On the Demand for Economic Growth and the Supply of All Inclusive Capital

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Abstract
In this paper we investigate whether and in what sense there is an equilibrium rate of economic growth. Given an ongoing expansion of productive possibilities arising from technical progress, we proceed to examine the implicit demand for growth opportunities and its reflection in the supply of capital consisting of all forms, and the determination of a net, or equilibrium rate of increase in per capita income as a joint outcome of supply and demand for growth.

Disciplines
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On the Demand for Economic Growth and the Supply of All Inclusive Capital

by

James D. Adams

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1. Introduction

In this paper we investigate whether and in what sense there is an equilibrium rate of economic growth. Given an ongoing expansion of productive possibilities arising from technical progress, we proceed to examine the implicit demand for growth opportunities, its reflection in the supply of capital consisting of all forms, and the determination of a net, or equilibrium rate of increase in per capita income as a joint outcome of supply and demand for growth. In addition, the paper sets out to show that finite length of life creates a rising supply curve of all inclusive capital under stationary conditions, the definiteness of this conclusion standing in sharp contrast to the usual result from capital theory [see Friedman (1962), Ch. 13; Hirshleifer (1970), Ch. 6].

Section 2 of the paper sets forth the structure of the model; section 3 derives equilibrium conditions for the individual, the rising supply curve of all inclusive capital, and exposit the concept of net economic growth, section 4 performs some comparative statics exercises using the model; and section 5 concludes the paper.

2. Structure of the Model

The Samuelson-Diamond model of overlapping generations is utilized to show how interdependent preferences expressed through wealth transfers can create a
demand for economic growth.\(^1\) Since finite life, in combination with imperfect substitutability between consumption of different generations produces this result, the model of overlapping generations is convenient for this purpose because it embodies both these facts while making the simplest possible assumptions about the turnover of generations. The model in outline is as follows. Each generation lives two periods, undergoing an initial period of dependence while young, when it does not work but acquires human capital, and a final period of maturity when it does work, supplying one unit of labor service per unit of constant-quality labor, regardless of the wage rate.\(^2\) Members of the succeeding generation are born at the start of the second period in the life of the earlier generation, and the fertility decision is incorporated into the model. Life cycle labor force participation and schooling patterns are stylized in an extreme form by assuming initial and final participation rates of 0 and 100\% respectively, and positive schooling only in the initial period.

Production is carried on using labor services augmented by human capital and services of physical capital. Constant returns to scale production functions for physical capital and consumption goods are assumed. Each utilizes productive factors in the same proportions, thereby collapsing the two goods into a single good.

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\(^1\) For a recent application, see Barro (1974).

\(^2\) Issues of elasticity of labor supply are thereby set aside, so that the wage rate is set by the interaction of current demand conditions and fertility decisions of the previous period.
By way of contrast, this assumption is not applicable to production of human capital, since it is assumed [as in Becker (1967)] that individual capacities which are fixed enter into its production, causing diminishing returns to set in. Therefore, constant costs apply only to production of physical capital and consumer goods.

Depreciation patterns of the two forms of capital are also assumed to differ markedly. Since human capital is embodied in the individual, it is produced in the initial period, yields income in the second, and is then eradicated by death. Depreciation on physical capital is ignored; its inclusion would introduce a difference between net and gross rates of return on capital, but that is not an essential effect for our purposes.

Given the above framework, particulars of the model can be described. The model basically divides into two parts over the life of a generation (denoted j). The first part describes the problem of individual choice faced by a member of the jth generation.

It consists of a system of five equations for a family spanning two generations: a utility function for a head of household of the jth generation; a budget line expressing the identity of wealth with consumption and gross transfers to generation j+1 by a head of household of the jth generation; an equation for offspring in the (j + 1)th generation stating the equality of j + 1 expenditures with endowments and transfers from the jth generation; an identity relating all net transfers to components of human capital and assets; and an identity relating all gross transfers to the expenditures on each of these components, which appear in the donor's budget constraint. Since the model assumes two forms of durable goods, two forms of inheritance may take place. In general, inheritance assumes as many forms as there are kinds of durable goods in a model.
identities join the quantity-quality approach used in the fertility literature with separability assumptions employed in the study of intergenerational relations. In symbols, the part of the model pertaining to an individual family of the $j$th and $(j+1)$th generations is

$$U_j = U_j \left[ V^{j+1}(C^{j+1}), n^{j+1}, V^j(C^j) \right] \quad (1)$$

$$E_{j+1} = n^{j+1} e^{j+1} = n^{j+1} \left( C^{j+1} - \frac{R_{j+1}}{n^{j+1}} \right) \quad (2)$$

$$G_{j+1} = n^{j+1} \left( P_{h}^{j+1} + P_{a}^{j+1} \right) \quad (3)$$

$$R_{j+1} = n^{j+1} \left( h^{j+1} + a^{j+1} \right) \quad (4)$$

$$e^j + \frac{R_{j}^j}{n^j} = C^j + G_{j+1} \quad (5)$$

Equation (1) assumes that the utility of a member of generation $j$ depends separably on its own consumption ($C^j$), per capita consumption of generation $j+1$ family members ($C^{j+1}$), and numbers of $j+1$ members in the family ($n^{j+1}$). Equation (2) states that the aggregate earnings endowment of such members ($E_{j+1}$) is the product of numbers and per capita earnings endowments ($e^{j+1}$), assumed to be the same for all members. Per capita endowments in turn are the difference between $j+1$ consumption per capita and per capita net transfers ($\frac{R_{j+1}}{n^{j+1}}$) from generation $j$. In equation (3) gross transfers by $j$ ($G_{j+1}$) are the product of numbers and expenditures per capital ($P_{h}^{j+1} + P_{a}^{j+1}$). These expenditures are the sum of gross outlays on human capital ($P_{h}^{j+1}$) and assets ($P_{a}^{j+1}$) per generation $j+1$ member. The equation for net outlays ($R_{j+1}$) is (4). Finally, equation (5) is the budget line for the head of the household expressing the equality between his wealth—which is the sum of his human capital endowment ($e^j$) and per capita net transfers from generation $j-1$ ($\frac{R_{j}^j}{n^j}$) -- and his expenditures, which consist of own consumption ($C^j$) plus gross transfers ($G^{j+1}$). This summarizes the problem of the individual donor in generation $j$.

As written, the individual part of the model is free of ability variations. The human capital earnings endowment is the same for all $j+1$ members of the family ($e^{j+1}$), and it is identical for all families in a generation. Therefore, though
the model has implications for income and wealth distribution, the role of ability
is excluded from the discussion primarily because ability variation is not crucial
for the purposes of this study, which is aimed at the mean position of succeeding
generations relative to the current one.

An additional point should be mentioned. The earnings of the head of any
household, and indeed of any individual, are related to his stock of human capital
through a wage per unit of services yielded (recall the assumption that a unit of
capital yields one unit of services per period). Symbolically,
\[ e_j^+ h_{j-1} = w^j q_j \]  
(6)
where \( h_{j-1} \) is the transfer of earnings from human capital by a head of household
in generation \( j-1 \) to a member of generation \( j \), \( e_j \) is the latter's endowment, \( W \) is
the wage per unit of services, and \( q_j \) is the number of service units embodied in
the generation \( j \) member. \( q_j \) in turn is the sum of endowed and inherited units of
service, where the former are identical for all persons.

The second part of the model is comprised of economy-wide equations for
clearance in the markets for assets and output, and an equation representing
product exhaustion. Hence,
\[ K(r, w) = A_{j-1}^+ + A_j^+ \]  
(7)
\[ Y = rK + wL \]  
(8)
\[ N_{j-1}^+ c_{j-1}^+ + N_j^+ c_j^+ + \Delta K = Y, \]  
(9)
where time is the initial period in the life of generation \( j \). Equation (7) states
that the market for physical capital must clear: the amount of capital demanded \( K \),
(a function of the interest rate \( r \) and the wage \( w \)) equals the supply, which is the
sum of asset holdings by the \((j-1)\)th and \( j \)th generations \((A_{j-1}^+ + A_j^+)\), where \( A_{j-1}^+ = \sum a_{j-1}^+ \), and \( A_j^+ = \sum a_j^+ \). Asset holdings by \( j-1 \) members in the final period of life
constitute a bequest to generation \( j \). Equation (8) represents exhaustion of the
product between physical capital and quality-adjusted units of labor service (L).  

On this view, education is productive through labor-augmentation. Finally, equation (9) states that spending equals the value of output, where spending consists of total consumption by generation j=1 individuals \( (N_{j-1}C_{j-1}) \), \( N_{j-1} = \Sigma n_{j-1} \), the population of j-1 individuals, and \( C_{j-1} \) = their mean consumption, total consumption by generation j individuals \( (N_{j}C_{j}) \), all terms defined in a similar manner to the above), and investment \( (\Delta k) \). The three equations with the generational index advanced by a unit of time are market clearing conditions for the second period in the life of generation j.

This presents the model in its main outline. Equations (1) through (5) are sufficient to determine numbers of the next generation for a family, per capita asset holdings in each period of the life of generation j by both generations j and j+1. Equations (7) through (9) with corresponding relations for the second period in the life of generation j, establish equilibrium in income, production, and the holding of physical assets. Clearance in holdings of human wealth is assumed to take place automatically, through equilibration of the demand and supply of transfers in each individual family, and hence for the economy as a whole. The sole remaining relation to be specified is the supply of growth opportunities. This is discussed in the course of the next section.

3. Equilibrium Conditions.

The individual problem defined by equations (1) through (5) is

\[
L = q^j N^j, \text{ where } q^j = \text{mean units of labor service per generation } j \text{ member.}
\]

Therefore, human capital in this model is labor-augmenting. For a study using this specification in a growth context, see Griliches (1970). The study finds the specification acceptable when inserted in Cobb-Douglas production functions estimated on time series data for several industries.
The Kuhn-Tucker conditions are then

\[
\frac{\partial u^j}{\partial y^{j+1}} \frac{\partial v^{j+1}}{\partial c^{j+1}} - \lambda n^{j+1} \frac{\partial c_{h}^{j+1}}{\partial h} \leq 0 \quad (11a)
\]

\[
\frac{\partial u^j}{\partial y^{j+1}} \frac{\partial v^{j+1}}{\partial c^{j+1}} - \lambda n^{j+1} \frac{\partial c_{a}^{j+1}}{\partial a} \leq 0 \quad (11b)
\]

\[
\frac{\partial u^j}{\partial n^{j+1}} - \lambda (P_{h}h^{j+1} + MC_{a}a^{j+1}) \leq 0 \quad (11c)
\]

\[
\frac{\partial u^j}{\partial n^{j+1}} \frac{\partial v^{j}}{\partial c^{j}} - \lambda \leq 0 \quad (11d)
\]

\[
e^j + h^j + a^j - c^j - n^{j+1} (P_{h}h^{j+1} + P_{a}a^{j+1}) = 0 \quad (11e)
\]

\[
\left( \frac{\partial u^j}{\partial y^{j+1}} \frac{\partial v^{j+1}}{\partial c^{j+1}} - \lambda n^{j+1} \frac{\partial c_{h}^{j+1}}{\partial h} \right) h^{j+1} +
\]

\[
\left( \frac{\partial u^j}{\partial y^{j+1}} \frac{\partial v^{j+1}}{\partial c^{j+1}} - \lambda n^{j+1} \frac{\partial c_{a}^{j+1}}{\partial a} \right) a^{j+1} +
\]

\[
\left[ \frac{\partial u^j}{\partial n^{j+1}} - \lambda (P_{h}h^{j+1} + MC_{a}a^{j+1}) \right] n^{j+1} +
\]

\[
\left( \frac{\partial u^j}{\partial y^{j}} \frac{\partial v^{j}}{\partial c^{j}} \right) c^j = 0 \quad (11f)
\]

\[
h^{j+1} \geq 0, a^{j+1} \geq 0, n^{j+1} \geq 0, c^j \geq 0 \quad (11g)
\]

Inequalities (11a) through (11c) express the quantity-quality equilibrium conditions: (11a) and (11b) show that marginal costs (MC) of per capita contributions to generation j+1 are a multiple of "unit" marginal costs, because they apply equally to all recipients, while (11c) shows that the marginal cost of increasing generation j+1 members of the family (the fertility decision) depends on the value of per capita contributions, since these contributions
are applied to an equal extent to the incremental generation \( j+1 \)
member.\(^5\)

\(^5\)In these expressions, marginal cost of the \( i \)th transfer (\( MC_{i}^{j+1} \)) is defined as:
\[
MC_{i}^{j+1} = P_i + i^{j+1} \frac{\partial P_i}{\partial i^{j+1}}, \quad \text{where} \ i = a, h.
\]

The transfer price of a dollar in discounted earnings derived from human
capital is treated as invariant with respect to numbers \( n^{j+1} \) [hence \( P_h = P_h \)
\((h^{j+1})\)], because diminishing returns, which are otherwise reflected in rising
price, originate in fixity of human capacity. But an expansion of \( n^{j+1} \), per capita
transfers constant is equivalent to a proportionate expansion of all inputs into
human capital production, hence there are constant returns to scale and an
invariant transfer price.

The transfer price on a dollar earned from assets rises because of factors
such as estate and gift tax progressivity. In contrast to human capital
transfers, it is sensitive to increases in numbers as well as per capita
contributions [hence \( P_a = P_a \) \((n^{j+1} a^{j+1})\)], implying that taxes depend on
total contributions. Thus a tax on asset transfers at death would be an
estate tax rather than an inheritance tax.
Equation (11d) is the equilibrium for own consumption of generation \( j \), which is likely to be on the interior. Indeed, interior solutions are likely in all cases except (11b), since children are born to most families and transfers are typically made in that event. Following Becker (1967), at low levels of per capita human capital transfers the rate of return on investment is high and in excess of the return on tangible capital. Therefore, in cases with positive transfers and generation \( j+1 \) members, initially

\[
\frac{\partial U^j}{\partial V^j} \frac{\partial V^j}{\partial C^j} \lambda n^j = MC^j_a < MC^j_h
\]

leading to \( a^j = 0 \). Only in the case where diminishing returns have set in sufficiently to raise \( MC^j_h \) into equality with \( MC^j_a \) is an interior solution for \( a^j \) attained. In that event, equality signs replace the more general formulation of equations (11a) and (11b).

We represent the supply of growth opportunities in this paper by an equiproportionate increase in the marginal productivity of physical capital and quality-adjusted labor applicable to all commodities. Since the price of physical capital is fixed by assumption in terms of consumer goods, the increase in physical capital's productivity is reflected in an equivalent rise in the rate

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6. Underlying the passage is an assumption that transfers are normal.

7. Since in the present formulation human capital is labor-augmenting, its productivity by implication also increases in this proportion.

A more complex formulation would stress the origins of growth opportunities; in particular, technological progress would be induced; in some formulations the neutral case we have assumed would follow only if marginal costs of "improving" marginal products were proportional to expenditures on productive factors. See Becker (1971), pp. 129-234.
of interest. Where \( P_k \) = price of a unit of physical capital (using the price of consumer goods as consumeraire), \( MP_k \) = marginal product of capital, and \( r \) = rate of interest, and since by our assumptions capital is infinitely long-lived,

\[
P_k = \frac{MP_k}{r}
\]  

(13)

Since \( P_k \) is constant, \( EP_k - EMP_k - Er = 0 \), implying that \( EMP_k = Er \), where \( E \) is the differential log operator. Furthermore, since productivity in human capital "manufacture" and employment rises by the common proportion, the increase in the rate of return on physical capital would be precisely matched at input levels prevailing prior to technical progress, since the rate of transformation is unaffected and therefore the price of human versus physical capital unaffected evaluated at that point.

Since all marginal products rise in the same proportion for all goods a neutral gross or exogenous growth rate in per-capita output exists, defined by

\[
\psi = \frac{y_{j+1}^{j+1} - y_j^j}{N_{j+1}^j - N_j^j},
\]

(14)

where \( \psi \) holds stocks of productive factors constant. By way of distinction, the net or achieved rate of growth depends on achieved rates of growth in population and the stocks of productive factors as well as \( \psi \). The growth rate in population, which emerges from the collectivity of decisions for any one individual characterized by equations (1) through (5), is

\[
g_N = \frac{N_{j+1}^j - N_j^j}{N_j^j}.
\]

(15)
Increments in capital stocks per head are defined by
\[ g_k = \frac{\Delta K^j}{N^j_j} \quad \text{and} \quad g_q = \frac{\Delta q^j}{q^j_j} \]
(16)

The achieved growth rate of output per head is then
\[ g = \frac{Y^{j+1}}{N^{j+1}} - \frac{Y^j}{N^j} = \psi + \eta_q g_q + \eta_k g_k, \]
where the partial elasticities of output with respect to per capita inputs (i = q, k).

To demonstrate a difference between exogenous and achieved growth rates would require showing that \( \frac{\partial g}{\partial \psi} \) (which might be termed the growth acceleration coefficient) were not equal to unity, where the expression for \( \frac{\partial g}{\partial \psi} \) is
\[ \frac{\partial g}{\partial \psi} = 1 + \eta_q \frac{\partial g_q}{\partial \psi} + \eta_k \frac{\partial g_k}{\partial \psi}, \]
where the approximation consists in assuming constant partial elasticities of output. We will try to show that \( \frac{\partial g}{\partial \psi} < 1 \) so that
\[ \eta_q \frac{\partial g_q}{\partial \psi} + \eta_k \frac{\partial g_k}{\partial \psi} < 0, \]
(19)
or in other words growth rates in factor stocks per head respond negatively to exogenous accelerations in growth. In addition we will try to show that \( \frac{\partial g}{\partial \psi} \) is bounded between zero and unity, and is positive as long as per capita intergenerational transfers are normal in the preferences of the preceding generation. Since the growth rate of expenditures per capita, which are the sum of per capita own consumption plus gross transfers (denoted \( \bar{c}_j^j = \bar{c}_j^j + \bar{c}_j^{j+1} \)) is equal to the achieved growth rate of income \( g \), if it can be shown that \( \bar{c}_j^j \) grows at a slower rate the greater the rate of exogenous growth, the sign of expression
(19) will have been proven. The growth rate in per capita expenditures is defined by

\[ g_j = \frac{\Delta \bar{C}_j^j}{\bar{C}_j^j} = \frac{\Delta C^j}{C^j} + \frac{\Delta G^{j+1}}{G^{j+1}} \]  

(20)

Since \[ \frac{G^{j+1}}{N^j} = \frac{N^{j+1}}{N^j} \left( \bar{P}_h e^{j+1} + \bar{P}_h n^{j+1} + \bar{P}_a a^{j+1} \right), \]

where barred variables represent economy-wide averages as usual, we have

\[ \Delta \bar{G}^{j+1} = \frac{N^{j+1}}{N^j} \left( \bar{MC}_h \Delta h^{j+1} + \bar{MC}_a \Delta a^{j+1} \right) \]

\[ = (1 + \bar{r}_N) \left( \bar{MC}_h \Delta h^{j+1} + \bar{MC}_a \Delta a^{j+1} \right). \]

Substituting (22) into (20), and rearranging terms, we obtain

\[ \frac{\Delta \bar{C}_j^j}{\bar{C}_j^j} = \alpha_C \delta_C + \alpha_G \gamma_h \delta_h + \gamma_a \delta_a, \]

(23)

where

\[ \alpha_C = \frac{\bar{C}^j}{\bar{C}^j + \bar{G}^{j+1}}, \]

\[ \alpha_G = \frac{\bar{G}^{j+1}}{\bar{C}^j + \bar{G}^{j+1}}, \]

\[ \delta_C = \frac{\Delta \bar{C}^j}{\bar{C}^j}, \]

\[ \delta_h = \frac{\Delta \bar{h}^{j+1}}{\bar{h}^{j+1}}, \]

\[ \delta_a = \frac{\Delta \bar{a}^{j+1}}{a^{j+1}}, \]

mean expenditure shares of own consumption and gross transfers, jth generation.

mean growth rates in own consumption, human capital transfers and asset transfers.
\[ \begin{align*}
\gamma_h &= \frac{MC_h \bar{h}}{MC_h \bar{h} + MC_a \bar{a}}, \\
\gamma_a &= \frac{MC_a \bar{a}}{MC_h \bar{h} + MC_a \bar{a}},
\end{align*} \]

mean shares of human capital and physical capital in all transfers, valued at the average of marginal costs.

Therefore, showing that the growth acceleration coefficient \( \delta^* \) is less than unity comes down to showing that \( \frac{\Delta C_t}{C_t} \) increases less than proportionately in response to an increase in \( \Psi \), or in other words

\[ \frac{3}{\partial \Psi} \left[ \frac{\Delta C_t}{C_t} \right] = \alpha \delta \frac{\Delta C_t}{\partial \Psi} + \bar{a}_G \left( \delta \frac{\Delta C_t}{\partial \Psi} + \gamma_a \frac{\Delta C_t}{\partial \Psi} \right) < 1, \quad (24) \]

This result follows if mean generation \( j \) transfers per generation \( j+1 \) recipient decrease in response to an increase in the wealth of the next generation, the present generation's wealth held constant. Put differently, an increase in the level of mean endowments of the next generation corresponds to an increase in the growth rate of income per capita. \(^9\) From the budget constraint of the representative individual, \( \varepsilon^j + r^j = c^j \), where \( r^j = \frac{R^j}{n^j} = h^j + a^j \), and \( c^j = c^j + c^{j+1} \), we obtain

\[ \varepsilon^j + r^j = c^j. \quad (25) \]

\(^9\) Clearly, from equation (14),

\[ \frac{d}{\partial \Psi} \left( \frac{\gamma^j}{N^j} \right) = \frac{\gamma^j + 1}{N^j+1} = \frac{\gamma^j + 1}{N^j+1}. \quad (a) \]

From \( \varepsilon^j + \frac{R^j}{n^j+1} = c^j+1 \), and the assumption of neutral growth, it is apparent that

\[ \frac{d}{\partial \Psi} \left( \frac{\gamma^j + 1}{N^j+1} \right) = \varepsilon^j+1. \quad (b) \]
The exogenous increase in opportunities is represented by \( e^j = e^{j+1} - e^j \), the change in recipient wealth, transfers per head held constant. The term \( \Delta r^j \) reflects the change in net transfers per head. In terms of proportional changes,

\[
\frac{\Delta c^j}{c^j} = g = \frac{e^j}{c^j} \psi - \frac{r^j}{c^j} \frac{\Delta r^j}{r^j},
\]

since \( \Delta e^j = \psi \), and \( \Delta r^j = g \), as we have shown. Hence

\[
g = \left( \frac{e^j}{c^j} + \frac{r^j}{c^j} \right) \eta_r \psi \psi,
\]

where \( \eta_r \psi = \frac{\Delta r^j}{r^j} \frac{e^j}{\Delta e^j} \).

In the Appendix it is shown that \( \eta_r < 0 \), and since \( \frac{e^j}{c^j} < 1 \), \( g < \psi \).

Furthermore, if \( \eta_r \psi \) is constant, (27) shows that

\[
\frac{e^j}{c^j} + \frac{r^j}{c^j} \eta_r \psi = \frac{\partial g}{\partial \psi},
\]

the growth acceleration coefficient.

That \( \eta_r \psi < 0 \) is shown by a simple chain of reasoning. With interdependent preferences, the social wealth of an individual consists of his wealth of recipients, the latter valued at the marginal cost of transfers. An increase in social wealth comprised entirely of an increase in recipient's wealth per head, as in the case of exogenous growth, implies an increase in own consumption and a corresponding decrease in personal transfers, hence \( \frac{\Delta r^j}{\Delta e^j} < 0 \). In other words, since the marginal propensity to contribute to generation \( j+1 \) is less than the increment in wealth, and the increment in wealth consists of "contributions by nature", own contributions decline. It is possible for price
effects to modify this conclusion under conditions different from those assumed in this paper, but notice that neutral growth precludes changes in capital prices at the position prior to intergenerational reallocation of resources.

More generally, when \( \eta_{\mathbf{rp}} \) is not constant, the growth acceleration coefficient

\[
\frac{\partial g}{\partial y} = \frac{e^j}{C^j_N} + \frac{e^j}{C^j_N} \eta_{\mathbf{rp}} + \frac{e^j}{C^j_N} \eta_{\mathbf{rp}} \frac{\partial \eta_{\mathbf{rp}}}{\partial y}
\]

(28)

It might appear that the indeterminacy of sign to be attached to the third term makes the bounding of \( \frac{\partial g}{\partial y} \) impossible. However, since \( \frac{dy}{N^j} = \frac{-e^j}{N^j} \), the argument presented above, which signs \( \frac{\Delta r^j}{\Delta e^j} \), also tells us \( \frac{\Delta r^j+1}{\Delta e^j} < 0 \).

Therefore, since

\[
dg = \frac{d\bar{c}^j+1}{\bar{c}^j_N} = \frac{e^j}{\bar{c}^j_N} \frac{de^j+1}{e^j} + \frac{e^j}{\bar{c}^j_N} \frac{dr^j+1}{r^j},
\]

or

\[
\frac{\partial g}{\partial y} = \left( \frac{e^j}{\bar{c}^j_N} + \frac{e^j}{\bar{c}^j_N} \eta_{\mathbf{rp}} \right)
\]

(30)

Where \( \eta_{\mathbf{rp}} = \frac{\Delta r^j+1}{\Delta e^j} \) or the elasticity marginal to \( \eta_{\mathbf{rp}} \). Furthermore, so long as the next generation's consumption is normal, \( \frac{\partial \bar{c}^j+1}{\partial e^j} = \frac{de^j+1}{\partial e^j} + \frac{dr^j+1}{\partial r^j} > 0 \) and \( \frac{de^j+1}{\partial r^j} > \frac{dr^j+1}{\partial r^j} \). Hence \( 0 < \frac{\partial g}{\partial y} < 1 \), which we set out to show.

Furthermore, at the position after exogenous growth has taken place, the amount of all-inclusive capital to society has increased at constant prices. The supply price of capital on the other hand, must have increased, since the demand for transfers has declined. This implies that the supply price of capital by
...savers rises as the stock of capital increases proportionately in all forms. Indirectly, the simplicity of this conclusion depends on the neutral growth assumption, which isolates the analysis from the separate question of changes in the relative prices of various forms assumed by capital.

The analysis on a market level of the effects we have been describing will incorporate repercussive changes in relative capital prices. Figure 1 illustrates the (non-market) equilibrium in accumulation of human capital for a Type I family, which transfers wealth in human earning power. The locus $P^h_S$ represents the demand curve for human capital prior to exogenous growth by members of generation $j+1$. The locus slopes downward, reflecting diminishing returns. The $CS^h$ locus on the other hand has positive but less than infinite slope up to the prevailing price on physical capital, and is thereafter vertical, reflecting the redirection of transfers into financial form, where $h^*$ is the maximum human capital attainment at the prevailing market price of physical capital. Exogenous growth shifts

\[ \text{FIGURE 1}\]

EQUILIBRIUM IN HUMAN CAPITAL ACCUMULATION FOR A TYPE I FAMILY

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The reader is reminded that this is a per capita analysis. In the following passage it is implicitly assumed that physical capital is not an important input into the human capital production process, hence $P^h_S$ is independent of movements in physical capital.
the productive locus to \( PS^h \); at the price of human capital prevailing before growth, human capital becomes \( h_0 = (1 + \psi)h_0 \). The rising \( CS^h \) locus establishes an intermediate level of human capital \( h_2 = (1 + g_h)h_0 \). This analysis for a family intramarginal in its transfers of human capital would be little altered for a marginal family. In the latter case, expansion in \( h \) would simply be truncated by the vertical section of \( CS^h \).

The analysis of the physical capital market is similar, though not identical, since account must be taken of movements in \( h \). Here \( CS^k \) is again upward-sloping and reflects the accumulation of capital by generation \( j \) members who acquire additional wealth only in financial form (Type II families).\(^{12}\) The \( PS^k \) locus is drawn as downward sloping, reflecting the diminishing marginal rate of transformation of present into future goods as \( k \) alone varies, symbolized in the diagram.

\(^{10}\) The price of human capital is \( P_h = \frac{1 + r}{MP_h} \), where \( MP_h \) = marginal product of \( h \) in consumer goods production.

\(^{11}\) Strictly speaking, constancy of \( CS^h \) requires that an increase in generation \( j + 1 \) wealth be compensated by a decline generation \( j \) wealth so as to hold social wealth of generation \( j \) constant.

\(^{12}\) Once again, \( P_k \) has the dimension of the inverse of present value, and is defined as \( P_k = \frac{r_k}{MP_k} \).
by fixity of h at h₀. After exogenous growth takes place, PS^k shifts rightward to PS^h. The intersection of CS^k and PS^h does not chart the final position, however, since equilibrium in human capital accumulation occurs at h₂, less than h₁. So long as k and h are production function complements, the new locus lies between PS^k and PS^h, at PS^h (h₂), and the equilibrium position occurs at k₂ = (1 + gₜ)k₀. Once again the achieved growth rate is less than the exogenous growth rate. As a secondary effect, the rise in price of physical capital shifts the vertical section of CS^h to the right, creating AECSh in place of ADCSh (see Figure 1). The main implication of the shift is to cause some families to accumulate more in human capital than they otherwise would.

All of this analysis has been conducted in per capita terms. Our conclusion that capital supply curves slope upward applies only to the capital labor ratio. The per capita analysis is certainly meaningful, both from the standpoint of economic welfare and for reasons of comparison with the literature on stationary states and moving equilibria. Still, no implications follow about the total supply curve of capital in all forms, defined as

\[ S = N^{j+1} (h^{j-1} + \frac{p^a}{P_h} \cdot a^{j+1}), \]  

(31)

since N^{j+1} is positively related to movements in ¥, as we shall show in the next section, among other things. Again let it be emphasized, however, that the important variables to be analyzed are per capita variables.

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13The stock of capital per head becomes k₁ = (1 + ¥)k₀.
4. Comparative Statics.

The number of family members increases because the social wealth of generation j members has risen due to growth, while numbers of children are initially unaffected. Furthermore, the marginal cost of increasing family size is lower because per capita contributions decline. Hence $\frac{\partial g_{n}}{\partial n} > 0$.\(^{15}\)

Equation (31) indicates that S tends to rise or fall at constant relative factor prices as $n^{j+1}$ is elastic or inelastic on average with respect to per capita contributions.

The demonstration that population growth is furthered by acceleration in economic growth also means that more capital is accumulated in human form, since diminishing returns are delayed with the increased expansion of human capacities as population grows.\(^{16}\) Therefore, the faster the rate of exogenous growth, the larger the share of human capital accumulation in all capital accumulated.

In addition, the nature of capital formulation is not independent of the level of present wealth, since physical capital transfers are more wealth-elastic than human capital transfers, implying that increases in the rate of economic growth have an attenuated effect on the form of capital accumulation the higher is mean wealth per family in the initial period. The reason for this connectedness in effects on the nature of capital formation is that a larger % of families are Type II, the greater is mean wealth, permitting lesser scope of adjustment in the direction of human capital accumulation.

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\(^{14}\)The analysis in this section is heavily indebted to Becker and Tomes (1976).

\(^{15}\)See the Appendix, and equation (15).

\(^{16}\)We have a curious asymmetry. The distinction between number of persons and hours of work per person in production is generally ignored [see Rosen (1968)]. A more complete analysis would incorporate it here.
5. Conclusion

The principal purpose of this paper has been to show that the supply of capital per head in all forms is rising. This result leads to a distinction between achieved and exogenous growth rates, where both level and change in the achieved rate lags behind the behavior of the exogenous rate. More importantly, the theory of capital supply leads to a concept of equilibrium growth; the supply of growth opportunities is matched with a full-fledged demand schedule for growth, and the theory of zero growth in per capita income and wealth becomes a theory of the conditions under which reductions in intergenerational savings exactly offset the supply of growth opportunities. This situation presumably is more likely to occur the lower the exogenous growth rate and hence the rate of interest; the lower the wealth elasticity of demand for consumption of the next generation, and hence the greater the reduction in capital supply for equivalent growth; and the greater the elasticity of substitution between consumption of adjacent generations, and hence the greater the reduction in capital supply for equivalent reductions is the rate of interest and growth.
Allen (1938), Section 19.2 shows that for a constrained maximum under a nonlinear constraint, the bordered Hessians which alternate in sign are

\[
\begin{vmatrix}
  fx & \phi x & fxy - \frac{fx}{\phi x} & \phi y \\
  fxy - \frac{fx}{\phi x} & \phi y & fyx - \frac{fx}{\phi x} & \phi x \\
  \phi x & \phi y & fyx - \frac{fx}{\phi x} & \phi y \\
  \phi x & \phi y & \phi x & \phi y \\
\end{vmatrix}
\]

\[\phi x < 0, \quad \phi y > 0, \quad (a.1)\]

and so forth where \( \phi \) is the constraint and \( f \) the function to be maximized.

We shall utilize this fact here, since the budget constraint is nonlinear in our model.

Assuming an interior solution for all goods except \( a^{j+1} \), we have a Type I family, the first order conditions for which are (where \( U_{j+1} = \frac{\partial U}{\partial y^{j+1}} \), \( V_{j+1} = \frac{\partial V}{\partial C^{j+1}} \), etc.):

\[
\begin{align*}
U_{j+1}V_j + \lambda n^{j+1}\mu c^{j+1} &= 0 \\
U_n - \lambda p_h^{j+1} &= 0 \\
U_jv_j - \lambda &= 0 \\
e^j + h^j + a^j - c^j - n^{j+1}p_h^{j+1} &= 0.
\end{align*}
\]

and

\[e^j + h^j + a^j = I^j\]
Second order conditions of the problem are that determinants similar to (a.1) alternate in sign:

\[
\begin{vmatrix}
U_{j+1}, j+1 V_{j+1}^2 + U_{j+1} V_{j+1}, j+1 - \frac{U_{j+1} V_{j+1}}{MC_{j+1}^j} \frac{dMC_{j+1}^j}{dh_{j+1}} & U_{j+1}, j V_{j+1} V_{j} - n^{j+1} MC_{j+1}^j & -1 > 0 \\
U_{j+1}, j V_{j+1} V_j & U_{jj} V_j^2 + U_{jj} V_j & -1 < 0 \\
-n^{j+1} MC_{j+1}^j & -1 & 0 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
U_{j+1} V_{j+1}^2 + U_{j+1} V_{j+1}, j+1 - \frac{U_{j+1} V_{j+1}}{MC_{j+1}^j} \frac{dMC_{j+1}^j}{dh_{j+1}} & U_{j+1}, j V_{j+1} V_{j} - n^{j+1} MC_{j+1}^j & -1 > 0 \\
U_{j+1}, j V_{j+1} V_j & U_{jj} V_j^2 + U_{jj} V_j & -1 < 0 \\
-n^{j+1} MC_{j+1}^j & -1 & 0 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
U_{j+1}, j V_{j+1} V_j & U_{j} V_{j} - 1 \\
U_{j+1}, n V_{j+1} - \frac{U_{j+1} V_{j+1}}{n^{j+1}} & U_{jn} V_j \\
-n^{j+1} MC_{j+1}^j & -1 & (a.3) \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
U_{j+1}, j V_{j+1} V_j & U_{j} V_{j} - 1 \\
U_{j+1}, n V_{j+1} - \frac{U_{j+1} V_{j+1}}{n^{j+1}} & U_{jn} V_j \\
-n^{j+1} MC_{j+1}^j & -1 & (a.3) \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
U_{j+1}, j V_{j+1} V_j & U_{j} V_{j} - 1 \\
U_{j+1}, n V_{j+1} - \frac{U_{j+1} V_{j+1}}{n^{j+1}} & U_{jn} V_j \\
-n^{j+1} MC_{j+1}^j & -1 & (a.3) \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
U_{j+1}, j V_{j+1} V_j & U_{j} V_{j} - 1 \\
U_{j+1}, n V_{j+1} - \frac{U_{j+1} V_{j+1}}{n^{j+1}} & U_{jn} V_j \\
-n^{j+1} MC_{j+1}^j & -1 & (a.3) \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
U_{j+1}, j V_{j+1} V_j & U_{j} V_{j} - 1 \\
U_{j+1}, n V_{j+1} - \frac{U_{j+1} V_{j+1}}{n^{j+1}} & U_{jn} V_j \\
-n^{j+1} MC_{j+1}^j & -1 & (a.3) \\
\end{vmatrix}
\]
Let $D$ be the matrix corresponding to the second determinant of (a.3), and $x$ and $y$ be the vectors

$$x = \begin{bmatrix} dC_{j+1}^x \\ dC_j^x \\ dn_{j+1} \\ d\lambda \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -MC_{j+1}^n \cdot j+1 \cdot de_{j+1} \end{bmatrix}$$

respectively. Then totally differentiating (a.2) for $de_{j+1} > 0$ yields

$$D \cdot x = y \quad (a.3)$$

No exogenous price changes take place, and hence the upper three entries of $y$ are zeroes; endogenous price effects are incorporated in $D$. Notice that for constant transfers $P_h$, hence $MC_{j+1}^h$, remain constant. Solving for $dC_{j+1}^x$,

$$dC_{j+1}^x = -\frac{|D_{j+1}|}{|D|} \cdot MC_{j+1}^h \cdot j+1 \cdot de_{j+1} \quad (a.4)$$

where $-|D_{j+1}| = \frac{\partial C_{j+1}^x}{\partial I^j}$. Therefore,

$$0 < \frac{\partial C_{j+1}^x}{\partial e_{j+1}^1} = \frac{\partial C_{j+1}^x}{\partial I^j} \cdot MC_{j+1}^h \cdot n_{j+1} < 1, \quad (a.5)$$

since the marginal propensity to consume $C_{j+1}^x$ is less than unity and $j+1$ consumption is normal. But

$$\frac{\partial h_{j+1}}{\partial e_{j+1}^1} = \frac{\partial C_{j+1}^x}{\partial e_{j+1}^1} - 1 < 0. \quad (a.6)$$

Furthermore,

$$\frac{\partial n_{j+1}}{\partial e_{j+1}^1} = \frac{\partial n_{j+1}}{\partial I^j} \cdot MC_{j+1}^h \cdot n_{j+1} > 0, \quad (a.7)$$

if the number of children is also a normal good.
The analysis for a Type II family is quite similar; first order conditions are

\[
\begin{align*}
U_{j+1}V_{j+1} - \lambda n_{j+1} MC_{a}^{j+1} &= 0 \\
U_{n} - \lambda (MC_{a}^{j+1} + P_{h}^{j+1}) &= 0 \\
U_{j}V_{j} - \lambda &= 0
\end{align*}
\]

where \( h_{a}^{j+1} \) is a constant reflecting maximum human capital accumulation at the prevailing price of a unit of physical capital. Second order conditions are also quite similar, and hence are omitted.
REFERENCES


Becker, G.S., 1967, Human Capital and the Personal Distribution of Income (University of Michigan, Ann Arbor).


Friedman, M., 1962, Price Theory (Aldine, Chicago).


