1993

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Bernhard Kawohl

Universitat Heidelberg

Howard A. Levine

Iowa State University, halevine@iastate.edu

Waldemar Velte

Universitat Wurzburg

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BUCKLING EIGENVALUES FOR A CLAMPED PLATE EMBEDDED IN AN ELASTIC MEDIUM AND RELATED QUESTIONS*

BERNHARD KAWOHL†, HOWARD A. LEVINE‡, AND WALDEMAR VELTE§

Abstract. This paper considers the dependence of the sum of the first $m$ eigenvalues of three classical problems from linear elasticity on a physical parameter in the equation. The paper also considers eigenvalues $\gamma_i(a)$ of a clamped plate under compression, depending on a lateral loading parameter $a$; $\Lambda_i(a)$, the Dirichlet eigenvalues of the elliptic system describing linear elasticity depending on a combination $a$ of the Lamé constants, and eigenvalues $\Gamma_i(a)$ of a clamped vibrating plate under tension, depending on the ratio $a$ of tension and flexural rigidity. In all three cases $a \in [0, \infty)$. The analysis of these eigenvalues and their dependence on $a$ gives rise to some general considerations on singularly perturbed variational problems.

Key words. eigenvalue, asymptotic, parameter dependence, plate equation, elasticity, singular perturbation

AMS(MOS) subject classifications. 35J50, 35J55, 35P15, 49G05, 49G20

Introduction. Let, for $i = 1, 2, \ldots$, $\gamma_i$ be the eigenvalues of the equation for the clamped plate under compression, $\Gamma_i$ be the eigenvalues for the equations of linear elasticity, and $\Lambda_i$ be the eigenvalues for the equation for the vibrating clamped plate under tension. Briefly, our first result says that $\sum_{i=1}^{m} \gamma_i(a)$ and $\sum_{i=1}^{m} \Gamma_i(a)$ are strictly concave functions of $a$, while $\sum_{i=1}^{m} \Lambda_i(a)$ is concave. Moreover and in particular

$$\lim_{a \to \infty} \gamma_1(a) = +\infty \quad \text{and} \quad \lim_{a \to \infty} \frac{\gamma_1(a)}{\sqrt{a}} = 2,$$

$$\lim_{a \to \infty} \Lambda_1(a) < \infty,$$

$$\lim_{a \to \infty} \Gamma_1(a) = +\infty \quad \text{but} \quad \lim_{a \to \infty} \frac{\Gamma_1(a)}{a} = \lambda_1.$$

(Here $\lambda_1$ is the first Dirichlet eigenvalue for the Laplacian which is also known as the first eigenvalue for the fixed membrane.) The graphs of these functions are sketched in Figs. 1, 2, and 4 along with the previously known upper and lower bounds. The plan of the paper is as follows. In §1 we discuss $\gamma_1(a)$, $\Lambda_1(a)$ and $\Gamma_1(a)$. We use some ideas of [10], [11], [12] to obtain some of our results. In §§2 and 3 we consider generalizations, first to abstract linear problems and then to nonlinear problems. Throughout the paper $\{\lambda_i\}_{i \in N}$ denotes the ordered sequence of eigenvalues of the problem

$$\Delta \psi + \lambda \psi = 0 \quad \text{in} \quad \Omega,$$

$$\psi = 0 \quad \text{on} \quad \partial \Omega,$$

while $\{\psi_j\}_{j \in N}$ denotes the corresponding sequence of orthonormal eigenfunctions.

*Received by the editors July 15, 1991; accepted for publication (in revised form) June 9, 1992.
†SBF 123, Universität Heidelberg, Im Neuenheimer Feld 294, D 6900 Heidelberg, Germany. This author was supported in part by the Deutsche Forschungsgemeinschaft (DFG) via SBF 123 and a Heisenberg award.
‡Mathematics Department, Iowa State University, Ames, Iowa 50011. This author was supported by the Deutsche Forschungsgemeinschaft (DFG) via SBF 123 as well as National Science Foundation grant DMS-8822788.
§Institut für Angewandte Mathematik, Universität Würzburg, D8700 Würzburg, Germany.
1. The first eigenvalue of a clamped plate under compression. Let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary and let $a \geq 0$ be a parameter. Consider the eigenvalue problem

$$\begin{align*}
\Delta \Delta u + au + \gamma(a) \Delta u &= 0 \quad \text{in } \Omega, \\
u = \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(1)

where $a > 0$ is given and represents the elasticity constant of a medium surrounding the plate. The function $u$ stands for the transverse displacement \[9\] and $\gamma_1(a)$ is the minimal compression at which the plate exhibits buckling. Payne established the following inequality in \[9\]

$$\max\{\gamma_1(0), 2\sqrt{a}\} \leq \gamma_1(a) \leq \gamma_1(0) + \frac{a}{\lambda_1}.$$  

(Indeed, he showed that (2) holds for each eigenvalue $\gamma_i(a)$ of (1).) Moreover, Levine and Protter derived the lower bound

$$\sum_{i=1}^{m} \gamma_i(a) \geq \frac{4\pi^2 N m^{(1+2/N)}}{(N+2)(\omega_N V)^{2/N}}$$

in \[6\]. Here $\omega_N$ denotes the surface area of the unit ball in $\mathbb{R}^N$ and $V$ denotes the volume of $\Omega$.

Theorem 1. The function $F(a) = \sum_{i=1}^{m} \gamma_i(a)$ is strictly concave and strictly increasing in $a$ on $[0, \infty)$.

For the proof we use the variational characterization of $F(a)$. It is well known that the Rayleigh quotient associated with (1) is given by

$$R_a(v) = \frac{\int_{\Omega} (\Delta v)^2 dx + a \int_{\Omega} v^2 dx}{\int_{\Omega} |\nabla v|^2 dx},$$

(4)

where $v \in H_0^2(\Omega)$. Let us first prove that $F(a)$ is concave by establishing that for any $a_0 \in [0, \infty)$ there exists $M \in \mathcal{R}$ such that

$$F(a) - F(a_0) \leq M(a - a_0) \quad \text{for any } a \in [0, \infty).$$

(5)

From the min–max characterization of eigenvalues (see \[1, \text{Vol. 1}\]) we know that

$$F(a) \leq \sum_{i=1}^{m} R_a(v_i)$$

(6)

for every orthonormal system \(\{v_1, \ldots, v_m\}\) of admissible functions in $H_0^2(\Omega)$. Here \(\{v_1, v_2, \ldots, v_m\}\) are orthonormal with respect to $\int_{\Omega} \nabla v_i \nabla v_j \, dx$.

Let $u_i^a$ denote the $i$th eigenfunction associated with (1), normalized to $||\nabla u_i^a||_{L^2(\Omega)} = 1$. Set $v_i = u_i^{a_0}$. Then (6) implies

$$F(a) - F(a_0) \leq (a - a_0) \sum_{i=1}^{m} \int_{\Omega} v_i^2 \, dx.$$

Therefore (5) holds with a positive $M$ and $F(a)$ is strictly increasing and concave. To prove that $F(a)$ is strictly concave, suppose the contrary. Then there exists an interval $[b, c] \subset [0, \infty)$ such that

$$F(tb + (1 - t)c) = tF(b) + (1 - t)F(c)$$

holds for every $T \in [0, 1]$. In particular, setting $t = \frac{1}{2}$ and $a = (b + c)/2$

(7) \quad \quad F(a) = \frac{1}{2}F(b) + \frac{1}{2}F(c).

We observe that

(8) \quad \quad \gamma_i(b) = \mathcal{R}_b(u_i^b) \leq \mathcal{R}_b(u_i^c),

and that the same inequality holds with $b$ replaced by $c$. Therefore we have from (7) and (8)

$$F(a) \leq \frac{1}{2} \sum_{i=1}^{m} \mathcal{R}_b(u_i^b) + \frac{1}{2} \sum_{i=1}^{m} \mathcal{R}_c(u_i^c)$$

$$= \sum_{i=1}^{m} \mathcal{R}_a(u_i^a) = F(a).$$

But now equality must hold in (8) for every $i = 1, \ldots, m$. In particular $\gamma_1(b) = \mathcal{R}_b(u_1^b)$ and $\gamma_1(c) = \mathcal{R}_c(u_1^c)$; that is, $u = u_i^b$ is an eigenfunction corresponding to both $\gamma_1(b)$ and $\gamma_1(c)$. Subtraction of the corresponding differential equations (1) yields:

(9) \quad \quad (\gamma_1(c) - \gamma_1(b))\Delta u + (c - b)u = 0.

If $\gamma_1(c) = \gamma_1(b)$ then (9) implies $c = b$ as desired. Otherwise $u$ is an eigenfunction to the Laplace operator on $\Omega$ and satisfies both $u = 0$ and $\partial u/\partial n = 0$ on $\partial \Omega$, a contradiction to Hopf's second lemma. Therefore $c = b$ and this completes the proof of Theorem 1. \quad \square

Remark 1. In [10, p. 286ff] Polya and Schiffer proved concavity of sums of eigenvalues for some Neumann problems. Our result and proof are inspired by theirs. One might conjecture that each of the eigenvalues is concave in $a$ separately. Numerical results in [5] indicate that in general this is not the case. Notice, however, that $\gamma_1(a)$ is strictly concave.

From now on we concentrate on the first eigenvalue $\gamma_1(a)$.

Corollary 2. Inequalities (2) are strict for $a > 0$.

Proof. By Theorem 1, equality cannot hold on the right-hand side of (2) for $a > 0$, nor on the left-hand side when $0 < a < \gamma_1^2(0)/4$. Moreover, we have after integration by parts in the denominator and by Schwarz's inequality

$$\gamma_1(a) = \mathcal{R}_a(u_1^a) = \frac{\int_{\Omega}(\Delta u_1^a)^2dx + a \int_{\Omega}(u_1^a)^2dx}{\int_{\Omega} |\nabla u_1^a|^2dx}$$

$$\geq \left( \frac{\int_{\Omega}(\Delta u_1^a)^2dx}{\int_{\Omega}(u_1^a)^2dx} \right)^{1/2} + a \left( \frac{\int_{\Omega}(\Delta u_1^a)^2dx}{\int_{\Omega}(u_1^a)^2dx} \right)^{1/2} \geq 2\sqrt{a}.$$

Thus $\gamma_1(a) = 2\sqrt{a}$ if and only if

$$\frac{\int_{\Omega}(\Delta u_1^a)^2dx}{\int_{\Omega}(u_1^a)^2dx} = a.$$
and
\[ \int_\Omega (\Delta u_i^a)^2 dx \cdot \int_\Omega (u_i^a)^2 dx = \left( \int_\Omega u_i^a \Delta u_i^a dx \right)^2, \]
and equality holds if and only if \( \Delta u_i^a + \sqrt{a} u_i^a = 0 \). But \( u_i^a = 0 = \partial u_i^a / \partial n \) on \( \partial \Omega \), so that \( u_i^a = 0 \). Thus, the strict inequalities
\[
\max\{\gamma_1(0), 2\sqrt{a}\} < \gamma_1(a) < \gamma_1(0) + \frac{a}{\lambda_1}
\]
hold. This completes the proof of Corollary 2. \( \square \)

Of particular interest is the asymptotic behavior of the eigenvalue \( \gamma_1(a) \) and the associated eigenfunction \( u_i^a \) as \( a \to \infty \). We can give the following partial answer to this problem.

**Theorem 3.** Let \( u_i^a \) be a first eigenfunction, normalized so that \( ||\nabla u_i^a||_{L^2(\Omega)} = 1 \), and \( \gamma_1(a) \) the first eigenvalue of (1). Then \( ||u_i^a||_{L^2(\Omega)} \to 0 \) and \( \gamma_1(a) / \sqrt{a} \to 0 \) as \( a \to \infty \).

The proof of Theorem 3 will proceed in several steps. The results of Theorems 1, 3, and inequality (2) are illustrated in Fig. 1.

**Fig. 1.** \( \gamma_1(a) \).

**Lemma 4.** (a) If \( \Omega \) is starshaped with respect to zero, then \( \gamma_1(a) / \sqrt{a} \) is decreasing, i.e.,
\[
\frac{\gamma_1(a)}{\sqrt{a}} > \frac{\gamma_1(b)}{\sqrt{b}} \quad \text{for } 0 < a < b.
\]
(b) If \( \Omega \) is a bounded domain, then there exists a constant \( M \in (2, \infty) \) such that
\[
\frac{\gamma_1(a)}{\sqrt{a}} < M \quad \text{as } a \to \infty.
\]

It should be remarked that Rother [12] proved (12) under the assumptions of Lemma 4(a). To prove Lemma 4(a), recall that
\[
\gamma_1(a) = \min_{u \in H_0^2(\Omega)} \mathcal{R}_a(u),
\]
with $R_a(u)$ defined by (4). Using the transformation

$$y_j = a^{1/4}x_j \quad \text{for } j = 1, \ldots, N; \quad v(y) = u(x(y)), \quad \frac{\partial u}{\partial x_j} = a^{1/4} \frac{\partial v}{\partial y_j},$$

the expression $R_a(u)$ is converted into

$$\tilde{R}_a(v) = \sqrt{a} \cdot \frac{\int_{\Omega_a} (\Delta v)^2 dx + \int_{\Omega_a} v^2 dx}{\int_{\Omega_a} |\nabla v|^2 dx},$$

where $\Omega_a = a^{1/4}\Omega$ is the image of $\Omega$ under the above transformation. Therefore $\gamma_1(a)/\sqrt{a}$ can be characterized through

$$\gamma_1(a)/\sqrt{a} = \min_{u \in H^2_0(\Omega_a)} \frac{\int_{\Omega_a} (\Delta v)^2 dx + \int_{\Omega_a} v^2 dx}{\int_{\Omega_a} |\nabla v|^2 dx},$$

and (13) is equivalent to the original characterization of $\gamma_1$. At this point the star-shapedness of $\Omega$ enters into the proof, because for starshaped domains we have

$$\Omega_a \subset \Omega_b \quad \text{for } 0 < a < b.$$ 

Since functions from $H^2_0(\Omega_a)$ can be continued by zero in $\overline{\Omega_b} \setminus \Omega_a$, property (11) follows from the well-known monotone dependence of eigenvalues on the domain $\Omega$. This completes the proof of Lemma 4(a).

To prove Lemma 4(b) let $B$ be a ball contained in $\Omega$. Without loss of generality we may assume that $\Omega$ contains zero and that $a > 1$. Let $\tilde{u}_1^a$ be a first eigenfunction associated to the first eigenvalue $\tilde{\gamma}_1(a)$ on $B$. Then by the monotone dependence of $\gamma_1$ on $\Omega$ and by (11) we have

$$\frac{\gamma_1(a)}{\sqrt{a}} \leq \frac{\tilde{\gamma}_1(a)}{\sqrt{a}} \leq \tilde{\gamma}_1(1),$$

and this completes the proof of Lemma 4. $\square$

Now we can prove the first statement of Theorem 3. Relation (14) implies

$$||u_1^a||^2_{L^2(\Omega)} \leq \frac{\tilde{\gamma}_1(1)}{\sqrt{a}},$$

so that $u_1^a \to 0$ in $L^2(\Omega)$ of order $a^{-1/4}$. To complete the proof of Theorem 3 it suffices to combine (14) with the following result.

**Lemma 5.** If $\Omega$ is a ball or rectangular parallelepiped, then

$$\lim_{a \to \infty} \frac{\gamma_1(a)}{\sqrt{a}} = 2.$$

To show (16), first in the one-dimensional case, we take $\{a_n\}_{n \in N} = \{\lambda_n\}_{n \in N} = \{n^2\pi^2\}_{n \in N}$ and $\overline{\Omega} = [0, 1]$. With $\psi_n = c_n \sin(n\pi x)$, it is easy to see that there are constants $d_1, d_2 > 0$ such that for all $n$, with $\Omega_n = (0, 1/n) \cup (1 - 1/n, 1)$

$$\int_{\Omega_n} |\nabla \psi_n|^2 \, dx \leq \frac{d_1}{\lambda_n^{1/2}},$$
$$\int_{\Omega_n} |\psi_n|^2 \, dx \leq \frac{d_2}{\lambda_n^{3/2}},$$
where we normalize the $\psi_n$ so that

$$\int_0^1 |\psi_n|^2 \, dx = \lambda_n \int_0^1 \psi_n^2 \, dx = 1.$$ 

The eigenfunctions are uniformly oscillating on (0,1). The functions $\psi_n$ are not admissible for the Rayleigh quotient (4), since they are not in $H_0^2(\Omega)$. In order to modify them near the boundary, we construct functions of the form $\phi_n = \psi_n \eta_e$, where $\varepsilon = 1/n$ and $\eta_e$ is given by

$$\eta_e(x) = \begin{cases} \phi_e(x) & 0 \leq x \leq \frac{1}{n}, \\ 1 & \frac{1}{n} < x < 1 - \frac{1}{n}, \\ \phi_e(1 - x) & 1 - \frac{1}{n} \leq x \leq 1, \end{cases}$$

where, for $x \in (0, \varepsilon)$,

$$\phi_e(x) = \frac{\int_0^x e^{-\varepsilon^2/(\varepsilon^2 - y^2)} \, dy}{\int_0^\varepsilon e^{-\varepsilon^2/(\varepsilon^2 - y^2)} \, dy}.$$

$\eta_e$ is of class $C^2$ in (0,1), $\eta = \eta' = 0$ at $x = 0, 1$, and there are constants $d_3, d_4$, independent of $\varepsilon$ such that $\max |\eta'_e| \leq d_3/\varepsilon$, $\max |\eta''_e| \leq d_4/\varepsilon^2$. Thus a tedious, but routine, calculation yields, using $\mu = \mu_1/n$ for notation,

$$\frac{\mathcal{R}_{\sqrt{\lambda_n}}(\psi_n \eta)}{\sqrt{\lambda_n}} \leq \frac{2 - \int_{\Omega_n} 2(\psi_n \eta' + \psi_n \eta'') \, dx + \lambda_n^2 \int_{\Omega_n} (\psi_n \eta'' + \psi_n \eta')^2 \, dx}{1 - \int_{\Omega_n} [(1 - \eta^2)(\psi_n' + \eta_n'' \eta)^2 \, dx \leq \frac{2 + d_5 \lambda_n^{-1/2}}{1 - d_6 \lambda_n^{-1/2}}$$

with constants $d_5, d_6$ independent of $n$. (Note that $\varepsilon = 1/n = \pi/\lambda_n^{1/2}$.) To verify (16) for arbitrary domains in higher dimensions, it is necessary to have good estimates for the local $L^2$ norms of the eigenfunction and its gradient near the boundary. For $N$-dimensional rectangles, however, the one-dimensional example is easily modified. If $\Omega$ is the unit ball in $\mathbb{R}^N$, then the radially symmetric eigenfunctions are given by

$$\psi_j(x) = c_j r^{-(N-2)/2} J_{(N-2)/2}(\sqrt{\lambda_j} r),$$

where $J_\nu$ is the usual Bessel function of order $\nu$, the numbers $\lambda_j^{1/2}$ are the roots of $J_\nu$ in increasing order and the $c_j$'s are normalizing constants chosen so that

$$\int_{\Omega} |
abla \psi_j|^2 \, dx = \lambda_j \int_{\Omega} \psi_j^2 \, dx = 1,$$

or

$$\int_0^1 (\psi_j'(r))^2 r^{N-1} \, dr = \lambda_j \int_0^1 \psi_j^2(r) r^{N-1} \, dr = \omega_N^{-1}.$$ 

Precisely, we have

$$c_j^2 = \frac{2}{\lambda_j J_{(N-2)/2}^2(\sqrt{\lambda_j})},$$

with

$$\sqrt{\lambda_j} = (j + (N - 3)/4)\pi + O\left(\frac{1}{j}\right).$$
and, for any index \( \nu > -1 \) as \( r \to +\infty \) and some constant \( C_\nu \),

\[
|J_\nu(r)| = C_\nu r^{-\nu} \left( 1 + O \left( \frac{1}{r} \right) \right).
\]

From these estimates we easily see that there are constants \( d_1, d_2 > 0 \) such that for all \( j >> 1 \),

\[
\int_{1-1/\sqrt{\lambda_j}}^{1} \psi_j^2(r) r^{N-1} \, dr \leq d_1 \lambda_j^{-3/2}
\]

and

\[
\int_{1-1/\sqrt{\lambda_j}}^{1} (\psi_j'(r))^2 r^{N-1} \, dr \leq d_2 \lambda_j^{-1/2}.
\]

Thus, if we take \( \eta = \eta_e(r) \) to be one on the ball of radius \( 1 - \varepsilon \), satisfy \( 0 < \eta < 1 \) in the annular region \( \{ 1 - \varepsilon < r < 1 \} \) with \( \eta(1) = \eta'(1) = 0 \) and with \( \varepsilon = \lambda_j^{-1/2} \), we see that with \( \phi_j = \eta_e \psi_j \), we again have

\[
\frac{\mathcal{R}_{\sqrt{\lambda_j}}(\eta_e \psi_j)}{\sqrt{\lambda_j}} \leq \frac{2 + d_5 \lambda_j^{-1/2}}{1 - d_6 \lambda_j^{-1/2}}
\]

for computable constants \( d_5, d_6 \). In fact we can choose \( \eta_e \) so that for some \( d_3, d_4 \) the following estimates hold: \( \max |\eta_e'| \leq d_3 \lambda_j^{-1/2} \) and \( \max |\eta_e''| \leq d_4 \lambda_j^{-1} \). This together with (10) completes the proof of Lemma 5 and thus of Theorem 3.

Remark 2. The limiting process in Theorem 3 can be recast as the singular perturbation problem of minimizing

\[
I_\varepsilon(v) = \int_{\Omega} \varepsilon (\Delta v)^2 + v^2 \, dx \quad \text{over} \quad \left\{ v \in H^2_0(\Omega) \mid \int_{\Omega} |\nabla v|^2 \, dx = 1 \right\}.
\]

The formal limit problem for \( \varepsilon = 0 \) has no solution, but as the proof of Lemma 4 shows, for certain domains \( \Omega \) there exists a minimizing sequence \( v_\varepsilon \) for \( I_0 \) such that \( I_0(v_\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Moreover, \( v_\varepsilon \) is highly oscillatory and the oscillations of \( v_\varepsilon \) are equidistributed. A similar qualitative behaviour has been observed by Müller in [8]. He minimized

\[
\tilde{I}_\varepsilon(v) = \int_0^1 [\varepsilon (v_{xx})^2 + (v_x^2 - 1)^2 + v^2] \, dx \quad \text{over} \quad H^2_0(0, 1),
\]

and showed that minimizers \( v_\varepsilon \) of \( \tilde{I}_\varepsilon \) are rapidly and regularly oscillating and converge to zero in \( L^2(\Omega) \). Moreover, the formal limit problem for \( \varepsilon = 0 \) has no solution, either. Theorem 3 shows that oscillatory behavior of this nature is not restricted to nonlinear problems, but can just as well occur for solutions of classical linear problems. In fact, physical intuition tells us that the buckled state of the plate should oscillate while its amplitude decreases as the ambient medium gets stiffer and stiffer.

Remark 3. Linear elasticity system. The above results were inspired by the paper [11] of W. Rother, who investigated the dependence of the first eigenvalue \( \Lambda_1(a) \) of Lamé's operator on a parameter \( a = (\lambda + \mu)/\mu \), where \( \lambda \) and \( \mu \) are the Lamé constants. This eigenvalue can be characterized by

\[
\Lambda_1(a) = \min \{ \|\nabla u\|^2 + a \|\text{div } u\|^2 \mid u \in [H^1_0(\Omega)]^N, \|u\|_{L^2(\Omega)^N} = 1 \},
\]

(17)
see, e.g., [2]. The associated system reads

\[ \Delta u + a \text{grad} \text{div } u + \Lambda_1 u = 0 \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{on } \partial \Omega. \]

Problem (17) is related to the so-called fundamental Stokes' eigenvalue:

\[ m_1 = \min \{ \| \nabla u \|^2 \mid u \in [H^1_0(\Omega)]^N, \text{div } u = 0, \| u \|_{L^2(\Omega)}^2 = 1 \}. \]

It was shown in [2] that \( m_1 \) is an upper bound for \( \Lambda_1(a) \). In [11] Rother showed that \( \Lambda_1(a) \) is increasing in \( a \). Using the ideas above it can easily be shown that in fact \( \sum_{i=1}^{m} \Lambda_i(a) \) is concave in \( a \). The lower bound

\[ \sum_{i=1}^{m} \Lambda_i(a) \geq \frac{3}{5} \left( \frac{2\pi^2}{V} \right)^{2/3} m^{5/3} \]

was derived in [6]. Under the technical assumption that \( \partial \Omega \in C^{0,1} \), Rother showed that the upper bound \( m_1 \) is optimal in the sense that

\[ \lim_{a \to \infty} \Lambda_1(a) = m_1. \]

In [4], it was shown that

\[ \lambda_1 \leq \Lambda_1(a) \left( 1 + \frac{a}{3} \right) \lambda_1. \]

See Fig. 2 for a graphical summary of the discussion of the results for \( \Lambda_1(\cdot) \).

The smoothness assumption on \( \partial \Omega \) was used in Rother's proof because he decomposed the eigenfunctions orthogonally into divergence free and remaining components, and he then applied some results for the divergence operator. We can avoid these difficulties (and thus derive (19) without any regularity assumption on \( \partial \Omega \)) as follows: let \( u_n \) be a sequence of eigenvectors associated with the eigenvalue \( \Lambda_1(n) \) and suppose that \( n \to \infty \). Since \( \Lambda_1(n) \leq m_1 \) we know that \( u_n \) is uniformly bounded in \( [H^1_0(\Omega)]^N \) and that \( \text{div } u_n \to 0 \) as \( n \to \infty \). Therefore, after possibly passing to a subsequence,
$u_n$ has a weak limit $u_\infty$ in $[H^1_0(\Omega)]^N$. Moreover, $u_\infty$ has unit length in $L^2(\Omega)^N$. Since the map $v \mapsto \int_\Omega (\text{div } v)^2 \, dx$ is convex and hence weakly lower semicontinuous in $[H^1_0(\Omega)]^N$, we conclude that $\text{div } u_\infty = 0$. Finally it should be noted that $u_n$ converges strongly in $[H^1_0(\Omega)]^N$ to $u_\infty$, since $||u_n|| \to ||u_\infty||$ and $u_n$ converges weakly. Therefore (19) must hold.

Remark 4. Clamped plate under tension. Instead of (1) consider the eigenvalue problem

$$
\Delta \Delta u - a \Delta u - \Gamma u = 0 \quad \text{in } \Omega,
$$

(20)

$$
u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega,
$$

where $a = T/D$ is given. $T$ describes tension and $D$ the flexural rigidity of the plate. The eigenvalues $\Gamma_i$ of (20) are characterized by means of the following Rayleigh quotient on $H^1_0(\Omega)$

$$
R_{\alpha}(v) = \frac{\int_\Omega \{(\Delta v)^2 + a|\nabla v|^2\} \, dx}{\int_\Omega v^2 \, dx}.
$$

(21)

As in the proof of Theorem 1 it can be shown that $\sum_{i=1}^m \Gamma_i(a)$ is strictly concave and strictly increasing in $a$. The following estimates hold for $\Gamma_1(a)$:

$$
\Gamma_1(0) + a \lambda_1 \leq \Gamma_1(a) \leq \Gamma_1(0) + a \sqrt{\Gamma_1(0)} \quad \text{for } a > 0;
$$

(22)

see [9]. Here again $\lambda_1$ is the lowest eigenvalue of the corresponding fixed membrane problem and $\sqrt{\Gamma_1(0)}$ is the fundamental frequency of a clamped plate in the absence of tension. Notice that (22) is sharp for $a = 0$, and that (22) implies that the curve $(a, \Gamma_1(a))$ stays inside a certain cone. A consequence of our results is that the inequalities in (22) are necessarily strict, an assertion not claimed by Payne. The numerical results of [14] indicate that $\Gamma_1(a)$ is a concave function with an asymptote whose slope is not smaller than $\lambda_1$, see Figs. 3 and 4.

We claim that the eigenfunction $u_a$ associated to $\Gamma_1(a)$ converges to $\psi_1$ and $\Gamma_1(a)$ converges to $\lambda_1$ as $a \to \infty$; see Fig. 4. Indeed, $\Gamma_i(a) \to \lambda_i$ for $i = 1, 2, \ldots$, as $a \to \infty$. This is easy to see once we realize that letting $a \to \infty$ is equivalent to letting the flexural rigidity of the plate tend to zero. Thus, in the limit the plate should behave like a membrane. Setting $\varepsilon = 1/a$ we can rewrite the differential equation in (20) as

$$
\varepsilon \Delta \Delta u - \Delta u - \lambda(\varepsilon) u = 0,
$$

and view this differential equation as a singular perturbation of the membrane equation given at the end of the Introduction. In fact, asymptotic expansions for $\lambda_i(\varepsilon)$ and its corresponding eigenfunctions are well known and recorded, for instance, in [2, p. 392], [3], [15].

2. More general results, the linear case. The above result can be generalized in several ways: For example, let $H$ be a Hilbert space and $D_1, D_2 \,(D_2 \subset D_1 \subset H)$ be dense linear subspaces on which the nonnegative, selfadjoint operators $E_1, E_2$ are defined respectively. We shall assume that $E_2$ is strictly positive, i.e., $(x, E_2 x) > 0$ unless $x = 0$. Let $(\cdot, \cdot)$ and $|| \cdot ||$ denote the scalar product and corresponding norm on $H$. We let $\overline{D_i}$ be the completion of $D_i$ in the norm $\{(x, E_i x) + ||x||^2\}^{1/2}$.

(A.1) For every $a \geq 0$ there exists $y_a \in D_2$ such that

$$
\Gamma(a) = \inf \{(x, E_2 x) + a(x, E_1 x) \mid ||x|| = 1, \, x \in \overline{D_2} \}
$$

$$
= (y_a, E_2 y_a) + a(y_a, E_1 y_a) =: J_a(y_a).$$
THEOREM 6. Suppose that (A.1) holds.

(i) If $a < b$, then

\begin{equation}
(y_b, E_1 y_b) < \frac{\Gamma(b) - \Gamma(a)}{b-a} < (y_a, E_1 y_a),
\end{equation}

and $\Gamma(a)$ is a monotone nondecreasing function of $a$.

(ii) $\Gamma(a)$ is a concave function of $a$.

(iii) $\Gamma(a)$ is strictly increasing on an interval $(\alpha, \beta) \subset (0, \infty)$ if and only if

$(y_a, E_1 y_a) > 0$ for all $a \in (\alpha, \beta)$.

(iv) $\Gamma(a)$ is strictly concave if $E_1$ and $E_2$ have no common eigenvector.

COROLLARY 7. (i) $|\Gamma(b) - \Gamma(a)| \leq (y_0, E_1 y_0) |b-a|$. ($\Gamma(a)$ is Lipschitz continuous.)
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(ii) \( \bar{F}(a) = (y_a, E_1y_a) \) is a nonincreasing function of \( a \).

(iii) \( \Gamma(0) + a \lim_{b \to \infty} (y_b, E_1y_b) \leq \Gamma(a) \leq \Gamma(0) + a(y_0, E_1y_0) \).

The proof of Corollary 7 is immediate. Let us prove Theorem 6. Statement (i) is straightforward, since \( \Gamma(b) = J_b(y_b) \leq J_a(y_a) \) and \( \Gamma(a) = J_a(y_a) \leq J_a(y_b) \). To prove concavity, and thus statement (ii), we have \( \Gamma(a) = tJ_a(u_a) \leq tJ_a(y_t) \) and \( (1 - t)\Gamma(b) = (1 - t)J_b(y_b) \leq (1 - t)J_b(y_t) \) for any \( t \in (0, 1) \), where \( y_t \) is shorthand for \( y_{ta+(1-t)b} \). Adding these inequalities, we have

\[
(24) \quad t\Gamma(a) + (1 - t)\Gamma(b) \leq \Gamma(ta + (1 - t)b).
\]

In order to prove (iii) we need only observe from (23) that \( (y_a, E_1y_a) > 0 \) in \( (a, b) \) if and only if \( \Gamma(a) \) is strictly increasing in \( (a, b) \). To prove (iv) notice that equality holds in (24) if and only if \( \Gamma(b) = J_b(y_b) = J_a(y_t) \) and \( J_a(y_a) = J_a(y_t) = \Gamma(a) \). The latter is equivalent to

\[
E_2y_t + bE_1y_t - \Gamma(b)y_t = 0
\]

and

\[
E_2y_t + aE_1y_t - \Gamma(a)y_t = 0.
\]

Upon subtraction we see that

\[
(25) \quad E_1y_t = \frac{\Gamma(b) - \Gamma(a)}{b - a} y_t,
\]

so that \( y_t \) is an eigenvector of the operator \( E_1 \). But now, using (25) or (26) it can be seen that \( y_t \) is an eigenvector of \( E_2 \), too. This proves Theorem 6. \( \square \)

Remark 5. One particular consequence of (23) or Corollary 7(iii) is the following:

If \( (y_a, E_1y_a) = 0 \) for some \( a > 0 \), then \( \Gamma(a) \) is constant on \( (a, \infty) \) and \( (y_b, E_1y_b) = 0 \) for \( b \in (a, \infty) \).

In order to obtain information on the limit \( a \to \infty \) we need more assumptions about the relationship between \( E_1 \) and \( E_2 \), e.g., the following assumption.

(A.2) There exists \( \alpha \in (0, 1] \) such that \( (x, E_1x) \leq (x, E_2x)^\alpha \) for all \( x \in D_2 \), \( ||x|| = 1 \).

Then

\[
(27) \quad (y_a, E_1y_a) \leq (y_0, E_1y_0) \leq (y_0, E_2y_0)^\alpha = \Gamma(0)^\alpha.
\]

For example, for the plate under tension, \( E_2u = \Delta \Delta u \), \( E_1u = \Delta u \) on \( H^2_0(\Omega) \) and \( H_0^1(\Omega) \) respectively, property (A.2) holds with \( \alpha = \frac{1}{2} \). Or for the Lamé operator \( E_2 = \Delta \) and \( E_1 = \nabla(\text{div}) \) on \( [H^1_0(\Omega)]^n \) property (A.2) holds with \( \alpha = 1 \). Also, by unique continuation the hypothesis of Theorem 6(iv) holds for this example.

\[
(28) \quad \Gamma(a) \leq \inf \{(x, E_2x) \mid ||x|| = 1, \ x \in D_2, \ (x, E_1x) = 0\}.
\]

One has to distinguish two cases: (1) The infimum in (28) is taken over an empty set. (2) The infimum in (28) is taken over a nonempty set.

In both cases the family \( \{(y_a, E_1y_a)\}_{a>0} \) is bounded in view of (27). In the second case, however \( (x, E_1x)^{1/2} \) is only a seminorm, since there are vectors for which \( (x, E_1x) = 0 \) and consequently the kernel \( \ker E_1 = \{x \mid E_1x = 0\} \) is not trivial. If the
infimum in (28) is taken over an empty set, it is \( \infty \) by convention and we assume the following.

(A.3.1) If \( \text{Ker} E_1 = \{0\} \), then \( \overline{D}_1 = \overline{D}_2 \) and sequence which is bounded in \( \overline{D}_1 \) possesses a subsequence which converges strongly in \( H \) and weakly in \( \overline{D}_1 \). Equivalently, \( \overline{D}_1 \) is compactly embedded in \( H \).

If the infimum is taken over a nonempty set, \( \Gamma(a) \leq M < \infty \) for all \( a \) and some \( M \). In that case we assume (A.3.2)

(A.3.2) If \( \text{Ker} E_1 \neq \{0\} \), then every sequence which is bounded in \( \overline{D}_2 \) possesses a subsequence which converges strongly in \( H \) and weakly in \( \overline{D}_1 \).

For reasons that will become obvious in the proof of Theorem 8(iii), we need an additional assumption, namely, (A.4).

(A.4) Let \( \mathcal{E}_a = \{y_a \mid J_a(y_a) = \Gamma(a)\} \). For every \( a \geq 0 \) there exists \( \tilde{y}_a \in \mathcal{E}_a \) such that

\[
(y_a, E_1 \tilde{y}_a) = \inf\{(y_a, E_1 y_a) \mid y_a \in \mathcal{E}_a\} =: F(a).
\]

We can now establish an analogue to (19) or Remark 4.

**Theorem 8.** Suppose that (A.1), (A.2), and (A.3) hold. Then

(i) \( \lambda_1 = \lim_{a \to \infty} (y_a, E_1 y_a) \) is the slope of the linear asymptote of \( \Gamma(a) \) and \( \lambda_1 \leq (y_0, E_1 y_0) \leq \Gamma'(0)^a \).

(ii) Moreover \( \lambda_1 \) is the smallest eigenvalue of \( E_1 \) and the family \( \{u_n\} \) contains a sequence \( \{u_{n_a}\} \) which converges strongly in \( H \) and weakly in \( \overline{D}_1 \) to an element of the first eigenspace of \( E_1 \) as \( a_n \to \infty \).

(iii) Whenever \( \Gamma'(a) \) exists and (A.4) holds, then \( \Gamma'(a) = \inf\{(y_a, E_1 y_a) \mid y_a \in \mathcal{E}_a\} \).

The proof of (i) follows from (27). To prove (ii) we notice that (i) and (A.3) imply the existence of a sequence \( y_{n_a} \) which converges weakly in \( \overline{D}_1 \) and strongly in \( H \) to a limit as \( a_n \to \infty \) we distinguish the above two cases.

(1) If \( \text{Ker} E_1 = \{0\} \) the set \( \{(y_a, E_1 y_a)\} \) is uniformly bounded and \( \{y_a\} \) possesses a sequence which converges strongly in \( \overline{D}_1 \) and weakly in \( \overline{D}_1 \) to an element of \( \overline{D}_1 \).

(2) If \( \text{Ker} E_1 \neq \{0\} \) the set \( \{(y_a, E_2 y_a)\} \) is uniformly bounded and \( \{y_a\} \) possesses a sequence which converges strongly in \( H \) and weakly in \( \overline{D}_1 \) to an element of \( \overline{D}_1 \). Let us call the limit element \( y_\infty \). We have

\[
\frac{(y_a, E_2 \phi)}{a} + (y_a, E_1 \phi) - \frac{\Gamma(a)}{a} (y_a, \phi) = 0
\]

for every \( \phi \in \overline{D}_2 \), so that \( y_\infty \) is an eigenfunction for \( E_1 \):

\[
(y_\infty, E_1 \phi) - m(y_\infty, \phi) = 0
\]

with \( d = \lim_{a \to \infty} \Gamma(a)/a \). Notice that \( ||y_\infty|| = 1 \) by assumption (A.3). By definition of \( \lambda_1, d \geq \lambda_1 \). It remains to show that \( d > \lambda_1 \). Clearly \( E_1 \) has a smallest (nonnegative) eigenvalue \( \mu_1 \leq \lambda_1 \) and some associated eigenfunction \( \psi_1 \in \overline{D}_1 \). We claim \( d = \lambda_1 = \mu_1 \). Since \( D_2 \) is dense in \( H \) and \( D_2 \subset D_1 \subset H \), we can approximate \( \psi_1 \in \overline{D}_1 \) with a function \( \phi_\varepsilon \in D_2 \) such that \( ||\phi_\varepsilon|| = 1 \) and \( (\phi_\varepsilon, E_1 \phi_\varepsilon) < \mu_1 + \varepsilon \). But this contradicts the choice of \( \varepsilon \) because

\[
d \leq \lim_{a_n \to \infty} \frac{1}{a_n} (\phi_\varepsilon, E_2 \phi_\varepsilon) + (\phi_\varepsilon, E_1 \phi_\varepsilon) \leq \mu_1 + \varepsilon,
\]

and thus concludes the proof of (ii). To prove (iii) we assume (A.4). Then for any decreasing sequence \( a_n \to a \) there exists a number \( M \) such that \( J_a(\tilde{y}_a) \leq J_{a_n}(\tilde{y}_{a_n}) \leq
\]
$J_{a_1}(y_{a_1}) \leq M$. Therefore $\{y_{a_n}\}$ has a subsequence, still denoted by $\{y_{a_n}\}$ with a limit $y_{a_\infty}$. We obtain

$$\Gamma(a) \leq J_{a}(y_{a_\infty}) \leq \liminf \Gamma(a_n) \leq \Gamma(a)$$

from the definition and the continuity properties of $\Gamma$. Therefore $y_{a_\infty} \in E_a$. Furthermore, due to (27) and the monotonicity of $F$,

$$F(a) \leq (y_{a_\infty}, E_1 y_{a_\infty}) \leq \liminf (y_{a_n}, E_1 y_{a_n}) = \liminf F(a_n) \leq F(a).$$

This proves that $F(a)$ equals the one-side derivative of $\Gamma$ from the right. Since $\Gamma'(a)$ is assumed to exist, the proof of Theorem 8 is complete.  \[ \square \]

3. More general results, the nonlinear case. For $i = 0, 1, 2$, let $J_i : X_i \to \mathbb{R}^+$ be a nonnegative weakly lower semicontinuous functional on a separable Banach space $X_i$.

**Theorem 9.** Suppose that $X_2 \subset X_1 \subset X_0$, and that there exists a unique minimizer $u_0$ in $X_2$ of $J_1(v)$ in $X_1 \cap \{v \mid J_0(v) = 1\}$. Let $u_\varepsilon$ be a minimizer of $J_\varepsilon(v) := \varepsilon J_2(v) + J_1(v)$ on $X_2 \cap \{v \mid J_0(v) = 1\}$.

(i) Then $\Gamma(\varepsilon) = J_\varepsilon(u_\varepsilon)$ is monotone nondecreasing and concave in $\varepsilon$.

(ii) If $X_1$ is compactly embedded in $X_0$ and if $J_1$ is coercive, then $u_\varepsilon$ converges to $u_0$ weakly in $X_1$ and strongly in $X_0$.

(iii) If $X_2$ is compactly embedded in $X_1$, then $u_\varepsilon$ converges to $u_0$ weakly in $X_2$ and strongly in $X_1$ and $X_0$.

The proof is straightforward if we use ideas from the proofs of Theorems 1 and 8. As an application for Theorem 9 consider the eigenvalue problem

$$\varepsilon \Delta \Delta u - \text{div} (|\nabla u|^{p-2} \nabla u) - \Gamma|u|^{p-2} u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega,$$

for $1 < p < 2n/(n - p)$. Here $J_2(v) = ||\Delta v||_{L^2(\Omega)}^2$, $J_1(v) = ||\nabla v||_{L^p(\Omega)}^2$ and $J_0(v) = ||v||_{L^p(\Omega)}$, while $X_2 = H_0^2(\Omega)$, $X_1 = W_0^{1,p}(\Omega)$ and $X_0 = L^p(\Omega)$. Then, as $\varepsilon \to 0$, the solutions of (29) converge to the (unique) ground state of the formal limit problem

$$\text{div} (|\nabla u|^{p-2} \nabla u) + \lambda|u|^{p-2} u = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega.$$

For more details on this eigenvalue problem see, e.g., [7], [13].

**Notes added in proof.** Professor F. Goerisch has kindly informed us of [16], in which it is shown that the entire spectrum of the elasticity operator converges to the spectrum of the Stokes operator. Therefore Remark 3 of this paper extends to all eigenvalues.

In [17], the author re-establishes the results of [11] in three dimensions. The author's method of proof relies on the decomposition of the Lamé operator using quaternions and a generalized Cauchy–Riemann operator. His result thus appears to be restricted to three dimensions. However, no regularity of the boundary is required.

**Acknowledgment.** We thank L. Frank for bringing references [2], [3], [15] to our attention.
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