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Extensions of dynamical systems

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Extensions of dynamical systems

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Iowa State University, 1987
Extensions of dynamical systems

Timothy James Pennings

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# TABLE OF CONTENTS

**INTRODUCTION**

I. PRELIMINARIES 4
   I.1. C*-Algebra Theory 4
   I.2. Dynamical Systems, Minimality, Topological Transitivity 7
   I.3. More Dynamical Properties 11
   I.4. An Irrational Rotation of the Circle 15

II. EXTENSIONS WHICH PRESERVE MINIMALITY AND TOPOLOGICAL TRANSITIVITY 18
   II.1. Introduction 18
   II.2. Forming an Extension 19
   II.3. An Alternative Description of $\hat{X}$ 23
   II.4. The Main Result 25

III. AN EXTENSION OF THE IRRATIONAL ROTATION OF THE CIRCLE 34
   III.1. Introduction 34
   III.2. Dynamical Properties 35

IV. CONCLUDING REMARKS 40

V. REFERENCES 42

VI. ACKNOWLEDGEMENTS 43
INTRODUCTION

My father once asked me to try to explain to him the relationship between mathematics and the physical sciences. After some thought, I explained it this way: The sciences are like a brick wall which is being built. To give the wall support, a tree (mathematics) is planted in the soil of logic and language (which in turn rests on reasoning and experience). The tree grows up and provides a solid support on which the wall can lean. However, once the tree has served that purpose, there is nothing which keeps it from growing and branching still further. In fact, without the prunings of 'usefulness' and 'applicability', the growth and ramifications are great indeed. Ironically, since the wall is continually being revised and extended, some of the previously 'useless' branches suddenly find themselves supporting an integral part of the new developments in science.

The analogy may also help in presenting the particular branch of mathematics called topological dynamics or dynamical systems.

The theory of dynamical systems, motivated especially by problems of celestial mechanics, grew up out of the theory of ordinary differential equations in the late 19th century with the pioneering work of Henri Poincaré and A. M. Lyapunov. Lyapunov investigated the stability of a motion (solution) for a system of \( n \) first order differential equations. He rigorously defined the concepts of stability, asymptotic stability, and instability; and provided a means to analyze the stability properties of a solution of an ordinary differential equation [3].
Poincaré studied topological properties of solutions of autonomous ordinary differential equations in the plane. His introduction of the concept of trajectory allowed him to study the qualitative behavior of differential equations and to formulate and solve them as topological problems. Consequently, Poincaré paved the way for the abstract notion of a dynamical system by A. A. Markov and H. Whitney. These authors studied trajectories generated by a general one-parameter topological transformation group acting on a (suitable) space [3].

G. D. Birkhoff took up the reins in the early 20th century, and it is he who is considered the founder of the theory of dynamical systems. His work is the foundation out of which has grown the two main branches of work in dynamical systems, namely topological theory and ergodic theory.

More recently, the concept of a dynamical system has been generalized still further to topological transformation groups. R. Ellis and H. Furstenberg, among others, have made progress in this area [3].

Returning to the analogy of the wall and the tree, it appears the theory of dynamical systems has grown from its original purpose of modeling celestial mechanics to take on a life and meaning of its own. In fact, R. Ellis comments [6, ix-x]: "The relevence of the abstract theory ... [is that] the topological tools used are refined and the essence of the theorems displayed. However, it is not clear that the problems which are germane to abstract topological dynamics will have application to differential equations. ... [For] not only is the
differential structure ignored but the topological properties of the reals are not made essential use of."

This thesis is clearly in the realm of abstract topological dynamics. Hopefully it will serve not only in answering questions, but also in raising still others.
I. PRELIMINARIES

I.1. C*-Algebra Theory

This section is devoted to introducing and developing the C*-algebra theory which we use in constructing extensions of dynamical systems. Since the results are included in the main body of functional analysis [4] we omit the proofs.

**Definition I.1.1.** A Banach algebra is an algebra \( \mathcal{A} \) over a field \( F (=\mathbb{C}\text{ or }\mathbb{R}) \) that has a norm \( \| \cdot \| \) relative to which \( \mathcal{A} \) is a Banach space and such that for all \( f, g \in \mathcal{A}, \| fg \| < \| f \| \| g \|. \) If \( \mathcal{A} \) has an identity, \( 1, \) it is assumed that \( \| 1 \| = 1. \) If \( f \in \mathcal{A} \) and \( \mathcal{A} \) has an identity, we say that \( f \) is invertible if there is a \( g \in \mathcal{A} \) with \( fg = gf = 1. \)

**Note.** We will assume henceforth that \( \mathcal{A} \) is taken over the complex field, \( \mathbb{C}. \)

**Remark I.1.2.** If \( h \) is invertible and \( f, g \in \mathcal{A} \) satisfy
\[ hg = 1 = fh, \]
then \( f = f 1 = f(hg) = (fh)g = 1g = g. \) So if \( h \) is invertible, there is a unique element, \( h^{-1}, \) which satisfies
\[ hh^{-1} = h^{-1}h = 1. \]

**Definition I.1.3.** An ideal of an algebra, \( \mathcal{I}, \) is a subalgebra \( \mathcal{M} \) such that \( fg \in \mathcal{M} \) and \( gf \in \mathcal{M} \) whenever \( f \in \mathcal{A}, g \in \mathcal{M}. \) A maximal ideal is a proper ideal which is contained in no larger proper ideal.
Definition I.1.4. If $A$ is a Banach algebra and $f \in A$, the spectrum of $f$ denoted by $\text{spec}(f)$ is the set $\{a \in \mathbb{C} : f-a \text{ is not invertible}\}$.

Definition I.1.5. A homomorphism, $\hat{x}$, from $A$ into $\mathbb{C}$ is a linear functional with the added property that $\hat{x}(fg) = \hat{x}(f)\hat{x}(g)$ for $f, g \in A$. The kernel of $\hat{x}$ is the set $\{f \in A : \hat{x}(f) = 0\}$ and is denoted by $\ker(\hat{x})$.

Proposition I.1.6. If $A$ is a Banach algebra, then the closure of a proper ideal is itself a proper ideal. Consequently, a maximal ideal is closed.

This is used in the following

Proposition I.1.7. If $A$ is an abelian Banach algebra and $\mathfrak{m}$ is a maximal ideal, then there is a homomorphism $\hat{x} : A \to \mathbb{C}$ such that $\mathfrak{m} = \ker(\hat{x})$. Conversely, if $\hat{x} : A \to \mathbb{C}$ is a nonzero homomorphism, then $\ker(\hat{x})$ is a maximal ideal. Moreover, this correspondence $\hat{x} + \ker(\hat{x})$ between homomorphisms and maximal ideals is bijective.

Corollary I.1.8. If $A$ is an abelian Banach algebra and $\hat{x} : A \to \mathbb{C}$ is a homomorphism, then $\hat{x}$ is continuous and $\|\hat{x}\| = 1$.

Definition I.1.9. If $A$ is an abelian Banach algebra, the spectrum of $A$, which we denote by $\text{spec}(A)$ or $\hat{X}$, is the set of all nonzero homomorphisms of $A \to \mathbb{C}$ with the weak* topology it inherits from $A^*$. 
Theorem 1.1.10. If \( \mathcal{A} \) is an abelian Banach algebra, then \( \hat{X} \) is a compact Hausdorff space. Moreover, if \( f \in \mathcal{A} \), then 
\[ \text{spec}(f) = \{ \hat{x}(f) : x \in \hat{X} \}. \]

Suppose \( X \) is a compact space and \( x \in X \). Take \( \delta_x : C(X) \to \mathbb{C} \) defined by \( \delta_x(f) = f(x) \). It is straightforward to show that \( \delta_x \) is a homomorphism on the algebra \( C(X) \). The next theorem shows that the functions \( \delta_x \) exhaust all homomorphisms.

Theorem 1.1.11. If \( X \) is compact, then the map \( x \mapsto \delta_x \) is a homeomorphism of \( X \) onto \( \text{spec}(C(X)) \).

Definition 1.1.12. Let \( f \in \mathcal{A} \), an abelian Banach algebra. The **Gelfand transform of \( f \)** is the function \( \hat{f} : \hat{X} \to \mathbb{C} \) defined by \( \hat{f}(x) = \hat{x}(f) \).

Theorem 1.1.13. If \( f \) belongs to the abelian Banach algebra \( \mathcal{A} \), then \( \hat{f} \) is an element of \( C(\hat{X}) \). The map \( f \mapsto \hat{f} \) is a continuous homomorphism from \( \mathcal{A} \) into \( C(\hat{X}) \) of norm 1 and its kernel is 
\( \bigcap \{ \mathfrak{m} : \mathfrak{m} \text{ is a maximal ideal of } \mathcal{A} \}. \) Furthermore, for each \( f \in \mathcal{A} \), 
\[ \|f\|_\infty = \lim_{n \to \infty} \|f^n\|^{1/n}. \] The homomorphism \( f \mapsto \hat{f} \) is called the **Gelfand transform of \( \mathcal{A} \)**.

We now turn our attention to a particular class of Banach algebras called \( C^* \)-algebras. Theorem 1.1.13 will have a counterpart (Theorem 1.1.18) which together with Theorem 1.1.11 will be used extensively in the formation of extensions of dynamical systems.
Definition 1.1.14. If $\mathcal{U}$ is a Banach algebra, an involution is a map $f \mapsto f^*$ of $\mathcal{U}$ into $\mathcal{U}$ which satisfies the following

$(f, g \in \mathcal{U}, \alpha \in \mathbb{C})$:

(i) $(f^*)^* = f$;
(ii) $(fg)^* = g^*f^*$;
(iii) $(\alpha f + g)^* = \alpha f^* + g^*$.

Definition 1.1.15. A $C^*$-algebra is a Banach algebra $\mathcal{U}$, with an involution such that for every $f \in \mathcal{U}$, $\|f^*f\| = \|f\|^2$.

Example 1.1.16. If $X$ is a compact space, $B(X)$, the collection of all bounded functions on $X$, is a $C^*$-algebra with the supremum norm and $f^* = \overline{f}$. $C(X)$, being a closed subspace of $B(X)$, is also a $C^*$-algebra.

Definition 1.1.17. An isometric $^*$-isomorphism from $\mathcal{U}$ to another $C^*$-algebra $\mathcal{B}$ is a surjective linear and multiplicative isometry which preserves the involution (i.e. $\nu(f^*) = \nu(f)^*$).

Theorem 1.1.18. If $\mathcal{U}$ is an abelian $C^*$-algebra then the Gelfand transform $\gamma : \mathcal{U} \to C(\hat{X})$ is an isometric $^*$-isomorphism.

I.2. Dynamical Systems, Minimality, Topological Transitivity

In this section we give the needed background material from the theory of topological dynamics and dynamical systems. The definition of
a dynamical system varies somewhat in the literature. The following will suffice for our purposes.

**Definition 1.2.1.** A dynamical system is a triple \((X, \phi, \Sigma)\) where \(X\) is a compact Hausdorff space, \(\Sigma\) is the semigroup \(\mathbb{N} = \{0,1,2,\ldots\}\) or \(\mathbb{Z}\), and \(\phi : X \rightarrow X\) is a continuous mapping \((\Sigma = \mathbb{N})\) or homeomorphism \((\Sigma = \mathbb{Z})\) so that \(\phi^k\) is continuous for all \(k \in \Sigma\).

**Note.** To avoid trivialities, we will assume that all compact Hausdorff spaces under consideration have infinite cardinality.

**Definition 1.2.2.** Given \(x \in X\), the set \(\{\phi^k(x) : k \in \Sigma\}\) is called the \(\phi\)-orbit of \(x\) and is denoted by \(\mathcal{O}_\phi(x)\). If there is an \(x \in X\) such that \(\mathcal{O}_\phi(x)\) is dense in \(X\), then \((X, \phi, \Sigma)\) is said to be **topologically transitive**. If \(\mathcal{O}_\phi(x)\) is dense in \(X\) for all \(x \in X\), then \((X, \phi, \Sigma)\) is called **minimal**.

The following propositions show when these dynamical properties have equivalent formulations [13].

**Proposition 1.2.3.** If \(\Sigma = \mathbb{Z}\), the following are equivalent.

(i) \(\phi\) is minimal.

(ii) The only closed subsets \(E\) of \(X\) with \(\phi(E) = E\) are \(\emptyset\) and \(X\).

(iii) For every nonempty open subset \(U\) of \(X\), we have 
\[
\bigcup_{n=0}^{\infty} \phi^n(U) = X.
\]
Proof. (i) ⇒ (ii). Suppose \( \phi \) is minimal and let \( E \) be closed, \( E \neq \emptyset \) and \( \phi(E) = E \). If \( x \in E \), then \( \phi_\phi(x) \subseteq E \) so \( X = \overline{\phi_\phi(x)} \subseteq E \).

Hence \( X = E \).

(ii) ⇒ (iii). If \( U \) is nonempty and open, then \( E = X \setminus \bigcup_{n}^{\infty} \phi_\phi(U) \) is closed and \( \phi(E) = E \). Since \( E \neq X \), we have \( E = \emptyset \).

(iii) ⇒ (i). Let \( x \in X \) and let \( U \) be any nonempty open subset of \( X \). By (iii) \( x \in \phi_\phi(U) \) for some \( n \in \mathbb{Z} \), so \( \phi^{-n}(x) \in U \) and \( \phi_\phi(x) \) is dense in \( X \).

Analogously, we have the

**Proposition 1.2.4.** If \( \Sigma = \mathbb{N} \), the following are equivalent.

(i) \( \phi \) is minimal.

(ii) The only closed subsets \( E \) of \( X \) with \( \phi(E) \subseteq E \) are \( \emptyset \) and \( X \).

(iii) For every nonempty open subset \( U \) of \( X \) we have

\[
\bigcup_{n=0}^{\infty} \phi^{-n}(U) = X.
\]

Proof. Similar to the preceding proof.

The next two propositions require that \( X \) be \( 2^\circ \) countable. This is not a severe restriction since one is often working with compact metric spaces.

**Proposition 1.2.5.** If \( X \) is \( 2^\circ \) countable and \( \Sigma = \mathbb{Z} \), the following are equivalent.
(i) $\phi$ is topologically transitive.

(ii) Whenever $E$ is a closed subset of $X$ and $\phi(E) = E$, then either $E = X$ or $E$ is nowhere dense.

(iii) Whenever $U$ is an open subset of $X$ with $\phi(U) = U$, then $U = \emptyset$ or $U$ is dense.

(iv) Whenever $U, V$ are nonempty open sets then there exists $n \in \mathbb{Z}$ with $\phi^n(U) \cap V \neq \emptyset$.

(v) $\{ x \in X : \overline{\phi(x)} = X \}$ is a dense $G_\delta$.

Proof. (i) $\Rightarrow$ (ii). Suppose $\overline{\phi(x_0)} = X$ and let $E \neq \emptyset$, $E$ closed and $\phi(E) = E$. Suppose $U$ is open and $U \subseteq E$, $U \neq \emptyset$. Then there exists $p$ with $\phi^p(x_0) \in U \subseteq E$ so that $\overline{\phi^p(x_0)} \subseteq E$. Thus $\overline{\phi(x_0)} = X \subseteq E$ and $X = E$. Therefore either $E$ has no interior or $E = X$.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (iv). Suppose $U, V \neq \emptyset$ are open sets. Then $\bigcup_{n=1}^{\infty} \phi^n(U)$ is open and $\phi(\bigcup_{n=1}^{\infty} \phi^n(U)) = \bigcup_{n=1}^{\infty} \phi^n(U)$, so it is dense by (iii). Hence $\bigcup_{n=1}^{\infty} \phi^n(U) \cap V \neq \emptyset$.

(iv) $\Rightarrow$ (v). Let $U_1, \ldots, U_n$ be a countable base for $X$. Then $\{ x \in X : \overline{\phi(x)} = X \} = \bigcap_{n=1}^{\infty} \bigcup_{m=-\infty}^{\infty} \phi^m(U_n)$ and each $\bigcup_{m=-\infty}^{\infty} \phi^m(U_n)$ is dense by (iv). The result follows since $X$ is a Baire space.

(v) $\Rightarrow$ (i). Clear. $\diamondsuit$
**Proposition I.2.6.** If $X$ is $2^\sigma$ countable and $\Sigma = \mathbb{N}$, the following are equivalent.

(i) $\phi$ is topologically transitive.

(ii) Whenever $E$ is a closed subset of $X$ and $\phi(E) \subseteq E$ then either $E = X$ or $E$ is nowhere dense.

(iii) Whenever $U$ is an open subset of $X$ and $\phi^{-1}(U) \subseteq U$ then $U = \emptyset$ or $U$ is dense.

(iv) Whenever $U, V$ are nonempty open sets there exists $n > 1$ with $\phi^{-n}(U) \cap V \neq \emptyset$.

(v) $\{ x \in X : \overline{\phi(x)} = X \}$ is a dense $G_\delta$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose $\phi(x_0)$ is dense in $X$, and suppose $E$ is closed and $\phi(E) = E$. If $U$ is a nonempty open set with $U \subseteq E$, then $\phi^n(x_0) \in U$ for some $n > 0$, so $\{ \phi^n(x_0) : n > p \} \subseteq E$. Therefore $\{x_0, \phi(x_0), \ldots, \phi^{-1}(x_0)\} \cup E = G_\phi(x_0) = X$. By applying $\phi^n$ to each side, we get $E = X$. Thus if $E$ has an interior, then $E = X$.

The other implications are similar to the preceding proof. \(\diamondsuit\)

### I.3. More Dynamical Properties

We go on to define and discuss four more dynamical properties - expansiveness, topological entropy, distallity and proximality. It is possible to define each of these properties either with or without reference to a metric [6], [13]. However, we will be applying them only to metric spaces so we will restrict ourselves to the (for our purposes
more useful) definitions which rely on a metric. Of course, since there are equivalent definitions which rely only upon the topology (and not the particular metric being used), this shows that the properties are independent of the metric—assuming the metric gives the topology. We also take \( \Sigma = \mathbb{Z} \) in this section since the properties depend on \( \phi \) being a homeomorphism.

**Note.** In this section we assume the space \( X \) is metrizable as well as compact.

**Definition 1.3.1.** \( \phi : X \rightarrow X \) is said to be **expansive** if there is a \( \delta > 0 \) with the property that if \( x \neq y \) then there exists an \( n \in \mathbb{Z} \) with \( d(\phi^n(x), \phi^n(y)) > \delta \). We call \( \delta \) an **expansive constant** for \( \phi \).

**Remark 1.3.2.** Even though expansiveness is independent of the metric, the expansive constant may change.

**Proposition 1.3.3.** if \( \phi : X \rightarrow X \) is expansive then \( \phi \) has only a finite number of fixed points.

**Proof.** Let \( \delta \) be an expansive constant for \( \phi \), and suppose \( x, y \in X, \ x \neq y \) are fixed points. Then there is an \( n \) such that \( d(\phi^n(x), \phi^n(y)) > \delta \). Thus \( d(x, y) > \delta \). It follows that the number of fixed points is finite since \( X \) is a compact metric space. \( \diamond \)

We leave expansiveness by noting that there are no expansive homeomorphisms on the unit circle (with its usual topology) [13, p. 145].
Defining the entropy of \( \phi \) requires some preliminary notions. Let \( d(\cdot, \cdot) \) be a metric which induces the topology on \( X \). Take
\[
d_n : X \times X \to [0, \infty) \text{ to be } d_n(x, y) = \max \{d(\phi^i(x), \phi^i(y)) : 0 < i < n-1\}
\]
where \( n \) is a positive integer. \( d_n(\cdot, \cdot) \) is then a metric and we can make the following

**Definition I.3.4.** Let \( n \in \mathbb{Z}^+, \varepsilon > 0 \), and take \( K \subset X \) to be compact. A subset \( F \) of \( X \) is said to span \( K \) with respect to \( \phi \) if for all \( x \in K \), there is a \( y \in F \) such that \( d_n(x, y) < \varepsilon \). We denote by \( r_n(\varepsilon, K, \phi) \) (or just \( r_n(\varepsilon, K) \)) the smallest cardinality of any \((n, \varepsilon)\)-spanning set of \( K \) with respect to \( \phi \). Finally, take
\[
r(\varepsilon, K, \phi) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon, K).
\]

**Remark I.3.5.** \( r_n(\varepsilon, K) < \infty \) since \( K \) is compact, however the value of \( r(\varepsilon, K, \phi) \) could be \( \infty \) [13, p. 170].

**Definition I.3.6.** Let \( h(\varepsilon; K) = \lim_{\varepsilon \to 0} r(\varepsilon, K, \phi) \). We then define the topological entropy of \( \phi \) to be \( h(\phi) = \sup \{h(\varepsilon; K) : K \subset X, K \text{ compact}\} \).

If we want to emphasize \( d \), we will write \( h_d(\phi) \).

**Proposition I.3.7.** If the topology of \( X \) is induced by \( d \) and \( d' \), then \( h_d(\phi) = h_{d'}(\phi) \).

**Proof.** The mappings \( i_d : (X, d) \to (X, d') \) and \( i_d' : (X, d') \to (X, d) \) are continuous. Since \( X \) is compact, \( i \) and \( i' \) are uniformly continuous. So let \( \varepsilon_1 > 0 \). Choose \( \varepsilon_2 > 0 \) such that \( d'(x, y) < \varepsilon_2 = d(x, y) \). Also choose \( \varepsilon_3 > 0 \) such that \( d(x, y) < \varepsilon_3 = d'(x, y) < \varepsilon_2 \). Let \( K \) be compact. Then \( r_n(\varepsilon_1, K, d) < r_n(\varepsilon_2, K, d') \) and
\[ r_n(\varepsilon_2, K, d') < r_n(\varepsilon_3, K, d). \] Hence \( r(\varepsilon_1, K, \phi, d) < r(\varepsilon_2, K, \phi, d') \) \(< r(\varepsilon_3, K, \phi, d). \) If \( \varepsilon_1 + 0, \) then \( \varepsilon_2 + 0 \) and \( \varepsilon_3 + 0 \) so we have \( h_d(\phi, K) = h_d(\phi, K). \) \( \diamond \)

The preceding proposition of course shows directly that the topological entropy is independent of the metric. It should also be noted that to find \( h(\phi) \) it is enough to find \( h(\phi; X). \) This is because if the set \( F (n, \varepsilon) - \text{spans} \ X, \) then \( F \) is also an \((n, \varepsilon)\)-spanning set for \( K. \) Thus \( r_n(\varepsilon, K) < r_n(\varepsilon, X). \) This proves the

**Remark 1.3.8.** \( h(\phi) = h(\phi; X). \)

In addition to the above, Walters [13, p. 179] proves that if \( \phi \) is a homeomorphism on the unit circle then \( h(\phi) = 0. \) (We will prove a special case of this in the next section.) Another interesting result is that an expansive homeomorphism has finite topological entropy [13, p. 177].

**Proposition 1.3.9.** \( h(\phi^m) = mh(\phi) \) for \( m \in \mathbb{Z}^+. \)

**Proof.** Since \( r_n(\varepsilon, K, \phi^m) < r_{mn}(\varepsilon, K, \phi), \) we have \( \frac{1}{n} \log r_n(\varepsilon, K, \phi^m) < m \left( \frac{1}{mn} \log r_{mn}(\varepsilon, K, \phi) \right). \) It follows that \( h(\phi^m) < mh(\phi). \) On the other hand, since \( \phi \) is uniformly continuous, given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( d(x, y) < \delta \) implies \( \max\{d(\phi^j(x), \phi^j(y)) : 0 < j < m-1\} < \varepsilon. \) So an \((n, \varepsilon)\)-spanning set for \( K \) with respect to \( \phi^m \) is also an \((mn, \varepsilon)\)-spanning set for \( K \) with respect to \( \phi. \) Thus \( r_{mn}(\varepsilon, K, \phi) < r_n(\delta, K, \phi^m), \) so \( \frac{m}{mn} r_{mn}(\varepsilon, K, \phi) < \frac{1}{n} r_n(\delta, K, \phi^n). \)
Therefore $mr(\rho, K, \phi) < r(\delta, K, \phi^N)$. Letting $\varepsilon + 0$ (so $\delta + 0$), we get $mh(\phi) < h(\phi^m)$. ☐

Finally, we mention the properties of distallity and its counterpart proximality [10].

**Definition 1.3.10.** $(X, \phi, \Sigma)$ is **distal** if given distinct points $x, y \in X$, there is an $\varepsilon > 0$ such that $d(\phi^k(x), \phi^k(y)) > \varepsilon$ for all $k \in \mathbb{Z}$. Two points $x, y \in X$ are said to be **proximal** if for any $\varepsilon > 0$ there is a $k \in \mathbb{Z}$ such that $d(\phi^k(x), \phi^k(y)) < \varepsilon$.

**Remark 1.3.11.** It is clear from the definition that $(X, \phi, \Sigma)$ is distal if and only if $X$ contains no pair of proximal points.

We will not investigate these properties further, but will mention that there are some interesting relationships involving these properties which have been studied [6], [13]. For example, a distal homeomorphism $\phi : X \to X$ can be decomposed into minimal pieces, i.e., $X = \bigcup_{i \in I} X_i$ where $I$ is some index set, the sets $X_i$ are closed and pairwise disjoint, $\phi(X_i) = X_i$ and $\phi|_{X_i}$ is minimal and distal.

### I.4. An Irrational Rotation of the Circle

We conclude this chapter with an example which illustrates many of the definitions and results of the preceding sections.

Given $[0,1]$ with its usual topology, let $X = [0,1]/\{0,1\}$ be the quotient space obtained by identifying 0 and 1. (So $X$ is
homeomorphic to the circle.) $X$ is compact, and the metric
\[ \rho(x, y) = \min(|x-y|, 1-|x-y|) \]
gives the topology on $X$. Define
\[ \phi : X \to X \text{ by } \phi(x) = x + \alpha \pmod{1} \]
where $\alpha$ is some fixed irrational in $X$. $\phi$ is a homeomorphism, so $(X, \phi, \Sigma)$ is a dynamical system, $\Sigma = \mathbb{N}$ or $\mathbb{Z}$.

**Proposition 1.4.1.** $\phi$ is minimal.

**Proof.** Since $\phi$ is a translation, it is enough to show that
\[ \overline{\phi}(\alpha) = X. \]

**Fact (1):** There are infinitely many distinct rational numbers,
\[ \frac{p}{q}, \ (p, q) = 1 \text{ such that } |\alpha - \frac{p}{q}| < \frac{1}{2^q} \quad [9, \text{ p. 129}]. \]

**Fact (2):** If $(p, q) = 1$, then \( \{np \pmod{q} : 1 \leq n \leq q\} = \{0, 1, 2, \ldots, q-1\} \), so $\frac{np}{q} = \left\{ \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q}{q} \right\}$ in $X$.

**Fact (3):** Given $x \in X$, there exists a sequence of rationals
\[ \frac{p_n}{q_n}, \ (p_n, q_n) = 1 \text{ with } \frac{p_n}{q_n} + X \text{ such that } q_n < q_{n+1}. \]
(Hence the $q_n$'s get arbitrarily large.)

Choose $x \in X$ and let $\varepsilon > 0$. By (1) and (3), we can find $q$ and $m$ such that $(m, q) = 1$, $\frac{1}{q} < \frac{\varepsilon}{2}$ and $|\alpha - \frac{m}{q}| < \frac{1}{q^2}$. By (2) there is an $n$, $0 < n < q$ such that $|\frac{nm}{q} - x| < \frac{1}{q}$. Furthermore,
\[ |na - \frac{nm}{q}| < \frac{1}{q^2}, \text{ so } |na - x| < |na - \frac{nm}{q}| + |\frac{nm}{q} - x| < \frac{2}{q} < \varepsilon; \]
i.e., $x \in \overline{\phi}(\alpha)$. ◊
As we noted (but did not prove) in the preceding section, any homeomorphism on the circle is nonexpansive and has zero topological entropy. We now show this to be true for $\phi$.

**Proposition I.4.2.** $(X,\phi,Z)$ has zero topological entropy.

**Proof.** Let $\varepsilon > 0$ be given. Let $F$ be a set of $\lceil \frac{1}{\varepsilon} \rceil + 1$ points evenly spaced in $X$. Then given $x \in X$, there is a $y \in F$ such that $d(x,y) < \varepsilon$, so $F$ is a $(1,\varepsilon)$-spanning set. Furthermore, $d(\phi^k(x), \phi^k(y)) = d(x+k\alpha, y+k\alpha) = d(x,y) < \varepsilon$ for all $k \in \mathbb{N}$; in particular, $\max\{d(\phi^k(x), \phi^k(y)) : 0 < k < n-1\} < \varepsilon$. Thus,

$r_n(\varepsilon,X) < \lceil \frac{1}{\varepsilon} \rceil + 1$, so $r(\varepsilon,X,\phi) = 0$. It follows that $h(\phi) = 0$ by Remark I.3.8. ◊

**Proposition I.4.4.** $(X,\phi,Z)$ is nonexpansive and is distal (so has no pair of proximal points).

**Proof.** The proof follows from the fact that a translation preserves the distance between points. ◊
II. EXTENSIONS WHICH PRESERVE MINIMALITY AND TOPOLOGICAL TRANSITIVITY

II.1. Introduction

**Definition II.1.1.** Given the dynamical systems \((X,\phi,\Sigma)\) and \((Y,\psi,\Sigma)\) if there exists a continuous surjection, \(p : Y \to X\) for which the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Y \\
\downarrow p & & \downarrow p \\
X & \xrightarrow{\phi} & X
\end{array}
\]

commutes, then \((Y,\psi,\Sigma)\) is called an extension of \((X,\phi,\Sigma)\) while \((X,\phi,\Sigma)\) is called a factor of \((Y,\psi,\Sigma)\).

When are the dynamical properties enjoyed by a system carried over to an extension or factor? It is easy to show that topological transitivity and minimality are passed from a dynamical system to a factor. (For if \(y \in Y\) has a dense orbit in \(Y\), then the orbit of \(p(y)\) is dense in \(X\).) However, these two properties will in general fail to carry over to extensions.

In this chapter we first construct extensions of \((X,\phi,\Sigma)\) obtained from function algebras. We then investigate when the extensions will inherit the properties of topological transitivity and minimality.
II.2. Forming an Extension

Let \((X,\phi,\Sigma)\) be a dynamical system and take \(B(X)\) to be the set of all bounded complex-valued functions on \(X\); recall \(B(X)\) is a \(C^*\)-algebra with the supremum norm. Define the mapping \(T : B(X) \to B(X)\) by \(Tf = f \circ \phi\). Let \(\mathcal{U}\) be a \(C^*\)-algebra of functions on \(X\) with \(C(X) \subseteq \mathcal{U} \subseteq B(X)\) and such that \(T^k(\mathcal{U}) \subseteq \mathcal{U}, \forall k \in \Sigma\); i.e., \(f \circ \phi^k \in \mathcal{U}, \forall f \in \mathcal{U}\). (We say \(\mathcal{U}\) is \(\Sigma\)-invariant.)

If \(\hat{X}\) is the spectrum of \(\mathcal{U}\), and \(\hat{f}\) the Gelfand transform of \(f \in \mathcal{U}\), then define \(\hat{T} : C(\hat{X}) \to C(\hat{X})\) to make the diagram commute; i.e., define \(\hat{T}(f) = \hat{Tf} = \hat{f} \circ \hat{\phi}\). This suggests the question: Does there exist a function \(\hat{\phi} : \hat{X} \to \hat{X}\) which satisfies \(\hat{f} \circ \hat{\phi} = \hat{\phi} \circ \hat{f}\)?

Since \((\hat{f} \circ \hat{\phi})(\hat{x}) = \hat{\phi}(\hat{x})f\) and \(\hat{\phi} \circ \hat{f}(\hat{x}) = \hat{x}(f \circ \phi)\), we define \(\hat{\phi}\) by \(\hat{\phi}(\hat{x})f = \hat{x}(f \circ \phi), f \in \mathcal{U}\).

**Remark II.2.1.** \(\hat{\phi}\) is well-defined.

**Proof.** \(\hat{\phi}(\hat{x})(af+g) = \hat{x}((af+g) \circ \phi) = \hat{x}(af \circ \phi + g \circ \phi) = \hat{x}(af \circ \phi) + \hat{x}(g \circ \phi) = a\hat{\phi}(\hat{x})f + \hat{\phi}(\hat{x})g\), so \(\hat{\phi}(\hat{x})\) is linear. Similarly, \(\hat{\phi}(\hat{x})(fg) = \hat{x}((fg) \circ \phi) = \hat{x}(f \circ \phi \circ \phi) = \hat{\phi}(\hat{x})f \cdot \hat{\phi}(\hat{x})g\), so \(\hat{\phi}(\hat{x})\) is multiplicative. Thus \(\hat{\phi}(\hat{x}) \in \hat{X}\). \(\diamondsuit\)
**Proposition II.2.2** \( \hat{\phi} \) is continuous.

**Proof.** Let \( \{x_n^*\} \) be a net which converges in the weak* topology on \( \hat{X} \) to an element \( x \in \hat{X} \).

\[
\begin{align*}
\hat{x}_n \xrightarrow{wk^*} \hat{x} &= \hat{x}_n(f) + \hat{x}(f) \quad \forall f \in \mathbb{U} \\
&= \phi(x_n(f)) + \phi(x)f \quad \forall f \in \mathbb{U} \\
&= \phi(x_n) + \phi(x)f \\
&= \phi(x_n) \xrightarrow{wk^*} \phi(x).
\end{align*}
\]

**Proposition II.2.3.** If \( \phi \) is surjective, then \( \hat{\phi} \) is surjective.

**Proof.** Suppose \( \hat{\phi}(X) \subseteq \hat{X} \). Since \( \hat{X} \) is compact and \( \phi \) is continuous, \( \hat{\phi}(X) \) is closed in \( \hat{X} \). Thus \( \hat{X} \setminus \hat{\phi}(X) \) is a nonempty open set of \( \hat{X} \). By Urysohn's lemma, there is an \( \hat{f} : C(\hat{X}) \rightarrow \mathbb{C}, \hat{f} \neq 0, \hat{f}|_{\hat{X} \setminus \hat{\phi}(X)} \equiv 0 \). That is \( f \neq 0 \), but \( \hat{f} \circ \hat{\phi} \equiv 0 \). Since \( \gamma^{-1} \) is an isometric *-isomorphism, \( \gamma^{-1}(\hat{f}) = f \neq 0 \), but \( f \circ \phi \equiv 0 \). This contradicts that \( \phi \) is a surjection.

**Proposition II.2.4.** If \( \phi \) is bijective, \( \hat{\phi} \) is bijective.

**Proof.** Let \( f \in \mathbb{U} \) and \( \phi \) be bijective. Then \( f \circ \phi^{-1} : X \rightarrow \mathbb{C} \) is well-defined and since \( X \) is compact \( \phi \) is a homeomorphism. In particular, \( \phi^{-1} \) is continuous, so \( f \circ \phi^{-1} \in \mathbb{U} \). Then

\[
T(f \circ \phi^{-1}) = (f \circ \phi^{-1}) \circ \phi = f \text{ so } T \text{ is surjective. Hence}
\]
\[ \hat{\phi}(x) = \hat{\phi}(y) = \hat{\phi}(x)f = \hat{\phi}(y)f \quad \forall f \in \mathcal{U} \]
\[ = \hat{x}(Tf) = \hat{y}(Tf) \quad \forall f \in \mathcal{U} \]
\[ = \hat{x}(f) = \hat{y}(f) \quad \forall f \in \mathcal{U}, \text{ since } T \text{ is surjective.} \]
\[ = \hat{x} = \hat{y}. \quad \diamond \]

Remark II.2.5. If \( \hat{\phi} \) is bijective, then \( \hat{\phi} \) is a homeomorphism for \( \hat{\phi} \) is continuous and \( \hat{X} \) is compact.

The results above show that \((\hat{X}, \hat{\phi}, \Sigma)\) is a dynamical system. We continue by showing that \((\hat{X}, \hat{\phi}, \Sigma)\) is an extension of \((X, \phi, \Sigma)\). We use the fact that \(X\) is the spectrum of \(C(X)\) (Theorem I.1.11).

Take \(x_0 \in \hat{X}\). Since \(x_0\) is a homomorphism on \(\mathcal{U}\), \(x_0|_{C(X)}\) is a homomorphism on \(C(X)\). Thus there is a unique element of \(X\), call it \(x_0\), such that \(x_0(f) = \delta_{x_0}(f) = f(x_0)\) for all \(f \in C(X)\). Let \(p : \hat{X} \times X\) be the mapping which associates elements of \(\hat{X}\) and \(X\) in this way; i.e., \(p(x_0) = x_0\).

Proposition II.2.6. \(p : \hat{X} \times X\) is a continuous surjection.

Proof. To show \(p\) is surjective, we must show that given a homomorphism, \(x\), on \(C(X)\) we can find a homomorphism \(\hat{x}\) on \(\mathcal{U}\) such that \(\hat{x}|_{C(X)} = x\). Equivalently, can \(x \in \text{spec}(C(X))\) be extended to \(\hat{x} \in \text{spec}(\mathcal{U})\)? The answer is 'yes' [5, Lemma 2.10.1.11] since the pure states of a commutative \(C^*\)-algebra are the homomorphisms of the algebra [5, Defn. 2.5.2].
Now let \( \{x_n\} \) be a net of homomorphisms of \( \mathbb{U} \) which converge to \( x \in \mathbb{U} \). Then

\[
\begin{align*}
\hat{x} & \xrightarrow{\text{wk}^*} \hat{x} \\
x_n & \xrightarrow{\text{wk}^*} x \\
x_n(f) + x(f) & \forall f \in \mathbb{U} \\
x_n(f) + x(f) & \forall f \in C(X) \\
\delta_{x_n}(f) + \delta_x(f) & \forall f \in C(X) \\
f(x_n) + f(x) & \forall f \in C(X) \\
x_n + x & \text{by Urysohn's lemma} \\
p(x_n) + p(x) & .
\end{align*}
\]

Note. Henceforth the symbol \( \hat{x} \) (\( \in \hat{X} \)) will be understood to be an element of \( \mathbb{p}^{-1}(x) \).

**Proposition II.2.7.** The diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\phi}} & \hat{X} \\
\downarrow p & & \downarrow p \\
X & \xrightarrow{\phi} & X
\end{array}
\]

commutes.

**Proof.** We must show that given \( \hat{x} \in \hat{X} \), \( p(\hat{\phi}(\hat{x})) = \phi(p(\hat{x})) \). We think of \( x \in X \) as being a homomorphism on \( C(X) \), so we show

\[
\langle f, p(\hat{\phi}(\hat{x})) \rangle = \langle f, \phi(p(\hat{x})) \rangle \quad \text{for all } f \in C(X).
\]

Now
\[\hat{p}(y) = y \Rightarrow \hat{y}(f) = f(y). \] Since \[\hat{\phi}(x) f = \hat{x}(f \circ \phi) = f(\phi(x))\] we have that \[p(\hat{\phi}(x)) = \phi(x).\] So \[\langle \xi, p(\hat{\phi}(x)) \rangle = \langle \xi, \phi(x) \rangle = \langle \xi, \phi(p(\hat{x})) \rangle. \]

We have then that \((\hat{X}, \hat{\phi}, \Sigma)\) is an extension of \((X, \phi, \Sigma)\). However, we do not yet have a 'workable understanding' of the space \(\hat{X}\). The next section provides an alternative description of \(\hat{X}\) which is useful in answering questions about the system \((\hat{X}, \hat{\phi}, \Sigma)\).

II.3. An Alternative Description of \(\hat{X}\)

Let \(Y = \text{cl}\{ (x, f(x)) : x \in X \} \subseteq X \times \prod_{f \in \mathcal{U}} \overline{f(X)}\); so \(Y\) is compact Hausdorff. Define the maps \(p : Y \rightarrow X\) and \(p_f : Y \rightarrow \overline{f(X)} \subseteq \mathcal{C}\) as follows: If \(y = (x, t_f) \in \mathcal{U}\), then \(p(y) = x\) and \(p_f(y) = t_f\).

**Proposition II.3.1.** The maps \(p, p_f\) are continuous surjections.

**Proof.** The surjectivity is obvious. Let \(q : X \times \prod_{f \in \mathcal{U}} \overline{f(X)} \rightarrow X\) be the projection to the first coordinate. Then \(q\) is continuous and \(p = q|_Y\). Hence \(U\) open in \(X\) implies \(q^{-1}(U)\) is open so it follows that \(p_f^{-1}(U) = q^{-1}(U) \cap Y\) is open. The continuity of \(p_f\) is shown similarly. \(\diamond\)

Now the maps \(p_f\) separate the points of \(Y\): For suppose \(y_1 \neq y_2\) where \(y_1 = (x_1, t_f) \in \mathcal{U}, y_2 = (x_2, s_f) \in \mathcal{U}\). If \(x_1 \neq x_2\), then there is a continuous function \(f_0 \in C(X) \subseteq \mathcal{U}\) for which \(f_0(x_1) \neq f_0(x_2)\). Since \(t_f = f_0(x_1)\) and \(s_f = f_0(x_2)\), it follows
If \( x_1 = x_2 \), then for some \( f_1 \in \mathfrak{U}, \ t_{f_1} \neq s_{f_1} \), so \( p_{f_1}(y_1) \neq p_{f_1}(y_2) \).

Thus the set \( \{ p_f : f \in \mathfrak{U} \} \) is an algebra of continuous, complex-valued functions on \( Y \) which separates the points, contains the constant functions, and is closed under complex conjugation. Furthermore,

\[
\| p_f \| = \sup \{|p_f(y)| : y \in Y\} = \sup \{|t| : t \in f(X)\} = \sup \{|f(x)| : x \in X\}
\]

= \( \| f \| \). It follows that \( f + p_f \) is an isometric \( * \)-isomorphism of \( \mathfrak{U} \) with a dense subalgebra of \( C(Y) \) (by Stone-Weierstrass). But \( \mathfrak{U} \) is complete by hypothesis, so \( \{ p_f : f \in \mathfrak{U} \} \) is complete and hence equal to \( C(Y) \). This proves the

**Proposition II.3.2.** \( Y = \text{spec}(\mathfrak{U}) \).

Henceforth we will often use \( \hat{f} \) (in addition to \( p_f \)) to denote the Gelfand transform of \( f \in \mathfrak{U} \).

**Remark II.3.3.** Instead of realizing \( \hat{X} \) as the closure of the graph

\[
\{(x, (f(x))_{f \in \mathfrak{U}}) : x \in X \} \subset X \times \prod_{f \in \mathfrak{U}} f(X) \]

as above, it is also possible to realize \( \hat{X} \) as the closure of the graph

\[
\{(x, (f(x))_{f \in E}) : x \in X \} \subset X \times \prod_{f \in E} f(X) \]

where \( E \) is any \( * \)-invariant subset of the unit ball of \( \mathfrak{U} \) which topologically generates \( \mathfrak{U} \); i.e., the algebraic span of \( E \) is dense in \( \mathfrak{U} \).

**Proof.** Let \( Y_E = \text{cl}\{(x, (f(x))_{f \in E}) : x \in X \} \subset X \times \prod_{f \in E} f(X) \) and let \( Y_S = \text{cl}\{(x, (f(x))_{f \in S}) : x \in X \} \subset X \times \prod_{f \in S} f(X) \) where \( S \) is the
algebraic span of $E$. If $y = (x, (t^f)_{f \in E})$ and $w = (x, (g^g)_{g \in S})$ then we let $y \mapsto w$ be the mapping obtained by choosing $s^g = p(t^f ; f \in E)$ where $g = p(E)$, i.e., $p$ is a polynomial of finitely many elements of $E$. In particular, if $x$ is well-defined and bijective. Furthermore, the composition $\tau^{-1}$ is continuous since it is merely the restriction of the projection mapping. Thus $\tau$ is a homeomorphism.

Similarly, if $z = (x, (r^h)_{h \in \mathfrak{U}})$, then the mapping $w \mapsto z$ defined by $r^h = \lim g^g_n$ where $g^g_n + h$ is a homeomorphism; so $Y_E \simeq Y_3 \simeq Y$. 

**Remark 11.3.4.** If $E$ (as above) has countable cardinality, then $C(Y)$ is separable so $Y$ is metrizable [4, p. 144, Thm. 6.6]. A metric is given by $d(y_1, y_2) = \sum_{n=1}^{\infty} \frac{1}{2n} |\xi_n(y_1) - \xi_n(y_2)|$ where $E = \{\xi_n : n = 1, 2, \ldots\}$.

**Proof.** Given $y \in Y_E$ and the $\epsilon$-ball $B_d(y, \epsilon)$, a basic open neighborhood, $U$, of $y$ can be found such that $U \subset B_d(y, \epsilon)$. Thus the identity mapping $Y_E + (Y_E, d)$ is continuous bijection. Since $Y_E$ is compact, the mapping is a homeomorphism.

**Remark 11.3.5.** The action of $\hat{\phi}$ on $Y$ is $\hat{\phi}(y) = (\phi(x), (t^f\circ\phi)_{f \in \mathfrak{U}})$ since $\hat{\phi}(y)(f) = y(f\circ\phi)$. For then $\hat{\phi}(y)(f) = p_f(\hat{\phi}(y)) = p_f\circ\phi(y) = y(f\circ\phi)$ for all $f \in \mathfrak{U}$.

**II.4. The Main Result**

**Definition II.4.1.** Let $f \in B(X)$, $x \in X$. We define the **limit set** of $f$ at $x$ to be \( \{z \in \mathfrak{U} : \text{there exists a net } (x_i) \subset X, x_i \rightarrow x, \)
such that \( f(x_1) + z \), and denote it by \( \Lambda(f;x) \). The same notation will be used for vector-valued functions \( \tilde{f} : X \to \mathbb{C}^n \).

**Lemma II.4.2.** Let \( f \in \mathcal{U}, \ x \in X \).

(i) \( \hat{f}(p^{-1}(x)) = \Lambda(f;x) \cup \{f(x)\} \).

(ii) If \( f \) is continuous at \( x \), \( \hat{f}(p^{-1}(x)) = \{f(x)\} \).

**Proof.** (i) Let \( \hat{x} \in p^{-1}(x) \). Then either \( \hat{x} = (x, (f(x))_{j} \in \mathcal{U}) \), in which case \( \hat{f}(\hat{x}) = f(x) \), or else \( \hat{x} = \lim(x_j, (f(x_j))_{j} \in \mathcal{U}) \), where \( x_j \to x, \ x_j \neq x, \) and \( f(x_j) \to f(x) \in \Lambda(f;x) \). (ii) follows immediately from (i).

As mentioned earlier, we are after a result which indicates when the topological transitivity or minimality of \( (X,\phi,\Sigma) \) is inherited by \( (X,\hat{\phi},\Sigma) \). The following examples illustrate what can go awry.

**Examples II.4.3.**

(i) Let \( \chi \) be the characteristic function of an orbit \( \{\phi^k(x_0) : k \in \mathbb{Z}\} \); so \( \chi \circ \phi = \chi \). Take \( \mathcal{U} \) to be the \((\Sigma\text{-invariant}) \ C^*\) algebra generated by \( C(X) \) and \( \chi \). Since \( f + \hat{f} \) is an isometric isomorphism, \( \hat{\chi}^2 = \hat{\chi} \) with range \( \{0,1\} \); thus \( \hat{X}_0 = \hat{\chi}^{-1}(0) \) and \( \hat{X}_1 = \hat{\chi}^{-1}(1) \) are nonempty open sets whose union is \( \hat{X} \). It follows from \( \hat{\chi} = \chi \circ \hat{\phi} = \chi \circ \phi \hat{\phi} \) that \( \hat{X}_0 \) and \( \hat{X}_1 \) are \( \mathbb{Z} \)-invariant. In particular \( (\hat{X},\hat{\phi},\mathbb{Z}) \) is not topologically transitive.
(ii) Suppose \( U \) contains the function \( e_{x_0} = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases} \), then \( y_0 = (x_0, [f(x_0)]_f \in U) \in Y = \hat{X} \) is an isolated point as is each \( \phi^k(y_0), k \in \mathbb{Z} \). It follows that \( (X, \phi, \Sigma) \) is not minimal, for \( Y \) being compact implies there is a \( y_1 \in Y \setminus \overset{\phi}{\Theta}(y_0) \). Then \( y_0 \notin \overset{\phi}{\Theta}(y_1) \), so \( y_0 \notin \overset{\phi}{\Theta}(y_1) \) since \( y_0 \) is isolated.

(iii) Let \( \mathcal{B} \) denote the \( C^* \)-algebra generated by the bounded, complex-valued functions on \( X \) which are continuous on a cofinite set (i.e., the complement of a set of finite cardinality). (Of course \( \mathcal{B} \) will also contain functions with infinitely many points of discontinuity.) Suppose that \( (X, \phi, \Sigma) \) is minimal and \( \phi \) has the property that \( \phi^{-1}(x) \) is of finite cardinality for each \( x \in X \). (Then \( f \circ \phi^k \in \mathcal{B} \) whenever \( f \in \mathcal{B} \).) Taking \( U = \mathcal{B} \), \( (X, \phi, \Sigma) \) will not even be topologically transitive. For given \( x_1 \in X \), there is an \( x_0 \in X \setminus \overset{\phi}{\Theta}(x_1) \). Then the function \( e_{x_0} \in \mathcal{U} \) gives rise, as above, to an isolated point in \( \hat{X} \setminus \overset{\phi}{\Theta}(x_1) \) where \( x_1 \in \phi^{-1}(x_1) \).

**Theorem II.4.4.** Let \( (X, \phi, \Sigma) \) be a dynamical system, and let \( \mathcal{U} \supset C(X) \) be a \( \Sigma \)-invariant \( C^* \)-algebra which is topologically generated by a subset, \( E \), of functions each of which is continuous on a cofinite set. Let \( \hat{X} \) be the spectrum of \( \mathcal{U} \) and \( (\hat{X}, \hat{\phi}, \hat{\Sigma}) \) the resulting extension with \( p : \hat{X} + X \) the continuous surjection. Consider the following conditions.

(i) \( f(x) \in A(f; x) \) for all \( f \in \mathcal{U}, x \in X \).

(ii) \( \mathcal{U} \) contains no characteristic function \( e_x \) (\( x \in X \)).
(iii) If the orbit \( \Theta \phi(x_0) \) is dense in \( X \), then the orbit \\
\( \Theta \phi(x_0) \) is dense in \( \hat{X} \) for every \( x_0 \in p^{-1}(x_0) \), \\
\( x_0 \in X \).

Conditions (i) and (ii) are equivalent, and each implies (iii). If we 
also assume that \( \phi^{-1}(x) \) is a finite set for each \( x \in X \), \( X \) contains 
no isolated points, and there is (at least) one point in \( X \) with a dense 
orbit, then (iii) implies (i) and (ii).

Following immediately is the

**Corollary II.4.5.** Suppose \( \mathcal{U} \) satisfies either (i) or (ii). Then 

(a) If \( (X, \phi, \Sigma) \) is topologically transitive, \( (\hat{X}, \hat{\phi}, \hat{\Sigma}) \) is also 
topologically transitive.

(b) If \( (X, \phi, \Sigma) \) is minimal, \( (\hat{X}, \hat{\phi}, \hat{\Sigma}) \) is also minimal.

**Proof of Theorem II.4.4.** Clearly (i) \( \Rightarrow \) (ii). Suppose (i) fails to 
hold, so there is a \( g \in \mathcal{U} \) and \( x_0 \in X \) with \( g(x_0) \notin \Lambda(g; x_0) \). Set

\[
g_1 = g - g(x_0); \quad g_1(x_0) = 0 \]
\[
g_2 = g_1 g_1; \quad g_2(x_0) = 0, \quad \text{range}(g_2) \subseteq [0, \infty) \]
\[
g_3 = g_2 / \|g_2\|; \quad g_3(x_0) = 0, \quad \text{range}(g_3) \subseteq [0, 1] \]
\[
g_4 = 1 - g_3; \quad g_4(x_0) = 1, \quad \text{range}(g_4) \subseteq [0, 1].
\]
Since \( 1 \notin \Lambda(g_4; x_0) \), there is an \( \varepsilon > 0 \) and an open neighborhood \( U \) of \( x_0 \) such that \( g_4(x) < 1 - \varepsilon \) for \( x \in U \setminus \{x_0\} \). By Urysohn's lemma there is an \( h \in C(X) \), \( 0 < h < 1 \), \( h(x_0) = 1 \), \( \text{supp}(h) \subseteq U \). Let \( g_5 = hg_4 \). Then \( g_5 \in \mathcal{U} \) and \( g_5(x_0) = 1 \), while \( 0 < g_5(x) < 1 - \varepsilon \) for \( x \in X \setminus \{x_0\} \). Then \( e_{x_0} = \lim_{n \to \infty} g_5^n \in \mathcal{U} \). Thus \((ii) \Rightarrow (i)\).

To show \((i) \Rightarrow (iii)\) we need the following

**Lemma II.4.6.** Suppose \( g(x_0) \notin \Lambda(g;x_0) \) for all \( x_0 \in X \) and for all \( g \in C^*(f_1, f_2, \ldots, f_n) \) where \( C^*(f_1, f_2, \ldots, f_n) \) is the \( C^* \)-algebra generated by \( f_1, \ldots, f_n \in B(X) \). Then \( \hat{f}(x_0) \notin \Lambda(\hat{f};x_0) \) where \( \hat{f} = (f_1, \ldots, f_n) \) maps \( X \) to \( C^n \).

**Proof.** Suppose \( \hat{f}(x_0) \notin \Lambda(\hat{f};x_0) \). Let \( K \subseteq C^n \) be compact such that \( \hat{f}(X) \subseteq K \). Since \( \Lambda(\hat{f};x_0) \) is closed, by Urysohn's lemma there is a continuous function \( u : K \to [0,1] \) such that \( u| = 1 \).

Consider the coordinate functions \( q_1 : K \to C, q_1(z_1, \ldots, z_n) = z_1 \).

By Stone-Weirstrass, the algebra over \( C \) generated by \( \{q_1, \ldots, q_n, \bar{q}_1, \ldots, \bar{q}_n, 1\} \) is dense in the continuous functions on \( K \). It follows that \( h = u(f_1, \ldots, f_n) \notin C^*(f_1, \ldots, f_n) \). But then \( h(x_0) \notin \Lambda(h;x_0) \), a contradiction. \( \Diamond \)

\((i) \Rightarrow (iii)\). By Remark II.3.3 we may consider \( \hat{X} \subseteq X \times \prod_{f \in E} \mathcal{F}(X) \) where each \( f \in E \) is continuous on a cofinite set. Let \( U \subseteq \hat{X} \) be a nonempty basic open set; i.e., given, say \( f_1, \ldots, f_n \in E \), let \( U \) be
open in \( f(X) \). Then \( U = X \cap (U' \times U'' \times \prod_{f \in E \setminus \{f_1, \ldots, f_n\}} f(X)) \)
where \( U' \) is open in \( X \) and \( U'' = U_1 \times \cdots \times U_n \).

Let \( \hat{f} = (f_1, \ldots, f_n) : X \to \mathbb{R}^n \), then \( \hat{f} \) is continuous on a cofinite subset \( X_0 \subset X \). Since by Lemma II.4.6 \( \hat{f} \) satisfies the limit set condition, the nonempty set \( \hat{f}^{-1}(U'') \cap U' \) has infinite cardinality. Thus the set \( \hat{f}^{-1}(U'') \cap U' \cap X_0 \) is nonempty. But the latter is \( (\hat{f}|X_0)^{-1}(U'') \cap U' \cap X_0 \equiv V \) which is open in \( X_0 \) since \( \hat{f}|X_0 \) is continuous. It follows that \( V \) is also open in \( X \).

By hypothesis (iii) of the theorem, the orbit of \( x_0 \) is dense in \( X \), so there is a \( k \in \Sigma \) with \( \hat{f}^k(x_0) \in V \). Choose an arbitrary \( \hat{x}_0 \in p^{-1}(x_0) \). Since \( p(\hat{x}_0) \in V \), \( \hat{f}^k(\hat{x}_0) \in U \), and we conclude that the orbit \( \Theta_{\hat{f}}(\hat{x}_0) \) is dense in \( \hat{X} \).

(iii) \Rightarrow (ii). Suppose \( e_{x_0} \in U \) and \( x_1 \) has a dense orbit in \( X \).

Case 1: \( x_0 = x_1 \). Since \( x_0 \) is not an isolated point, \( p^{-1}(x_0) \) contains at least two points \( y_0 \) and \( y'_0 \) where \( y_0 = (x_0, (f(x_0))_f \in U) \) and \( y'_0 = \lim(x_1, (f(x_1))_f \in U) \times x_0, x_1 \neq x_0 \). If \( \hat{f}^k(y'_0) = y_0 \), then \( \hat{f}^k(x_0) = y_0 \) which contradicts that \( \Theta_{\hat{f}}(x_0) \) is dense in \( X \); hence \( y_0 \notin \Theta_{\hat{f}}(y'_0) \). Since \( y_0 \) is an isolated point, \( y_0 \notin \Theta_{\hat{f}}(y'_0) \).

Case 2: \( x_0 \in \Theta_{\hat{f}}(x_1) \). Then \( x_0 = \hat{f}^k(x_1) \), so \( e_{x_0} \circ \hat{f}^k \in U \) is the characteristic function of the finite set \( \hat{f}^{-k}(x_0) \), which includes the point \( x_1 \). By Urysohn's lemma, there is a function \( h \in C(X) \) such that
h(x_1) = 1 and h \big|_{\phi^{-k}(x_0) \setminus \{x_1\}} = 0. Then h \cdot (e_0 \circ \phi^k) = e_{x_1} \in \mathbb{U}.

This is Case 1.

Case 3: \( x_0 \notin \phi(x_1) \). Since \( y_0 \) is isolated, \( y_0 \notin \phi(x_1) \) for any \( x_1 \in \phi^{-1}(x_1) \).

Corollary II.4.7. Let \((X, \phi, \Sigma)\) be a dynamical system such that \( X \) has no isolated points and \( \phi \) is an open map. Let \( \{x_i\} \subseteq X \) be a collection of distinct points and \( \{\xi_i\} \subseteq \mathcal{B}(X) \) a collection of functions satisfying

(a) \( \xi_i \) is continuous on \( X \setminus \{x_i\} \), \( i \in I \);

(b) the orbits \( \phi_i(x_i), \phi_j(x_i) \) are disjoint for \( i \neq j \);

(c) \( \phi^{-k}(x_i) \) is a finite set for \( k \in \mathbb{N}, i \in I \).

Let \( \mathcal{U} \) be the \( C^* \)-subalgebra of \( \mathcal{B}(X) \) generated by \( C(X) \) and the functions \( \xi_i \circ \phi^k, i \in I, k \in \mathbb{N} \). Then Theorem II.4.4 applies to \( \mathcal{U} \), and condition (i) may be rephrased as

\[(i') \quad \xi_i(x_1) \in \Lambda(\xi_i; x_1) \quad \text{for all} \quad i \in I.\]

Proof. If \( \phi \) is a homeomorphism, \( \xi_i \circ \phi^k \) has at most one point of discontinuity, \( \phi^{-k}(x_i) \), and clearly \( \xi_i \) satisfies the limit set condition at \( x_1 \) iff \( \xi_i \circ \phi^k \) satisfies the condition at \( \phi^{-k}(x_i) \).

Suppose \( \Sigma = \mathbb{N} \) and \( \xi_i(x_1) \in \Lambda(\xi_i; x_1) \). Let \( \{t_n\} \) be a net converging to \( x_1 \) such that \( \xi_i(x_1) = \lim_{n \to \infty} \xi_i(t_n) \). For a fixed \( k \in \mathbb{N} \) let
s ∈ φ^k(x_i). Given an open neighborhood, U_j, of s, φ^k(U_j) is an open neighborhood of x_i, so there is an s_j ∈ U_j such that

φ^k(s_j) = t_n_j. Then the net s_j + s and

\[ \lim_j \phi^k(s_j) = \lim_j \xi_i(t_n_j) = \xi_i(x_i) = \xi_i \circ \phi^k(s). \]

It follows from this argument and (b) that f(x_i) ∈ A(f;x_i) ∀f ∈ \mathbb{U}, ∀I ∈ I. Since X has no isolated points, f(x) ∈ A(f;x) ∀f ∈ \mathbb{U}, ∀x ∈ X.

Finally, it follows from condition (c) that the functions \( \xi_i \circ \phi^k \), which together with \( C(X) \) topologically generate \( \mathbb{U} \), are each continuous on a cofinite set.  

**Remark II.4.8.** It would be interesting to have a theorem analogous to Theorem II.4.4 which applied to arbitrary \( \Sigma \)-invariant \( C^* \)-function algebras between \( C(X) \) and \( B(X) \). Indeed, the equivalence of conditions (i) and (ii) does hold in general. However, without some restriction (i) does not imply (iii). For let \( (X,\phi,\Sigma) \) be a minimal dynamical system and take \((\hat{X},\hat{\phi},\hat{\Sigma})\) to be the extension described in Example II.4.3.i. This system is not even topologically transitive even though the limit set condition is satisfied for every \( f ∈ \mathbb{U} \).

Finally, we see that an extension can be topologically transitive even if condition (i) of Theorem II.4.4 is not satisfied.

**Proposition II.4.9.** Let \((X,\phi,\Sigma)\) be a dynamical system and let \( \{x_i\}, \{\xi_i\} \) be as in Corollary II.4.7. Fix an index \( i_0 = 0 ∈ I \), and assume that \( \xi_0(x_0) \notin A(\xi_0;x_0) \); otherwise assume \( \xi_1(x_i) ∈ A(\xi_1;x_1) \).
Suppose that $\phi(x_0)$ is dense in $X$, and let $y_0 = (x_0, (f(x_0))_f \in \mathcal{U})$. Then

(i) if $\Sigma = \mathbb{N}$, $y_0$ is the unique point of $\hat{x}$ with a dense orbit;

(ii) if $\Sigma = \mathbb{Z}$, then $y$ has a dense orbit if and only if $y \in \phi(y_0)$.

Sketch of Proof: Since the points $\phi(y_0)$ are open, no point lying outside this orbit can itself have a dense orbit. The fact that $\phi(y_0)$ is dense is essentially the same as in Theorem II.4.4.

We conclude this chapter with a specific example.

Example II.4.10. Take $(X, \phi, \Sigma)$ be the dynamical system described in Chapter I.4. Let

$$s(x) = \begin{cases} \sin\left(\frac{1}{x - \frac{1}{2}}\right) & x \neq \frac{1}{2} \\
\beta & x = \frac{1}{2} \end{cases}$$

and let $\mathcal{U}_\Sigma$ be the $C^*$-algebra generated by $C(X)$ and the functions $s \circ \phi^k$, $k \in \Sigma$. Let $\hat{X}_\Sigma$ be the spectrum of $\mathcal{U}_\Sigma$. If $\beta \in [-1,1]$, then $(\hat{X}_\Sigma, \hat{\phi}, \Sigma)$ is minimal. If $\beta \notin [-1,1]$, then $(\hat{X}_\Sigma, \hat{\phi}, \Sigma)$ has a countably infinite number of points with dense orbits (namely, the points $(\phi^k(-\frac{1}{2}), (f(\phi^k(-\frac{1}{2})))_f \in \mathcal{U}_\Sigma)$, $k \in \mathbb{Z}$), $(\hat{X}_\mathbb{N}, \hat{\phi}, \mathbb{N})$ has exactly one point with a dense orbit, $(\frac{1}{2}, (f(-\frac{1}{2}))_f \in \mathcal{U}_\mathbb{N})$, and $(\hat{X}_\mathbb{Z}, \hat{\phi}, \mathbb{N})$ has no point with a dense orbit.
III. AN EXTENSION OF THE IRRATIONAL
ROTATION OF THE CIRCLE

III.1. Introduction

We now turn our attention to the dynamical properties discussed in
Chapter I.3. One can show - using an alternate but equivalent definition
of topological entropy [13, p. 166] - that the entropy of any extension
of \((X, \phi, \mathbb{Z})\) is greater than or equal to the entropy of
\((X, \phi, \mathbb{Z})\). This is, of course, to be expected. A more interesting - and
challenging - question is: When is the entropy of an extension equal to
that of the original system? Likewise, when and how do the other
topological properties carry over to extensions? We do not answer these
questions in any generality, but we will prove the following result.

Let \((X, \phi, \mathbb{Z})\) be the dynamical system discussed in Chapter I.4. Let
\(\xi_0 : X \to \mathbb{C}\) be a bounded function (without loss of generality we assume
\(\|\xi_0\| = 1\)) which is continuous on \(X \setminus \{0\}\), and discontinuous at 0 (so
that we get a proper extension). Take \(\mathcal{A}\) to be the \(C^\ast\)-algebra generated
by \(C(X)\) and the functions \(\xi_k = \xi_0 \circ \phi^k, k \in \mathbb{Z}\) (so \(\xi_k\) is
discontinuous at \(-ka\)). With \(\hat{X}\) and \(\hat{\phi}\) as in Chapter II, the dynamical
system \((\hat{X}, \hat{\phi}, \mathbb{Z})\) preserves the zero topological entropy of \((X, \phi, \mathbb{Z})\),
differs from \((X, \phi, \mathbb{Z})\) in that \((\hat{X}, \hat{\phi}, \mathbb{Z})\) has proximal points (so is not
distal), and may or may not be expansive depending on the function \(\xi_0\).

We will use \(\Delta(z, r) \subseteq \mathbb{C}\) to denote the open disc with center \(z\) and
radius \(r\). Also, if \(U_1, U_2, \ldots, U_n\) are open covers of some space \(W\),
then \( \bigvee_{i=1}^{n} U_i = \left\{ \bigcap_{i=1}^{n} V_i : V_i \text{ open in } U_i \right\} \) is itself an open cover of \( W \) and is called the **join** of \( U_1, U_2, \ldots, U_n \).

### III.2. Dynamical Properties

It follows from Remark II.3.4 that \( \hat{X} \) is metrizable and that a metric is given by

\[
d(y_1, y_2) = \sum_{k \in Z} \frac{1}{2|k|} |\hat{\xi}_k(y_1) - \hat{\xi}_k(y_2)| + \rho(p(y_1), p(y_2))
\]

where we recall that \( \hat{\xi}_k \) is the Gelfand transform of \( \xi_k \) and \( \rho(\cdot, \cdot) \) is the usual metric on the circle.

**Proposition III.2.1.** The topological entropy of \((\hat{X}, \hat{\phi}, \hat{Z})\) is zero.

**Proof.** Let \( \varepsilon > 0 \) be given. Choose \( M \) such that

\[
\sum_{|k| \geq M} \frac{1}{2|k|} < \frac{\varepsilon}{4}.
\]

Let \( q = \left[ \frac{3}{\varepsilon} \right] + 1 \). Partition \( X \) as \( \bigcup_{j=1}^{q} I_j \) where \( I_j = \left[ \frac{j}{q}, \frac{j+1}{q} \right] \). By [11, p. 29, 39] we can choose \( N(\varepsilon) \) so large that

\[
n > N \Rightarrow \text{card} \left\{ \{k\alpha : |k| < n \} \cap I_j \right\} < \frac{3n}{q} \quad \text{for all } 1 \leq j \leq q.
\]

**Remark III.2.2.** Suppose the set \( T = \{y_1, y_2, \ldots, y_t \} \subset Y \) satisfies: For each \( y \in Y \), there is a \( y_j \in T \) such that

\[
|\hat{\xi}_k(y) - \hat{\xi}_k(y_j)| < \frac{\varepsilon}{6}, \quad k \in \{-M, -M+1, \ldots, M+n-1\}.
\]

This is equivalent to: For each \( y \in Y \), there is a \( y_j \in T \) such that

\[
|\hat{\xi}_k(\hat{\phi}^m(y)) - \hat{\xi}_k(\hat{\phi}^m(y_j))| < \frac{\varepsilon}{6}
\]

where \( k \in \{-M, -M+1, \ldots, M+n-1\} \), \( m \in \{0,1, \ldots, n-1\} \). This in turn implies that: For each \( y \in Y \), there
is a $y_j \in T$ such that $\sum_{k \in \mathbb{Z}} \frac{1}{2|k|} |\hat{\phi}_k^n(y) - \hat{\phi}_k^n(y_j)| < \frac{\varepsilon}{2}$,

$m = 0, 1, \ldots, n-1$. In other words, $T$ is an $(n, \varepsilon)$-spanning set for

$(Y, \phi, Z)$ with respect to $\sum_{k \in \mathbb{Z}} \frac{1}{2|k|} |\hat{\phi}_k(y) - \hat{\phi}_k(y')|$. 

We now calculate the number of points needed for an $(n, \varepsilon)$-spanning

set where $n > N$. Let $I$ be the union of three consecutive intervals,

$I_j$; say $I = I_1 \cup I_2 \cup I_3$. (Notice the length of $I$ is $3q < \varepsilon$.) Let

$A = \{k : |k| < n + M \text{ and } ka \in I\}$. Since $n > N$, $\text{card}(A) < \frac{9(n+M)}{q}$.

Take $\{\Delta(z_i, \frac{1}{2q}) : z_i \in S\}$ to be a covering of $\varepsilon_0(X)$ where

$\text{card}(S) < Bq^2$ for some constant $B$. Then

$\text{card}(S) \cup \bigcup_{i=1}^{\text{card}(S)} \left( p^{-1}(\Delta(z_i, \frac{1}{2q})) \cap p^{-1}(I) \right) = p^{-1}(I)$ for each $k \in A$, so if

$C_1 \equiv \bigvee_{k \in A} \left( p^{-1}(\Delta(z_i, \frac{1}{2q})) \cap p^{-1}(I) \right) \quad 1 \leq i \leq \text{card}(S)$,

then

$\bigcup C_1 = p^{-1}(I)$ and $\text{card}(C_1) < (Bq^2)^{\frac{9(n+M)}{q}}$.

Now divide $I_2$ into $D(\varepsilon)$ intervals of equal length,

$J_0, J_1, \ldots, J_D$, where $D$ is taken large enough that if $-ma \notin I$, then

$|\hat{\phi}_m^n(y) - \hat{\phi}_m^n(y')| < \frac{\varepsilon}{12}$ for $y, y' \in J_j$. Then $C_2 \equiv \{p^{-1}(J_j), 1 \leq j \leq D\}$

covers $p^{-1}(I)$ and so $C_3 \equiv C_1 \bigvee C_2$ covers $p^{-1}(I)$ and has at most

$D(Bq^2)^{\frac{9(n+M)}{q}}$ elements. Take $T$ to be a collection of points gained

by choosing one from each element of $C_3$. Then $T$ has at most

$D(Bq^2)^{\frac{9(n+M)}{q}}$ points, and by Remark III.2.1, $T$ is an $(n, \varepsilon)$-spanning

set for $I_2$ with respect to $\sum_{k \in \mathbb{Z}} \frac{1}{2|k|} |\hat{\phi}_k(y) - \hat{\phi}_k(y')|$. We may add

$\left[ \frac{2}{\varepsilon} \right]$ points to $D$ if necessary to ensure that $T$ is an $(n, \varepsilon)$-spanning

set for $I_2$ with respect to the metric $d$. 

We find an \((n,e)\)-spanning set for each of the \(I_j\), \(1 < j < q\) in the same way. The union of these \(q\) sets is an \((n,e)\)-spanning set for \(Y\) and has at most \(qD(Bq^2)2^{9(n+M)/q}\) points. Thus
\[ r_n(e,Y) < qD(Bq^2)2^{9(n+M)/q}, \]
so
\[ \log_2 r_n(e,Y) = \log q + \log D + \frac{9(n+M)}{q} \log B + 2 \log q \]. It follows that
\[ r(e,Y) = \frac{9q}{q} [\log B + 2 \log q] \] and so \(h(\phi;Y) = \lim_{q \to \infty} \frac{18}{q} \log q = 0\).
Finally, \(h(\phi) = 0\) by Remark I.3.8. 

We now turn to the property of expansiveness. Our result depends on the following

**Lemma III.2.3.** Let \(\Lambda^- (\xi_0;0)\) and \(\Lambda^+ (\xi_0;0)\) denote the left-sided and right-sided limit sets of \(\xi_0\) at \(0\). Then \(\Lambda^- (\xi_0;0) \subseteq C\) and \(\Lambda^+ (\xi_0;0) \subseteq C\) are each connected sets.

**Proof.** Suppose \(\Lambda^+ (\xi_0;0)\) is not connected. Since \(\xi_0\) is bounded, there are open sets \(U_1, U_2 \subseteq \Lambda(0,R)\) (where \(\Lambda(0,R)\) contains \(\text{range}(\xi_0)\)), such that \(\Lambda^+ (\xi_0;0) \cap U_i \neq \emptyset\) \(i = 1,2\), and \(\Lambda^+ (\xi_0;0) \subseteq U_1 \cup U_2\). Let the set \(\{x_i\}_{i=1}^{\infty}\) approach \(0\) monotonically from the right such that \(\xi_0(x_{2n}) \in U_1\) and \(\xi_0(x_{2n+1}) \in U_2\). Since \(\xi_0\) is continuous on \([x_{2n+1}, x_{2n}]\), there is a \(t_n \in [x_{2n+1}, x_{2n}]\) with \(\xi_0(t_n) \in D = \Lambda(0,R) \setminus (U_1 \cup U_2)\). Since \(\Lambda(0,R)\) is compact and \(D\) is closed, the set \(\{\xi_0(t_n)\}_{n=1}^{\infty}\) has a limit point, \(z_0\), in \(D\). But then \(z_0 = \lim_{n \to \infty} \xi_0(t_n)\) so \(z_0 \in \Lambda^+ (\xi_0;0)\), contradiction. 

Proposition III.2.4. \((\hat{x}, \hat{\phi}, \mathbb{C})\) is expansive if and only if the set 
\(\Lambda(\xi_0; 0) \subseteq \mathbb{C}\) has no limit points.

Proof. Suppose \(\Lambda(\xi_0; 0)\) has a limit point. Then given \(\delta > 0\) we 
can find \(y_1, y_2 \in \mathbb{C}^{-1}(0)\) such that \(|\hat{\xi}_0(y_1) - \hat{\xi}_0(y_2)| < \delta\). Since 
\(|\hat{\xi}_0(y_1) - \hat{\xi}_0(y_2)| = d(y_1, y_2)\), \(d(y_1, y_2) < \delta\). Moreover,
\[
\frac{1}{1 - k} |\hat{\xi}_k(y_1) - \hat{\xi}_k(y_2)|
\]
\[
= \frac{1}{1 - k} |\hat{\xi}_0(y_1) - \hat{\xi}_0(y_2)| < \delta \quad \text{for } k \in \mathbb{Z}, \quad \text{so } \hat{\phi} \text{ is nonexpansive.}
\]

Conversely, suppose \(\Lambda(\xi_0; 0)\) has no limit point. Since \(\Lambda(\xi_0; 0)\)
is compact, \(\Lambda(\xi_0; 0)\) contains only finitely many points. It follows 
from Lemma III.2.2 that \(\Lambda(\xi_0; 0)\) contains only two points, namely
\[
\lim_{x \rightarrow 0} \xi_0(x) \quad \text{and} \quad \lim_{x \rightarrow 0} \xi_0(x).
\]
Let 
\[
\eta = \min \{d(y, y') : y \neq y', y, y' \in \mathbb{C}^{-1}(0)\}
\]
Since each one sided 

limit exists, \(\xi_0\) will be uniformly continuous as seen on the 
interval \([0,1]\). Thus there is a \(\delta > 0\) such that if 
\[
|x_1 - x_2| < \delta \quad (x_1, x_2 \in [0,1]), \quad \text{then } |\xi_0(x_1) - \xi_0(x_2)| < \frac{\eta}{4}.
\]

We claim that \(\varepsilon = \frac{1}{2} \min(\eta, \delta)\) is an expansive constant for \(\hat{\phi}\). For 
if \(y_1, y_2 \in \mathbb{C}^{-1}(-k\alpha)\), then 
\[
d(\hat{\xi}_k(y_1), \hat{\xi}_k(y_2))
\]
\[
|\hat{\xi}_0(\hat{\phi}^k(y_1)) - \hat{\phi}^k(y_2)| = 2\varepsilon. \quad \text{On the other hand, if}
\]
x_1 \equiv p(y_1) \neq p(y_2) \equiv x_2, \quad \text{then if } \rho(x_1, x_2) > \delta \text{ we are done. If}
\[
\rho(x_1, x_2) < \delta, \quad \text{then there is some } -k\alpha \text{ contained in the interval}
\]
(x_1, x_2) whose length is less than \(\delta\). Thus \(0 \in X\) is contained in the 
interval \((\phi^k(x_1), \phi^k(x_2))\). Hence \(\rho(\phi^k(x_1), 0) < \delta, \quad i = 1, 2\) so
It follows that $d(\hat{\phi}^k(y_1), \hat{\phi}^k(y_2)) > \frac{n}{2} > \varepsilon$. \hfill \Box

**Proposition III.2.5.** \((X, \hat{\phi}, Z)\) is not distal.

**Proof.** Let \(y_1, y_2 \in p^{-1}(0)\). Then \(y_1\) and \(y_2\) are proximal points, for $d(y_1, y_2) = |\hat{\varepsilon}_0(y_1) - \hat{\varepsilon}_0(y_2)|$, and

$$d(\hat{\phi}^k(y_1), \hat{\phi}^k(y_2)) = \frac{1}{2|k|} |\hat{\varepsilon}_k(\hat{\phi}^k(y_1)) - \hat{\varepsilon}_k(\hat{\phi}^k(y_2))| = \frac{1}{2|k|} |\hat{\varepsilon}_0(y_1) - \hat{\varepsilon}_0(y_2)| + 0 \text{ as } |k| \to \infty.$$ The same argument works for \(y_1, y_2 \in p^{-1}(na), n \in \mathbb{Z}\). \hfill \Box
IV. CONCLUDING REMARKS

In the introduction, we commented that the results contained herein would raise new questions. We conclude by listing several of these which arose naturally in the course of our research.

1. Can a condition which implies iii) of Theorem II.9 be found when \( \mathbb{U} \) is not restricted by \( \mathbb{U} \subseteq \mathcal{A} \)? We experimented with:
\[ \Lambda(f;x) = \Lambda(g;s) \Rightarrow f(x) = g(x). \]

2. Are there other useful equivalent conditions which can be added to Theorem II.9?

3. We showed that the topological entropy of \( (\hat{X}, \hat{\phi}, Z) \) is zero when \( \phi \) is an irrational rotation of the circle and \( \hat{X} \) is the spectrum of \( C^*(C(X), \xi_k : k \in \mathbb{Z}) \). Is the entropy of \( \hat{\phi} \) zero when \( \phi \) is taken to be a general homeomorphism of the circle (which has zero entropy). Does the entropy of \( \hat{\phi} \) change (where \( \phi \) is either a translation or homeomorphism) if the \( C^* \)-algebra is enlarged? Can it be bounded if not found exactly?

4. On a more ambitious scale, what relationships exist between \( h(\phi) \) and \( h(\hat{\phi}) \) when \( X \) is not the circle. One might generalize to a compact metric space, or even forego the metrizability and formulate the definition of entropy in terms of a uniformity.

5. Expansiveness, distallity and proximality are closely related. These terms are defined only when \( \phi \) is a homeomorphism. Can the
definitions be naturally extended to the case when \( \phi \) is a continuous mapping (i.e., \( \Sigma = N \))? 

6. As with topological entropy, we investigated these properties on the system \((\hat{X}, \hat{\phi}, Z)\) with \( \phi \) being an irrational rotation on the circle and \( \hat{X} \) the spectrum of \( \mathbb{C}^*(C(X), \xi_k; k \in \mathbb{Z}) \). We found that the expansiveness of \( \hat{\phi} \) depends on whether the set \( \Lambda(\xi_0; 0) \subset \mathbb{C} \) has any limit points. Does this criterion hold in more general settings? What if \( \phi \) is not necessarily an irrational rotation so that it may contain fixed points or periodic points. What can be said if \( \phi \) is a general homeomorphism on the circle? How does the result change if \( X \) is taken to be a compact metric space? What happens if the \( \mathbb{C}^* \)-algebra (hence the space \( \hat{X} \)) is enlarged? Similar questions can be asked about distality.
V. REFERENCES


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[Signature]