

8-2009

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Recommended Citation

Su, Hai-Jun; Dorozhkin, Denis V.; and Vance, Judy M., "A Screw Theory Approach for the Type Synthesis of Compliant Mechanisms With Flexures" (2009). *Mechanical Engineering Conference Presentations, Papers, and Proceedings*. 43.
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Abstract

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Keywords

VRAC, Screws, Compliant mechanisms

Disciplines

Computer-Aided Engineering and Design | Mechanical Engineering

DETC2009-86684

A SCREW THEORY APPROACH FOR THE TYPE SYNTHESIS OF COMPLIANT MECHANISMS WITH FLEXURES

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ABSTRACT

This paper presents a screw theory based approach for the type synthesis of compliant mechanisms with flexures. We provide a systematic formulation of the constraint-based approach which has been mainly developed by precision engineering experts in designing precision machines. The two fundamental concepts in the constraint-based approach, constraint and freedom, can be represented mathematically by a wrench and a twist in screw theory. For example, an ideal wire flexure applies a translational constraint which can be described a wrench of pure force. As a result, the design rules of the constraint-based approach can be systematically formulated in the format of screws and screw systems. Two major problems in compliant mechanism design, constraint pattern analysis and constraint pattern design are discussed with examples in details. This innovative method paves the way for introducing computational techniques into the constraint-based approach for the synthesis and analysis of compliant mechanisms.

NOMENCLATURE

Ω A 3D vector presenting the angular velocity of a twist
 \mathbf{V} A 3D vector presenting the linear velocity of a twist
 p A scalar representing the pitch of a twist
 \hat{T} A 6D general twist representing an allowable motion
 Π_T A twist matrix representing the allowable motion space
 \mathbf{F} A 3D vector presenting the force part of a wrench

\mathbf{M} A 3D vector presenting the couple part of a wrench
 q A scalar representing the pitch of a wrench
 \hat{W} A 6D general wrench representing a constraint
 Π_W A wrench matrix representing the constraint space

1 Introduction

Compared with traditional rigid body mechanisms, compliant mechanisms [1] or flexures have many advantages, such as high precision and a simplified manufacturing and assembly process. However the design and analysis of compliant mechanisms is complex due to the nonlinearity of deformation of the flexible members. Researchers in two isolated fields, kinematics and mechanisms and precision engineering, have independently made major contributions to compliant mechanism design.

In the kinematics and mechanisms community, research has focused on applying computational techniques to determine the dimensions and/or topologies of compliant mechanisms to achieve a pre-specified design objective. The two most often used approaches in this field are the Pseudo-Rigid-Body-Model (PRBM) [2] approach and the topological synthesis approach [3–8]. The former approach models a compliant mechanism as a rigid body with one or more springs. These springs impose approximated lumped compliance to the rigid link models of compliant mechanisms. This allows the theories and methodologies developed for rigid body mechanisms [9, 10] to be used to design compliant mechanisms. Because of this simplification in the modeling process, it is necessary to evaluate the designs to ensure

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the validity of the PRBM. The topological synthesis approach models the compliant linkage as a network of link members of different sizes which together achieve a specified objective function such as geometric advantage and mechanical advantage. The result is a compliant mechanism of complex topology with distributed compliance. This complexity results in mechanisms that are difficult to manufacture and produce non-intuitive motions.

In parallel, the precision engineering community have been using the constraint-based approach for the design of compliant instruments with flexures. The foundations of the constraint-based method were developed by Maxwell [11] in the 1880s. It was recently revisited by Blanding [12] and several researchers at the MIT Precision Engineering Labs [13–15] for the design of fixtures, rigid body machines and flexure systems. The fundamental premise of the constraint-based method is that all motions of a rigid body are determined by the position and orientation of the constraints (constraint topology) which are placed upon the body. The method is attractive because it is based upon motion visualization and is therefore well-suited to conceptual development; however the proficiency in using the constraint-based methods for designing compliant mechanisms requires commitment to a steep learning curve and development of “hands-on” experience to understand the stiffness characteristics of alternate designs. Hence the design process is not systematical and may not necessarily lead to the optimal design, especially when the designer is inexperienced.

In this paper, a mathematical formulation of the constraint-based approach based upon screw theory is presented. A screw is the geometric entity that underlies the foundation of statics and instantaneous (first-order) kinematics. Many authors have made contributions to screw theory. The two fundamental concepts in screw theory are “*twist*” representing a general helical motion of a rigid body about an instantaneous axis in space, and “*wrench*” representing a system of force and moment acting on a rigid body. These two concepts are often called duality [16] in kinematics and statics. Ball [17] was the first to establish a systematical formulation for screw theory. Hunt [18] and Phillips [19,20] further developed the mathematical and geometrical representation of screws and screw systems. Their focus lies on the application of screw theory to the analysis and synthesis of mechanisms. Lipkin and Pattern [21–23] systematically investigated the screw theory and its applications to compliance or elasticity analysis of robot manipulators. Huang and Schimmels [24,25] studied the realization of a prescribed stiffness matrix with serial or parallel elastic mechanisms. Other applications of screw theory include mobility analysis [26], assembly analysis [27,28] and topology synthesis [29]. Recently Kim [30] studied the characterization of compliant building blocks by utilizing the concept of eigentwists and eigenwrenches based on screw theory.

A constraint and a degree-of-freedom in the constraint-based approach can be described by a wrench and a twist in screw theory respectively. Therefore, all the rules in constraint pat-

tern analysis and design can be explained and mathematically represented using screw theory. The result is a powerful tool that is capable of systematically finding intuitive design topologies. The problem of analyzing and synthesizing spatial stiffness/compliance of a general elastic mechanism with spring elements has been investigated extensively by Lipkin [21–23], Schimmels [24,25] and others. As far as the authors’ knowledge, applying screw theory to the type synthesis of compliant mechanisms with flexure elements is a new contribution to the field. This linking of screw theory to constraint-based compliant mechanism design allows a wide variety of computational techniques developed in the kinematics field [31] to be combined together with the constraint-based design approach to achieve design automation for compliant mechanisms.

The rest of the paper is organized as follows. Section 2 provides a review of screw theory. Section 3 describes a screw representation of the basic concepts including constraint, constraint space, freedom and freedom space. Also in this section, the general steps for constraint pattern analysis and design are presented and illustrated with examples. Section 5 presents conclusions and discussion.

2 Screw Theory Review

This section provides a concise review of general screw theory.

2.1 Twists and Wrenches

It is well known that screw theory underlies the foundation of both instantaneous kinematics and statics. In kinematics, a general spatial motion of a rigid body is a screw motion (a rotation and a translation) about a line in space called *screw axis*. The rotation and translation is further coupled by a scalar quantity called *pitch*. The screw in kinematics also called a twist is formed by a pair of three dimensional vectors, namely angular velocity $\boldsymbol{\Omega}$ and linear velocity \mathbf{V} , written as

$$\text{Twist: } \hat{T} = (\boldsymbol{\Omega} \mid \mathbf{V}) = (\omega \mathbf{s} \mid \mathbf{c} \times \omega \mathbf{s} + v \mathbf{s}) \quad (1)$$

where vectors \mathbf{s} and \mathbf{c} denote the direction of and a point on the twist axis respectively, scalars ω and v are the magnitude of angular velocity and partial linear velocity along the axis. The pitch is defined as the ratio of the linear velocity to angular velocity, i.e. $p = v/\omega$. As special cases, a pure rotation and a pure translation in space are represented by a twist of zero pitch and infinite pitch respectively, written as,

$$\text{Pure Rotation}(p = 0) : \hat{T} = (\omega \mathbf{s} \mid \mathbf{c} \times \omega \mathbf{s}) \quad (2)$$

$$\text{Pure Translation}(p = \infty) : \hat{T} = (\mathbf{0} \mid v \mathbf{s}) \quad (3)$$

Note that a translation can be also viewed as a rotation with axis at infinity. The screw axis of planar motion degenerates to a point on the plane called the instantaneous center [32], displacement pole or virtual pivot. The pitch of a planar twist is always zero.

Similarly in statics, a general screw or a wrench consists of two vectors representing a force \mathbf{F} and a couple (moment) \mathbf{M} acting on a rigid body, written as,

$$\text{Wrench: } \hat{W} = (\mathbf{F} \mid \mathbf{M}) = (f\mathbf{u} \mid \mathbf{r} \times f\mathbf{u} + m\mathbf{u}) \quad (4)$$

where vectors \mathbf{u} and \mathbf{r} are the direction of and a point on the wrench axis respectively, scalars f and m are magnitude of the force and partial moment along the axis, coupled by a pitch $q = m/f$. Similar to the case of twist, a pure force and a pure couple are represented by a wrench of zero pitch and infinite pitch respectively, written as,

$$\text{Pure Force}(q = 0) : \hat{W} = (f\mathbf{u} \mid \mathbf{r} \times f\mathbf{u}) \quad (5)$$

$$\text{Pure Couple}(q = \infty) : \hat{W} = (\mathbf{0} \mid m\mathbf{u}) \quad (6)$$

Figure 1 illustrates a general twist \hat{T} and a general wrench \hat{W} in space.

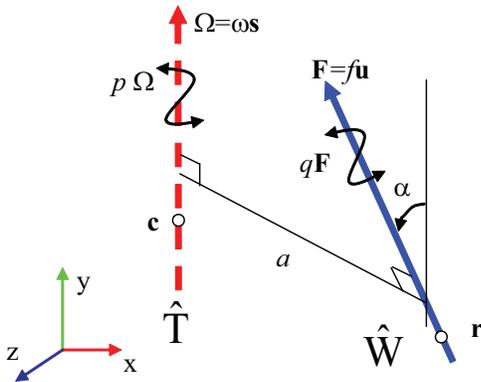


Figure 1. A general wrench \hat{W} does work on a body with the motion defined by a general twist \hat{T}

2.2 Reciprocity of Screws

Let us denote the normal distance and skew angle of axes of a general wrench \hat{W} and a general twist \hat{T} by a and α respectively (Fig. 1). The virtual power of wrench \hat{W} acting on a moving body with motion \hat{W} is given by the reciprocal product of the twist and

the wrench, calculated as,

$$\begin{aligned} \hat{T} \circ \hat{W} &= \mathbf{F} \cdot \mathbf{V} + \mathbf{M} \cdot \boldsymbol{\Omega} \\ &= [(fv + m\omega)(\mathbf{s} \cdot \mathbf{u}) + f\omega(\mathbf{c} - \mathbf{r}) \cdot (\mathbf{s} \times \mathbf{u})] \\ &= [(fv + m\omega) \cos \alpha - f\omega a \sin \alpha] \end{aligned} \quad (7)$$

A twist is called *reciprocal* to a wrench when their reciprocal product is zero. To find all possible reciprocal conditions, we consider nontrivial (nonzero) twist \hat{T} and nontrivial wrench \hat{W} .

1. If $f = 0 (q = \infty)$ and $\omega = 0 (p = \infty)$, \hat{T} is always reciprocal to \hat{W} . This says that a pure couple is always reciprocal (does no work) to a pure translation.
2. If $f = 0, \omega \neq 0$ or $f \neq 0, \omega = 0$, $\hat{T} \circ \hat{W} = 0$ is satisfied if and only if $\cos \alpha = 0$ i.e. the twist axis and the wrench axis are perpendicular to each other.
3. If $f \neq 0 (q \neq \infty)$ and $\omega \neq 0 (p \neq \infty)$, Eq.(7) is reduced to

$$\hat{T} \circ \hat{W} = f\omega[(p + q) \cos \alpha - a \sin \alpha], \quad (8)$$

where we have substituted the definition of pitches. By considering the values of the pitches p and q in Eq.(8), we can obtain the following observations that can be very useful as thumb rules in design practice.

- (a) If $p + q = 0$ (including the case $p = q = 0$), the two screws are reciprocal if either $a = 0$ or $\sin \alpha = 0$. This situation occurs when the twist axis and wrench axis are coplanar, i.e., intersecting or parallel to each other
- (b) If the two screw axes are perpendicular $\cos \alpha = 0$, their reciprocity is independent of their pitches. This can occur only when $a = 0$, i.e. two axes intersect.

Reciprocal product is considered a linear operation on either twist or wrench separately. For instance, the reciprocal product of a twist \hat{T} with a linear combination of two wrenches \hat{W}_1, \hat{W}_2 can be expressed as the linear combination of the reciprocal product of \hat{T} with each of the two wrenches, that is,

$$\hat{T} \circ (k_1 \hat{W}_1 + k_2 \hat{W}_2) = k_1 \hat{T} \circ \hat{W}_1 + k_2 \hat{T} \circ \hat{W}_2 \quad (9)$$

where the coefficients k_1 and k_2 are arbitrary constants. This linearity property is very important in the freedom and constraint pattern analysis and synthesis.

3 The Screw Theory Representation of the Constraint-Based Design Approach

This section relates concepts and design rules in constraint-based design approach to screw theory.

3.1 Freedom and Constraint of a Rigid Body

Constraint and freedom are key concepts in the constraint-based design approach. In screw theory, a *degree-of-freedom* (dof) is represented by a general twist. As special cases, a rotational freedom or a translational freedom can be represented by a twist of pure rotation or pure translation shown in Eq.(2) and Eq.(3) respectively.

In this paper, all vectors are considered row vectors. For example, $\boldsymbol{\Omega} \in \mathbb{R}^3$ is denoted by $\boldsymbol{\Omega} = (\Omega_x \ \Omega_y \ \Omega_z)$. Also a general twist is essentially a six dimensional row vector, i.e. $\hat{T} = (\Omega_x \ \Omega_y \ \Omega_z \ | \ V_x \ V_y \ V_z) \in \mathbb{R}^6$. An unconstrained rigid body has six dof in space, i.e. three rotations and three translations along three orthogonal axes (Fig 2) denoted by principle twists, written as

$$\begin{cases} \hat{T}_{Rx} = (1 & 0 & 0 & | & 0 & 0 & 0) \\ \hat{T}_{Ry} = (0 & 1 & 0 & | & 0 & 0 & 0) \\ \hat{T}_{Rz} = (0 & 0 & 1 & | & 0 & 0 & 0) \\ \hat{T}_{Tx} = (0 & 0 & 0 & | & 1 & 0 & 0) \\ \hat{T}_{Ty} = (0 & 0 & 0 & | & 0 & 1 & 0) \\ \hat{T}_{Tz} = (0 & 0 & 0 & | & 0 & 0 & 1) \end{cases} \quad (10)$$

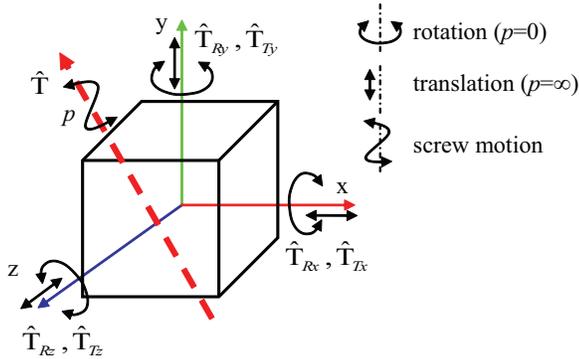


Figure 2. An unconstrained rigid body has three translations and three rotations represented by six principle twists.

When a mechanical connection is built between the rigid body and a reference base in such a way that the number of dof of the rigid body is reduced, this body is said to be *constrained*. The number of reduced dof is defined as *degree-of-constraint* (doc) of the mechanical connection. In screw theory, a constraint can be described by a general wrench. A constraint that eliminates translation along a line is called a *translational constraint* which can be represented by a wrench of pure force shown in Eq.(5). A constraint that eliminates rotation about a line is called a *rotational constraint* represented by a wrench of pure couple shown in Eq.(6).

In the constraint-based design approach, an ideal constraint is a slender structural member that is infinitely stiff along its axis but is infinitely compliant perpendicular to its axis. The ideal constraint is essentially a nontrivial translational constraint represented by a wrench of pure force, expressed as

$$\begin{aligned} \hat{W} = (\mathbf{F} \ | \ \mathbf{M}) &= (F_x \ F_y \ F_z \ | \ M_x \ M_y \ M_z), s.t. \\ F_x M_x + F_y M_y + F_z M_z &= 0, F_x^2 + F_y^2 + F_z^2 \neq 0. \end{aligned} \quad (11)$$

In real designs, the ideal constraint can usually be approximated by a rigid link with two ball joints at both ends or a compliant link (wire flexure) that is much stiffer along the direction of its axis than along the direction perpendicular to the axis [12]. Figure 3(a) shows a rigid body constrained by an ideal wire flexure which does not allow compression or stretch in its axial direction but is fully compliant in the perpendicular directions. Without losing generality, let us assume the wire axis aligns along the x axis. The constraint representing the wire flexure is denoted by a wrench $\hat{W} = (1 \ 0 \ 0 \ | \ 0 \ 0 \ 0)$, shown in Figure 3(b). The freedoms subject to the constraint can be obtained by requiring a general twist \hat{T} reciprocal to the wrench. This allows us to obtain from Eq.(7):

$$\hat{T} \circ \hat{W} = 0, \Rightarrow V_x = 0 \quad (12)$$

Clearly this constraint removes the translational freedom along the x axis.

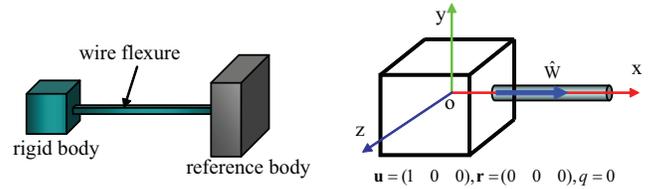


Figure 3. An ideal wire flexure imposes on a rigid body an ideal constraint which removes the translational freedom in the axial direction of the wire flexure

Even though it is quite straightforward to design a translational constraint, it is not trivial to design a rotational constraint (infinite pitch) or a general constraint (finite pitch). Typically a complex structure formed through cascading intermediate bodies must be used. Hunt [18] provided a “wrench support” that applies a general constraint to a body. Blanding [12] showed an example of rotational constraint realized by using two pulleys and a cable. Since building rotational and general constraints is relatively complicated (arguably not cost effective), it is preferable to use translational (ideal) constraints when possible in compliant mechanism design.

3.2 Freedom Space

The freedom space (topology) of a rigid body represents all of its allowable motion in space. Mathematically the freedom space can be described by a twist matrix Π_T formed by f independent twists $\hat{T}_j (j = 1, \dots, f)$, written as

$$\Pi_T = \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \vdots \\ \hat{T}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega}_1 & | & \mathbf{V}_1 \\ \boldsymbol{\Omega}_2 & | & \mathbf{V}_2 \\ \vdots & & \vdots \\ \boldsymbol{\Omega}_f & | & \mathbf{V}_f \end{bmatrix} \quad (13)$$

And f is called the dimension of the freedom space. \hat{T}_j are the basis twists that span the freedom space. Any motion in the freedom space can be denoted by a linear combination of the basis twists,

$$\hat{T} = \sum_{j=1}^f k_j \hat{T}_j \quad (14)$$

where k_j are arbitrary constants and cannot be zero simultaneously. If the rank of the twist matrix Π_T is less than f , twists \hat{T}_j are redundant, meaning that some twists can be written as linear combinations of others.

Freedom space sometimes can have a geometric representation in space. For instance, a one dimensional freedom space is a single line in space. A two dimensional freedom space is a surface generated by two lines. And three dimensional space is a volume, e.g. a solid sphere generated by three rotational twists intersecting at the same point. Recently Hopkins and Culpepper [15] systematically enumerated the topologies of freedom and constraint space and provided a graphical illustration for each case.

Let us take a look at a two dimensional freedom space spanned by two pure rotations. Physically this freedom space can be generated by a serial chain of two revolute joints. Depending on whether the two rotation axes are parallel, intersecting or skew, the freedom space is a plane, a disk or a cylindroid respectively.

Figure 4(a) shows a plane generated by two parallel rotational twists $\hat{T}_1 = (\boldsymbol{\Omega} \mid \mathbf{c}_1 \times \boldsymbol{\Omega})$ and $\hat{T}_2 = (\boldsymbol{\Omega} \mid \mathbf{c}_2 \times \boldsymbol{\Omega})$. Any parallel lines on the plane can be represented by

$$\begin{aligned} \hat{T} &= k_1 \hat{T}_1 + k_2 \hat{T}_2 \\ &= ((k_1 + k_2)\boldsymbol{\Omega} \mid (k_1 \mathbf{c}_1 + k_2 \mathbf{c}_2) \times \boldsymbol{\Omega}), \end{aligned} \quad (15)$$

where the coefficients k_1, k_2 can be viewed as the angular speeds of the joints if the freedom space is generated by a serial chain

of two revolute joints. If the angular speeds are constrained such that $k_1 + k_2 = 0$, twist (15) shows a pure translation in the normal direction of the plane formed by the two parallel lines. This result tells us that a serial chain of two parallel revolute joints can generate an instantaneous translation by driving the two joints with same angular velocity but in the opposite direction.

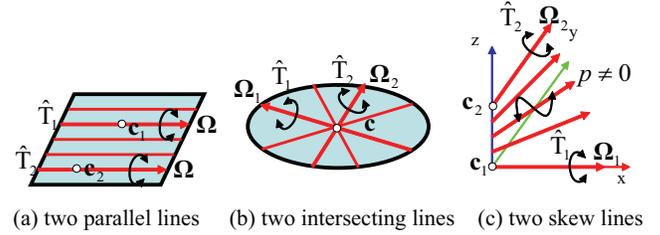


Figure 4. Two dimensional freedom space generated by (a) two parallel lines, (b) two intersecting lines and (c) two skew lines.

Figure 4(b) shows a disk generated by two rotational twists intersecting at a point c . Any freedom in the space can be expressed by

$$\begin{aligned} \hat{T} &= k_1 \hat{T}_1 + k_2 \hat{T}_2 \\ &= (k_1 \boldsymbol{\Omega}_1 + k_2 \boldsymbol{\Omega}_2 \mid \mathbf{c} \times (k_1 \boldsymbol{\Omega}_1 + k_2 \boldsymbol{\Omega}_2)), \end{aligned} \quad (16)$$

Clearly every freedom in this space is still a pure rotation about a line through the same point c and in the direction $k_1 \boldsymbol{\Omega}_1 + k_2 \boldsymbol{\Omega}_2$.

Figure 4(c) shows a cylindroid generated by two skew rotational twists. Any freedom in the space can be described by a general twist

$$\begin{aligned} \hat{T} &= k_1 \hat{T}_1 + k_2 \hat{T}_2 \\ &= (k_1 \boldsymbol{\Omega}_1 + k_2 \boldsymbol{\Omega}_2 \mid \mathbf{c}_1 \times k_1 \boldsymbol{\Omega}_1 + \mathbf{c}_2 \times k_2 \boldsymbol{\Omega}_2), \end{aligned} \quad (17)$$

One can see that all motions in the space except \hat{T}_1, \hat{T}_2 are in the form of screw motion since the pitch of twist in (15) is nonzero in general.

The freedom space of four and five dimensions cannot be represented graphically in general. Only in some special cases, could we use the combination of lower dimensional spaces to describe a higher dimensional space. For these cases, the twist matrix is more preferable to describe higher dimensional freedom space.

3.3 Constraint Space

The constraint space (topology) of a rigid body represents all the forbidden motions of the body subject to a constraint ar-

arrangement. In screw theory, a constraint space can be represented by a *wrench matrix* formed by c independent wrenches $\hat{W}_i (i = 1, \dots, c)$, written as

$$\prod_W = \begin{bmatrix} \hat{W}_1 \\ \hat{W}_2 \\ \vdots \\ \hat{W}_c \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 & | & \mathbf{M}_1 \\ \mathbf{F}_2 & | & \mathbf{M}_2 \\ \vdots & & \vdots \\ \mathbf{F}_c & | & \mathbf{M}_c \end{bmatrix} \quad (18)$$

where c is called the dimension of the constraint space. \hat{W}_i are the basis wrenches that span the whole constraint space. If matrix \prod_W does not have a full rank, the wrenches \hat{W}_i are redundant, meaning that removing some of the constraints does not affect the mobility of the constrained body. Similar to freedom space, any constraint in the constraint space can be represented by a linear combination of the basis wrenches,

$$\hat{W} = \sum_{i=1}^c k_i \hat{W}_i \quad (19)$$

Figure 5(a) shows a rigid body constrained by an ideal sheet flexure. Since its thickness is much smaller than its width and length, an ideal sheet flexure allows rotations about any line on the sheet plane and translating along the normal direction. It prohibits the rotation about the normal direction and translations in the plane. In the constraint-based design approach [12], an ideal sheet flexure applies three ideal constraints on the sheet plane. In our screw approach, these three ideal constraints can be represented by a set of any three independent wrenches on the sheet plane. An example set of three constraints is shown in Fig. 5(a). Wrenches \hat{W}_1 and \hat{W}_2 are parallel to the y axis and intersecting the x axis at $\mathbf{r}_1 = (1 \ 0 \ 0)$ and $\mathbf{r}_2 = (-1 \ 0 \ 0)$ respectively. And the wrench \hat{W}_3 aligns with the x axis. They are written as,

$$\begin{aligned} \hat{W}_1 &= (0 \ 1 \ 0 \ | \ 0 \ 0 \ 1) \\ \hat{W}_2 &= (0 \ 1 \ 0 \ | \ 0 \ 0 \ -1) \\ \hat{W}_3 &= (1 \ 0 \ 0 \ | \ 0 \ 0 \ 0) \end{aligned} \quad (20)$$

Any other ideal constraint on the sheet plane in the direction $\mathbf{F} = (F_x \ F_y \ 0)$ through a point $\mathbf{r} = (r_x \ r_y \ 0)$ is written as

$$\hat{W} = (\mathbf{F} \ | \ \mathbf{r} \times \mathbf{F}) = (F_x \ F_y \ 0 \ | \ 0 \ 0 \ r_x F_y - r_y F_x), \quad (21)$$

which can be expressed a linear combination of $\hat{W}_1, \hat{W}_2, \hat{W}_3$ as

$$\hat{W} = \left(\frac{r_x F_y - r_y F_x + F_y}{2} \right) \hat{W}_1 + \left(\frac{F_y - r_x F_y + r_y F_x}{2} \right) \hat{W}_2 + F_x \hat{W}_3 \quad (22)$$

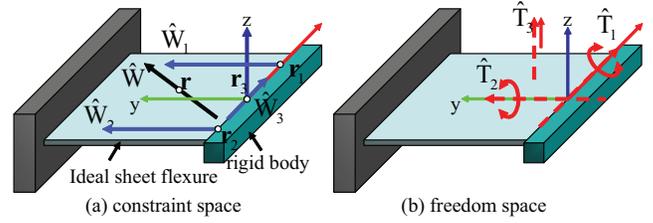


Figure 5. Constraint and freedom space of a rigid body constrained by an ideal sheet flexure.

3.4 Rule of Complementary Patterns

The design rules in the constraint-based design approach are mostly illustrated in the form of subjective statements. The most important one is *Rule of Complementary Patterns* which state: “when a pattern of c nonredundant constraints is applied between an object and a reference body, the object will have $f = 6 - c$ independent degrees-of-freedom.” This rule can be explained in screw theory as follows. Let wrenches $\hat{W}_i (i = 1, \dots, c)$ denote c nonredundant constraints and twists $\hat{T}_j (j = 1, \dots, f)$ denote nonredundant f degrees-of-freedom. And all wrenches are reciprocal to all twists

$$\hat{T}_j \circ \hat{W}_i = 0, \quad i = 1, \dots, c, j = 1, \dots, f, c + f = 6 \quad (23)$$

There are two major categories of design problems of compliant mechanisms using the constraint-based design approach. One is called *constraint pattern analysis* which studies the mobility of a rigid body subject to a pattern of constraints. The other is called *constraint pattern design* which seeks for a pattern of constraints to achieve a specified pattern of freedoms. We discuss each in the following.

Constraint Pattern Analysis Here let us use the sheet flexure shown in Fig 5(a) as an example to demonstrate the steps for constraint pattern analysis.

Step 1: write all constraints in the format of wrenches which are written in Eq.(20)

Step 2: require a general twist $\hat{T} = (\Omega_x \ \Omega_y \ \Omega_z \ | \ V_x \ V_y \ V_z)$ reciprocal to all wrenches in Eq.(20) and yield a system of linear equations,

$$\begin{aligned} \hat{T} \circ \hat{W}_1 &= 0 \Rightarrow 1V_y + 1\Omega_z = 0 \\ \hat{T} \circ \hat{W}_2 &= 0 \Rightarrow 1V_y - 1\Omega_z = 0 \\ \hat{T} \circ \hat{W}_3 &= 0 \Rightarrow 1V_x = 0 \end{aligned} \quad (24)$$

where the reciprocal product is explicitly expressed to show the consistence in units.

Step 3: solve the above linear system to obtain $V_x = V_y = \Omega_z = 0$ and write the complementary freedom space in the format of a general twist,

$$\hat{T} = (\Omega_x \quad \Omega_y \quad 0 \quad | \quad 0 \quad 0 \quad V_z)$$

Step 4: find independent basis twists from the above twist system. For this example, simply setting one of Ω_x, Ω_y, V_z to be nonzero and the other two to be zero yields three basis twists:

$$\begin{aligned} \hat{T}_1 &= (1 \quad 0 \quad 0 \quad | \quad 0 \quad 0 \quad 0) \\ \hat{T}_2 &= (0 \quad 1 \quad 0 \quad | \quad 0 \quad 0 \quad 0) \\ \hat{T}_3 &= (0 \quad 0 \quad 0 \quad | \quad 0 \quad 0 \quad 1) \end{aligned}$$

Clearly they represent rotation around x axis, rotation around y axis and translation along z axis respectively. See Fig. 5(b). Note the step of choosing basis twists is not unique. Any three independent twists in the freedom space should span the same freedom space.

It should be pointed out that a general freedom is usually in the form of screw motion represented by a general twist (finite pitch). However for the sake of intuition, we prefer to use rotational or translational twists as basis twists whenever possible to represent the freedom space. An interesting question is “can we always find $f = 6 - c$ rotational freedom for a pattern of c constraints?” Unfortunately the answer is NO even when all the constraints in an arrangement are ideal, i.e. $q = 0$. A counter example is shown in Fig. 6 where a rigid body is subject to four ideal constraints. The physical arrangement of this constraint pattern and corresponding constraint space are shown in Fig. 6(a) and Fig. 6(b) respectively. Wrenches \hat{W}_1 and \hat{W}_2 are parallel to the y axis and intersect with the x axis at $+1$ and -1 respectively. Wrench \hat{W}_3 aligns the z axis. And the last wrench \hat{W}_4 is a skew line on the plane $y = -1$ and has an angle of 45° with both x and z axis. They are written as

$$\begin{aligned} \hat{W}_1 &= (0 \quad 1 \quad 0 \quad | \quad 0 \quad 0 \quad 1) \\ \hat{W}_2 &= (0 \quad 1 \quad 0 \quad | \quad 0 \quad 0 \quad -1) \\ \hat{W}_3 &= (0 \quad 0 \quad 1 \quad | \quad 0 \quad 0 \quad 0) \\ \hat{W}_4 &= (1 \quad 0 \quad 1 \quad | \quad -1 \quad 0 \quad 1) \end{aligned} \quad (25)$$

Following the constraint pattern analysis steps yields two independent twists

$$\begin{aligned} \hat{T}_1 &= (0 \quad 1 \quad 0 \quad | \quad 0 \quad 0 \quad 0) \quad p_1 = 0 \\ \hat{T}_2 &= (1 \quad 0 \quad 0 \quad | \quad 1 \quad 0 \quad 0) \quad p_2 = 1 \end{aligned} \quad (26)$$

They represent a pure rotation ($p_1 = 0$) around y axis and a screw motion around x axis with pitch $p_2 = 1$ respectively. These two independent twists form the basis of the two dimensional freedom space that defines the allowable motion of the constrained body.

Let us have a look at the freedom space to see if there exists any rotational freedom other than \hat{T}_1 . According to the definition of freedom space, any freedom $\hat{T} = (\boldsymbol{\Omega} \quad | \quad \mathbf{V})$ in the freedom space can be expressed as linear combination of \hat{T}_1, \hat{T}_2 , written as

$$\hat{T} = k_1 \hat{T}_1 + k_2 \hat{T}_2 = (k_2 \quad k_1 \quad 0 \quad | \quad k_2 \quad 0 \quad 0) \quad (27)$$

If \hat{T} is a rotation, we must have

$$\boldsymbol{\Omega} \cdot \mathbf{V} = 0 \Rightarrow k_2^2 = 0 \Rightarrow k_2 = 0. \quad (28)$$

However this means that \hat{T} is simply a multiple of \hat{T}_1 . Hence this proves that no other rotation line exists in this freedom space. In other words, all freedoms except the rotation \hat{T}_1 are in the form of screw motion.

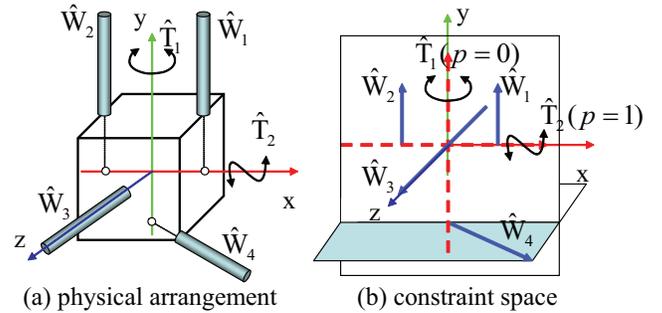


Figure 6. A body constrained by a pattern of ideal constraints can have a screw motion in space.

Constraint Pattern Design The constraint pattern design starts with specifying a freedom space described by a set of f twists $\hat{T}_j (j = 1, \dots, f)$. The objective is to find the complementary constrained space, that is to find $c = 6 - f$ independent constraints denoted by wrenches $\hat{W}_i (i = 1, \dots, c)$.

Here we use an example to demonstrate the constraint pattern design steps. Suppose we want to design a compliant mechanism with two allowable motions: a pure rotation around z axis and a pure translation along the direction of $(0 \quad 1 \quad 1)$. See Fig 7. We seek an arrangement of ideal wire flexures which constrain a rigid body and allows the prescribed motion. The constraint space is found by the following steps.

Step 1: denoting all specified freedoms in twists yields

$$\begin{aligned} \hat{T}_1 &= (0 \ 0 \ 1 \ | \ 0 \ 0 \ 0) \quad p_1 = 0, \\ \hat{T}_2 &= (0 \ 0 \ 0 \ | \ 0 \ 1 \ 1) \quad p_2 = \infty \end{aligned} \quad (29)$$

Step 2: requiring a general wrench $\hat{W} = (F_x \ F_y \ F_z \ | \ M_x \ M_y \ M_z)$ reciprocal to both twists yields

$$\hat{T}_1 \circ \hat{W} = 0 \Rightarrow M_z = 0 \quad (30)$$

$$\hat{T}_2 \circ \hat{W} = 0 \Rightarrow F_y + F_z = 0 \Rightarrow F_z = -F_y \quad (31)$$

Since we are only interested in ideal constraints (design with wire flexures), we also want

$$F_x M_x + F_y M_y + F_z M_z = 0, F_x^2 + F_y^2 + F_z^2 \neq 0. \quad (32)$$

Step 3: substituting Eqs.(30,31) into Eq.(32), we can write a general wrench in the complementary constraint space as

$$\begin{aligned} \hat{W} &= (F_x \ F_y \ -F_y \ | \ M_x \ M_y \ 0), s.t. \\ F_x M_x + F_y M_y &= 0, F_x^2 + 2F_y^2 \neq 0. \end{aligned} \quad (33)$$

Step 4: categorize the condition Eq.(33) and find independent subcases. For the sake of intuitiveness, we prefer to find constraints parallel to coordinate axes or plane whenever possible. This can be done by assigning one or more force elements to be zero and solve the other elements by Eq.(33). For this example, the following subcases are obtained,

$$\begin{cases} F_x \neq 0, F_y = 0, M_x = 0, M_y = 0 \\ F_x \neq 0, F_y = 0, M_x = 0, M_y \neq 0 \\ F_x = 0, F_y \neq 0, M_x = 0, M_y = 0 \\ F_x = 0, F_y \neq 0, M_x = 0, M_y \neq 0 \end{cases} \quad (34)$$

Step 5: write the subcases in (32) in the form of wrenches,

$$\begin{cases} \hat{W}_1 = (1 \ 0 \ 0 \ | \ 0 \ 0 \ 0) \\ \hat{W}_2 = (1 \ 0 \ 0 \ | \ 0 \ 1 \ 0) \\ \hat{W}_3 = (0 \ 1 \ -1 \ | \ 0 \ 0 \ 0) \\ \hat{W}_4 = (0 \ 1 \ -1 \ | \ 1 \ 0 \ 0) \end{cases} \quad (35)$$

By checking the rank of the wrench matrix, we find that the four wrenches in Eq.(35) are independent. Wrenches \hat{W}_1, \hat{W}_2 are parallel to the x axis and intersecting the twist \hat{T}_1 . Both wrenches \hat{W}_3, \hat{W}_4 are perpendicular with \hat{T}_2 and lie on the yz plane. See Fig 7.

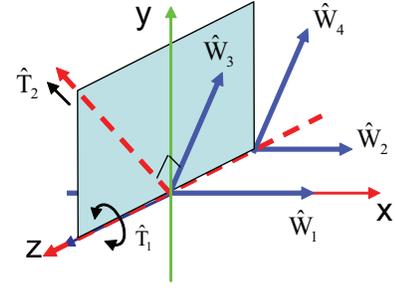


Figure 7. Four ideal constraints found for given a pattern of two degree-of-freedom.

In general the pitches of wrenches in the complementary constraint space may be zero, infinite or nonzero finite which correspond to translational (ideal), rotational and general constraint respectively. Because of the relatively low cost of building ideal constraints in compliant mechanism design, we would prefer to find as many ideal constraints in the constraint space as possible. As in constraint pattern analysis, one should be aware that it is NOT always possible to find $6 - f$ independent ideal constraints for arbitrary f freedoms. For instance, if two or three of the given freedoms are translational, by going through the design steps, one can find that there do not exist $6 - f$ complementary ideal constraints. Apparently these kind of cases are not rare. As a matter of fact, most of spatial freedom space cannot be realized without using non-ideal constraints. For these design cases, one has to use cascaded complex structures to provide rotational or general constraints.

4 Type Synthesis of Compliant Prismatic Joints

In this section, we show how to apply the proposed approach to the design of a compliant prismatic joint using wire or sheet flexures. Without losing generality, let us assume that the given single freedom is a translational motion along the x axis, denoted by a twist $\hat{T} = (0 \ 0 \ 0 \ | \ 1 \ 0 \ 0)$. By following the constraint design steps, we find $f_x = 0$, i.e. all wrenches representing ideal constraints in the constraint space must have the form

$$\hat{W} = (0 \ f_y \ f_z \ | \ m_x \ m_y \ m_z) \quad (36)$$

To design the constraint pattern, we only need to find five independent constraints in the constraint space. Suppose we want to use five wire flexures (ideal constraints) to achieve the design. This means that Eq.(36) is subject to

$$f_y m_y + f_z m_z = 0, f_y^2 + f_z^2 \neq 0. \quad (37)$$

If $m_x = 0$, we substitute either $f_y = 0$ or $f_z = 0$ (Note f_y and f_z cannot be zero simultaneously) into Eq.(37) and obtain the

following four subcases:

$$\begin{cases} f_y \neq 0, f_z = 0, m_y = 0, m_z = 0 \\ f_y \neq 0, f_z = 0, m_y = 0, m_z \neq 0 \\ f_y = 0, f_z \neq 0, m_y = 0, m_z = 0 \\ f_y = 0, f_z \neq 0, m_y \neq 0, m_z = 0 \end{cases} \quad (38)$$

which lead us the following four independent wrenches,

$$\begin{cases} \hat{W}_1 = (0 \ 1 \ 0 \ | \ 0 \ 0 \ 0) \\ \hat{W}_2 = (0 \ 1 \ 0 \ | \ 0 \ 0 \ 1) \\ \hat{W}_3 = (0 \ 0 \ 1 \ | \ 0 \ 0 \ 0) \\ \hat{W}_4 = (0 \ 0 \ 1 \ | \ 0 \ 1 \ 0) \end{cases} \quad (39)$$

They represent two constraint lines parallel to the y axis and two constraint lines parallel to the z axis respectively (Fig. 8). And if $m_x \neq 0$, setting $f_z = m_y = m_z = 0$ yields the last ideal constraint,

$$\hat{W}_5 = (0 \ 1 \ 0 \ | \ 1 \ 0 \ 0) \quad (40)$$

This constraint is a line in the direction $\mathbf{F}_5 = (0 \ 1 \ 0)$ and passing through a point $\mathbf{r}_5 = (0 \ 0 \ 1)$. By checking the rank of the wrench matrix of the constraint space, we verify that the five wrenches are not redundant.

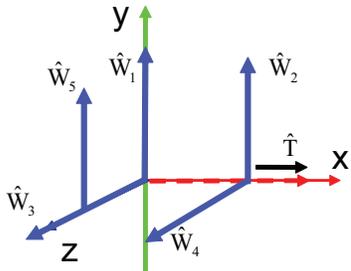


Figure 8. The constraint space for a compliant prismatic joint.

The design of the compliant prismatic joint using five wire flexures is shown in Fig. 9(a). The compliant prismatic joint allows the rigid body translate along the x direction only. As shown in Section 3.2, a single sheet flexure provides three ideal constraints whose axes lie on the same plane. An alternative physical arrangement is to use a single sheet flexure to replace the wire flexures $\hat{W}_1, \hat{W}_3, \hat{W}_5$. This design is shown Fig. 9(b).

Furthermore let us manually add a sixth constraint \hat{W}_6 which is parallel to \hat{W}_2 and intersects \hat{W}_4 at the point $(1 \ 0 \ 1)$ (Fig. 10(a)). Since \hat{W}_6 is redundant (can be written as a linear combination of $\hat{W}_i (i = 1, \dots, 5)$), it does not apply extra constraint to

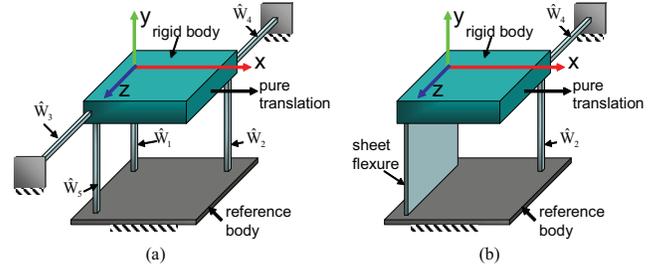


Figure 9. Design of a compliant prismatic joint using (a) five wire flexures or (b) one flexure sheet and two wire flexures.

the rigid body. However this allows us to use a second flexure sheet to apply the constraints $\hat{W}_2, \hat{W}_4, \hat{W}_6$. The result is the well known parallel flexure design of compliant prismatic joint. See Fig. 10(b).

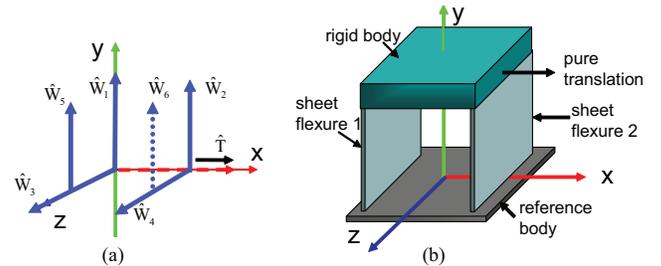


Figure 10. Design of a compliant prismatic joint using two parallel flexure sheets.

As one can see, the choice of the basis constraints is not unique which means that there are multiple constraint patterns which will achieve a given freedom pattern. Even for the same constraint pattern, there may be multiple physical arrangements (designs). Apparently this is beneficial to the designers as they have multiple ways to achieve the same design goal. An interesting future research task is to systematically find all possible physical arrangements for a given freedom pattern.

5 Conclusion

A screw theory based approach for the design of compliant mechanisms is introduced in this paper. This approach presents a mathematical representation of the constraint-based design approach which is typically characterized by subjective statements. A constraint and a freedom in the constraint-based design approach can be mathematically denoted by a wrench and a twist respectively. The constraint topology or space or pattern formed by a system of constraints acting on a rigid body is essentially a linear space spanned by a system of independent wrenches. Sim-

ilarly a freedom space that describes the possible instantaneous motion of a rigid body is represented by a system of independent twists. These two spaces are complementary and can be found from each other using linear algebra. Two major compliant mechanism design problems, constraint pattern analysis and constraint pattern design, are elaborated with examples of compliant mechanisms and flexures. The proposed analysis/design framework is beneficial for the early design stages, such as type synthesis of compliant mechanisms.

Acknowledgment

All authors acknowledge funding provided by the National Science Foundation grant CMMI-0457041 for this research. The first author acknowledges support provided by National Science Foundation CAREER grant.

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