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Locational Marginal Pricing Basics for Restructured Wholesale Power Markets

Haifeng Liu
California ISO

Leigh Tesfatsion
Iowa State University, tesfatsi@iastate.edu

A.A. Chowdhury
California ISU

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Haifeng Liu, *Member, IEEE*, Leigh Tesfatsion, *Member, IEEE*, A. A. Chowdhury, *Fellow, IEEE*

***Abstract*-- Although Locational Marginal Pricing (LMP) plays an important role in many restructured wholesale power markets, the detailed derivation of LMPs as actually used in industry practice is not readily available. This lack of transparency greatly hinders the efforts of researchers to evaluate the performance of these markets. In this paper, different AC and DC optimal power flow (OPF) models are presented to help understand the derivation of LMPs. As a byproduct of this analysis, the paper provides a rigorous explanation of the basic LMP and LMP-decomposition formulas (neglecting real power losses) presented without derivation in the business practice manuals of the U.S. Midwest Independent System Operator (MISO).**

***Index Terms*-- Locational marginal pricing, wholesale power market, AC optimal power flow, DC optimal power flow, U.S. Midwest Independent System Operator (MISO).**

I. INTRODUCTION

IN an April 2003 White Paper [2] the U.S. Federal Energy Regulatory Commission (FERC) proposed a market design for common adoption by U.S. wholesale power markets. Core features of this proposed market design include: central oversight by an independent market operator; a two-settlement system consisting of a day-ahead market supported by a parallel real-time market to ensure continual balancing of supply and demand for power; and management of grid congestion by means of Locational Marginal Pricing (LMP), i.e., the pricing of power by the location and timing of its injection into, or withdrawal from, the transmission grid.

Versions of FERC's market design have been implemented (or scheduled for implementation) in U.S. energy regions in the Midwest (MISO), New England (ISO-NE), New York (NYISO), the mid-Atlantic states (PJM), California (CAISO), the southwest (SPP), and Texas (ERCOT). Nevertheless, strong criticism of the design persists [3]. Part of this criticism stems from the concerns of non-adopters about the suitability of the design for their regions due to distinct local conditions (e.g., hydroelectric power in the northwest). Even in regions adopting the design, however, criticisms continue to be raised about market performance.

One key problem underlying these latter criticisms is a lack of full transparency regarding market operations under FERC's design. Due in great part to the complexity of the market design in its various actual implementations, the business practices manuals and other public documents released by market operators are daunting to read and difficult to comprehend. Moreover, in many energy regions (e.g., MISO), data is only posted in partial and masked form with a significant time delay [4]. The result is that many participants are wary regarding the efficiency, reliability, and fairness of market protocols (e.g., pricing and settlement practices). Moreover, university researchers are hindered from subjecting FERC's design to systematic testing in an open and impartial manner.

One key area where lack of transparency prevents objective assessments is determination of LMPs. For example, although MISO's Business Practices Manual 002 [5] presents functional representations for LMPs as well as an LMP decomposition for settlement purposes, derivations of these formulas are not provided. In particular, it is unclear whether the LMPs are derived from solutions to an AC optimal power flow (OPF) problem or from some form of DC OPF approximation. Without knowing the exact form of the optimization problem from which the LMPs are derived, it is difficult to evaluate the extent to which pricing in accordance with these LMPs ensures efficient and reliable market operations.

This paper provides readers interested in the operation of wholesale power markets with complete and mathematically rigorous derivations, as follows:

- derivation of the "full-structured" DC OPF model from the "full-structured" AC OPF model and derivation of the "reduced-form" DC OPF model from the "full-structured" DC OPF model;
- derivation of LMP and LMP components based on the OPF models, the LMP definition, and the envelope theorem;
- derivation and explanation of the basic LMP and LMP-decomposition formulas (neglecting real power losses) presented without derivation in the MISO Business Practices Manual 002 for Energy Markets [5].

To the best of the authors' knowledge, such self-sufficient derivations are not available in complete form in any existing publication.

The paper is organized as follows. Section II presents a

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H. Liu and A. A. Chowdhury are with the Division of Market & Infrastructure Development, California ISO, Folsom, CA 95630 USA (e-mail: hliu@caiso.com; achowdhury@caiso.com).

L. Tesfatsion is with the Economics, Mathematics, and ECpE Departments, Iowa State University, Ames, IA 50011 USA (e-mail: tesfatsi@iastate.edu).

full-structured AC OPF model for LMP calculation. The LMPs are derived from the full-structured AC OPF model based on the definition of an LMP and the envelope theorem. Section III first derives a full-structured DC OPF model from the full-structured AC OPF model, together with corresponding LMPs. A reduced-form DC OPF model is then derived from the full-structured DC OPF model, and it is shown that the LMPs derived from the reduced-form DC OPF model are the same as those derived from the full-structured DC OPF model. As a byproduct of this analysis, the paper provides a rigorous explanation of the basic LMP and LMP-decomposition formulas (neglecting real power losses) presented without derivation in the MISO Business Practices Manual 002 for Energy Markets [5]. Numerical results are discussed in Section IV. Section V concludes.

II. LMP Calculation under AC OPF

The concept of an LMP (also called a spot price or a nodal price) was first developed by Schweppe et al. [6]. LMPs can be derived using either an AC OPF model or a DC OPF model ([7], [8], [9], [10], [11], [12], [13], [14]).

The AC OPF model is more accurate than the DC OPF model, but it is prone to divergence. Also, the AC OPF model can be up to 60 times slower than the DC OPF model [15]. The DC OPF model (or the linearized AC OPF model) has been used for LMP calculation for power market operation [16], [17]. Several commercial software tools for power market simulation such as Ventyx Promod IV®, ABB GridView™, Energy Exemplar PLEXOS® and PowerWorld use the DC OPF model for power system planning and LMP forecasting ([18], [19], [20]).

There are two forms of DC OPF models, "full structured" ([13], [21], [22]) and "reduced form" ([16], [17], [20], [23], [24], [25]). The full-structured DC OPF model has a real power balance equation for each bus. This is equivalent to imposing a real power balance equation for all but a "reference" bus, together with a "system" real power balance equation consisting of the sum of the real power balance conditions across all buses. The reduced-form DC OPF model solves out for voltage angles using the real power balance equations at all but the reference bus, leaving the system real power balance equation.

In this paper, real power load and reactive power load are assumed to be fixed and a particular period of time is taken for the OPF formulations, e.g., an hour. Given a power system with N buses, G_{ij} + jB_{ij} is the ij^{th} element of the bus admittance matrix, Y, of the power system. Let the bus voltage in polar form at bus i be denoted as follows:

$$\dot{V}_i = V_i \angle \theta_i \tag{1}$$

where V_i denotes the voltage magnitude and θ_i denotes the voltage angle.

The N buses are renumbered as follows for convenience:

- Non-reference buses are numbered from 1 to N-1;
- The reference bus is numbered as bus N. Only the differences of voltage angles are meaningful in power flow calculation. Therefore, following standard practice, the voltage angle of the reference bus is set to 0.

A. Power Balance Constraint

The power flow equations (equality constraints) in the AC OPF problem are as follows:

$$f_{pk}(x) + [\xi_k + D_k] - \sum_{i \in I_k} p_i = 0 \quad \text{for } k=1,\ldots,N \tag{2}$$

$$f_{qk}(x) + Q_{load\_k} - \sum_{i \in I_k} q_i = 0 \quad \text{for } k=1,\ldots,N \tag{3}$$

Here,

- $x = [\theta_1 \; \theta_2 \cdots \theta_{N-1} \; V_1 \; V_2 \cdots V_N]^{\mathrm{T}}$ is a vector of voltage angles and magnitudes;
- $f_{pk}(x)$ is the real power flowing out of bus k:

$$f_{pk}(x) = \sum_{i=1}^{N} V_k V_i [G_{ki} \cos(\theta_k - \theta_i) + B_{ki} \sin(\theta_k - \theta_i)] \tag{4}$$

- $f_{qk}(x)$ is the reactive power flowing out of bus k:

$$f_{qk}(x) = \sum_{i=1}^{N} V_k V_i [G_{ki} \sin(\theta_k - \theta_i) - B_{ki} \cos(\theta_k - \theta_i)] \tag{5}$$

- I_k is the set of generators connected to bus k;
- p_i is the real power output of generator i;
- D_k is the given real power load at bus k;
- Q_{load_k} is the given reactive power load at bus k;
- q_i is the reactive power output of generator i; and
- ξ_k is an auxiliary parameter associated with bus k that is set to zero. Changes in ξ_k will later be used to parameterize the real load increase at bus k in order to derive the real power LMP at bus k.

B. Network Constraints

In general, the network constraints for an AC OPF problem formulation include:

- branch (transmission line and transformer) power flow limits, and
- voltage magnitude and angle limits.

The complex power flowing from bus i to bus j on the branch ij is:

$$\tilde{S}_{ij} = P_{ij} + jQ_{ij} = \dot{V}_i \overset{*}{I}_{ij} = \dot{V}_i \left[\frac{\dot{V}_i - \dot{V}_j}{r_{ij} + jx_{ij}} \right]^* \tag{6}$$

$$= \frac{[V_i^2 - V_i V_j \cos\theta_{ij} - jV_i V_j \sin\theta_{ij}][r_{ij} + jx_{ij}]}{r_{ij}^2 + x_{ij}^2}$$

where I_{ij} is the current flowing from bus i to bus j, $\theta_{ij} = \theta_i - \theta_j$, and r_{ij} and x_{ij} are the resistance and reactance of branch ij, respectively. Therefore, the real power flowing from bus i to bus j is:

$$P_{ij}(x) = \frac{[V_i^2 - V_i V_j \cos\theta_{ij}] r_{ij} + [V_i V_j \sin\theta_{ij}] x_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{7}$$

The reactive power flowing from bus i to bus j is:

$$Q_{ij}(x) = \frac{[V_i^2 - V_i V_j \cos\theta_{ij}] x_{ij} - [V_i V_j \sin\theta_{ij}] r_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{8}$$

The magnitude of the complex power flowing from bus i to bus j is:

$$S_{ij}(x) = \left|\tilde{S}_{ij}(x)\right| = \sqrt{P_{ij}^2(x) + Q_{ij}^2(x)} \tag{9}$$

The power system operating constraints include:

Branch power flow constraints:

$$0 \le S_{ij}(x) \le S_{ij}^{\max} \quad \text{for each branch } ij \tag{10}$$

Bus voltage magnitude constraints:

$$V_k^{\min} \le V_k \le V_k^{\max} \quad \text{for } k=1,2,\ldots,N \tag{11}$$

To simplify the illustration, a general form of constraints is used to represent the above specific inequality constraints (10) and (11), as follows:

$$g_m^{\min} \le g_m(x) \le g_m^{\max} \quad \text{for } m=1,\ldots,M \tag{12}$$

C. Generator Output Limits

Generator real power output limits for the submitted generator supply offers are assumed to take the following form:

$$p_i^{\min} \le p_i \le p_i^{\max} \quad \forall i \in I \tag{13}$$

Similarly, generator reactive power output limits are assumed to take the following form:

$$q_i^{\min} \le q_i \le q_i^{\max} \quad \forall i \in I \tag{14}$$

D. Objective Function of the Market Operator

According to MISO's business practices manuals and tariff [26] and [27], the supply (resource) offer curve of each generator in each hour h must be either a step function or a piecewise linear curve consisting of up to ten price-quantity blocks, where the price associated with each quantity increment (MW) gives the minimum price (\$/MWh) the generator is willing to accept for this quantity increment. The blocks must be monotonically increasing in price and they must cover the full real-power operating range of the generator.

Let $C_i(p_i)$ denote the integral of generator i's supply offer from $p_i^{\min}$ to p_i. For simplicity of illustration, $C_i(p_i)$ will hereafter be assumed to be strictly convex and non-decreasing over a specified interval.

In this study the Independent System Operator (ISO) is assumed to solve a centralized optimization problem in each hour h to determine real power commitments and LMPs for hour h conditional on the submitted generator supply offers and given loads (fixed demands) for hour h; price-sensitive demand bids are not considered. As will be more carefully explained below, this constrained optimization problem is assumed to involve the minimization of total reported generator operational costs defined as follows:

$$\sum_{i \in I} C_i(p_i) \tag{15}$$

where $C_i(p_i)$ is generator i's reported total costs of supplying real power p_i in hour h, and I is the set of generators. Since for each generator supply offer the unit of the incremental energy cost is \$/MWh and the unit of the operating level is MW, the unit of the objective function (15) is \$/h.

E. AC OPF Problem

The overall optimization problem is as follows:

$$\min_{p_i, q_i, x} \sum_{i \in I} C_i(p_i) \tag{16}$$

s.t.

Real power balance constraints for buses k=1,…,N:

$$f_{pk}(x) + \left[\xi_k + D_k\right] - \sum_{i \in I_k} p_i = 0 \tag{17}$$

Reactive power balance constraints for buses k=1,…,N:

$$f_{qk}(x) + Q_{load\_k} - \sum_{i \in I_k} q_i = 0 \tag{18}$$

Power system operating constraints for m=1,..,M:

$$g_m^{\min} \le g_m(x) \le g_m^{\max} \tag{19}$$

Generator real power output constraints for generators $i \in I$:

$$p_i^{\min} \le p_i \le p_i^{\max} \tag{20}$$

Generator reactive power output constraints for generators $i \in I$:

$$q_i^{\min} \le q_i \le q_i^{\max} \tag{21}$$

The endogenous variables are p_i, q_i and x. The exogenous variables are ξ_k, D_k and Q_{load_k}. The above optimization problem is also called the *AC OPF problem.*

F. LMP Calculation Based on AC OPF Model

The Lagrangian function for the AC OPF problem is as follows:

$$\begin{aligned}
\ell = & \sum_{i \in I} C_i(p_i) \quad \text{Total cost} \\
& - \sum_{k=1}^{N} \pi_k \left[-[\xi_k + D_k] - f_{pk}(x) + \sum_{i \in I_k} p_i \right] \text{Active power balance constraint} \\
& - \sum_{k=1}^{N} \lambda_k \left[-f_{qk}(x) - Q_{load_k} + \sum_{i \in I_k} q_i \right] \text{Reactive power balance constraint} \\
& - \sum_{m=1}^{M} \hat{\mu}_m \left[g_m^{\max} - g_m(x) \right] \text{Power system operating constraint upper limit} \\
& - \sum_{m=1}^{M} \check{\mu}_m \left[g_m(x) - g_m^{\min} \right] \text{Power system operating constraint lower limit} \\
& - \sum_{i \in I} \hat{\tau}_i \left[p_i^{\max} - p_i \right] \text{ Generator real power output upper limit} \\
& - \sum_{i \in I} \check{\tau}_i \left[p_i - p_i^{\min} \right] \text{ Generator real power output lower limit} \\
& - \sum_{i \in I} \hat{\omega}_i \left[q_i^{\max} - q_i \right] \text{ Generator reactive power output upper limit} \\
& - \sum_{i \in I} \check{\omega}_i \left[q_i - q_i^{\min} \right] \text{ Generator reactive power output lower limit}
\end{aligned} \tag{22}$$

LMP Definition: The *Locational Marginal Price (LMP)* of electricity at a location (bus) is defined as the least cost to service the next increment of demand at that location consistent with all power system operating constraints ([27], [28]).

Assume the above AC OPF problem has an optimal solution, and assume the minimized objective function J^*(exogenous variables) is a differentiable function of ξ_k for each $k = 1,...,N$. Using the envelope theorem [29], the LMP at each bus k can then be calculated as follows:

$$LMP_k = \frac{\partial J^*}{\partial \xi_k} = \left.\frac{\partial \ell}{\partial \xi_k}\right|_{\chi^*} = \pi_k \text{, for } k=1, 2, .., N \quad (23)$$

Here,

- J^* is the minimized value of the total cost objective function (15), also referred to as the indirect objective function or optimal value function;
- χ^* is the solution vector consisting of the optimal values for the decision variables.

It follows from (23) that the real power LMP at each bus k is simply the Lagrange multiplier associated with the real power balance constraint for that bus.

III. LMP Calculation and Decomposition under DC OPF

A. DC OPF Approximation in Full-Structured Form

The AC OPF model involves real and reactive power flow balance constraints and power system operating constraints, which constitute a set of nonlinear algebraic equations. It can be time consuming to solve AC OPF problems for large power systems, and convergence difficulties can be serious. The DC OPF model has been proposed to approximate the AC OPF model for the purpose of calculating real power LMPs [15].

In the DC OPF formulation, the reactive power flow equation (3) is ignored. The real power flow equation (2) is approximated by the DC power flow equations under the following assumptions ([15], [30], [31], [32]):

a) The resistance of each branch r_{km} is negligible compared to the branch reactance x_{km} and can therefore be set to zero.
b) The bus voltage magnitude is equal to one per unit.
c) The voltage angle difference $\theta_k - \theta_m$ across any branch is very small so that $\cos(\theta_k - \theta_m) \approx 1$ and $\sin(\theta_k - \theta_m) \approx \theta_k - \theta_m$.

Purchala *et al.* in [33] show that the resulting DC OPF model is acceptable in real power flow analysis if the branch power flow is not very high, the voltage profile is sufficiently flat, and the r_{km}/x_{km} ratio is less than 0.25. The DC OPF model itself does not include the effect of the real power loss on the LMP due to assumption a). Li *et al.* in [25] propose an iterative approach to account for the real power loss in the DC OPF-based LMP calculation. In the present study, however, real power loss is neglected in conformity with standard DC OPF treatments.

Given assumption a), it follows that $G_{km} = -\frac{r_{km}}{r_{km}^2 + x_{km}^2} = 0$ for $k \neq m$, $G_{kk} = \frac{r_{k0}}{r_{k0}^2 + x_{k0}^2} + \sum_{m=1, m\neq N}^{N} \frac{r_{km}}{r_{km}^2 + x_{km}^2} = 0$ and $B_{km} = \frac{x_{km}}{r_{km}^2 + x_{km}^2} = \frac{1}{x_{km}}$ for $k \neq m$, $B_{kk} = \sum_{m=1}^{N} \frac{-x_{km}}{r_{km}^2 + x_{km}^2}$. Given assumption b), it follows that $V_k = V_m = 1$. Given assumption c), it follows that $\sin(\theta_k - \theta_m) \approx \theta_k - \theta_m$ Therefore, (2) reduces to:

$$\sum_{m=1,\, m\neq k}^{N} \left[\frac{1}{x_{km}} (\theta_k - \theta_m) \right] = \sum_{i \in I_k} p_i - D_k = P_k - D_k \quad \text{for } k=1\ldots N \quad (24)$$

Therefore, the net injection $P_k - D_k$ of real power flowing out of any bus k can be approximated as a linear function of the voltage angles.

From (7) and based on the assumptions a), b) and c), the real power flowing from bus k to bus m is as follows

$$P_{km}(x) = \frac{\theta_k - \theta_m}{x_{km}} \quad (25)$$

Therefore, this branch real power flow can be approximated as a linear function of the voltage angle difference between bus k and bus m.

From (8) and based on the assumptions a), b) and c), the reactive power flowing from bus k to bus m is as follows:

$$Q_{km}(x) = 0 \quad (26)$$

From (9), the magnitude of the complex power flow $S_{km}(x)$ is:

$$S_{km}(x) = \sqrt{P_{km}^2(x) + Q_{km}^2(x)} = \sqrt{P_{km}^2(x)} \quad (27)$$

Therefore, the branch power flow constraint becomes:

$$F_{km}^{\min} \leq P_{km}(x) \leq F_{km}^{\max} \quad (28)$$

There are no voltage magnitude constraints because all voltage magnitudes are assumed to be 1.0 p.u.

For a power system consisting of N buses, the DC power flow equation for each bus k is shown in (24). The corresponding matrix form for the full system of equations is as follows:

$$\mathbf{P} - \mathbf{D} = \mathbf{B}\theta \quad (29)$$

Here,

- $\mathbf{P} = [P_1\, P_2 \cdots P_N]^T$ is the $N\times1$ vector of nodal real power generation for buses 1, …,N.
- $\mathbf{D} = [D_1\, D_2 \cdots D_N]^T$ is the $N\times1$ vector of nodal real power load for buses 1, …,N.
- $\mathbf{B}$ is an $N \times N$ matrix (independent of voltage angles) that is determined by the characteristics of the transmission network as follows: $B_{kk} = \sum_m 1/x_{km}$ for each diagonal element kk, and $B_{km} = -1/x_{km}$ for each off-diagonal element km.
- $\theta = [\theta_1\, \theta_2 \cdots \theta_N]^T$ is the $N\times1$ vector of voltage angles for buses 1, …, N.

The system of equations (29) is called the *full-structured DC power flow model*.

The voltage angle at the reference bus N is usually normalized to zero since the real power balance constraints and real power flow on any branch are only dependent on voltage angle differences, as seen from (24) and (25). We follow this convention here, therefore:

$$\theta_N = 0 \quad (30)$$

Given (30), the system of real power balance equations for buses 1, …, N-1 (29) can be expressed in reduced matrix form as follows:

$$\mathbf{P}' - \mathbf{D}' = \mathbf{B}'\theta' \tag{31}$$

Here,

- $\mathbf{P}' = [P_1\ P_2 \cdots P_{N-1}]^T$ is the (N-1)×1 vector of real power generation for buses 1, …, N-1.
- $\mathbf{D}' = [D_1\ D_2 \cdots D_{N-1}]^T$ is the (N-1)×1 vector of real power load for buses 1, …, N-1.
- **B**' is the "B-prime" matrix of dimension (N-1)×(N-1), independent of voltage angles, that is determined by the characteristics of the transmission network. The **B**' matrix is derived from the **B** matrix by omitting the row and column corresponding to the reference bus.
- $\theta' = [\theta_1\ \theta_2 \cdots \theta_{N-1}]^T$ is the (N-1)×1 vector of voltage angles for buses 1, …, N-1.

In a lossless transmission system (i.e. r_{km}=0), consideration of conservation of power gives the following (see page 358 of [34] for details):

$$P_N - D_N = -\mathbf{e}^T[\mathbf{P}' - \mathbf{D}'] \tag{32}$$

where $\mathbf{e}^T = [1\,1 \cdots 1]$ is an $1 \times (N-1)$ row vector with each element equal to 1.

In the DC OPF model, the real power flow on any branch km is given in (25). Letting M denote the total number of distinct transmission network branches for the DC OPF model, it follows that the real power flow on all M branches can be written in a matrix form as follows:

$$\mathbf{F} = \mathbf{X}\theta \tag{33}$$

Here,

- $\mathbf{F} = [F_1(x)\ F_2(x) \cdots F_M(x)]^T$ is the M×1 vector of branch flows.
- $\mathbf{X} = \mathbf{H} \times \mathbf{A}$ is a M×N matrix, which is determined by the characteristics of the transmission network.
- **H** is an M×M matrix whose non-diagonal elements are all zero and whose kkth diagonal element is the negative of the susceptance of the k^{th} branch.
- **A** is the M×N adjacency matrix. It is also called the node-arc incidence matrix, or the connection matrix.

Inverting (31) yields:

$$\theta' = [\mathbf{B}']^{-1}[\mathbf{P}' - \mathbf{D}'] \tag{34}$$

Substitution of (34) into (33) yields:

$$\mathbf{F} = \mathbf{X}\theta = \mathbf{X}\begin{bmatrix}\theta' \\ \theta_n\end{bmatrix} = \mathbf{X}\begin{bmatrix}\mathbf{B}'^{-1}[\mathbf{P}' - \mathbf{D}'] \\ 0\end{bmatrix} = \mathbf{X}\begin{bmatrix}\mathbf{B}'^{-1} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{P}' - \mathbf{D}' \\ P_N - D_N\end{bmatrix} \tag{35}$$

$$= \mathbf{X}\begin{bmatrix}\mathbf{B}'^{-1} & 0 \\ 0 & 0\end{bmatrix}[\mathbf{P} - \mathbf{D}]$$

Let

$$\mathbf{T} = \mathbf{X}\begin{bmatrix}\mathbf{B}'^{-1} & 0 \\ 0 & 0\end{bmatrix} \tag{36}$$

Here,

- **T** is a M×N matrix,
- $T_{mN} = 0$ for m=1, …, M.

Therefore, the branch power flows in terms of bus net real power injections can be expressed as:

$$\mathbf{F} = \mathbf{T}[\mathbf{P} - \mathbf{D}] \tag{37}$$

The system of equations (37) is called the *reduced-form DC power flow model* because it directly relates branch real power flows to bus net real power injections.

The real power flow on branch l in (37) is as follows:

$$F_l = \sum_{k=1}^{N} T_{lk}[P_k - D_k] = \sum_{k=1}^{N-1} T_{lk}[P_k - D_k] \text{ for } l=1, \ldots, M \tag{38}$$

Assume P_k is increased to $P_k + \Delta P_k$ while P_1, P_2, …, P_{k-1}, P_{k+1}, …, P_{N-1} and D_1, D_2, …, D_N remain fixed. Then, according to (38), the increase in the real power flow on branch l, ΔF_l, is as follows:

$$\Delta F_l = T_{lk}\Delta P_k \tag{39}$$

By (32), note that the change in the real power injection at bus k is exactly compensated by an opposite change in the real power injection at the reference bus N, given by $P_N - \Delta P_k$. Therefore, T_{lk} in (39) is a *generation shift factor*.

More precisely, it is clear from (35) that the branch power flows are explicit functions of nodal net real power injections (generation less load) at the non-reference buses. It follows from (32) that the generation change at bus k will be compensated by the generation change at the reference bus N assuming the net real power injections at other buses remain constant. Thus, the lkth element T_{lk} in the matrix **T** in (37) is equal to the generation shift factor a_{lk} as defined on page 422 of [31], which measures the change in megawatt power flow on branch l when one megawatt change in generation occurs at bus k compensated by a withdrawal of one megawatt at the reference bus.

The full-structured DC OPF model is derived from the full-structured AC OPF model in Section II based on the three assumptions a), b), and c) in Section III.A, as follows:

$$\min_{p_i, \theta_k} \sum_{i \in I} C_i(p_i) \tag{40}$$

s.t.

Real power balance constraint for each bus $k = 1, \cdots, N$:

$$\sum_{i \in I_k} p_i - [\xi_k + D_k] = \sum_{m=1,\, m \neq k}^{N} \left[\frac{1}{x_{km}}(\theta_k - \theta_m)\right] \tag{41}$$

Real power flow constraints for each distinct branch km:

$$\frac{1}{x_{km}}[\theta_k - \theta_m] \le F_{km}^{\max} \tag{42}$$

$$\frac{1}{x_{km}}[\theta_k - \theta_m] \ge F_{km}^{\min} \tag{43}$$

Real power generation constraints for each generator:

$$p_i^{\min} \le p_i \le p_i^{\max} \quad \forall i \in I \tag{44}$$

The endogenous variables are p_i and θ. The exogenous variables are D_k and ξ_k.

The optimal solution is determined for the particular parameter values $\xi_k = 0$ in (41). Changes in these parameter values are used below to generate LMP solution values using envelope theorem calculations.

The Lagrangian function for the optimization problem is:

$$\begin{aligned}\ell = &\sum_{i\in I} C_i(p_i) \\ &-\sum_{k=1}^{N}\pi_k\left[\sum_{i\in I_k} p_i - \sum_{m=1,\, m\neq k}^{N}\left[\frac{1}{x_{km}}(\theta_k-\theta_m)\right]-[\xi_k+D_k]\right] \\ &-\sum_{km}\hat{\mu}_{km}[F_{km}^{\max}-\frac{1}{x_{km}}[\theta_k-\theta_m]] \\ &-\sum_{km}\check{\mu}_{km}[\frac{1}{x_{km}}[\theta_k-\theta_m]-F_{km}^{\min}] \\ &-\sum_{i\in I}\hat{\tau}_i(p_i^{\max}-p_i) \\ &-\sum_{i\in I}\check{\tau}_i(p_i-p_i^{\min})\end{aligned} \tag{45}$$

Assume the above DC OPF problem has an optimal solution and the optimized objective function J*(exogenous variables) is a differentiable function of ξ_k for each k = 1,...,N. Based on the envelope theorem and using the auxiliary parameter ξ_k, we can calculate the LMP at each bus k as follows:

$$LMP_k = \frac{\partial J^*}{\partial \xi_k} = \left.\frac{\partial \ell}{\partial \xi_k}\right|_{\chi^*} = \pi_k^*\text{, for } k=1, 2, .., N \tag{46}$$

It follows from (46) that the LMP at each bus k is the Lagrange multiplier corresponding to the real power balance constraint at bus k, evaluated at the optimal solution.

B. DC OPF Approximation in Reduced Form

The reduced-form DC OPF model can be derived directly from the full-structured DC OPF model in Section III.A by applying the following three steps:

a) Replace the real power balance equation at the reference bus N by the sum of the real power balance equations across all N buses. This is an equivalent formulation that will not change the optimal solution of the DC OPF problem. Since there is no real power loss in the DC power flow model, the sum of the net real power injections across all buses is equal to zero; see (32). Therefore, the system real power balance constraint (in parameterized form) can be expressed as in (48), below.

b) Solve the voltage angles at the N-1 non-reference buses as functions of the net real power injections at the N-1 non-reference buses as shown in (34).

c) Replace the voltage angles in the branch flow constraints as functions of the net real power injections at the non-reference buses as shown in (35) and (38). Since the above transformation is based on equivalency and only eliminates internal variables (i.e. voltage angles at non-reference buses), the optimal solution and the corresponding Lagrange multipliers of the branch power flow constraints are the same for the two DC OPF models.

The resulting *reduced-form DC OPF model* is then as follows:

$$\min_{p_i}\sum_{i\in I} C_i(p_i) \tag{47}$$

s.t.

System real power balance constraint:

$$\sum_{k=1}^{N}(P_k-(D_k+\xi_k))=0 \tag{48}$$

Branch real power flow constraint for each branch l:

$$\sum_{k=1}^{N-1}T_{lk}\left[P_k-D_k-\xi_k\right]\le F_l^{\max}\quad\text{for } l=1,\ldots,M \tag{49}$$

$$F_l^{\min}\le\sum_{k=1}^{N-1}T_{lk}\left[P_k-D_k-\xi_k\right]\quad\text{for } l=1,\ldots,M \tag{50}$$

Real power output constraint for each generator i:

$$p_i^{\min}\le p_i\le p_i^{\max}\qquad\forall i\in I \tag{51}$$

The Lagrangian function for this optimization problem is:

$$\begin{aligned}\ell = &\sum_{i\in I} C_i(p_i) \\ &-\pi\sum_{k=1}^{N}\left[P_k-D_k-\xi_k\right] \\ &-\sum_{l=1}^{M}\hat{\mu}_l\left[F_l^{\max}-\sum_{k=1}^{N-1}T_{lk}\left[P_k-D_k-\xi_k\right]\right] \\ &-\sum_{l=1}^{M}\check{\mu}_l\left[\sum_{k=1}^{N-1}T_{lk}\left[P_k-D_k-\xi_k\right]-F_l^{mim}\right] \\ &-\sum_{i\in I}\hat{\tau}_i\left[p_i^{\max}-p_i\right] \\ &-\sum_{i\in I}\check{\tau}_i\left[p_i-p_i^{\min}\right]\end{aligned} \tag{52}$$

Assume the reduced-form DC OPF problem has been solved. Based on the envelope theorem, using the auxiliary parameter ξ_k, we can calculate the LMPs for all buses as follows:

$$\begin{aligned}LMP_k &= \frac{\partial J^*}{\partial \xi_k}=\left.\frac{\partial \ell}{\partial \xi_k}\right|_{\chi^*}=\pi-\sum_{l=1}^{M}\hat{\mu}_l T_{lk}+\sum_{l=1}^{M}\check{\mu}_l T_{lk} \\ &=\mathrm{MEC}_N+\mathrm{MCC}_k,\ \forall k\neq N\end{aligned} \tag{53}$$

$$LMP_k = \frac{\partial J^*}{\partial \xi_k}=\left.\frac{\partial \ell}{\partial \xi_k}\right|_{\chi^*}=\pi=\mathrm{MEC}_N,\ k=N \tag{54}$$

Here,

- $\mathrm{MEC_N}$ = π is the LMP component representing the marginal cost of energy at the reference bus N.
- $\mathrm{MCC}_k=-\sum_{l=1}^{M}\hat{\mu}_l T_{lk}+\sum_{l=1}^{M}\check{\mu}_l T_{lk}$ is the LMP component representing the marginal cost of congestion at bus k relative to the reference bus N.

The derived marginal cost of energy, MEC, in (53) and (54) is the same as that in Equations (4-1) and (4-2) on page 35 of the MISO's Business Practices Manual for Energy Markets [5]. Recall that T_{lk} is equal to the Generation Shift Factor (GSF_{lk}), which measures the change in megawatt power flow on flowgate (branch) l when one megawatt change in generation occurs at bus k compensated by a withdrawal of one megawatt at the reference bus. From (52), $\hat{\mu}_l-\check{\mu}_l$ is the Flowgate Shadow Price (FSP_l) on flowgate l, which is equal to the reduction in minimized total variable cost that results from an increase of 1 MW in the capacity of the flowgate l. Therefore, the marginal congestion component MCC can be expressed as,

$$\text{MCC}_k = -\sum_{l=1}^{M} \text{GSF}_{lk} \times \text{FSP}_l \quad (55)$$

The derived marginal cost of congestion, MCC, in (55) is the same as that in Equation (4-3) on page 36 of the MISO's Business Practices Manual for Energy Markets [5].

IV. Numerical Results

As depicted in Fig. 1, we use a three-bus system with two generators and one fixed load to illustrate LMP calculations based on the full-structured DC OPF model. For the purpose of illustration, assume: (i) the reactance of each branch is equal to 1 p.u.; (ii) the capacity of branch 2-1 is 50 MW; (iii) there are no capacity limits on branches 2-3 and 3-1; (iv) the demand at Bus 1 is fixed at 90 MW; (v) the real power operating capacity limit for generator 2 and for generator 3 is 100MW; (vi) the indicated marginal costs \$5/MWh and \$10/MWh for Generator 2 and Generator 3 are constant over their real power operating capacity ranges; (vii) the time period assumed for the DC-OPF formulation is one hour; and (viii) the objective of the market operator is the constrained minimization of the *total variable costs of operation* (\$/h), i.e., the summation of the variable costs of operation (marginal cost times real power generation) for Generator 2 and Generator 3.

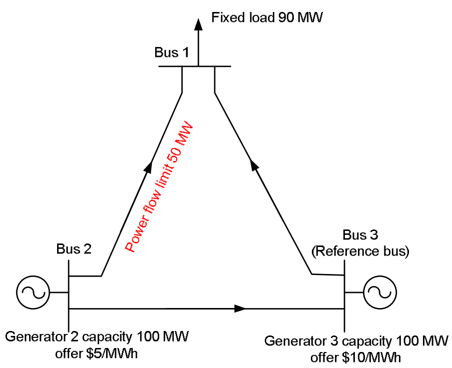

Fig. 1. LMP calculation for the full-structured DC OPF model

In the following calculations, all power amounts (generator outputs, load demand, and branch flows) and impedances are expressed in per unit (p.u.). The base power is chosen to be 100 MW. The objective function for the DC OPF problem is expressed in per unit terms as well as the constraints. The variable cost of each generator i is expressed as a function of per unit real power P_{Gi}, i.e., as $100 \times \text{MC}_i \times P_{Gi}$, where MC_i denotes the marginal cost of Generator i. Note that the per unit-adjusted total variable cost function is then still measured in dollars per hour (\$/h).

Given the above assumptions, the market operator's optimization problem is formulated as follows:

$$\min_{\theta_1,\theta_2,P_{G2},P_{G3}} \; 500P_{G2} + 1000P_{G3} \quad (56)$$

s.t.

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.9 \\ P_{G2} \\ P_{G3} \end{bmatrix} \quad (57)$$

$$-\begin{bmatrix} F_{21}^{\max} \\ F_{31}^{\max} \\ F_{23}^{\max} \end{bmatrix} \le \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ 0 \end{bmatrix} \le \begin{bmatrix} F_{21}^{\max} \\ F_{31}^{\max} \\ F_{23}^{\max} \end{bmatrix} \quad (58)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \le \begin{bmatrix} P_{G2} \\ P_{G3} \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (59)$$

The solution to this optimization problem yields the following scheduled power commitments for Generators 2 and 3 and LMP values for Buses 1 through 3:

- P_{G2} = 0.6 p.u. = 60 MW, P_{G3} = 0.3 p.u. = 30 MW
- LMP_1 = \$15/MWh, LMP_2 = \$5/MWh, LMP_3 = \$10/MWh

In the following, we use the same three-bus system to illustrate the calculation of LMP solution values based on the reduced-form DC OPF model. First, the optimization problem is formulated as follows:

$$\min_{P_{G2},P_{G3}} \; 500P_{G2} + 1000P_{G3} \quad (60)$$

s.t.

$$P_{G2} + P_{G3} - 0.9 = 0 \quad (61)$$

$$-\begin{bmatrix} F_{21}^{\max} \\ F_{31}^{\max} \\ F_{23}^{\max} \end{bmatrix} \le \begin{bmatrix} -1/3 & 1/3 & 0 \\ -2/3 & -1/3 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} -0.9 \\ P_{G2} \\ P_{G3} \end{bmatrix} \le \begin{bmatrix} F_{21}^{\max} \\ F_{31}^{\max} \\ F_{23}^{\max} \end{bmatrix} \quad (62)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \le \begin{bmatrix} P_{G2} \\ P_{G3} \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (63)$$

The optimal real power commitments for Generators 2 and 3 are the same as those obtained for the full-structured DC OPF model:

- P_{G2} = 0.6 p.u. = 60 MW, P_{G3} = 0.3 p.u. = 30 MW

The Lagrange multiplier corresponding to the system real power balance constraint, π, is \$10/MWh and the Lagrange multiplier corresponding to the inequality constraint for branch 2-1, μ, is \$15/MWh. The LMPs can then be calculated based on (53) and (54) as

- $\text{LMP}_1 = MEC_3 + MCC_1 = \pi - \mu(T_{11}) = 10 - 15(-1/3) =$ \$15/MWh
- $\text{LMP}_2 = MEC_3 + MCC_2 = \pi - \mu(T_{12}) = 10 - 15(1/3) =$ \$5/MWh
- $\text{LMP}_3 = MEC_3 = \pi =$ \$10/MWh

These LMP solution values are the same as those obtained using the full-structured DC OPF model. Moreover, the marginal cost of congestion at Bus 1 relative to the reference Bus 3, MCC_1, is \$5/MWh, and the marginal cost of congestion at Bus 2 relative to the reference Bus 3, MCC_2, is -\$5/MWh.

V. Conclusion

Locational marginal pricing plays an important role in many recently restructured wholesale power markets. Different AC and DC optimal power flow models are carefully presented and analyzed in this study to help understand the determination of LMPs. In particular, the paper shows how to derive the full-structured DC OPF model from the full-structured AC OPF model, and the reduced-form

DC OPF model from the full-structured DC OPF model. Simple full-structured and reduced-form DC OPF three-bus system examples are presented for which the LMP solutions are derived using envelope theorem calculations. These examples are also used to illustrate that LMP solution values derived for the full-structured DC OPF model are the same as those derived for the reduced-form DC OPF model. As a byproduct of this analysis, the paper provides a rigorous explanation of the basic LMP and LMP-decomposition formulas (neglecting real power losses) presented without derivation in the MISO Business Practices Manual 002 for Energy Markets.

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VII. Biographies

Haifeng Liu (M'08) received the B.S. and M.S. degrees in electrical engineering from Zhejiang University, Hangzhou, China, in 2000 and 2003, respectively. He received the Ph.D. degree in electrical engineering and the M.S. degree in economics from Iowa State University, Ames, in 2007 and 2008, respectively. Currently, he is with the Division of Market & Infrastructure Development at California ISO, Folsom, California. His research interests include power system planning and power markets.

Leigh Tesfatsion received her Ph.D. degree in Economics from the University of Minnesota in 1975. She is currently Professor of Economics, Mathematics, and Electrical and Computer Engineering at Iowa State University, Ames, Iowa. Her principal research area is Agent based Computational Economics (ACE), the computational study of economic processes modeled as dynamic systems of interacting agents, with a particular focus on restructured electricity markets. She is an active participant in IEEE Power Engineering Society working groups and task forces focusing on power economics issues and a co-organizer of the ISU Electric Energy Economics (E3) Group. She serves as associate editor for a number of journals, including the Journal of Energy Markets.

A. A. Chowdhury (F '05) received his MSc degree with honors in electrical engineering from the Belarus Polytechnic Institute in Minsk, Belarus. His MSc and PhD degrees in electrical engineering with specialization in power systems reliability and security were earned at the University of Saskatchewan in Canada, and his MBA degree from the St. Ambrose University in Davenport, USA. Dr. Chowdhury is currently the Director of Planning and Infrastructure Development at the California Independent System Operator, Folsom, California.

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