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THE ROLE OF CRITICAL EXPONENTS IN BLOWUP THEOREMS*

HOWARD A. LEVINE†

This paper is dedicated to Lawrence E. Payne in honor of his 66th birthday.

Abstract. In this article various extensions of an old result of Fujita are considered for the initial value problem for the reaction-diffusion equation \( u_t = \Delta u + u^p \) in \( \mathbb{R}^N \) with \( p > 1 \) and nonnegative initial values. Fujita showed that if \( 1 < p < 1 + 2/N \), then the initial value problem had no nontrivial global solutions while if \( p > 1 + 2/N \), there were nontrivial global solutions. This paper discusses similar results for other geometries and other equations including a nonlinear wave equation and a nonlinear Schrödinger equation.

Key words. critical exponents, parabolic, hyperbolic equations, Schrödinger equations, global existence, global nonexistence

AMS(MOS) subject classifications. 35B30, 35B35, 35K15, K55, K65, L15, L70

0. Introduction. Over the course of the last fifteen to twenty years there has been an explosion (pun intended) of interest in blowup theorems for solutions of nonlinear evolution equations. In addition to being of interest to workers in partial differential equations, blowup theorems are often relevant to workers in such diverse areas as chemical reactor theory [60], [64], [65], [76], [82], [89], [113], quantum mechanics and fluid mechanics [95], [131]. Recently, [39], singularity formation in the solutions of nonlinear Schrödinger equations has been proposed as a mechanism for studying the onset of turbulent flows. See also [37], [38]. The question of finite time singularity formation in the boundary layer equations of fluid flow is currently under active investigation by aeronautical engineers, continuum mechanists, and many others. See, for example, [73], [119], [122]–[129], [135], and some of the references cited therein.

In the study of evolution equations, the terms “global” and “local” refer to the existence of the solution on the entire half line \( t > 0 \) or on some finite interval to the right of zero, respectively. For the purpose of this paper, we shall use the term “blowup” as a pseudonym for “global nonexistence,” i.e., as a pseudonym for the statement, “The maximal interval of existence is bounded.” Although the latter concept is much more general in some sense because it allows for the loss of regularity of the solution in finite time without the solution actually becoming unbounded in that time, such regularity loss manifests itself in the finite time blowup of some derivative of the solution. A well-known example of the latter phenomenon is given by the following initial value problem for Burgers’ equation:

\[
\begin{align*}
\frac{u_t + uu_x}{(B)} &= 0, & x \in \mathbb{R}, & t > 0 \\
\quad f(x), & x \in \mathbb{R}.
\end{align*}
\]

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The solution is given implicitly for $t > 0$ by

$$u(x, t) = f(x - tu(x, t)).$$

If $f$ is a bounded, continuously differentiable function, the (continuously differentiable) solution remains bounded as long as it exists. However, if $f'(x)$ is somewhere negative, then from an easy calculation using implicit differentiation, we see that $u_t$ and $u_x$ both become unbounded in finite time.

Therefore, we shall take the phrase “finite time blowup” to mean that either the solution or some derivative of the solution becomes unbounded in some norm in finite time. Blowup in the classical sense will be referred to as “pointwise blowup,” by which we mean that the solution itself becomes unbounded at some point in the spatial domain (possibly including the “point at infinity” if the domain is unbounded).

Our purpose here is to survey the recent literature on the role the size of the nonlinearity plays in determining whether or not blowup occurs. As an example of the type of results we wish to survey, let us recall the classical result of Fujita [7]: He considered, for $p > 1$ the following initial value problem (where $\Delta$ denotes the $N$-dimensional Laplace operator):

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N, \quad t > 0.$$  

(F)

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N.$$  

His interest was in nonnegative solutions which, for fixed $t$, decay at infinity. Thus the initial values are nonnegative and the nonlinear term is well defined. He proved the following result.

**Theorem (F).** Let $pc(N) = 1 + 2/N$.

A. If $1 < p < pc(N)$, then the only nonnegative global (in time) solution of (F) is $u=0$.

B. If $p > pc(N)$, then there exist global positive solutions of (F) if the initial values are sufficiently small (less than a small Gaussian).

Several comments about this result are in order. First, when the solution fails to exist globally, it actually does blow up pointwise [3], [33]. Secondly, the interest here is in small initial values. Indeed, the author has shown [15] that whenever the initial values are so large that

$$\int_{\mathbb{R}^N} u_0^{p+1}(x) dx > \left(\frac{1}{2}\right) \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx,$$

then $u$ fails to be global in the sense that $u$ cannot remain in $L^{p+1} \cap H^1$ for all time (for any $p > 1$). ($\nabla_x$ denotes the usual gradient.)

The number $pc(N)$ defined in the statement of the theorem, is called, as its name suggests, the critical exponent for (F). Now critical exponents appear in other contexts in partial differential equations (for example, in the context of Sobolev spaces and imbedding theorems) so that perhaps we ought to call $pc(N)$ the critical blowup exponent. However, since we will only be considering critical blowup exponents here, no confusion should arise.

As Fujita himself observed, the result is to be expected for small data. The intuitive explanation is that if $p$ is “large” and the initial values are “small” then the tendency of the solution to blow up (which it would do if the forcing term, $u^p$, were the only term in the equation) is inhibited by the dissipative effect of the Laplace
operator. On the other hand, if \( p \) is close to one, then no matter how small the initial values are, the dissipative effects of the Laplacian are insufficient to prevent blowup.

The first statement of the theorem is called the blowup case while the second statement is called the global existence case. Sometimes this terminology is applied directly to the inequalities on \( p \). A natural question arises. To which case does \( pc(N) \) belong?

Several authors have furnished the answer [1], [11], [14], [32] but the most elegant proof that \( pc(N) \) belongs to the blowup case was given by Weissler [32]. We shall have more to say about his arguments later.

Four possible directions for extending Fujita's result come readily to mind. We could consider (F) on other domains; for example, when \( \mathbb{R}^N \) is replaced by the exterior of a bounded domain or by a domain with the property that both it and its complement are unbounded. The whole issue in the case of bounded domains should also be considered. Secondly (perhaps in combination with the first), more general parabolic equations including those with nonlinear dissipative terms could be studied. A third avenue of investigation, again possibly in tandem with the first two, is to consider what happens when we change the form of the nonlinear term (often called the reaction term). Systems of equations could also be considered. Finally, initial value problems for which the evolution equation is no longer parabolic could also be examined, for example, a nonlinear Schrödinger equation or a nonlinear wave equation.

In these last two cases, the solutions will be complex or real valued, respectively. This will necessitate defining the nonlinearity in a suitable manner off of the nonnegative real axis.

What is known for other geometries is restricted essentially to parabolic problems. For the other types of equations discussed above, arguments involving the Maximum Principle cannot be employed. For such equations, our discussion is therefore limited to the pure initial value problem. Thus there remain several interesting and challenging open questions in the case of nonparabolic equations.

Intuitively, we might expect the critical exponent to depend upon geometry as well as upon the equation itself. For example, in the case of bounded domains, the eigenvalues of \(-\Delta\) increase as the size of the domain decreases. (A simple argument with the Rayleigh quotient shows us that this is indeed so. See [71, I, Chap. 6].) This amounts to increasing the effect of dissipation on the solution and we would correspondingly expect blowup to be inhibited. Direct calculation of critical numbers for unbounded domains also shows that as the size of the domain increases, the critical number does so also. Both of these observations are often summarized in the rough intuitive statement that “small domains are more stable than large domains.” Unfortunately, the fact that the eigenvalues of \(-\Delta\) decrease with increasing domain cannot be used as an intuitive explanation for the increase of critical number with domain if the domains are unbounded because, in general, the Laplacian has only continuous spectra for unbounded domains. Therefore, we offer the following loose explanation: The larger the domain (bounded or not) the easier it is for the reaction term to have a large positive maximum without the diffusion term (the Laplacian of the solution) to be sufficiently negative to cancel it. It is in this sense that we also understand the statement that “large domains are less dissipative than small domains.”

In view of the nondissipative character of the associated linear problems, we might also expect that for the nonlinear Schrödinger equation (NLS) and the nonlinear wave equation, the critical exponent should be larger than for (F) and this will turn out to
be the case. In the case of (NLS) the result takes a peculiar twist. This is getting ahead of the story, however.

We caution the reader that the discussion of the results of the references here is necessarily incomplete due to considerations of length. Mention of many interesting extensions and special cases has been therefore omitted. The author apologizes to the authors of the referenced and (unintentionally) unreferenced articles for any omissions.

A word about notation: Since the critical exponent can depend on many variables (geometry, dimension, parameters in the equation, etc.), our notation for it may seem somewhat confusing. For example, we have denoted Fujita's original number, \(1 + 2/N\) by \(pc(N)\). We could equally well use the notation \(pc(R^N)\) to indicate that this is the critical exponent for the initial value problem or the notation \(pc(F)\) to indicate that it is the critical number corresponding to Problem (F). Similar remarks hold for the other problems. We shall adopt the following (rough) convention. When we wish to emphasize the role of geometry, the argument in \(pc(.)\) will be the spatial domain. When the geometry is understood (as for example in initial value problems), we will list the parameters in the equation as the arguments of \(pc(.)\).

OUTLINE OF THE REMAINDER OF THE PAPER

1. Extensions of Problem (F)
   1.1. In other geometries, with various (linear) dissipative terms or with other reaction terms.
   1.2. Nonlinear dissipative terms.
   1.3. Problems on bounded domains.
   1.4. Systems of equations.
2. Nonlinear Schrödinger equations.
3. Nonlinear wave equations.

1. Extensions of Problem (F).

1.1 Other geometries, various linear dissipative terms or other reaction terms. Suppose \(D \subset R^N\) is any domain, bounded or unbounded. In place of (F) we consider the initial boundary value problem:

\[
\begin{align*}
  u_t &= \Delta u + u^p, & (x,t) \in D \times (0,T) \\
  u(x,0) &= u_0(x), & x \in D \\
  u(x,t) &= 0, & (x,t) \in \partial D \times [0,T)
\end{align*}
\]

where again our interest is focused only on the long time behavior of nonnegative solutions of (D). Moreover, we shall only concern ourselves with classical solutions of the problem. We also assume that when D is not bounded, u “vanishes at infinity,” although this assumption can be weakened.

Rather than discuss the results in this section historically, which would make for a somewhat disjointed presentation, we will follow a more systematic development.

In [25], Meier considered the following initial boundary value problem:

\[
  u_t = \sum_{i,j=1}^N (a_{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^N b_i(x,t)u_{x_i} + u^p \quad (p \geq 1),
\]
(GD) \[ u(x, 0) = u_0(x), \quad x \in D \]
\[ u(x, t) = 0, \quad x \in \partial D, \quad t > 0 \]

where the coefficients of the linear operator on the right-hand side are uniformly bounded in \( D \times [0, \infty) \) and are such that the operator is uniformly elliptic on the same set. Meier then proved the following result.

**Theorem (GD).** There exists a critical exponent \( pc(GD) \geq 1 \), with the following properties:

A. If \( pc(GD) > 1 \) and \( 1 < p < pc(GD) \), then (GD) has no global, positive solutions.

B. If \( p > pc(GD) \), then (GD) has global, positive solutions for sufficiently small initial values.

While the existence of \( pc(GD) \) is asserted by Meier’s result, it is not an easy number to find explicitly. Also Meier’s result makes no claim as to which case the critical exponent belongs. We can view the Theorem (F) and the result of Weissler as an attempt to answer the questions raised by Meier’s result although this is not correct from a historical perspective.

In some of the references, the nonlinearity \( u^p \) is replaced by \( t^k|x|^\sigma u^p \). See [2], [3], [23]–[35]. The critical numbers are changed to reflect the dependence on \( k \) and \( \sigma \). For example, for the initial value problem (F) with this replacement, in place of \( pc(N) \) we have

\[ pc(N, k, \sigma) = 1 + \frac{(2 + 2k + \sigma)}{N}. \]

(In some cases, \( \sigma > -2 \) is allowed [3]. However, in all cases, \( k \geq 0 \).) We might intuitively expect that such nonlinearities would have a greater destabilizing influence on solutions of the above initial boundary value problems than those for which \( k = \sigma = 0 \) and, from the literature cited above, we are not disappointed. In some of these cases, it is not known whether or not the critical number belongs to the blowup case. (See below.)

However, in all of these results, the domain is unbounded. If the domain is bounded, then for all of these nonlinearities, \( pc = 1 \). (See below.)

It is also possible to have \( pc(D) = +\infty \). For example, if \( D = \{(x, y) | x \geq 0, y \geq 0\} \), then (with \( r^2 = x^2 + y^2 \)) the initial boundary value problem (with homogeneous Dirichlet data) for

\[ u_t = u_{xx} + u_{yy} + \left( \frac{4}{r^3} + \frac{1}{r^2} - \frac{1}{r} \right) (xu_x + yu_y) + u^p \]

can be shown not to have global solutions whenever it has local solutions. The local existence question for this problem is open.

In [3], Bandle and Levine considered two special cases of (D), the case in which \( D \) is the complement of a bounded domain in \( \mathbb{R}^N \) (sometimes called the exterior problem) and the case in which \( D \) is a cone in \( \mathbb{R}^N \). We shall have more to say about the second case later. They were able to prove that for the exterior problem we also have

\[ pc(D) = 1 + \frac{2}{N} = pc(N). \]

Thus the critical exponent is constant on domains with bounded complement. This answered an old conjecture of Fujita. They did not show that, in this case, \( pc(D) \) was in the blowup case. An attempt to show this by Levine and Meier using the arguments of [19], [32] was unsuccessful.
Next we turn to the question of how the critical exponent behaves when the domain and its complement are both unbounded.

As a precursor to the discussion of results for cones, let us consider an early result of Meier. In [23], [24], he calculated $pc(D)$ for the case that $D$ was an “orthant.” That is, suppose $k$ is a nonnegative integer in $[0, N]$. Define

$$D_k = \{ x \in \mathbb{R}^N \mid x_1 > 0, \cdots, x_k > 0 \}.$$  

(When $k = 0$, $D_k = \mathbb{R}^N$.) Then

$$pc(D_k) = 1 + \frac{2}{(k + N)}$$

which is also smaller than $pc(N)$. In [19], he found $pc(D)$ the result for other types of domains for which he could find the Green’s function for the heat equation explicitly. He also left unanswered the result for the critical exponent itself.

By a cone in $\mathbb{R}^N$ with vertex at the origin we mean the following: Let $\Omega$ be an open subset of the unit sphere $S^{N-1}$ and represent a point in $\mathbb{R}^N$ in “polar” coordinates in the form

$$x = (r, \tilde{\theta})$$

where $\tilde{\theta}$ is a point on the sphere. Then $D$ is a cone with vertex at the origin (by definition) if and only if we can write

$$D = (0, \infty) \times \Omega.$$  

Let the boundary of $\Omega$, $\partial \Omega$, be smooth enough to allow us to integrate by parts on $\Omega$ and let $\Delta_\delta$ denote the Laplace–Beltrami operator on $\Omega$. Let $\omega_1$ denote the first Dirichlet eigenvalue for this operator on $\Omega$ and let $\gamma_-$ be the negative root of the quadratic

$$x(N - 2 + x) = \omega_1.$$  

The next result shows that for cones:

$$pc(D) = 1 + \frac{2}{(2 - \gamma_-)}.$$  

Precisely, we have the following theorem.

**Theorem (D).** Let $D$ be a cone with vertex at the origin. Then $pc(D) = 1 + 2/(2 - \gamma_-)$. That is, the following statements are true:

A. If $1 < p \leq 1 + 2/(2 - \gamma_-)$, $(D)$ has no nontrivial, global, positive solutions.

B. If $p > 1 + 2/(2 - \gamma_-)$ then global, positive, small data solutions do exist.

The first statement with strict inequality was proved in [3] where an upper bound for $pc(D)$ was also given. Later, in [19], Levine and Meier showed that $pc(D)$ is indeed the critical exponent. More recently, they showed that it belongs to the blowup case [20]. (Weissler’s arguments had to be modified to show this as the $L^1$ norm of the Green’s function is no longer constant on such conical domains. The modification of his arguments is not trivial.)

Notice that as $\Omega$ increases, $\omega_1$ decreases (as is seen from variational arguments) and consequently the principle that $pc$ increases with increasing domain holds here.

Let us briefly sketch the proof of Theorem (D) in the case $p \neq pc(D)$.  

For part (A), we modify the old argument of Kaplan [12]. We let $\psi$ be the first eigenfunction of $\Delta_\delta$ corresponding to $\omega_1$ on $\Omega$. We let

$$\varphi(r, \theta) = C^{-1} r^m e^{-kr} \psi(\theta)$$

where $C$ is such that

$$\int_D \varphi \, dx = 1,$$

and where $m, k > 0$. We set

$$F(t) = \int_D u \varphi \, dx.$$

Then we can show that for $\lambda > 0$ and $\lambda, k$, sufficiently small,

$$\Delta \varphi \geq -\lambda \varphi$$

$$F'(t) \geq -\lambda F(t) + F^p(t)$$

and

$$F(0) > \lambda^{-1/(p-1)}$$

provided $m, k, \lambda, p$ satisfy the following inequalities:

(i) $(k^2 + \lambda)(m^2 + (N-2)m - \omega_1) \geq k^2 (m + \frac{1}{2}(N-1))^2$

(ii) $m^2 + (N-2)m - \omega_1 > 0$

(iii) $2 - N - \gamma_0 < m < 2/(p-1) - N$.

We can choose $\lambda, k^2$ such that (i) holds with equality, i.e.:

$$\lambda \frac{k^2}{k^2} = \frac{m + \omega_1 + \frac{1}{4}(N-1)^2}{m^2 + (N-2)m - \omega_1}.$$

With $m, k, \lambda$, and $p$ so restricted, a simple quadrature shows that $F(t)$ must blow up in finite time $T$, where

$$T \leq \int_{F(0)} (\sigma^p - \lambda \sigma)^{-1} d\sigma.$$

The solution of (D) need “vanish at infinity” only in the sense that for all $k > 0$, and $t$ fixed:

$$\lim_{r \to \infty} e^{-kr} \int_{\Omega} (|u(r, \theta, t)| + r|\partial_r u(r, \theta, t)|) \, dS_{\theta} = 0.$$

The proof for (B) is based on the construction of a global supersolution if $p > pc(D)$. We look for a supersolution $\bar{u}$ of the form

$$\bar{u}(x, t) = \beta(t) w(x, t)$$

where $w$ is a positive solution of $w_t = \Delta w$ in $D \times [0, \infty)$. $\bar{u}$ will be a global supersolution of (D) if

$$\beta'(t) = \beta^p(t) \left[ \sup_{x \in D} w(x, t) \right]^{p-1},$$

$$W_{\infty} = \int_0^\infty \left[ \sup_{x \in D} w(x, t) \right]^{p-1} \, dx < \infty$$

and if
0 < \beta(0) < ((p - 1)W_\infty)^{-1/(p-1)}.

The correct choice of \( w \) is

\[
w(r, \theta, t) = (t + t_0)^{-1}r^{-\frac{1}{2}}(N-2)I_{\nu} \left( \frac{r}{2(t + t_0)} \right) e^{-\left( r^2 + 1 \right)/4(t + t_0)} \psi(\theta)
\]

where \( I_{\nu} \) is the modified Bessel function of order \( \nu \) where

\[
\nu = \left[ \omega_1 + \frac{1}{4}(N-2)^2 \right]^{1/2}
\]

The function \( w \) is essentially the first term of the Fourier–Bessel expansion of the Green’s function for \( u_t = \Delta u \) with Dirichlet boundary conditions in the cone \( D \). It also plays a crucial role in the proof of (A) when \( p = p_c(D) \).

Escobedo and Kavian have taken a somewhat different approach to the study of (F) [6], [13]. They study blowup properties through the use of self-similar solutions, an approach taken also by Giga and Kohn [92], [93]. They allow real valued solutions of (F) so that they consider

\[
\begin{align*}
  u_t &= \Delta u + |u|^{p-1}u, & x &\in D, & t &> 0 \\
  u(x, 0) &= u_0(x), & x &\in D, \\
  u(x, t) &= 0, & x &\in \partial D, & t &> 0.
\end{align*}
\]

They let

\[
K(x) = \exp \left( |x|^2/4 \right)
\]

and introduce the weighted Hilbert spaces

\[
L^2(K, D) = \{ f \| f \|_K < \infty \}
\]

and

\[
H^1_0(K, D) = \{ f | f, \nabla_x(f) \in L^2(K, D) \text{ and } f = 0 \text{ on } \partial D \}
\]

where

\[
(f, g)_K = \int_D f(x)g(x)K(x)dx
\]

denotes the scalar product on the first space. The requirement that for each \( t \geq 0 \), \( u(., t) \) is in the weighted Sobolev space \( H^1_0(K,D) \) with rapidly growing weight replaces the condition that the domain be bounded and the solution vanish on the boundary.

They then make the following change of variables in (K):

\[
t \rightarrow \exp(t) - 1, \quad x \rightarrow \exp(t/2)x, \quad u \rightarrow \exp \left( \frac{t}{(p - 1)} \right) u
\]

and define the operator (in the weak sense)

\[
L_f = -\Delta f - \frac{1}{2}x \cdot \nabla_x(f)
\]

on \( H^1_0(K,D) \). In this space, they consider the nonlinear evolutionary equation

\[
\begin{equation}
\tag{K}
u_t = -Lu + (p - 1)^{-1}u + |u|^{p-1}u.
\end{equation}
\]

Then, in order to study (K) they employ the potential well arguments of Sattinger [118] and Payne and Sattinger [26]. We may apply potential well arguments here because, in some sense, the rapid decay at infinity of the elements of \( H^1_0(K,D) \) and \( L^2(K,D) \) allow us to have a Poincaré inequality and a compact imbedding theorem.
just as for bounded domains when \( K = 1 \). If we let \( \Lambda_1 \) denote the smallest positive eigenvalue of \( L \), we see that the linear operator (where \( I \) is the identity operator):
\[
L - (p - 1)^{-1}I
\]
is symmetric (in fact self-adjoint). It has its spectrum in the right half line if and only if
\[
pc(K) := 1 + \frac{1}{\Lambda_1} > p.
\]
Thus if we linearize (K) around \( u = 0 \) and if \( p \leq pc(K) \), the null solution will be unstable, whereas if \( p > pc(K) \) the null solution will be stable. (Here is another illustration of the principle mentioned in the Introduction that “large domains are more unstable than small domains” applied to \( L \) rather than \( \Delta \). As \( D \) increases \( \Lambda_1 \) decreases as can be seen from variational arguments.) Therefore, for the nonlinear problem, we would expect that if \( p \leq pc(K) \), (K) cannot have global, nontrivial solutions while if \( p > pc(K) \), (K) will have both nontrivial, global, small data solutions and nonglobal solutions. Indeed, it is not too hard to show that for cones and for the exterior problem for (F)
\[
pc(K) = pc(D).
\]
(In fact, it is easy to show that \( \Lambda_1 = \frac{1}{2}(2 - \gamma^-) \).) This is precisely the result that the authors of [6], [13] obtain. However, the methods used to prove the result require some additional conditions on the geometry and upon \( p \). First, in the case of cones, they require that the cone be convex. Secondly, they restrict \( p \) to the interval
\[
\left(1, \frac{(N + 2)}{(N - 2)}\right).
\]
They do not consider other nonlinearities (as far as we know) nor do they consider the question of global existence for \( p \geq (N + 2)/(N - 2) \). (This is a defect of the Hilbert space approach.) Also, they do not consider the case of variable coefficients in the differential equations. Finally, we remark that the global nonexistence result that they obtain for large initial values is in fact a special case of that of [15] applied to (K) in the weighted Hilbert spaces.

By contrast, the results of [3], [19], [20] place no convexity condition on the cone or any upper bound on \( p \). The solutions are required to live in the space
\[
\left\{ f : \overline{D} \rightarrow \mathbb{R} | \forall k > 0, \int_D e^{-k|x|} \left( |f(x)| + |\nabla f(x)| \right) dx < \infty \text{ and } f = 0 \text{ on } \partial D \right\}
\]
which strictly contains \( H^1_0(D) \). A discussion of related results for the equation
\[
u_t = \Delta u + |x|^\sigma u^p
\]
in cones is also included in [3], [19], [20]. There is also a result which states that if
\[
1 + 2/(-\gamma^-) < p < \begin{cases} \frac{(N + 1)}{(N - 3)}, & N \geq 4 \\ \infty, & N = 2, 3 \end{cases}
\]
then singular (at the vertex) positive stationary solutions of the form \( r^{-2/(p-1)}\alpha(\theta) \) and any solution with initial values pointwise smaller than such a singular solution is global while if
\[
1 + 2/(-\gamma^-) < p < \begin{cases} \frac{(N + 2)}{(N - 2)}, & N \geq 3 \\ \infty, & N = 2 \end{cases}
\]
such a solution decays to zero. Both sets of papers contain results for global solutions. Under some conditions, the decay results of [6], [13] are superior to those of [19], [20].

For example, they show that for all global solutions (with $p_{c(K)} < p < (N + 2)/(N - 2)$), the $C^2$ norm decays like $t^{-1/(p-1)}$. On the other hand, in [19] it is shown that for a wide class of initial values, solutions decay in $L^\infty$ like $t^{-1/(p_{c(K)}-1)}$ which is clearly faster than $t^{-1/(p-1)}$. Also this result only requires that $p > p_{c(K)}$.

In spite of the technical restrictions placed on the nonlinearity, the coefficients of the differential operator, and the geometry, this method of attack allows us great insight into the structure of the solution set of (F).

Recently, Bandle and Levine [4] investigated (GD) when $D$ was assumed to be the exterior of a bounded region, which, without loss of generality, may be assumed to contain the origin. Additionally, it was necessary to assume that the coefficients in the partial differential equation were time independent. The purpose of the investigation was to obtain upper and lower bounds on $p_{c(GD)}$ in terms of the coefficients of the linear operator.

In order to describe the results they obtained, we need some notation. We let

$$a(x) := (a_{ij}(x))_{N \times N},$$

$$b(x) := (b_1(x), \ldots, b_N(x))_{1 \times N},$$

$$\text{tr}(a) := \sum_{i=1}^{N} a_{ii}(x),$$

$$\rho(x) := \sum_{i,j=1}^{N} a_{ij}x_i x_j/|x|^2,$$

and assume, without loss of generality, since $a(x)$ is uniformly bounded in $D$, that there is a constant $\nu_0$ such that

$$0 < \nu_0 < \rho(x) \leq 1.$$ 

We write $c_{ij}$ for $\partial c/\partial x_i$. We also define

$$\ell(x) := [a_{ij,j}(x) + b_i(x)] x_i$$

and finally

$$\ell^*(x) := [a_{ij,j}(x) - b_i(x)] x_i.$$ 

Then we have the following results (where we have used the notation (GE) to remind ourselves that this is an exterior problem).

**Theorem (GE).** Let $D$ have bounded complement, and suppose the coefficients in (GD) do not depend on $t$. We have:

A. Suppose that on $D$, we have

$$\text{div}(b(x)) \leq 0$$

(B-1)

$$\rho(x) \leq \frac{1}{2} [\text{tr}(a(x)) + \ell^*(x)].$$

(C-1)

If $1 < p < p_{c(N)}$, then there are no nontrivial, global positive solutions of (GE).

B. Suppose that (B-1) holds and that

$$2\gamma_0 := \inf [\text{tr}(a(x)) + \ell^*(x)] > 0.$$ 

(G-1)

If $p > 1 + 1/\gamma_0$, then (GE) has nontrivial, positive, global solutions.
Therefore, if (B-1), (C-1), (G-1) hold,

\[ pc(N) \leq pc(\text{GE}) \leq 1 + \frac{1}{\gamma_0}. \]

The result is optimal in the sense that if we specialize it to the Dirichlet problem for the Laplace operator, we see that the upper and lower bounds for \( pc(\text{GE}) \) are equal.

### 1.2. Nonlinear dissipative terms.

For general nonlinear dissipative operators, \( A(u) \), there is no known result for the equation

\[ u_t = A(u) + u^p \]

which is analogous to the general result of Meier (Theorem (GD)) of the preceding section. The notation \( pc(\cdot) \) thus has a somewhat different meaning than it did above. That is, here we will define \( pc \) for each problem we consider below. In this section, we shall look at some special cases of the operator \( A(u) \). We begin with the following initial value problem:

\[ u_t = \text{div} \left\{ \frac{\nabla_x(u)}{[1 + |\nabla_x(u)|^2]^{1/2}} \right\} + u^p, \quad x \in \mathbb{R}^N, \quad t > 0 \]

(MC)

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N. \]

(MC = "Mean Curvature.") Again the interest is in nonnegative solutions. The rather strange-looking operator that replaces the Laplace operator in (F) arises in differential geometry and is known as the mean curvature operator [94]. The question of the existence of stationary solutions of equations of the form (MC) (with other reaction terms) has been under active investigation of late. See [104], [111], [112], [114], [116] for example. In [18], a somewhat more general problem is considered, namely (GMC = "Generalized Mean Curvature")

\[ u_t = \text{div} \left\{ \psi \left[ (1 + |\nabla_x(u)|^2)^{1/2} \right] \nabla_x(u) \right\} + u^p, \quad x \in \mathbb{R}^N, \quad t > 0 \]

(GMC)

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N. \]

The function \( \psi \) is assumed be in an appropriate Hölder space on the half line \([1, \infty)\), be positive at 1, and to satisfy, for all \( s > 1 \),

\[ 0 \leq s \psi'(s) + \psi(s) \leq (1 + \theta)\psi(s) \]

for some \( \theta > 0 \) and all \( s > 1 \) and be uniformly bounded on the half line \([1, \infty)\). (This last condition assures us that the "GMC" operator is elliptic and that it does not grow faster than \( |\nabla u|^{\theta} \) for large \( |\nabla u| \).) For example, \( \psi(s) = 1/s \) satisfies these conditions. Then we have that \( pc(\text{GMC}) = pc(N) \). More precisely, we have the following theorem.

**Theorem (GMC).** We have \( pc(\text{GMC}) = pc(N) \), i.e.;

A. If \( 1 < p < pc(N) \), there are no nontrivial positive solutions of (N).
B. If \( p > pc(N) \), then there exist both positive global solutions of (N) and solutions of (N) which blow up in finite time.
No claim is made when $p = pc(GMC)$.
The following problem was considered by Galaktionov \[8\]:
\[
    u_t = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( |\nabla_x(u)|^\sigma \frac{\partial u}{\partial x_j} \right) + w, \quad x \in \mathbb{R}^N, \quad t > 0.
\]  
(SL)
\[
    u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N.
\]
Here $\sigma \geq 0$ and $p > 1$. (SL = "sigma Laplacian.") He showed that
\[
    pc(SL) = 1 + \frac{2}{N} + \sigma \left(1 + \frac{1}{N}\right).
\]
That is,

Theorem (SL). Let $pc(SL)$ be given by the above expression.

A. If $1 < p < pc(SL)$, then (SL) has no global, nonnegative, nontrivial solution.

B. If $p > pc(SL)$, then (SL) does have such solutions (which in fact decay in the uniform norm like $t^{-1/(p-1)}$).

No statement is claimed when $p = pc(SL)$ although this value of $p$ most likely belongs to the blowup case.

It was proved in [114] and by an alternate method in [117] that global nonexistence occurs for large initial values as long as $p > \sigma + 1$.

When $N = 1$, Galaktionov [8] observed the following concerning $\sigma + 1$: If $1 < p < \sigma + 1$, then every nontrivial solution of (PM) blows up at every $x \in \mathbb{R}^1$ (no localization). If $p \geq \sigma + 1$, then there is strong numerical evidence to support the claim that solutions which blow up do so (at least for some initial values) only on compact subsets or even single points. He has recently shown this (rigorously) to be the case if $p > \sigma + 1$ and $u_0$ decays at $\infty$ like $|x|^{-1/p}$ and is even $[136]$.

Additionally, in [9], Galaktionov considered the following initial value problem:
\[
    u_t = \nabla \cdot (u^\sigma \nabla u) + w, \quad x \in \mathbb{R}^N, \quad t > 0
\]  
(PM)
\[
    u(x, 0) = u_0(x) \quad (\geq 0), \quad x \in \mathbb{R}^N
\]
where again $\sigma > 0$, $p > 1$. (We are to think of (PM) as a forced porous medium equation. Hence the notation "PM.") He proved that
\[
    pc(PM) = 1 + \frac{2}{N} + \sigma.
\]
That is,

Theorem (PM). Let $pc(PM)$ be given by the above expression.

A. If $1 < p < pc(PM)$, then (PM) has no positive global solutions.

B. If $p > pc(PM)$, then there exist global, positive solutions which decay like $t^{-1/(p-1)}$.

Again no statement is asserted when $p = pc(PM)$.

It was shown in [22] that if
\[
    \frac{1}{p + \sigma + 1} \int_{\mathbb{R}^N} (u_0(x))^{p+\sigma+1} dx > \frac{1}{2(\sigma + 1)^2} \int_{\mathbb{R}^N} |\nabla u_0^{\sigma+1}|^2 dx
\]
then (PM) has no global solutions as long as \( p > \sigma + 1 \). The arguments there are similar to those used in [17]. In [9], a similar result was obtained (when \( p > \sigma + 1 \)).

For (PM) when \( N = 1, \sigma + 1 \) is a second "critical exponent" in the following sense [9], [10]: If \( 1 < p < \sigma + 1 \), all nontrivial solutions blow up at every point \( x \) in finite time \( T = T(u_0) \). If \( p > \sigma + 1 \), solutions which blow up may do so at a single point or on a set of zero measure (single point blowup). When \( p = \sigma + 1 \), blowup occurs but on a set of finite positive measure. In [86], Galaktionov has obtained a more precise estimate of how the blowup occurs in the case \( p = \sigma + 1 \).

Both results for (SL), (PM) are to be expected. Increasing \( \sigma \) weakens dissipation near \( u = 0 \). Thus, the set of \( p \)'s for which we have blowup should increase as \( \sigma \) increases.

It should be possible to obtain extensions of Theorems (SL), (PM) for some of the other geometries considered earlier.

Reference [28] contains many of the details of the above results as well as numerous computational results and an extensive bibliography.

1.3. Problems on bounded domains. Let \( \psi \) be the first Dirichlet eigenfunction corresponding to the first eigenvalue, \( \lambda_1 \), of the membrane problem, i.e.,

\[
\Delta \psi + \lambda_1 \psi = 0, \quad x \in D,
\]

\[
\psi = 0, \quad x \in \partial D,
\]

then, \( \psi > 0 \) in \( D \) [71] and it is not too hard to show that there is a positive global supersolution of (D) of the form

\[
U(x, t) = w(t)\psi(x)
\]

provided \( w(0) \) is sufficiently small and \( p > 1 \). Standard arguments [115], [120], [134] then allow us to conclude that if

\[
u_0(x) < w(0)\psi(x)
\]
on \( D \), then \( u \) is a global solution. Therefore

\[
pc(D) = 1 \quad (D \text{ bounded.})
\]

In this case, the critical exponent also belongs to the global existence case. (The equation is linear and comparison with a suitable supersolution can be used to show this.)

As we remarked earlier, if for bounded domains we replace \( u^p \) by \( |x|^\sigma t^k u^p \), we still have \( pc = 1 \).

Suppose we change the problem slightly. For example, if we consider, for positive constants \( c, b \), the following initial boundary value problem on a bounded domain \( D \):

\[
\begin{align*}
&u_t = \Delta u + c \exp(bt)u^p, \quad x \in D, \quad t > 0, \\
&(BD) \quad u(x, t) = 0, \quad x \in D, \quad t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \partial D
\end{align*}
\]

then we have the following result of Meier [25] (BD = "bounded domain"): Theorem (BD). We have \( pc(b, D) = 1 + b/\lambda_1 \). More precisely,

A. If \( 1 < p < 1 + b/\lambda_1 \) then (BD) has no nontrivial, positive global solutions.

B. If \( p > 1 + b/\lambda_1 \), then (BD) has both nontrivial, global, small data solutions and solutions which blow up in a finite time.
In [20], it was shown that \( pc(b,D) \) belongs to the blowup case. This result also clearly illustrates the observation that, for bounded domains, larger domains are more unstable than smaller domains. (See the Introduction.)

A second result along these lines and in the spirit of the preceding section is also worth mentioning. In [30], Tsutsumi considered the following initial boundary value problem (again on a bounded domain):

\[
\begin{align*}
  &u_t = \sum_{i=1}^{N} (|u_{x_i}|^{q-2}u_{x_i})_{x_i} + u^p, \quad x \in D, \quad t > 0, \\
  &u(x,t) = 0, \quad x \in \partial D, \quad t > 0, \\
  &u(x,0) = u_0(x), \quad x \in D.
\end{align*}
\]

(Q)

Here the interest is again in nonnegative solutions. Tsutsumi proved the following interesting theorem:

**Theorem (Q).** We have \( pc(Q) = q - 1 \). That is

A. If \( p < q - 1 \), then for all nonnegative initial values in \( W^{1,q}_0(D) \), (Q) has nonnegative, nontrivial, global weak solution which is unique if \( q > N \).

B. If \( p > q - 1 \) and \( q > N \) or if \( N \leq q \) and \( q-1 < p < \left(\frac{(N+1)(q-1)+1}{q-N}\right) \), then there are both nontrivial, nonnegative global solutions and solutions which blow up in finite time.

The blowup result for \( p > q - 1 \) was also proved in [21].

When \( N = 1 \), this critical exponent is the same as the second critical exponent discovered by Galaktionov [8] for (SL) above \( \sigma + 1 = q - 1 \). This parallel between (Q) and (SL) persists (in a more rigorous form) between (PM) and (BPM) below.

Galaktionov [10], also considered the following Dirichlet problem (BPM = "bounded domain porous medium operator"):

\[
\begin{align*}
  &u_t = \Delta(u^{\sigma+1}) + u^p, \quad (x,t) \in D \times [0,T), \\
  &u(x,t) = 0, \quad x \in \partial D, \quad t > 0, \\
  &u(x,0) = u_0(x), \quad x \in D.
\end{align*}
\]

(BPM)

Again \( \sigma \geq 0 \) and \( p > 1 \).

He proved that \( pc(BPM) = 1 + \sigma \) was the critical exponent in the following sense.

**Theorem (BPM).** We have \( pc(BPM) = 1 + \sigma \). That is:

A. If \( 1 < p < 1 + \sigma \), then (BPM) always has global solutions for any nonnegative initial values with \( u_0^p \in H^1_0(D) \).

B. If

\[
1 + \sigma < p < \begin{cases} 
  \infty, & N = 1, 2 \\
  (\sigma + 1)\left(\frac{N+2}{N-2}\right), & N \geq 3
\end{cases}
\]

then there are small data, positive global solutions of (BPM).

If

\[
1 + \sigma < p < 1 + \frac{4}{N} + \sigma \left(1 + \frac{2}{N}\right)
\]
then there are nonglobal solutions of (BPM).

Notice that this critical exponent is the same as the second critical exponent for (PM) when $N = 1$.

The upper bound on $p$ in (B) can be relaxed [22]. Precisely, if $p > 1 + \sigma$, there are nonglobal solutions of (BPM). (Galaktionov also observed that this restriction could be removed by using arguments of [12].)

Notice that statement (A) in Theorem (BD) is of the same flavor as Fujita's original statement, while statement (A) in Theorems (Q), (BPM) is the complete reverse of Fujita's statement. That is, these latter statements are global existence assertions for "all" initial values.

The case $p = pc(BPM) = \sigma + 1$ is very interesting: We have the following, where again $\lambda_1$ is the first Dirichlet eigenvalue for the Laplacian on $D$:

If $\sigma = 0$, then all solutions are global.

If $\sigma > 0$, then [116] we have:

A. If $p = 1 + \sigma$ and $\lambda_1 < 1$ no solution is global (if $u_0 \neq 0$).

B. If $p = 1 + \sigma$ and $\lambda_1 > 1$ there exist global nontrivial solutions.

Galaktionov makes no assertion in the case $p = pc(BPM)$ and $\lambda_1 = 1$. However, Sacks [27] has shown that in this case $pc$ belongs to the global existence case. The results again reflect the principle that large domains ($\lambda_1 < 1$) are more unstable than small domains ($\lambda_1 > 1$).

For problems (Q) when $q = 2$ and (BPM) when $\sigma = 0$, we see that $pc = 1$ and we obtain the result for (D) when $D$ is bounded which we discussed at the beginning of this section.

1.4. Systems of equations. Until recently, nothing was known about systems from the point of view of critical exponents. This is not to say that there are no blowup results for systems of parabolic equations. See, for example, [15], [87], [88]. One system in particular, has been well studied:

(S) \[
\begin{align*}
\begin{cases}
u_t &= \Delta u + v^p \\
v_t &= \Delta v + u^q
\end{cases}
\end{align*}
\]

$x, t \in R^N \times (0, T)$

$u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \ u_0, v_0 \geq 0$ \quad $x \in R^N$.

See [75], [87], [88]. This (weakly coupled) system is the subject of [75] from the point of view of critical exponents. Here $p > 0, q > 0$, and $u_0, v_0$ are bounded and continuous. In [7] the following is proved where $\gamma = \text{max}\{p, q\}$.

THEOREM S. If $0 < pq \leq 1$, all solutions of (S) are global.

A. If $1 < pq \leq 1 + \frac{2}{N} (\gamma + 1)$, (S) has no global, nontrivial solution.

B. If $pq > 1 + \frac{2}{N} (\gamma + 1)$, (S) has both global solutions and nonglobal solutions.

This result should be extendable in several directions. For example, the arguments of [3], [19], [20] are amenable to modification for (S) in the case that $R^N$ is replaced by a cone with vertex at the origin. See [137]. The case of exterior domains is also discussed there.

More generally, it would be nice to have a result analogous to that of Meier (Theorem (GD)).

If, on the other hand, we are dealing with a strongly coupled system, as for example

(SS) \[
\begin{align*}
u_t &= \alpha \Delta u + \beta \Delta v + v^p \\
v_t &= \beta \Delta u + \delta \Delta v + u^q
\end{align*}
\]
THE ROLE OF CRITICAL EXPOSENTS IN BLOWUP THEOREMS 277

where \( \alpha, \delta > 0, \alpha \delta > \beta^2 \), no such analogous result seems to be known.

These and other related questions are currently under investigation by the author and his students.

In [87], [88], the system

\[
\begin{align*}
(PMS) & \quad u_t = \Delta (u^{\nu+1}) + u^p \\
& \quad v_t = \Delta (v^{\mu+1}) + u^q \\
& \quad u = v = 0, \\
& \quad u(x, 0) \geq u_0(x), \quad v(x, 0) = v_0(x), \quad x \in D
\end{align*}
\]

with \( u_0 \geq 0, v_0 \geq 0 \) is studied on a bounded domain, \( D \). (PMS = “porous medium system.”) Several interesting results are established. However, we shall only record one of them here (the analogue of Theorem (BPM)).

**Theorem (PMS).** Assume \( p > 1, q > 1, \mu > 0, \nu > 0 \).

A. If \( pq < (1 + \nu)(1 + \mu) \), then all solutions of (PMS) with continuous, bounded initial values are global.

B. If \( pq > (1 + \nu)(1 + \mu) \), then there exist both nontrivial global solutions and nonglobal solutions of (PMS).

When \( pq = (1 + \nu)(1 + \mu) \) the situation is not as clear as it was for (BPM). However, if \( p = \mu + 1 \) and \( q = \nu + 1 \) then

A. If \( \lambda_1 < 1 \), there are no nontrivial global solutions of (PMS).

B. If \( \lambda_1 > 1 \), all solutions of (PMS) are global.

Otherwise, when \( pq = (1 + \mu)(1 + \nu) \), in [88] it is shown that when the diameter of the domain is sufficiently small, all solutions are global. When the domain is sufficiently large, we might expect no global solutions (except \( u = v \equiv 0 \)) but the author could find no references to this fact.

Unlike the case of (BPM), the case \( \lambda_1 = 1 \) remains open.

It would be of interest to have a result for the Cauchy (initial value) problem for (PMS) analogous to that for (S).

2. **Nonlinear Schrödinger equations.** For \( u \neq 0 \), we can define \( u^p \) as \( |u|^{p-1}u \) or as \( |u|^p \). In this section we use the former definition. If we were to replace \( t \) by \( it \) (where \( i = \sqrt{-1} \)) in (F) and allow \( u \) to be complex valued, then we would be led to the following initial value problem:

\[
\begin{align*}
(NLS) & \quad iu_t + \Delta u + |u|^{p-1}u = 0, \quad x \in R^N, \quad t > 0, \\
& \quad u(x, 0) = u(x), \quad x \in R^N.
\end{align*}
\]

This is the initial value problem for a nonlinear Schrödinger equation. This formal observation is intended to motivate rather than trivialize this important equation. Indeed, there is an extensive literature on this equation, some of which we shall discuss shortly. The following result is a distillation of the works of several people [34]-[36], [43]. See also [70]. (Again there is no theorem analogous to that of Meier for these and the wave equation problems discussed below.)

**Theorem (NLS).** Let \( pnls(N) = 1 + 4/N \).

A. If \( 1 < p < pnls(N) \), then every (weak) solution of (NLS) which is initially in \( H^1(R^N) \) is global and remains in this Sobolev space for all \( t > 0 \).

B. If \( p \geq pnls(N) \) then both global and nonglobal solutions exist.
Recall that $H^1(R^N)$ can be thought of as the space of functions which, together with their gradients, are square integrable on $R^N$. The norm on this space is given by

$$\|u\|_1^2 = \int_{R^N} [|\nabla_x(u)|^2 + |u|^2] \, dx$$

where $u$ is in the space.

The first result of the theorem (with strict inequality) was established by Ginibre and Velo in [34], [35] (who proved a more general result) while the second statement is due to Glassey [36]. Tsutsumi [42] obtained an extension of Glassey's result to equations of the form:

$$iu_t + \Delta u + q(|u|)u - \frac{iau}{2} = 0.$$ 

The result for $p = p_{NLS}(N)$ is due to Weinstein [43]. Notice that the behavior at criticality is different for (NLS) as well.

For solutions of (NLS), it is fairly easy to see that the $L^2$ norm of the solution is constant in time. Moreover, the potential energy

$$\mathcal{H}(u) = \frac{1}{2} \int_{R^N} |\nabla_x u(x,t)|^2 \, dx - \frac{1}{p+1} \int_{R^N} |u|^{p+1} \, dx$$

is also constant. The blowup, when it occurs, is in the Dirichlet norm,

$$\mathcal{D}(u) := \int_{R^N} |\nabla_x (u(x,t))|^2 \, dx.$$ 

This suggests a "focusing" of the solution in the blowup case. Several authors are currently investigating this question [37], [38], [41], [44].

The conclusion of Theorem (NLS)(A), is quite different from that of Theorem (F)(A) and that of Theorem (W)(A). We might expect the critical exponents to be different but not the conclusion of the statements. There is an intuitive explanation for this apparent conundrum.

It is to be found in the rough statement that for small $p$, the wave-like nature of the solution of (NLS) introduces enough cancellation to prevent the growth of the supremum norm of the solution. We can see this more precisely from the form of the nonlinear integral equations that the solutions of (F) and (NLS) satisfy. Let, for $x \in R^N$,

$$G(x, t) = (4\pi t)^{-N/2} \exp \left( -\frac{|x|^2}{4t} \right)$$

denote the standard heat kernel and let

$$H(x, t) = G(x, it).$$

Because $G$ is the fundamental solution of the ordinary heat equation, $H$ is therefore the fundamental solution of the classical Schrödinger equation in a perfect vacuum. Solutions of (F) satisfy

$$u(x, t) = (G * u_0)(x, t) + \int_0^t (G * u^p)(x, t - s) \, ds$$

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THE ROLE OF CRITICAL EXPONENTS IN BLOWUP THEOREMS

where \( \ast \) denotes convolution in \( \mathbb{R}^N \). On the other hand, solutions of (NLS) satisfy

\[
   u(x, t) = (H \ast u_0)(x, t) + \int_0^t (H \ast |u|^{p-1}u)(x, t - s)ds.
\]

(These may be thought of as "variation of parameters" formulas.)

From these two equations, we see the following: In the first, the time integral has a strictly positive integrand (since \( G > 0 \)) if the initial values are positive. This integral could (and does) increase with time. Thus for small \( p \) the second term might be expected to swamp the first term as \( t \) gets large. In the second equation, the kernel \( H \) is oscillatory in time. Therefore, it is reasonable to expect that the condition \( p > 1 \) coupled with the oscillation could produce sufficient cancellation to prevent blowup. This cancellation is so strong that the range of \( p \) for which we have global existence for all data in (NLS) is larger than the range of \( p \) for which we have global nonexistence for all data for (F). This is not the whole story, so perhaps we should sketch the proof.

The proof of this result is worthy of discussing for two reasons. First, it is much different from that of Theorem (D) and Theorem (F) because we cannot rely on the maximum principle and the positivity of the solution. Second, the result for (B) when \( p = p_{\text{NLS}} \) is very sharp in the sense that there is a number \( M > 0 \) such that whenever \( \|u_0\|_{L^2} \leq M \), then \( u \) is global.

Our discussion is taken from [34], [36], [43]. Let \( \sigma = (p - 1)/2 \). It is easy to see that \( \mathcal{N}(u) = \mathcal{N}(u_0) \) and \( \mathcal{H}(u) = \mathcal{H}(u_0) \). (\( \mathcal{N}(u) \) is the \( L^2 \) norm of \( u \).)

To prove (A) it suffices to establish an a priori bound on

\[
   \|u(t)\|_{H^1} \equiv \int_{\mathbb{R}^N} [\|u\|^2 + \|\nabla u\|^2] \, dx
\]

which is independent of \( t \) when \( N\sigma < 2 \) [34]. Now there is, if \( 0 < \sigma < 2/(N - 2) \), a positive constant \( C = C(\sigma, N) \) such that

\[
   \|f\|_{L^2}^2 \leq C^2 \|\nabla f\|_2^2 \|f\|_2^{2+\sigma(N-2)}
\]

if \( f \in H^1(\mathbb{R}^N) \). (This is a standard Sobolev inequality. See [71], [94], for example.) \( (\|f\|_q = L^q \text{ norm of } f. ) \) Then from \( \mathcal{H}(u) = \mathcal{H}(u_0) \) we have

\[
   \|\nabla u\|_2^2 \leq \mathcal{H}(u_0) + \frac{C^{p+1}}{p+1} \|\nabla u\|_2^2 \|u\|_{L^2}^{2+\sigma(N-2)}.
\]

Now, using the facts that \( \sigma N < 2 \) and \( \mathcal{N}(u) = \mathcal{N}(u_0) \) we have

\[
   \|\nabla u\|_2^2 \leq \mathcal{H}(u_0) + \frac{2C^{p+1}}{p+1} \|u_0\|_2^{2+\sigma(N-2)} \|\nabla u\|_{L^N}^\sigma
\]

and this implies \( \|\nabla u\|^2 \leq M \) for some \( M > 0 \) and all time. Notice that \( M = M(C, u_0, \sigma) \).

To prove part (B) consider the case \( N\sigma = 2 \). Then we see that if \( \|u_0\|_2 \) is so small that

\[
   1 > \frac{2C^{p+1}}{p+1} \|u_0\|_2^{2(n-1)/N}
\]

we again obtain an a priori time independent bound on the \( H^1 \) norm. Weinstein gives a somewhat better bound but this conveys the flavor of his result. He shows (using the best or smallest constant for (\( *) \)) that global existence holds in the critical case if...
\[ \|u_0\|_2 < \|\psi\|_2 \] where \( \psi \neq 0 \) forces equality in (\( * \)). (\( \psi \) is the minimizer of the quotient
\[ \|\nabla f\|_{L^2}^{N\sigma} \|f\|_{L^2}^{2+\sigma(N-2)} / \|f\|_{L^2}^{2+2\sigma} \].
To prove the rest of part (B), we prove a simplified version of Theorem 4.2 of [43]
which was due to Glassey.

We first observe that if \( |x|f \) and \( \nabla f \) are in \( L^2(\mathbb{R}^N) \) then
\[ \|f\|_2^2 \leq \frac{2}{N} \|\nabla f\|_2 \|x|f\|_2. \]
This follows from integrating \( \text{Re} \int_{\mathbb{R}^N} \vec{x} \cdot \vec{\nabla} (|f|^2) \, dx \) by parts and applying Schwarz's
inequality. ("Re" = "real part.") Now, using the conservation law \( \mathcal{H}(u) = \mathcal{H}(u_0) \), a
routine calculation shows that
\[ \frac{d^2}{dt^2} \int_{\mathbb{R}^N} \rho|u(x, t)|^2 \, dx = 2\mathcal{H}(u_0) + \frac{2N}{(p+1)} \left( \frac{2}{N} - \sigma \right) \int_{\mathbb{R}^N} |u|^{p+1} \, dx. \]
So if \( \mathcal{H}(u_0) < 0 \) and \( \sigma \geq 2/N \), we must have that \( F(t) \equiv \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \) satisfies
\[ F(t) \leq \mathcal{H}(u_0)t^2/2 + tF'(0) + F(0). \]
Consequently, \( F \) vanishes in finite time. However,
\[ \|u_0\|_2^2 = \|u\|_2^2 \leq \frac{2}{N} \|\nabla u\|_2^2 F(t). \]
Therefore, \( \|\nabla u\|_2^2 \) blows up in finite time.

The idea of showing that some functional of the solution is concave in order to
show blowup (global nonexistence) was also used in [12] for parabolic problems where
it was shown (for (D) for example) that if
\[ \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 \, dx - \frac{1}{(p+1)} \int_{\Omega} w_0^{p+1} \, dx > 0, \]
then \( u \) could not be global. A similar argument can be used to show that whenever
the potential energy is negative, solutions of the initial value problem for
\[ u_{tt} = \Delta u + |u|^{p-1}u \]
will also blow up in finite time. See [49].

For all three problems (NLS), (G) and the aforementioned initial value problem,
blowup is possible even when \( \mathcal{H}(u_0) > 0 \) (under some further restrictions).

In [45] Weinstein remarks that Theorem (NLS) can be extended to the following
initial value problem:
\[ (\text{NLS})_m \quad iu_t = (-\Delta)^m u - |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad t > 0, \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \]
where \( m \) is a positive integer. The critical exponent is then
\[ p_{\text{nlsm}}(N, m) = 1 + 4m/N. \]
The author is not aware of any similar results for higher-order versions of (F) or of
problem (W) below.

3. Nonlinear wave equations. Perhaps the first attempt to extend Fujita's
result (Theorem (F)) to other types of initial value problems was made by F. John
and later, by others [46], [47], [50]–[53], who considered the following initial value problem where now $u(x, t)$ is a real valued function:

$$u_{tt} = \Delta u + |u|^p, \quad x \in \mathbb{R}^N, \quad t > 0$$

(W)

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N,$$

$$u_t(x, 0) = v(x), \quad x \in \mathbb{R}^N.$$  

Notice that now $u^p$ has been extended to be an even function of $u$ for real $u$. The initial values are taken to be infinitely differentiable with compact support so that the corresponding local (in time) solution is at least twice continuously differentiable in all variables and the solution is classical.

Let $pw(N)$ denote the larger root of the quadratic equation

$$(N - 1)p^2 - (N + 1)p - 2 = 0.$$  

That is

$$pw(N) = \frac{(N + 1) + \sqrt{(N + 1)^2 + 8(N - 1)}}{2(N - 1)}.$$  

Strauss [40] conjectured that $pw(N)$ is the critical number for (W). As Sideris [53] mentions, the main technical difficulty in proving blowup for small data solutions is the fact that the Riemann function for the wave equation is no longer positive in more than three dimensions. Sideras gets around this difficulty by showing first that for $N \geq 4$, the time average of the Riemann function results in a positive operator. This is also the main difficulty to be overcome in considering the global existence question for large $p$. The following result is a synopsis of the works of several people.

**Theorem (W).**

A. If $1 < p < pw(N)$ or if $p = pw(N)$ and $N = 2$ or $N = 3$, (W) has no nontrivial global solutions.

B. If $N = 2, 3$ and $p > pw(N)$, then there exist nontrivial global small data solutions.

Statement (A) with strict inequality was proved by Sideris [53] for any $N$. It was established by F. John when $N = 3$, [48] and by Glassey when $N = 2$, [47]. Glassey also established the second statement for $N = 2$ while John established it for $N = 3$ (in the same works). The case of equality (when $N = 2, 3$) was established by Shaeffer [52].

Let

$$pl(N) = \frac{(N^2 + 3N - 2)}{N(N - 1)}.$$  

In [55] it was shown that if $p > pl(N)$, then there are nontrivial global small data solutions of (W). However, $pl(N) > pw(N)$ so that at least as far as the open literature is concerned, for $N > 3$, it is not known whether or not $pw(N)$ is the critical exponent for (W). However, Sideris informs me that he has shown that for $p > pw(N)$ but $p - pw(N)$ is small, (W) admits global, nontrivial, radial solutions.

Notice that we have the following inequalities:

$$pw(N) > pc(N) \quad \text{for} \quad N = 2, 3, \ldots$$  

with

$$pw(1) = pc(1) = 3.$$  

These inequalities tell us that for $N > 1$ (assuming that $pw(N)$ is indeed the critical number for $(W)$), our intuition that $(W)$ is a more unstable problem than $(F)$ is confirmed.

4. Concluding remarks. A more detailed study of the set of solutions of the various initial and initial boundary value problems discussed above would begin with an investigation into the question of the existence of stationary solutions or time periodic solutions of these or related initial or initial boundary value problems. Such a study is already well underway. See, for example, [3], [62], [90], [109]–[112], [114], [116], [120]. This would be followed by an examination of the questions of the stability or instability of such solutions (and in which norms we have such stability and instability) and whether or not unstable solutions blow up in finite time. Although there is already a growing body of literature on such questions, including the recent literature on the general topic of inertial manifolds—not only for the problems mentioned above, but also for many other interesting problems, some of which arise in the applications which we briefly mentioned in the introduction—research in this area is far from complete.

Many problems must still be studied on a case-by-case basis. This is particularly true in the case of nonlocal problems and problems which involve convection in the dynamical equations as well as for systems of equations. Also of more than passing interest are dynamical problems for which the boundary conditions contain nonlinear terms or for which the source terms are singular at a finite point (quenching problems). The literature on such problems comes under the general heading of dynamical systems and is quite extensive.

In addition to an examination of the role of stationary or time periodic solutions play in evolution equations, there are other interesting questions which arise in the study of such equations. For example, when blowup occurs, just what is the nature of the blowup set? That is, does it occur at a single point in the domain or can we have blowup at every point in the domain (complete blowup, a term coined in [63]). How does the set of blowup points depend upon the initial values? What is the nature of the approach to “infinity” as the time parameter approaches its limit (finite or infinite)? In the case of systems of equations or equations in which there are nonlocal terms or convection terms, there is also relatively little known.

In particular the interaction between the terms that cause pointwise blowup and those that cause finite time breakdown in the derivatives (shock formation) is incompletely understood. This question is of some applied interest. See [73], [119], [123]–[129], [135] and the references cited therein. Here again questions of blowup behavior are little explored when the nonlinear terms occur in the boundary condition. Here, too, there is a rapidly exploding literature, although the focus of most current research on these questions appears be mostly on nonlinear parabolic equations and systems of equations.

The topics and questions discussed in the preceding paragraphs are more appropriate as subjects for books and monographs rather than for short survey articles such as this. Any attempt on the author’s part to give the reader a complete survey of the literature on these topics (or even one which pretends to be reasonably complete) would not only be pretentious but also foolhardy. Therefore, he has decided to simply give a few representative references to the literature here and again offer his apologies for the omissions.

For the role that stationary solutions play in dynamical problems, see [58], [96]–[99], [108]–[110], [113], [120], [130] for example. The nature of blowup has been dis-
cussed by several workers. We refer the reader to [61], [63], [84], [91], [92]. These works will serve as starting points for further readings for the interested reader. For problems for which nonlinear terms appear in the boundary condition, see [59], [74], [103], [105], [106]. The author surveyed the literature on quenching problems in [101]. However, he is preparing a survey of recent advances in this topic since interest in it has certainly not been quenched [103].

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THE ROLE OF CRITICAL EXPONENTS IN BLOWUP THEOREMS


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