\) and \(K(k)\) respectively. \(E(k)\) and \(K(k)\) are then replaced by the Chebyshev series approximations presented by Cody (1965). It should be noted that the presence of the elliptic integrals does not alter the solution theory, their only impact is to complicate the integration algorithms.
Determination of unknown boundary values

It was stated earlier that for elliptic partial differential equations, such as Poisson's equation, it is necessary to specify only half of the boundary information to obtain an unique solution. The remaining information is obtained by rearranging the discrete or finite element approximations of their respective integral solutions. The development of these finite element formulas are covered in detail in the next chapter. For present purposes it is sufficient to state that all subject integrals in this research can be approximated by one or more finite sum of terms which are the product of nodal values of a distributed variable and kernel functions. For example Eq. 2.11 can be approximated as

\[ u_k = \sum_{i=1}^{N_I} (g_{i}K_{gki}) + \sum_{j=1}^{N_B} (u_{j}K_{ukj}) + \sum_{j=1}^{N_B} (u'_{j}K_{u'kj}) \]

where \( N_I \) is the number of interior nodes, \( N_B \) the number of boundary nodes, and \( k \) goes form 1 to \( N_B \). This expressions may be rearranged to form two boundary sums one containing \( N_k \) known boundary nodal values and one containing \( N_u \) unknown boundary nodal values where \( N_B = N_k + N_u \).

\[ a u_k + \sum_{i=1}^{N_I} (g_{i}K_{gki}) - \sum_{i=1}^{N_k} (x_{i}K_{gki}) = \sum_{j=1}^{N_u} (x_{j}K_{ukj}) + (1-a)u_k. \]
In the above equation, \( x \) may be either \( u \) or \( \partial u / \partial n \) (\( u' \)). Also \( \alpha \) equals 1 when \( u_k \) is known and 0 when \( u_k \) is one of the \( x_j \) unknowns. All of the terms on the left of the equal sign are known or constant resulting in a linear equation with \( N_u \) unknown boundary values. Using this format it is possible to generate a linear set of \( N_u \) equations with the same set of unknowns. Such a set can be written in convenient matrix form as \( \mathbf{C} = [A] \mathbf{B} \) where \( \mathbf{C} \) is a known vector, \( [A] \) is a matrix of \( K_{k,j} \), and \( \mathbf{B} \) is a vector of unknown boundary values.

When a boundary value problem is well-posed, i.e., with the correct type and number of boundary values specified, the linear matrix equation resulting from the described collocation procedure is well-conditioned with a stable inverse matrix. These can be solved easily by any of several methods. Perhaps the most commonly used general procedure is Gaussian elimination though there are several that may be more efficient especially for sparsely populated matrices.

In real engineering problems it is often the case that not enough boundary information is known to formulate a well-posed problem. Under these conditions it is common to obtain an approximate solution by designing some type of iterative procedure which incorporates other available information and/or governing equations. Actually this type of problem may be more common than the neat well-posed problem.
An interesting alternative to iterative methods for solving Poisson's velocity equation with ill-posed boundary conditions is made possible by Eq. 2.12. Using the discrete approximation of this equation it is possible to generate by collocation a linear set of equations for several problems with ill-posed boundary conditions. An example which is discussed throughout the thesis is that of developing entrance flow in a pipe. Here, none of the outflow velocities are known but both $v_x$ and $v_z$ are known on the pipe wall making it possible to have an equal number of known and unknown nodal boundary values.

Though it is easy to generate a linear set of equations for such ill-posed problems it is not easy to solve them. For such problems the resulting coefficient matrix $[A]$ will be ill-conditioned which means that its inverse, $[A]^{-1}$, is unbounded or undefined. This makes it impossible to use inverse methods such as Gaussian elimination to solve the linear set of equations. This situation is unfortunate but it is completely consistent with the solution theory of boundary value problems governed by elliptic partial differential equations.

A possible approach to obtaining an approximate solution to such ill-posed elliptic partial differential equations starts with defining a new problem which is well-posed. This is done by incorporating additional qualitative or
quantitative information into the original governing equation. When done carefully collocation using the new equation will yield a matrix equation with well-conditioned matrix, \([A]\), with a stable inverse which can be solved with common inverse methods for the solution vector \(\mathbf{B}\). This procedure is called regularization and a discussion of how it is applied to ill-conditioned linear equations is given by Tikhonov and Arsenin (1977).

Phillips (1962) and Twomey (1963) also presented early work on regularization as it applies to the solution of Fredholm integrals of the first kind. They both primarily addressed the mechanics of a solution method. A similar discussion as it applies to non-destructive evaluation is given by Wing (1984). Optimization of the regularization technique for problems with known error statistics is discussed by Vinokurov (1972), Wahba (1977), Marti (1978), and Groetsch (1982).
CHAPTER III. BOUNDARY ELEMENT
SOLUTION OF POISSON'S EQUATION FOR VELOCITY

In Chapter II, there were described two integral solutions of Poisson's equation for velocity, Eq. 2.8. One integral solution was developed using a strongly mathematical approach by applying the standard direct BEM equation, Eq. 2.11, to the scalar component forms of Eq. 2.8. The other, presented by Wu (Eq. 2.12), was developed more from a fluid mechanics point of view although the resulting integrals have similar characteristics and present the same integration difficulties as all BEM integrals. Therefore in the literature this formulation is usually classified as a BEM integral solution. This chapter outlines the development of numerical solution algorithms based on both of these formulations.

The Biot-Savart Integral Solution

The integral solution for Poisson's velocity equation, which is analogous to the Biot and Savart law in magnetic theory, is given again for reader convenience. The equation is given in vector form so the first step is to derive the scalar component integral equations. For axisymmetric problems the appropriate coordinate system is cylindrical polar \((r, \theta, z)\) and the component velocities are \(v_r\) and \(v_z\).
The development of the component integral equations requires the completion of the following steps:

1. Write each of the integrand vectors in the form
   \[ \mathbf{U} = U_x \hat{i} + U_y \hat{j} + U_z \hat{k} \]
   where \( \hat{i} \), \( \hat{j} \), and \( \hat{k} \) are the unit vectors in the \( x \), \( y \), and \( z \) directions respectively, and \( U_x \), \( U_y \), and \( U_z \) are the component magnitudes.

2. Perform all of the vector operations as defined for the rectangular coordinate system.

3. Separate the rewritten vector integral equation into three scalar integral equations and define \( dV \) and \( dS \) appropriately in cylindrical polar coordinates.

The vectors contained in Eq. 2.12 are the velocity vector, \( \mathbf{V} \), the vorticity vector \( \mathbf{\omega} \), and two position vectors, \( \mathbf{R}_0 \) and \( \mathbf{R} \). Though it is theoretically correct to define each in the cylindrical polar form, \( \mathbf{U} = u_r \hat{r} + u_\theta \hat{\theta} + u_z \hat{z} \), the vector operations are much more straightforward for rectangular coordinates. However, all scalar magnitudes are still written in terms of \( r \), \( \theta \), and \( z \).

Consider Fig. 3.1 which depicts an arbitrary position vector, \( \mathbf{R} \), in 3 dimensional space where \( \mathbf{R} = \Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k} \). The scalar magnitudes can be rewritten as desired by
superimposing a cylindrical polar coordinate system such that its z axis is coincident with the rectangular z axis. This allows the position vector, \( \vec{R} \), to be written as follows:

\[
\vec{R} = (r \cos \theta) \hat{i} + (r \sin \theta) \hat{j} + z \hat{k}.
\]

Similarly the position vector \( \vec{R}_0 \) may be written

\[
\vec{R}_0 = (r_0 \cos \theta_0) \hat{i} + (r_0 \sin \theta_0) \hat{j} + z_0 \hat{k}.
\]

In Eq. 2.12, \( \vec{R} \) and \( \vec{R}_0 \) are the position vectors relative to the origin of the field point and load point respectively. Therefore \( (\vec{R}_0 - \vec{R}) \) is the relative position vector between the two points and \( \theta \) may be set equal to 0 without effecting the solution. \( |\vec{R}_0 - \vec{R}|^3 \) is the cube of the magnitude of the relative position vector and can be evaluated as follows:

\[
\begin{align*}
\vec{R}_0 - \vec{R} &= (r_0 \cos \theta_0 - r) \hat{i} + (r_0 \sin \theta_0) \hat{j} + (z_0 - z) \hat{k} \\
|\vec{R}_0 - \vec{R}| &= \sqrt{(r_0 \cos \theta_0 - r)^2 + (r_0 \sin \theta_0)^2 + (z_0 - z)^2} \\
|\vec{R}_0 - \vec{R}|^3 &= (r_0 \cos \theta_0 - r)^3 + (r_0 \sin \theta_0)^3 + (z_0 - z)^3
\end{align*}
\]

Figure 3.1. Transformation of a position vector
A development similar to the one for position vectors can be made performed the velocity and vorticity vectors. Figure 3.2 shows an arbitrary velocity vector in 3 dimensional space where \( \vec{V} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \). As before, it is desired to write the scalar magnitudes, which for velocity vectors are \( v_x, v_y, \) and \( v_z \), in terms of the axisymmetric coordinates. The results of this transformation are

\[
\begin{align*}
  v_x &= (v_r \cos \theta - v_\theta \sin \theta) \\
  v_y &= (v_r \sin \theta + v_\theta \cos \theta) \\
  v_z &= v_z
\end{align*}
\]

These relationships are further simplified by recognizing that for axisymmetric flow \( v_\theta = 0 \) thus allowing the velocity vector to be written \( \vec{V}_o = (v_r \cos \theta) \hat{i} + (v_r \sin \theta) \hat{j} + v_z \hat{k} \).
A similar development can be made for the vorticity vector for which it can be shown that \( \omega_r \) and \( \omega_z \) both equal zero. Therefore \( \vec{\omega}_0 = (-\omega_\theta \sin \theta_\theta) \hat{i} + (\omega_\theta \cos \theta_\theta) \hat{j} + 0 \hat{k} \). In future equations the subscript, \( \theta \), will be omitted from \( \omega_\theta \) since it is the only non-zero vorticity component.

With all vectors defined as component vectors it is possible to perform all vector operations found in Eq. 2.12. The following is an abbreviated version of this procedure.

1. \( \vec{\omega}_0 \times (\vec{R}_0 - \vec{R}) = \omega (\cos \theta_\theta (z_0 - z) \hat{i} + \sin \theta_\theta (z_0 - z) \hat{j} + (r_0 - r \cos \theta_\theta) \hat{k}) \)

2. \( \vec{V}_0 \cdot \vec{n}_\theta = v_r n_r \cos^2 \theta_\theta + v_z n_z \sin^2 \theta_\theta + v_z n_z = v_r n_r + v_z n_z \)
   define \( V_1 = (v_r n_r + v_z n_z) \)

3. \( (\vec{V}_0 \cdot \vec{n}_\theta)(\vec{R}_0 - \vec{R}) = ((r_0 \cos \theta_\theta - r) \hat{i} + (r_0 \sin \theta_\theta) \hat{j} + (z_0 - z) \hat{k}) V_1 \)

4. \( \vec{V}_0 \times \vec{n}_\theta = (\sin \theta_\theta \hat{i} - \cos \theta_\theta \hat{j} + 0 \hat{k})(v_z n_r - v_r n_z) \)
   define \( V_2 = (v_z n_r - v_r n_z) \)

5. \( (\vec{V}_0 \times \vec{n}_\theta) \times (\vec{R}_0 - \vec{R}) = (\cos \theta_\theta (z_0 - z) \hat{i} + \sin \theta_\theta (z_0 - z) \hat{j} + (r \cos \theta_\theta - r_0) \hat{k}) V_2 \)

The final step in developing the scalar integral equations is the definition of \( dV \) and \( dS \). For \( dV \) the simple relationship of \( dV = r d\theta dr dz \) is applicable throughout the domain. The universal definition of \( dS \) is \( dS = r d\theta ds \) where the form of \( ds \) will depend on the orientation of the surface of integration. For surfaces perpendicular to the centerline (z axis) \( ds = dr \) and for surfaces parallel to the centerline \( ds = dz \).

For other surface orientations \( ds \) is usually expressed as a function of the unit normal, \( n_\theta \), or the direction
cosines of the integration surface. The general expression is somewhat simplified in axisymmetric flows since, by definition, \( n_\theta = 0 \).

When integration is done numerically, as in this research, the surface of integration is usually approximated by a set of straight line segments. Figure 3.3 shows such a line segment with its orientation defined by \( n_0 \) and having a length of \( ds \).

In order to complete a surface integration with \( ds \) defined as \( dz/n_r \) it is necessary to write \( r \) as a function of \( z \). For a straight line segment as shown in Fig. 3.3 this is a simple problem in geometry. Similarly, if \( ds = dr/n_z \) was used, \( z \) would be defined as a function of \( r \).

\[
|n_0| = 1 = (n_r^2 + n_z^2)^{1/2}
\]
\[
ds^2 = dr^2 + dz^2
\]
\[
0 < n_r < 1 \quad \text{or} \quad ds = dz/n_r
\]
\[
n_r = 0 \quad ds = dr
\]
\[
n_r = 1 \quad ds = dz
\]

Figure 3.3. Arbitrary line segment on integration surface
For the simple example problems used in this research it was necessary to consider only horizontal and vertical surfaces, i.e., \((n_r=1\) and \(n_z=0\)) or \((n_r=0\) and \(n_z=1\)). This permits the simple expression \(ds = n_r\,dz + n_z\,dr\) which is a form easily translated to computer code. Using this definition of \(ds\), it follows that \(dS = rd\theta(n_r\,dz + n_z\,dr)\).

At this point Eq. 2.12 has been completely transformed into a scalar component equation of the form \(u_x\,i + u_y\,j + u_z\,k\) where \(i, j,\) and \(k\) are the unit vectors defining a rectangular coordinate system. This may be confusing since it has been stated repeatedly that for axisymmetric problems it is appropriate to use the cylindrical coordinate system. With the help of Fig. 3.1 this apparent inconsistency can be resolved.

Earlier in this chapter, the angle \(\theta\) was arbitrarily set equal to 0, primarily for simplification. This was permissible since: 1. absolute position of the load and field points is not important, only the relative position \((\vec{R}_o-\vec{R})\) appears in Eq. 2.12 and 2. by definition the solution of \(\vec{V}(\vec{R})\) does not depend on \(\theta\). Therefore, even though the scalar equations were developed for \(\theta=0\) they are valid for all values of \(\theta\). Also at \(\theta=0\), \(n_r=n_x\), \(n_\theta=n_y\), and \(n_z=n_z\) which leads directly to the relationships \(v_r=v_x\), \(v_\theta=v_y\), and \(v_z=v_z\) and Eqs. 3.1-3.3.
\[ v_r(\vec{R}) = \frac{1}{D_k} \int_{V_0} \frac{\omega \cos \theta \,(z_0 - z)}{|\vec{R}_0 - \vec{R}|^3} \, dV_0 + \frac{1}{D_k} \int_{S_0} \frac{V_1(r_0 \cos \theta - r) + V_2 \cos \theta \,(z_0 - z)}{|\vec{R}_0 - \vec{R}|^3} \, dS_0 \]  

(3.1)

\[ v_\theta(\vec{R}) = \frac{1}{D_k} \int_{V_0} \frac{\omega \sin \theta \,(z_0 - z)}{|\vec{R}_0 - \vec{R}|^3} \, dV_0 + \frac{1}{D_k} \int_{S_0} \frac{V_1 r_0 \sin \theta + V_2 \sin \theta \,(z_0 - z)}{|\vec{R}_0 - \vec{R}|^3} \, dS_0 \]  

(3.2)

\[ v_z(\vec{R}) = \frac{1}{D_k} \int_{V_0} \frac{\omega (r_0 - \cos \theta)}{|\vec{R}_0 - \vec{R}|^3} \, dV_0 + \frac{1}{D_k} \int_{S_0} \frac{V_1 (z_0 - z) + V_2 (r_0 \cos \theta - r_0)}{|\vec{R}_0 - \vec{R}|^3} \, dS_0 \]  

(3.3)

**Reduction of 3 dimensional equations to 2 dimensions**

Equations 3.1-3.3 involve integration over \( dr_0, d\theta_0, \) and \( dz_0 \). However since neither the load function or field points are functions of \( \theta \) it is possible to take the load function outside the \( \theta \) integral and integrate. After the \( \theta \) integration the integrand is a function only of \( r_0 \) and \( z_0 \) which in effect reduces the original 3 dimensional vector equation to 2 dimensional scalar equations, as is consistent with the definition of axisymmetry.

The reduction of the problem dimension yields great savings in computation time and storage requirements but it
does not come without penalty. The integrands of all axisymmetric BEM formulations contain elliptic functions of $\theta$. These functions cannot be integrated analytically in closed form and conventional numerical techniques in 3 dimensions cannot be performed with acceptable accuracy. The difficulties presented by the elliptic functions are compounded by the presence of singularities, which are a part of all BEM formulations, imbedded within the elliptic functions themselves.

The preferred approach to the integration of elliptic integrals, and the one used in this research, involves expressing each $\theta$ integral in terms of the standard complete elliptic integrals of the first and second kind, $K(k)$ and $E(k)$ respectively. This makes it possible to use proven series approximations for $K(k)$ and $E(k)$ with acceptable accuracy. In addition, this rewriting naturally isolates the singularities and allows them to be put in a form that can be integrated with the same techniques used in the simpler 2 dimensional planar formulations.

The task of expressing each elliptic integral in terms of $E(k)$ and $K(k)$ is completed by rewriting such that the limits are changed from $0$ to $2\pi$ to $0$ to $\pi/2$ and the denominator has the form $(1-k^2\sin\theta_0)^0$. This procedure is outlined in Appendix A which also includes the results of all
such transformations needed for the BEM integral solutions in this research.

To illustrate this procedure the transformation of the interior integral in Eq. 3.1 is detailed below.

\[
\frac{1}{D_k} \int_{V_0} \frac{\omega(z_0-z)}{|r_0-r|^3} \, dz_0 = \frac{1}{D_k} \int_{z_0}^{2\pi} \frac{\omega(z_0-z)}{(A-B\cos\theta)} \frac{\cos\theta}{(A-B\cos\theta)^{3/2}} r_0 d\theta_0 dr_0 dz_0
\]

This integral can be expressed in terms of \( E(k) \) and \( K(k) \) by referring to Appendix A.

\[
= \frac{1}{D_k} \left[ \frac{4\omega(z_0-z)}{(A+B)^{1/2}} \left( \frac{A E(k) - K(k)}{B(A-B)} \right) \right] r_0 dr_0 dz_0
\]

The final integrand is now a function of only \( r \) and \( z \) and can be integrated using quadratures common in 2 dimensional planar BEM solutions. One result in Appendix A worth noting is that all integrands with \( \sin\theta \) in the numerator integrate to 0. This makes \( v_\theta = 0 \) (Eq. 3.2) as it should be for axisymmetric flow.

In the following final 2 dimensional axisymmetric expressions for \( v_r \) and \( v_z \) the terms \( J_0 \) and \( J_1 \) are defined for simplification.
\[ v_T(\vec{R}) = \frac{1}{D_k} \left[ \frac{\omega_0 (z_0 - z) r_0 J_1 dA_0 + 1}{D_k} \right] \left[ \frac{(V_1 r_0 + (z_0 - z) V_2) J_1 - V_1 r J_0}{S_0} \right] r_0 ds \]  
\[ v_z(\vec{R}) = \frac{1}{D_k} \left[ \frac{\omega_0 (r_0 J_0 - J_1) r_0 dA_0 + 1}{D_k} \right] \left[ \frac{(V_1 (z_0 - z) - r_0) J_0 + V_2 r J_1}{S_0} \right] r_0 ds \]  
A_0 = dr_0 dz_0 
ds = (n_x dr_0 + n_x dz_0)

\[ J_0 = \left\{ \frac{2 E(k)}{(A+B)^{1/2}(A-B)} \right\} \]  
\[ J_1 = \left\{ \frac{-2}{B(A+B)^{1/2}} \right\} K(k) + \frac{A}{B} J_0 \]

where \( A = r^2 + r_0^2 + (z_0 - z)^2 \)
\( B = 2r_0 r \)
\( k^2 = \frac{2B}{A+B} \)

**Direct BEM Formulation**

A second BEM formula for the solution of Poisson's velocity equation is possible by applying the standard direct BEM integral solution, Eq. 2.11. This general solution was developed for scalar Poisson's equation of the form \( \nabla^2 v = g \) and can be applied to 1, 2, or 3 dimensional problems by appropriate definition of \( dV_0, dS_0, \) and the principal solution \( P. \)

The equation to be solved is a vector differential equation, \( \nabla^2 \vec{V} = -\vec{v} \times \vec{\omega}, \) therefore the first step is to replace it with the correct scalar or component differential equations. This is the first departure from the first
Poisson solution which started with a vector integro-differential equation.

The selection of the coordinate system and thus the component equation form is primarily determined by the problem geometry and dimension. Axisymmetric problems are true 3 dimensional problems but can be expressed in 2 dimensions when cylindrical polar coordinates are used. Developing the component equations in this system would require performing all vector operations as defined for cylindrical polar coordinates and using the principal solution, P, specific to axisymmetric potential problems (Banerjee and Butterfield, 1981, p. 129). Though this approach is valid and indeed would seem the natural choice it was dismissed in favor of a slightly different method which seemed more straightforward but yields the same 2 equation solution set.

The method used for developing the component integral equations starts by performing all vector operations contained in Eq. 2.10. as defined for 3 dimensional rectangular coordinates (x y z). Next a cylindrical polar coordinate system (r θ z) is superimposed on the rectangular system. If this is done judiciously by aligning the x and r axis and the two z axis it is possible to express the rectangular quantities in terms of cylindrical polar quantities. The three-dimensional component integral
equations may then be reduced to two axisymmetric integrals by defining \( dV_0 = r_0 d\theta_0 dr_0 dz_0 \) and \( dS_0 = r_0 d\theta_0 ds \) and integrating over \( \theta_0 \).

The derivation of the desired component integral equations is more tedious than challenging but it is important to get them exactly correct. Also the same basic method is used in the solution development of the remaining Poisson's equations, Eq. 2.8 and Eq. 2.9. For these reasons, all steps are described in detail.

Derivation of component integral equations in 2 dimensions

BEM integrals contain two point functions, i.e., the field point and load point. The field point has coordinates \((x,y,z)\) or \((r,\theta,z)\) in rectangular and polar coordinates respectively and is located by the position vector \( \mathbf{R} \).

Similarly, the load point has the coordinates \((x_0,y_0,z_0)\) or \((r_0,\theta_0,z_0)\) and is located by the position vector \( \mathbf{R}_0 \). Using these definitions the original Poisson's velocity is correctly given as \( \nabla^2 \mathbf{v}(\mathbf{R}) = -\mathbf{v}_0 \times \omega(\mathbf{R}_0) \) and the component differential equations become

\[
\begin{align*}
\nabla^2 v_x &= -(\partial \omega_{z_0} / \partial y_0 - \partial \omega_{y_0} / \partial z_0) \\
\nabla^2 v_y &= -(\partial \omega_{x_0} / \partial z_0 - \partial \omega_{z_0} / \partial x_0) \\
\nabla^2 v_z &= -(\partial \omega_{y_0} / \partial x_0 - \partial \omega_{x_0} / \partial y_0).
\end{align*}
\]

For 3 dimensional solutions these three scalar Poisson's equations would be solved directly using Eq. 2.11. However to develop the desired 2 equation axisymmetric form, a
transformation to cylindrical polar coordinates is necessary. In this transformation, it is defined that the z axes are parallel, the origins are coincident, and that \( \theta \) is angle between the rectangular x axis and the polar r axis.

The following are the relationships between rectangular and cylindrical polar quantities associated with the load point as indicated by the \( \circ \) subscript. Analogous relationships for the field point quantities are quickly written by omitting this subscript. In both sets it is assumed that \( \partial / \partial \theta = v_\theta = n_\theta = 0 \).

\[
\begin{align*}
x_0 &= r_0 \cos \theta_0 & n_x &= n_r \cos \theta_0 & v_x &= v_r \cos \theta_0 & \omega_x = -\omega_\theta \sin \theta_0 \\
y_0 &= r_0 \sin \theta_0 & n_y &= n_r \sin \theta_0 & v_y &= v_r \sin \theta_0 & \omega_y = \omega_\theta \cos \theta_0 \\
z_0 &= z_0 & n_z &= n_z & v_z &= v_z & \omega_z = 0
\end{align*}
\]

Remembering that for axisymmetric problems the solution is independent of the \( \theta \) coordinate, the relationships for the field point may be greatly simplified without qualification by setting \( \theta = 0 \). These simplified transformation relationships for the field point quantities are as follows

\[
\begin{align*}
x &= r & n_x &= n_r & v_x &= v_r \\
y &= 0 & n_y &= 0 & v_y &= 0 \\
z &= z & n_z &= n_z & v_z &= v_z
\end{align*}
\]

Substitution of these relationships in the 3 component differential equations yields the following set of two
equations suitable only to axisymmetric flow. Note that the \( \theta_0 \) subscript has been dropped from \( \omega_{\theta_0} \) since \( \omega \) is never a field quantity and only the \( \omega_\theta \) component is non-zero.

\[
v^2 v_r = \frac{\partial (\omega \cos \theta_0)}{\partial z_0} \quad v^2 v_z = -\frac{\partial (\omega \cos \theta_0)}{\partial x_0} - \frac{\partial (\omega \sin \theta_0)}{\partial y_0}
\]

The next step in the development is to express all rectangular partial derivatives in terms of cylindrical partial derivatives. This is accomplished by using the chain rule and the following Jacobian elements. All elements not listed equal 0.

\[
\frac{\partial r}{\partial x} = \cos \theta, \quad \frac{\partial r}{\partial y} = \sin \theta, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}, \quad \frac{\partial z}{\partial z} = 1
\]

The first partial derivatives to be transformed are the source or load terms of the component differential equations. These are the terms to the right of the equal sign in Eqs. 3.6 and 3.8.

\[
\frac{\partial (\omega \cos \theta_0)}{\partial z_0} = \omega \frac{\partial (\cos \theta_0)}{\partial z_0} + \cos \theta_0 \frac{\partial \omega}{\partial z_0} = \cos \theta_0 \frac{\partial \omega}{\partial z_0}
\]

\[
\frac{\partial (\omega \cos \theta_0)}{\partial x_0} = \omega \frac{\partial \cos \theta_0}{\partial r_0} + \cos \theta_0 \frac{\partial \omega}{\partial x_0} + \cos \theta_0 \frac{\partial \omega}{\partial \theta_0} + \cos \theta_0 \frac{\partial \omega}{\partial x_0} = \frac{\partial \omega}{\partial r_0}
\]

\[
\frac{\partial (\omega \sin \theta_0)}{\partial y_0} = \omega \frac{\partial \sin \theta_0}{\partial r_0} + \sin \theta_0 \frac{\partial \omega}{\partial y_0} + \sin \theta_0 \frac{\partial \omega}{\partial \theta_0} + \sin \theta_0 \frac{\partial \omega}{\partial y_0} = \frac{\partial \omega}{\partial \theta_0}
\]

\[
\frac{\partial (\omega \sin \theta_0)}{\partial y_0} = \omega \frac{\partial \sin \theta_0}{\partial y_0} + \sin \theta_0 \frac{\partial \omega}{\partial y_0} + \sin \theta_0 \frac{\partial \omega}{\partial \theta_0} + \sin \theta_0 \frac{\partial \omega}{\partial y_0} = \frac{\partial \omega}{\partial \theta_0}
\]
\[= \omega \frac{\cos^2 \theta_0}{r_0} + \sin^2 \theta_0 \frac{\partial \omega}{\partial r_0} \]

\[
-\frac{\partial \left(\omega \cos \theta_0\right)}{\partial x_0} \frac{\partial \left(\omega \sin \theta_0\right)}{\partial y_0} = -\frac{\omega (\sin^2 \theta_0 + \cos^2 \theta_0) - (\sin^2 \theta_0 + \cos^2 \theta_0)}{r_0} \frac{\partial \omega}{\partial r_0}
\]

\[= -\frac{1}{r_0} \frac{\partial (r_0 \omega)}{\partial r_0} \]

In the direct BEM solution integral, Eq. 2.11, there are two normal derivatives which must be written in terms of \(\partial r_0\) and \(\partial z_0\), they are the component velocities and principal solution. The transformation of the \(v_{z_0}\) normal derivative is trivial since the z axes are parallel but \(v_{x_0}\) is a function of \(r_0\) and \(\theta_0\) and therefore its normal derivative transformation is more involved.

\[
\frac{\partial (v_{x_0})}{\partial n_0} = \frac{\partial (v_{r_0} \cos \theta_0)}{\partial n_0} = \cos \theta_0 \left( \frac{\partial v_{r_0}}{\partial n_0} + v_{r_0} \frac{\partial \cos \theta_0}{\partial n_0} \right)
\]

\[
\frac{\partial \cos \theta_0}{\partial n_0} = v_0 \cos \theta_0 \cdot \hat{n}_0 \left\{ \frac{\partial \cos \theta_0}{\partial x_0} i + \frac{\partial \cos \theta_0}{\partial y_0} j + \frac{\partial \cos \theta_0}{\partial z_0} k \right\} \cdot (n_{x_0} i + n_{y_0} j + n_{z_0} k)
\]

\[
= \left( \frac{\partial \cos \theta_0}{\partial \theta_0} \frac{\partial \theta_0}{\partial x_0} i + \frac{\partial \cos \theta_0}{\partial \theta_0} \frac{\partial \theta_0}{\partial y_0} j + \frac{\partial \cos \theta_0}{\partial \theta_0} \frac{\partial \theta_0}{\partial z_0} k \right) \cdot (n_{r_0} \cos \theta_0 i + n_{r_0} \sin \theta_0 j + n_{z_0} k)
\]

\[= (-\sin^2 \theta_0 \cos \theta_0 n_{r_0} + \sin^2 \theta_0 \cos \theta_0 n_{r_0}) / r_0 = 0 \]
The last necessary variable transformations in Eq. 2.11 are in the normal derivative of the principal solution, \( P_0 \).

The principal solution for 3 dimensional potential problems is \((K|\mathbf{R}_0 - \mathbf{R}|)^{-1}\) where \(|\mathbf{R}_0 - \mathbf{R}|\) is the absolute magnitude of the relative position vector between the field point and load point. In rectangular coordinates this scalar quantity is written \(((x_0-x)^2+(y_0-y)^2+(z_0-z)^2)^{1/2}\) and in cylindrical polar coordinates with \( \theta = \theta \) it is \((r_0^2+r^2-2r_0r\cos \theta + (z_0-z)^2)^{1/2}\). \( K \) is \(4\pi\) for interior integrals and equals the interior angle in surface integrals, \(2\pi\) for straight boundaries for example.

\[
\frac{\partial P_0}{\partial n_0} = v_0 P_0 \cdot \mathbf{n}_0 = \left( \frac{\partial P_0}{\partial x_0} \hat{i} + \frac{\partial P_0}{\partial y_0} \hat{j} + \frac{\partial P_0}{\partial z_0} \hat{k} \right) \cdot (n_{x_0} \hat{i} + n_{y_0} \hat{j} + n_{z_0} \hat{k})
\]

\[
\frac{\partial P_0}{\partial x_0} n_{x_0} = \left( \frac{\partial P_0}{\partial r_0} \frac{\partial r_0}{\partial x_0} + \frac{\partial P_0}{\partial \theta_0} \frac{\partial \theta_0}{\partial x_0} \right) (n_{r_0} \cos \theta_0)
\]

\[
= \frac{n_{r_0} \cos \theta_0}{K} \left\{ \frac{(r_0 - r \cos \theta_0) \cos \theta_0 + (r_0 \sin \theta_0)(-\sin \theta_0/r_0)}{(r_0^2 + r^2 - 2r_0r \cos \theta_0 + (z_0-z)^2)^{3/2}} \right\}
\]

\[
= \frac{n_{r_0} (r \cos \theta_0 - r_0 \cos^2 \theta_0)}{K(r_0^2 + r^2 + (z_0-z)^2 - 2r_0r \cos \theta_0)^{3/2}}
\]
\[ \frac{\partial P_0}{\partial y_0} n_{y_0} = \left\{ \frac{\partial P_0}{\partial r_0} \frac{\partial r_0}{\partial y_0} + \frac{\partial P_0}{\partial \theta_0} \frac{\partial \theta_0}{\partial y_0} \right\} (n_{r_0} \sin \theta_0) \]

\[ = \frac{n_{r_0} \cos \theta_0}{-K} \left[ \frac{(r_0 - r \cos \theta_0) \cos \theta_0 + (r_0 r \sin \theta_0)(-\sin \theta_0/r_0)}{(r_0^2 + r^2 + (z_0 - z)^2 - 2r_0 r \cos \theta_0)^{3/2}} \right] \]

\[ = \frac{n_{r_0} \cos \theta_0}{-K} \frac{n_{r_0} (-r_0 \sin^2 \theta_0)}{(r_0^2 + r^2 + (z_0 - z)^2 - 2r_0 r \cos \theta_0)^{3/2}} \]

\[ \frac{\partial P_0}{\partial n_0} n_{z_0} = \frac{-n_{z_0} (z_0 - z)}{K(r_0^2 + r^2 + (z_0 - z)^2 - 2r_0 r \cos \theta_0)^{3/2}} \]

The final expression for the normal derivative of the principal solution is obtained by adding all components just developed and defining \( A = r_0^2 + r^2 + (z_0 - z)^2 \) and \( B = 2r_0 r \).

\[ \frac{\partial P_0}{\partial n_0} = \frac{n_{r_0} (r \cos \theta_0 - r_0 \cos^2 \theta_0 - r_0 \sin \theta_0) - n_{z_0} (z_0 - z)}{K(A - B \cos \theta_0)^{3/2}} \]

\[ = \frac{n_{r_0} (r \cos \theta_0 - r_0)}{K(A - B \cos \theta_0)^{3/2}} \frac{n_{z_0} (z_0 - z)}{K(A - B \cos \theta_0)^{3/2}} \]

This completes all of the variable transformations necessary to develop direct BEM solutions for velocity in terms of the desired cylindrical polar coordinate system. Equation 2.11 may now be used to solve independently for the
3 dimensional component velocities \( v_r \) and \( v_z \) which are equivalent to the axisymmetric components \( v_r \) and \( v_z \) since \( \theta=0 \) by definition. In the integrals reported below \( AB=A-B\cos\theta_c \).

\[
\begin{align*}
\mathbf{V}_r &= \frac{\partial \omega}{\partial z_0} \cos \theta_c \, dV_0 - \frac{\partial v_{r_0}}{\partial \theta_c} \cos \theta_c \, dS_o + \frac{v_{r_0} \cos \theta_c \{(r \cos \theta_c - r_o) n_{r_0} - (z_0 - z) n_{z_0}\}}{K(\text{AB})^{3/2}} \, dS_o \\
\mathbf{V}_z &= \frac{1}{r_o} \frac{\partial (r_0 \omega)}{\partial \theta_c} \, dV_0 + \frac{\partial v_{z_0}}{\partial \theta_c} \, dS_o - \frac{v_{z_0} \{(r \cos \theta_c - r_o) n_{r_0} - (z_0 - z) n_{z_0}\}}{K(\text{AB})^{3/2}} \, dS_o
\end{align*}
\] (3.9) (3.10)

The preceding equations are valid BEM solutions for the axisymmetric component velocities \( v_r \) and \( v_z \) but their present form would require a 3 dimensional integration algorithm. The reduction to 2 dimensions is accomplished as before with the Wu integrals, by defining \( dV_0 = r_o d\theta_o dr_o dz_0 \) and \( dS_o = r_o d\theta_o ds_o \) and integrating over \( d\theta_o \) with limits 0 to 2\( \pi \). Also as before, the integrands are elliptic functions of \( \theta_o \) and therefore cannot be integrated in closed form but must be approximated by expressions containing \( E(k) \) and \( K(k) \). This is done by expressing each integral as the sum of simpler integrals which are expressed in terms of \( E(k) \) and \( K(k) \) in Appendix A.
\[
\int_0^{2\pi} \frac{f(r, z) \, d\theta_o}{(A-B\cos\theta_o)^{1/2}} = \frac{4f(r, z)}{(A+B)^{1/2}} K(k)
\]

\[
\int_0^{2\pi} \frac{f(r, z) \cos\theta_o \, d\theta_o}{(A-B\cos\theta_o)^{1/2}} = \frac{4f(r, z)}{(A+B)^{1/2}} \left[ \frac{A}{B} K(k) - \frac{A+B}{B} E(k) \right]
\]

\[
\int_0^{2\pi} \frac{v_{r_0}\cos\theta_o((r\cos\theta_o-r_0)n_{r_0}-(z_0-z)n_{z_0})}{(A-B\cos\theta_o)^{3/2}} \, d\theta_o =
\]

\[
\frac{4E(k)}{(A+B)^{1/2}} \left[ \frac{2rn_{r_0}(A+B)}{B^2} - \frac{(Ar_0+rB)n_{r_0}+A(z_0-z)n_{z_0}}{B(A-B)} \right] +
\]

\[
\frac{4K(k)}{(A+B)^{1/2}} \left[ \frac{(r_0-2Ar/B)n_{r_0}+(z_0-z)n_{z_0}}{B} \right]
\]

\[
\int_0^{2\pi} \frac{v_z((r\cos\theta_o-r)n_{r_0}-(z_0-z)n_{z_0})}{(A-B\cos\theta_o)^{3/2}} \, d\theta_o =
\]
After the integrations over $\theta$, the integrands are functions only of $r_0$ and $z_0$ for a given field point. These 2 dimensional forms still cannot be integrated exactly but they can be integrated numerically using the same methods used in the more common 2 dimensional planar problems.

In addition to being the same dimension, axisymmetric and planar solutions also contain the same singularities though they are much more transparent in the planar problems. The principal solution in the 2 dimensional Poisson's equation contains a $\ln r$ singularity, sometimes called a weak or apparent singularity. The normal derivative of the principal solution contains a $1/r$ type or strong singularity.

The axisymmetric solution integrands contain the same singularities as the planar integrands but they cannot be isolated for integration until after the original 3 dimensional integrands have been expressed in terms of the elliptic integrals $E(k)$ and $K(k)$. Since $A-B=|\hat{R}_o-\hat{R}|^2$ all terms containing $(A-B)$ in the denominator will behave as $1/r$, i.e., a strong singularity as the field point approaches the load point. The $\ln r$ or weak singularities are found in the series approximations for $K(k)$ which are discussed in Chapter IV. The finite element integration as well as all techniques
for handling singularities are described in detail in Chapter IV.

The integral solutions for \( v_r \) and \( v_z \) based on the Wu or Biot-Savart and the direct BEM solutions for \( v_r \) and \( v_z \) as well as pressure and vector potential can be integrated with 2 dimensional finite element methods. The details of the algorithm development is the subject of Chapter IV.

\[ D\text{etermination of Unknown Boundary Quantities} \]

In Chapter IV, it is shown that all integral equations in this research can be approximated with finite sums, i.e.,

\[ u_i = \sum_{j=1}^{N_I} g_j K_{ij} + \sum_{k=1}^{N_B} b_k K_{ik}. \]

In this equation \( u_i \) is the field quantity to be calculated, \( g_j \) is the nodal value of the source function, \( K_{ij} \) is the integrated product of the interpolation and kernel functions, and \( b_k \) is a nodal boundary quantity. This boundary quantity may be either a nodal \( u \) value such as \( v_r \), \( v_z \), pressure, or vector potential or its normal derivative.

In the solution theory of potential boundary value problems it is stated that the problem will be well posed with a unique solution when either the boundary variable or its normal derivative is specified at every point on the boundary. The application of this theorem to most direct BEM solutions is obvious since both the variable and its normal
derivative appear explicitly. However in the Biot-Savart formulation for velocity these conditions are modified by the relationship between $v_r$, $v_z$, and vorticity.

The type and number of boundary conditions specified will of course depend on the problem and available information. A discussion of common conditions for the velocity equation is the next topic in this chapter with similar discussions relating to the pressure and vector potential equations appearing in later chapters covering their respective solutions.

For some boundary value problems it may be possible to specify all boundary quantities which is permissible providing they are kinematically compatible. In these rare cases it is possible to solve immediately for the dependent variable $u$ at any point in the computational region. However for most problems some of the boundary quantities will be unknown and so before the dependent variable can be calculated the unknown quantities must be determined.

For well posed problems the determination of boundary quantities not specified as boundary conditions is a straightforward procedure which ultimately requires the solution of a matrix equation. The first step is to separate the $N_k$ known and $N_u$ unknown boundary quantities in the original finite element summation equation, where $N_B = N_k + N_u$. 
This type of linear equation is written for $u$ at each boundary node where there is an unknown boundary quantity. If $u_i$ is the unknown $\mu_i=1$ otherwise $\mu_i=0$. This set of $N_u$ equations can be written more compactly as a matrix equation of the form $\mathbf{C} = [\mathbf{A}] \mathbf{B}$ where

$$
C_i = u_i(1-\mu_i) - \sum_{j=1}^{N_i} g_{ij} K_{ij} - \sum_{k=1}^{N_k} b_k K_{ik} = \sum_{n=1}^{N_n} b_n (K_{in} - \mu_i)
$$

$A_{iu} = K_{iu} - \mu_i$

$B_n = b_n$.

If the problem is well-posed the matrix $[A]$ will be well-conditioned with a stable inverse. This permits the matrix equation to be solved for $\mathbf{B}$ with any common inverse method using $[A]^{-1}$ such as Gaussian elimination. The boundary quantities of the solution vector, $\mathbf{B}$, can then be used in the original finite element summation to calculate $u_i$ at any specified field point.

This general procedure for determining unknown boundary quantities is called collocation and can be applied in some form to all integral solutions of Poisson's equations in this research. The exact form needed to most efficiently use the boundary information for each particular problem will be described as needed.
Boundary conditions for poisson's velocity equation

Two forms of integral equations have been presented for calculating the component velocities in axisymmetric laminar flows, \( v_r \) and \( v_z \). The first is analogous to the Biot and Savart law in magnetism, Eq. 3.4 and Eq. 3.5, and the second is the direct BEM solution of the component differential equations, Eq. 3.10 and Eq. 3.11. Both equations sets are valid solutions to the vector Poisson's equation for velocity since they both contain interior integrations of the vorticity field and their solutions will completely define the velocity field for the given vorticity field and boundary conditions. However there is a difference in the type and number of boundary quantities needed for a unique solution.

As was pointed out in the previous section integral equations for velocity developed with direct BEM methods contain both the component velocity and its normal derivative in the boundary or surface integral. In the equation for \( v_r \) a well-posed problem with a unique solution would be assured by specifying either \( v_{r\circ} \) or \( \partial v_{r\circ}/\partial n_{\circ} \) at every point on the boundary. The equation for \( v_z \) would have analogous boundary condition requirements.

In the Biot-Savart formulation the integral solutions for \( v_r \) and \( v_z \) both contain \( v_{r\circ} \) and \( v_{z\circ} \) but do not contain any normal derivatives. This makes it possible to obtain a complete velocity field for a given vorticity field by
specifying only \(v_{r_0}\) or \(v_{z_0}\) at every point on the boundary. This is half the number of boundary quantities required in the direct BEM formulations. Since the two methods are solutions to the same potential boundary value problem it would seem that one of the two formulations is incorrect.

The confusion surrounding the boundary requirements of the two methods is resolved by taking a more comprehensive and physical approach to the solution Poisson's vector equation of velocity. Though it is possible to solve for \(v_r\) and \(v_z\) independently using the direct BEM equations, physically the two component velocity fields are not independent. They are related by the definition of vorticity and the continuity equation. These relationships are usually expressed as partial derivatives of \(r_0\) and \(z_0\) but for present purposes it is more illustrative when expressed terms of tangential and normal partial derivatives.

**Definition of vorticity**

\[
\omega = -\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}
\]

\[
= \frac{\partial v_z}{\partial t} - \frac{\partial v_z}{\partial r} \frac{\partial}{\partial n} + \frac{\partial v_r}{\partial t} \frac{\partial}{\partial r} - \frac{\partial v_r}{\partial z} \frac{\partial}{\partial n}
\]

**Continuity equation**

\[
0 = \frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z}
\]

\[
= \frac{v_r}{r} + \frac{\partial v_r}{\partial t} \frac{\partial}{\partial r} + \frac{\partial v_r}{\partial n} \frac{\partial}{\partial n} + \frac{\partial v_z}{\partial t} \frac{\partial}{\partial z} + \frac{\partial v_z}{\partial n} \frac{\partial}{\partial n}
\]
When using the direct BEM solutions for $v_r$ and $v_z$ it is essential that the boundary conditions specified satisfy the above relationships, whereas the Biot-Savart formulation satisfies them automatically by specifying either $v_r$ or $v_z$.

For some boundary types, such as the no-slip boundary, specification of compatible $r$ and $z$ boundary conditions is trivial. For problems where the choice is not obvious it is possible to specify one boundary quantity for one component equation and calculate the kinematically compatible quantity for the remaining component equation. Such a solution procedure might proceed as follows:

1. Specify $v_{z0}$ at every node on the boundary.
2. Collocate to determine $\partial v_{z0}/\partial n_0$ at all boundary nodes.
3. Use the direct BEM solution to calculate the $v_{z0}$ field.
4. Use finite difference approximations or differentiate the velocity interpolation function to determine $\partial v_{z0}/\partial t_0$ on the boundary.
5. Use the definition of vorticity and the continuity equation to express $v_{r0}$ in terms of $\partial v_{r0}/\partial n_0$, $\partial v_{z0}/\partial t_0$, $\partial v_{z0}/\partial n_0$, and $r_0$.
6. Collocate to determine $\partial v_{r0}/\partial n_0$.
7. Use the direct BEM equation to calculate the $v_r$ field.

As long as the sufficient attention is given to the specification of boundary conditions the Biot-Savart and direct BEM formulations are equivalent and equally valid.
solutions to the vector Poisson's equation for velocity. The choice of formulation then becomes mainly a function of the type of boundary information available to the analyst. The differences in the form of the kernel function and load terms do not present any clear advantage for either formulation.

**Known velocity profile** For the laminar flow test problems and most real engineering problems flow restrictions at the boundaries can be described by a composite of a few simple conditions. The simplest is the known velocity profile. The profile information may be the result of previous numerical procedures or an assumed profile such as the uniform or fully developed parabolic which are simple to model mathematically. The Biot-Savart and direct BEM formulations are equally well suited for incorporation of this type of boundary condition.

**Normal derivative specification** For flow boundaries where neither the \( v_r \) or \( v_z \) velocity profile are known it is necessary to specify the normal derivative of one of the velocities. A common application of this condition is for boundaries where it can be assumed that the velocity profile is not changing or is changing very slowly in the normal direction, i.e., \( \partial v_i / \partial n = 0 \). This condition is much easier to incorporate in the direct BEM formulation.

The flow conditions leading to the 0 specification of the velocity normal derivatives will often be accompanied by
a similar condition on vorticity. A discussion of how this condition is incorporated in the solution of the vorticity transport equation is covered in a later chapter.

**No-slip condition**  In the interior flows of viscous fluids there is almost always one boundary where fluid flows in contact with a non-porous solid. This type of boundary is called a no-slip boundary and is characterized by having normal and tangential component velocities equal to 0. All velocity tangential derivatives are also 0 but the normal derivatives are unknown. Using the direct BEM formulation the unknown velocity normal derivatives on a no-slip boundary can be determined by collocation since $\tilde{V}=0$.

Also unknown on the no-slip boundary is the vorticity which is sometimes called the extraneous vorticity. For time dependent problems the change in interior vorticity is described by the vorticity transport equation but the growth of vorticity on a no-slip boundary, $\omega_{\text{ns}}$, is not a transport phenomenon. Instead the extraneous vorticity must be determined such that it is kinematically compatible with the new vorticity field and velocity boundary conditions.

Several methods of determining the extraneous vorticity were reviewed in the introductory chapters all of which used finite difference approximations of the definition of vorticity.
An alternative to the finite difference approximations of the no-slip vorticities is based on the integral solutions for \( v_r \) and \( v_z \). Numerically, this explicit method is a simple extension of the collocation procedure already discussed for determining unknown boundary quantities. However the analytical form of the solution is changed since the vorticity field is no longer assumed to be continuous. The original integral representing the influence of vorticity on the solution is split into two integral. One integral contains all of the free vorticity nodes whose kinetic behavior is governed by the vorticity transport equation and the second covers the thin region separating the no-slip boundary and the row of interior nodes adjacent to it. This separation is illustrated below where \( K \) represents the appropriate kernel function.

\[
\int_{V_0} \omega(\vec{R}_0)K \, dV_0 = \int_{V_0-V_r} \omega(\vec{R}_0)K \, dV + \int_{V_r} \omega(\vec{R}_0)K \, dV_r
\]

Next, it is assumed that all of the vorticity contained in the thin region adjacent to the no-slip boundary, \( V_r \), is concentrated on the no-slip surface. This is done by incorporating the concept of a vortex sheet, \( \zeta \). The strength of the vortex sheet, which has the units of velocity, is defined such that the total vorticity represented by \( \zeta(\vec{R}_0) \) on the no-slip boundary is the same as is contained in \( V_r \). As illustrated in Fig. 3.4 this requirement is expressed on a
differential level as $\omega dh dS_0 = \zeta dS_0$ and $dV_\tau = dh dS_0$ where $h$ is the normal distance between the no-slip surface and the part of the interior region containing the free vorticity nodes.

Using the definitions for $\zeta$ and $dV_\tau$ the integral over $V_\tau$ can be expressed as a surface integral by replacing $\omega(\vec{R}_0)$ with $\zeta(\vec{R}_0)$ which is the only unknown function of vorticity since $\omega(\vec{R}_0)$ in the free vorticity region, $V_0-V_\tau$, is defined by the vorticity transport equation or specified initially.

$$\int_{V_\tau} \omega(\vec{R}_0) K dh dS_0 = \int_{S_0} \zeta(\vec{R}_0) K dS_0$$

![Diagram showing concentration of vorticity in a vortex sheet](image)

Figure 3.4. Concentration of vorticity in a vortex sheet
When this identity is substituted in the appropriate interior integral, the result is a Fredholm integral equation which, at least for the problems in this research, has a stable inverse. This permits a discretization and matrix solution for all nodal values of $\zeta$ similar to the collocation procedure used to determine other boundary quantities. The nodal values of vorticity on the no-slip boundary are then equated to the new nodal values of $\zeta$ such that the total vorticity in $V_T$ is unchanged and the assumed vorticity distribution is satisfied. In a numerical study done by Wu (1976), $\omega_{ns}$ was determined such that $\omega$ was uniformly distributed over one half the distance between the no-slip boundary and the adjacent node on the free vorticity boundary. In this research, a 2 dimensional linear interpolation function is used for the finite element integration vorticity therefore if a similar distribution is assumed in $V_T$, $\omega_{ns}$ may determined directly without first collocating for the $\zeta$ nodal values.

The treatise in the preceding paragraphs is a rather lengthy analytical justification for using a relatively simple and numerically intuitive procedure. The technique was originally introduced via the Biot-Savart formulation but it can also be implemented using the direct BEM formulation though it will be somewhat modified by the presence of normal derivatives in the boundary integrals.
It was mentioned earlier that for a no-slip type boundary, in addition to the vorticity the normal derivatives of both component velocities are also unknown. Thus when using the direct BEM formulation there are 3 unknown boundary quantities on a no-slip boundary but only 2 equations to use for collocation. This problem is resolved by utilizing the definition of vorticity given in terms of tangential and normal derivatives. Since by definition both $\partial v_r/\partial t$ and $\partial v_z/\partial t$ are 0 on a no-slip boundary the definition becomes:

$$\omega = - \frac{\partial v_z}{\partial n} \frac{\partial n}{\partial r} + \frac{\partial v_r}{\partial n} \frac{\partial n}{\partial z}.$$  

Incorporation of this identity reduces the number of unknown quantities to 2, $\omega$ and $\partial v_r/\partial n$ or $\partial v_z/\partial n$, which can be determined by collocation. The resulting algorithm may become tedious if developed for all possible surface orientations but in this research either $\partial n/\partial r$ or $\partial n/\partial z$ equal 0, i.e., all boundaries are either horizontal or vertical. This permits the independent solution of $v_r$ and $v_z$ which reduces the size of the matrix equation to be solved in collocation process. Whereas in the Biot-Savart algorithm they are always solved simultaneously.

Testing the Solutions of Poisson's Velocity Equation

One of the most unsettling aspects in the development of BEM solutions to partial differential equations is the number
of places critical errors can be made. Probably the most
fundamental is the incorrect derivation of the solution
integrals which are especially complicated for axisymmetric
formulations. Adding to the anxiety is the fact that there
is no way to make intermediate quantitative checks during the
development process.

Another common trouble area in the development of BEM
solutions is the numerical integration of singular
integrands. When troubleshooting these integration
algorithms there are some intermediate qualitative checks
which can help detect derivation or programming errors.
These diagnostics as well as the development of the numerical
integration techniques are covered later in this chapter.

Regardless of how many intermediate checks can be made
during the development of a numerical technique its validity
will remain in doubt until it has been successfully applied
to a problem with a known solution. To verify the integral
solutions of the vector Poisson's equation for velocity
developed in this chapter flow problems are needed which have
known compatible vorticity and velocity fields. Another
desirable feature of such problems is that the known fields
be completely non-zero.

Selection of test flow problems

The two axisymmetric flow problems most suited for the
present task are: 1. impulsively started pipe flow and 2.
fully developed or Poiseuille flow. The vorticity and velocity fields and boundary conditions for these flow problems are illustrated in Fig. 3.5 and Fig. 3.6. One unfortunate shortcoming of both problems is that \( v_\tau \) and \( \partial v_\tau / \partial n \) equal 0 everywhere. Consequently the integrals involving these distributed quantities in both the Biot-Savart and direct BEM formulations cannot be verified by these test problems. The verification of these integrals must wait until the time dependent or kinetic part of the Navier Stokes solution is completed. The test problems for these equations have non-zero values of \( v_\tau \) and \( \partial v_\tau / \partial n \).

An indirect test of the integrals containing \( v_\tau \) and \( \partial v_\tau / \partial n \) in the direct BEM formulation will come in the test problems for the direct BEM solutions of the Poisson's equation for vector potential. This is possible since the two integral solutions have identical kernel functions although the form of the load functions are slightly different.

The solution of the impulsively started pipe flow or slug flow problem is defined to be the velocity and vorticity fields which exist immediately after a fluid volume located in the interior of a circular pipe is impulsively started from rest. The assumed solution vorticity and velocity fields are as follows: 1. \( v_\tau \) is everywhere 0, 2. \( v_z \) is uniform except for the nodes immediately adjacent to and on
the no-slip boundary where it varies quadratically from the uniform value $v_m$ to 0 on the boundary, and 3. the vorticity is 0 everywhere except on the no-slip boundary where it equals $\omega_m = -\partial v_z / \partial r$.

Figure 3.5. Impulsively started pipe flow

Figure 3.6. Fully developed or Poiseuille pipe flow
As in the impulsively started pipe flow problem, the solution to the fully developed flow is not dependent on $z$ and both $v_r$ and $\partial v_r/\partial n$ equal 0 everywhere. The component velocity $v_z$ is 0 on the no-slip boundary and varies quadratically to $v_m$ at the centerline. The vorticity is 0 at the centerline and varies linearly to $\omega_m$ on the no-slip boundary.

Test procedure

Using the two defined flow problems as a foundation, simple procedures were outlined to test the solution algorithms based on both the Biot-Savart and direct BEM formulations. The procedures for the two are slightly different due to the different boundary quantities in the boundary integrals but the same philosophy was used in both. Briefly, the procedure for all tests was: 1. specify boundary conditions and interior vorticities, 2. collocate to determine the unspecified boundary quantities, and 3. use the integral solutions to calculate both the $v_r$ and $v_z$ component velocity fields.

Specification of boundary conditions

For all problems there are 3 boundaries to be defined. They are: 1. inlet, 2. outlet, and 3. no-slip. To shorten the tests it was assumed for both problems that all quantities were known at the inlet boundary. At the outlet boundary it was felt that the most severe test of the collocation procedure would
be afforded by specifying quantities known to be 0. This meant that for the Biot-Savart formulation, \( v_r \) is specified and for the direct BEM formulation, \( \partial v_z / \partial n \) was specified. At the extraneous boundary both \( v_r \) and \( v_z \) are 0 so either may be specified as boundary conditions in the Biot-Savart formulation. In the direct BEM formulation for \( v_z \), \( v_z \) is specified to be 0.

**Collocation** The procedure for determining the unknown boundary quantities is essentially the same for both test problems but there are differences associated with the formulation. Since the Biot-Savart formulations do not contain velocity normal derivatives, \( \omega_{ns} \) is the only unknown on the no-slip boundary. However, when using the direct BEM formulations, it is necessary to solve for both \( \omega_{ns} \) and \( \partial v_z / \partial n \) by collocation made possible by utilizing the definition of vorticity.

**Calculation of velocity fields** The final step in the tests for both problems is to substitute the boundary quantities determined in the collocation procedure in the integral solutions for \( v_r \) and \( v_z \). When calculating the \( v_z \) component field with the direct BEM formulation, all types of kernels are tested since each problem contains some non-zero values of \( \omega \), \( v_z \), and \( \partial v_z / \partial n \). However, in the direct BEM formulation for \( v_r \), only the vorticity kernel is tested. In both the \( v_r \) and \( v_z \) Biot-Savart formulations, the kernel in
the integral containing $v_r$ is not tested since that component is always 0.

**Error terms**

Two types of error terms have been used to evaluate the accuracy of a complete solution procedure. They are: 1. problem specific and 2. physical law compatibility. The first is simply the deviation of a numerical solution from a known solution and therefore has limited application in industry but can be useful in algorithm development. When used in this research such error terms will report the sum of the error squared with the subscript indicating the variable being tested. For example, in the solution error $e_{v_z}$, $v_z^*$ is the known value of $v_z$ at the $i$th node.

$$e_{v_z} = \sum_{i=1}^{N} (v_z^*(i) - v_z(i))^2$$  \hspace{1cm} (3.11)

The second type of error is a measure of to what extent a specified physical law is violated and can be applied to any flow problem for which the law applies. Since the second type does not require a known solution, it has much greater potential for use in engineering.

There are two error terms of the second type used, both of which are derived from the conservation of mass. One is an expression of this law on the control volume level and states that across any defined enclosed boundary the net mass
flow, or the net volume flow since \( \rho \) is assumed constant, must equal 0. Mathematically this law is stated as

\[
\int_S (\vec{V} \cdot ds) = 0.
\]

The error term based on this law reflects to what degree the new solution violates the balance of mass. It is the mass balance error, \( e_{mb} \), and is calculated

\[
e_{mb} = \frac{\int_{in} (\vec{V} \cdot ds) - \int_{out} (\vec{V} \cdot ds)}{\int_{in} (\vec{V} \cdot ds)}.
\tag{3.12}
\]

The integrals appearing in the above error expression are approximated numerically using the same assumed boundary velocity distribution and interpolation functions as used in the BEM integrations.

On the differential level, the conservation of mass, given that the fluid is incompressible, becomes the continuity equation, \( \nabla \cdot \vec{V} = 0 \). The error term based on this expression is

\[
e_C = \sum_{j=k_1}^{k_2} \sum_{i=n_1}^{n_2} \frac{(v \cdot \vec{V})_i,j}{2}.
\tag{3.13}
\]

This is approximated by a summation of finite difference representations of \( v \cdot \vec{V} \) at a specified number of element nodes. The order and form of the finite difference
representation and number of nodes can be changed to suit the particular application.

Test Results and Conclusions

The formulations, Biot-Savart and direct BEM, for the solution of Poisson's velocity equation will be put to severe tests in the following two chapters but for now a simple preliminary test will be offered as proof of their validity. The test flow problems, impulsively started and Poiseuille flow, are described along with the test procedure on pp. 99-100. The reported error terms which are used extensively throughout this dissertation are defined on pp. 101-102.

The results of the test flow solutions are reported in Table 3.1. The intention of this table is not to establish

<table>
<thead>
<tr>
<th>Problem</th>
<th>Formulation</th>
<th>$e_{vz}$</th>
<th>$e_{mb}$</th>
<th>$e_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>impulsively started flow</td>
<td>Biot-Savart</td>
<td>0.0029</td>
<td>-0.148%</td>
<td>0.031</td>
</tr>
<tr>
<td></td>
<td>direct BEM</td>
<td>0.0001</td>
<td>-0.009%</td>
<td>0.932</td>
</tr>
<tr>
<td>Poiseuille flow</td>
<td>Biot-Savart</td>
<td>0.0000</td>
<td>-0.024%</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>direct BEM</td>
<td>0.0002</td>
<td>0.029%</td>
<td>0.665</td>
</tr>
</tbody>
</table>
which formulation is more accurate but only to demonstrate that both have passed a first test of validity. In fact such a comparison would seldom be used to identify the preferred formulation for a particular type of problem. As was described earlier this will usually be determined by the type of available boundary information since the two formulations use different boundary conditions.

The performance of the two formulations for solution of Poisson's velocity equation is as dependent on the accuracy of the kernel integrations as on the formulation itself. However the effects of variations in the quadrature algorithm, such as density of quadrature points, will not be investigated in this chapter. Test problems in this chapter were solved with what was judged to be convergent values of the kernel functions. A discussion of the integration fine tuning process which produced these convergent values is given in Chapter IV.
CHAPTER IV. NUMERICAL EVALUATION
OF BOUNDARY ELEMENT INTEGRALS

Theoretically correct and eloquent integral equations will produce good engineering solutions only when the methods used to evaluate them are of comparable quality. This is especially true with integrands that become singular within the integration limits. For some integrals having singular integrands, it is possible to simply ignore the singularity and proceed with simple numerical quadratures. However the nature of integral equations based on the use of fundamental solutions to solve partial differential equations are such that such crude procedures are almost always unacceptable. A major part of this chapter outlines numerical techniques which have proved successful at integrating the types of singular integrands common in BEM integrals.

Since all of the integrals to be evaluated are from axisymmetric BEM solutions the integrands all contain elliptic functions. In Chapter III, a procedure was given for expressing elliptic integrals of a general form in terms of the standard complete elliptic integrals of the first and second kind, $E(k)$ and $K(k)$ respectively. This makes the accurate calculation of $E(k)$ and $K(k)$ a crucial part of the successful evaluation of the integrals developed in this research. Also one of the singularities which must be dealt
with is imbedded in $K(k)$. For these reasons, the method used for evaluating the standard complete elliptic integrals will be presented prior to the section on numerical integration.

Evaluation of Elliptic Integrals

The method used for numerical evaluation of $E(k)$ and $K(k)$ is the Chebyshev approximations presented by Cody (1965) which was selected for two main reasons. First of all the number of terms used can be varied to satisfy the accuracy requirements which change as a function of the modulous, $k$. Secondly, in the series approximation for $K(k)$, it is easy to identify and isolate the terms which are responsible for the singular behavior as $k$ approaches 0.

The expressions used to approximate $K(k)$ and $E(k)$ are Eq. 4.1 and Eq. 4.2, respectively. In these series equations, $n$ may be increased from 2 to 10 to improve accuracy. $\eta$ is the square of the complementary modulous, $\eta = 1 - k^2$. The values of all the coefficients, $a_i$, $b_i$, $c_i$, and $d_i$, for $n=2$ to 10 are listed in the Cody paper.

$$K(k) = \ln 4 + \sum_{i=1}^{n} (a_i \eta^i) + \ln(1/\eta) \left[ \frac{1}{2} + \sum_{i=1}^{n} (b_i \eta^i) \right]$$ \hspace{1cm} (4.1)

$$E(k) = 1 + \sum_{i=1}^{n} (c_i \eta^i) + \ln(1/\eta) \left[ \sum_{i=1}^{n} (d_i \eta^i) \right]$$ \hspace{1cm} (4.2)
The value of a complete elliptic integral is uniquely determined by \( \eta \) which may take on values from 0 to 1. As \( \eta \) approaches 0 \( K(k) \) approaches 0 but \( E(k) \) is finite for all values of \( \eta \). These statements can be quickly verified by substituting \( \eta=0 \) and \( \eta=1 \) into Eqs. 4.1 and 4.2.

The physical significance of \( \eta \) can be exposed by expressing \( \eta \) as follows:

\[
\eta = \frac{r^2+r_0^2-2rr_0+(z_0-z)^2}{r^2+r_0^2+2rr_0+(z_0-z)^2} = \frac{(r_0-r)^2+(z_0-z)^2}{(r_0+r)^2+(z_0-z)^2}
\]

The denominator of the final expression for \( \eta \) is always greater than 0 except for the unimportant case when both field point and load point are located at the origin. More importantly the numerator is equal to the magnitude squared of the relative position vector \( \vec{R}_0-\vec{R} \) when expressed as 2 dimensional axisymmetric vectors. Therefore the physical interpretation of \( \eta \to 0 \) is that the field point is approaching the load point, which has already been identified as a problem area in BEM solutions. Besides being helpful in relating the numerics and physics of the problem, the new expression for \( \eta \) will prove very helpful when developing the numerical integration formulas.
Numerical Integration

The character of the integrals developed in this research are best studied as the product of two terms. One term is the load function which is a function of either vorticity or velocity. The second term quantifies the influence of each load at prescribed load point coordinates on the solution at prescribed field point coordinates. This term, called the kernel, is a 2 point function, i.e., it is a function of both the field point and load point.

The denominators of all BEM kernels are functions of the magnitude \(|\mathbf{R}_0 - \mathbf{R}|\) where \(\mathbf{R}_0\) is the load point position vector and \(\mathbf{R}\) is the field point position vector. Therefore as the load point approaches the field point during numerical integration the kernel will approach infinity. However, the interpretations made in the previous paragraph suggest that such mathematic behavior would not accurately model the physical behavior. If it was accurate, it would mean that a finite non-zero load located at the point coincident with the field point would result in an infinite velocity at that point, and this does not make physical sense.

The apparent discrepancy between the mathematical model and physical problem when \(|\mathbf{R}_0 - \mathbf{R}|\rightarrow 0\) is not caused by incorrect theory or derivation, only by the necessity to integrate numerically. In fact, the singularities have already been removed analytically in a process which is the
foundation of any solution method based the use of fundamental solutions. The result of this analytical limiting process are integrals which are finite valued but contain singular integrands. The challenge then is to develop integration algorithms which remove the singular behavior. The method appropriate for this task will depend of the type of singularity involved.

In the BEM solution of Poisson's equation there are three general types of singularities encountered. One is the \( \ln r \) type, sometimes called a "weak" singularity, which appears in the approximation of \( K(k) \) (Eq. 4.1). When \( r=0 \) \( \ln r \) is \( \infty \) but the integral over the limits of 0 to \( r \) is finite. When integrated numerically a variable transformation to local coordinates must be made such that \( \ln r \) is replaced by finite valued function, or use a special Gaussian quadrature that has already incorporated such a transformation.

A second form of apparent singularity occurs when the integrand can be rearranged to form the product of two terms, one which behaves like \( 1/r \) and one which behaves like \( r \). When left unchanged, integration by simple quadrature would force the computer to try to perform the limiting process as \( r \to 0 \), something it is not very good at. To avoid this, it is advisable to make a variable transformation, if necessary, and perform the algebraic cancellations in the model equations.
The third type of singularity encountered is the strong or true singularity of the form $1/r$. In this case both the integrand and the ordinary integral value containing $r=0$ are infinite, thus requiring something more than a variable transformation. The key to resolving this problem is found in the development of the BEM theory incorporating the concept of a fundamental solution. An important step in this derivation is the removal of these $1/r$ type singularities, which occur only in surface integrals. This is done by integrating around them at a radius of $\varepsilon$, evaluating in the limit as $\varepsilon \to 0$, and subtracting the result from the total surface integral. Using this procedure in the derivation justifies the integration of these singularities in the Cauchy principal value sense. Usually, this will first require a transformation to local coordinates to get the integral in suitable form to apply the technique.

The goal of this section is to ultimately replace each continuous BEM integral with a discretized sum of $N$ terms. Each of these terms will be the product of a nodal load value and a kernel function which depends only on geometry. This will start with the transformation to a local coordinate system since this affects the form of all formula. Following this is a derivation of the load variable interpolation functions. The last two topics are the actual integration
algorithms, one for singular terms and one for non-singular
terms.

Transformation from global to local coordinate systems

The numerical integration in this research will be
performed by finite element methods which, simply stated,
approximate a continuous integral by the summation of N
elemental integrations. It is common practice to define a
new coordinate system for each integration rather than using
the same global system. These local coordinate systems are
usually employed for the sake of standardization but for the
current problem they are a key step in the removal of
apparent singularities.

After the θ integration, axisymmetric problems appear
identical to 2 dimensional planar problems, though the
governing equations are still different. Figure 4.1 shows a
2 dimensional r-z plane with a superimposed spherical polar
local coordinate system (ρ,α,φ). In the local coordinate
system, α is defined equal to 0 and the origin, ρ=0, is
defined coincident with the field point (r,z). This results
in the relationships (r₀-r)=ρsinφ and (z₀-z)=ρcosφ. The
convenience of this transformation is that the magnitude of
the relative position vector, |R₀-R|, now simply equals |ρ|.

The transformation is completed by defining the
differential area, dA₀, and boundary line segment, ds, in the
local coordinates. Throughout the domain the general
relationship $dA_0 = \rho \phi d\rho$ can be used. The specialized expression for $ds$, valid only for horizontal or vertical boundaries, transforms simply to $ds = d\rho$.

When using some quadrature schemes with predetermined nodal coordinates and weights it is necessary to make additional changes to normalize the limits of integration, usually to 0 to 1 or -1 to +1. These simple transformations will be explained as they are needed.

**Interpolation functions**

**Vorticity** One of the key steps in finite element type integration is the approximation of the continuous distribution of the load variable. This is done in a

![Figure 4.1. Transformation from global to local coordinates](image)
discretized manner by expressing the distribution over each element as a summation of the nodal values of the load function contained in that element. For example, the distribution of vorticity over an axisymmetric interior element containing $n$ nodes would be approximated as follows:

$$\omega_j(r_0, z_0) = \sum_{i=1}^{n} \omega_{i,j} L_i, j(r_0, z_0).$$

(4.3)

In Eq. 4.3, $\omega_j(r_0, z_0)$ is the vorticity distribution over the $j$th element and $\omega_{i,j}$ is the nodal vorticity at the $i$th node in the $j$th element. $L_i, j(r_0, z_0)$ is the interpolation function which defines the contribution, as a function of $r_0$ and $z_0$, of the $i$th nodal vorticity to $\omega_j(r_0, z_0)$.

The form of the interpolation function and the required number of nodes is determined by the assumed distribution of vorticity over the $j$th element. In this research, vorticity was assumed to vary linearly in both the $r$ and $z$ directions for all elements, i.e., $\omega_j(r_0, z_0) = a_j + b_j r_0 + c_j z_0$. Since this distribution contains three unknown coefficients, each element must contain at least three nodes to determine them, which in turn requires the use of triangular elements. Interpolation functions of this form are easily found in elementary texts, but they are also quickly derived with collocation methods. This is not only instructive but allows them to be adapted to the particular needs of the problem.
The interpolation functions for vorticity are derived below where \((r_i, z_i)\) are the coordinates of the \(i\)th node.

\[
\begin{align*}
\omega_{1,j} &= a_j + b_j r_1 + c_j z_1 \\
\omega_{2,j} &= a_j + b_j r_2 + c_j z_2 \\
\omega_{3,j} &= a_j + b_j r_3 + c_j z_3
\end{align*}
\]

or

\[
\begin{bmatrix}
1 & r_1 & z_1 \\
1 & r_2 & z_2 \\
1 & r_3 & z_3
\end{bmatrix}
\begin{bmatrix}
a_j \\
b_j \\
c_j
\end{bmatrix}
=
\begin{bmatrix}
\omega_{1,j} \\
\omega_{2,j} \\
\omega_{3,j}
\end{bmatrix}
\]

Solution of the above system of equations yields \(a_j\), \(b_j\), and \(c_j\) which can then be substituted into the assumed elemental vorticity distribution. The result is Eq. 4.4 which is the approximate distribution of \(\omega_j\) for a 3 node triangular element.

\[
\omega_j(r_0, z_0) = \omega_1 L_1(r_0, z_0) + \omega_2 L_2(r_0, z_0) + \omega_3 L_3(r_0, z_0)
\]  \hspace{1cm} (4.4)

where:

\[
L_1(r_0, z_0) = \frac{(r_2 z_3 - r_3 z_2) + (z_2 - z_3) r_0 + (r_3 - r_2) z_0}{D}
\]

\[
L_2(r_0, z_0) = \frac{(r_3 z_1 - r_1 z_3) + (z_3 - z_1) r_0 + (r_1 - r_3) z_0}{D}
\]

\[
L_3(r_0, z_0) = \frac{(r_1 z_2 - r_2 z_1) + (z_1 - z_2) r_0 + (r_2 - r_1) z_0}{D}
\]

\[
D = r_2 z_3 + r_1 z_2 + r_3 z_1 - r_2 z_1 - r_1 z_3 - r_3 z_2
\]

**Velocity**  
The velocity along each boundary element or segment was assumed to vary quadratically rather than linearly since the 2nd order distribution is more compatible with the linear vorticity distribution. In explanation, consider a boundary which is perpendicular to the z axis and on which \(v_r = 0\), \(\partial v_r / \partial r = 0\), and \(\partial \omega / \partial r = C\). From the definition of
vorticity, \( \omega = -\frac{\partial v_z}{\partial r} \), it is evident that \( v_z(r) \) must be at least 2nd order in \( r \) so that \( \frac{\partial \omega}{\partial r} \) is non-zero.

For a general 2nd order boundary segment the velocity distribution is \( v_j(r_o, z_o) = a_j + b_j r_o + c_j r_o^2 + d_j z_o + e_j z_o^2 \). This expression contains five unknown coefficients which would require a 5 node boundary segment. However, this expression is simplified by the already stated restriction of considering only vertical or horizontal boundaries. Now the 2nd order velocity distribution is expressed by \( v_j(x_o) = a_j + b_j x_o + c_j x_o^2 \), where \( x_o = r_o \) for vertical surfaces and \( x_o = z_o \) for horizontal surfaces. This form requires only a 3 node element which significantly reduces the computing time and storage requirements.

For the axisymmetric flow problem there is justification for using two different interpolation functions for velocity. The first interpolation applies to all boundary segments except those adjacent to the centerline. The derivation of the first function proceeds exactly as the vorticity interpolation derivation, by collocation on three nodal values. The following derivation can be applied to either vertical or horizontal surfaces by proper definition of \( x \), also the velocity, \( v \), can be either \( v_r \) or \( v_z \).

\[
\begin{align*}
v_{1,j} &= a_j + b_j x_1 + c_j x_1^2 \\
v_{2,j} &= a_j + b_j x_2 + c_j x_2^2 \quad \text{or} \\
v_{3,j} &= a_j + b_j x_3 + c_j x_3^2
\end{align*}
\]

\[
\begin{bmatrix}
1 & x_1 & x_1^2
\end{bmatrix}
\begin{bmatrix}
a_j \\
b_j \\
c_j
\end{bmatrix}
= 
\begin{bmatrix}
v_{1,j} \\
v_{2,j} \\
v_{3,j}
\end{bmatrix}
\]
Solution of the above system for the coefficients $a_j$, $b_j$, and $c_j$ results in the elemental velocity distribution in the desired form, Eq. 4.5

\[ v_j(x_0) = v_1L_1(x_0) + v_2L_2(x_0) + v_3L_3(x_0) \]  \hspace{1cm} (4.5)

where:

\[ L_1(x_0) = \frac{(x_2x_3^2-x_3x_2^2)+(x_2^2-x_3^2)x_0+(x_3-x_2)x_0^2}{D} \]

\[ L_2(x_0) = \frac{(x_3x_1^2-x_1x_3^2)+(x_3^2-1^2)x_0+(x_1-x_3)x_0^2}{D} \]

\[ L_3(x_0) = \frac{(x_1x_2^2-x_2x_1^2)+(x_1^2-x_2^2)x_0+(x_2-x_1)x_0^2}{D} \]

\[ D = x_2x_3^2+x_1x_2^2+x_3x_1^2-x_2x_1^2-x_1x_3^2-x_3x_2^2 \]

In axisymmetric flows there are two special velocity boundary conditions which must meet at the centerline ($r=0$):

1. $v_r=0$ and 2. $\partial v_z/\partial r=0$. Given the definition of vorticity, $\omega$, these two conditions also mean that $\omega=0$ at $r=0$. However, the partial with respect to $r$ of the velocity distribution function, Eq. 4.3 with $v_j(x_0)=v_{z,j}(r_0)$, is not identically equal to 0, as it must be to satisfy this condition. For this reason a third order interpolation function is used for boundary segments adjacent to the centerline.

The third order interpolation function contains one more coefficient which permits the incorporation of the centerline vorticity condition. This is demonstrated in the following derivation where node 1 is on the centerline.
\[ v_z = a_j + b_j r_0 + c_j r_0^2 + d_j r_0^3 \]

\[ \omega(0) = \partial v_z / \partial r = 0 + b_j + 2c_j r_1^2 + 3d_j r_1^2 \]

The above equation set can be greatly simplified since \( r_1 = 0 \) which yields immediately \( a_j = v_{z1} \) and \( b_j = 0 \). Solution of the reduced set of equations leads to the following expression of \( v_z \) for boundary segments adjacent to the centerline.

\[ v_{z,j}(r_0) = v_{z1,L1}(r_0) + v_{z2,L2}(r_0) + v_{z3,L3}(r_0) \tag{4.6} \]

where:

\[ L_1(r_0) = 1 + \frac{(r_2^3 - r_3^3) r_0^2 + (r_3^2 - r_2^2) r_0^3}{(r_2^2 r_3^3 - r_2^3 r_3^2)} \]

\[ L_2(r_0) = \frac{r_3^3 r_0^3 - r_3^2 r_0^3}{(r_2^2 r_3^3 - r_2^3 r_3^2)} \]

\[ L_3(r_0) = \frac{-r_2^3 r_0^2 - r_2^2 r_0^3}{(r_2^2 r_3^3 - r_2^3 r_3^2)} \]

Development of Eqs. 4.4–4.6 completes the first step of the discretization of the BEM integral equations. It is now possible to replace each continuously distributed load variable over an interior element or boundary with a discrete approximation.
Numerical grid

The next step in the finite element integration is the development of the numerical grid. This essentially means establishing: 1. the number of elements, 2. the number of nodes per element, and 3. the coordinates of each node. The number of nodes per element has already been set by the assumed elemental load distributions. The linear interior vorticity distribution dictates a 3-node element and the simplified 2nd order velocity distribution dictates a 3 node boundary segment. The density of the nodes and the orientation of the elements are subject to such considerations as domain geometry, existence of large gradients such as sharp corners, and desired accuracy. These are weighed against the practical considerations of computing time and memory requirements.

In this research, the grid selection was most influenced by the desire to minimize the computer storage requirements. The result is the simplest grid which could still accommodate the assumed load distributions; a rectangular domain composed of triangular elements with uniform node spacing in both the r and z directions. The simplicity and symmetry of this grid design not only saves computer memory but eliminates the need to develop a grid generation algorithm. Figure 4.2 is an unscaled drawing of the numerical grid used for all problems.
Integration algorithms

There are two types of integration algorithms used, one for non-singular integrands and one for singular integrands. The majority of the elemental integrations involve non-singular integrals but their total contribution to the solution is much less than that of the singular integrals since the singular kernels are much larger.

Non-singular Non-singular integrals by definition contain well-behaved functions permitting the simplest of quadrature methods to be used. In this research a Gaussian quadrature with predetermined coordinates and weights was used. Since there are no singularities, no variable transformations are necessary though they can be used in the interest of standardization.

For the integration of vorticity loads over interior elements Gaussian integration formula for triangles was used.
with the coordinates given in normalized area coordinates, 
\(a_k = (r_k, z_k)\) (Banerjee and Butterfield, 1981, p. 441). To demonstrate how the finite element interpolation functions are implemented, the numerical approximation of a integral over \(A_0\), for the \(j\)th element, has been written below.

\[
\int_{A_0, jth \ element} \omega_0 g(r_0, z_0) \, dA_0 = \sum_{i=1}^{n} \omega_{i,j} \int_{A_0} L_{i,j}(r_0, z_0) g(r_0, z_0) \, dA_0
\]

In some finite element integrations the above integral over \(A_0\) could be integrated numerically. This method is called semi-analytical integration and is quite common. However, the functions \(g(r_0, z_0)\) in this research, primarily due to the presence of elliptic integrals, cannot be integrated analytically. Instead, a numerical quadrature must be employed over \(A_0\) as shown in Eq. 4.7.

\[
\int_{A_0, jth \ element} \omega_0 g(r_0, z_0) \, dA_0 = \sum_{i=1}^{n} \omega_{i,j} \left\{ \sum_{k=1}^{N} (L_{i,j}(a_k) g(a_k) w_k(a_k)) \right\}
\]

where: 
- \(a_k\) = area coordinates of integration points 
- \(w(a_k)\) = Gaussian weight at \(a_k\) 
- \(N\) = number of integration points per triangle 
- \(n\) = number of nodes per element 
- \(g(r_0, z_0) = (z_0-z)r_0J_1/2\pi\)
Numerical integrations similar to the ones used for vorticity can be applied to the non-singular boundary segment integrations which have velocity loads. The only difference is that quadrature formula for straight lines are used (Banerjee and Butterfield, 1982, p. 439).

**Singular elements** As has already been explained the first step in formulating the singular integrations is the transformation to a local coordinate system. This permits the algebraic simplifications necessary to remove the troublesome apparent singularities. For the vorticity load integrations over $A_o$, this is sufficient to transform the integrals to a form suitable for simple Gaussian integration formula, such as was described for the non-singular integrations. To demonstrate the steps necessary to arrive at this form, the area integral over $A_o$ in Eq. 3.4 will be transformed.

The integrands in this research are actually the sum of several terms. To facilitate the transformations to local coordinates, it is necessary to first isolate the terms which are singular from the terms which are non-singular. For example, consider the integrand of the area integral in Eq. 3.4, $\omega _0 (z_0 - z) r_0 J_1 dA_0$. In this integral, the problem terms are contained in both $J_0$ and $J_1$, previously defined as:
\[ J_0 = \left( \frac{2(A+B)^{1/2}}{(A^2-B^2)} \right) E(\eta) \quad J_1 = \left( \frac{-2}{B(A+B)^{1/2}} \right) K(\eta) + \frac{A}{B} J_0 \]

where \( K(\eta) \) and \( E(\eta) \) are approximated by Eq. 4.1 and Eq. 4.2 respectively.

In the approximation for \( K(\eta) \) the term \( \ln(1/\eta) \) will approach \( 0 \) as \( \eta \) approaches 0. As was explained earlier, this is tantamount to the physical condition of the field point approaching the load point, i.e., \( |\hat{R}_o - \hat{R}| \to 0 \). Transformation to the local system \( (\rho,\phi) \), as illustrated in Fig. 4.1, redefines the singular behavior to occur as \( \rho \to 0 \). When the term containing \( \ln(1/\eta) \) is isolated and written in terms of the local coordinates and \( dA_\phi \) is replaced by \( \rho d\phi d\rho \) the problem is resolved. In its local coordinate form, shown below, it is clear that as \( \rho \to 0 \) the integrand also goes to 0.

\[
\left\{ \frac{-1}{B(A+B)^{1/2}} \right\} \ln(1/\eta) dA_\phi = \left\{ \frac{-1}{B(A+B)^{1/2}} \right\} \ln \left( \frac{(A+B)}{\rho} \right) \rho d\rho d\phi
\]

where \( r_o = r + \rho \sin\phi \) and \( z_o = z + \rho \cos\phi \).

The other algebraic form of singularity is found in \( J_0 \).

Since \( E(\eta) \) is finite for all values of \( \eta \), the only problem term is \( (A^2-B^2) \) or \( (A-B)(A+B) \) which appears in the numerator. Using the previous definitions of \( A=r^2+r_o^2+(z_o-z)^2 \) and
\( B = 2r_0 r, \ (A-B) \) is quickly written as \((r_0 - r)^2 + (z_0 - z)^2\) or in local coordinates as \(r^2\). As before, replacing \(dA_0\) with \(\rho d\phi d\rho\) and \((z_0 - z)\) with \(\rho \cos \phi\) allows the isolated singular term to be transformed, the result of this term is as follows:

\[
\begin{align*}
\text{global coordinates} & \quad \frac{A}{B} \left[ \frac{2(A+B)^{1/2}}{(A^2-B^2)} \right] (z_0 - z) E(\eta) dA_0 \\
\text{local coordinates} & \quad \frac{A}{B} \left[ \frac{2(A+B)^{1/2}}{\rho^2(A+B)} \right] \rho \cos \phi E(\eta) \rho d\phi d\rho
\end{align*}
\]

The transformation of the above expression is completed by performing the algebraic cancellation of \(\rho^2\) which appears in both the denominator and numerator. In this final computational form, the weak singularity as \(\eta \to 0\) is removed and normal Gaussian quadrature formula may be used in the integration algorithm.

The end result of all the transformations is an integral which is rather complicated but free of any type of singularity. To integrate these equations, it was desired to use a method which conformed to and took advantage of the variable transformation and also for which it was easy to vary the number of quadrature points. The method developed is actually a combination of two types of integration formula, a straight line Gaussian quadrature used for the integration over \(d\rho\) and the Trapezoidal rule used to integrate over \(d\phi\).
Figure 4.3. Integration of triangular elements

Figure 4.3 depicts an arbitrary 3 node triangular element and a field point $F(r,z)$, as is used in all area integrations required in the finite element integrations. To be integrated over this element is a known function $g=g(r,z,r_0,z_0)$ or $g(r,z,\rho,\phi)$ in local coordinates.

Rather than developing an algorithm which integrates directly over $\Delta_{123}$ it is more efficient from a programming standpoint to evaluate the integral as the sum

$$
\int_{\Delta_{123}} g \, dA = \int_{\Delta_{F12}} g \, dA + \int_{\Delta_{F23}} g \, dA - \int_{\Delta_{F13}} g \, dA.
$$

This simplifies the integration subroutine since the only input is the coordinates of the 3 element nodes and the field point. Also all of the necessary logic controlling algebraic signs, etc., can be done elsewhere in the program.
The first step in the numerical integration of $g$ over an arbitrary triangle, $\Delta_{F_{23}}$ for example, is to subdivide into $N_\phi$ smaller triangles all containing the field point $F$. For this development it is assumed that all $\Delta \phi$ are equal, i.e., $\phi_U - \phi_L = N_\phi \Delta \phi$, but this is not mandatory. This automatically defines $N_\phi + 1$ straight lines all of which start at $F$ and end at their intersection of $r_{23}$.

The function $g$ is integrated along each of the $N_\phi + 1$ lines using Gaussian quadrature formula for straight lines with $N_\rho$ quadrature points. This integration of $g$ over $d\rho$ can now be thought of as a new function $f$ where

$$ f_1 = \int g \, d\rho = \sum_{j=1}^{N_\rho} g(r, z, \rho, \phi_1) \, w_j. $$

For the integration over $d\phi$ it is assumed that the defined function $f$ is uniformly distributed over each $\Delta \phi$ with a value of $f = (f_1 + f_{1-1})/2$. This allows the integral over the entire triangle to be approximated as follows:

$$ \int g \, d\rho \, d\phi = \sum_{l=0}^{N_\phi} \left[ \sum_{k=1}^{N_\rho} \left( g(r, z, \rho_k, \phi_l) \, w_k \right) \right] \Delta \phi $$

$$ K_1 = \begin{cases} 0.5, & l=0 \\ 0.5, & l=N_\phi \\ 1, & \text{otherwise} \end{cases} $$

When this integration formula is used to integrate Eq. 3.4 over $\Delta_{F_{23}}$ of the jth element, the complete summation equation is
\[
\int \omega g(r_0, z_0) \, dA_0 = \sum_{i=1}^{n} \omega_i \sum_{j=0}^{N_\phi} \sum_{k=1}^{N_\rho} \left\{ \sum_{l=1}^{N_\rho} \left[ \sum_{j=1}^{J} (a_k) g(a_k) w_k \right] \right\} \Delta F_{23}
\]

where \( a_k = a_k(r, z, \rho_k, \phi) \).

The equation above is only for the integration over one of the triangles, \( \Delta F_{23} \). Two more similar equations for \( \Delta F_{13} \) and \( \Delta F_{12} \) must be incorporated to complete the integration over the entire jth element.

In Eq. 4.8 there are two parameters which control the number of integration points used in the quadrature scheme. \( N_\rho \) is the number of points in a selected straight line Gaussian quadrature. Weights and coordinates have been tabulated for schemes for \( N_\rho \) greater than 100, which is sufficient for most integrations. \( N_\phi \) is the number of angular subdivisions in the trapezoidal integration and can be set to any desired number. In a later section the impact of the value of \( N_\phi \) on convergence is investigated.

Integration of the line integrals over \( S_\phi \) in the boundary integrals also requires the proper handling of singular terms. Besides being more complicated than the singular terms found in the area integrals, they are also a different type. One is the ln \( r \) or weak singularity the second is \( 1/r \) strong singularity and the third is an apparent singularity. As will be demonstrated, the transformation required to prepare the expressions for numerical
integrations are very similar to those done for the area integrals, but the final forms are significantly different.

Once again the integrals of Eq. 3.4 will be used to demonstrate the necessary transformations and typical resulting algebraic forms. In all derivations it is assumed that $n_r=0$ and $n_z=1$ which also means that: 1. $ds=dr_0$, 2. $V_1=v_z$, and 3. $V_2=0$. These assumptions simplify the algebra in the example derivations but do not subtract from their illustrative value. For these conditions the integral over $S_0$ in Eq. 3.4 may be written

$$
\frac{1}{2\pi} \int_{r_0}^{r_{o2}} v_z(r_0J_1-rJ_0)r_0 dr_0 - \frac{1}{2\pi} \int_{r_0}^{r_{o2}} v_r(z_0-z)r_0J_1 dr_0 .
$$

For $n_z=1$ the second integral will equal 0 since $(z_0-z)=0$, however transformation of the first integral is sufficient to demonstrate all of the typical numerical techniques. Using the previous definitions for $J_0$, $J_1$, $A$, and $B$ this integral becomes:

$$
\frac{1}{2\pi} \int_{r_0}^{r_{o2}} -2v_z r_0 K(\eta) dr_0 + \frac{1}{2\pi} \int_{r_0}^{r_{o2}} 2v_z r_0 (A\eta - Br)(A+B)^{1/2} E(\eta) dr_0 . \quad (4.9)
$$

Due to the presence of $K(\eta)$, the first integral contains a ln $r$ type singularity. To integrate this function properly it is necessary to isolate the singular term by algebraic
manipulation and variable transformation. Substituting Eq. 4.1 for $K(\eta)$ identifies the term with singular behavior to be

$$\frac{1}{2\pi} \int_{r_0}^{R_0} \frac{-vzr_0^2 \ln(1/\eta)}{B(A+B)^{1/2}} dr_0.$$  

When this integral is transformed to the same local coordinate system used for the area integrals, $(\rho, \phi)$ in Fig. 4.1, the result is

$$\frac{1}{2\pi} \int_{\rho_0}^{\rho_1} \frac{-vzr_0^2 \ln(\eta)}{B(A+B)^{1/2}} d\rho + \frac{1}{2\pi} \int_{\rho_0}^{\rho_1} \frac{vzr_0^2 \ln(\rho)}{B(A+B)^{1/2}} d\rho.$$  

This step further isolates the singularity. The first integral above is non-singular and may be integrated conventionally. The second integral is singular and is integrated with special Gaussian formula for functions containing $\ln x$ when $x=0$ is within the limits of integration. These formula are developed for integrals having the general form

$$\int_{0}^{1} f(\rho) \ln(\rho) d\rho.$$  

To recast the integral in this form a new variable is defined, $\rho' = \rho/DS$, where $DS$ is the distance from 0 to $\rho_1$ and will be constant for a given integral. The substitution of $\rho'$ for $\rho$ and $DSd\rho'$ for $d\rho$ yields
This completes the transformation of the line integral Eq. 3.4 containing the ln r singularity. The final computational form of this integral is below.

\[
\frac{1}{2\pi} \int_0^1 -v_z r_0^2 \ln(DS) \, d\rho' + \frac{1}{2\pi} \int_0^1 v_z r_0^2 \ln(\rho') \, d\rho'.
\]

where \( M = \frac{v_z r_0^2}{\pi B(A+B)^{1/2}} \) and \( K^*(\eta) = K(\eta) - 0.5 \ln(1/\eta). \)

The first of the above integrals contains the ln r singularity and is in the proper form to integrate using the Gaussian formulas for such integrals. The remaining three are well-behaved and non-singular and may be integrated using conventional Gauss formula.

The second integral of Eq. 4.9, which is part of the boundary integral in Eq. 3.4, contains the 1/x type or strong singularity. A similar singular term appeared in the area integrals and the necessary transformations are nearly identical, but in this case the singular behavior cannot be removed. This situation usually means one of two things: 1. the model assumptions are not valid in the limiting case or 2. the integral is meant to be evaluated in the principal value sense. Fortunately the second is true, as explained
earlier the singularity was removed analytically thus justifying this interpretation.

One way to numerically integrate a function in the Cauchy principal value sense is to use special Gauss-type formula (Davis and Rabinowitz, 1975, p. 185). As is prerequisite to using special Gaussian formula for $\ln x$ singularities, principal value integrals must be cast in a standard form with normalized limits. The correct standard form for the $1/x$ singularity is

$$\int_{-1}^{1} \frac{f(x)}{x} \, dx.$$

This is a legitimate approach but is somewhat limited by the small "size" of the published schemes, i.e., the small number of integration points. A more flexible approach is to recast the integral such that conventional Gaussian quadratures can be used, which is the method used in this research. This is done by subtracting the singularity from the integrand, yielding one finite integral and one principal value integral which can usually be integrated in closed form. This technique is illustrated below for a simple principal value integral which is singular at $x=x_0$ where $a<x_0<b$. 
\[
\int_a^b f(x) \, dx = \int_a^b \frac{f(x) - f(x_0)}{x} \, dx + \int_a^b \frac{f(x_0)}{x} \, dx
\]

Since \( f(x_0) \) is a constant the far right principal value integral equals \( f(x_0) \log(-b/a) \) which equals 0 when the limits are symmetric about \( x_0 \), i.e., \( b-x_0=x_0-a \). The first integral on the right side is now finite and may be numerically integrated with conventional Gauss formula.

To prepare for integration the strongly singular integral extracted from Eq. 3.4, the now familiar algebraic manipulations and variable transformations are again employed. In the following example derivation, the integrals are constructed such that the limits are symmetric about the singularity which, by definition, occurs at the field point therefore \( r_0^2=r+DR \) and \( r_0^1=r-DR \).

\[
P \frac{1}{2\pi} \int_{r_0^1}^{r_0^2} \frac{2\sqrt{z}r_0^2(Ar_0-Br)(A+B)^{1/2}E(\eta)}{B(A-B)(A+B)} \, dr_0 = P \int_{r_0^1}^{r_0^2} \frac{M((r_0-r)(r_0+r)+(z_0-z)^2)}{B(A-B)(A+B)} \, dr_0
\]

where \( M = \frac{\sqrt{\pi}r_0^2E(\eta)}{B(A+B)^{1/2}} \). Transformation to local coordinates by \( r_0=r+\rho \sin\phi \) and \( z_0=z+\rho \cos\phi \) converts the integral on the right of the equal sign to

\[
P \int_{\rho}^{+\rho} M\rho^2 \sin^2(\rho \sin\phi \rho + r) \, d\rho + \int_{-\rho}^{-\rho} M\rho^2 \cos^2(\rho \cos\phi) \, d\rho.
\]
The transformation has removed the apparent singularity in the second integral since \( \rho^2 \) may be cancelled, permitting integration by conventional Gauss formula. The first is still singular and is converted to a standard form by normalizing the limits. This is done by defining \( \rho' = \rho/DS \) and \( d\rho' = d\rho/DS \) where \( DS = |\rho| \). Substitution for \( \rho \) and \( d\rho \) and subtracting the singularity yields

\[
\begin{align*}
\int_{-1}^{+1} \frac{M \sin \phi (r_0 + r) - G}{\rho'} d\rho' + \int_{-1}^{+1} \frac{G}{\rho'} d\rho'
\end{align*}
\]

where \( G \) is the quantity \( M \sin \phi (r_0 - r) \) evaluated at the field point, i.e., \( r_0 = r \) and \( z_0 = z \). The remaining principal value integral is 0 as will always be the case. In this way all principal value integrals evolving from Eq. 2.12 may be converted to finite proper integrals.

The presence of the strong \( 1/x \) singularity is common to all integral solutions of PDEs which are based on the use of fundamental solutions or Green's Functions. Therefore numerical techniques similar, at least in principle, to the ones just outlined have become common place in recent years. There is, however, another singular condition which is more subtle and not as well documented. The condition also occurs in the integration of the \( 1/x \) type term but in this case
variable transformations do not remove the singularity and principal value integration is not appropriate.

In all preceding discussions of singular integrations, the load variable has been left in its continuous form rather than substituting its approximate sum, as in Eq. 3.11. However this last problem is the result of discretization and therefore it is logically necessary to make this substitution to both explain the apparent singularity and eliminate it.

For 3 node elements, when the continuous load variable is replaced by its approximate sum each elemental integration becomes the sum of three integrals. Each integral

![Diagram](image)

\[ \int_{-1}^{1} f(x) \, dx \]

Figure 4.4. Second order interpolation functions
will contain a nodal load value, which is constant, and the associated interpolation function. Figure 4.4 shows how each 2nd order interpolation function varies over a three node boundary element. A key feature is that the interpolation function for the ith node will be 1 at the ith node and 0 at the other two nodes.

Consider the numerical integration of the product of two terms, one which behaves like \(1/x\) near node 1 and one which behaves like \(x\) near node 1, where \(x\) is the distance from the integration point to node 1. When the integrand is unchanged the computer must try to perform a limiting process which it does not do well. This is what happens when the velocity load node and the field node are in the same element but not coincident. Near the field node the interpolation function behaves like \(\rho\) which is multiplied by another function which behaves like \(1/\rho\).

The solution to this problem is to perform the familiar transformation to local coordinates \((\rho, \phi)\) on both the singular term and the interpolation function. The transformation of the singular term has already been explained. The transformation of the interpolation functions as defined for Eq. 3.11 is demonstrated below where the field node is specified to be at node 1, \(r_0 = r_1 + \rho\).

\[
L_1(x_0) = \frac{R + (r_2^2 - r_3^2 - 2r_1(r_3 - r_2))\rho + (r_3 - r_2)^2\rho^2}{D}
\]
\[ R = r_1^2(r_3 - r_2) + x_2^2(r_1 - r_3) + r_3^2(r_2 - r_1) \]
\[ L_2(x_0) = \frac{(r_1 - r_3)^2 \rho + (r_1 - r_3) \rho^2}{D} \]
\[ L_3(x_0) = \frac{(r_1 - r_2)^2 \rho - (r_1 - r_2) \rho^2}{D} \]
\[ D = r_2 r_3^2 + r_1 r_2^2 + r_3 r_1^2 - r_2 r_1^2 - r_1 r_3^2 - r_3 r_2^2 \]

Substituting a discretized approximation for \( v_z \) using the above interpolation functions in the integral
\[
P \int_{-\rho}^{+\rho} \frac{v_z r_\rho^2 E(\eta) \rho \sin(\phi) r_\rho + r_\rho d\rho}{\pi B(A+B)^{1/2} \rho^2} \]
yields three integrals. One will be a strongly singular principal value integral, which has already been discussed and the other two will now be non-singular.

**Assembly of the coefficient matrix**

Once all of the element integrations have been completed for a given grid geometry they may be stored and used when needed. To simplify the storage and retrieval algorithm and reduce the required computer memory, it is desirable to assemble the integrated values. In this research, assembly means simply adding together all integration results having common load points and field points. For example, an interior nodal vorticity will influence the vorticity distribution over the four adjacent elements. Therefore the solution for
any given field point will require four separate integrations for each interior nodal vorticity. These four terms may be "assembled" to yield one term for each nodal load.

Assembly of the finite element integrations allows Eq. 3.5 to be expressed as the following simple discretized sums where $N$ is the number of vorticity nodes and $NB$ is the number of boundary nodes:

$$v_{zi} = \sum_{j=1}^{N} w_{j} K(w_{j}, z(i,j)) + \sum_{k=1}^{NB} v_{zK} K(z(i,k)) + \sum_{p=1}^{NB} v_{rp} K(r(i,p))$$

$$v_{ri} = \sum_{j=1}^{N} w_{j} K(\rho(i,j)) + \sum_{k=1}^{NB} v_{zK} K(r(i,k)) + \sum_{p=1}^{NB} v_{rp} K(r(i,p))$$

Testing and Refinement of the Integration Algorithms

In order to reduce Eq. 2.12 to 2 dimensions and ensure the proper handling of all singularities, the original eloquent integral equation has been dissected into several working component parts. As explained, for some of the parts direct application of simple Gauss integration formula will work but others will contain singular integrals requiring variable transformation and/or specialized Gauss-type formula. Once they are properly formulated the next step is to translate each of the component integrations, using proper integration formula, into usable computer algorithms. Since the writing of computer code is rather personal, and extremely uninteresting, the details of the computer program
developed will not be discussed. However there are some refinement procedures which were used that might be helpful to those developing similar programs.

To complete the development of BEM integration algorithms there are two areas which must be addressed. First, regardless of the attention to detail, it is still not known for certain if any of the component integrals are correct, as there are many opportunities for error. Secondly, assuming they are correct, an acceptable level of accuracy must be established. Unfortunately neither of these questions can be answered definitively without completely developing the procedure and testing on a problem with a known solution. While this is a test which must eventually be performed, waiting until the algorithm is complete for any "reward" or feedback can prove to be inefficient not to mention unnerving.

Fortunately there are some qualitative intermediate checks which can be made on each local or elemental integration result. These tests will not ensure success but they will detect the presence of some types of derivation or programming errors. The remainder of this section is an explanation of these intermediate local tests and a discussion of how they relate to the global tests for two test problems with known solutions.
The intermediate integration tests performed in this research originate from the desire to develop mathematics which are consistent with the physics of the problem. First of all for smooth boundaries all integrands should be "well behaved" functions, that is there should be no sharp jumps when the integrand is plotted over the element of integration. As explained in an earlier section, such a mathematical discontinuity could only be caused by a physical irregularity such as a sharp corner, which for now has been avoided. Also, as the number of integration points is increased, the integrated value should converge. For non-singular integrands this convergence will be almost immediate, singular integrals typically require much denser integration schemes.

Figure 4.5. Plot of surface integrand. [a) discontinuity before transformation; b) after transformation]
To illustrate how these qualitative guidelines can be used to test integration algorithms, an actual problem encountered in this research will be discussed. The subject integral happens to be associated with a 1/x type singularity but similar results are obtained using other types. Figure 4.5a is a plot of a singular integrand along a line connecting the element node at the local coordinate \( \rho = -1 \) with the node coincident with the load node located at \( \rho = 0 \). The plot shows a sharp jump near \( \rho = -1 \) which violates the first guideline. Also for this particular integrand it was discovered that as the number of integration points was increased the integrated value seemed to oscillate rather than converge, thus violating the second guideline.

These intermediate tests eventually led to the variable transformation of the interpolation function as outlined in the discussion of the last type of singularity in the previous section. The plot of the transformed integrand is presented in Fig. 4.5b. As shown in the plot, the jump near the origin is now gone, producing an integrand which converges with a much smaller number of integration points.

Another type of intermediate quantitative test was employed in the development of the algorithm used to integrate over the interior integral containing singularities. As was described (pp. 126-127), the integration density is a function of \( N_\phi \), the number of
angular divisions. The parameter should be made large enough to yield sufficient accuracy without requiring unnecessarily large computing time.

In order to establish a workable range for $N_{\phi}$ integrated results were obtained for several integrands for increasing values of $N_{\phi}$. The plotted result of one of these integrands is given in Fig. 4.6. Here it can be seen that as $N_{\phi}$ approaches $\approx 20$ the rate of convergence slows considerably. Based on these results it is concluded that using $N_{\phi}$ greater than 30 would be inefficient use of computer time. This particular integral was taken from the Biot-Savart formulation for velocity but the results are representative of all interior B.E.M. integrals developed in this research.

![Figure 4.6. Typical convergence in the numerical integration of interior integrals](image)

Figure 4.6. Typical convergence in the numerical integration of interior integrals
After the Biot-Savart algorithm for velocity was completely developed, a more global effect of the size of $N_\phi$ on solution accuracy was investigated. This was done by calculating the % mass balance error at different values of $N_\phi$ for both the impulsively started and fully developed or Poiseuille flow problems. The plotted results in Fig. 4.7 show that as $N_\phi$ approaches 25 the error gradient decreases sharply for both test problems.

The conclusions based on the results in Fig. 4.7 are very similar to those based on the intermediate convergence tests. This lends additional credence to the use of intermediate or local elemental tests to establish acceptable numerical accuracy.

Figures 4.7. The effect of angular divisions, $N_\phi$, on the % mass balance error for two test flow problems
Comments and Conclusions

The focus of this chapter has been on the accurate finite element integration of the boundary element integrals used in this research. A critical part of the procedure is the isolation and integration of the singularities which are a characteristic of all integral solutions of partial differential equations based on the concept of fundamental solutions or Green's functions.

The techniques used to handle the singularities are very similar to those in 2 dimensional planar solutions which are quite common in the literature. However the actual implementation of the techniques for 2 dimensional axisymmetric problems are more complicated and less documented due mainly to the presence of elliptic functions in all kernels. By first approximating these elliptic functions with Chebyshev polynomials it is has been demonstrated that it is possible to write each axisymmetric integral as the sum of several planar form integrals.

In the development of the numerical integration algorithms one concern is to establish an efficient integration density for integrals containing singularities. For the surface or boundary integrals, it was discovered that as long as proper attention was given to the singularities very modest densities produced convergent integrated values.
This was also true for non-singular area or interior integrals.

For the singular interior integrals integration density proved more critical. The was especially true for the impulsively started flow problem as is illustrated in Fig. 4.7. Though this problem appears trivial it is quite a severe test of integration accuracy. This is true because of the gradients in the vorticity field which are approximately 100 times the vorticity gradients in the fully developed pipe flow problem.

Based on the intermediate convergence and preliminary complete flow tests the integration density for singular interior integrals was established by setting the number of angular divisions, $N_\phi$, at 30 for all subsequent solutions. This produced satisfactory results for both formulations of the Poisson's velocity equation as well as the pressure and stream function equations covered in upcoming chapters.
The solution of scalar Poisson type boundary value problems of the general form $v^2 u = g$ requires that either $u$ or $\partial u/\partial n$ be specified at every point on the boundary. According to the solution theory of this class of problems, these boundary conditions accompanied by the complete definition of $g$ will ensure a well-posed problem with a unique solution. For these problems the collocation equations used to determine the unspecified boundary values will contain well-posed integral equations. The boundary element solution techniques in Chapter III were developed for such well-posed problems.

Unfortunately, in fluids engineering applications of numerical methods, it is rarely the case that even the minimum number of boundary quantities are known with certainty. It is usually necessary to estimate at least part of the boundary values before the numerical methods designed for well-posed problems can be used. Often the solution thus obtained will then be used to improve the boundary conditions estimate, which in turn are used to obtain an improved complete solution. This iterative procedure is repeated until the convergence criterion selected for the problem is satisfied.
In this chapter, an alternative procedure is presented for the solution of some ill-posed Poisson's boundary value problems. As will be explained, the new method does not require the estimation of boundary quantities but instead incorporates additional qualitative or quantitative boundary information in the integral solution for the unspecified boundary values. When done correctly, the modified integral equation will be well-posed and may be solved with the collocation techniques developed in Chapter III.

The algorithm for the numerical solution of the new integral equation is rather straightforward and very similar to the one used for well-posed equations. However it would be irresponsible to present the numerics without first establishing the analytical foundation. This will be done by first discussing some of the characteristics of the integral equations involved and how these characteristics effect the solution process. Next, the discretized equivalents of the integral equations are studied. These matrix equations are the final link between the mathematical problem and the computer algorithm and it is important to understand what characteristics they must have to yield a "good" engineering solution.
The integral equation which is the subject of this chapter is derived by the collocation procedure outlined in Chapter III (pp. 84-86) using the Biot-Savart formulation for velocity (Eq. 2.12). In this procedure, the integrated values of the interior integrals and the surface integrals containing known boundary quantities can be combined as one known function (g(x) below). This permits the integral equation to be written and analyzed as a Fredholm integral equation of the first kind (Eq. 5.1).

\[ g(x) = \int K(x,y)f(y) \, dy \]  

In this common integral equation g(x) is a known function, K(x,y) is the kernel function which is a known function of geometry, and f(y) is the unknown function. For the particular integral equation being studied g(x) is the value of all known integrals and f(y) is the distributed boundary quantities to be determined by the collocation process.

The solution difficulties of the Fredholm integral equation of the first kind, and thus its discretized counterpart, are well documented. The ability or inability to solve these integral equations is directly related to the characteristics of the integral operator, \( \kappa \), when defined as \( \kappa = \int K(x,y)^* \, dy \) and specifically to if the inverse of \( \kappa \) is
bounded or unbounded. In general, an operator $T$ defined such that $g = T*f$ has a bounded inverse if for every $g$ there is exactly one $f$ (Collantz, 1966, p. 61). Using this definition it is often difficult to establish the existence of an inverse of a particular operator. However Wing (1984, pp. 27-42) states that except for special cases the inverse of the operator $\kappa$, as defined here, does not exist. One of the special cases occurs when the problem formulation is symmetric which results in a symmetric $\kappa$.

Symmetric formulation requires that every nodal velocity used for collocation is also the location of unknown nodal boundary value. Perfect symmetry also requires that the numerical grid is uniform and symmetric but experience indicates that slight deviations will still yield a $\kappa$ with a bounded inverse. An example of such a symmetric formulation is the collocation integral associated with the solution of a well-posed boundary value problem. This explains why the procedure for determining unknown boundary quantities in Chapter III was successful.

The Matrix Equation

When solved numerically the Fredholm integral equation, Eq. 5.1, is replaced by a matrix equation of the general form $\tilde{u} = [A] \tilde{z}$ where $\tilde{u}$, $[A]$, and $\tilde{z}$ are the discretized counterparts of $g(x)$, $\kappa$, and $f(y)$ respectively. In the
discretization process the characteristics and thus the solution difficulties of the integral equation are inherited by the matrix equation.

For example, the problems originally associated with the boundedness of the inverse of $k$ are now embodied in $[A]^{-1}$ which is the inverse of the matrix $[A]$. The difficulties caused by having an unbounded operator inverse are more conspicuous in the context of matrix equations. The direct or inverse solution of $\tilde{u} = [A] \tilde{z}$ is obtained by multiplying both sides by the inverse of $[A]$, $[A]^{-1}$, or $\tilde{z} = [A]^{-1} \tilde{u}$. Clearly, if the inverse $[A]^{-1}$ is unbounded a solution is not possible using this method.

The condition of the matrix operator

Unfortunately for $[A]^{-1}$ to be unbounded, the determinant of $[A]$ must be exactly 0 but except for the simplest grid geometries this will never happen owing mainly to discretization and computer round off error. Consequently, a computer algorithm can almost always produce a solution without overflowing, even for ill-posed integral equations.

For this reason the investigation of boundedness is necessarily replaced with investigation of the condition of the matrix $[A]$. This is done in a relative sense by calculating the condition number of $[A]$. The general definition of condition number is $\text{cond. } A = ||A|| * ||A^{-1}||$ where $|| ||$ represents a selected matrix norm (Forsythe and
Table 5.1. Condition numbers of ill-posed and well-posed impulsively started flow problems

<table>
<thead>
<tr>
<th></th>
<th>Well-posed</th>
<th>Ill-posed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Condition number</td>
<td>66.071</td>
<td>1.065 x 10^{10}</td>
</tr>
</tbody>
</table>

Moler, 1967, p. 20). In this research, the matrix norm \( \sum (A_{i,j})^2 \) is used. In general a small condition number indicates a bounded operator inverse and a large condition number should alert the analyst that the solution is suspect.

When judging the quality of a matrix solution by condition number there are no well defined acceptable limits for the condition number. However it was found that for the type of boundary value problems in this research the difference between condition numbers for ill- and well-posed problems was so great that there was no doubt as to which solutions were valid. Table 5.1 contains the condition numbers as defined in the previous paragraph for the pipe entrance flow problem with ill- and well-posed boundary conditions.

The Concept of a Well-Posed Equation

The objective of this chapter is to develop a procedure for determining the unspecified boundary quantities in a
boundary valued problem with ill-posed boundary conditions. The equation derived in Chapter III for this purpose has the general form \( u = \Lambda z \) where the operator, \( \Lambda \), is \( \kappa \) in the integral form and \([A]\) in the matrix form. For either form to produce a good solution it is necessary that the operator have a stable inverse and this will be true only if the problem is well-posed. Tikhonov and Arsenin (1977, pp. 7-9) give the following required conditions for well-posedness of an equation with the form \( u = \Lambda z \).

1. For every \( u \in U \) there exists a solution \( z \) in the space \( F \).
2. The solution is unique.
3. The problem is stable on the spaces \((F,U)\).

The concept of space is unavoidable in a formal discussion of ill-posedness. A rigorous mathematical discussion of various spaces is given by Collantz (1966, pp. 15-44) but for present purposes a much simpler definition will suffice.

Definition: A space is a set of \( n \)-dimensional \((n>1)\) elements from which a \( n \)-dimensional element may be selected.

The importance of the proper specification of space is found in the fact that an ill-posed problem can be made well-posed simply by changing or restricting the solution space. For problems in this research the data set, \( U \), will be
assumed to be from the "natural" space which imposes only
typical engineering restrictions such as being real and
continuous. Only the solution space, \( F \), will be further
restricted to facilitate solution. As will be explained
later this restriction is done indirectly by replacing the
matrix operator \( \Lambda \).

For a problem to be stable on the spaces \((F,U)\), it is
required that a small change in \( \hat{u} \), contained in \( U \), results in
a small change in \( \hat{z} \), contained in \( F \). The degree to which
this is satisfied is determined by the boundedness of the
operator inverse. An operator which satisfies the
requirement is called a stable operator on the spaces \((F,U)\).

The concept of a stable operator is similar to that of a
stable transfer function used in the frequency analysis of an
electro-mechanical servo-system. In control theory, a
transfer function is unstable if a small change in the input
signal produces a large unpredictable change in the output
signal. In the direct solution of a matrix equation, the
"input signal", \( \hat{u} \), is operated on by \( [A]^{-1} \) to produce an
"output signal", \( \hat{z} \).

In summary, the well-posedness of an equation of the
general form \( u = \Lambda z \) is directly related to the boundedness
of the operator inverse. For equations with an unbounded
operator inverse conventional numerical techniques will
typically produce poor solutions with unpredicted
oscillations. For the matrix form, ill-posed equations can be identified by relatively large condition numbers.

For some types of ill-posed equations an approximate solution can be obtained explicitly by replacing the unstable operator with a stable or regularization operator. The derivation of this operator and replacement is the foundation of a solution technique for ill-posed integral equations called regularization.

Regularization

The solution technique developed in this section for ill-posed equations of the general form \( u = A z \) can be applied to both the integral and matrix forms. However since all solutions are to be obtained numerically, the technique will be developed specifically for the solution of ill-conditioned systems of linear algebraic equations represented in matrix form by \( \hat{u} = [A] \hat{z} \).

In this method, the ill-conditioned matrix operator \([A]\) which has an unstable inverse, \([A]^{-1}\), is replaced with one that is well-conditioned and has a stable inverse. Tikhonov and Arsenin (1977) call this procedure Regularization and call the new stable operator a regularization operator, \( R \).

The regularization operator should be selected such that the new solution space which it defines is sufficiently restricted so as to satisfy the condition of well-posedness.
given by Tikhonov and Arsenin. As stated earlier, this will ensure an operator with a stable inverse. This is accomplished by the incorporation of additional qualitative or quantitative information about the solution into the original governing matrix equation. To demonstrate how this is done it is helpful to first recast the equation \( \hat{u} = [A] \hat{z} \) in a form which is more flexible.

Instead of the common deterministic matrix equation form consider the alternative equation \( ||[A] \hat{z} - \hat{u}|| < \epsilon \) where \( || \cdot || \) is a vector norm and \( \epsilon \) is a small number. In this form the solution for \( \hat{z} \) is defined to be the set of all \( \hat{z} \)'s from the solution space \( F \) which satisfy the inequality. From this solution set it is common engineering practice to select the \( \hat{z} \) which results in the minimum \( \epsilon \). A problem thus defined is an extremum problem of the calculus of variations where the functional to be minimized is \( M = ||[A] \hat{z} - \hat{u}|| \).

If the vector norm \( || \cdot || = \sum \left( (A_i j z_i) - u_i \right)^2 \) is selected, it can be shown that \( M \) is minimized by \( \hat{z} = [A]^{-1} \hat{u} \). This of course is the familiar direct solution to the original deterministic equation which does not produce good solutions for ill-posed problems. This result contradicts the conventional engineering thought which associates good solutions with small \( \epsilon \). For the problems of interest, it is necessary to use more comprehensive methods for evaluating the merits of solutions.
At this point it may seem that the new functional solution form has little advantage over the conventional inverse solution. Actually, however, the development is but one step away from obtaining a regularization or stable operator.

There exists a class of extremum problems which is very well suited for deriving the desired regularization operator. These are variational problems involving a conditional extremum (Elsgolts, 1970, p. 389). As suggested by the name they are problems for which it is required to find an extremum (max or min) of a functional, $M$, when certain constraints are imposed on the functions which determine $M$. For the present ill-posed problem the "certain constraint" is the additional information about the solution which is to be incorporated into the equations. The solution for the extremum of the new functional will produce the desired regularization operator.

The general form of the conditional functional is

$$M^* = M + \alpha_i \Omega_i(\hat{z})$$  \hspace{1cm} (5.2)

where:

- $M$ = original unconditional functional
- $\alpha_i$ = arbitrary or regularization constant(s)
- $\Omega_i(\hat{z})$ = mathematical expression of constraint(s).

The index $i$ is equal to the number of distinct and physically continuous boundary groups contained in the solution vector $\hat{z}$. For the purpose of developing the
technique it will be assumed that $i=1$. Formulations for $i>1$
are developed in the next chapter. The constraints placed on
each group, $\Omega_i(z)$, are subject to few mathematical
restrictions. It is only necessary that they can be
expressed approximately in a discreet or nodal manner as a
function of $\dot{z}$. The engineering restriction is more critical;
the constraint must be consistent with the physical laws and
equations governing the problem. A detailed discussion of
several constraints and their mathematical expressions is
presented in a later section.

**Regularization constant**

From a theoretical point of view the $\alpha_i$s are very similar
to Lagrange multipliers since their values establish the
mathematical character of the functional. From an
engineering point of view they serve as weighting terms by
determining the relative influence of the original
functional, $M$, and the constraints, $\Omega(z)$, on the solution.
For $\alpha_i$s near 0 the character of the solution will be
dominated by $f(z)$ with $\varepsilon$ close to its minimum and if the
problem is ill-posed the solution will have unstable
oscillations. As $\alpha_i$ is increased the influence of the
constraint will increase as will $\varepsilon$.

The success or correctness of a solution obtained by
regularization is then very dependent on the selection of the
$\alpha_i$s. This situation is somewhat of a paradox as it does
complicate the problem by introducing another set of variables but it also provides the analyst with a means of fine tuning a solution. From either perspective the determination of the regularization constant is a very important step in the technique.

For some types of ill-posed equations involving experimental or measurement error $\delta$ with known limits, it is possible to make an estimate of the minimum $\delta = \delta_m$. The optimum set of regularization constants is defined to be the set of $\alpha_i$ which results in $\delta_m$. In other problems where the statistics of the error are known the $\alpha$s may be optimized by a method called general cross validation (Wahba, 1977).

In this research a more functional or engineering definition of an optimum $\alpha$ is used. A range for each $\alpha$ is determined which results in acceptable engineering solutions as judged by calculation of the error terms defined in Eqs. 3.11-3.13. This is done in a straightforward way by plotting each error term for several values of each $\alpha_i$ and defining a range which appears to produce acceptable error levels.

**Constraints**

The mathematical constraints, $\Omega_i$, used in the regularization method must of course result in a matrix operator with a stable inverse. While this is the most fundamental requirement of the technique experience indicates
that it is the easiest to satisfy. It seems that any reasonable constraint will result in a dramatic reduction in the matrix condition number to a value usually associated with an operator with a stable inverse.

A more critical and interesting requirement, at least from an engineering point of view, is that the constraint be consistent with all quantitative and qualitative information about the solution. In the following paragraphs, three types of constraints tested in this research are described. Included are the corresponding mathematical expressions and brief comments on their suitability to the test flow problems.

**Approximate solution** For some types of problems, it may be possible to obtain an approximate solution before implementing the procedure for unknown boundary values. One such method is to use a simplified set of governing equations such as the boundary layer equations. Another example is time dependent flows where an appropriate guess would be the solution at the previous time step, assuming the time step is not to large.

If the approximate solution for $\hat{z}$ is called $\hat{z}_0$, then a conditional functional can be defined in discrete form as follows where $N$ is the number of unknown elements:

$$\alpha(N) \equiv \alpha \sum_{j=1}^{N} (z_j - z_{0,j})^2. \quad (5.3)$$
This functional alone tries to force the solution to the approximate solution. This conditional constraint is added to the sum of the squared error to produce the functional $M^a$ where, for simplification, $i$ has been set equal to 1. As before the definition of the solution space is completed by requiring $M^a$ to be less than $\varepsilon$.

$$M^a = \sum_{i=1}^{N} \sum_{j=1}^{N} (\Sigma(A_{i,j}z_{j}-u_{i})^2 + \alpha \Sigma(z_{j}-z_{0,j})^2 < \varepsilon \ (5.4)$$

Smooth curve A characteristic which engineers often require of solutions is that they be smooth. While this is not always a justifiable restriction, for the simple model problems in this research it is consistent with qualitative solution information. This requirement can be expressed mathematically as follows:

$$\Omega(\ddot{z}) = \alpha \int_{S} \left( \frac{\partial^{\alpha} {Z}}{\partial s^{\alpha}} \right)^2 ds < \varepsilon, \ n > 1 \ (5.5)$$

The value of $n$ determines the degree of the restriction on the curve slope. For example, $n=1$ would try to force the solution to a straight line. For engineering problems this is overly restrictive and also unnecessary to produce a regularization operator. Preliminary test results for $n=2$ were promising and are reported in a later section.

To incorporate a smooth curve constraint, it is necessary to approximate the integral and derivative in
discrete form as a function of $\dot{z}$. This is done by summing the finite difference approximation of each nodal derivative. The specific finite difference formula will depend on $n$ and the desired order of truncation error. To demonstrate the procedure, a 2nd order centered formula for $n=2$ has been used to develop a discretized form of this constraint.

$$\Omega(\dot{z}) = \Delta s \left( \frac{z_{i+1} - 2z_i + z_{i-1}}{2\Delta s} \right)^2$$

Notice that the above equation cannot be used to approximate the derivative at the end nodes. This is not a serious problem since the regularization technique can be easily modified to accommodate any endpoint conditions or approximations appropriate for the problem. For example consider the outflow boundary of the pipe entrance flow problem. The unknowns are the velocity vectors which approximate the velocity profiles on that boundary which is defined to be perpendicular to both the centerline and pipe wall. For this problem there are several particular conditions which can be incorporated into the smooth curve constraint.

The most obvious are the boundary velocity values specified as boundary conditions. On the no-slip boundary, i.e., pipe wall and $i=N$, $v_r$ and $v_z$ equal 0. Also since the problem is axisymmetric $v_r=0$ at the centerline, $i=1$. These
boundary values can be incorporated in finite difference formulas for the nodal derivatives in the smooth curve constraint. These types of restrictions have been named "anchors" since they tie the restrictions on profile slope to known nodal velocities.

The axisymmetric geometry also leads to other restrictions related to the necessary symmetry at the centerline. This symmetry requires first of all that the slopes of both the $v_r$ and $v_z$ profiles be 0 at the centerline. It also requires that the nodal velocities be symmetric about the centerline.

**Conservation of mass** This constraint is not suitable for all boundary value problems but is a logical choice for the pipe flow problems in this chapter. For the constant density assumption, this constraint is satisfied by flow conservation. The constraint is formulated to restrict the unknown boundary normal velocities such that the equation

$$\oint_S (\vec{v} \cdot ds) = 0$$

is satisfied. In discretized form, this restriction can be written as Eq. 5.7 where $N_k$ is the number of known boundary normal velocities, $N$ is the number of unknown boundary normal velocities and $K$ is the integrated value of the velocity interpolation function.

$$\sum_{i=1}^{N_k} (v_n)_{ik} K_i - \alpha \sum_{j=1}^{N} z_j K_j < \epsilon$$

(5.7)
The details of all the constraints and variations are not important in terms of understanding the method of regularization. However they do serve to demonstrate the flexibility that the method affords. As long as the constraint does not violate any fundamental laws and the condition number of R is relatively low the method will produce a stable solution.

**Minimization of the functional M**α

The next step is to solve for the \( \tilde{z} \) which minimizes the functional \( M^\alpha \). This will require the simultaneous solution of \( N \) equations which are generated by taking the partial of \( M^\alpha \) with respect to each \( z_i \) and setting each result equal to 0, \( \partial M^\alpha / \partial z_1 = 0, \partial M^\alpha / \partial z_2 = 0, \partial M^\alpha / \partial z_3 = 0, \) etc.

The results of the minimizing procedure can be expressed as a simple matrix equation, \( \tilde{c} = [R] \tilde{z} \). In this equation, \( [R] \) is the stable regularization operator which replaces the original unstable operator \( [A] \), \( \tilde{c} \) is a constant vector which replaces \( \tilde{u} \), and \( \tilde{z} \) is identical to \( \tilde{z} \) in the original ill-posed equation. Both \( \tilde{c} \) and \( [R] \) are functions of the quantities they replace.

In order to simplify and generalize the computing algorithm, each element of \( \tilde{c} \) and \( [R] \) may be written as the sum of two terms. The first term, denoted by \( \tilde{f} \), is derived from the original deterministic functional and the second term, denoted by \( \tilde{g} \), from the constraint. In this way, if it
is desired to test a different constraint or regularization constant it is necessary to recalculate only the second terms in each element of \( \bar{c} \) and \([R]\). For example, the terms in the functional using the approximate solution constraint are:

\[
c_i = c_i^f + c_i^0 \quad \text{where} \quad c_i^f = \sum_{k=1}^{N} \sum_k A_{k,i} u_k \quad \text{and} \quad c_i^0 = -\alpha
\]

\[
R_{i,j} = R_{i,j}^f + R_{i,j}^0 \quad \text{where}
\]

\[
R_{i,j}^f = \sum_k A_{k,i} A_{k,j} \quad \text{and} \quad R_{i,j}^0 = \begin{cases} \alpha & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
\]

**Testing the Regularization Algorithms**

The first objective of the testing is to demonstrate the stabilizing effect of the regularization technique on the solution of an ill-conditioned system of linear algebraic equations. This will initially be done subjectively with no attempt to optimize the solution by constraint type or regularization constant, \( \alpha \). The system of equations tested are all the discretized or matrix form of the integral equation developed in the collocation process using the Biot-Savart formulation for the solution of Poisson's velocity equation.

The second major objective is to investigate the relative effectiveness of two types of physical constraints
and three options for developing the stabilizing functionals. Comparisons are made via the error terms defined in Eqs. 3.11-3.13 which were calculated for each regularized velocity solution.

Based on the results of the second phase of tests the most promising stabilizing functionals were selected for refinement. Solutions were obtained and error terms calculated for each functional for a wide range of regularization constant (α) values. These results were then plotted to identify acceptable α ranges as determined by each error term.

A secondary objective of the α optimization tests is to investigate the validity of using the physical law based error terms, e_mb and e_c, to evaluate the quality of solution. This was done in a non-rigorous way by comparing the graphical results of e_mb and e_c vs α with that of the solution error e_vz vs α.

Objectives summary

1. Demonstrate the stabilizing effect of a regularization procedure on the solution of an ill-conditioned system of linear algebraic equations.

2. Investigate the suitability of three types of stabilizing functionals on the test problem.

3. Identify acceptable α ranges for several types of stabilizing functionals.
4. Investigate the usefulness of the physical law based error terms $e_{mb}$ and $e_c$.

**Test problems**

All tests of the regularization method will be performed using the velocity and vorticity fields specified for the impulsively started and fully developed pipe flow problems described in Fig. 3.5 and Fig. 3.6 respectively. The first problem was chosen because it has a known solution and also because it is the initial conditions for time dependent problem developing flow at a pipe entrance which is studied in Chapter VI.

To facilitate the tests, boundary conditions are specified such that the governing partial differential equation is ill-posed. Specifically, all interior vorticities are assumed known as are all boundary quantities, except for $v_z$ on the outflow boundary which is to be determined. This will produce a matrix equation for the unspecified boundary quantities which has an ill-conditioned matrix operator, $[A]$, and also an unstable operator inverse, $[A]^{-1}$. This ill-posedness is artificial since the solutions are known but in the temporal studies in Chapter VI these are realistic boundary conditions.

**Comparison of conventional and regularized solutions**

In order to demonstrate the stabilizing effects of regularization the selection of stabilizing functional and $\alpha$
value are not critical. To accomplish this the 2nd order smoother with $\alpha=0.01$ was used which was the first regularization constraint tried during study. The results of applying this regularization algorithm to the output velocity profile for fully developed pipe flow is reported in Table 5.2. The table also contains the non-regularized solution velocity, the regularized solution velocity, and the known velocity.

Table 5.2 clearly illustrates the type of solution obtained using an ill-conditioned matrix operator. The velocity profile contains large irregular oscillations whereas the regularized solution is smooth as a result of the additional constraints placed on the solution.

Comparison of constraint methods

Two types of physical constraints and three options have been described for formulating the stabilizing functionals used in $M^\alpha$. For the approximate solution constraint the initial guess was specified to be contained in a $\pm 5\%$ envelope of the known nodal velocity. For the smoothing functional only the 2nd order ($n=2$) constraint is reported since preliminary trials showed this to be the best choice. Also, the constraint based on the conservation of mass was not tested alone but rather was used as an optional addition to the other two. In all tests $\alpha=0.01$. 
Table 5.2 Non-regularized and regularized solutions for the output velocity in fully developed pipe flow

<table>
<thead>
<tr>
<th>Radius (in.)</th>
<th>Non-regularized velocity</th>
<th>Regularized velocity</th>
<th>Known velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>-1.049 x 10^8</td>
<td>9.895</td>
<td>10.000</td>
</tr>
<tr>
<td>0.025</td>
<td>-3164194.000</td>
<td>9.763</td>
<td>9.975</td>
</tr>
<tr>
<td>0.050</td>
<td>-3338612.000</td>
<td>10.046</td>
<td>9.900</td>
</tr>
<tr>
<td>0.075</td>
<td>-637779.700</td>
<td>9.706</td>
<td>9.775</td>
</tr>
<tr>
<td>0.100</td>
<td>8450466.000</td>
<td>9.632</td>
<td>9.600</td>
</tr>
<tr>
<td>0.125</td>
<td>2009322.000</td>
<td>9.367</td>
<td>9.375</td>
</tr>
<tr>
<td>0.150</td>
<td>-9840744.000</td>
<td>9.021</td>
<td>9.100</td>
</tr>
<tr>
<td>0.175</td>
<td>2148371.000</td>
<td>8.943</td>
<td>8.775</td>
</tr>
<tr>
<td>0.200</td>
<td>-1685787.000</td>
<td>8.321</td>
<td>8.400</td>
</tr>
<tr>
<td>0.225</td>
<td>-139632.100</td>
<td>7.990</td>
<td>7.975</td>
</tr>
<tr>
<td>0.250</td>
<td>674104.000</td>
<td>7.544</td>
<td>7.500</td>
</tr>
<tr>
<td>0.275</td>
<td>286818.000</td>
<td>7.105</td>
<td>6.975</td>
</tr>
<tr>
<td>0.300</td>
<td>757463.600</td>
<td>6.418</td>
<td>6.400</td>
</tr>
<tr>
<td>0.325</td>
<td>-705968.800</td>
<td>5.676</td>
<td>5.775</td>
</tr>
<tr>
<td>0.350</td>
<td>-1476243.000</td>
<td>4.994</td>
<td>5.100</td>
</tr>
<tr>
<td>0.375</td>
<td>1469708.000</td>
<td>4.349</td>
<td>4.375</td>
</tr>
<tr>
<td>0.400</td>
<td>-1232478.000</td>
<td>3.547</td>
<td>3.600</td>
</tr>
<tr>
<td>0.425</td>
<td>16670.210</td>
<td>2.854</td>
<td>2.775</td>
</tr>
<tr>
<td>0.450</td>
<td>384.362</td>
<td>1.859</td>
<td>1.900</td>
</tr>
<tr>
<td>0.475</td>
<td>-4141.209</td>
<td>1.003</td>
<td>0.975</td>
</tr>
</tbody>
</table>

The resulting output velocity profiles for the impulsively started flow using two types of regularization methods are reported in Figs. 5.1 and 5.2. It should be noted that for axisymmetric problems it is necessary to solve for only half of the profile across the diameter. However in the interest of clarity, the complete profile is plotted.

For the approximate solution constraint the conservation of mass option was tested. For the 2nd order smoother
constraint the conservation of mass, centerline symmetry, and no-slip anchor options were tested. In all profile plots the dotted line is the known profile and the solid line is the solution profile. The calculated error terms associated with each solved profile are reported in Table 5.3.

Figure 5.1. Output $V_z$ profile for the impulsively started flow problem using the approximate solution regularization constraint
Figure 5.2. Output $V_z$ profile for the impulsively started pipe flow problem using the 2nd order smoother regularization constraint
Table 5.3. Computed errors for several regularized solutions of the impulsively started pipe flow problem

<table>
<thead>
<tr>
<th>Type of constraint</th>
<th>options</th>
<th>$e_{vz}$</th>
<th>$e_{mb}$</th>
<th>$e_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>approximate</td>
<td>a) none</td>
<td>0.36</td>
<td>-0.423%</td>
<td>1.084%</td>
</tr>
<tr>
<td>solution</td>
<td>b) conservation</td>
<td>0.34</td>
<td>-0.075%</td>
<td>1.062</td>
</tr>
<tr>
<td></td>
<td>of mass</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2nd order</td>
<td>a) none</td>
<td>4.24</td>
<td>0.866%</td>
<td>1.612</td>
</tr>
<tr>
<td>smoother</td>
<td>b) no-slip anchor</td>
<td>277.33</td>
<td>-3.332%</td>
<td>22.967</td>
</tr>
<tr>
<td></td>
<td>c) conservation</td>
<td>4.85</td>
<td>0.420%</td>
<td>2.667</td>
</tr>
<tr>
<td></td>
<td>of mass</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>d) centerline</td>
<td>4.24</td>
<td>0.870%</td>
<td>1.605</td>
</tr>
<tr>
<td></td>
<td>zero slope</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>e) b &amp; d</td>
<td>123.43</td>
<td>-3.222%</td>
<td>17.472</td>
</tr>
<tr>
<td></td>
<td>f) b &amp; c</td>
<td>240.08</td>
<td>-2.924%</td>
<td>21.512</td>
</tr>
<tr>
<td></td>
<td>g) c &amp; d</td>
<td>3.61</td>
<td>0.544%</td>
<td>2.254</td>
</tr>
<tr>
<td></td>
<td>h) all options</td>
<td>110.85</td>
<td>-2.850%</td>
<td>16.604</td>
</tr>
</tbody>
</table>

*Error terms for the randomly generated profile used in the approximate solution regularization constraint.

The same trial regularized solutions for the output velocity in fully developed pipe flow were also performed. The resulting profiles are reported in Figs. 5.3 and 5.4 and the associated error terms in Table 5.4.
Table 5.4. Computed errors for several regularized solutions of the fully developed or parabolic flow problem

<table>
<thead>
<tr>
<th>Type of constraint</th>
<th>options</th>
<th>$e_{vz}$</th>
<th>$e_{mb}$</th>
<th>$e_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>approximate solution</td>
<td>a) none</td>
<td>0.18$^a$</td>
<td>-0.070%$^a$</td>
<td>0.583$^a$</td>
</tr>
<tr>
<td></td>
<td>b) conservation of mass</td>
<td>0.34</td>
<td>-0.011%</td>
<td>0.577</td>
</tr>
<tr>
<td>2nd order smoother</td>
<td>a) none</td>
<td>16.23</td>
<td>0.841%</td>
<td>4.628</td>
</tr>
<tr>
<td></td>
<td>b) no-slip anchor</td>
<td>5.42</td>
<td>-0.023%</td>
<td>2.371</td>
</tr>
<tr>
<td></td>
<td>c) conservation of mass</td>
<td>11.40</td>
<td>0.510%</td>
<td>3.752</td>
</tr>
<tr>
<td></td>
<td>d) centerline zero slope</td>
<td>0.51</td>
<td>0.470%</td>
<td>1.004</td>
</tr>
<tr>
<td></td>
<td>e) b &amp; d</td>
<td>0.04</td>
<td>-0.081%</td>
<td>0.293</td>
</tr>
<tr>
<td></td>
<td>f) b &amp; c</td>
<td>5.44</td>
<td>-0.021%</td>
<td>2.376</td>
</tr>
<tr>
<td></td>
<td>g) c &amp; d</td>
<td>0.23</td>
<td>0.297%</td>
<td>0.657</td>
</tr>
<tr>
<td></td>
<td>h) all options</td>
<td>0.04</td>
<td>-0.072%</td>
<td>0.281</td>
</tr>
</tbody>
</table>

$^a$Error terms for the randomly generated profile used in the approximate solution regularization constraint.
Figure 5.3. Output $V_z$ profile for the fully developed pipe flow problem using the 2nd order smoother regularization constraint.
Figure 5.4. Output $V_z$ profile for the fully developed pipe flow problem using the approximate solution regularization constraint

Using the tabulated errors and output velocity profiles, one algorithm based on the approximate solution constraint and one based on the 2nd order smoother constraint were selected for refinement. It was desired that the selected algorithms produce good solutions for both the impulsively started and fully developed pipe flow problems. The reason being that they respectively represent gradients of the initial condition and steady state solution of the time dependent pipe flow problems studied in Chapter VI.

For the approximate solution constraint, the profile plots are of little help in identifying a preferred technique. However the plots do clearly illustrate the tendency for the solution to follow the randomly generated approximate solution. The magnitude of the deviation is a
function of the regularization coefficient, $\alpha$, as will be demonstrated later.

A reasonable qualification of an approximate solution constraint is that all calculated errors be less than for the randomly generated profile. As seen in Tables 5.2 and 5.3, both algorithms for the impulsively started problem satisfy this condition but for the fully developed flow problem only the % mass balance error is less for both algorithms. This less than desirable situation may be related to the small solution envelope which was ± 5% and/or characteristics of randomly generated solutions. A more complete picture would be provided by looking at different envelopes for an ensemble of random solutions. For the present, however, the algorithm with the conservation of mass option will be selected based on the % mass balance error.

For the second order smoother constraint, the profile plots in Figs. 5.3 and 5.4 are very useful in evaluating the eight possible algorithms. They clearly illustrate the negative effect of the no-slip anchor option on the impulsively started problem. This is also reflected in the error terms in Table 5.2. Though the no-slip anchor does not effect the fully developed problem in the same way, it still must be eliminated from consideration as a algorithm for use in time dependent studies.
The profile plots also illustrate the positive effect of the centerline zero slope option on all regularized solutions. In all cases, this option, as it should, forces the solution slope to be zero at the centerline which is a necessary condition of axisymmetric flows.

Further evaluation of the second order smoother constraint must be based on the errors in Tables 5.2 and 5.3. After eliminating all algorithms containing the no-slip anchor option the best remaining algorithm is the option combination conservation of mass and centerline zero slope.

Based on the profile plots and tabulated errors two algorithms have been selected for further refinement. They are: 1. approximate solution constraint with the conservation of mass option and 2. second order smoother constraint with the option combination of conservation of mass and centerline zero slope. These two algorithms will be used in the functional optimization of the regularization coefficient in the next section.

Functional Optimization of the Regularization Coefficient

In the first phase of regularization testing, the coefficient $\alpha$ found in Eq. 5.4 was set at 0.01 for all algorithms. Based on preliminary results, not reported in this dissertation, this value produced representative results for all regularization methods studied. For the two most
promising algorithms, tests were conducted to determine optimum values of \( \alpha \) or more realistically acceptable ranges of \( \alpha \) values.

The optimization or selection procedure used in this research does not have an eloquent theoretical foundation as in some methods discussed in the review of literature. Rather, it is based on the simple functional requirement that the selected \( \alpha \) produce a "good" engineering solution. The "goodness" or quality of each solution is evaluated subjectively with profile plots and objectively with the three error terms: 1. known solution or \( V_z \) error, \( e_{Vz} \), 2. mass balance error, \( e_{mb} \), and 3. continuity error, \( e_c \).

To investigate the effect of the \( \alpha \) value on solution quality, the error terms were calculated for \( \alpha \)s ranging from \( 10^{-7} \) to \( 10^{-1} \). These tests were conducted for the impulsively started and fully developed flow problems using the two regularization methods selected from previous results. The plotted results of this study are presented in Figs. 5.5, 5.6, and 5.7.

One point that is clearly illustrated by the plots is the poor performance of the 2nd order smoother constraint in the impulsively started flow problem. This is the result of forcing a smooth solution on a velocity profile which has a sharp corner near the no-slip wall. These results reflect those indicated by the errors reported in Table 5.2.
Figure 5.5. $V_z$ solution error, $e_{Vz}$, vs the log of the regularization coefficient, $\alpha$

Figure 5.6. % mass balance error, $e_{mb}$, vs the log of the regularization coefficient, $\alpha$
In contrast, the 2nd order smoother constraint works quite well for the fully developed pipe flow problem for a wide range of $\alpha$ values. This result is predictable since the known profile is a second order or parabolic curve. These results suggest that the approximate solution constraint should be used whenever the flow is likely to contain large gradients as in the impulsively started flow problem. The degree to which this is true depends on the accuracy of the approximate solution and the order of smoother constraint.

Disregarding the poor impulsively started 2nd order smoother regularizations, all plotted error terms show a flat response for a wide range of regularization coefficient.
values. Based on these results, it is concluded that any $\alpha$ between $10^{-6}$ and $10^{-3}$ would produce error very near the minimum but it would be difficult to identify a single optimum value of $\alpha$ with such tests.

Studying the relationship between the value of regularization coefficient and the resulting errors may also help shed some light on the role that $\alpha$ plays in the solution of the functional $M^\alpha$. As $\alpha$ gets very small, the solution should approach the non-regularized or ill-posed solution which is characterized by high frequency oscillations and large errors. As $\alpha$ is increased, the influence of the constraint on the solution is also increased. For example, in the approximate solution constraint, large $\alpha$s should produce errors close to those of the approximate solution.

![Figure 5.8. Expanded plot of continuity error vs $\alpha$](image)
The effects of very large and very small $\alpha$s are illustrated in the expanded plot of continuity error vs $\alpha$ in Fig. 5.8.

A secondary objective of the $\alpha$ optimization tests is to further evaluate the two physical law based error terms $e_{mb}$, % mass balance error, and $e_c$, continuity error, as tools for judging the quality of solutions to realistic flow problems. Since the solution is not known these will be the only errors available for this task. The validity of each error measurement is tested in a non-rigorous way by comparing them to the solution error, $e_{vz}$, which was assumed to be the best measure of true error in the two test flow problems.

As illustrated in the plots in Figs. 5.5, 5.6, and 5.7, the continuity error is best at paralleling the trends in solution quality indicated by $e_{vz}$. The continuity error defines approximately the same lower and upper limits on the acceptable range of $\alpha$ as does $e_{vz}$. The % mass balance error is a more sensitive measure of the $\alpha$ upper limit but goes to 0 for small $\alpha$ which is not consistent with the true solution error.

In summary, it appears that for the types of flow problems encountered in this research a comprehensive error such as the continuity error is most appropriate for evaluating the quality of solutions to real engineering problems. This information will be put to use in the solution of the time-dependent or kinetic problems which are
the subject of Chapter VI. For these problems the mass balance error will not be abandoned since it still may provide useful information for some flow conditions.

Comments and Conclusions

A well-posed boundary valued problem governed by the equation \( \nabla^2 u = g \) requires that either \( u \) or \( \partial u / \partial n \) is known at every point on the boundary. For these problems the remaining boundary quantities may be determined by collocation which involves the solution of a linear set of algebraic equations. When expressed in the matrix form \( \tilde{C} = [A]\tilde{B} \) the solution is \( \tilde{B} = [A^{-1}]\tilde{C} \).

For problems which do not satisfy the conditions of well-posedness other methods must be employed to determine a complete set of kinematically compatible boundary values. This is often done by estimating the unknown values and then iterate to improve the estimates using other available solution information. An alternative method outlined in this chapter is called regularization.

It is important to emphasize that regularization does not alter the original governing equation. Rather it formulates a new problem by combining the original governing equation with other solution information. The result is a conditional functional which is solved explicitly for the regularized solution. In addition to being explicit, the new
method is much more flexible than the iterative procedure which it replaces.

To develop the conditional functional, $M^a$, two types of constraints were tested with three different options appropriate to axisymmetric flows. Based on the results of these tests two regularization methods were selected for further refinement. They were: 1. approximate solution constraint with the conservation of mass option and 2. 2nd order smoother constraint with the option combination conservation of mass and centerline zero slope.

The refinement of the regularization method essentially involves the selection of an acceptable range of regularization coefficient values. The tests designed to accomplish this task indicated that an $a$ in the range of $10^{-6}$ to $10^{-3}$ should produce acceptable results. This result shall be tested in chapter VI.

One shortcoming of the development of the regularized solution of ill-posed Poisson's velocity equation should be addressed before concluding this chapter. In the time-dependent problems studied in Chapter VI most flows will contain two component velocities, $v_r$ and $v_z$. However in all test solutions thus far only the $v_z$ component has appeared. This unfortunate situation was unavoidable since there are no analytical solutions known to the author for problems where both component velocities are non-zero.
Because of the lack of appropriate test problems it will be necessary to refine the $v_r$ regularization during the time-dependent studies in Chapter VI. In these problems the same constraints and options will be tested along with the additional option based on the axisymmetric condition that $v_r$ must equal 0 at the centerline.
CHAPTER VI. FINITE DIFFERENCE SOLUTION
OF THE VORTICITY TRANSPORT EQUATION

When laminar incompressible flows are solved using the derived variable of vorticity, \( \vec{\omega} \), the governing vector Navier-Stokes equation is replaced by separate kinematic and kinetic equations. The kinematic relationship between velocity and vorticity is described by the vector Poisson's velocity equation (Eq. 2.10). Two boundary element formulations presented for solution of this equation were derived in Chapter III and developed in Chapters IV and V. The kinetic or time-dependent part of the complete solution is governed by the vorticity transport equation. The numerical solution of this equation using finite difference methods is the subject of this chapter.

Derivation of the Vorticity Transport Equation

The vorticity transport equation in vector form is derived by taking the curl of the conservation of momentum or Navier-Stokes equations and incorporating the definition of vorticity, \( \vec{\omega} = \vec{v} \times \vec{V} \). The details of this vector operation procedure using cylindrical polar coordinates are outlined in Appendix B.

The complete 3 dimensional result of the procedure is a set of 3 scalar equations describing the time advancement of
vorticity in the r, θ, and z component directions. For 2
dimensional axisymmetric flows, the identities ∂/∂θ=0 and
v_θ=0 may be used to eliminate both the r and z component
equations. This reduces the original vector governing
equation to one scalar equation, Eq. 6.1 which is the θ
component equation.

\[
\frac{\partial \omega}{\partial t} = -\frac{\partial (v_r \omega)}{\partial r} - \frac{\partial (v_z \omega)}{\partial z} + \nu \left( \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{\omega}{r^2} + \frac{\partial^2 \omega}{\partial z^2} \right)
\] (6.1)

Finite Difference Representations

When studying the numerical solution of partial
differential equations, there is one interesting fact that
quickly surfaces. That is the relatively small number of
model equations necessary to represent mathematically the
many governing equations found in engineering. The model
equation most appropriate for the vorticity transport
equation is the complete non-linear Burger's equation. The 2
dimensional form of this equation is given in Eq. 6.2 where u
is the dependent variable which is a function of time, t, and
two spacial coordinates, x and y (Anderson, Tannehill, and
Pletcher, 1984, p. 166).

\[
\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \lambda \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]
\] (6.2)
The first step in adapting the model equation, Eq. 6.2 to Eq. 6.1 is to substitute $\omega$, $r$, and $z$ for $u$, $x$, and $y$ respectively and replace $\lambda$ with $\nu$, the kinematic viscosity. Also $F$ and $G$ are replaced by $v_r\omega$ and $v_z\omega$ when the vorticity transport equation is left in the presented form which is the conservation law form. If for some reason the non-conservative form is desired, $F$ and $G$ will change accordingly and the convective term $(v_r\omega)/r$ must be added.

There are also two additional diffusion terms which must be added to correctly model the transport of vorticity in axisymmetric flows, they are $\frac{1}{r}\frac{\partial \omega}{\partial r}$ and $\omega/r^2$. The partial derivative in the first term was modeled with the same finite difference representations used for the convective terms. Both terms are functions of $1/r$ and therefore become singular at the centerline. Fortunately, a necessary condition of axisymmetric flows is that $\omega$ is always 0 at the $r=0$. Therefore it is not necessary to solve the vorticity transport equation at $r=0$ which eliminates this possible singularity as a concern.

From the literature covering the 2 dimensional solutions of the non-linear Burger's equation, two finite difference methods were selected for use on the axisymmetric vorticity transport equation. They are: 1. time-split MacCormack (MacCormack, 1971) and 2. the alternate direction implicit (ADI) method developed by Douglas and Gunn (Douglas and Gunn,
Both methods advance the solution in the two directions $(r,z)$ in separate steps. The complete difference representations for each method are given in Appendix C.

**Time-split MacCormack**

This explicit multi-step method was formulated by applying the 1 dimensional MacCormack method separately to each direction. Each directional step is composed of a predictor step and a corrector step. In both the predictor and corrector equations, forward time differencing is used and 2 dimensional central differencing is used for the 2nd order diffusion terms. In the predictor step, forward differencing is used on the convection terms and the 1st order diffusion term. In the corrector step, these first order derivatives are modeled with backward differencing.

In the time-split MacCormack method the predictor-corrector equation pairs may be applied to each direction more than one time. The only restriction is that the total time advanced be the same in both directions. The scheme outlined in Appendix C for use in this research, uses 3 separate predictor-corrector pairs, 2 in the $r$ direction and 1 in the $z$ direction. The result is a finite difference solution which is 2nd order accurate in both time and space.

Since the MacCormack method is explicit, numerical stability must be considered when selecting a time step size. Unfortunately analytical procedures appropriate to 1
dimensional equations can not be easily adapted to two dimensions. For some 2 dimensional cases, it is possible to apply the 1 dimensional time step formula separately to each direction.

ADI procedure of Douglas and Gunn

Like the time-split MacCormack the ADI method advances the solution in each direction in separate steps. Unlike the time-split MacCormack the ADI method is implicit and thus requires the solution of a tridiagonal system of equations for each step. The scheme outlined in Appendix C is first order accurate in time and 2nd order accurate in space.

The literature states that ADI methods are unconditionally stable for the linear case (Anderson et al., 1984, p. 185). This would imply that for non-linear problems, such as the vorticity transport equation, stability is not assured. This is a area which will be investigated in the tests of the ADI method.

Testing the Complete Navier-Stokes Solution

The solution of the vorticity transport equation alone produces little useful information. It is valuable only when used in conjunction with a solution of the Poisson's equation for velocity. This pair of equations constitute a complete solution for Navier-Stokes or laminar Newtonian flows. For this reason, all tests of the vorticity transport solution
algorithm must necessarily include solution of the velocity equation. In this research, the natural choice for the later is one of the BEM formulations developed earlier is this dissertation.

The complete Navier-Stokes solution may be used on two types of problems with different accuracy requirements. The most obvious, but unfortunately the most difficult, is to investigate the time-dependance of a laminar flow for a given set of boundary and initial conditions. These problems require a numerical method with good temporal accuracy. Of the two methods used in this research, the time-split MacCormack would appear to be best suited for these problems.

The second type of problem, which is probably more common, is the determination of a steady state velocity field when a well-posed set of boundary conditions are known. For steady state problems temporal accuracy is not critical, it is only necessary that the solution converge. Since steady state problems are more stable and less sensitive to error they were studied first.

Stability considerations

Before any numerical solutions of Burger's equation are attempted it is necessary to first establish a workable time step envelope. As stated earlier, for simple one dimensional linear problems the stability criterion, i.e., maximum time step, may be determined analytically. However, for 2
dimensional problems the task is more complicated and nearly impossible if the complete solution involves a system of equations, as is the case for Navier-Stokes flow. For this reason, a more empirical and practical approach was used to establish a stable time step size.

To establish time step size limits, a simple flow problem with a known solution was solved. The inlet and outlet boundary velocity and vorticity distributions were defined to be those of fully developed pipe flow. These boundary conditions were input in the complete Navier-Stokes solution which was iterated to the steady state velocity and vorticity distributions in the interior.

Steady state solutions to the test problem were obtained using both the Biot-Savart and direct BEM velocity solutions and for both the time-split MacCormack and alternate direction implicit, ADI, vorticity transport solutions. Each algorithm was applied using several values of $\Delta t$. The procedure follows these steps:

1. Specify boundary velocity and vorticity distributions.
2. Determine the unknown boundary quantities using collocation methods.
3. Determine all remaining velocities using the selected velocity solution.
4. Solve the vorticity transport equation for the updated vorticity field.
5. Check convergence: a. return to step 2 if the solution has not converged or b. stop if it has converged or is clearly divergent.

The results of these tests for the Biot-Savart velocity formulation is reported in Fig. 6.1 and in Fig. 6.2 for the direct BEM formulation. The time steps reported define the experimentally determined meta-stable range. The larger $\Delta t$ is the smallest value tested which produced a unstable algorithm and the smaller $\Delta t$ is the largest value tested which produced a stable algorithm. These results were used as a guideline in forthcoming solution procedures.

![Figure 6.1. Stability test results using the Biot-Savart velocity formulation](image)
Both plots clearly illustrate another important point. When the ADI finite difference representation is used in the complete solution of Navier-Stokes flow it is not unconditionally stable. Given this fact and that it is only first order accurate in time, more emphasis was placed on developing the time-split MacCormack method which is 2nd order accurate in time.

Steady state fluid flow problems

By definition, for steady state problems $\partial \omega / \partial t = 0$ which transforms the parabolic vorticity transport equation into an elliptic partial differential equation similar to Poisson's...
equation. However, attempts to determine steady state vorticity fields with this form have previously proven unsuccessful due to numerical instability. Results from unreported preliminary tests in this research concurred.

A more reliable alternative to the elliptic form is to use the parabolic form and iterate to a steady state solution. This method was tested on three fluid flow problems. All of the problems are formulated using the uniform rectangular grid previously described in Fig. 4.2.

Pipe entrance flow The first steady state solution obtained was the velocity and vorticity fields in the entrance of a pipe. The inlet velocity and vorticity distributions were assumed to be those defined by impulsively started pipe flow as in Fig. 3.5. Preliminary calculations were made to select fluid properties and velocities such that the outlet velocity and vorticity distributions were those of fully developed pipe flow as in Fig. 3.6.

The desire to model the entire transition length of the developing flow and the small numerical grid already in place, did put a limit on the allowable Reynolds number. Based on the average velocity at the outlet and the pipe diameter, the modeled flow had a Re ≈ 100. While from an engineering perspective this is not a very interesting flow, the model still served the purpose of verifying the derived integral equations which are the algorithm's foundation.
Figure 6.3. Steady state $v_z$ profile plots for entrance flow using four different algorithms
Using $\Delta t = 0.0075$ sec, steady state entrance flow solutions were obtained using the Biot-Savart and direct BEM velocity solutions and the two vorticity transport solutions, time-split MacCormack and ADI. The resulting $v_z$ profile plots are shown in Fig. 6.3. In these plots and subsequent profile plots, the dotted lines represent boundary conditions and the solid lines represent solutions. All four algorithms converged per the continuity error with similar error values.

The profile plots work nicely for reporting solutions of simple flows dominated by one component velocity. However they are not convenient for describing multidimensional flows. For the entrance flow problem, the $v_r$ component is relatively small so not much information is lost but for more complicated flows profile plots are not acceptable.

For these flows, it is more efficient to determine the streamlines which are a function of all velocity components. The calculation of streamlines for axisymmetric flows is covered in the next chapter but the results of this operation will be used to report solutions in this chapter. An example is Fig. 6.4 which are the streamlines for the steady state entrance flow using the Biot-Savart velocity formulation and the time-split MacCormack finite difference vorticity transport solution.

Realistic modeling of more complicated flows is difficult with no experimental boundary information and the
Figure 6.4. Streamlines for steady state entrance flow using Biot-Savart velocity and time-split MacCormack vorticity transport solutions.

given numerical grid. This is especially true for flows which have significant gradients in the axial direction since the resolution in the radial direction is four times that in the axial direction, i.e., $\Delta z = 4\Delta r$. Still, for small Re the algorithm produced solutions with acceptable errors. Based on previous results it was decided to develop only the time-split MacCormack vorticity transport solution. Preliminary attempts using the ADI method were satisfactory but offered no justification for the additional time and effort required to refine them at this time.

Preliminary tests also surfaced problems with the direct BEM velocity solution. The algorithms using this method either converged to solutions with unacceptable errors or did
not converge at all. Despite these problems, it is thought they are not the result of a fundamental deficiency of the formulation but are caused by the lack of detailed boundary information. The direct BEM formulation requires normal derivative information and care must be taken to specify compatible $v_r$ and $v_z$ boundary values. The Biot-Savart formulation is easier to use since it does not contain derivatives and compatibility of $v_r$ and $v_z$ is inherent.

The one remaining Navier-Stokes solution algorithm is the combination of the Biot-Savart velocity formulation and the time-split MacCormack finite difference vorticity transport solution. This algorithm was used to obtain solutions for the two remaining steady state problems.

_Rotary coupler flow_ For the next steady state flow problem it was desired to select a flow geometry which would better test the $v_r$ component equations without a great increase in complexity. The selected flow is similar to that in a rotary coupler which communicates fluid from a stationary hydraulic prime mover to a rotating hydraulic component. Basically, the flow enters (or exits) radially and exits (or enters) axially. The streamlines of the steady state solution obtained are shown in Fig. 6.5.
Figure 6.5. Streamlines of the steady state solution for the rotary coupler flow problem

Modeled valve flow  The final steady state problem solved is the closest this research came to directly addressing the type of problems which were the stated motivation. The general flow geometry of the model problem is the same as spool valve flow. The fluid flow both enters and exits radially via annuli usually cast in the valve body.

The most critical shortcoming of the model problem is the large valve openings made necessary by the poor axial resolution. The axial resolution also made it necessary to use small uninteresting fluid velocities. Another difference, although not as distorting, is the absence of the spool stem. This would replace the centerline as the lower boundary, which need not be modeled, with a no-slip boundary which must be modeled.
Still, there are enough similarities to make the defined model valve problem interesting and suitable as a first approximation of the actual valve flow. The end result of trial runs to numerous to report is the flow field defined by the plotted streamlines in Fig. 6.6.

In the trial and error process of obtaining the reported solution, many problems were encountered and satisfactorily resolved. Most of them were related to the proper specification of boundary conditions. As was mentioned earlier, this was expected since the numerical region was so small and many assumptions had to be made about the character of the flow leading into and out of the region.

There was another problem encountered which is a function of flow geometry and not the quality of the
specified boundary conditions. In the collocation process to determine unspecified boundary quantities, the resulting system of linear algebraic equations is ill-conditioned with a condition number \( \approx 2 \times 10^6 \). This occurred despite the fact that the problem is well-posed in terms of proper specification of boundary conditions. When left in this form, the solution obtained is unstable and divergent much like the ill-posed problems discussed in Chapter V. The system condition could possibly be improved with a higher resolution numeric grid but is not affected by the quality or accuracy of the boundary information.

Since the valve kinematic linear system displayed the same characteristics as an ill-posed system, the regularization methods developed for them were tried. Though there are many questions to be answered and definitely deserves more attention, the results were quite promising. The solution reported in Fig. 6.6 is a regularized solution.

One of the legitimate concerns with regularized solutions to engineering problems is how to measure the solution quality from a physical perspective. This concern was partially addressed in this research by monitoring the error terms used previously. The record of continuity error during the iteration process is shown in Fig. 6.7. The plot shows acceptable error values after 20 iterations. What it does not show is that the solution does not clearly converge.
Figure 6.7. Record of continuity error during iteration of the model valve problem

**Time dependent flow problems**

By definition, the vorticity transport equation describes the advancement of a vorticity field in time for a given set of initial and boundary conditions. The solution accuracy of this equation is dependent on the form of the numerical scheme and also on the time step size. In general, a scheme which is at least 2nd order accurate in time should be used with as small a time step as feasible. This is unlike steady state problems where numerical stability was the only criterion for selecting a time step size.
Time dependent problems are much more sensitive than steady state problems to errors in the initial and boundary conditions, numerical grid resolution, and the condition of the kinematic system. Despite the difficulties and restrictions of these problems already reported in the steady state problems there was one kinetic or time dependent fluid flow which was solved satisfactorily without further grid refinement or experimental work.

Developing flow at a pipe entrance  The flow solved is that of developing flow in the entrance of a pipe. The solution procedure was successful because: 1. the boundary conditions are simple and well-defined and 2. the dominant velocity gradients paralleled the grid's highest resolution.

At t=0 it was assumed that everywhere in the flow domain the velocity and vorticity distributions were those of impulsively started flow as shown in Fig. 3.5. This is also the boundary condition, 0<t<∞, at the inlet boundary. This leaves two boundaries to be determined at each time step: 1. the no-slip boundary and 2. the outlet boundary.

The velocity vector, \( \vec{V} \), on a no-slip boundary is by definition equal to 0 at all time. Therefore both \( v_x \) and \( v_z \) are 0 for all time steps. The no-slip or extraneous vorticity is determined in the collocation process. This includes the vorticity located at the intersection of the no-slip and outlet boundary.
The outlet boundary is much more difficult to model since all boundary quantities, \( v_r, v_z, \) and \( w, \) are unknown. For these types of boundaries the best approach to boundary specification will depend on the flow geometry, numerical method and flow parameters such as the Reynolds number. In all cases, however, the goal is the same: to successfully communicate to the numerical region what is happening beyond the grid boundaries.

The vorticity at the outlet boundary is an interior vorticity therefore its time dependence is a transport phenomenon and is governed by the vorticity transport equation. However since it is on the grid boundary, the interior representations cannot be used. For the diffusion terms the 2nd order centered difference scheme must be replaced with a 2nd order backwards scheme. This is tantamount to the assumption that \( \partial^3\omega/\partial z^3 = 0 \) which is not very restrictive and thus proved acceptable.

The interior convection terms are modeled with forward differencing in the predictor step and backward differencing in the corrector step. On the outlet boundary the convection terms must be modeled with backward differencing for both steps. It has been reported that when used exclusively this approach often results in artificial diffusivity and an over damped solution. For present purposes, however, it was
assumed that such limited use of complete backward differencing would have little adverse affect.

According to the solution theory of boundary valued problems, for a well-posed problem with a unique solution it is necessary to specify either \( v_r \) or \( v_z \) everywhere on the outlet boundary. Unfortunately, both of these velocity components are unknown. Confronted with this situation, the analyst could either abandon the solution or, as was the approach in this research, proceed with carefully and judiciously chosen flow assumptions.

As demonstrated by the steady state solution for pipe entrance flow, the velocity vector in this problem is dominated by the \( v_z \) component everywhere in the region. Given this fact, a reasonable and very convenient assumption would be that \( v_r = 0 \) everywhere on the outlet boundary. This permits the solution to be obtained as if the problem were actually well-posed. For this problem, this basic approach was used successfully with one modification made necessary by an unfortunate quirk of the axisymmetric formulation.

With the given \( v_r \) assumption, the matrix assembled in the collocation process will contain one row of terms which quantify the influence of \( v_z(0<r<R) \) on \( v_r(r=0) \). Though not immediately apparent from Eq. 3.4, for the given flow and grid geometry, these terms are identically 0. From a matrix algebra perspective, this is incapacitating and a solution is
not possible. Two approaches were taken to overcome this idiosyncrasy of axisymmetric formulations. Both methods will be described but only the one with the best results is reported.

The first method utilized the regularization methods developed in Chapter V for ill-posed problems. The form of the collocation was not changed but the resulting linear system was stabilized or conditioned by adding an approximate solution constraint where the solution was assumed to change slowly with time. This approach did converge but the constraint required a very small time step, about 1/100th the time step of the method reported.

In the most successful approach it was assumed that for the each time step \( v_z \) at \( r=0 \) did not change. This removed this one nodal velocity from the collocation procedure which eliminated the row of zeros and stabilized the solution. After collocation \( v_z \) at the centerline was recalculated with the newly collocated boundary values and interior vorticities. The resulting profile plots at four times during modeling are reported in Fig. 6.8. A time step size of \( t=0.0025 \) sec was used in the model.
Figure 6.8. Profile plots at four intervals for time dependent developing flow at a pipe entrance
Comments and Conclusions

The focus of this research is the derivation of original BEM solutions of the vector Poisson's velocity equation. This equation is the governing equation of the kinematic part of the complete Navier-Stokes solution. In the development of these solutions the resulting algorithms were tested on simple pipe flow problems with well defined velocity and vorticity fields. For these problems, it was not necessary to solve the kinetic part of the solution.

However for more complicated and interesting flows with unknown solutions, it is necessary to also solve the vorticity equation, which models the kinetic effects. Consequently the objective of the vorticity transport solution section of this research was not to develop new numerical solutions but rather to adapt proven techniques to facilitate further testing of the new kinematic solutions. To this end, two finite difference solutions were selected, time-split MacCormack and alternate direction implicit, and were modified from 2 dimensional planar formulations to the required axisymmetric formulations.

The two vorticity transport solutions were combined with the two velocity equation solutions to create four algorithms for the complete solution of Navier-Stokes flows. After a functional determination of acceptable time step size, the
four algorithms were used to obtain solution fields for steady state entrance pipe flow.

Results from this test problem showed that the implicit vorticity solution is not unconditionally stable as it is for simpler applications. The time-split MacCormack scheme also was not unconditionally stable but it is 2nd order accurate in time as compared to the 1st order accuracy of the implicit scheme. For this reason, only the time-split MacCormack finite difference vorticity transport solution was used in subsequent tests.

Also not considered for further development was the direct BEM solution of the velocity equation. This decision was precipitated by the difficulties, both programming and conceptual, associated with the determination and/or specification of normal derivatives at boundary corners. A corner is defined to be a boundary node located at the junction of two boundary segments which are not in a straight line.

The distribution of the normal derivative across corner nodes is not continuous and therefore two nodal derivative values are required to correctly and completely model the distribution. This of course complicates the algorithm and increases necessary program size. For the present, these problems were simply avoided by using only the Biot-Savart formulation which does not contain normal derivatives.
However there is no reason to believe that if more flexible algorithms were developed the direct BEM formulation would not perform as well as the Biot-Savart.

The one remaining algorithm was then used to obtain solutions for two more steady state flow problems. The first problem modeled the flow in a power hydraulic rotary flow coupler and the second modeled the flow in a spool valve. Acceptable solutions were obtained and reported for both but the process brought to surface one serious programing deficiency and one area of conceptual uncertainty.

Both the rotary coupler and spool valve flow models have regions where \( \mathbf{v} \) is dominated by radial velocities and axial velocity gradients. Unfortunately the simple numerical grid being used was designed with axial resolution four times that of the radial resolution. While it was not considered necessary to validate the new BEM velocity the addition of a interactive grid generating algorithm would surely improve the performance of the numerical techniques. The total impact of this recommended software upgrading will not be known until it is actually implemented.

The solution of the valve model flow surfaced another area of concern of a more theoretical or conceptual nature. Boundary conditions where specified such that the problem was well-posed. Still, the resulting kinematic system matrix had a very large condition number and the solution had
unexplainable oscillations with large errors. Specifically, the no-slip vorticity distribution on the vertical surface adjacent to the inlet had alternately large negative and large positive values.

Part of the blame for this situation could be placed on the already maligned numerical grid. There is some evidence which suggests that the condition number of such linear systems could be improved by grid modification. While this is an area which certainly merits investigation, it appears unlikely that the numerical difficulties would be completely eliminated with a new higher resolution grid.

To stabilize the linear system developed during collocation, the regularization techniques developed in Chapter III for ill-posed problems were employed. The vorticity distribution of the new solution no longer contained oscillations and the continuity and mass balance error terms were in an acceptable range. These results are all positive and based on them alone the solution would probably be accepted. Still, there are many questions to be answered and facets of the problem which need to be better understood.

For example, the condition number of the regularized kinematic system was approximately $1 \times 10^6$. This was more than a 50% reduction from the non-regularized system but is still much larger than the condition numbers usually
associated with stable linear systems. This contradiction between the large condition number and the stable appearing solution also should be further investigated.

It would also be interesting to see how the ill-conditioned character of the BEM kinematic system is translated to other numerical methods such as finite element or finite difference. This information would help in determining if the numerical difficulties are intrinsic to the problem or caused by the selected numerical method.

Finally, there is one underlying or fundamental question to be addressed. That is what effect the use of numerical stabilizers on ill-conditioned systems has on the relationship between the physical problem and mathematical problem. It has been demonstrated that solutions which satisfy conventional criterion can be easily obtained to some ill-conditioned problems but it is still not known how well solutions thus obtained predict the "real" solution. To answer this question will ultimately require a project with coordinated experimental and numerical components.

The final fluid flow problem modeled was developing flow at the entrance of a pipe. This is a simple problem in terms of the boundary condition specification but is still an interesting study. The solution obtained using the one remaining algorithm shows clearly how the initially blunt velocity profile changes to the steady state profiles seen in
earlier results. The error terms recorded during the time-dependent modeling were well within an acceptable range.

In summary, despite the problems encountered in the complete Navier-Stokes solutions they did adequately serve the purpose of verifying the two solutions of the vector Poisson's velocity equation. The one addition which would do the most to improve the solutions is an interactive grid generation algorithm.

The grid not only decreased performance but also created a low upper limit on the flow Reynolds number. Numerical solutions of laminar flows all seem to have a natural upper Reynolds number limit. This limit for the developed numerical solutions could not be determined because of the Reynolds number limits imposed by the grid.
CHAPTER VII. BOUNDARY ELEMENT SOLUTION OF POISSON'S VECTOR POTENTIAL AND POISSON'S PRESSURE EQUATIONS

When Navier-Stokes flows are solved in terms of the derived variable of vorticity, the pressure variable is eliminated from the governing equations. If pressure information is required a secondary equation must be solved, the Poisson's pressure equation. Another secondary operation that is often useful is the calculation of the stream function field which is used for improved flow visualization. For axisymmetric flows, the stream function is simply related to the vector potential which is obtained by solving the Poisson's vector potential equation.

Both of these Poisson's equations may be solved using the direct BEM equation, Eq. 2.11, following procedures very similar to those used to develop solutions for Poisson's velocity equation. The derivation of these integral solutions is the focus of this chapter.

Solution of Poisson's Vector Potential Equation

For 3 dimensional flows a vector potential, \( \mathbf{B} \), exists such that \( \mathbf{V} = \nabla \times \mathbf{B} \) and \( \nabla \cdot \mathbf{B} = 0 \). The vector potential equation is derived by taking the curl of both sides of \( \mathbf{V} = \nabla \times \mathbf{B} \) and employing a common vector identity. The result is \( \nabla^2 \mathbf{B} = -\mathbf{\omega} \), formerly given as Eq. 2.9.
The first step in developing the BEM solution of the vector potential equation is to write the rectangular components of the vectors involved in terms of the axisymmetric components. As in the velocity development, \( \theta \) is the field point angular location and \( \theta_0 \) is the load point angular location.

\[
\begin{align*}
\vec{B}_\theta &= (-B_\theta \sin \theta) \hat{i} + (B_\theta \cos \theta) \hat{j} + 0 \hat{k} \\
\vec{B}_\theta^o &= (-B_{\theta_0} \sin \theta_0) \hat{i} + (B_{\theta_0} \cos \theta_0) \hat{j} + 0 \hat{k} \\
\vec{\omega}_\theta^o &= (-\omega_{\theta_0} \sin \theta_0) \hat{i} + (\omega_{\theta_0} \cos \theta_0) \hat{j} + 0 \hat{k}
\end{align*}
\]

As for all axisymmetric solutions \( \theta \) may be set equal to zero since the solution is not a function of the field point angular position. This leaves \( B_\theta \hat{j} \) as the only non-zero component of \( \vec{B} \). The original vector equation is now reduced to one scalar Poisson equation which is the correct form for using the direct BEM integral solution.

From this point on, the vector potential development parallels almost exactly the \( v_x \) solution development and therefore is not repeated. The final integral solution for the one vector potential component is Eq. 7.1. In Eq. 7.1 the \( \theta \) subscripts on \( \omega \) and \( B \) have been dropped. \( K \) and \( AB \) are defined as in Chapter III, pps. 79 and 81 respectively.

\[
B = \left\{ \begin{array}{c}
-\frac{\omega \cos \theta_0}{K (AB)^{1/2}} \\
-\frac{\partial B_\theta}{\partial n_0} \cos \theta_0 \\
\int_{V_0} \frac{1}{K (AB)^{1/2}} dV_0 + \int_{S_0} \frac{1}{K (AB)^{3/2}} \frac{B_0 \cos \theta_0}{K (AB)^{1/2}} (r \cos \theta_0 - r_0) n_{r_0} - (z_0 - z) n_z \end{array} \right\} \\
\int_{S_0} \frac{1}{K (AB)^{3/2}} dS_0
\]

(7.1)
The kernels in the surface integrals of Eq. 7.1 are exactly the same as the ones for the solution of \( v_r \) in Eq. 3.9. This is very convenient if the direct BEM solution is used for velocity. One of the benefits is reduction in necessary computer storage and another is that it provides another check of the kernels. The interior or volume kernel is slightly different but the integrations necessary are of the same form and present no complications.

**Implementation of the vector potential solution**

The solution of Eq. 7.1 requires knowledge of the complete vorticity field and either \( B \) or \( \partial B / \partial n \) at every point on the boundary. The vorticity field is determined explicitly by the complete Navier-Stokes solution. The necessary \( B \) boundary information is not determined explicitly but can be expressed in terms of the boundary values of \( v_r \) and \( v_z \), which are known.

From the definition of the vector potential, \( v_r \) and \( v_z \) can be calculated from the vector potential field with Eqs. 7.2 and 7.3 and substituted in the definition of \( \partial B / \partial n \), Eq. 7.4.

\[
v_r = \frac{1}{r} \frac{\partial (-rB)}{\partial z} = - \frac{\partial B}{\partial z} \tag{7.2}
\]

\[
v_z = - \frac{1}{r} \frac{\partial (-rB)}{\partial r} = B + \frac{\partial B}{\partial r} \tag{7.3}
\]

\[
\frac{\partial B}{\partial n} = \hat{n} \cdot \nabla B = n_r \frac{\partial B}{\partial r} + n_z \frac{\partial B}{\partial z} \tag{7.4}
\]
These expressions can be rearranged and used in a collocation procedure to determine all of the necessary boundary values of \( B \) and \( \frac{\partial B}{\partial n} \). The remainder of the vector potential field can now be calculated explicitly with Eq. 7.1. Finally, the stream function field, \( \psi \), can be calculated using the simple relationship \( \psi = -rB \).

The stream function field is most often used as a media element to aid in the visualization of multi-dimensional flow solutions. In addition, however, it can also serve as a diagnostic tool in the development of new solution methods.

Example results of this procedure to determine the stream function field were used in Chapter VI to report the results of three complete Navier-Stokes solutions. The streamline plots for the entrance pipe flow, rotary coupler, and model valve problems are shown in Fig. 6.4, Fig. 6.5, and Fig. 6.6 respectively.

Solution of Poisson's Pressure Equation

Poisson's pressure equation is derived by taking the divergence of the vector form of the primitive variable Navier-Stokes equation, Eq. 2.1. The result is \( \nabla^2 p = -\rho G(\vec{V}) \) where \( G(\vec{V}) \) equals \( \nabla \cdot (\vec{V} \cdot \nabla \vec{V}) \) and is valid for all orthogonal coordinate systems. The axisymmetric form of \( G(\vec{V}) \) is derived by performing the vector operations as defined for cylindrical polar coordinates. Incorporation of the
continuity equation, \( \nabla \cdot \mathbf{V} = 0 \), produces a simple expression containing \( v_r \) and velocity derivatives. The following is an abbreviated account of this procedure.

\[
G(V) = \frac{1}{r} \left( r v_r \frac{\partial v_r}{\partial r} + r v_z \frac{\partial v_z}{\partial z} \right) + \frac{\partial}{\partial z} \left( v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right)
\]

\[
\left( \frac{\partial v_r}{\partial r} \right)^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 + 2 \frac{\partial v_r}{\partial z} \frac{\partial v_z + v_r}{\partial r} \frac{\partial}{\partial r} + v_r + r^2 \frac{\partial}{\partial z} \left( \frac{\partial v_r + \partial v_z + v_r}{\partial r} \right)
\]

\[
= \left( \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r - v_r}{r} \right)^2 + 2 \left[ \frac{\partial v_r}{\partial z} \frac{\partial v_z}{\partial r} - \frac{\partial v_r}{\partial r} \frac{\partial v_z}{\partial z} \right] + \frac{v_r^2}{r^2}
\]

\[
= 2 \left[ \frac{\partial v_r}{\partial z} \frac{\partial v_z}{\partial r} - \frac{\partial v_r}{\partial r} \frac{\partial v_z}{\partial z} + \frac{v_r^2}{r^2} \right]
\]

Since Poisson's pressure equation is scalar, it may be substituted directly into the direct BEM equation, Eq. 2.11, without any component transformation. The resulting integral solution for pressure in axisymmetric Navier-Stokes flows is given in Eq. 7.5.
The kernels in the surface integrals of Eq. 7.7 are identical to the surface kernels in the direct BEM solution for \( v_z \), Eq. 3.10. This duplication, as was the duplication of the \( v_r \) kernels in the vector potential solution, is convenient from a programming perspective and can also serve as a diagnostic tool when investigating new types of boundary conditions. The volume kernels in Eq. 7.7 are unique but may be accurately integrated using previously developed algorithms.

**Implementation of the Poisson pressure equation solution**

Since the determination of the pressure field using Eq. 7.7 is a secondary operation the velocity field will be completely defined. This permits the determination of the interior integral load term, \( G(\vec{V}) \), in a postprocessing procedure to any desired accuracy. For the example pressure solutions in this research first order finite difference approximations of the derivatives proved adequate.

Solution of Eq. 7.7 also requires that at least one nodal pressure on the boundary be specified. For typical engineering flows boundary pressure information may be
available from pressure measurements and/or supplementary calculations based on component loads and conduit losses. In the example solutions, however, this information was not available making it necessary to arbitrarily specify a pressure at one boundary node. The remaining nodal pressures are determined in the collocation process.

For the boundary nodes where pressure has not been specified it is necessary to specify the normal derivative of pressure $\partial p/\partial n$. Again the most appropriate approach will depend on the available information and the type of problem. One possible approach, and the one used exclusively in this research, is to rearrange the $r$ and $z$ component equations of the primitive variable Navier-Stokes equations, Eq. 2.3, to yield Eqs. 7.6 and 7.7 which are approximations for $\partial p/\partial r$ and $\partial p/\partial z$ respectively. These expressions are evaluated at the desired boundary nodes and used in Eq. 7.8 to determine $\partial p/\partial n$.

\[
\frac{\partial p}{\partial r} = \rho \left\{ \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right\} - \rho \nu \left( \frac{v^2 v_r - \nu_r}{r^2} \right) \]  

(7.6)

\[
\frac{\partial p}{\partial z} = \rho \left\{ \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right\} - \nu v^2 v_z \]  

(7.7)

\[
\frac{\partial p}{\partial n} = \mathbf{n} \cdot \mathbf{v} = n_r \frac{\partial p}{\partial r} + n_z \frac{\partial p}{\partial z} \]  

(7.8)
These three equations can be incorporated in a simple postprocessing procedure to determine the necessary pressure normal derivatives for use in the collocation procedure. Again for the problems in this research first order finite difference approximations were used. After collocation the interior pressure field can then be calculated explicitly using Eq. 7.5.

This procedure was used to determine representative pressure fields for the three problems solved in Chapter VI. The results are reported with plots of constant pressure or isobars in Figs. 7.1, 7.2, and 7.3.

Figure 7.1. Plot of constant pressure lines for the pipe entrance flow problem
Figure 7.2. Plot of constant pressure lines for the rotary coupler flow problem

Figure 7.3. Plot of constant pressure lines for the model valve problem
Comments and Conclusions

In this chapter, two Poisson's equations, the vector potential equation and pressure equation were solved using the direct BEM integral formulation. Both equations are used as secondary operations since they are not essential to the complete Navier-Stokes solution but are solved only when the additional information is desired. The vector potential equation is solved to construct flow streamlines which aid in flow visualization and the pressure equation is solved only when detailed pressure information is desired.

In the development of the integral solutions, it was discovered that the surface kernels of the secondary equations were the same as the ones already evaluated in the direct BEM velocity solution. The vector potential solution used the $v_r$ surface kernels and the pressure solution used the $v_z$ surface kernels. Given the size of kernel files for even modest grids, this an important consideration when deciding if these secondary operations should or should not be included in the solution software.

In this research, the volume or interior kernels of both secondary equations were calculated and stored separately. However, in retrospect, it appears that if postprocessing of the vorticity fields in the velocity solution was permitted, the $v_r$ kernels could be used in the vector potential solution and the $v_z$ kernels in the pressure equation.
The solution procedures for both secondary equations were designed to be used in conjunction with the solution of Poisson's velocity equation and the vorticity transport equation. This meant that the complete velocity field was available for use in the postprocessing procedures for approximating the boundary normal derivatives and $G(\vec{V})$ in the pressure equation. If it was desired to use the solutions in other types of applications, new boundary specification procedures would have to be developed.

The velocity solution of the valve model problem had an ill-conditioned kinematic system matrix which required conditioning procedures before solution. However, both the vector potential and pressure solutions were stable and their kinematic system matrices had small condition numbers. Although certainly not conclusive, these observations indicate that the secondary equations do not necessarily inherit all unfavorable characteristics of the associated velocity problem.

One problem which must be addressed in direct BEM formulations is the proper handling of the normal derivatives on the boundary. This problem was already discussed in the summary of Chapter VI as it related to the velocity solution. In the velocity solution, these difficulties were avoided by using only the Biot-Savart BEM integral formulation which does not contain normal derivatives. However, there is not
an alternative BEM solution for the pressure equation. Instead, in order to proceed it was necessary to assume that one of boundary derivatives was zero. This approach was acceptable for present purposes but for future more critical applications this algorithm deficiency would have to be corrected.
CHAPTER VIII. COMPREHENSIVE SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

The development of the complete solutions to axisymmetric Navier-Stokes flows presented in this dissertation required research of both basic and applied nature. The basic research component was in the area of computational fluid mechanics. Specifically this component was the derivation and numerical integration of six boundary element integral equations; two for each of the component velocities, \( v_r \) and \( v_z \), and one each for pressure and component vector potential.

The applied research consisted of using the six BEM integral equations in Navier-Stokes solutions of simulated engineering flows. The objective of this component was to verify the form of the integral equations and accuracy of the numerical integration algorithms; and to assess their ability to handle boundary and initial conditions typical of the subject flows. Both the positive and negative results of these tests were discussed in Chapters VI and VII.

In addition to the BEM integral derivations and integration algorithms, the complete Navier-Stokes solution required several other research components. Some of these components involved the derivation original numerical formulations and others were simply the writing of computer
algorithms based on existing numerical techniques. All of these necessary components are listed below.

1. boundary element integral equations
2. numerical integration of the boundary element integrals
3. problem definition algorithms
   a. grid generation and connectivity information
   b. input of boundary and initial conditions
4. collocation procedures
   a. solution of simultaneous equations
   b. stabilization of ill-conditioned systems
5. solution of vorticity transport equation

This list is an accurate account of the components necessary in Navier-Stokes solutions developed in the research. However, it was assembled with the benefit of hindsight and does not reflect the evolution of the solutions in the research. The following section contains a summary of this evolution, starting with the original engineering motivations for the research. It also clarifies the development starting point of each research component as defined by available literature.

Comprehensive Summary

The stated motivation of this research was to provide detailed fluid velocity and pressure information for flows common in power hydraulic components to aid in the design of these components. For laminar flows of Newtonian fluids, i.e., Navier-Stokes flows, this information is provided by
solution of the vector Navier-Stokes equation. In spite of the Reynolds number limitations of these governing equations and thus on the flow regimes appropriate for modeling, the numerical solution of this equation was the selected focus of this research.

The scope of the research was further narrowed by the decision to consider only two dimensional axisymmetric solutions and also to use only the vorticity transport form of the Navier-Stokes equation. The axisymmetric formulation was chosen since it is the most efficient way to describe a particular flow geometry which is common in hydraulic components. These are flows which are true 3 dimensional flows but may be described in 2 dimensions when the cylindrical polar coordinate system is used.

This reduction in problem dimension made it advantageous to use the vorticity transport form of the governing equations rather than the primitive variable form. In 2 dimensions the primitive variable form requires the simultaneous solution of two multi-variable equations. The vorticity transport form requires the solution two single variable equations which may be solved independently. The independence is afforded by the separation of the kinematic and kinetic parts of the Navier-Stokes solution.

The kinematic part of Navier-Stokes flow is governed by the vector Poisson's velocity equation. The original
intention was to select appropriate finite difference, finite element, and boundary element solutions of this equation from the literature and compare their suitability to the flows of interest. This approach was quickly abandoned when the literature search produced no numerical methods of the appropriate form. Consequently it was decided to develop new boundary element or BEM solutions to this equation.

The first BEM solutions developed were derived from the fluid mechanics counterpart of the Biot and Savart formula for the magnetic effect of a current. This equation is published in 3 dimensional vector form, therefore the first step was to derive the component forms in cylindrical polar coordinates. The resulting integral solutions for \( v_r \) and \( v_z \) for the axisymmetric interior flows were similar to those published for the planar exterior flows.

To facilitate the use of the integral equations in a computer solution, it was necessary to develop a problem definition algorithm. This is an interactive algorithm which gives the analyst the flexibility to define different flow problems by inputing different geometries and boundary and initial conditions. The problem geometry is used to generate, automatically or interactively, the discretized numerical grid. In this research, a simple rectangular grid was generated automatically.
For well-posed boundary valued problems, at least half of the nodal boundary quantities must be specified as boundary conditions. The remaining boundary quantities are then determined by collocation. The problem definition algorithm is used by the analyst to control this process.

After the numerical grid was generated, it was then possible to approximate each integral equation as a set of finite summations. The coefficients of these summations are generated with finite element type integration algorithms. As a result of the axisymmetric formulation, these integrals all contain elliptic integrals which cannot be evaluated in closed form. Accurate numerical integration requires that these integrals first be transformed to integrals containing standard elliptic integrals of the first and second kinds.

The standard elliptic integrals were approximated successfully with proven Chebyshev Polynomials. The expressions containing these polynomials were then broken down to isolate the singular terms characteristic of 2 dimensional boundary element integrands. Once in this form, the numerical integrations could proceed using the same techniques available in the literature for 2 dimensional planar problems.

In the process of developing the BEM solution to the velocity equation, a problem was encountered which presented formidable conceptual and numerical hurdles. The problem was
the solution of the ill-conditioned linear systems of equations produced in the collocation procedure of some flow problems.

The ill-conditioned systems were first encountered in problems which were ill-posed due to incorrect specification of boundary conditions but were also later encountered in a well-posed problem. This was to be expected for the ill-posed problems but it is much less obvious why it occurs for some well-posed problems.

For both cases the numerical instabilities and large solution errors characteristic of ill-conditioned linear systems were greatly reduced with a solution algorithm developed in this research. The algorithm stabilizes the matrix operator of the system by incorporating additional constraints in the solution.

The solution technique developed for ill-conditioned systems is based on the theory and numerical methods of Tikhonov regularization as it applies to linear algebraic systems. Using these methods the original unstable linear systems were modified by adding additional solution constraints. Constraints were developed which are appropriate for the types of boundary conditions and geometries typical of the types of flows being studied. In spite of the promising numerical results produced with
this technique, there are still questions to be answered concerning its use on engineering problems.

The kinetic part of the Navier-Stokes solution is described by the vorticity transport equation. The combination of a solution of this equation and a solution of Poisson's velocity equation constitutes a complete Navier-Stokes solution. In this research, it was decided to adapt available 2-dimensional planar solutions of the vorticity transport equation to the axisymmetric geometry rather than develop new methods. Finite differences solutions were chosen since they were the most proven and easiest to program, especially for the simple numerical grid being used.

Before complete Navier-Stokes solutions were attempted, another BEM formulation was developed for the kinematic part of the problem. This second solution to the velocity equation started explicitly with the vector Poisson's velocity equation. This equation was solved using the direct BEM equation for the solution of scalar rectangular boundary valued problems governed by Poisson's equation.

The rectangular component forms of the velocity equation were expressed in terms of the components of the cylindrical polar coordinate system. The terms of each of these scalar equations were then substituted in the direct BEM equation. The solution for each component velocity was completed by
integrating in cylindrical polar coordinates using previously developed integration algorithms.

The vorticity transport form of the Navier-Stokes solution will not yield pressure information. If pressure is required it is necessary to solve a secondary equation, the Poisson's pressure equation. Another secondary equation which may be useful is Poisson's vector potential equation. Solution of this equation yields the vector potential field which can be used to determine the flow streamlines. Both of these secondary Poisson's equations were solved using direct BEM methods which paralleled the derivation of the velocity solutions.

When the solution algorithms were completed for all the primary and secondary equations three steady state problems and one time-dependent problem were solved. The problems were designed to simulate the axisymmetric flows often found in power hydraulic components. For each problem a complete Navier-Stokes solution provided the velocity field and the pressure field and streamlines were provided by the solution of the two secondary Poisson's equations.

The successful solution of these test flow problems was an important step in the research. However the importance is not found in the specific velocity and pressure information obtained. The importance or real value of solving the tests problems is the contribution it made to the verification of
all the derived integral equations used in the boundary element numerical methods. For it is these numerical methods which were the goals of this research and not the solutions to specific flow problems.

Recommendations for Future Work

All of the components of the complete Navier-Stokes solution listed at the beginning of this chapter would benefit in some degree from additional work. Some of the work is only refinement and some is essential to facilitate useful engineering applications. Also, some of the work is research of a basic nature and some is improvements of existing computer algorithms.

The boundary element integrals developed in this research are the foundations of the BEM solutions. These equations along with the numerical integration procedures are considered to be the major contributions of this research. Based on the intermediate convergence tests and test flow problems, it is concluded that the integral equations are correct and that the integration procedures are sufficiently accurate. However it may be possible improve the efficiency of the integration algorithms both in terms of computing time and storage requirements.

In the present integration of interior or volume integrals, all singular and near-singular elements are
integrated using very dense Gauss type quadratures. It is possible that for some near-singular elements it is much more dense than necessary. Optimizing this integration may reduce computing time considerably. Also it may be possible to use 1st order velocity interpolations instead of the present 2nd order. This would reduce the storage required for the surface integration kernels.

In a design engineering application software it would not be practical to include code for both the Biot-Savart and direct BEM velocity solutions. However at this point it is not clear which is the preferred formulation.

In the flow problems studied thus far, the Biot-Savart proved the most convenient due primarily to the absence of normal derivatives. Yet the direct BEM formulation uses the same boundary kernels as the pressure and vector potential solutions. If these secondary solutions are desired, using the direct BEM form would greatly reduce the computing and/or storage requirements.

The problem definition algorithm is the interface between the physical problem as defined by the analyst and the numerical problem solved by the computer. The more flexible and sophisticated these algorithms are, the more complex the flow problems can be. In this research, only very simple problems were permitted because it was not possible to program a complex grid. Incorporating an
interactive grid algorithm is absolutely necessary to solve real engineering flows and would also facilitate some interesting basic research.

As stated in Chapter VI, the vorticity transport solutions were based on existing 2 dimensional planar finite difference formulations. These same methods would probably also be sufficient for more complicated flow problems although the algorithms will be more complicated for the necessary irregular grids with varying resolutions. At some point it may be worthwhile to investigate using BEM integral solutions. These tend to be more flexible and easier to adapt to special boundary shapes.

One serious shortcoming of the present problem definition algorithm which effects only direct BEM formulations is the way it handles normal derivatives. Specifically, it cannot correctly model corner normal derivatives which have two components. This was one reason why the Biot-Savart velocity solution performed better that the direct BEM solution. This software shortcoming also presented problems in the pressure solution. This would have to be corrected in future software development.

The collocation algorithms are necessary to determine the unspecified boundary quantities as part of each BEM solution. In this procedure, it is necessary to solve a system of linear simultaneous equations. A simple Gaussian
elimination algorithm was used successfully in this research but for larger grids more efficient existing methods may be advisable.

Another part of the collocation procedures used in this research is the recognition, stabilization, and solution of ill-conditioned linear systems. The numerical aspects of this process are relatively simple and well developed. However, as discussed in detail in Chapter VI, additional basic research is necessary before the solution procedures developed can be used with confidence for engineering flows.

The perceived order of importance of these recommendations for future work will depend on the interests and objectives of the funding institution. An engineering management would of course be interested in the work necessary to transform the numerical tools developed into a useful engineering design tool. A research institute would more interested in the work which would increase the understanding of the relationship between the physical problem and numerical problem. The objectives of both concerns, however, would both be served by first developing a new, or adapting an existing, interactive grid generating algorithm in future software.

With the present state of solution development engineering solutions can be obtained with confidence only for well-posed and well-conditioned kinematic problems.
Although this would eliminate some interesting flows, it is actually a common qualification and is implicit in most numerical solutions of boundary valued problems.

Also, if pressure and stream function information is not desired it would definitely be advisable to use the Biot-Savart velocity formulation. This approach would require the least amount of additional developmental work. If this information is desired it would probably be worth the time and effort to conduct the applied research necessary to refine the direct BEM solution of Poisson's velocity equation.

Without doubt the most interesting and potentially rewarding future basic research is the continued study of the ill-conditioned flow problems. As stated in Chapter VI a comprehensive study of this topic would necessarily involve a combined experimental and numerical effort. This problem is interesting not only because of the applied mathematics involved but more importantly because of its potential to be used immediately to solve real engineering problems.
BIBLIOGRAPHY


APPENDIX A.
TRANSFORMATION OF ELLIPTIC INTEGRALS TO STANDARD FORM

The following is an outline of a procedure for expressing integrals common in axisymmetric BEM formulations containing general elliptic integrals in terms of the standard complete elliptic integrals of the first and second kind, \( K(k) \) and \( E(k) \) respectively.

By definition:

\[
E(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \quad \text{and} \quad K(k) = \int_0^{\pi/2} (1-k^2\sin^2\alpha)^{1/2} d\alpha
\]

where \( k^2 \) is called the modulus and \( 0<k^2<1 \). The general integrals to be transformed have the form:

\[
\text{I.} \quad \int_0^{2\pi} \frac{\sin^\nu \theta \cos^\mu \theta}{(A-B\cos\theta)^\rho} \, d\theta \quad \forall, \mu, \rho > 0
\]

Define \( \alpha = (\theta - \pi)/2 \) and \( d\theta = 2d\alpha \) and substitute in I.

\[
\text{II.} \quad \int_{-\pi/2}^{\pi/2} \frac{\sin^\nu(\pi + 2\alpha) \cos^\mu(\pi + 2\alpha)}{(A+B-B(1+\cos(\pi + 2\alpha)))^\rho} (2d\alpha)
\]
Substitute in the denominator the identities \( \cos(\pi+2\alpha) = -\cos 2\alpha = 2\sin^2\alpha - 1 \) and define \( k^2 = \frac{2B}{(A+B)} \).

III. 
\[
\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^\nu(\pi+2\alpha)\cos^\mu(\pi+2\alpha)}{(A+B)^\rho} \frac{(A+B)^\rho}{(A+B-2B\sin^2\alpha)^\rho} (2d\alpha)
\]

IV. 
\[
2(-1)^{\nu+\mu} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^\nu 2\alpha \cos^\mu 2\alpha}{(A+B)^\rho} \frac{1}{(1-k^2\sin^2\alpha)^\rho} d\alpha
\]

The denominator is now in the proper form for using integral tables such as Gradshteyn and Ryzhik (1980). All integrals in this research have denominators of the same form as Eq. IV. The final form of each particular integral will depend on the values of the coefficients \( \mu, \nu, \) and \( \rho \). The final forms of the integrals found in this research are given next.

\( \nu=0, \mu=0, \rho=1/2 \)

\[
\int_{0}^{2\pi} \frac{d\theta}{(A-B\cos\theta)^{1/2}} = \frac{4}{(A+B)^{1/2}} \int_{0}^{\pi/2} \frac{d\alpha}{(1-k^2\sin^2\alpha)^{1/2}} = \frac{4}{(A+B)^{1/2}} K(k)
\]
\[ \begin{align*}
\mu = 0, \ \nu = 0, \ \rho = 3/2 \\
\int_{0}^{2\pi} \frac{d\theta}{(A-B\cos\theta)^{3/2}} &= \frac{4}{(A+B)^{3/2}} \int_{0}^{\pi/2} \frac{d\alpha}{(1-k^2\sin^2\alpha)^{3/2}} = \frac{4 \ E(k)}{(A-B)(A+B)^{1/2}} \\
\mu = 1, \ \nu = 0, \ \rho = 1/2 \\
\int_{0}^{2\pi} \frac{\cos\theta \ d\theta}{(A-B\cos\theta)^{1/2}} &= \frac{4}{(A+B)^{1/2}} \int_{0}^{\pi/2} \frac{(1-2\cos^2\alpha)}{(1-k^2\sin^2\alpha)^{1/2}} \ d\alpha \\
&= \frac{4}{(A+B)^{1/2}} \left\{ \frac{A \ K(k) - A+B \ E(k)}{B} \right\} \\
\mu = 1, \ \nu = 0, \ \rho = 3/2 \\
\int_{0}^{2\pi} \frac{\cos^2\theta \ d\theta}{(A-B\cos\theta)^{3/2}} &= \frac{4}{(A+B)^{3/2}} \int_{0}^{\pi/2} \frac{(1-2\cos^2\alpha)}{(1-k^2\sin^2\alpha)^{3/2}} \ d\alpha \\
&= \frac{4}{(A+B)^{1/2}} \left\{ \frac{A \ E(k) - K(k)}{B(A-B)} \right\} \\
m = 2, \ \nu = 0, \ \rho = 3/2 \\
\int_{0}^{2\pi} \frac{\cos^2\theta \ d\theta}{(A-B\cos\theta)^{3/2}} &= \frac{4}{(A+B)^{1/2}} \int_{0}^{\pi/2} \frac{(1+4\cos^4\alpha-4\cos^2\alpha)}{(1-k^2\sin^2\alpha)^{3/2}} \ d\alpha
\end{align*} \]
m=0, \nu=1, \rho=3/2

\[\int_{0}^{2\pi} \frac{\sin\theta}{(A-B\cos\theta)^{3/2}} d\theta = \int_{-\pi/2}^{\pi/2} \frac{2\sin\alpha \cos\alpha}{(A+B)^{1/2} (1-k^2\sin^2\alpha)^{3/2}} d\alpha = 0\]
APPENDIX B.
DERIVATION OF THE AXISYMMETRIC VORTICITY TRANSPORT EQUATION

The vorticity transport equation is derived by taking the curl of the Navier-Stokes equations. The axisymmetric form is formulated by performing all the vector operations as defined for the cylindrical polar coordinate system. This development starts with the below generalized vector form of the Navier-Stokes equation. All steps use the simplifying assumptions $v_\theta=0$ and $\partial/\partial \theta=0$ which are appropriate for axisymmetric problems.

\[
\frac{\partial \vec{V}}{\partial t} - \nabla \times \vec{\omega} = -\nabla (\mu/\rho + V^2/2 + \omega) - \nu (\nabla \times \vec{\omega})
\]

\[
v \times \frac{\partial \vec{V}}{\partial t} = \frac{1}{r} \left| \begin{array}{ccc}
r & r\theta & \hat{z} \\
\partial \theta & \partial \theta & \partial z \\
\partial r & \partial \theta & \partial z \\
\partial r & \partial \theta & \partial z \\
\partial r & \partial \theta & \partial z \\
\partial r & \partial \theta & \partial z \\
\partial \theta & \partial \theta & \partial \theta \\
\partial \theta & \partial \theta & \partial \theta \\
\partial \theta & \partial \theta & \partial \theta \\
\partial \theta & \partial \theta & \partial \theta \\
\partial \theta & \partial \theta & \partial \theta \\
\partial \theta & \partial \theta & \partial \theta
\end{array} \right|
\]

\[
v \times \frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \hat{\theta} = \frac{\partial \omega}{\partial t}
\]

\[
v \times \vec{V} \times \vec{\omega} = v \times \left( \left(-v_z \omega_r \right) \hat{r} + \left(v_r \omega_z \right) \hat{z} \right)
\]
\[
\begin{align*}
\mathbf{v} \times \mathbf{V} \times \omega &= \frac{1}{r} \begin{vmatrix}
\mathbf{r} & \mathbf{r}\theta & \mathbf{z} \\
\frac{\partial}{\partial \mathbf{r}} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\
-\mathbf{v}_z \omega & 0 & \mathbf{v}_r \omega
\end{vmatrix} \\
\mathbf{v} \times \mathbf{V} \times \omega &= \frac{\partial}{\partial z} (-\mathbf{v}_z \omega) - \frac{\partial}{\partial \mathbf{r}} (\mathbf{v}_r \omega) \\
\mathbf{v} \times \mathbf{V} \times \omega &= \frac{\partial \omega}{\partial z} \mathbf{r} + \frac{1}{r} \frac{\partial (\mathbf{r} \omega)}{\partial \mathbf{r}} \mathbf{z} \\
\mathbf{v} \times \mathbf{V} \times \omega &= \frac{1}{r} \begin{vmatrix}
\mathbf{r} & \mathbf{r}\theta & \mathbf{z} \\
\frac{\partial}{\partial \mathbf{r}} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\
-\frac{\partial \omega}{\partial z} & 0 & \frac{1}{r} \frac{\partial (\mathbf{r} \omega)}{\partial \mathbf{r}}
\end{vmatrix} \\
\mathbf{v} \times \mathbf{V} \times \omega &= \frac{\partial^2 \omega}{\partial \mathbf{r}^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{1}{r} \frac{\partial \omega}{\partial \mathbf{r}} - \frac{\omega}{\mathbf{r}} \\
\mathbf{v} \times -\nabla (p/\rho + \nabla^2/2 + \omega) &= 0 \quad \text{since all quantities in the brackets are scalar}
\end{align*}
\]

Combining the results of all the vector operations yields the axisymmetric vorticity transport equation in conservation form.

\[
\begin{align*}
\frac{\partial \omega}{\partial t} &= \frac{\partial (\mathbf{v}_r \omega)}{\partial \mathbf{r}} - \frac{\partial (\mathbf{v}_z \omega)}{\partial z} + \nu \left( \frac{\partial^2 \omega}{\partial \mathbf{r}^2} + \frac{1}{r} \frac{\partial \omega}{\partial \mathbf{r}} - \frac{\omega}{\mathbf{r}} + \frac{\partial^2 \omega}{\partial z^2} \right)
\end{align*}
\]
APPENDIX C.

FINITE DIFFERENCE SOLUTION OF THE VORTICITY TRANSPORT EQUATION

In this research two finite difference solutions of the complete 2 dimensional Burger's equation were used to solve the vorticity transport equation. Both formulations were originally in planer form and had to be adapted to the axisymmetric geometry. The resulting coefficients are presented two tables in this appendix.

A rectangular grid with uniform radial and axial grid spacings was used for all problems. Nodal vorticites, $\omega$, are located by $I$ and $J$ the respective radial and axial indices. All finite difference representations use a five node pattern. Also the convective terms use a shorthand notation for the nodal velocities involved. Both the node pattern and velocity shorthand used are given below.

\[
\begin{align*}
(I+1,J) & \quad \omega^* = \text{unknown or new vorticity} \\
(I,J-1) & \quad \text{node pattern abbreviations used in coefficients} \\
(I,J) & \quad w^c \quad (I,J) \quad v_r(I,J) = VRc \quad v_z(I,J) = VZc \\
(I,J+1) & \quad (I,J) \quad v_r(I+1,J) = VRn \quad v_z(I,J+1) = VZe \\
(I-1,J) & \quad (I,J) \quad v_r(I-1,J) = VRs \quad v_z(I,J-1) = VZw
\end{align*}
\]

five node pattern

All coefficients given are for interior vorticites. Coefficients for boundary nodes must be changed as dictated by the flow being solved.
Time-Split MacCormack

This is an explicit multiple time increment method with predictor and corrector steps in each time increment. Each time increment advances the solution in one direction. Any number of time increments may be used as long as the total time advanced in the two directions are equal. In this research three time increments were used in the following order: 1. radial, 2. axial, 3. radial. The resulting finite difference solution is 2nd order accurate in time.

**radial direction, predictor step**

\[
\begin{align*}
\omega^*(I,J) & \quad \omega (I,J) & \quad \omega (I+1,J) & \quad \omega (I-1,J) \\
1 & \quad \frac{2 \Delta T \nu - \Delta T \nu - \Delta T \nu + \Delta T \nu c}{\Delta R^2 \ R^* \Delta R \ R^2} & \quad \frac{\Delta T \nu + \Delta T \nu - \Delta T \nu c}{\Delta R^2 \ R^* \Delta R \ \Delta R} & \quad \frac{\Delta T \nu}{\Delta R^2} \\
\end{align*}
\]

**radial direction, corrector step**

\[
\begin{align*}
\omega^*(I,J) & \quad \omega (I,J) & \quad \omega (I+1,J) & \quad \omega (I-1,J) \\
1 & \quad \frac{-2 \Delta T \nu + \Delta T \nu - \Delta T \nu - \Delta T \nu c}{\Delta R^2 \ R^* \Delta R \ R^2} & \quad \frac{\Delta T \nu - \Delta T \nu c}{\Delta R^2} & \quad \frac{\Delta T \nu - \Delta T \nu + \Delta T \nu s}{\Delta R^2 \ R^* \Delta R \ \Delta R} \\
\end{align*}
\]

**axial direction, predictor step**

\[
\begin{align*}
\omega^*(I,J) & \quad \omega (I,J) & \quad \omega (I,J+1) & \quad \omega (I,J-1) \\
1 & \quad \frac{-2 \Delta T \nu - \Delta T \nu V z c}{\Delta Z^2 \ \Delta Z} & \quad \frac{\Delta T \nu - \Delta T \nu V z e}{\Delta Z^2} & \quad \frac{\Delta T \nu}{\Delta Z^2} \\
\end{align*}
\]

**axial direction, corrector step**

\[
\begin{align*}
\omega^*(I,J) & \quad \omega (I,J) & \quad \omega (I,J+1) & \quad \omega (I,J-1) \\
1 & \quad \frac{-2 \Delta T \nu - \Delta T \nu V z c}{\Delta Z^2 \ \Delta Z} & \quad \frac{\Delta T \nu}{\Delta Z^2} & \quad \frac{\Delta T \nu + \Delta T \nu V z e}{\Delta Z^2 \ \Delta Z} \\
\end{align*}
\]
Alternate Direction Implicit

The ADI method is an implicit two step method which advances the solution in one direction in each step. Each step requires the solution of a several sets of simultaneous linear equations with total number depending on the size of the grid. All ADI solutions to Burger's equation are 1st order accurate in time.

radial direction

<table>
<thead>
<tr>
<th>$\omega^*(I,J)$</th>
<th>$\omega^*(I+1,J)$</th>
<th>$\omega^*(I-1,J)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + \frac{2<em>DT</em>\nu + DT*\nu}{DR + R^2}$</td>
<td>$\frac{DT<em>V_{Rn} - DT</em>\nu - DT*\nu}{2<em>DR + DR^2 + 2</em>\nu<em>R</em>DR}$</td>
<td>$\frac{-DT<em>V_{Rs} - DT</em>\nu + DT*\nu}{2<em>DR + DR^2 + 2</em>\nu<em>R</em>DR}$</td>
</tr>
<tr>
<td>$\omega^*(I,J)$</td>
<td>$\omega^*(I,J+1)$</td>
<td>$\omega^*(I,J-1)$</td>
</tr>
<tr>
<td>$1 - \frac{2<em>DT</em>\nu}{DZ^2}$</td>
<td>$\frac{-DT<em>V_{Ze} + DT</em>\nu}{2*DZ + DZ^2}$</td>
<td>$\frac{DT<em>V_{Zw} + DT</em>\nu}{2*DZ + DZ^2}$</td>
</tr>
</tbody>
</table>

axial direction

<table>
<thead>
<tr>
<th>$\omega^*(I,J)$</th>
<th>$\omega^*(I,J+1)$</th>
<th>$\omega^*(I,J-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + \frac{2<em>DT</em>\nu}{DZ^2}$</td>
<td>$\frac{DT<em>V_{Ze} - DT</em>\nu}{2*DZ + DZ^2}$</td>
<td>$\frac{-DT<em>V_{Zw} - DT</em>\nu}{2*DZ + DZ^2}$</td>
</tr>
<tr>
<td>$\omega^*(I,J)$</td>
<td>$\omega^*(I+1,J)$</td>
<td>$\omega^*(I-1,J)$</td>
</tr>
<tr>
<td>$1 - \frac{2<em>DT</em>\nu - DT*\nu}{DR + R^2}$</td>
<td>$\frac{-DT<em>V_{Rn} + DT</em>\nu + DT*\nu}{2<em>DR + DR^2 + 2</em>\nu<em>R</em>DR}$</td>
<td>$\frac{DT<em>V_{Rs} + DT</em>\nu - DT*\nu}{2<em>DR + DR^2 + 2</em>\nu<em>R</em>DR}$</td>
</tr>
</tbody>
</table>
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