Compact posets and ramifiability of large cardinals

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Compact posets and ramifiability of large cardinals

Amin, Wael A. Ahmad, Ph.D.
Iowa State University, 1989
Compact posets and ramifiability
of large cardinals

by

Wael A. Ahmad Amin

A Dissertation Submitted to the
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1 INTRODUCTION

Prime ideals as well as ultrafilters of an ordered structure play an essential role in the possibility of embedding the ordered structure into a poset (short, for partially ordered set) of set-theoretic inclusion.

In Section 2, we first consider posets each having a maximum element 1 and satisfying the distributivity condition stating that if \( \sup\{x, a_i\} = 1 \) for \( i = 1, \ldots, n \) and if \( \sup\{x, \inf\{a_i\}\} \) exists then \( \sup\{x, \inf\{a_i\}\} = 1 \).

As a corollary we prove that in such a poset \( \mathcal{P} \) if every two elements have a supremum, then there exists a prime ideal of \( \mathcal{P} \) which contains a given subset of \( \mathcal{P} \) that has the finite supremum property.

We also show that without the assumption of the existence of 1 and under the condition \( \inf\{\sup\{x, a_i\}\} = \sup\{\inf\{x, a_i\}\} \) with \( i = 1, 2 \), corresponding to every two incomparable elements \( x \) and \( y \) of \( \mathcal{P} \) there exists a prime ideal of \( \mathcal{P} \) which separates \( x \) and \( y \).

As an application of results obtained in this Section, we introduce the notion of a subbase \( S \) of a poset \( \mathcal{P} \) with 1 and we prove the compactness of \( \mathcal{P} \) based on the existence of a finite subcover of a cover of 1 by elements of \( S \).
In Sections 3 and 4, a key Theorem and a novel technique of transfinite inductive proof are introduced for establishing directly the Tower and Complete Accumulation Point compactness of the product of compact topological spaces.

The key Theorem that we prove states that if $S$ is a subset of an $A$-inductive poset $\mathcal{P}$ such that every nonempty finite subset of $S$ has a least upper bound (in $\mathcal{P}$) then every nonempty subset of $S$ has a least upper bound (in $\mathcal{P}$). As a corollary, a poset $\mathcal{P}$ is complete if and only if $\mathcal{P}$ is $A$-inductive and every finite subset of $\mathcal{P}$ has a least upper bound.

The Tower compactness of a topological space $T$ is then proved to be equivalent to the $A$-inductivity of the poset of all proper open sets of $T$. We use this fact in establishing various other results.

The novel technique of transfinite induction (mentioned above) that we have introduce in Section 4 runs as follows.

Let $\prod_{i \in I} X_i$ be a product of compact topological spaces $X_i$ with $i \in I$ where $I$ is an infinite cardinal and let $A$ be an infinite subset of $\prod_{i \in I} X_i$. We construct a point $a = (a_i)_{i \in I}$ of $\prod_{i \in I} X_i$ by choosing $a_i$ in such a way that it has the following property:

for every finite subset $F_i$ of ordinals $< i$

and every $j \in F_i \cup \{i\}$ and for every neighborhood $V(a_j)$ of $a_j$

it is the case that

$$\overline{A \cap Y_i(a_i)} = \overline{A} \quad \text{with} \quad Y_i(a_i) = \prod_{m \in I} Z_m$$

where

$$Z_m = \begin{cases} V(a_j) & \text{if } m \in F_i \cup \{i\} \\ X_m & \text{otherwise} \end{cases}$$
We show that \( a = (a_i)_{i \in I} \) thus constructed, is a complete accumulation point of \( A \).

In Section 5, coordinatewise construction of complete accumulation points of the countable infinite product of the real unit interval \( I \) is given. The results are used in establishing the solvability of infinite system of linear equations each with at most a countable (finite or infinite) number of unknowns. The existence of a solution of such a system of linear equations over the reals is proved based on the compactness of the product topology and under the assumption that finite subsystem have uniformly bounded solutions.

In Section 6, we introduce the concept of Receding sequences of ordinals which serves as the basic motivation for the partition properties of infinite cardinals which in turn leads to the notion of the Ramifiability of infinite cardinals.

Let \( K \) be an infinite cardinal. A sequence \((S_i)_{i < K}\) of nonempty subsets \( S_i \) of \( K \) is called receding if and only if for every elements \( i, j, v > 0 \) of \( K \) it is the case that \( i < j \) implies \( S_i \supset S_j \) and \( \min(S_i) < \min(S_j) \), moreover, \( \cap_{i < v} S_i = S_w \) for some \( w < K \).

We prove a basic property of \((S_i)_{i < K}\) which states that a mapping \( \mathcal{F} \) from \( K \) into \( K \) such that

\[
\mathcal{F}(x) = \begin{cases} 
0 & \text{if } x \notin S_i \text{ for every } i < K \\
\min(\cap\{S_i : x \in S_i\}) & \text{otherwise}
\end{cases}
\]

has a fixed point \( c > 0 \) if and only if \( c = \min(S_w) \) for some \( w < K \). Clearly, \( \mathcal{F}(0) = 0 \). Based on this, we show that if \( D \) is a strongly \( K \)-complete
nonprincipal ultrafilter of $2^K$ such that $S_i \in D$ for every $i < K$ then
$\{\min S_i : i < K\} \in D$.

In Section 7, we consider the question of the existence of ramifiable cardinals
and various properties which characterizes them.

An infinite cardinal $\alpha > \omega$ is called ramifiable if and only if every tree
of rank $\alpha$ whose levels are of cardinality $< \alpha$ has a well ordered subset of
cardinality $\alpha$.

The existence of ramifiable cardinals cannot be proved in $ZFC$. However,
the existence of a $CAC$-ramifiable cardinal can be shown in $ZFC + MA + \neg CH$
(where $MA$ is the Martin’s axiom). This fact is proved in Section 7 in a rather
simple way.

In the absence of Martin’s axiom, however, various useful characterization of
ramifiable cardinals can be derived. To this end in Section 7, we have introduced
a special lexicographic order on a Hausdorff cardinal $\alpha$ which implies that $\alpha$
is ramifiable. On the other hand, the introduction of a special tree in Section 7
implies the converse provided $\alpha$ is strongly inaccessible.
2 EXISTENCE OF PRIME IDEALS AND ULTRAFILTERS IN
PARTIALLY ORDERED SETS

In the existing literature the algebraic notions of an Ideal and Prime ideal [1, p. 122] and the corresponding dual notions of a Filter and an Ultrafilter [1, p. 142] are predominantly generalized to the case of Lattices [2] and [3].

Here, we introduce these notions in partially ordered sets. An Ideal in partially ordered set can be defined in various (not necessarily pairwise equivalent) ways. The same is the case for the definitions of a Prime ideal, Filter and an Ultrafilter.

In what follows, we refer to a partially ordered set simply as a poset. Also, we introduce the following notions:

(2.1) \([x, y]\) for \(\sup \{x, y\}\)
(2.2) \((x, y)\) for \(\inf \{x, y\}\)

Based on the above notions, we introduce:

**DEFINITION 2.1.** A nonempty proper subset \(I\) of a poset \((\mathcal{P}, \leq)\) is called an ideal of \(\mathcal{P}\) iff
(2.3) \( x \in I \) and \( y \leq x \) imply \( y \in I \) for every \( x \) and \( y \in P \)

(2.4) \( x \in I \) and \( y \in I \) then \( [x, y] \in I \) for every \( x \) and \( y \in P \)

Moreover, an ideal \( D \) of \( P \) is called a prime ideal of \( P \) iff

(2.5) if \( (x, y) \in D \) then \( x \in D \) or \( y \in D \) for every \( x \) and \( y \in P \)

**Lemma 2.1.** Let \((\mathcal{P}, \leq)\) be a poset with a maximum \( 1 \). Then for every \( a, b, c \in \mathcal{P} \) it is the case that:

(i) If \( b \leq c \) and \([a, b] = 1\) then \([a, c] = 1\)

(ii) Let \([b, c]\) exist. Then \([a, [b, c]] = 1\) iff \([a, b, c] = 1\)

**Proof.** (i): Let \( b \leq c \) and \([a, b] = 1\). Then \( 1 \) is the only upper bound of \( \{a, b\} \). If \( x \) is an upper bound of \( \{a, c\} \) then \( x \) is an upper bound of \( \{a, b\} \). Thus \( x = 1 \) and consequently \( 1 \) is the only upper bound of \( \{a, c\} \). Hence \([a, c] = 1\).

(ii): Assume \([b, c]\) exists and \([a, [b, c]] = 1\). Again, \( 1 \) is the only upper bound of \( \{a, [b, c]\} \). If \( x \) is an upper bound of \( \{a, b, c\} \) then \( x \) is also an upper bound of \( \{a, [b, c]\} \). Thus \( x = 1 \). Consequently \( 1 \) is the only upper bound of \( \{a, b, c\} \), i.e., \([a, b, c] = 1\). Conversely, let \([b, c]\) exist and \([a, b, c] = 1\). Since \([b, c] \geq c\), by (i) we have \([a, b, [b, c]] = 1\). But \([b, c] \geq b\) and therefore, \( 1 = [a, b, [b, c]] = [a, [b, c]] \). Thus (ii) is established.
Let \((\mathcal{P}, \leq)\) be a poset with maximum 1 such that:

\[(2.6)\] The supremum of every two elements of \(\mathcal{P}\) exists.

And for every finite subset \(\{x, a_1, \ldots, a_n\}\) of \(\mathcal{P}\), the following distributivity condition holds:

\[(2.7)\] If \([x, a_1] = \cdots = [x, a_n] = 1\) and \([x, (a_1, \ldots, a_n)]\) exists then
\[ [x, (a_1, \ldots, a_n)] = 1. \]

Moreover, as usual, a subset \(\mathcal{A}\) of \(\mathcal{P}\) is said to have the finite supremum property iff:

\[(2.8)\] 1 is not the supremum of any finite subset of \(\mathcal{A}\).

**THEOREM 2.1.** Let \((\mathcal{P}, \leq)\) be a poset with a maximum 1 satisfying \((2.6)\) and \((2.7)\). Let \(D_0\) be a nonempty subset of \(\mathcal{P}\) satisfying \((2.8)\). Then there exists a subset \(D\) of \(\mathcal{P}\) such that:

(i) \(1 \notin D\) and \(D_0 \subseteq D\)

(ii) \(x \in D\) and \(y \leq x\) imply \(y \in D\) for every \(x\) and \(y\) \(\in \mathcal{P}\)

(iii) \([d_1, \ldots, d_n] \in D\) for every finite subset \(\{d_1, \ldots, d_n\}\) of \(D\)

(iv) If \((a_1, \ldots, a_n) \in D\) then \(a_i \in D\) for some 1 \(\leq i \leq n\).
PROOF. Let \( H' = \{ H : H \subseteq \mathcal{P} \text{ and } D_0 \subseteq H \text{ and } H \text{ satisfies (2.8)} \} \)

Clearly, \((H', \subseteq)\) is a nonempty partially ordered set since \( D_0 \in H' \). By Zorn's Lemma, it can be readily verified that \( H' \) has a maximal element \( D \). We show that \( D \) satisfies (i) to (iv).

Clearly, \( 1 \not\in D \) and \( D_0 \subseteq D \) so that \( D \) satisfies (i). Let us observe that by the maximality of \( D \) we have:

\[
(2.9) \quad \text{If } x \not\in D \text{ then } [x, d_1, \ldots, d_n] = 1 \text{ for some finite subset } \{d_1, \ldots, d_n\} \text{ of } D.
\]

Now, let \( x \in D \) and \( y \leq x \) and let \( y \not\in D \). By (2.9) \([y, d_1, \ldots, d_n] = 1\) and by (i) of Lemma 2.1 we derive that \([x, d_1, \ldots, d_n] = 1\) which is a contradiction since \( D \) satisfies (2.8). Hence \( y \in D \), i.e., (ii) is established.

Let \( \{t_1, \ldots, t_n\} \) be a subset of \( D \) and assume that \( t = [t_1, \ldots, t_m] \) is not an element of \( D \). Then \([t, d_1, \ldots, d_n] = 1\), by (2.9). But then (ii) of Lemma 2.1 implies \([t_1, \ldots, t_m, d_1, \ldots, d_n] = 1\) which a contradiction since \( D \) satisfied (2.8). Thus \( t \in D \). Hence (iii) is established.

To show (iv), let us assume on the contrary that \((a_1, \ldots, a_n) \in D \) and \( a_i \not\in D \) for every \( i \) with \( 1 \leq i \leq n \). Then by (2.9) we have:

\([a_i, d_{i1}, \ldots, d_{im_i}] = 1 \) for every \( 1 \leq i \leq n \). Now, let:

\[
x = [d_{11}, \ldots, d_{1m_1}, d_{21}, \ldots, d_{2m_2}, \ldots, d_{n1}, \ldots, d_{nm_n}].
\]

Then \( x \in D \) by (iii) and if \( y_t \) is an upper bound of \( \{x, a_i\} \) then \( y_t \) is an upper bound of \( \{d_{i1}, \ldots, d_{im_i}\} \). But \( 1 \) is the only upper bound of
Thus \( y_i = 1 \) and consequently \([x, a_i] = 1\). On the other hand, since \([x, (a_1, \ldots, a_n)]\) exists, then by (2.7) we have \([x, (a_1, \ldots, a_n)] = 1\) which is a contradiction since \(D\) satisfies (2.8). Thus \(a_i \in D\) for some \(1 \leq i \leq n\). Hence (iv) is established.

**REMARK 2.1.** We note that Theorem 2.1 implies that the subset \(D\) of the poset \(\mathcal{P}\) is a prime ideal of \(\mathcal{P}\). We observe that the same is true if the condition (2.7) is replaced by the weaker condition:

\[
(2.7)' \quad \text{If} \ [x, a_1] = [x, a_2] = 1 \quad \text{and} \quad [x, (a_1, a_2)] \quad \text{exists then} \quad [x, (a_1, a_2)] = 1
\]

The Theorem below which (in view of Remark 2.1) ensures the existence of a prime ideal of a poset as follows readily from Theorem 2.1.

**THEOREM 2.2.** Let \((\mathcal{P}, \leq)\) be a poset with a maximum \(1\) satisfying (2.6) and (2.7)'. Let \(D_0\) be a nonempty subset of \(\mathcal{P}\) satisfying (2.8). Then there exists a prime ideal \(D\) of \(\mathcal{P}\) such that \(D_0 \subseteq D\).

**REMARK 2.2.** We observe that for every nonmaximum element \(x\) of a poset \((\mathcal{P}, \leq)\) satisfying (2.6), the subset \(I(x)\) of \(\mathcal{P}\) given by:

\[
(2.10) \quad I(x) = \{ z : z \in \mathcal{P} \text{ and } z \leq x \}
\]

is an ideal of \(\mathcal{P}\).
As usual, $I(x)$ in (2.10) is called the principal ideal of $\mathcal{P}$ generated by $x$. Clearly, for every $x, y \in \mathcal{P}$ with $x \neq y$ there exists an ideal of $\mathcal{P}$ containing, say, $x$ but not $y$.

Next, we consider the case of the existence of a prime ideal of a poset without the maximum element. For this purpose we replace the distributivity condition (2.7)' by:

$$(2.11) \quad ([x, a_1], [x, a_2]) \leq [x, (a_1, a_2)]$$

with understanding that (2.11) holds whenever the right side of $\leq$ exists, and, this for every $x, a_1, a_2 \in \mathcal{P}$.

We observe that (2.11) does not hold in every poset. For instance, it fails in the poset $(\{e, a, b, c, m\}, \leq)\ e \leq a, \ e \leq b, \ e \leq c, \ a \leq m, \ b \leq m, \ c \leq m$.

**THEOREM 2.3.** Let $(\mathcal{P}, \leq)$ be a poset in which every two elements have a supremum and which satisfies (2.11). Let $x, y \in \mathcal{P}$ with $y \not\leq x$. Then there exists a prime ideal $D$ of $\mathcal{P}$ such that $x \in D$ and $y \not\in D$.

**PROOF.** From (2.10) it follows that $I(x)$ is an ideal of $\mathcal{P}$ and that $y$ is not the supremum of any finite subset of $I(x)$. This is because $x$ is an
upper bound of any subset of \( I(x) \) and \( y \leq x \).

Let \( H' \) be the set of all ideals \( H \) of \( P \) such that \( I(x) \subseteq H \) and \( y \) is not the supremum of any finite subset of \( H \). It is obvious that \( (H', \subseteq) \) is a nonempty poset. By Zorn’s Lemma it can be readily verified that \( H' \) has a maximal element \( D \).

We claim that \( D \) is a prime ideal of \( P \). Let us assume on the contrary, i.e., there exist \( a_1, a_2 \in P \) such that \( (a_1, a_2) \in D \) but \( a_1 \notin D \) and \( a_2 \notin D \).

Now, let us consider:
\[
(2.12) \quad D_i = D \cup \{z : z \in P \text{ and } z \leq [a_i, d] \text{ and } d \in D\} \quad \text{with } i = 1, 2.
\]

One of the following two cases must occur:

**Case 1.** \( D_i = P \) for some \( i \in \{1, 2\} \). For this case, from (2.12) we derive:
\[
(2.13) \quad y \leq [a_i, d_i] \text{ for some } d_i \in D
\]

**Case 2.** \( D_i \) is a proper subset of \( P \).

For this case we show that \( D_i \) is an ideal of \( P \) which contains \( D \) properly. Let \( t_1, t_2 \in D_i \), thus, from (2.12) it follows that \( t_1 \leq [a_i, d_3] \) and \( t_2 \leq [a_i, d_4] \) for some \( d_3, d_4 \in D \). Based on the hypothesis of Theorem, we let \( d = [d_3, d_4] \). Since \( D \) is an ideal of \( P \), we have \( d \in D \). Also, it can be readily verified that \( t_1 \leq [a_i, d] \) and \( t_2 \leq [a_i, d] \). Thus, \( [t_1, t_2] \leq [a_i, d] \) which by (2.12) implies that \( [t_1, t_2] \in D_i \). Hence, \( D_i \) satisfies (2.4). Now, let \( t \in D_i \) and \( r \leq t \) with \( r \in P \). But then, again from (2.12) it follows that \( r \in D_i \). Hence, \( D_i \) also satisfies (2.3). Consequently, \( D_i \) is an ideal of \( P \). However, the maximality of \( D \) implies that \( y \leq [a_i, d_i] \) for \( i \in \{1, 2\} \).
Thus, (2.13) holds in both of the mentioned cases. Let \( d = [d_1, d_2] \) which exists by the hypothesis of the theorem.

Clearly, \( y \) is a lower bound of \( \{[a_1, d], [a_2, d]\} \). Since \( [d, (a_1, a_2)] \) exists by the hypothesis of the theorem and since \( [d, (a_1, a_2)] \in D \) by (2.11) we have \( y \leq ([a_1, d], [a_2, d]) \leq [d, (a_1, a_2)] \in D \).

Since, \( D \) is an ideal of \( P \), by (2.3) we have \( y \in D \). But this contradicts that \( D \in H' \). Hence, our assumption is false and \( D \) is a prime ideal of \( P \).

The existence of prime ideals in structures related to order (e.g., semilattices, lattices, Boolean rings, etc.) has been considered under assumptions generally stronger than those stated in Theorem 2.3. In this connection reference is made to [4], [5], [6].

**REMARK 2.3.** We observe that the existence of prime ideals in posets is proved in Theorems 2.2 and 2.3 under the assumption that every two elements of the poset have a supremum. Next, we consider cases where this assumption is not satisfied by the poset. As shown below, for such cases we prove the existence of subsets of posets which will act almost like prime ideals.

**DEFINITION 2.2.** A nonempty proper subset \( D \) of a poset \( (P, \leq) \) is called a pseudo ideal of \( P \) iff:

\[(2.14) \quad x \in D \text{ and } y \leq x \text{ imply } y \in D \text{ for every } x \text{ and } y \in P\]
(2.15) if \( x, y \in D \) and \([x, y]\) exists then \([x, y]\) \( \in D \)

Moreover, a pseudo ideal \( D \) is called a pseudo prime ideal of \( \mathcal{P} \) iff:

(2.16) \((a, b) \in D \) implies \( a \in D \) or \( b \in D \)

Let \( \mathcal{P} \) be a poset with maximum \( 1 \) satisfying the distributivity condition:

(2.17) \([x_1, a_{11}, \ldots, a_{1n_1}] = [x_2, a_{21}, \ldots, a_{2n_2}] = 1\)

implies

\[
[(x_1, x_2), (x_1, a_{21}), \ldots, (a_{1n_1}, x_2), \ldots, (a_{1n_1}, a_{2n_2})] = 1
\]

for every \( x_1, x_2, a_{11}, \ldots, a_{1n_1}, a_{21}, \ldots, a_{2n_2} \in \mathcal{P} \)

THEOREM 2.4. Let \((\mathcal{P}, \leq)\) be a poset with the maximum \( 1 \) satisfying (2.17). Let \( D_0 \) be a nonempty subset of \( \mathcal{P} \) satisfying (2.8). Then there exists a pseudo prime ideal \( D \) of \( \mathcal{P} \) such that \( D_0 \subseteq D \).

PROOF. Let \( H' \) be the set of all subsets \( H \) of \( \mathcal{P} \) such that \( D_0 \subseteq H \) and \( H \) satisfies (2.8).

Clearly, \((H', \subseteq)\) is a nonempty poset and by Zorn's Lemma \( H' \) has a maximal element \( D \). We observe that \( D \) satisfies (2.9). We show that \( D \) is pseudo ideal of \( \mathcal{P} \). To show that \( D \) satisfies (2.14), we assume to the contrary that \( x \in D \) and \( y \leq x \) but \( y \not\in D \) for some \( y \in \mathcal{P} \). Then by (2.9) we have \([y, d_1, \ldots, d_n] = 1\) for some \( d_1, \ldots, d_n \in D \). Using (i) of Lemma 2.1, we obtain that \([x, d_1, \ldots, d_n] = 1\) which contradicts that \( D \in H' \).
and that $D$ satisfies (2.8). Hence, $\gamma \in D$. To show that $D$ satisfies (2.15), we assume to the contrary that for some $t_1, t_2 \in D$ it is the case that $t = [t_1, t_2]$ exists but $t \notin D$. Again, from (2.9) it then follows that $[t, d_1, \ldots, d_k] = 1$ for some $d_1, \ldots, d_k \in D$. Also, by (ii) of Lemma 2.1, we obtain that $[t_1, t_2, d_1, \ldots, d_k] = 1$ which again contradicts that $D$ satisfies (2.8). Hence, $t = [t_1, t_2] \in D$. Thus, $D$ is a pseudo ideal of $\mathcal{P}$.

Next, we show that $D$ is a pseudo prime ideal of $\mathcal{P}$. We assume to the contrary that $(a_1, a_2) \in D$ for some $a_1, a_2 \in \mathcal{P}$ but $a_1 \notin D$ $a_2 \notin D$. Then by (2.9) we have $[a_i, d_{i1}, \ldots, d_{i_n_i}] = 1$, for $i = 1, 2$ and some $d_{i1}, \ldots, d_{i_n_i} \in D$. But then from (2.17) it follows that

$$(2.18) \quad [(a_1, a_2), (a_1, d_{21}), \ldots, (d_{1n_1, a2}), \ldots, (d_{1n_1, d_{2n_2}})] = 1$$

Clearly, for every term such as $(a_i, d_{kj})$ which appears in (2.18) we have $(a_i, d_{kj}) \leq d_{kj} \in D$ and therefore, $(a_i, d_{kj}) \in D$ by (2.14). Also, by our assumption $(a_1, a_2) \in D$. Consequently, the entire left side of the equality sign in (2.18) is an element of $D$. But this contradicts that $D$ satisfies (2.8). Thus, our assumption is false and the pseudo ideal $D$ satisfies (2.16) and therefore $D$ is pseudo prime ideal of $\mathcal{P}$.

For ideas related to the prime ideals of a poset see [7] and [8].

**Lemma 2.2.** Let $(\mathcal{P}, \leq)$ be a poset with the maximum 1 satisfying (2.17). Then every finite subset of $\mathcal{P}$ has an infimum.
**PROOF.** Since every element of $\mathcal{P}$ is $\leq$ to every element of $\emptyset$, we see that $1$ is the infimum of $\emptyset$. Next we show that every two elements $a, b$ of $\mathcal{P}$ have an infimum. Since $[a, 1] = [b, 1] = 1$, by (2.17) we have $[(a, b), (a, 1), (b, 1), (1, 1)] = 1$. Thus, $(a, b)$ exists. But then by induction on the number of elements of a nonempty finite subset $S$ of $\mathcal{P}$. It can be readily shown that $S$ has an infimum in $\mathcal{P}$.

As an application of Theorem 2.4, we shall generalize the Alexander's subbase Theorem [9, p. 160] and [10, p. 139] and [20, p. 256], to posets which are not necessarily complete and join infinite distributive as $(T, \sqsubseteq)$ is, when $T$ is the set of all open sets of a topological space. Our generalization pertains to any poset which satisfies the hypothesis of the Theorem 2.4.

We use the familiar terminology in a poset with the maximum element $1$. We say that $1$ is covered by the elements of a subset $C$ of $\mathcal{P}$ (or, simply, $1$ is covered by $C$) iff $\sup(C) = 1$. A subset $H$ of $C$ is called a subcover of $1$ iff $\sup(H) = 1$. A subset $S$ of $\mathcal{P}$ is called a subbase of $\mathcal{P}$ iff every $p \in \mathcal{P}$ is the supremum of a set of infima of finitely many elements of $S$ (i.e.,

$$p = \sup_{i \in I} \left( \inf_{1 \leq j \leq n_i} (s_{ij}) \right)$$

with $s_{ij} \in S$). Based on the above notions, we prove:
THEOREM 2.5. Let \((\mathcal{P}, \leq)\) be a poset with the maximum element \(1\) satisfying (2.17). Let \(S\) be a subbase for \(\mathcal{P}\) such that every cover of \(1\) by elements of \(S\) has a finite subcover. Then every cover of \(1\) by the elements of \(\mathcal{P}\) has a finite subcover.

PROOF. Assume on the contrary that there exists a cover \(D_0\) of \(1\) by the elements of \(\mathcal{P}\) such that \(D_0\) has no finite subcover. Thus, \(D_0\) satisfies (2.8). Hence, by Theorem 2.4, there exists a pseudo prime ideal \(D\) of \(\mathcal{P}\) such that \(D_0 \subseteq D\).

Clearly, \(D\) satisfies (2.8) and (2.9). Also, since \(D_0 \subseteq D\) we have \(\sup(D) = 1\). Let \(D = \{d_i : i \in I\}\). Since \(S\) is a subbase for \(\mathcal{P}\), for every \(d_i \in D\) it is the case that:

\[
d_i = \sup_{j \in M_i} \left( \inf_{1 \leq k \leq n_j} (s_{jk}^i) \right)
\]

Now, obviously,

\[
\inf_{1 \leq k \leq n_j} (s_{jk}^i) \leq d_i \text{ for every } i \in I \text{ and } j \in M_j
\]

Therefore,

\[
\inf_{1 \leq k \leq n_j} (s_{jk}^i) \in D
\]

since \(D\) satisfies (2.14). But since \(D\) satisfies (2.15), by Lemma 2.2, we have \(s_{jk}^i \in D\) for some \(1 \leq k \leq n_j\). Let \(e_j^i\) denote that element \(s_{jk}^i \in D\).
Then for every \( i \in I \) and \( j \in M_j \) it is the case that

\[
\inf_{1 \leq k \leq n_j} (s^i_{jk}) \leq c^i_j
\]

By the Axiom of Choice, the set \( C = \{c^i_j : i \in I \text{ and } j \in M_j\} \) exists. Clearly, \( C \subseteq D \). We claim that \( 1 \) is the only upper bound of \( C \). Assume on the contrary that \( x < 1 \) is an upper bound of \( C \) (i.e., \( c^i_j \leq x \) for every \( i \in I \) and \( j \in M_j \)) then

\[
\inf_{1 \leq k \leq n_j} (s^i_{jk}) \leq x \text{ for every } i \in I \text{ and } j \in M_i
\]

However, since

\[
d_i = \sup_{j \in M_i} \left( \inf_{1 \leq k \leq n_j} (s^i_{jk}) \right)
\]

then \( d_i \leq x \) for every \( i \in I \). Hence, \( \sup(D) \leq x < 1 \) which contradicts that \( \sup(D) = 1 \). Thus, \( \sup(C) = 1 \). Consequently \( 1 \) is covered by the elements of the subbase \( S \) of \( \mathcal{P} \). But by the hypothesis of the Theorem, \( C \) has a finite subcover. Hence, \( 1 \) is covered by a finite number of the elements of \( D \) which is a contradiction. Therefore, our assumption is false and the Theorem is proved.

We conclude this section by observing that the existence of ultrafilters and pseudo ultrafilters in posets are ensured, as expected, by Theorems dual to Theorems 2.1, 2.3 and 2.4.
3 TOWER AND COMPLETE ACCUMULATION POINT
CHARACTERIZATION OF COMPACTNESS

In this section, we consider the question of compactness of product of compact
topological spaces based solely on the notions of tower and complete accumulation
point definitions of compactness.

First we prove the following lemma which will be needed in the sequel.

**Lemma 3.1.** Let $W$ be a well-ordered set and $S$ be a simply ordered
set. Let $F$ be an ordered preserving mapping from $W$ onto $S$. Then $S$
is a well ordered set.

**Proof.** Let $s$ be the smallest element of $F^{-1}(T)$ where $T$ is a nonempty
subset of $S$. Since $F$ is onto and order preserving, it follows that $F(s)$
is the smallest element of $T$. Thus $S$ is a well-ordered set.

We recall that a poset $(P, \leq)$ is A-inductive [11] iff every nonempty
well-ordered subset of $P$ has a supremum [cf.12,13,14].

**Theorem 3.1.** Let $(P, \leq)$ be an A-inductive poset. Let $S$ be a
subset of $P$ such that every nonempty finite subset of $S$ has a least upper
bound (in \(\mathcal{P}\)). Then every nonempty subset of \(S\) has a least upper bound (in \(\mathcal{P}\)).

**PROOF.** Assume on the contrary that there is a nonempty subset of \(S\) which has no supremum in \(\mathcal{P}\) and let \(H\) be such subset of smallest cardinality. Clearly, \(\overline{H} = \aleph_\mu \geq \aleph_\alpha\). Let \(H = \{ h_i : i \in \aleph_\mu \}\). We observe by our assumption that for every nonempty subset \(T\) of \(S\) with \(\overline{T} \leq \aleph_\mu\), it is the case that \(T\) has a supremum. Now, for every \(i \in \aleph_\mu\), let \(c_i\) denote the supremum of the subset \(\{ h_0, \ldots, h_i \}\) of \(H\). Clearly \(c_i\) exists, because \(\{h_0, \ldots, h_i\} \leq \aleph_\mu\).

Consider the set \(C = \{ c_i : i \in \aleph_\mu \}\). Obviously, \(C\) is a nonempty simply ordered set. Moreover, \(\mathcal{F}\) given by \(\mathcal{F}(i) = c_i\) is an order preserving mapping from \(\aleph_\mu\) onto \(C\). Thus from Lemma 3.1 it follows that \(C\) is a well-ordered subset of \(\mathcal{P}\). But since \(\mathcal{P}\) is \(\alpha\)-inductive, this implies that \(C\) has a supremum \(c\) in \(\mathcal{P}\). We claim that \(c\) is the supremum of \(H\).

Clearly, \(c\) is an upper bound of \(H\). Let \(d\) be an upper bound of \(H\). Since for every \(i \in \aleph_\mu\), we have \(c_i = \sup\{h_0, \ldots, h_i\}\), it follows that \(c_i \leq d\) for all \(i \in \aleph_\mu\). Thus \(d\) is an upper bound of \(C\) which implies that \(c\) is the supremum of \(H\), as desired.

**COROLLARY 3.1.** Let \(\mathcal{P}\) be an \(\alpha\)-inductive poset. If every nonempty finite subset of \(\mathcal{P}\) has a supremum, then every nonempty subset of \(\mathcal{P}\) has a supremum.

**PROOF.** It is sufficient to apply the previous Theorem 3.2 to the case
Using the above Corollary 3.3 we have:

**COROLLARY 3.2.** A poset $\mathcal{P}$ is complete iff $\mathcal{P}$ is $A$-inductive and every finite subset of $\mathcal{P}$ has a supremum.

In what follows by a tower of sets we mean a set well-ordered by inclusion.

**DEFINITION 3.1.** Let $(X, T)$ be a topological space. We say that $X$ is tower compact iff the union of every tower of proper open sets of $X$ is not equal to $X$.

**REMARK 3.1.** We note here that a topological space $(X, \tau)$ is tower compact iff $(\tau \setminus \{X\}, \subseteq)$ is $A$-inductive poset.

As an application of Theorem 3.1 based on the above notions we prove:

**THEOREM 3.2.** Let $(X, \tau)$ be a topological space. Then $X$ is compact iff $X$ is Tower compact.

**PROOF.** Let $X$ be compact, i.e., if no finite subfamily of a family $S$ of open sets of $X$ covers $X$, then $S$ does not cover $X$. Now, for any Tower $W$ of proper open subsets of $X$ it is the case that no finite subfamily of $W$ covers $X$. $\bigcup W$ is a proper open subset of $X$. Thus $X$ is Tower compact. Conversely, let $X$ be Tower compact. Then by Remark 3.1 we have $(\tau \setminus \{X\}, \subseteq)$ is $A$-inductive poset.

To show that $X$ is compact, let $S$ be any family of proper open set
of $X$ such that no finite subfamily of $S$ is a subset of the $A$-inductive poset \( \tau \setminus \{X\} \) such that every finite subset $F$ of $S$ has a supremum (i.e., $\bigcup F$) in $\tau \setminus \{X\}$. Therefore, by Theorem 3.2, we have $\sup S = \bigcup S \in \tau \setminus \{X\}$. Hence, $S$ does not cover $X$ and thus $X$ is compact.

The dual notion of Tower compactness [10, p. 163] can be also used to characterize compactness. Let $(X, \tau)$ be a topological space. A set $N = \{N_i : i \in I\}$ of nonempty closed subsets $N_i$ of $X$ is called Nested iff $\{X \setminus N_i : i \in I\}$ is a Tower in $(\tau, \subseteq)$.

Based on these notions we have:

**Corollary 3.3.** Let $X$ be a topological space. Then $X$ is compact iff every nested set of nonempty closed subsets of $X$ has a nonempty intersection.

**Remark 3.2.** Theorem 3.2 can be directly proved in the case of the special poset $(\tau, \subseteq)$ of all open sets of a topological space $(X, \tau)$. Let $X$ be tower compact and $S$ be any subset of $\tau \setminus \{X\}$ such that no union of finitely many elements of $S$ covers $X$. We show that $S$ does not cover $X$. Let $R = \{r_i : i \in R\}$ be a subset of $S$ of smallest cardinality with $\bigcup R = X$, i.e., for any subset $F$ of $S$ with $\overline{F} < \overline{R}$ we have $\bigcup F \neq X$. Let $R_i = \bigcup_{j < i} r_j$ for $i \in R$. Now $X = \bigcup R = \bigcup_{i \in R} R_i$ which contradicts that $X$ is Tower compact since $(R_i)_{i \in R}$ is a Tower of proper open sets of $X$.

Motivated by [16,17], [18, p. 6], [19, p. 605], we introduce the following
DEFINITION 3.2. Let $\mu$ be an infinite ordinal and $(s_i)_{i \in \mu}$ be a sequence in a topological space $X$. We call a point $c \in X$ a Complete Accumulation Point (CAP) of $(s_i)_{i \in \mu}$ iff for every neighborhood $V(c)$ of $c$ we have:

$$\{i : i \in \mu \text{ and } s_i \in V(c)\} = \bar{\mu}$$

DEFINITION 3.3. Let $(X, \tau)$ be a topological space. We say that $X$ is CAP-compact iff every infinite sequence of $X$ has a complete accumulation point.

Based on the above Definitions we have:

THEOREM 3.3. Let $(X, \tau)$ be a topological space. Then $X$ is compact iff $X$ is CAP-compact.

PROOF. Let $X$ be compact, and $S = (s_i)_{i \in \mu}$ be an infinite sequence of $X$. We show that $S$ has a complete accumulation point. Assume on the contrary, i.e., for every $x \in X$, there exists a neighborhood $V(x)$ such that

$$\{i : i \in \mu \text{ and } s_i \in V(x)\} < \bar{\mu}$$

Clearly, $X = \bigcup_{x \in X} V(x)$. But since $X$ is compact we have $X = \bigcup_{x \in F} V(x)$ for some finite subset $F$ of $X$. 
Thus \( S = \bigcup_{x \in F} (V(x) \cap S) \). That is:

\[
\bar{\mu} = \bigcup_{x \in F} (V(x) \cap S) \leq \sum_{x \in F} V(x) \cap S = \max_{x \in F} \{i : i \in \mu \text{ and } s_i \in V(x)\}
\]

which contradicts that:

\[
\{i : i \in \mu \text{ and } s_i \in V(x)\} < \bar{\mu} \quad \text{for every } x \in X
\]

Hence \( S \) has a \( \text{CAP} \) i.e., \( X \) is \( \text{CAP} \)-compact.

Conversely, Let \( X \) be \( \text{CAP} \)-compact and assume that \( X \) is not compact. Thus, there exists a cover \( C \) of \( X \) with no finite subcover.

Let \( S = \{s_i : i \in \overline{\alpha}\} \) be a subfamily of \( C \) with smallest cardinality such that \( X = \bigcup S \). That is, for any subfamily \( F \) of \( C \) with \( \overline{F} < \overline{S} \) it is the case that \( \bigcup F \neq X \). It is clear that \( \overline{S} \geq \aleph_0 \). Next, let \( S_i = \bigcup_{j<i} s_j \) for \( i \in \overline{S} \). If the \( S_i \)'s are not pairwise distinct let \( E_0 = S_0 \) and if \( i \) is the smallest ordinal such that:

\[
s_i = s_j \quad \text{for } i \leq j \text{ let } E_i = S_i \text{ and } E_{i+1} = S_{j+1}
\]

Thus, for some set \( A \) we have:

\[
\bigcup_{i \in \overline{A}} E_i = \bigcup_{j \in \overline{S}} S_j = X
\]

and the \( E_i \)'s are pairwise distinct. Moreover, \( \overline{A} = \overline{S} \) since \( S \) is a subfamily of \( C \) with smallest cardinality and covers \( X \).
Now, let \( R = \{ r_i : r_i \in E_{i+1} - E_i \text{ and } i \in \mathbb{S} \} \), then \( R \) is an infinite sequence of \( X \) and clearly, \( \overline{R} = \overline{\mathbb{S}} \). Since \( X \) is \( \mathcal{CAP} \)-compact, \( R \) has a complete accumulation point \( c \). Thus, \( c \in E_j \) for some \( j \in \mathbb{S} \). But

\[
\{ i : r_i \in R \cap E_j \} = \overline{j} \subset \overline{\mathbb{S}}
\]

which contradicts that \( c \) is a complete accumulation point of \( R \). Hence \( X \) is compact.

Next, based solely on the notion of Tower compactness, we consider the case of Tower compactness of the product topology, where each factor is Tower compact.

**THEOREM 3.4.** The product of two Tower compact topological spaces is Tower compact.

**PROOF.** Let \( T = \{ W_i : i \in I \} \) be a Tower of proper open sets \( W_i \) of the product topological space \( X_1 \times X_2 \), where \( X_1, X_2 \) are Tower compact topological spaces. For every \( x \in X_1 \), the subspace \( \{ x \} \times X_2 \) is a Tower compact, since it is homeomorphic to \( X_2 \). Assume on the contrary that:

\[
\bigcup_{i \in I} T = \bigcup_{i \in I} W_i = X_1 \times X_2
\]

We show then that the subspace \( \{ x \} \times X_2 \) is covered by an element, say, \( W_j \) of the Tower \( T \). Because otherwise, it is covered by \( T' = \{ \{ x \} \times X_2 \cap W_i : i \in I \} \).
which is a Tower in the subspace \( \{ x \} \times X_2 \). Thus \( \bigcup T' \subseteq \{ x \} \times X_2 \) since \( \{ x \} \times X_2 \) is Tower compact. Hence for every \( x \in X_1 \), the subspace \( \{ x \} \times X_2 \) is covered by an element \( W_j \) of \( T \). Now for every \( y \in X_2 \), there exist neighborhoods \( U_y(x) \) of \( x \) in \( X_1 \) and \( V(y) \) of \( y \) in \( X_2 \) such that

\[
U_y(x) \times V(y) \subseteq W_j
\]

(where in view of the above \( W_j \in T \) and contains \( \{ x \} \times X_2 \))

Thus:

\[
\{ x \} \times X_2 \subseteq \bigcup_{y \in X_2} U_y(x) \times V(y)
\]

If there exists a finite subset \( F_x = \{ y_1, \ldots, y_{m_x} \} \) of \( X_2 \) such that:

\[
\{ x \} \times X_2 \subseteq \bigcup_{y \in X_2} U_{y_k}(x) \times V(y_k)
\]

let \( \bigcap_{k=1}^{m_x} U_{y_k}(x) \). Then \( U \times X_2 \subseteq W_j \). However, if there exists no finite subset of \( X_2 \) such that the corresponding basic elements \( U_y(x) \times V(y) \) cover the subspace \( \{ x \} \times X_2 \), we let \( S_x \subseteq X_2 \) and \( S_x \) of smallest cardinality such that:

\[
\{ x \} \times X_2 \subseteq \bigcup_{y \in S_x} U_y(x) \times V(y)
\]

Clearly, \( \overline{S}_x \geq \aleph_0 \). Let:

\[
R_i' = \bigcup_{j<i} U_{y_j} \times V(y_j) \quad \text{with} \quad i \in \overline{S}_x \quad \text{and} \quad y_j \in S_x
\]

Let \( R_i = R_i' \cap \{ x \} \times X_2 \). Then \( R_i \) is a proper relatively open subset of \( \{ x \} \times X_2 \) and \( \bigcup_{i \in \overline{S}_x} R_i \supseteq \{ x \} \times X_2 \). However, \( \{ R_i \}_{i \in \overline{S}_x} \) is a Tower in \( \{ x \} \times X_2 \). Thus, it is the case that \( \bigcup_{i \in \overline{S}_x} R_i \subseteq \{ x \} \times X_2 \) since
\{x\} \times X_2 \text{ is Tower compact. Hence, we arrived at a contradiction and there is a finite subset } F_x = \{y_1, \ldots, y_{m_x}\} \text{ such that }

\{x\} \times X_2 \subseteq \bigcup_{k=1}^{m_x} U_{y_k}(x) \times V(y_k)

As in the above we have \( U_x \times X_2 \subseteq W_j \), where \( U_x \) is a neighborhood of \( x \) in \( X_1 \). Now \( X_1 \subseteq \bigcup_{x \in X_1} U_x \). Again if there is no finite subset \( F_1 = \{x_1, \ldots, x_n\} \) such that \( X_1 \subseteq \bigcup_{k=1}^{n} U_{x_k} \) we let \( S \subseteq X \) be of smallest cardinality such that \( X_1 \subseteq \bigcup_{z \in S} U_x \). Then for each \( i \in \overline{S} \), let \( S_i = \bigcup_{j<i} U_{x_j} \) with \( x_j \in S \).

Clearly, \( S_i \)'s are proper open subsets of \( X_1 \) and \( \bigcup_{i \in \overline{S}} S_i = X_1 \).

However, \((S_i)_{i \in \overline{S}}\) is Tower in \( X_1 \) and therefore, \( \bigcup_{i \in \overline{S}} S_i \subseteq X_1 \) since \( X_1 \) is Tower compact. This is a contradiction. Hence there exists a finite subset :

\[ F_1 = \{x_1, \ldots, x_n\} \text{ such that } X_1 = \bigcup_{k=1}^{n} U_{x_k} \]

Now \( X_1 \times X_2 = \bigcup_{k=1}^{n} U_{x_k} \times X_2 \). But \( U_{x_k} \times X_2 \subseteq W_{j_k} \), for some \( j_k \in I \). Let \( k \) be the maximum of \{\( j_1, \ldots, j_n \)\}. Then, \( X_1 \times X_2 \subseteq W_k \) which contradicts that \( W_k \) is a proper open subset of \( X_1 \times X_2 \). Thus, our assumption is false and the Theorem is proved.

Next, we show the \( \mathcal{CAP} \)-compactness of the topological product of two \( \mathcal{CAP} \)-compact spaces, solely based on the concept of \( \mathcal{CAP} \)-compactness.
THEOREM 3.5. The product of two CAP-compact topological spaces is CAP-compact.

PROOF. Let $I$ be an infinite ordinal and let $S = \{(s_{1i}, s_{2i}) : i \in I\}$ be a sequence in the product space $X_1 \times X_2$, where $X_1$ and $X_2$ are CAP-compact topological spaces. We show that $S$ has a complete accumulation point in the product topological space $X_1 \times X_2$.

Clearly, the projection $(s_{1i})_{i \in I}$ of $S$ on $X_1$ is an infinite sequence in $X_1$. Thus, since $X_1$ is CAP-compact the sequence $(s_{1i})_{i \in I}$ has a complete accumulation point $p$ in $X_1$.

Let us assume on the contrary that for every $x_2 \in X_2$ it is the case that $(p, x_2)$ is not a complete accumulation point of $S$. Hence, for every $x_2 \in X_2$ there exists a neighborhood $V(x_2)$ of $x_2$ in $X_2$ and a neighborhood $U_{x_2}(p)$ of $p$ in $X_1$ such that:

$$\{j : (s_{1j}, s_{2j}) \in U_{x_2}(p) \times V(x_2)\} < \aleph_0$$

Now, $X_2 = \bigcup_{x_2 \in X_2} V(x_2)$. Let $T = \{x_2 : x_2 \in X_2\}$ be a subset of the smallest cardinality such that $X_2 = \bigcup_{x_2 \in T} V(x_2)$.

Two cases may occur:

Case (i)

$$\aleph_0 < \aleph_0 \quad \text{say} \quad T = \{t_1, t_2, \ldots, t_m\}$$

Then $U = \bigcap_{k=1}^{m} U_{t_k}(p)$ is a neighborhood of $p$ in $X_1$ and $\{j : s_{1j} \in U\} = \aleph_0$. 


Hence:
\[ \{ j : (s_{1j}, s_{2j}) \in U \times X_2 \} = \overline{1} \]

But since \( U \times X_2 \subseteq \bigcup_{k=1}^{mp} U \times V(t_k) \) we have:
\[ \{ j : (s_{1j}, s_{2j}) \in U \times X_2 \} \leq \{ j : (s_{1j}, s_{2j}) \in \bigcup_{k=1}^{mp} U \times V(t_k) \} \leq \overline{\Sigma_{k=1}^{mp} \{ j : (s_{1j}, s_{2j}) \in U \times V(t_k) \}} < \overline{1} \]

which a contradiction

Case (ii)
\[ \overline{1} \geq \aleph_0 \]

Then for every \( j < \overline{1} \) let \( T_j = \bigcup_{k<j} V(t_k) \), we may assume that the \( T_j \)'s are pairwise distinct [See the proof of Theorem 3.3] Moreover, the \( T_j \)'s are proper open subsets of \( X_2 \), because of our choice (smallest) of the cardinality of \( T \). Now, let:
\[ R = \{ r_j : r_j \in T_{j+1} \setminus T_j \text{ and } j \in \overline{1} \} \]

Clearly, \( R \) is an infinite sequence of \( X_2 \) and \( \overline{R} = \overline{1} \). Since \( X_2 \) is \( C.A.P \)-compact, \( R \) has a complete accumulation point \( r \in X_2 \). Thus \( r \in T_j \) for some \( j \in \overline{1} \) because \( \bigcup T_j = \bigcup_{x_2 \in T} V(x_2) = X_2 \). But:
\[ \{ k : r_k \in R \cap T_j \} = j < \overline{1} \]

which contradicts that \( r \) is a complete accumulation point of \( R \). Hence, our assumption is false and therefore, there exists \( q \in X_2 \) such that \( (p, q) \) is a complete accumulation point of \( S \). Thus \( X_1 \times X_2 \) is \( C.A.P \)-compact.
4 INFINITE PRODUCT OF COMPACT SPACES AND
TRANSFINITE INDUCTION

In this section, we present novel methods of proofs of Tychonoff’s product
Theorem with respect to various definitions of compactness. Emphasis will be on
the use of Transfinite Induction.

In the existence literature there are two standard methods of proofs of
Tychonoff’s Theorem [10, p. 143] and [20, p. 256] and [21, p. 94] and [22,
p. 180].

However, both proofs are somewhat indirect. The first [10, p. 143] uses
Alexander’s Lemma, while the second [21, p. 94] and [22, p. 180] considers
the dual definition of compactness in terms of closed sets and then considers the
closure of the projections of closed sets. Here we give a most direct proof starting
with an open cover [cf. 23].

First, for sake of simplicity, we introduce a definition and prove two easy (but
essential) Lemmas.

DEFINITION 4.1. Let \( I \) be an ordinal and \( \{X_i\}_{i \in I} \) a family of
topological spaces. If $\mathcal{X} = \prod_{i \in I} X_i$ and $U$ is an open subset of the topological space $X_j$. Then:

$$E_j(U) = \{(x_i)_{i \in I} \in \mathcal{X} : x_j \in U\}$$

is called an elementary strip of type $j$ determined by $U$.

Clearly, an elementary strip is an open subset of $\mathcal{X}$.

**Lemma 4.1.** Let $\mathcal{X}$ be given as in Definition 4.1. Let

$$E = \{E_k(U_m) : k \in K \subseteq I \text{ and for every } m \in M_k \text{ (an index set)}$$

$U_m$ is an open set of $X_k\}$$

be a set of elementary strips $E_k(U_m)$ such that:

$$\bigcup_{k \in K, m \in M_k} E_k(U_m) = \mathcal{X}$$

Then there exists $k \in K$ such that $\mathcal{X} = \bigcup_{m \in M_k} E_k(U_m)$

(i.e., $\mathcal{X}$ is covered by one type of elementary strips)

**Proof.** Assume on the contrary. Then for every $j \in K$ there exists $(x_i)_{i \in I} \in \mathcal{X}$ such that $(x_i)_{i \in I} \not\in \bigcup_{m \in M_j} E_j(U_m)$. Thus, the set:

$$A_x = \{(y_i)_{i \in I} \in \mathcal{X} : y_j = x_j\} \not\subseteq \bigcup_{m \in M_j} E_j(U_m)$$
For every \( j \in K \) consider the set \( B_j = \{ x_j : A x_j \not\subset \bigcup_{m \in M_j} E_j(U_m) \} \). Then define

\[
Y = \prod_{i \in I} Y_i
\]

where

\[
Y_i = \begin{cases} 
B_j & \text{if } i \in K \\
X_i & \text{if } i \not\in K
\end{cases}
\]

By Axiom of Choice \( Y \) is nonempty since the \( Y_i \)'s are nonempty. Hence there exist \( z \in Y \subset \mathcal{X} \) and since for every \( j \in K \) we have \( z_j \in B_j \). Thus \( z \in A z_j \subset \bigcup_{m \in M_j} E_j(U_m) \) and, hence the Lemma is established •

**Lemma 4.2.** In the Definition 4.1, if the \( X_i \)'s are compact spaces and if \( \mathcal{X} \) is covered by a set \( \mathcal{E} \) of elementary strips, then \( \mathcal{X} \) is already covered by a finite number of elementary strips of \( \mathcal{E} \) which are all of the same type.

**Proof.** Without loss of generality we can take \( \mathcal{E} \) as in Lemma 4.1. Hence by Lemma 4.1, we have \( \mathcal{X} = \bigcup_{m \in M_k} E_k(U_m) \) where \( E_k(U_m) \in \mathcal{E} \). Clearly, \( \{U_m\}_{m \in M_k} \) is a cover of the compact space \( X_k \) and therefore, it has a finite subcover, say, \( X_k = \bigcup_{i=1}^n U_{m_i} \) with \( m_i \in M_k \). But, then it is clear that the subset

\[
E_n = \{ E_k(U_{m_i}) \in \mathcal{E} : 1 \leq i \leq n \text{ and } m_i \in M_k \}
\]

of \( \mathcal{E} \) is a finite cover of \( \mathcal{X} \), as claimed by the Lemma •

Now, we prove:
THEOREM 4.1. In the Definition 4.1 if the $X_i$'s are compact spaces, then $\mathcal{X} = \prod_{i \in I} X_i$ is compact with respect to the product topology.

PROOF. Let us assume on the contrary that $\mathcal{X}$ is not compact. Hence there exists an open cover $\mathcal{V}$ of $\mathcal{X}$ with no finite subcover. By Zorn's Lemma it can be readily verified that there is an open cover $\mathcal{M}$ of $\mathcal{X}$ such that $\mathcal{V} \subseteq \mathcal{M}$ and $\mathcal{M}$ is maximal with respect to the property of having no finite subcover. Because of the maximality of $\mathcal{M}$ it is clear that $\mathcal{M}$ has the following properties (see Theorem 2.1):

(i) If $S$ is an open subset of $\mathcal{X}$ such that $S \notin \mathcal{M}$, then the union of $S$ with a finite number of elements of $\mathcal{M}$ is equal to $\mathcal{X}$ and therefore, we have:

(ii) If $H \in \mathcal{M}$ and $B$ is an open set of $\mathcal{X}$ such that $B \subseteq H$ then $B \in \mathcal{M}$

(iii) If $\{E_1, \ldots, E_n\}$ is a finite set of open sets of $\mathcal{X}$ and if $\bigcap_{i=1}^{n} E_i \in \mathcal{M}$ then $E_i \in \mathcal{M}$ for some $1 \leq i \leq n$

Now, let $x \in \mathcal{X}$, then $x \in H \in \mathcal{M}$. Thus there exists (By the definition of the product topology) a finite set of indices $K \subseteq I$, which defines a basic open set $B$ of $\mathcal{X}$ given by

$$B = \{(x_i)_{i \in I} \in \mathcal{X} : x_i \in U_i \text{ if } i \in K \text{ and } U_i \text{ is an open set of } \mathcal{X}\}$$

Clearly, $x \in B \subseteq H \in \mathcal{M}$ which by (ii) implies that $B \in \mathcal{M}$. But, $B = \bigcap_{i \in K} E_i(U_i)$ where $E_i(U_i)$ is an elementary strip of type $i$ determine...
by $U_i \subset \mathcal{X}$. Using the property (iii) above, we have

$$E_i(U_i) \in \mathcal{M}$$

for some $i \in K$. Thus $x \in B \subseteq E_i(U_i) \in \mathcal{M}$ (i.e., every element $x \in X$ is covered by an elementary strip belonging to $\mathcal{M}$).

Consequently, by Lemma 4.2, we see that $X$ is already covered by a finite number of elements of $\mathcal{M}$. Which contradicts our assumption. Hence the Theorem is proved.

**REMARK 4.1.** Before proving Theorem 4.1 by transfinite induction, we prove the following Lemma as a motivation.

**LEMMA 4.3.** Let $C$ be a compact space and let $T$ be a topological space. Let $B$ be a cover of the product topological space $C \times T$ by a set of its basic elements. Then for every $t \in T$ there exists an elementary strip containing $C \times \{t\}$ which is covered by a finite number of the elements of $B$.

**PROOF.** Fix $t \in T$, then for every element $c \in C$ the point $(c, t)$ of the space $C \times T$ is covered by a basic element $U_c \times V_{ct}$ of $B$. Consider the set $\{U_c : c \in C\}$ which is a cover of $C$. Therefore, there exists a finite, say, $\{U_{c_1}, \ldots, U_{c_n}\}$ of $C$, since $C$ is compact. Obviously, the elementary strip

$$C \times \bigcap_{i=1}^{n} V_{c_i t}$$

contains the set $C \times \{t\}$ and is covered by the finite subset

$$\{U_{c_1} \times V_{c_1 t}, \ldots, U_{c_n} \times V_{c_n t}\}$$
of $B$. Hence, the Lemma is proved.

For the sake of simplicity, in what follows in the definition of $CAP$-compact space we consider infinite sets instead of infinite sequences.

**Lemma 4.4.** Let $T$ be a topological space and $C$ a $CAP$-compact space. Let $A$ be an infinite subset of the product space $C \times T$. Then there exists $c \in C$ such that for every open set $V(c)$ of $c$ in $C$ we have:

$$\overline{(V(c) \times T)} \cap A = \overline{A}$$

for every elementary strip $V(c) \times T$ of $C \times T$.

**Proof.** Assume on the contrary. Thus for every $x \in C$ there exists an elementary strip $V(x) \times T$ such that

$$\overline{(V(x) \times T)} \cap A < \overline{A}$$

Clearly, $\{V(x) : x \in C\}$ is a cover of $C$. Therefore, $C$ is covered by finitely many elements of the cover $\{V(x) : x \in C\}$, say, $\{V(x_1), \ldots, V(x_n)\}$ for some elements $x_1, \ldots, x_n \in C$. Hence

$$C \times T = \bigcup_{i=1}^{n} V(x_i) \times T$$

which implies that

$$\overline{A} = \bigcup_i (V(x_i) \times T) \cap A \leq \sum_{i=1}^{n} \overline{V(x_i) \times T) \cap A} < \overline{A}$$
which a contradiction •

Next, we use transfinite induction to prove:

**THEOREM 4.2.** Let $I$ be an ordinal and $\{X_i\}_{i \in I}$ a family of $CAP$-compact spaces. Then the cartesian product $\mathcal{X} = \prod_{i \in I} X_i$ is $CAP$-compact with the product topology defined on $\mathcal{X}$.

**PROOF.** Let $A$ be an infinite set of $\mathcal{X}$. We show that $A$ has a complete accumulation point $a = (a_i)_{i \in I}$ where $a_i \in X_i$ for every $i \in I$. To construct the coordinate $a_i$ of $a$, we choose $a_i$ in such a way that it has the following property:

$$(4.1) \text{ for every finite subset } F_i \text{ of ordinals } < i$$

and every $j \in F_i \cup \{i\}$ and for every neighborhood $V(a_j)$ of $a_j$

it is the case that

$$\overline{A \cap Y_i(a_i)} = \overline{A} \text{ with } Y_i(a_i) = \prod_{m \in I} Z_m$$

where

$$Z_m = \begin{cases} V(a_j) & \text{if } m \in F_i \cup \{i\} \\ X_m & \text{otherwise} \end{cases}$$

Now, if there is a point $a = (a_i)_{i \in I}$ in $\mathcal{X}$ whose coordinates $a_i$'s have the property (4.1), then it easy to verify that $a$ is a complete accumulation point of $A$. For, let $V$ be a basic neighborhood of $a$. Then from the way the product topology is defined $V = \prod_{m \in I} Z_m$ where $Z_m$ is a proper open subset of $X_m$ for at most finitely many $m$'s. Let $k$ be the maximum of
these finitely many $m$'s. But since $a_k$ satisfies the property (4.1) we have

$$A \cap V = A$$

as required.

We need to show that the existence of the coordinates $a_i$'s can be proved by transfinite induction. Assume on the contrary. Thus there exists a $k \in I$ (we can assume it the smallest) such that no point of $X_k$ have the property (4.1). That is, for every $x \in X_k$, there exists a neighborhood $U(x)$ of $x$ in $X_k$ and a finite set $F_x$ of indices $j < k$ and for every $j \in F_x$ there exists a neighborhood $V_x(a_j)$ of $a_j$ in $X_j$ such that

$$A \cap \bigcap_{x \in X_k} Y_k(x) < A$$

where $H(x) = \{t = (t_i)_{i \in I} \in \mathcal{X} : t_k \in U(x)\}$ and

$$Y_k(x) = \prod_{m \in I} Z_m \quad \text{with} \quad Z_m = \begin{cases} V_x(a_j) & \text{if } m \in F_x \\ X_m & \text{otherwise} \end{cases}$$

Now, $X_k$ is $\mathcal{CAP}$-compact. Hence there exists a finite subset, say, $\{x_1, \ldots, x_n\}$ of $X_k$ such that

$$\mathcal{X} = \bigcup_{i=1}^n H(x_n)$$

[Otherwise, we can construct an infinite subset of $X_k$ without complete accumulation point. See Theorem 3.3].

Hence we have:

$$\bigcup_{n=1}^m A \cap H(x_n) \cap Y_k(x_n) = \bigcup_{n=1}^m A \cap Y_k(x_n) \leq \sum_{n=1}^m \bigcap_{n=1}^m A \cap Y_k(x_n) < A$$
Let $F = \bigcup_{n=1}^{m} F_{x_n}$ and $Y_k = \bigcap_{n=1}^{m} Y_k(x_n)$. Clearly, $\overline{F} < \infty$ and if $p$ is the maximum of $F$, then $p < k$. However, $\overline{A \cap Y_k} < \overline{A}$ which contradicts that $a_p$ has the property (4.1). Thus the Theorem is proved.
5 COMPLETE ACCUMULATION POINTS IN $T^e$

In this section, we examine more closely the existence via "construction" of a complete accumulation point of an infinite subset of $T^e$ where $T$ is the closed real unit interval and $e$ is a nonzero ordinal (finite or infinite). The construction will pertain to $T^\omega$.

For the sake of simplicity we consider a complete accumulation point with respect to an infinite set and not with respect to an infinite sequence as introduced in Definition 3.3.

The case $e = 1$:
Let $S$ be an infinite subset of $T = [0,1]$. Clearly, there exists a digit $d_1$ and a corresponding subset $S_1$ of $S$ of the same cardinality as $S$ such that $S_1$ consists of all elements $x$ of $S$ with $d_1$ appearing as the first digit in the decimal representation of $x$, i.e.,

$S_1 = \{x : x \in S \text{ and } x = 0.d_1\cdots\}$ and $\overline{S_1} = \overline{S}$

This is because $S$ can be partitioned into 10 equivalence classes according to the (10 digits) of the first decimal digit in the decimal representation of the elements of $S$. 
In connection with (5.1) we make the following crucial observation

\[(5.2)\] The interval whose lowest vertex is \(0.d_1\)
and whose length is \(10^{-1}\) contains \(S_1\), i.e.,
\[S_1 \subseteq [0.d_1, 0.d_1 + 10^{-1}] \quad \text{and} \quad \overline{S_1} = \overline{S} \]

Similarly, there exists a digit \(d_2\) and a corresponding subset \(S_2\) of \(S_1\)
such that:
\[(5.3) \quad S_2 = \{x : x \in S_1 \quad \text{and} \quad x = 0.d_1d_2\ldots\} \quad \text{and} \quad \overline{S_2} = \overline{S_1} = \overline{S} \]

Again, in connection with (5.3) we make the following crucial remark:

\[(5.4)\] The interval whose lowest vertex is \(0.d_1d_2\)
and whose length is \(10^{-2}\) contains \(S_2\), i.e.,
\[S_1 \subseteq [0.d_1, 0.d_1 + 10^{-2}] \quad \text{and} \quad \overline{S_2} = \overline{S} \]

Continuing this process we construct a decimal representation of a real number \(c \in [0,1]\)

\[(5.5) \quad c = 0.d_1d_2\ldots d_n\ldots\]

The corresponding crucial remark:

\[(5.6) \quad \text{for every} \quad n \in \omega, \quad \text{the interval whose lowest vertex is} \quad 0.d_1d_2\ldots d_n \quad \text{and}
\text{whose length is} \quad 10^{-n} \quad \text{contains the subset} \quad S_n \quad \text{of} \quad S_{n-1}, \quad \text{i.e.,} \]
\[S_n \subseteq [0.d_1d_2\ldots d_n, 0.d_1d_2\ldots d_n + 10^{-n}] \quad \text{and} \quad \overline{S_n} = \overline{S} \]
Obviously, \( c \) given by (5.5) is a complete accumulation point of \( S \).

**The case \( e = 2 \):**

Let \( S \) be an infinite subset of \( I \times I \). Clearly, there exists a digit \( d_1 \) and the corresponding subset \( S_1 \) of \( S \) of the same cardinality as \( S \) such that:

\[
S_1 = \{(x, y) : (x, y) \in S \text{ and } x = 0.d_1 \cdots \} \text{ and } \overline{S_1} = \overline{S}
\]

In connection with (5.7) we have:

\[
(5.8) \quad \text{The box whose lowest vertex is } (0.d_1, 0) \text{ and whose dimensions are } 10^{-1} \text{ by } 1 \text{ contains } S_1. \text{ Also, there exists a digit } g_1 \text{ and a corresponding subset } S_{11} \text{ of } S_1 \text{ of the same cardinality as } S_1 \text{ such that:}
\]

\[
S_{11} = \{(x, y) : (x, y) \in S_1 \wedge x = 0.d_1 \cdots \wedge y = 0.g_1 \cdots \} \text{ and } \overline{S_{11}} = \overline{S}
\]

Again, in connection with (5.9) we have the following observation:

\[
(5.10) \quad \text{The box whose lowest vertex is } (0.d_1, 0.g_1) \text{ and whose dimensions are } 10^{-1} \text{ by } 10^{-1} \text{ contains } S_{11}.
\]

Continuing the digitwise construction, we construct the ordered pair \( c \in I \times I \)

\[
(5.11) \quad c = (0.d_1 d_2 \cdots d_p, 0.g_1 g_2 \cdots g_q)
\]

The crucial observation on (5.11) is that:

\[
(5.12) \quad \text{For any } p, q \in \omega \text{ the box whose lowest vertex is } (0.d_1 d_2 \cdots d_p, 0.g_1 g_2 \cdots g_q) \text{ and whose dimensions are } 10^{-p} \text{ by } 10^{-q} \text{ contains a subset } S_m \text{ (} m = \max \{p, q\} \text{ ) of cardinality } \overline{S}.
\]
Obviously, \( c \) given by (5.11) is a complete accumulation point of \( S \).

**The case \( e = \omega \):**

Let \( S \) be an infinite set of \( T^\omega \). To construct a complete accumulation point \( c \in T^\omega \) of \( S \) given by:

\[
(5.13) \quad c = (0.d_{11} \cdots d_{1n} \cdots, 0.d_{21} \cdots d_{2n} \cdots, \ldots)
\]

we use induction simultaneously on the digits of the first coordinates as well as the coordinates of \( c \).

Obviously, among the first digits of the first coordinate of elements of \( S \), there exists a digit \( d_{11} \) and a corresponding subset \( S_{11} \) of \( S \) of same cardinality as \( S \) such that:

\[
(5.14) \quad S_{11} = \{(x_1, \ldots, x_n, \ldots) : (x_1, \ldots, x_n, \ldots) \in S \text{ and } x_1 = 0.d_{11} \cdots\}
\]

Among the second digits of the first coordinate of the elements of \( S_{11} \), there exists a digit \( d_{12} \) and a corresponding subset \( S_{12} \) of \( S_{11} \) of the same cardinality as \( S_{11} \) such that:

\[
(5.15) \quad S_{12} = \{(x_1, \ldots, x_n, \ldots) : (x_1, \ldots, x_n, \ldots) \in S_{11} \text{ and } x_1 = 0.d_{11}d_{12} \cdots\}
\]

Also there exists a digit \( d_{21} \) and a corresponding subset \( S_{21} \) of \( S_{12} \) such that:

\[
(5.16) \quad S_{21} = \{(x_1, \ldots, x_n, \ldots) \in S_{12} : x_2 = 0.d_{21}, \cdots\} \text{ and } \overline{S_{21}} = \overline{S_{12}}
\]

In general, for any \( 0 < n \in \omega \), there exists a digit \( d_{1n} \) and a corresponding subset \( S_{1n} \) of \( S_{(n-1)1} \) such that:
(5.17) \[ S_{1n} = \{(x_1, \ldots, x_n, \ldots) \in S_{(n-1)1} : x_1 = 0.d_{11}d_{12} \cdots d_{1n} \cdots \} \]
and \[ \overline{S_{1n}} = \overline{S_{(n-1)1}} \]

Also, there exists a digit \( d_{2(n-1)} \) and a corresponding subset \( S_{2(n-1)} \) of \( S_{1n} \) such that :

(5.18) \[ S_{2(n-1)} = \{(x_1, \ldots, x_n, \ldots) \in S_{1n} : x_2 = 0.d_{21}d_{22} \cdots d_{2n-1} \cdots \} \]
and \[ \overline{S_{2(n-1)}} = \overline{S_{1n}} \]

Finally, there exists a digit \( d_{n1} \) and a corresponding subset \( S_{n1} \) of \( S_{(n-1)2} \) such that \[ \overline{S_{n1}} = \overline{S_{(n-1)2}} \]
and

(5.19) \[ S_{n1} = \{(x_1, \ldots, x_n, \ldots) \in S_{(n-1)2} : x_n = 0.d_{n1} \cdots \} \]

The crucial observation corresponding to \( n \in \omega \) is that the box with the lowest vertex \( v \in T^\omega \) given by :

(5.20) \[ v = (0.d_{11} \cdots d_{1n}, 0.d_{21} \cdots d_{2n-1}, \ldots, 0.d_{n1}, 0, 0, 0, \ldots) \]
and dimensions \( 10^{-n} \) by \( 10^{-n+1} \) by \( \cdots \) by \( 1 \) \( \cdots \) by \( 1 \) \( \cdots \) contains the subset \( S_{n1} \) of \( S \) with \[ \overline{S_{n1}} = \overline{S} \]

We claim that \( c \) given by (5.13) is a complete accumulation point of \( S \).

First, we remark that \( c \) has the following property :

(5.21) For any two finite subsets \( F_1 = \{n_1, \ldots, n_k\} \) and \( F_2 = \{j_1, \ldots, j_k\} \) of \( \omega \) the box with the lowest vertex \( v \) whose \( n_1, \ldots, n_k \) coordinates are the \( j_1, \ldots, j_k \) truncations of the \( n_1, \ldots, n_k \) coordinates of \( c \) and zero elsewhere, and with the corresponding dimensions \( 10^{-j_i} \) and \( 1 \) elsewhere with \( i = 1, \ldots, k \) contains a subset of \( S \) of cardinality \( \overline{S} \).
To prove (5.21) we observe the following:

The box given in (5.21) contains the box with lowest vertex

\[ v_m = (0.c_{11} \cdots c_{1m}, 0.c_{21} \cdots c_{2m-1}, \ldots, 0.c_{m1}, 0, 0, \ldots) \]

and corresponding dimensions

\[ 10^{-m} \text{ by } 10^{-m+1} \text{ by } \ldots \text{ by } 10^{-1} \text{ by } 1 \ldots \text{ by } 1 \ldots \]

where

\[ m = \max\{n_1 + j_1, \ldots, n_k + j_k\} \]. This is because for the coordinate \( n_i \in F \)

we have \( 10^{-(m-n_i)} \leq 10^{-j_i} \) and

\[ 0.a_{n_i1} \cdots a_{n_i(m-n_i)+1} + 10^{-(m-n_i)} \leq 0.a_{n_i1} \cdots a_{n_ij_i} \]

Now, for any neighborhood \( U \) of \( c \), there exists a basic neighborhood \( V \)
(In the product topology) such that \( c \in V \subseteq U \) and \( V = \prod_{i \in \omega} U_i \) where \( U_i \)
is an open subset of \( I \) and \( U_i \neq I \) for finitely many \( i \)'s, say, \( i_1, \ldots, i_n \).

Hence there exist truncations of the coordinates \( i_1, \ldots, i_n \) of \( c \) such that:

\[ (0.c_{i1} \cdots c_{ij}k_j, 0.c_{i1} \cdots c_{ij}k_j + 10^{-k_j}) \subseteq U_{i_j} \text{ for } j = 1, \ldots, n \]

Clearly, from (5.21) the box with the lowest vertex \( v \) whose \( i_1, \ldots, i_n \)
coordinates are the \( k_1, \ldots, k_n \) truncations of the \( i_1, \ldots, i_n \) coordinates of \( c \) and zero elsewhere, and, with the corresponding dimensions \( 10^{-k_j} \) and

1 elsewhere with \( j = 1, \ldots, n \) contains a subset of \( S \) with cardinality \( \aleph \).
Thus \( c \) is a complete accumulation point of \( S \).

The case \( c \) an infinite ordinal:

For this case the existence of a complete accumulation point \( c \) of an infinite
set \( S \) of \( I^\omega \) cannot be obtained by transfinite induction simultaneously on the
digits and the coordinates of \( c \). In fact, the existence of \( c \) can be secured
either by:

(i) Transfinite induction on the digits of the coordinates of \( c \).

OR

(ii) Transfinite induction on the coordinates of \( c \).

Method (i):
Assume the first digit was constructed for every coordinate \( i \) of a complete accumulation point \( c \) of an infinite set \( S \) of \( \mathcal{I}^\infty \), where \( i < \lambda \in e \). To construct the first digit of the \( \lambda \)th coordinate of \( c \) choose a digit so it has the following property:

For any finite set of coordinates \( i_1, \ldots, i_n \) where

\[ i_j \leq \lambda, \text{ for } j = 1, \ldots, n \]

the box with the lowest vertex

\[ v = (0, 0, \ldots, 0, c_{i_11}, 0, \ldots, 0, c_{i_21}, 0, \ldots, 0, c_{i_n1}, 0, \ldots) \]

and, of corresponding dimensions \( 10^{-1} \) and 1 elsewhere contains a subset of \( S \) of cardinality \( \overline{S} \).

If there is no first digit in the \( \lambda \)th coordinate of \( c \) satisfying the above property, then for every \( k = 0, \ldots, q \) there exists a finite set \( F_k \) of coordinates of \( c \) such that the corresponding box \( B_k \) contains a subset \( S_k \) with \( \overline{S_k} < \overline{S} \).

If \( m = \max \bigcup_{k=1}^q F_k \), then the box with vertex \( v_m \) whose coordinates are zero except possibly on the \( \bigcup F_k \) coordinates (where coordinates are given by the induction assumption) and, of corresponding dimensions contains a subset of \( S \) of cardinality less than that of \( S \). This contradicts the construction of the first digit in the \( m \)th coordinate.
In general, to construct the \( n \)th digit in the \( \lambda \)th coordinate of \( c \), choose a digit so that it has the following property:

For any finite set of coordinates, say, \( i_1, \ldots, i_m \) and any truncations (less than or equal to \( n \) if \( i_j < \lambda \) or less than or equal to \( n - 1 \) if \( i_j \geq \lambda \)) of the coordinates \( i_1, \ldots, i_m \) the box with vertex \( v \) whose coordinates are zero except possibly at the coordinates \( i_1, \ldots, i_m \) and, of corresponding dimensions contains a subset of \( S \) of cardinality \( \overline{\delta} \).

Again, if there is no \( n \)th digit in the \( \lambda \)th coordinate of \( c \) satisfying the above property, then as in the case of first digit construction, we can easily obtain a contradiction to the construction of the \( n \)th or the \((n - 1)\)th digit of some coordinate. Therefore, each coordinate of the point \( c \in \mathcal{I}^c \) is completely determined and in analogy to the case \( \epsilon = \omega \) it can be easily shown that \( c \) is in fact a complete accumulation point of the infinite set \( S \) of \( \mathcal{I}^c \).

Method (ii):

Assume all the digits of every coordinate \( i \) of \( c \) are completely determined, for \( i < \lambda \).

To construct the \( n \)th digit of the \( \lambda \)th coordinate of \( c \) choose a digit so that it has the following property:

For any finite set of coordinates, say, \( i_1, \ldots, i_m \) where \( i_j \leq \lambda \) for \( j = 1, \ldots, m \).
and, any truncations (less than or equal to \( n - 1 \) if \( i_j = \lambda \)) of the coordinates \( i_1, \ldots, i_m \) the box with vertex \( v \) given by

\[ v = (0, 0, \ldots, 0, 0.c_{i_11} \cdots c_{i_1k_1}, 0, \ldots, 0, 0.c_{i_21} \cdots c_{i_2k_2}, 0, \ldots, 0, 0.c_{i_m1} \cdots c_{i_mk_m}, 0, \ldots) \]

and, of corresponding dimensions \( 10^{-k_1}, \ldots, 10^{-k_m} \) elsewhere contains a subset \( S_{\lambda n} \) of \( S \) of cardinality \( \overline{S} \).

If there is no \( n \)-digit in the \( \lambda \)th coordinate of \( c \) satisfying the above property, then it is not difficult to verify that this would contradict the construction of the \((n - 1)\)-digit of the \( \lambda \)th coordinate. Hence all the digits of each coordinate of \( c \) are completely determined.

We show that \( c \) is in fact a complete accumulation point of the infinite set \( S \) of \( \mathcal{I}^e \). Let \( \mathcal{U} \) be any neighborhood of \( c \). Then there exists a basic neighborhood \( \mathcal{V} \) of \( c \) (in the product topology) such that \( c \in \mathcal{V} \subseteq \mathcal{U} \) and \( \mathcal{V} = \prod_{i \in I} U_i \) where \( U_i \) is an open subset of \( \mathcal{I} \) and \( U_i \neq \mathcal{I} \) for finitely many \( i \)'s, say, \( i_1, \ldots, i_n \). Hence there exist truncations of the coordinates \( i_1, \ldots, i_n \) of \( c \) such that:

\[ (0.c_{i_11} \cdots 0.c_{i_jk_j}, 0.c_{i_11} \cdots 0.c_{i_jk_j} + 10^{-k_j}) \subseteq U_{i_j} \text{ for } j = 1, \ldots, n \]

If \( m = \max\{i_1, \ldots, i_n\} \); say, \( m = i_3 \), then from the construction of the \( k_3 \)-digit of the \( i_3 \) coordinate of \( c \) the neighborhood \( \mathcal{V} \) contains the subset \( S_{i_3^i k_3} \) of \( S \) of cardinality \( \overline{S} \). Thus \( c \) is a complete accumulation point of \( S \).

For related ideas see [24 - 27].
As an application of the ideas related to $I^\omega$, we establish the solvability of an infinite system of equations each with at most a countable number of unknowns, provided that every finite subsystem is solvable.

First, we observe that if $\mathcal{F}$ is a continuous function from a topological space $X$ into the Reals $\mathbb{R}$, then the subset $\{x \in X : \mathcal{F}(x) = c\}$ is a closed subset of $X$.

**Lemma 5.1.** Let $\{f_i\}_{i \in \omega_\alpha}$ be a family of continuous functions $f_i$ from a compact space $X$ into the Reals $\mathbb{R}$. Then the system $(f_i(x) = c_i)_{i \in \omega_\alpha}$ of equations has a solution iff every finite subsystem has a solution.

**Proof.** Let $S_i$ be the set of all solutions of the equation $f_i(x) = c_i$. By the above observation $S_i$ is closed subset of $X$, for each $i \in \omega_\alpha$. Also by the hypothesis $\{S_i\}_{i \in \omega_\alpha}$ satisfies the finite intersection property. Hence $\bigcap_{i \in \omega_\alpha} S_i$ is nonempty, since $X$ is compact space. Therefore, the system $(f_i(x) = c_i)_{i \in \omega_\alpha}$ is solvable. The converse is obvious.

We recall that in a first-countable topological space $T$, the point $p$ is in the closure of a subset $A$ of $T$ iff there exists a sequence $\{a_i\}_{i \in \omega}$ in $A$ which converges to $p$. Also, the countable product of first-countable spaces is first-countable space [28, p. 191].

**Lemma 5.2.** Let $c$ be a real number and $\{a_i\}_{i \in \omega}$ be a sequence of real numbers such that $\sum_{i \in \omega} |a_i| < \infty$. Then the set $S$ of all solutions of
the equation:

\[(5.22) \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n + \cdots = c \quad \text{with} \quad x_i \in [-1,1] = A \]

is a closed subset of \(A^\omega\) (in the product topology).

**Proof.** Let \(x = (x_j)_{j \in \omega}\) be a limit point of \(S\) in \(A^\omega\). Since \(A^\omega\) is a countable product of a first-countable space \(A\), there exists a sequence \((x_{ij})_{(i,j) \in \omega \times \omega}\) in \(S\) which converges to \(x\) (in the product topology) which equivalent to the pointwise convergence of \((x_{ij})_{(i,j) \in \omega \times \omega}\) to \(x\). We claim that \(x\) is a solution of the equation (5.22). Notice:

\[
\sum_{j=m}^{n} a_j x_{ij} \leq \sum_{j=m}^{n} |a_j| \quad \text{for} \quad n \geq m \quad \text{and} \quad i \in \omega
\]

Let \(\epsilon > 0\) be given, then there exists an \(N \in \omega\) such that

\[
\sum_{j=N+1}^{\infty} |a_j| < \frac{\epsilon}{2}
\]

Hence

\[
|c - \sum_{j=1}^{N} a_j x_{ij}| < \frac{\epsilon}{2} \quad \text{for all} \quad i \in \omega
\]

But since

\[
\left(c - \sum_{j=1}^{N} a_j x_{ij}\right)_{i \in \omega} \text{ convergent to } \left(c - \sum_{j=1}^{N} a_j x_j\right)_{i \in \omega}
\]

we have for \(i > i_0\)

\[
|c - \sum_{j=1}^{N} a_j x_j| \leq \sum_{j=1}^{N} |a_j (x_{ij} - x_j)| + \left|c - \sum_{j=1}^{N} a_j x_{ij}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
Hence \( \sum_{j=1}^{\infty} a_j x_j = c \). Thus \( x = (x_j)_{j\in \omega} \) with \( x_j \in A = [-1,1] \), is a solution of the equation (5.22). Hence, \( S \) is closed subset of \( A^\omega \).

We observe that the proof of Lemma 5.2 did not use the continuity of the linear functional involved in (5.22) which is proved in the Lemma below:

**Lemma 5.3.** Let \( m > 0 \) be a real number and \( f \) be a function from the topological space \( M = [-m, m]^{\omega} \) into the Reals \( \mathbb{R} \) such that:

\[
(5.23) \quad f((x_i)_{i\in \omega}) = \sum_{i\in \omega} a_i x_i \quad \text{with} \quad x_i \in [-m, m] \quad \text{and} \quad \sum |a_i| = A < \infty
\]

Then \( f \) is continuous.

**Proof.** Since the space \( M \) is a first-countable space, it is enough to show that if a sequence \( ((x_{ij})_{i\in \omega})_{i\in \omega} \) in \( M \) converges to \( (x_j)_{j\in \omega} \), then \( f((x_{ij})_{i\in \omega}) \) converges to \( f((x_j)_{j\in \omega}) \) i.e.,

\[
\left( \sum_{j=1}^{\infty} a_j x_{ij} \right)_{i\in \omega} \quad \text{converges to} \quad \sum_{j=1}^{\infty} a_j x_j
\]

First, we notice that if \( (y_j)_{j\in \omega} \in M \) and \( \epsilon > 0 \) is given, then there exists \( J = J(\epsilon) \) such that:

\[
(5.24) \quad \left| \sum_{j=J+1}^{\infty} a_j y_j \right| \leq m \sum_{j=J+1}^{\infty} |a_j| < m \left( \frac{\epsilon}{2m} \right) = \frac{\epsilon}{2}
\]
Now, we claim that there exists an \( N \in \omega \) such that
\[
\left| \sum_{j=1}^{\infty} a_j(x_{n_j} - x_j) \right| < \varepsilon \quad \text{for all } n \geq N
\]

Let \( N = \max\{N_1, \ldots, N_J\} \), where for \( k = 1, \ldots, J \) \( N_k \) is chosen so that
\[
|x_{n_k} - x_k| < \frac{\varepsilon}{2^A} \quad \text{for all } n \geq N_k
\]

Clearly, we have the following:
\[
\left| \sum_{j=1}^{\infty} a_j(x_{n_j} - x_j) \right| \leq \left| \sum_{j=1}^{J} a_j(x_{n_j} - x_j) \right| + \left| \sum_{j=J}^{\infty} a_j(x_{n_j} - x_j) \right|
\]

Using (5.24) from the above we see that for all \( n \geq N \) it is the case that:
\[
\left| \sum_{j=1}^{\infty} a_j(x_{n_j} - x_j) \right| < \frac{\varepsilon}{2^A} \sum_{j=1}^{J} |a_j| + \frac{\varepsilon}{2}
\]

Therefore, \( \sum_{j=1}^{\infty} a_j(x_{n_j} - x_j) \) converges for all \( n \geq N \). Hence
\[
(\sum_{j=1}^{\infty} a_jx_{ij})_{i \in \omega} \text{ converges to } \sum_{j=1}^{\infty} a_jx_j
\]

**REMARK 5.1.** The continuity of the function \( f \) defined in (5.23) can be asserted in view of the following observation:

Since the linear functional \( f \) is an element of \( \ell^1 \) it follows that \( f \) is bounded. Hence \( f \) is continuous on every subset of \( \ell^\infty \) [29, p. 102].
LEMMA 5.4. Let $m > 0$ be a real number and $f$ be a function from the subspace (indicated below) of the topological space $\mathcal{M} = [-m, m]^\omega$ into the Reals $\mathbb{R}$ such that:

\[(5.25) \quad f((x_j)_{j \in \omega}) = \sum_{j \in \omega} a_j x_j \quad \text{with} \quad x_j \in [-m, m]\]

where $\sum |x_j| \leq m$ and $\lim_{i \to \infty} a_i = 0$. Then $f$ is continuous.

**Proof.** First, we observe that from (5.25) it follows that the domain of the function is closed subset of $\mathcal{M}$. But then as in Lemma 5.3, since $\mathcal{M}$ is first-countable space, it is enough to show if $((x_{ij})_{j \in \omega})_{i \in \omega}$ converges to $(x_j)_{j \in \omega}$ then:

\[\left(\sum_{j=1}^{\infty} a_j x_{ij}\right)_{i \in \omega} \quad \text{converges to} \quad \sum_{j=1}^{\infty} a_j x_j\]

Since $(a_i)_{i \in \omega}$ converges to zero it implies that for every $\epsilon > 0$, there exists $i_0 \in \omega$ such that $|a_i| < \frac{\epsilon}{4m}$ for all $i > i_0$. Let $N = \max\{N_1, \ldots, N_{i_0}\}$ where for $k = 1, \ldots, i_0$ the integer $N_k$ is chosen in such a way that for all $n \geq N_k$, we have:

\[|x_{ni} - x_i| < \frac{\epsilon}{2i_0 a} \quad \text{with} \quad a = \sup_{i \in \omega} |a_i|\]

Now, for all $n \geq N$, we then have:

\[\left| \sum_{j=1}^{\infty} a_j (x_{nj} - x_j) \right| \leq \left| \sum_{j=1}^{i_0} a_j (x_{nj} - x_j) \right| + \left| \sum_{j=i_0+1}^{\infty} a_j (x_{nj} - x_j) \right| < \]

\[\sum_{j=1}^{i_0} |a_j| \frac{\epsilon}{2i_0 a} + \frac{\epsilon}{4m} \sum_{j=i_0+1}^{\infty} |x_{nj} - x_j|\]
Which in view of \( \sum |x_j| \leq m \) implies \( \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \). Thus

\[
\left( \sum_{j=1}^{\infty} a_j x_{ij} \right)_{i \in \omega} \text{ converges to } \sum_{j=1}^{\infty} a_j x_j
\]

Therefore, \( f \) is a continuous function.

**REMARK 5.2.** Lemmas 5.3 and 5.4 established the continuity of the function \( f \) given by \( f((x_i)_{i \in \omega}) = \sum_{i \in \omega} a_i x_i \) for the following cases:

(i) \((a_i)_{i \in \omega} \in \ell^1 \) and \((x_i)_{i \in \omega} \in \ell^\infty \) with \( x_i \in [-m, m] \)

(ii) \((a_i)_{i \in \omega} \in c_0 \) and \((x_i)_{i \in \omega} \in \ell^1 \) with \( \sum |x_i| \leq m \)

where, \( c_0 \) the space of all scalar sequences that converges to 0 [30, p. 68].

We will establish the continuity of \( f \) for the case where \((a_i)_{i \in \omega} \in \ell^q \) and \((x_i)_{i \in \omega} \in \ell^p \) with \( q < 1 \) and \( \sum_{i \in \omega} |x_i|^p \leq m^p \) and \( \frac{1}{p} + \frac{1}{q} = 1 \)

**LEMMA 5.5.** Let \( m > 0 \) be a real number and \((a_i)_{i \in \omega} \in \ell^q \) with \( q > 1 \). Let \( f \) be a function from the subspace (indicated below) of the topological space \( M = [-m, m] \) into the Reals \( \mathbb{R} \) such that:

\[
(5.26) \quad f((x_i)_{i \in \omega}) = \sum_{i \in \omega} a_i x_i \quad \text{with} \quad x_i \in [-m, m]
\]

where \( \sum |x_i|^p \leq m^p \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).
Then \( f \) is continuous.

**PROOF.** As shown in Lemma 5.3, since the space \( M \) is first-countable, it is enough to show that if the sequence \( (x_{ij})_{j \in \omega} \) converges pointwise to \( (x_j)_{j \in \omega} \), then:

\[
\left( \sum_{j=1}^{\infty} a_j x_{ij} \right)_{i \in \omega} \text{ converges to } \sum_{j=1}^{\infty} a_j x_j
\]

We first note that the domain of the function \( f \) given in (5.26) is a closed subset of \( M \). This because \( |x_{ij}|^p \rightarrow |x_j|^p \) and the sequence of partial sums \( S_n = \sum_{j=1}^{n} |x_j|^p \) is bounded above by \( m^p \). Now, since \( (x_i)_{i \in \omega} \in \ell^q \), then it is the case that, given \( \epsilon > 0 \), there exists \( i_0 \in \omega \) such that:

\[
\left[ \sum_{i=i_0+1}^{\infty} |a_i|^q \right]^{\frac{1}{q}} < \frac{\epsilon}{4m}
\]

Hence by Hölder's inequality we have:

\[
\left| \sum_{i=i_0+1}^{\infty} a_i x_i \right| \leq \left[ \sum_{i=i_0+1}^{\infty} |a_i|^q \right]^{\frac{1}{q}} \left[ \sum_{i=i_0+1}^{\infty} |x_i|^p \right]^{\frac{1}{p}} < \frac{\epsilon}{4}
\]

Let \( N = \max\{N_1, \ldots, N_{i_0}\} \), where for \( k = 1, \ldots, i_0 \) the integer \( N_k \) is chosen in such a way that for all \( n \geq N_k \), it is the case that:

\[
|x_{ni} - x_j| < \frac{\epsilon}{2A} \text{ where } A = \sum_{i=1}^{i_0} |a_i|
\]

Thus, for all \( n \geq N \), we then have:

\[
\left| \sum_{j=1}^{\infty} a_j (x_{nj} - x_j) \right| \leq \left| \sum_{j=1}^{i_0} a_j (x_{nj} - x_j) \right| + \left| \sum_{j=i_0+1}^{\infty} a_j (x_{nj} - x_j) \right|
\]
The contents of the Lemmas 5.3, 5.4, 5.5 can be summarized as follows. Consider the $W \times \omega$ matrix $M$, where $W$ is an infinite ordinal and the vector $(a_i)_{i \in \omega}$, where the $a_i$'s are reals:

\[
< \sum_{j=1}^{i_0} |a_j| \frac{\epsilon}{2A} + \frac{\epsilon}{2} = \epsilon. \]

(1) \[
\begin{bmatrix}
  x_{11} & x_{12} & \cdots & x_{1j} & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  x_{i1} & x_{i2} & \cdots & x_{ij} & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\cdot
\begin{bmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_j \\
\end{bmatrix}
\cdot
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_j \\
\end{bmatrix}
\]

$W \times \omega$ \[ \omega \times 1 \]

Let $x_j$ be defined as:

(2) \[
x_j = \lim_{i \in W} x_{ij}
\]

We are interested in the existence of the inner product:

(3) \[
(x_1, x_2, \ldots, x_j, \ldots) \cdot
\begin{bmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_j \\
\end{bmatrix}
\]
and the existence of:

$\left( 4 \right) \quad \lim_{i \in W} c_i$

In fact, we are interested in the case where (3) exists and implies the existence of (4) and there equality. Let us observe that (3) can be interpreted as the sum of the limits of $N_0$ many convergent sequences:

$$\begin{pmatrix}
  x_{11} \\
  \vdots \\
  x_{i1}
\end{pmatrix} + a_1,

\begin{pmatrix}
  x_{12} \\
  \vdots \\
  x_{i2}
\end{pmatrix} = a_2, \ldots$$

Moreover, (4) can be interpreted as a possible limit of a sequence which is obtained by summing $N_0$ many convergent sequences appearing in (5).

Thus, we are interested in the question: "Under what conditions the sequence $S$ obtained by adding $N_0$ many sequences $S_i$ is convergent to the sum of the limits of the $S_i$'s".

The above question was answered in Lemmas 5.3, 5.4, 5.5.

Let us observe that the same question has an affirmative answer in the case of finitely many convergent sequences $S_i$ (instead of $N_0$ many convergent sequences) without any additional conditions.

As shown above the existence of (3) is secured in particular when:

(6) $\left( x_i \right)_{i \in \omega} \in \ell^p$ and $\left( a_i \right)_{i \in \omega} \in \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, allowing the case where $p = 1, q = \infty$ and vice versa.

Viewing $A = (a_1, a_2, \ldots)$ as a linear functional from a subset of $\mathbb{R}^\omega$. 
into the Reals $\mathbb{R}$, from (6) it follows that $A$ is a continuous function on $\ell^p$ in the $\ell^p$-norm (since $A$ is bounded). However, as (2) indicates we assume only the pointwise convergence of $((x_{ij})_{i \in \omega})$ in (1). Thus if we insist on treating $A$ as a linear operator from a subset of $\mathbb{R}^\omega$ in its product topology (and not necessarily in its norm topology defined on some subspace of $\mathbb{R}^\omega$).

The product (Tychonoff) topology was devised precisely to secure the convergence of the sequence $((x_{ij})_{i \in \omega})$ for every $j \in \omega$.

In fact, the following example shows that the product topology is the right topology in the context of the Lemmas 5.3, 5.4, 5.5:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
\frac{1}{2} \\
\frac{1}{3} \\
\vdots \\
0
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
\frac{1}{2} \\
\frac{1}{3} \\
\vdots \\
0
\end{bmatrix}
\]

(7)

We observe also that the existence of (3), the existence of (4) and their equality was established via the imposition of the following restrictions on the $x_i$'s and the $a_i$'s:

(i) $(x_i)_{i \in \omega} \in [-m, m]^\omega$, when $(a_i)_{i \in \omega}$

(ii) $\sum |x_i| \leq m$, when $\lim a_i = 0$

(iii) $\sum |x_i|^p \leq m^p$, when $(a_i)_{i \in \omega} \in \ell^q$
Below, we consider application of the Lemmas 5.3, 5.4, 5.5.

**THEOREM 5.1.** Let the infinite system (not necessarily countable) of linear equations in infinitely (countable) many unknowns \(x_1, x_2, \ldots\) be given as:

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1j} & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \ddots \\
  a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_i \\
  \vdots \\
\end{bmatrix}
= 
\begin{bmatrix}
  c_1 \\
  \vdots \\
  c_i \\
  \vdots \\
\end{bmatrix}
\]

\[W \times \omega \times \omega \times 1 \times W \times 1\]

where:

(9) The rows of the matrix \((W \times \omega)\) are elements of \(\ell^q\), and \(q > 1\)

(10) Each finite subsystem has a solution \(X\) as an element of \(\ell^p\), such that \(\|X\|_p \leq m\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Then the system (8) has a solution.

**PROOF.** First, we notice that \(\|X\|_p \leq m\) implies that \(X \in [-m, m]^\omega = \mathcal{M}\), which is a compact subset of \(\mathbb{R}^\omega\) in the product topology. Since \((a_{ij})_{j \in \omega} \in \ell^q\), by Lemma 5.5 we see that for each \(i \in W\) the function

\[
f_i((x_j)_{j \in \omega}) = \sum_{j \in \omega} a_{ij}x_j
\]

is continuous on the set \(T = \ell^p \cap \mathcal{M}\). Hence, the solution set \(S_i\) of the equation \(f_i((x_{ij})_{j \in \omega}) = c_i\) is a closed subset of \(T\). In view of (10) the family \((S_i)_{i \in W}\) satisfies the finite intersection property. Clearly, from the proof of Lemma 5.5 it follows that \(T\) is a closed subset of \(\mathcal{M}\). Thus, \(T\) is compact. Therefore, \(\bigcap_{i \in W} S_i\) is a nonempty subset of \(T\). Hence,
The system (8) has a solution, namely, any point in the intersection $\bigcap_{i \in W} S_i$.

**Theorem 5.2.** Consider the system (8) in Theorem 5.1, where:

1. \((a_{ij})_{j \in \omega} \in \ell^1\) for each \(i \in W\)
2. Each finite subsystem has a solution \(X\) as an element of \(\ell^\infty\), such that \(\|X\|_\infty \leq m\). Then the system (8) has a solution.

**Proof.** We invoke Lemma 5.3 and follow the step of the proof of Theorem 5.1.

**Theorem 5.3.** Consider the system of linear equations (8) in Theorem 5.1, where:

1. \(\lim_{j \in \omega} a_{ij} = 0\) for every \(i \in W\) and every finite subsystem of (8) has a solution \(X\) as an element of \(\ell^1\) such that \(\|X\|_1 \leq m\). Then the system (8) has a solution.

We invoke Lemma 5.4 and follow the steps of the proof of Theorem 5.1.

**Remark 5.3.** In Theorems 5.1, 5.2, 5.3, we assumed that the system (8) has the property that every finite subsystem of equations has a solution satisfying an appropriate condition.

Below, we give a necessary condition for the solvability of each finite subsystem of the system (8).
THEOREM 5.4. A necessary condition for the solvability in $\ell^p$, with solutions bounded by $m$ of the subsystem $(\sum_{j \in \omega} a_{i_k,j} x_j = c_{i_k})_{k \in n \in \omega}$ of the system (8) is:

$$
\left| \sum_{k \in n} r_k c_k \right| \leq m \left\| \left( \sum_{k \in n} r_k a_{i_k,j} \right)_{j \in \omega} \right\|_q
$$

for any finite set of real numbers $(r_k)_{k \in n}$ and $q \geq 1$, where

$$
\lim_{j \in \omega} a_{i_k,j} = 0 \quad \text{if} \quad q = \infty
$$

PROOF. Let $(r_k)_{k \in n}$ be any finite set of real numbers. Since $(a_{i_k,j})_{j \in \omega} \in \ell^q$ for $k \in n$, then it is obvious that $(\sum_{k \in n} r_k a_{i_k,j})_{j \in \omega} \in \ell^q$.

Let $(y_j)_{j \in \omega} \in \ell^p$ be a solution of the finite subsystem $(\sum_{j \in \omega} a_{i_k,j} x_j = c_{i_k})_{k \in n}$ with $\| (y_j)_{j \in \omega} \|_p \leq m$. By Hölder's inequality we have:

$$
\left| \sum_{j \in \omega} \left( \sum_{k \in n} r_k c_{i_k,j} \right) y_j \right| \leq m \left\| \left( \sum_{k \in n} r_k c_{i_k,j} \right)_{j \in \omega} \right\|_q
$$

Hence

$$
\left| \sum_{k \in n} r_k c_{i_k} \right| \leq m \left\| \left( \sum_{k \in n} r_k a_{i_k,j} \right)_{j \in \omega} \right\|_q
$$

Thus Theorem 5.4 is established.

In view of (14), we observe that the above necessary condition also implies the consistency of the subsystem mentioned in Theorem 5.4. This is because whenever $(\sum_{k \in n} r_k a_{i_k,j})_{j \in \omega} = 0$ then $(\sum_{k \in n} r_k c_{i_k}) = 0$.
Using Hahn-Banach Theorem [31, p. 102] it can be shown that the condition (15) is also a sufficient condition for the existence of the solution mentioned in Theorem 5.4 of the finite subsystem of the system (8).

Thus, in view of Theorems 5.1, 5.2, 5.3 we see that (14) is a necessary and sufficient for the system (8) to have m-bounded a solution under the given hypotheses. This can be done by taking the subspace $S$ generating by the rows of the $W \times \omega$ matrix of the system (8) and by defining the linear functional $f$ on $S$ as:

\[
\begin{align*}
(16) \quad f((a_{ij})_{j \in \omega}) &= c_i \quad \text{for each } i \in W
\end{align*}
\]

In view of (15) the function defined in (16) is bounded on $S$. Then by Hahn-Banach Theorem, there exists an extension of $f$ which can be represented as an element of $\ell^p$. Obviously, this extension is an $m$-bounded solution of the system (8).

For related ideas see [32, p. 196], [33, p. 15].
6 RECEDING SETS OF ORDINALS

In this section, we consider the properties and the consequences of Receding sets and in particular receding sets of ordinals. In a way, they are the duals of expanding sets of which we give some results to start with.

Let $T$ be a topological space. The following statements are pairwise equivalent:

(i) **Classical**: Every open cover of $T$ has a finite subcover.

(ii) **Cluster point**: Every net in $T$ has a cluster point.

(iii) **Tower**: No well-ordered (by $\subseteq$) set of proper open sets covers $T$.

First, we show (i) implies (ii):
Let $X = \{x_i \in T : i \in I\}$ be a net, where, $I$ is a directed set. Assume on the contrary, that $X$ has no cluster point in $T$. Hence, for every $t \in T$ there exists a neighborhood $U_t$ of $t$ and $\alpha_t \in I$ such that $x_j \notin U_t$ for all $j \geq \alpha_t$. Thus, $T = \bigcup_{t \in T} U_t$. Since $T$ is classical compact, we have $T = \bigcup_{i=1}^{n} U_{t_i}$. Let $\alpha$ be an upper bound of $\{\alpha_{t_1}, \ldots, \alpha_{t_n}\}$. Then, $x_\alpha \in \bigcup_{i=1}^{n} U_{t_i}$. Therefore, $x_\alpha \in U_{t_i}$ for some $i$ with $1 \leq i \leq n$, which contradicts that $\alpha \geq \alpha_{t_i}$. 
To show that (ii) implies (iii):

Let \( S = \{ S_i : i \in I \} \) be a well-ordered (by \( \subseteq \)) set of proper open subsets \( S_i \)'s. Assume on the contrary, that \( \bigcup S = T \). This implies that \( I \) is a limit ordinal. "Disjointify" \( S \) (by transfinite induction) as follows:

Let
\[
D_\mu = \left( \bigcup_{i < \mu} S_i \right) \cup \left( S_\mu - \bigcup_{i < \mu} S_i \right)
\]
for every \( \mu \in I \).

If \( S_\mu - \bigcup_{i < \mu} S_i = \emptyset \) then discard \( D_\mu \). It is easy to see that the well-ordered set \( D = \{ D_j : j \in J \} \) has the following properties:

1. \( D_j \subseteq S_j \)
2. \( \bigcup_{j \in J} D_j = \bigcup_{i \in I} S_i \)
3. no element \( D_j \) of \( D \) is the union of the preceding elements in \( D \).

From (1) and (2), we see that the well-ordered set \( D \) is a cover of \( T \) by proper open sets. Moreover, because of (3) above we can choose a point
\[
t_j \in D_j \text{ with } t_j \notin \bigcup_{i < j} D_i
\]
Then the set \( Y = \{ t_i : i \in J \} \) is a well-ordered set (by the well ordering of \( D \)). Since \( D \) is a cover of \( T \) by proper open sets, we see that \( J \) is a limit ordinal and by (ii) there exists \( t \in T \) which is a cluster point of \( Y \). Then \( t \in D_j \) for some \( j \in J \). Since \( D_j \) is an open set containing \( t \), we see that \( Y \) must be frequently in \( D_j \). But, this is a contradiction, because \( t_k \notin D_j \) for all \( k \geq j + 1 \).
Next, we show that (iii) implies (i):
Suppose \( \mathcal{A} = \{ A_i : i \in \omega \} \) is a cover of \( T \) with no finite subcover. Without loss of generality, we can assume that \( \mathcal{A} \) is a cover of the smallest cardinality with such property. Now, we construct the Tower \( D = \{ D_i : i \in \omega \} \) defined recursively as follows:

\[
D_0 = A_0
\]

and

\[
D_\mu = \left( \bigcup_{i < \mu} D_i \right) \cup \left( A_\mu - \bigcup_{i < \mu} D_i \right)
\]

The family \( D \) has the following properties:

1. If \( D_j \in D \), then \( D_j \) is a proper open subset, because of the smallest cardinality of \( \mathcal{A} \).
2. \( D \) is a well-ordered set (by \( \subseteq \))
3. \( \bigcup D = \bigcup \mathcal{A} \)

Now, since \( \mathcal{A} \) is a cover of \( T \) in view of (3) above we see that \( D \) is a cover of \( T \) which a contradiction to (ii).

We notice that (ii) can be replaced by the following Theorem due to (A. Abian)

**Theorem** (Abian). A topological space \( T \) is compact iff every well-ordered subset of \( T \) has a cluster point.

**Proof.** In proving (ii) implies (iii) above by assuming the negation of (iii) we constructed in fact a well-ordered set with no cluster point. This contradicts
the hypothesis of the Theorem. Hence, if every well-ordered (by $\subseteq$) set of proper open sets of $T$ covers $T$ The converse clearly holds, because of the equivalence of (i), (ii) and (iii) $\bullet$.

For the sake of completeness, it is also interesting to notice that the Tower definition of compactness directly implies the complete accumulation point definition of compactness.

**THEOREM.** In a topological space $T$ the following statements are equivalent:

(I) No well-ordered (by $\subseteq$) set of proper open sets of $T$ covers $T$.

(II) Every infinite subset of $T$ has a complete accumulation point.

**PROOF.** We show (I) implies (II):

Let $S$ be an infinite set with no complete accumulation point in $T$. Hence, for every $t \in T$, there exists a neighborhood $U_t$ of $t$ such that

$$\overline{U_t} \cap S < S$$

Clearly, $\mathcal{A} = \{U_t : t \in T \text{ and } \overline{U_t} \cap S < S\}$ is a cover of $T$ with no finite subcover. This is because no finite subfamily of $\mathcal{A}$ will cover $S$. Without loss of generality, we can assume $\mathcal{A}$ is of smallest cardinality with such property.

Now, the rest of the proof is the same as proving (iii) implies (i) for the converse see Theorem 3.3 $\bullet$. 
Next, we consider the questions related to the receding sequences of sets of ordinals.

Let \( \min S \) denote the minimum of the nonempty set \( S \) of ordinals.

**DEFINITION 6.1.** Let \( \kappa \) be an infinite cardinal. for every \( i < \kappa \) let \( S_i \) be a nonempty subset of \( \kappa \). The sequence \( (S_i)_{i<\kappa} \) is called receding if and only if for every element \( i, j, v > 0 \) of \( \kappa \)

\[
\begin{align*}
(6.1) & \quad i < j \implies \min S_i < \min S_j \\
(6.2) & \quad i < j \implies S_i \supset S_j \\
(6.3) & \quad \cap_{i<v} S_i = S_w \text{ for some } w \in \kappa
\end{align*}
\]

Based on the above definition we prove:

**LEMMA 6.1.** Let \( (S_i)_{i<\kappa} \) be a receding sequence. Let \( c < \kappa \). If \( c \in S_i \) for some \( i < \kappa \) then:

\[
\cap \{ S_i : c \in S_i \} = S_w \text{ for some } w \leq c
\]

**PROOF.** First, we claim that \( c \leq \min S_c \) for every \( c < \kappa \). Assume on the contrary and let \( c \) be the smallest ordinal such that \( c > \min S_c = m \).

Since \( m < c \), then by (6.1) above we have \( \min S_m < \min S_c = m \) which contradicts that \( c \) is the smallest with such property. Hence, \( c \leq \min S_c \) for every \( c \), in particular from (6.1) we can see that \( c \not\in S_{c+1} \) and if \( x \in S_i \), then \( x \geq i \) because \( x \geq \min S_i \geq i \).

From \( c \not\in S_{c+1} \) it follows that \( \cap_{i<\kappa} S_i = 0 \). Now, let \( v \) be the smallest ordinal such that \( c \not\in S_v \). Clearly, \( v \leq c+1 < \kappa \), moreover, if \( i < v \)
then \( c \in S_i \).

Conversely, because of (6.2) above, if \( c \in S_i \), then \( i < v \) but then (6.3) implies (6.4).

**Lemma 6.2.** Let \((S_i)_{i < \kappa}\) be a receding sequence, then (6.3) is equivalent to the following condition:

\[
\bigcap_{i < v} S_i = \begin{cases} 
  S_{v-1} & \text{if } v \text{ is a nonlimit ordinal} \\
  S_v & \text{if } v \text{ is a limit ordinal}
\end{cases}
\]

**Proof.** Assume (6.3) and let \( v \) be a nonlimit ordinal, then \( v \) has an immediate predecessor \((v-1)\). For \( i \leq v-1 < v \) by (6.2) \( S_{v-1} \subseteq S_i \) which implies \( S_{v-1} \subseteq \bigcap_{i < v} S_i \). Moreover, since \( v-1 < v \) we have \( \bigcap_{i < v} S_i \subseteq S_{v-1} \). Hence, \( \bigcap_{i < v} S_i = S_{v-1} \). Next, let \( v \) be a limit ordinal. Clearly, \( S_v \subseteq S_i \) for every \( i < v \). Thus, \( S_v \subseteq \bigcap_{i < v} S_i \). But, then by (6.2) it is the case that \( w < v \). Since \( v \) is a limit ordinal we have \( w + 1 < v \) which implies \( S_{w+1} \supseteq \bigcap_{i < v} S_i = S_w \) which contradicts (6.2). Thus, \( \bigcap_{i < v} S_i = S_v \).

Conversely, Assume (6.5) holds, then (6.3) holds obviously.

We see that (6.3) can be replaced by (6.5).

**Lemma 6.3.** Let \((S_i)_{i < \kappa}\) be a receding sequence. Let \( f \) be a
mapping from $\kappa$ into $\kappa$ such that

\[
(6.6) \quad f(x) = \begin{cases} 
0 & \text{if } x \notin S_i \text{ for every } i < \kappa \\
\min \cap \{S_i : x \in S_i\} & \text{otherwise}
\end{cases}
\]

Then $f(0) = 0$. Moreover, $c > 0$ is a fixed point of $f$ if and only if $c = \min S_w$ for some $w < \kappa$.

**Proof.** It is clear from (6.6) that $f(x) \leq x$ for every $x < \kappa$. Therefore, $f(0) = 0$. Moreover, let $c = \min S_w$. First, we notice that if $i \neq w$ and $c \in S_i$ then $i < w$, for if $w < i$ then by (6.1) we see that $\min S_w < \min S_i$ which contradicts that $c \in S_i$. Hence, $i < w$ and by (6.2), it is the case that $S_w \subset S_i$. Therefore, $\cap \{S_i : c \in S_i\} = S_w$ which by (6.6) we have

\[
f(c) = \min \cap \{S_i : c \in S_i\} = \min S_w = c
\]

Thus, indeed $c$ is a fixed point of $f$.

Conversely, let $f(c) = c > 0$, then from (6.6) it follows that $c \in S_i$ for some $i < \kappa$. Hence, $\{S_i : c \in S_i\} \neq \emptyset$ then from (6.4) we have $\cap \{S_i : c \in S_i\} = S_w$ for some $w < \kappa$. But, then from (6.6) it follows that $f(c) = \min S_w$. Thus the Lemma is proved.

Next, we recapitulate some of the principal properties of a receding sequence $(S_i)_{i<\kappa}$ of sets $S_i$'s of of an ordinal number. First, $\cap_{i<\kappa} S_i$ is empty, because for every $i < \kappa$ it is the case that $i \notin S_{i+1}$. Moreover, by (6.2) we have $S_i \supset S_j \neq \emptyset$ for every $i, j$ with $i < j < \kappa$. Hence, it is easy to see...
that each $S_i$ is cofinal to $\kappa$. Also, we notice that the set of all fixed points of the function $f$ given by (6.6) is cofinal to the cardinal $\kappa$ because for every $v < \kappa$ we see that $v \leq \min S_v = \min \bigcap_{i < v+1} S_i$, i.e.,

$$S_v = \bigcap_{i < v+1} S_i \quad \text{for every } v < \kappa$$

**DEFINITION 6.2.** We recall [34, p. 30] that a cardinal $\kappa > \omega$ is called (two-valued) measurable iff there exists a $\kappa$-complete nonprincipal ultrafilter $U$ in the field $2^\kappa$ of all subsets of $\kappa$. Thus, $U$ is a nonprincipal ultrafilter of $2^\kappa$ such that for every subset $S$ of $U$ if:

$$\overline{S} < \kappa \quad \text{then} \quad \bigcap S \in U$$

[cf. 37].

**DEFINITION 6.3.** Let $\kappa$ be a measurable cardinal [34, p. 33] A $\kappa$-complete nonprincipal ultrafilter $D$ of $2^\kappa$ is called a strongly $\kappa$-complete nonprincipal ultrafilter of $2^\kappa$ iff for every mapping $f$ from $\kappa$ into $\kappa$ we have:

$$\{x : f(x) < x\} \in D \quad \text{implies there exists a } t \in \kappa \quad \text{such that}$$

$$\{x : f(x) = t\} \in D$$

Based on the above definitions we prove

**LEMMA 6.4.** Let $\kappa$ be a measurable cardinal and $D$ be a strongly $\kappa$-complete nonprincipal ultrafilter of $2^\kappa$. Let $f$ be a mapping from $\kappa$ into $\kappa$
such that

$$A = \{x : f(x) \leq x\} \in \mathcal{D}$$

If

$$\{x : f(x) = t\} \notin \mathcal{D} \quad \text{for every} \quad t < \kappa$$

then

$$\{x : f(x) = x\} \in \mathcal{D}$$

**PROOF.** Since $$\{x : f(x) < x\} \cup \{x : f(x) = x\} = A \in \mathcal{D}$$ and

$$\{x : f(x) < x\} \cap \{x : f(x) = x\} = \emptyset$$

then

$$\{x : f(x) < x\} \notin \mathcal{D}$$

Because of the hypothesis and the fact that $$\mathcal{D}$$ is strongly $$\kappa$$-complete nonprincipal ultrafilter we have

$$\{x : f(x) = x\} \in \mathcal{D}$$

It is well known [34, p. 33] that if $$\kappa$$ is a measurable cardinal then there exists a strongly $$\kappa$$-complete nonprincipal ultrafilter in the field $$2^\kappa$$ of all subsets of $$\kappa$$ [cf. 38].

**THEOREM 6.1.** Let $$\kappa$$ be a measurable cardinal and $$\mathcal{D}$$ be a strongly $$\kappa$$-complete nonprincipal ultrafilter of $$2^\kappa$$. Let $$(S_i)_{i \in \kappa}$$ be a receding sequence.
such that:

(6.8) \( S_i \in \mathcal{D} \) for every \( i < \kappa \)

then

(6.9) \( \{ \min S_i : i \in \kappa \} \in \mathcal{D} \)

PROOF. Consider the mapping \( f \) from \( \kappa \) into \( \kappa \) as given by (6.6) Since \( f(x) \leq x \) for every \( x < \kappa \) we have

\[ \{ x : f(x) \leq x \} = \kappa \in \mathcal{D} \]

Now, we claim that

(6.10) \( \{ x : f(x) = t \} \notin \mathcal{D} \) for every \( t < \kappa \)

Assume on the contrary. Then the subset:

(6.11) \( H = \{ x : f(x) = h \} \in \mathcal{D} \) for some \( 0 < h < \kappa \)

Let \( x \in H \), then by (6.6) we have \( h = \min \cap \{ S_i : x \in S_i \} \) and by (6.4) it is the case that \( h = \min S_w \), for some \( w < \kappa \). Moreover, \( x \notin S_{w+1} \), because otherwise by (6.1), (6.2) it would follow that \( f(x) = \min S_{w+1} > h \). In short, if \( x \in H \), \( x \notin S_{w+1} \). Hence, \( H \cap S_{w+1} = \emptyset \) which contradicts that both \( H \) and \( S_{w+1} \) are elements of the ultrafilter \( \mathcal{D} \).

On the other hand, if \( h = 0 \), then either \( H \) is contained in \( S_0 \) but not in any \( S_i \) with \( o < i < \kappa \) (because \( 0 \notin S_i \) for all \( i > 0 \)) or else, \( H \cap S_i = \emptyset \) for every \( i < \kappa \). Thus, either of the two cases leads to a contradiction as in the above. Hence, our claim is established. But then by Lemma 6.4 the set \( \{ x : f(x) = x \} \) is an element of \( \mathcal{D} \). Moreover, by Lemma 6.3 the set of fixed
points of the mapping \( f \) mentioned in (6.6) is equal to \( \{ \min S_i : i < \kappa \} \)

Thus, the Theorem is proved.

As an application of the notion of receding sequences we give a novel proof of Ramsey's Theorem.

First, we introduce some notations and recall some definitions.

A partition of a set \( B \) is a pairwise disjoint family \( F = \{ X_i : i \in I \} \) such that \( \bigcup F = B \).

For any set \( B \) and any natural number \( n > 0 \)

\[ [B]^n = \{ X \subseteq B : |X| = n \} \]

is the set of all subsets of \( B \) that have exactly \( n \) elements.

**DEFINITION 6.3.** If \( \{ X_i : i \in I \} \) is a partition of \( [B]^n \) then a set \( H \subseteq B \) is homogeneous for the partition if for some \( i \in I \), \( [H]^n \subseteq X_i \).

**THEOREM 6.2.** Let \( n \) and \( k \) be natural numbers. Every partition \( \{X_1, \ldots, X_k\} \) of \( [\omega]^n \) into \( k \) pieces has an infinite homogeneous set.

**PROOF.** Clearly, the Theorem is true for \( n = 1 \).

Assume that the statement of the Theorem holds for \( n \) then we prove it for \( n+1 \).

Let \( \{X_1, \ldots, X_k\} \) be a partition of \( [\omega]^{n+1} \). \( [\omega \setminus \{0\}]^n \) be partition into \( k \) pieces in the following sense:
If \( x \in [\omega \setminus \{0\}]^n \) then \( x \in X_i \) iff \( x \cup \{0\} \in X_i \). But then by the induction hypothesis there exists an infinite set \( H_m \subseteq \omega \setminus \{0\} \) such that \([H_m \cup \{0\}] \subseteq X_i \) for some \( i \) with \( 1 \leq i \leq k \).

Let \( m_1 \) be the smallest element of \( H_m \). It is obvious that \( m_1 > m_0 = 0 \).

Let \( x \in [H_m \setminus \{m_1\}]^n \) then \( x \in X_i \) iff \( x \cup \{m_1\} \in X_i \). By the induction hypothesis there exists an infinite set \( H_1 \subseteq H_m \setminus \{m_1\} \) such that \([H_1] \subseteq X_i \) for some \( i \) with \( 1 \leq i \leq k \).

Let \( m_2 \) be the smallest element of \( H_1 \). Clearly, \( m_2 > m_1 > m_0 = 0 \).

Let \( M = \{m_i : i \in \omega \} \) where \( m_{i+1} \) is the smallest element of \( H_{m_i} \). Let \([M]^1 \) be partitioned into \( k \) pieces in the following sense:

\[
m_j \in X_i \iff [H_{m_j}] \subseteq X_i \text{ for some } i, 1 \leq i \leq k
\]

Therefore, there exists an infinite set \( \mathcal{H} \subseteq M \) such that \([\mathcal{H}]^1 \subseteq X_i \) for some \( t \) with \( 1 \leq t \leq k \).

We claim that \( \mathcal{H} \) is the desired homogeneous set for the partition \( \{X_1, \ldots, X_k\} \) of \([\omega]^{n+1} \). Let \( X = \{m_{i_0}, \ldots, m_{i_n}\} \in [\mathcal{H}]^{n+1} \) and assume \( m_{i_0} \) is the minimum of \( X \). Hence, \( m_{ij} \in H_{m_{i_0}} \) for \( j = 1, \ldots, n \) and we have \( \{m_{i_1}, \ldots, m_{i_n}\} \cup \{m_{i_0}\} \subseteq X_t \). Thus, the Theorem is proved •
We prove the following generalization of Theorem (6.2). In what follows \([S]^2\) shall denote the set of all 2-elements subsets of \(S\).

**THEOREM 6.3.** Let \(\kappa\) be a measurable cardinal and \(D\) a strongly \(\kappa\)-complete nonprincipal ultrafilter of \(2^\kappa\). Let \([\kappa]^2\) be partitioned into two (mutually disjoint) parts \(A\) and \(B\). Then there exists a subset \(H\) of \(\kappa\) such that \(H \in D\) and either \([H]^2 \subseteq A\) or else \([H]^2 \subseteq B\).

(We observe that \(H \in D\) implies \(\bar{H} = \kappa\))

**PROOF.** For elements \(\mu\) and \(\nu\) of \(\kappa\) we let

\[(6.12) \quad (\mu, S\nu, M) \text{ with } M \in \{A, B\}\]

denote the fact that:

\[(6.13) \quad S\nu \in D \text{ and } \{\{x, \mu\} : x \in S\nu\} \subseteq M\]

Triplets such as the given in (6.12) exist. For instance, let us consider the set \(S\) given by:

\[S = \{\{x, 0\} : x \in \kappa \setminus \{0\}\}\]

Clearly, by virtue of the partition \(\{A, B\}\) of \([\kappa]^2\), the set \(S\) is also partitioned into two corresponding parts which induce the obvious partition of \(\{x : \{x, 0\} \in S\}\) into two parts. However, since \(\kappa \setminus \{0\} \in D\), one and only one of the two parts, say, \(S_0\) of \(\{x : \{x, 0\} \in S\}\) must be an element of \(D\). But then either \(\{x, 0\} : x \in S_0\} \subseteq A\) or else \(\{x, 0\} : x \in S_0\} \subseteq B\). If, say, \(\{x, 0\} : x \in S_0\} \subseteq B\) then the triplet \((0, S_0, B)\) exists which satisfies (6.13) i.e., \(S_0 \in D\) and \(\{x, 0\} : x \in S_0\} \subseteq B\).

Again, starting with \(\min S_0\) instead of 0 and \(S_0\) instead of \(\kappa\) we
derive the existence of the triplet, say, \((\min S_0, S_1, A)\) with \(S_1 \subseteq S_0\) and \(\{x, \min S_0\} : x \in S_1 \subseteq A\). Since \(\mathcal{D}\) is a \(\kappa\)-complete ultrafilter, from the above it follows that for every \(i < \kappa\) there exists a triplet,

\[
(6.14) \quad (\min_{j < i} S_j, S_i, M) \quad \text{with} \quad M \in \{A, B\} \quad \text{such that} \quad \bigcap_{j < 0} S_j = \kappa
\]

and

\[
(6.15) \quad S_i = \{x : x \in \bigcap_{j < i} S_j \quad \text{and} \quad \{x, \min_{j < i} S_j\} \in M\} \in \mathcal{D}
\]

From (6.15) it follows that \((S_i)_{i < \kappa}\) is a receding sequence satisfying the hypothesis of Theorem 6.1 namely, (6.8). Hence in view of (6.10) we see that \(\{\min S_i : i < \kappa\} \in \mathcal{D}\). But then, as (6.14) shows, the partition \(\{A, B\}\) induces the obvious partition of \(\{\min S_i : i \in \kappa\}\) into two parts. However, since \(\{\min S_i : i < \kappa\} \in \mathcal{D}\) one and only one of the two parts, say, \(\mathcal{H}\) of \(\{\min S_i : i < \kappa\}\) must be an element of \(\mathcal{D}\). But then either \([\mathcal{H}]^2 \subseteq A\) or else \([\mathcal{H}]^2 \subseteq B\), as desired. Thus, the Theorem is proved.

**REMARK 6.1.** We recapitulate the essential points of the proof of Theorem 6.3. Let \(\kappa\) be a measurable cardinal and \(\mathcal{D}\) be a strongly \(\kappa\)-complete nonprincipal ultrafilter in the field \(2^\kappa\) of all subsets of \(\kappa\). Let \(\{A, B\}\) indicate a partition of \([\kappa]^{\leq}\).

We constructed a receding sequence \((S_i)_{i < \kappa}\) of subsets of \(\kappa\) based on
\{A, B\} as follows:

(i) \{\{x, m_i\}; x \in \bigcap_{j<i} S_j \text{ and } m_i = \min \bigcap_{j<i} S_j \text{ and } x \neq m_i\}

is completely a subset of A or completely a subset of B

(ii) \quad S_i \in \mathcal{D}\quad \text{for every} \quad i < \kappa

(iii) From (ii) and by Theorem 6.1 we have:

\[ M = \{ \min S_i : i < \kappa \} \in \mathcal{D} \]

Moreover, by (i) we see that M is partitioned by \{A, B\}. The subset of M which is in \mathcal{D} is the desired homogeneous subset of \kappa. From the above we see that the measurability of \kappa is used to ensure that \bigcap_{i<v} S_i \in \mathcal{D}, for \quad v \in \kappa.

Moreover, since the elements of the receding sequence are in \mathcal{D}, the set of minimums of the elements of the receding sequence is also in \mathcal{D}. We observe that the existence of a \kappa-complete nonprincipal ultrafilter is enough to ensure the existence of \kappa \rightarrow [\kappa]^2 (i.e., The existence of a homogeneous subset of \kappa of cardinality \kappa for the partition \{A, B\}) without implying that the homogeneous subset of \kappa is in ultrafilter.

We show below that for the existence of \kappa \rightarrow [\kappa]^2 it is not necessary that \kappa be a measurable cardinal. In fact, for the existence of \kappa \rightarrow [\kappa]^2 it is necessary and sufficient that \kappa be a Hausdorff cardinal (i.e., weakly compact cardinal, i.e., strongly inaccessible ramifiable cardinal) as defined below [cf. 40–42].
In this section, motivated by the notion of a ramifiable cardinal we define a $\mathcal{CAC}$-cardinal and show that such cardinals $> \aleph_0$ exist under the assumption of Martin's axiom.

Let $x$ be an element of a partially ordered set $(\mathcal{P}, \leq)$. Then we denote by $I(x)$ the initial segment of $\mathcal{P}$ determined by $x$, i.e., the set of all the predecessors of $x$ in $\mathcal{P}$. Moreover, if $I(x)$ is a well-ordered subset of $\mathcal{P}$ then we denote by $\text{rank}(x)$ the unique ordinal similar to $I(x)$.

Accordingly, we have:

$$I(x) = \{y : y \in \mathcal{P} \text{ and } y < x\} \text{ for every } x \in \mathcal{P}$$

We recall that a partially ordered set $(\mathcal{T}, \leq)$ is called a tree iff $I(x)$ is a well-ordered subset of $(\mathcal{T}, \leq)$, for every $x \in \mathcal{T}$. Moreover, the rank of the tree denoted by $\text{rank}(\mathcal{T})$ is defined to be:

$$\text{rank}(\mathcal{T}) = \sup \{\text{rank}(x) : x \in \mathcal{T}\}$$

An infinite cardinal $\alpha$ is called a ramifiable cardinal iff for every tree $(\mathcal{T}, \leq)$ (with a least element) of rank $\alpha$ and $\overline{\alpha} \gamma < \alpha$, for every $\gamma < \alpha$,
where

\[(7.1) \quad A_\gamma = \{a \in T : \text{rank}(a) = \gamma\}\]

There exists a subset \(B\) of \(T\) such that \(\overline{B} = \alpha\) and \(B\) well-ordered by \(\leq\).

**LEMMA 7.1.** \(\omega\) is ramifiable.

**PROOF.** Let \((T, \leq)\) (with a least element) of rank \(\omega\) and suppose \(\overline{\alpha_n} < \aleph_0\) for \(n \in \omega\). Clearly, \(\overline{T} = \aleph_0\), and because \(\text{rank}(T) = \omega\) we can see that \(\alpha_n \neq \emptyset\) for every \(n \in \omega\).

Let \(S\) be a subset of \(T\) such that \(x \in S\) iff \(x\) has infinitely many successors in \(T\). Clearly, the least element is in \(S\). Hence, \(S\) is a nonempty subset of \(T\). By Zorn's Lemma \(S\) contains a maximal chain \(C\). If \(C\) is finite, then it has a last element, say, \(t \in A_m\) for some \(m \in \omega\). Now, \(t\) has infinitely many successors. Because \(\overline{A_m} < \aleph_0\), then there exist \(x \in A_{m+1}\) such that \(x\) is a successor of \(t\) and \(x\) has infinitely many successors, which contradicts the maximality of \(C\). Hence, \(C\) is infinite, and it is not difficult to see that a chain in a tree is a well-ordered subset. Thus, the Lemma is proved.

**REMARK 7.1.** We observe that the above Lemma is proved in \(ZF\), However, it is interesting to notice that this proof will fail for the case of \(\omega_1\), because it might be the case that there exists an element in the tree (of rank \(\omega_1\)) for example, the least element, which has \(\aleph_1\) successors but any chain which contains this element (the least element) is at most countable.
Although, in the case of the cardinal \( \aleph_1 \), the initial segment of each element in the limit ordinal level (Say, \( A_\omega^2 \)) is a well-ordered set and because each level has elements of cardinality less than \( \aleph_1 \), which means that there exist elements at each level with \( \aleph_1 \) successors, but still there is no guarantee that the constructed chain [where each element of this chain has \( \aleph_1 \) successors] will be an initial segment of an element in the limit ordinal level.

In fact, if we consider the Model of \( ZF + (V = L) \) (i.e., Gödel's Model), then there exist a tree (Suslin tree) in which every chain (and anti-chain) is at most countable. Hence, \( \omega_1 \) is not ramifiable cardinal in \( ZF + (V = L) \).

On the other hand, if we consider \( ZF + MA + \neg CH \) where \( MA \) is the Martin's axiom [43] and \( CH \) is the continuum hypothesis (i.e., \( 2^{\aleph_0} = \aleph_1 \)) then we can show that any cardinal less than \( 2^{\aleph_0} \) is \( \text{CAC} \)-ramifiable (Countable Anti-chain Condition ramifiable) cardinal as defined below. To prove this assertion we need the following preliminaries:

Let \( (\mathcal{P}, \leq) \) be a partially ordered set. Elements \( x \) and \( y \) of \( \mathcal{P} \) are called compatible iff \( x \) and \( y \) have nonzero (i.e., not the least element) lower bound in \( \mathcal{P} \) otherwise, \( x \) and \( y \) are called incompatible. \( \mathcal{P} \) is said to satisfy \( \text{CIC} \) (the Countable Incompatibility Condition) iff every subset of \( \mathcal{P} \) whose elements are pairwise incompatible is countable. Clearly, if a pair of nonzero elements of \( \mathcal{P} \) are incompatible, then they are incomparable, but not conversely. Thus if \( \mathcal{P} \) satisfies \( \text{CAC} \) (the Countable Anti-chain Condition, i.e., any set of pairwise incomparable elements of \( \mathcal{P} \) is countable) then \( \mathcal{P} \) satisfies
A subset $F$ of $P$ is called a filter of $P$ iff:

(i) The elements of $F$ are pairwise compatible in $F$

(ii) $x \in F$ and $y \in P$ and $y \geq x$ imply $y \in F$

A subset $H$ of $P$ is called a dense subset of $P$ iff for every $x \in P$ there exists $y \in H$ such that $y \leq x$. We notice that $P$ has finitely many or else continually many dense subsets [44]. Moreover, it has been shown [45, p. 72] that $ZF + MA + \neg CH$ consistent.

An infinite cardinal $\alpha$ is called CAC-ramifiable iff for every tree $(T, \leq)$ (with a least element) of rank $\alpha$ which satisfies CAC, there is a subset $B$ of $T$ such that $\overline{B} = \alpha$ and $B$ is well-ordered by $\leq$.

**Lemma 7.2.** $\kappa$ be an infinite cardinal and $(T, \leq)$ be a tree of rank $\kappa$ and $\overline{A_\gamma} \leq \aleph_0$ for every $\gamma < \kappa$. Let $x \in T$ be such that $x$ has $\kappa$-many successors. Then for every $\gamma < \kappa$, with $\gamma > \text{rank}(x)$ there exists $y \in A_\gamma$ such that:

(i) $y$ is a successor of $x$

and

(ii) $y$ has $\kappa$-many successors

**Proof.** To prove (i), suppose that there exists $\gamma_0 < \kappa$, such that for every $y \in A_{\gamma_0}$ it is the case that $y$ is not a successor of $x$. It then follows
that $x$ has at most $\aleph_0 \gamma_0 < \kappa$ successors, which a contradiction. Hence, (i) is established.

Next, we show that (ii) holds. Assume on the contrary, i.e., there exists $\gamma_0 < \kappa$ such that none of the successors of $x$ in $A\gamma_0$ has $\kappa$-many successors. Now:

$$\bigcup_{\alpha \leq \gamma_0} A_{\alpha} \leq \aleph_0 \gamma_0 < \kappa$$

Hence, $x$ has $\kappa$-many successors in $T \setminus \bigcup_{\alpha \leq \gamma_0} A_{\alpha}$ and since $\overline{A_{\gamma_0}} \leq \aleph_0$ then this would imply that one of the successors of $x$ in $A\gamma_0$ has $\kappa$-many successors, which contradicts our assumption. Therefore, there is a successor $y \in A_\gamma$ of $x$ which has $\kappa$-many successors, for every $\alpha < \kappa$. Thus the Lemma is proved.

Based on the consideration we prove:

**Theorem 7.1.** If $\kappa$ is an infinite cardinal less than $2^{\aleph_0}$, then there is a Model of $\mathsf{ZF} \mathsf{C}$, where $\kappa$ is $\mathsf{CAC}$-ramifiable.

**Proof.** Let $(M, \in)$ be a Model of $\mathsf{ZF} + MA + \neg CH$. We show that $\kappa$ is $\mathsf{CAC}$-ramifiable in $(M, \in)$. Let $(T, \leq)$ be a tree (with a least element) of rank $\kappa$ and $\overline{A_\gamma} \leq \aleph_0$, for every $\gamma < \kappa$ such that $T$ satisfies $\mathsf{CAC}$. Let $S$ be a subset of $T$ such that $x \in S$ iff $x$ has $\kappa$-many successors in $T$. 
Clearly, the least element of $T$ is in $S$. Hence, $S$ is a nonempty subset of $T$. Let us reverse the order in $S$. Thus, in $(S, \geq)$ every element has $\kappa$ predecessors.

Now, given $\gamma < \kappa$, by Lemma 7.2 it is the case that:

$$H_\gamma = \bigcup_{\gamma < \alpha < \kappa} A_\alpha \cap S$$

is a dense subset of $(S, \geq)$. Hence, $H = (H_\gamma)_{\gamma < \kappa}$ is a list of $\kappa$ many dense subsets of $(S, \geq)$. Since no two incomparable elements of $(T, \leq)$ have an upper bound, it follows that in $(S, \geq)$ no two elements have a lower bound. Thus, every set of incomparable elements of $(S, \geq)$ is a set of incompatible elements of $(S, \geq)$. But, since $(S, \geq)$ is $CAC$, then $(S, \geq)$ is $CIC$. Consequently, we may apply Martin's axiom to $(S, \geq)$ and $H = (H_\gamma)_{\gamma < \kappa}$ and conclude that there exists a filter $F$ of $(S, \geq)$ such that:

$$F \cap H_\gamma \neq \emptyset \text{ for every } \gamma < \kappa.$$  Hence $\overline{F} = \kappa$.

But, since no two incomparable elements of $(S, \geq)$ have a lower bound, it then follows that $F$ must be a (well-ordered) chain in $(S, \geq)$ and therefore, also a (well-ordered) chain in $(S, \leq)$.

**DEFINITION 7.1.** An infinite cardinal $\kappa$ is called a Hausdorff cardinal iff for every linear order $\preceq$ on $\kappa$ there is a subset of $\kappa$ of cardinality $\kappa$ which is either well-ordered or anti-well-ordered $(\succeq)$ by $\preceq$.

**LEMMA 7.3.** A Hausdorff cardinal $\kappa$ is a regular cardinal.
PROOF. Assume on the contrary, i.e., $\text{cf}(\kappa) < \kappa$. Partition $\kappa$ into $\text{cf}(\kappa)$ equivalence classes $A_\gamma$ with $\overline{A_\gamma} < \kappa$.

Hence:

$$\kappa = \bigcup_{\gamma < \text{cf}(\kappa)} A_\gamma \quad \text{and} \quad A_\gamma \cap A_\delta = \emptyset \quad \text{for} \quad \gamma \neq \delta$$

Define a linear ordering on $\kappa$ ($\leq$) as follows:

For all $\eta, \eta' \in \kappa$ we say $\eta \leq \eta'$ iff either:

(i) there are $\gamma, \delta$ such that $\gamma < \delta < \text{cf}(\kappa)$ and $\eta \in A_\gamma$ and $\eta' \in A_\delta$

or

(ii) there is $\gamma < \text{cf}(\kappa)$ such that $\eta, \eta' \in A_\gamma$ and $\eta' < \eta$

(where, $<$ is the natural order on $\kappa$)

[For instance, if we consider $\kappa = \aleph_\omega$, then $\text{cf}(\kappa) = \omega$. Consider the partition on $\aleph_\omega$ such that:

$$A_n = \{x \in \aleph_\omega : \aleph_{n-1} \leq x < \aleph_n\} \quad \text{for} \quad n \in \omega$$

Then an example of linear order $\preceq$ defined on $\aleph_\omega$ can be given as follows:

$$\begin{array}{cccccccc}
\hline
0 & & \aleph_\omega & | & \ast & \ast & | & \ast & | & \cdots \\
\hline
\end{array}$$

$b \preceq a$ since $a, b \in A_1$ and $a < b$

$a \preceq c$ and $b \preceq c$ since $a, b \in A_1$ and $c \in A_2$]
Now, if $A$ is a well-ordered subset of $\kappa$, then we claim that:

$$\bar{A} \cap A_\gamma < \omega \text{ for } \gamma < cf(\kappa)$$

Assume $\bar{A} \cap A_{\gamma_0} \geq \aleph_0$ for some $\gamma_0 < cf(\kappa)$. Since $(\bar{A} \cap A_{\gamma_0}, \preceq)$ is a well-ordered subset of $A_{\gamma_0}$, then $\bar{A} \cap A_{\gamma_0}$ contains an infinite subset $B$ similar (i.e., order-isomorphic) to $\omega$, therefore, $B$ has no maximum element. But, $(A_{\gamma_0}, \preceq)$ is anti-well-ordered, hence, it is the case that $B$ has a maximum element which is a contradiction. Thus, $A \cap A_\gamma$ is finite, for all $\gamma < cf(\kappa)$ and

$$\bar{A} = \sum_{\gamma < cf(\kappa)} \bar{A} \cap A_\gamma \leq \omega \cdot [cf(\kappa)] < \kappa$$

If $A$ anti-well-ordered, then the maximum element $m$ of $A$ is in $A_{\gamma_0}$, for some $\gamma_0 < cf(\kappa)$. Because of the ordering $\preceq$ of $\kappa$, we have $A \cap A_\delta = \emptyset$ for $\delta$, with $\gamma_0 < \delta < cf(\kappa)$ and therefore, we have $A \subseteq \bigcup_{\gamma < \gamma_0} A_\gamma$. Hence,

$$\bar{A} \leq \sum_{\gamma < \gamma_0} \bar{A}_\gamma < \kappa$$

Thus, $\kappa$ is not Hausdorff cardinal, contradiction. Hence, our assumption is false and therefore, $\kappa$ is regular.

Next, we show that a Hausdorff cardinal $\kappa$ is e-inaccessible, i.e., $\beta < \kappa$ implies $2^\beta < \kappa$. However, first we prove the following:
LEMMA 7.4. Let \( 2^\alpha \) (i.e., the set of all dyadic sequences) be linearly ordered lexicographically. Let \( W \) be a well-ordered subset of \( 2^\alpha \). Then \( \overline{W} \leq \alpha \).

PROOF. Assume on the contrary that \( \overline{W} > \alpha \). Let \( (w_0, w_1, w_2, \ldots) = (w_i)_{i < \overline{W}} \) indicate the well-ordering of \( W \).

For every \( i < \overline{W} \), let \( j < \alpha \) be the smallest ordinal such that:

\[
(7.2) \quad w_i(j) = 0 \quad \text{and} \quad w_{i+1}(j) = 1
\]

for the pair

\[
(7.3) \quad (w_i(j), w_{i+1}(j))
\]

Since there are \( \overline{W} \) pairs as given in (7.3), and since \( \overline{W} > \alpha \) there is an ordinal \( s < \alpha \) such that for some \( A \subseteq \overline{W} \) we have:

\[
(7.4) \quad (w_i(s), w_{i+1}(s))_{i \in A} \quad \text{with} \quad A = \overline{W}
\]

Clearly, (7.4) shows that for some ordinal \( s \) there exists an alternating sequence \( (w_i(s))_{i \in A} \) of 0's and 1's of type \( \overline{W} \). Let \( m \) be the smallest such an ordinal \( s \). Let \( (w_i(m))_{i \in B} \) with \( B \subseteq \overline{W} \) and \( \overline{B} = \overline{W} \) be a corresponding such an alternating sequence.

We observe that because of the lexicographic ordering for every \( i \in B \) if \( w_i(m) = 1 \) then for some \( n < m \) it must be the case that \( w_i(n) = 0 \) and \( w_{i^*}(n) = 1 \) (where \( i^* \) is the immediate successor of \( i \) in \( B \)).

Next, we consider the pairs:
and by replacing in our reasoning above (7.4) by (7.5) we derive the existence of an ordinal \( k < m \) such that there exists an alternating sequence \((w_i(k))_{i \in C}\) with \( \overline{C} = \overline{W} \). But, this contradicts the choice of \( m \). Thus our assumption is false and the Lemma is proved.

**LEMMA 7.5.** A Hausdorff cardinal \( \alpha \) is a strong limit cardinal.

(\( i.e., 2^\beta < \alpha \) for all \( \beta < \alpha \))

**PROOF.** Let \( \beta \) be a cardinal such that \( \alpha < 2^\beta \), we show that \( \alpha \leq \beta \).

Let \( A \subseteq 2^\beta \) such that \( \overline{A} = \alpha \) and consider \( (A, \preceq) \), where \( \preceq \) is the lexicographic ordering on \( 2^\beta \). Since \( \alpha \) is a Hausdorff cardinal, then there is a subset \( B \subseteq A \) such that \( \overline{B} = \alpha \) and \( B \) is either well-ordered or anti-well-ordered. But, according to the Lemma 7.4 a well-ordered or an anti-well-ordered subset of \( \{0, 1\}^\beta \) has cardinality of at most \( \beta \) which imply that \( \alpha = \overline{B} \leq \beta \). Thus the Lemma is proved.

Next, we show a Hausdorff cardinal is a ramifiable cardinal. If \( \alpha \) is a cardinal and \( \beta < \alpha \), then we set:

\[
(7.6) \quad ts(\beta) = \{ \gamma \in \alpha : \beta \leq \gamma \}
\]

The following special lexicographic order imposed on a tree \( T = (\alpha, \preceq) \), will yield a well-ordered branch in \( (\alpha, \preceq) \), provided that \( \alpha \) is a Hausdorff cardinal. In what follows we assume \( 0 \) is the least element of \( T \).
Let $T \equiv (\alpha, \leq)$ be a tree and $\overline{A_\gamma} < \alpha$ for $\gamma < \alpha$ [see (7.1)], assign for an element $x$ of the tree $T$ in [the level] $A_\gamma$ the $(\gamma + 1)$-tuple $(0, x_1, \ldots, x_\gamma)$ such that:

(7.6a) If $y$ is a predecessor of $x$ in [the level] $A_\delta$, then the $(\delta + 1)$-tuple of $y$ is the first $\delta + 1$ coordinates of the $(\gamma + 1)$-tuple of $x$.

(7.6b) The $(\gamma + 1)$-coordinate $x_\gamma$ of the element $x$ is an ordinal of $\alpha$ and if $t_i \in \alpha$ is the $i$th coordinate of an assigned $\delta$-tuple then $x_\gamma \neq t_i$.

We observe that if two elements $X \equiv (0, x_1, \ldots, x_\gamma)$ and $Y \equiv (0, y_1, \ldots, y_\delta)$ of the tree $T \equiv (\alpha, \leq)$ we say that $X \lesssim Y$ iff either:

(7.7a) $X \leq Y$

or

(7.7b) if $X$ and $Y$ are treewise incomparable then $x_j(X, Y) < y_j(X, Y)$.

It is not difficult to see that the relation $\lesssim$ given by (7.7a) and (7.7b) is reflexive and anti-symmetric linear relation on the tree $T$.

To show that $\lesssim$ is transitive relation, we use

\[
\begin{array}{c}
Y \\
\uparrow
\end{array}
\begin{array}{c}
X
\end{array}
\quad \text{(to mean $X \leq Y$)}
\]

and

\[
\begin{array}{c}
Y \\
\downarrow
\end{array}
\begin{array}{c}
X
\end{array}
\quad \text{(to mean $X$ and $Y$ are treewise incomparable)}
\]
For three distinct elements \( X, Y, Z \) of the tree \( T \equiv (\alpha, \leq) \) the possible cases for \( X, Y, Z \ in \ T \) are given as follows:

I

\[
\begin{array}{ccc}
X & Y & X \\
\uparrow & \uparrow & \uparrow \\
Z & X & Y \\
\uparrow & \uparrow & \uparrow \\
X & Z & Z \\
\end{array}
\]

II

\[
\begin{array}{ccc}
Z & X \\
\uparrow & \uparrow \\
X & Z & Z \\
\uparrow & \uparrow & \uparrow \\
Y & Y & Y \\
\end{array}
\]

III

\[
\begin{array}{ccc}
Z & Y & X \\
\uparrow & \uparrow \\
X \\
\end{array}
\]

IV

\[
\begin{array}{ccc}
Z \\
\uparrow \\
Y & Z & Y \\
\uparrow & \uparrow \\
X & X \\
\end{array}
\]

V

\[
\begin{array}{ccc}
Y & Z \\
\uparrow \\
X & Y & X \\
\end{array}
\]
Now, given that $X \preceq Y$ and $Y \preceq Z$, suppose that $X, Y, Z$ are distinct, then consider (I) above, since $Z \preceq Y$ we have $Z \preceq Y \preceq Z$ which imply that $Z = Y$. This is impossible to occurs in the tree. Also, by similar reasoning (II) is impossible to occurs in the tree too. Next, consider the first case of (III) above we have

$$x_j(X,Y) < y_j(X,Y) \quad \text{also} \quad j(X,Y) = j(Z,Y)$$

since the representative tuple of $Z$ is "physical extension" of the representative tuple of $X$ (i.e., $x_j(X,Y) = z_j(X,Y)$). Hence

$$z_j(Z,Y) = z_j(X,Y) = x_j(X,Y) < y_j(X,Y) = y_j(Z,Y)$$

which contradicts that $Y \preceq Z$ Thus

$$\begin{array}{c}
Z \\
\uparrow \\
X \quad Y
\end{array}$$

is impossible to occur in the tree (provided that $X \preceq Y \preceq Z$). For the second case of (III) above, we have

$$x_j(X,Y) < y_j(X,Y) \quad \text{also} \quad j(X,Y) = j(Z,Y)$$
But

\[ y_j(X,Y) = y_j(Z,Y) < z_j(Z,Y) = x_j(Z,Y) = x_j(X,Y) < y_j(X,Y) \]

which is a contradiction.

If (IV) occurs in the tree, then clearly, we have \( X \preceq Z \). Suppose the first case of (V) above occurs in the tree, then \( j(X, Z) = j(Y, Z) \) because \( Z \) is incomparable treewise with \( X \) and \( Y \). So, \( j(X, Z) \) and \( j(Y, Z) \) are defined. Since \( y_j(Y,Z) < z_j(Y,Z) \), then \( x_j(X,Z) = x_j(Y,Z) = y_j(Y,Z) \) and therefore, \( X \preceq Z \). Similarly, if the second case of (V) above occurs in the tree it is not difficult to show that \( j(X,Y) = j(X,Z) \) and it is the case that

\[ x_j(X,Y) < y_j(X,Y) = y_j(X,Z) = z_j(X,Z) \]

thus \( X \preceq Z \)

Finally, if (VI) is the case in the tree, then under the hypothesis that

\[ x_j(X,Y) < y_j(X,Y) \text{ and } y_j(Y,Z) < z_j(Y,Z) \]

we claim that \( x_j(X,Z) < z_j(X,Z) \)

To this end, we consider the following cases:

(i) \( j(X,Y) = j(Y,Z) \), then \( j(X,Y) = j(X,Z) \) and

\[ x_j(X,Z) = x_j(X,Y) < y_j(X,Y) = y_j(Y,Z) < z_j(Y,Z) = z_j(X,Z) \]

Thus, \( X \preceq Z \)

(ii) \( j(X,Y) < j(Y,Z) \), then \( j(X,Z) = j(X,Y) \) and

\[ x_j(X,Z) = x_j(X,Y) < y_j(X,Y) = z_j(X,Y) = z_j(X,Z) \quad \text{hence} \quad X \preceq Z \]

(iii) \( j(X,Y) > j(Y,Z) \), then \( j(X,Z) = j(Y,Z) \) and

\[ x_j(X,Z) = x_j(Y,Z) = y_j(Y,Z) < z_j(Y,Z) = z_j(X,Z) \quad \text{thus} \quad X \preceq Z \]
therefore, \( \preceq \) is a transitive relation on \( T \) and thus \( \preceq \) is a linear order on \( T \).

As a consequence of the above linear ordering \( \preceq \) on the tree \( T = (\alpha, \preceq) \) we prove

**LEMMA 7.6.** Let \( B \preceq C \preceq D \) and \( A \preceq B \) and \( A \preceq D \), then \( A \preceq C \) [i.e., if \( A \) is a treewise lower bound of \( B \) and \( D \), then \( A \) is a treewise lower bound of the interval (in the linear order \( \preceq \)) \( [B, D] \).]

**PROOF.** Assume on the contrary that \( A \not\preceq C \) then we claim that \( C \) is treewise incomparable with \( A, B, D \). It is clear that none of the following can occur in the tree under the assumption that \( A \not\preceq C \):

\[
\begin{array}{c}
D & & B & & C & & D \\
\text{\searrow} & & \nearrow & & \nearrow & & \text{\searrow} \\
A & & D & & B & & C \\
\text{\uparrow} & & \text{\searrow} & & \text{\nearrow} & & D \\
\text{\uparrow} & & \text{\nearrow} & & \text{\uparrow} & & \text{\nearrow} \\
C & & A & & B & & C & & A \\
\end{array}
\]

Next, suppose that the following is the case in the tree:
Then it is the case that

\[ j(A, C) = j(B, C) = j(D, C) \quad \text{and} \quad a_j(A, C) = b_j(B, C) = d_j(D, C) \]

But

\[ b_j(B, C) < c_j(B, C) < d_j(D, C) \]

which is a contradiction. Hence, \( A \leq C \) and the Lemma is established.

Next, we show that every well-ordered subset \((V, \preceq)\) of the linear ordered set \(T\) such that \(\overline{V} = \alpha\) contains a unique element \(v_\gamma\) at each level \(A_\gamma\) provided that \(v_\gamma\) has \(\alpha\) treewise successors in \(V\):

**Lemma 7.7.** Let \( T \equiv (\alpha, \preceq) \) be a tree of rank \( \alpha \) and \( A_\beta < \alpha \) [see (7.1)] for \( \beta < \alpha \). Let \( L \equiv (\alpha, \preceq) \) be the simply ordering of \( T \) as defined by (7.7a) and (7.7b). Then if \((B, \preceq)\) is a well-ordered subset of \( \alpha \) of cardinality \( \alpha \), then

\[ \{ a \in A_\gamma : \overline{ts(a)} \cap B = \alpha \} \]

is a singleton for every \( \gamma < \alpha \).
PROOF. Assume on the contrary that there is $\gamma_0 < \alpha$ such that $ts(a) \cap B < \alpha$, for every $a \in A_{\gamma_0}$. But, since

$$B = \left[ \bigcup_{\gamma < \gamma_0} (B \cap A_\gamma) \right] \cup \left[ \bigcup_{a \in A_{\gamma_0}} (ts(a) \cap B) \right]$$

we have

$$\overline{B} \leq \sum_{\gamma < \gamma_0} \overline{A_\gamma} + \sum_{a \in A_{\gamma_0}} \overline{(ts(a) \cap B)}$$

Moreover, $\overline{A_\gamma} < \alpha$ and $\overline{ts(a) \cap B} < \alpha$, which implies that $\overline{B} < \alpha$, contradiction. Therefore,

$$S_\gamma \equiv \{ a \in A_\gamma : \overline{ts(a) \cap B} = \alpha \} \neq \emptyset \text{ for every } \gamma < \alpha$$

Next, we show that $S_\gamma$ contains at most one element. Assume on the contrary that there are $a, a' \in A_{\gamma_0}$ and $S_{\gamma_0} = \{a, a'\}$ for some $\gamma_0 < \alpha$. Without loss of generality, suppose $a \preceq a'$. If $(B, \preceq)$ is order-isomorphic (similar) to $\alpha$ then $ts(a) \cap B$ is cofinal in $(B, \preceq)$, since

$$\overline{ts(a) \cap B} = \alpha$$

Thus, there is $b \in ts(a) \cap B$ such that $a' \preceq b$. Hence, $a \preceq a' \preceq b$, and it follows from Lemma 7.6 that $a \preceq a'$. But, since $a, a' \in A_{\gamma_0}$ we have $a = a'$. Thus

$$\{ a \in A_\gamma : \overline{ts(a) \cap B} = \alpha \} \text{ is a singleton}$$
REMARK 7.2. The case that if \((B, \preceq)\) is anti-well-ordered, then the above Lemma 7.7 is still holds (with roles of \(a, a'\) reversed).

Based on the above we prove:

THEOREM 7.2. A Hausdorff cardinal \(\alpha\) is a ramifiable cardinal.

PROOF. Consider the notation of Lemma 7.7, then \((\alpha, \preceq)\) has a well-ordered or an anti-well-ordered subset \((B, \preceq)\) with cardinality \(\alpha\), because \(\alpha\) is a Hausdorff cardinal. Hence, \(\{a \in A_\gamma : ts(a) \cap B = \alpha\}\) is a singleton for every \(\gamma < \alpha\). Consider the set \(\mathcal{C}\) where

\[
\mathcal{C} = \{a_\gamma : ts(a_\gamma) \cap B = \alpha\ \text{and} \ \gamma < \alpha\}
\]

Let \(\delta < \gamma < \alpha\), if \(b_\delta\) is the predecessor of \(a_\gamma\) in \(A_\delta\), then we have

\[
\text{ts}(a_\gamma) \cap B \subseteq \text{ts}(b_\delta) \cap B \quad \text{and} \quad \text{ts}(b_\delta) \cap B = \alpha
\]

Hence, by Lemma 7.7 \(b_\delta = a_\delta\), or , \(a_\delta \leq a_\gamma\). Therefore, \(\mathcal{C}\) is a well-ordered subset \((\mathcal{C}, \preceq)\) of the tree \(T \equiv (\alpha, \preceq)\). Thus, \(\alpha\) is ramifiable cardinal. Hence, the Theorem is established.

REMARK 7.3. We recall that an infinite cardinal \(\kappa\) is Hausdorff iff:

(7.8) Every simple ordering of \(\kappa\) has a well-ordered (or anti-well-ordered) subset of cardinality \(\kappa\)

A natural question arises:

(7.9) "Can the simple ordering in (7.8) be replaced by another condition \(C\)"
Obviously, not every partial ordering of \( \kappa \) can serve as \( C \) in (7.9). Also, not every tree on \( \kappa \) can serve as \( C \) in (7.9). Moreover, not every tree of height (rank) \( \kappa \) and levels \( A_\gamma \) of cardinality less than \( \kappa \) can serve as \( C \) in (7.9). Furthermore, not even every tree on \( \kappa \) of rank \( \kappa \) and levels \( A_\gamma \) of cardinality less than \( \kappa \) which produces a well-ordered branch of cardinality \( \kappa \) (i.e., \( \kappa \) is ramifiable cardinal) can serve as \( C \) in (7.9).

However, as shown below, it is the case that a tree on \( \kappa \) of rank \( \kappa \) and levels \( A_\gamma \) of cardinality less than \( \kappa \) which also produces a branch of cardinality \( \kappa \) such that \( \kappa \) also a strongly inaccessible cardinal can serve as \( C \) in (7.9).

To show the above we first prove that if \( \kappa \) is ramifiable strongly inaccessible cardinal, then \( \kappa \rightarrow [\kappa]^2 \).

To this end, For a cardinal \( \kappa \) we define a tree \( T \) on subsets of \( \kappa \) (by reverse inclusion) with levels \( A_\mu \) as follows:

(7.10a) \( A_0 = \{\kappa\} \) (the 0-th level)

(7.10b) If \( S \) is a nonempty subset of \( \kappa \) in the level \( A_\mu \) then \( S \) has at most two immediate successors \( S', S'' \) in the level \( A_{\mu+1} \) such that

\[ S' \cap S'' = \emptyset \text{ and } S = S' \cup S'' \cup \{\min S\} \]

(7.10c) For the nonzero limit ordinal \( \mu < \kappa \), we let

\[ A_\mu = \left\{ \bigcap_{\nu < \mu} A_\nu : A_\nu \in A_\nu \right\} \]
LEMMA 7.8. Let $\kappa$ be a cardinal and $T$ be given as in (7.10a), (7.10b), (7.10c), then:

(i) $A_\mu$ is a family of pairwise disjoint subsets of $\kappa$ and $\overline{A_\mu} \leq 2^\mu$

(ii) $T$ is a tree

(iii) If $A_\mu = \emptyset$ for some $\mu < \kappa$, then for every $t \in \kappa$ it is the case that $t = \min S$ for some $S \in T$. Moreover, $\bigcup_{\nu < \mu} A_\nu = \kappa$.

PROOF. By transfinite induction we prove (i):

First, if $\mu = 0$ then (i) is obvious. If $\mu = \nu + 1$, then by the induction hypothesis $A_\nu$ is a family of pairwise disjoint subsets of $\kappa$ and since every $S \in A_\nu$ has at most two disjoint immediate successors (namely, $S'$, $S''$) in $A_{\mu=\nu+1}$. Thus, the elements of $A_\mu$ are pairwise disjoint and $\overline{A} = 2^\mu$.

If $\mu$ is a limit ordinal and $S \in A_\mu$ then $S = \cap_{i<\mu} S_i$, where $S_i \in A_i$ for some $i < \mu < \kappa$. Assume that the $S_i$'s are not pairwise comparable, then $S$ is the empty set (by the induction hypothesis). Now, let $S_1$ and $S_2$ be a nonempty subsets in the level $A_\mu$ such that

$$S_1 = \bigcap_{i<\mu} S_{1i} \quad \text{and} \quad S_2 = \bigcap_{i<\mu} S_{2i}$$

Since $S_{1i} \cap S_{2i} = \emptyset$ then, clearly, $S_1 \cap S_2 = \emptyset$. Moreover, if $S \in A_\nu$ with $\nu < \mu$, then $S$ has at most $2^\mu$ successors. Thus $\overline{A_\mu} \leq 2^\mu$ (since there
are $2^\mu$ dyadic sequences of type $\mu$.

To prove (ii): Let $A \in \mathcal{A}_\mu$, clearly from (i) for every $i < \mu$ there is a unique $A_i \in \mathcal{A}_i$ such that $A \supset A_i$. Let $I(A)$ denote the set of all predecessors of $A$

( i.e., $I(A) = \{ A_i : A_i \supset A \text{ and } A_i \in \mathcal{A}_i \text{ for } i < \mu \} $)

If $S$ is a nonempty subset of $I(A)$, then let $j$ be the smallest ordinal such that $A_j \in S$, clearly, $A_j$ is the minimum element of $S$. Thus, $I(A)$ is a well-ordered subset of $T$. Hence, (ii) is established.

To prove (iii): Let $t \in \kappa$ be given and assume that $t$ is not the minimum of any element of the tree $T$, then we claim that for every $\nu < \mu$ we have $t$ an element of some $A_\nu$ in $\mathcal{A}_\nu$. Suppose this is not the case, then let $\nu_0$ be the smallest ordinal such that $t \in A_{\nu_0}$, but $t \notin A_\nu$ for every $A_\nu \in \mathcal{A}_\nu$ with $\nu_0 < \nu < \mu < \kappa$. By (7.10b) we have

$$A_{\nu_0} = A_{\nu_0}' \cup A_{\nu_0}'' \cup \{ \min A_{\nu_0} \}$$

But, since $A_{\nu_0}'$ and $A_{\nu_0}''$ are in $\mathcal{A}_{\nu_0} + 1$, clearly, $t = \min A_{\nu_0}$ which contradicts our assumption. Thus, for every $\nu < \mu$ we have $t \in A_\nu$ for some $A_\nu \in \mathcal{A}_\nu$.

Now, if $\mu = \nu + 1$ and since $t \neq \min A_\nu$ for every $A_\nu \in \mathcal{A}_\nu$, then $t \in A_\mu$ for some $A_\mu \in \mathcal{A}_\mu$ which contradicts that $A_\mu = \emptyset$.

If $\mu$ is a limit ordinal, then $t \in A_\mu$ for some $A_\mu \in \mathcal{A}_\mu$, where $A_\mu = \cap_{\nu < \mu} A_\nu$ and $t \in A_\nu$. Thus, $A_\mu \neq \emptyset$, contradiction. Hence, our
assumption is false and therefore, \( t = \min S \), for some \( S \in T \).

Next, define

\[
m : \bigcup_{\nu < \kappa} A_\nu \rightarrow \kappa \quad \text{by} \quad m(A) = \min A
\]

Now, for \( A \) and \( B \) in \( \bigcup_{\nu < \kappa} A_\nu \) if \( A \cap B = \emptyset \), then \( \min A \neq \min B \) and if \( A \subseteq B \), then \( \min B < \min A \). Thus \( m \) is a one-one order preserving mapping. Moreover, if \( A_\mu = \emptyset \) for some \( \mu < \kappa \), then \( m \) is onto and therefore,

\[
\bigcup_{\nu < \mu} A_\nu = \kappa
\]

Thus (iii) is established.

**REMARK 7.4.** We observe that if \( \kappa \) in Lemma 7.8 is strongly inaccessible cardinal, then \( A_\mu \neq \emptyset \) for every ordinal \( \mu < \kappa \) and hence the tree \( T \) as given in (7.10a)–(7.10c) has height \( \kappa \). Moreover, if in addition \( \kappa \) is ramifiable cardinal, then, also, the tree \( T \) contains a well-ordered branch of cardinality \( \kappa \).

Based on Lemma 7.8 and Remark 7.4 it is easy to show that if \( \kappa \) is a ramifiable strongly inaccessible cardinal, then \( \kappa \rightarrow [\kappa]^2 \) as follows:

Consider the partition \( \{A, B\} \) of the set of all doubletons \( [\kappa]^2 \) of \( \kappa \). Based on the partition \( \{A, B\} \) of \( [\kappa]^2 \) we define a tree \( T \) on subsets of \( \kappa \) as in (7.10a)–(7.10b) where corresponding to \( S \) in (7.10b) we now define
\( S' \) and \( S'' \) as follows:

\[(7.11a) \quad S' = \{ x : x \in (S \setminus \{\text{min}S\}) \text{ and } \{x, \text{min}S\} \in A \} \]
\[(7.11b) \quad S'' = \{ x : x \in (S \setminus \{\text{min}S\}) \text{ and } \{x, \text{min}S\} \in B \} \]

We observe that if \( 0 \neq S' \), then \( \{\text{min}S', \text{min}S''\} \in A \). By Lemma 7.8 and Remark 7.4 we see that \( T \) has a well-ordered chain (branch) \( H \) of cardinality \( \kappa \). If \( S \in H \) then either \( S' \) or \( S'' \) in \( H \). Let \( C = \{\text{min}S : S \in H\} \) define

\[ f : C \to \{A, B\} \text{ such that for } S \in H \]
\[ f(\text{min}S) = \begin{cases} A & \text{if } S' \in H \\ B & \text{if } S'' \in H \end{cases} \]

Now, since \( \bar{C} = \kappa \), then \( f^{-1}(A) \) or \( f^{-1}(B) \) has cardinality \( \kappa \). Without loss of generality, say, \( f^{-1}(A) \) has cardinality \( \kappa \). We observe that for every \( S \) and \( Q \) elements of \( H \) with \( f(\text{min}S) = A \) and \( \text{min}S < \text{min}Q \), it is the case that \( \{\text{min}S, y\} \in A \), for every \( y \in Q \). Hence, for \( \{x_1, x_2\} \subseteq f^{-1}(A) \) such that \( x_1 < x_2 \), we have \( x_1 = \text{min}S \) and \( x_2 = \text{min}Q \) \( Q \subseteq S' \) [see \( (7.11a) \)]. Thus, \( \{x_1, x_2\} \in A \). Therefore, \( f^{-1}(A) \) is a homogeneous set for the partition \( \{A, B\} \). Hence, \( \kappa \to [\kappa]^2 \), as desired.

Next, we show that a Ramsey cardinal [cf. 46, 47] (i.e., a ramifiable strongly inaccessible cardinal) \( \kappa \) is a Hausdorff cardinal.

To this end, Let \( (\kappa, \preceq) \) be a simple ordering of \( \kappa \) and we set:
(7.12a) \[ A = \{ \{ \alpha, \beta \} : \text{either } \alpha < \beta < \kappa \text{ and } \alpha \leq \beta \text{ or } \beta < \alpha < \kappa \text{ and } \beta \leq \alpha \} \]

(7.12b) \[ B = \{ \{ \alpha, \beta \} : \text{either } \alpha < \beta < \kappa \text{ and } \beta \leq \alpha \text{ or } \beta < \alpha < \kappa \text{ and } \alpha \leq \beta \} \]

We observe that \( A \) is the set of all doubletons \( \{ \alpha, \beta \} \) of \( \kappa \) where the usual ordering of \( \kappa \) and the imposed simple ordering of \( \kappa \) agree, while, \( B \) is the set of all doubletons \( \{ \alpha, \beta \} \) where the two orderings disagree.

Now, we have \( [\kappa]^2 = A \cup B \). But, since \( \kappa \rightarrow [\kappa]^2 \), then there exists a subset \( C \) of \( \kappa \) such that \( \overline{C} = \kappa \) and either \( [C]^2 \subseteq A \) or else, \( [C]^2 \subseteq B \).

Clearly, if \( [C]^2 \subseteq A \), then \( C \) is well-ordered by \( \preceq \). If \( [C]^2 \subseteq B \), then \( C \) is anti-well-ordered by \( \preceq \). Thus, \( \kappa \) is a Hausdorff cardinal, as desired.
8 BIBLIOGRAPHY


