Direct and Inverse Problems Pertaining to the Scattering of Elastic Waves in the Rayleigh (Long Wavelength) Regime

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Abstract
It is well known that in the scattering of elastic waves from localized inhomogeneities the scattering amplitude \( A \) is proportional to the square of the frequency \( \omega \) in the Rayleigh (long wavelength) regime, i.e., \( A \propto \omega^2 + \ldots \). This talk deals with the problem of (1) extracting \( A^2 \) from experimental scattering data, (2) calculating \( A^2 \) for an assumed scatterer and (3) deducing the properties of the scatterer from a set of values of \( A^2 \) measured for various transducer configurations. A review of experimental and theoretical results for \( A^2 \) will be presented for the case of spheroidal voids and the remaining discrepancies between the two kinds of results will be discussed. The inverse problem (i.e., deducing the scatterer properties from the scattering measurements) will be discussed in detail. The probabilistic inverse problem, which provides the appropriate framework for the interpretation of real data, will be covered at greater length. In the case in which it is assumed that the scatterer is an ellipsoid void, whose size, shape and orientation are unknown a priori, a number of computational results involving best estimates and associated measures of significance will be given. Analogous results will be derived for parameters related to fracture mechanics.

Keywords
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DIRECT AND INVERSE PROBLEMS PERTAINING TO THE SCATTERING
OF ELASTIC WAVES IN THE RAYLEIGH (LONG WAVELENGTH) REGIME

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ABSTRACT

It is well known that in the scattering of elastic waves from localized inhomogeneities the scattering
amplitude $A$ is proportional to the square of the frequency $\omega$ in the Rayleigh (long wavelength) regime,
i.e., $A = A_0 \omega^2 + ...$. This talk deals with the problem of (1) extracting $A_0$ from experimental scattering
data, (2) calculating $A_0$ for an assumed scatterer and (3) deducing the properties of the scatterer from a
set of values of $A_0$ measured for various transducer configurations. A review of experimental and theoretical
results for $A_0$ will be presented for the case of spheroidal voids and the remaining discrepancies
between the two kinds of results will be discussed. The inverse problem (i.e., deducing the scatterer
properties from the scattering measurements) will be discussed in detail. The probabilistic inverse prob­
lem, which provides the appropriate framework for the interpretation of real data, will be covered at
greater length. In the case in which it is assumed that the scatterer is an ellipsoid void, whose size,
shape and orientation are unknown a priori, a number of computational results involving best estimates
and associated measures of significance will be given. Analogous results will be derived for parameters
related to fracture mechanics.

INTRODUCTION

A number of techniques have recently emerged
for the determination of fracture related
parameters of defects from measured ultrasonic
fields. One of the newest of these, and perhaps
the most unexpected, is the observation that
considerable information can be derived from
ultrasonic scattering measurements in which the
wavelength is large with respect to the flaw
size. From the familiar concept of resolution of
an image, one would expect to obtain little useful
information under such conditions. However, the
elastic nature of the ultrasound-flaw interaction
leads directly to results that are quite in
contrast to this overly simple point of view. For
elementary, the long wavelength scattering of elastic
waves depends upon 22 parameters representing
properties of a general scatterer. This is very
different from the situations in the scalar wave
scattering case.

Here, we summarize recent progress on the
demonstration of the feasibility and usefulness of
low frequency scattering of elastic waves in the
context of nondestructive evaluation. Here we
attempt a partial "vertical integration" to show,
at least theoretically, that the results of such
measurement can be interpreted in terms of the
central concepts of fracture mechanics. This gives
an indication that the remaining steps in an
overall NDE decision process could be taken without
significant difficulties.

Before considering the detailed results, it
is important to ask: What advantages would such an
approach have relative to other approaches for
defect characterization? The following points can
be made in its favor:

(1) The theory of the scattering of elastic
waves at low frequencies is well
established for the case of ellipsoidal
inclusions and voids. Thus, the inverse
scattering problem for this class of
scatterers is quite tractable. At
higher frequencies, this is not the
case.

(2) Low frequency measurements are sensitive
only to the overall shape and size of the
defect and not to small textural
details. This is also the information
of importance in fracture.

(3) Low frequency scattering measurements
are particularly sensitive to cracks
compared with other scatterers (e.g.,
inclusions of the same volume or even
the same area). In particular, the
scattering measurements are signifi­
cantly more sensitive to a large crack
than to a number of small cracks with
the same total area.

(4) The elastic processes involved in low
frequency scattering are intimately
related to those involved in the early
stages of the fracture process (at least
in most metals) as has been pointed out
by Budiansky and Rice. A further
advantage is that the relevant stress
intensity factor is proportional to the
$1/6$ power of the scattering amplitude,
yielding thereby a substantial reduction
of variance in the estimation process, a
fact emphasized by Kino.

(5) Another advantage is the fact that long
wavelength scattering is insensitive to
the position of the scatterer and thus
precise location of the scatterer is
unimportant.

Of course, there are also disadvantages. Some
of these are:

(1) Relatively complex post-experiment data
processing is involved in deducing the
low frequency scattering characteristics.
However, the main problems appear
not yet confronted, is the isolation of each domi­
nant scatterer from competing scatterers
in taking the low frequency limit.
It is useful to direct the reader's attention, at least temporarily, to the overall NDE decision process of which the present topic is a part. In Fig. 1 we illustrate a typical NDE decision process in the case of a metal in which a possible failure process involves conventional fracture mechanics. A similar decision structure has been discussed by Evans at this meeting.

Starting at the upper left we show the NDT apparatus (involving long wavelength longitudinal-to-longitudinal scattering in the present discussion) in bilateral interaction with the test piece. The results of measurement are fed into another box whose function is to provide a good estimate of the state $x$ of a particular scatterer. The state is a set of parameters that provides a sufficiently good characterization for the purpose at hand. This box also provides estimates of the standard deviations (a posteriori) of the components of $x$. Although it is not shown, this estimation process also involves the a priori probability of the state. The final stage of the decision process on the first row of the block diagram is involved in producing corresponding estimates of a relevant fracture mechanics parameter and its standard deviation (a posteriori).

The next row of the block diagram involves calculation of the probability of failure for a given state $x$ and the final collation of probabilistic inputs (relating to failure, NDT measurement and a priori probability of the state). The final stage of the decision process on the first row of the block diagram is involved in producing corresponding estimates of a relevant fracture mechanics parameter and its standard deviation (a posteriori).

The subsequent discussion deals only with the first row of boxes in the figure.

**THEORY OF SCATTERING OF ELASTIC WAVES AT LOW FREQUENCIES**

The longitudinal-to-longitudinal scattering of elastic waves from an arbitrary scatterer is described by the scalar scattering amplitude $A = A(\hat{e}_s, \hat{e}_i, \omega)$ where $\hat{e}_s$ is the scattered (observer) direction $\hat{e}_i$ is the incident direction, and $\omega$ is the frequency (expressed in radians per unit time). This scattering amplitude can be expanded in a power series in $\omega$ in the following form:

$$A = A_0^2 + A_2^2 + A_4^2 + A_6^2 + \ldots$$  \hspace{1cm} (1)

where the $A_n = A_n(\hat{e}_s, \hat{e}_i)$. The vanishing of $A_0$ and $A_2$ is a general property of localized scatterers. If the scatterer has inversion symmetry about the origin, then $A_2$ also vanishes, but this question need not concern us here. The absolute value of $A$ can also be expanded in powers of $\omega$ but here only even powers will enter, namely

$$|A| = a_0^2 + a_2^2 + a_4^2 + \ldots$$  \hspace{1cm} (2)

where, of course, $a_0 = |A_2|$. In Fig. 2 these relationships are illustrated for the case of longitudinal-to-longitudinal backscatter from a spherical $B_4C$ inclusion in a SiC matrix.

In the experiments only the absolute magnitude $|A_2|$ is yielded. Since it is known theoretically that $A_2 > 0$ for spheroidal voids, the absence of the sign of $A_2$ in the experimental output is of no consequence. However, this may be a serious lack in the case of more general scatterers.

In Fig. 3 we illustrate the logical equations involved in obtaining experimental values of $A_2$.
We will not give a detailed discussion here of the theoretical treatment of the low frequency scattering of elastic waves from general spheroidal inclusions. It will suffice here to present a description of the input and output of the computer program LOWSCATEL. Actually, the input is presently given in a form suitable for general ellipsoidal inclusions of isotropic material even though the internal algorithm has not yet been extended beyond the spheroidal case. In setting up a framework for the description of the input, we use a Cartesian coordinate system \((x,y,z)\) with the associated unit vectors \(\hat{e}_x, \hat{e}_y, \hat{e}_z\). The principal axes of the ellipsoid are defined by the mutually perpendicular vectors \(\hat{u}, \hat{v}, \hat{w}\) and the corresponding semi-axis lengths are directed by \(a, b, c\). The material properties of the host material are the density and the two Lame constants \(\rho, \lambda\) and \(\mu\), and the corresponding properties of the inclusion are denoted by \(\rho^+, \lambda^+, \mu^+\). In the case of a void we set \(\rho = \rho^+\), \(\lambda = -\lambda^+\), and \(\mu = -\mu^+\). In the case of a void we set \(\rho = \rho^+\), \(\lambda = -\lambda^+\), and \(\mu = -\mu^+\). Finally, we must include the set of incident and scattered wave directions defined by the unit vectors \(\hat{e}_1^i, \hat{e}_s^i\), respectively, in order to specify the configurations of interest. The output of the computer program is simply \(I_{22} = I_{22}(\hat{e}_1, \hat{e}_s)\) for the case of longitudinal-to-longitudinal scattering of elastic waves.

In the particular cases of interest here, we took
\[
\hat{e}_1 = \hat{e}_x, \quad \hat{e}_s = \hat{e}_y, \quad \hat{w} = \hat{e}_z
\]
\(a = 0.04\) cm and \(c = 0.02\) cm
\(\rho = 4.42\) gm cm\(^{-3}\), \(c_t = 0.634\) cm \(\mu\)sec\(^{-1}\)
\(c_t = 0.303\) cm \(\mu\)sec\(^{-1}\)

The \(\lambda\) and \(\mu\) for the host material (titanium) were determined from the above values of the host material longitudinal and transverse propagation velocities, \(c_t\) and \(c_t\), respectively. The selected sets of incident and scattered directions will be indicated in the next section.

**COMPARISON OF THEORY AND EXPERIMENT**

We turn now to a comparison of the theoretical and experimental results. Both pitch-catch and pulse-echo types of scattering measurements are considered. In Figs. 4 and 5 the geometries and associated notation pertaining to these types are presented.

Figure 4 shows the geometrical setup in which the incident beam propagates in the negative \(z\)-direction (the \(z\)-axis is chosen as the axis of symmetry of the spheroid). All of the scattering (i.e., observer) directions chosen in the experiments are co-planar with each other and with the incident direction (i.e., there is a single scattering plane common to all experiments). The scattered direction is defined by the polar angle \(\theta\) as shown. Clearly, in the case of a spherical void, all incident directions are equivalent.

In Fig. 5 the geometry of the pulse-echo type of measurement is shown. Here the common angular position of the "points" of entry and exit of the incident and scattered waves, respectively, is defined by the polar angle \(\theta\). As in the previous case the measurements are confined to a single scattering plane.

We first discuss the pitch-catch measurements obtained by Tittmann and Morris. The absolute value of deconvolved experimental results (appropriately desensitized) were extrapolated from a range of frequencies for which they were valid, to low frequencies to obtain a quantity that is proportional to \(|A_2|\). The proportionality factor enters because of the calibration experiments used to normalize the data for variation in the transducer efficiency have slightly different diffraction properties than the scattering measurement. Assuming that the proportionality factor is the same for all experiments, we can obtain this factor by comparison of a set of control measurements with theory. For the latter, scattering from a spherical void of 400 \(\mu\)m diameter was chosen. The results are presented in Table I for the configurations of Fig. 4 corresponding to...
Table 1. Scattering from a spherical void determination of experimental factor $\beta$.

<table>
<thead>
<tr>
<th>$\theta$ (deg)</th>
<th>$A_0$ (cm $\mu$sec$^{-1}$)</th>
<th>$\beta A_0$</th>
<th>$\beta$</th>
<th>$A_0$ (cm $\mu$sec$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.125 x 10$^{-4}$</td>
<td>7.1 x 10$^{-5}$</td>
<td>3.17 x 10$^{-4}$</td>
<td>0.25 x 10$^{-4}$</td>
</tr>
<tr>
<td>20</td>
<td>0.200</td>
<td>6.8</td>
<td>3.06</td>
<td>0.206</td>
</tr>
<tr>
<td>30</td>
<td>0.189</td>
<td>6.2</td>
<td>3.05</td>
<td>0.188</td>
</tr>
<tr>
<td>40</td>
<td>0.171</td>
<td>5.7</td>
<td>3.00</td>
<td>0.172</td>
</tr>
<tr>
<td>50</td>
<td>0.154</td>
<td>5.4</td>
<td>2.85</td>
<td>0.163</td>
</tr>
</tbody>
</table>

$\theta$ = 35°, 45°, 55°, 65°, and 75°. The results are denoted by $\beta A_2$ where $\beta$ is the experimental proportionality factor. Actually, as stated before, the absolute value $|\beta A_2|$ is measured, however, since $A_2$ is known to be positive, the absolute value symbol $|\beta|$ will be dropped. The experimental values $\beta A_2$ given in the third column of Table 1 are divided into the theoretical values of $A_2$ given in the second column to yield the values of $\beta$ given in the fourth column. The average value of these last results is 3.02 x 10$, a value used for converting all experimental results into meaningful values of $A_2$ expressed in the units: cm $\mu$sec$^{-1}$.

The comparison of the experimental and theoretical values of $A_2$, given in the second and fifth columns of Table 1, is shown graphically in Fig. 6. The agreement of the sample-average values is, of course, tautological. However, the agreement of trends, which is not tautological, can be seen to be quite satisfactory.

It is worthy of note that the experimental proportionality factor $\beta$ can be determined theoretically with the result:

$$\beta = \frac{R}{2\pi}$$

in which $R$ is the radius of the sphere in Fig. 4 and where a factor of $(2\pi)^2$ in the denominator comes from the conversion of frequency in cycles per unit time to radians per unit time. Since $R = 1.1$ in $= 1.79$ cm, we obtain $\beta = 0.035$ cm which compares surprisingly well with the experimental value 0.003.

In Table II we give the experimental results $\beta A_2$ and the corrected results $A_2$ for the configurations of Fig. 1 with $\theta = 35°, 45°, 55°, 65°, 75°, 85°$, and $90°$ for an oblate spheroidal void. It is assumed that the same value of $\beta$ applies. The comparison with the theoretical values of $A_2$ is very good if the points for $\theta = 35°$ and 45° are omitted. The rather significant deviations at the latter values of $\theta$ are believed to involve a substantial systematic component which is presumably due to the spurious propagation effects discussed by Elsley. The comparison is also shown graphically in Fig. 6.

We turn now to a discussion of the pulse-echo measurements obtained by Elsley and Nadler. Their results are compared with theory in Table III and Fig. 7. The comparison is surprisingly good with a relative error of only 3.9%. It must be emphasized that we have used the old value of the experimental factor $\beta$, namely 0.0303. An adjustment of this value could bring down the relative error to 2.8%. It is clear that these measurements are less vulnerable to the kinds of systematic error involved in the earlier pitch-catch measurements.

Table 2. Scattering from a spheroidal void (configurations of Fig. 4(a))

<table>
<thead>
<tr>
<th>$\theta$ (deg)</th>
<th>$A_0$ (cm $\mu$sec$^{-1}$)</th>
<th>$\beta A_2$</th>
<th>$A_2$ (cm $\mu$sec$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.402 x 10$^{-4}$</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>35</td>
<td>1.197</td>
<td>47.8 x 10$^{-4}$</td>
<td>1.45 x 10$^{-4}$</td>
</tr>
<tr>
<td>45</td>
<td>1.085</td>
<td>40.3</td>
<td>1.22</td>
</tr>
<tr>
<td>55</td>
<td>0.968</td>
<td>31.0</td>
<td>0.94</td>
</tr>
<tr>
<td>65</td>
<td>0.856</td>
<td>28.8</td>
<td>0.87</td>
</tr>
<tr>
<td>75</td>
<td>0.759</td>
<td>28.1</td>
<td>0.85</td>
</tr>
<tr>
<td>85</td>
<td>0.650</td>
<td>22.0</td>
<td>0.67</td>
</tr>
<tr>
<td>90</td>
<td>0.558</td>
<td>21.2</td>
<td>0.64</td>
</tr>
</tbody>
</table>

Fig. 6 Scattering in configuration of Fig. 4 for spherical void and spheroidal void.
Table 3. Scattering from a spheroidal void pulse-echo case

<table>
<thead>
<tr>
<th>Theory</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_x ) (cm ( \mu )sec(^{-1} ))</td>
<td>( \Delta^{-1} A_y ) (dB)</td>
</tr>
<tr>
<td>0</td>
<td>1.402 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>15</td>
<td>1.374</td>
</tr>
<tr>
<td>30</td>
<td>1.296</td>
</tr>
<tr>
<td>45</td>
<td>1.189</td>
</tr>
<tr>
<td>60</td>
<td>1.080</td>
</tr>
<tr>
<td>75</td>
<td>0.999</td>
</tr>
<tr>
<td>90</td>
<td>0.969</td>
</tr>
</tbody>
</table>

Fig. 7 Pulse-echo measurements of scattering from spheroidal void.

Fig. 8 The inverse scattering procedure.

We will pursue a probabilistic approach in which we start with a statistical ensemble of scatterer properties and measurement errors and then remove the members inconsistent with the scattering data obtained from the measurements. The best estimates of the geometrical properties of the spheroidal void are then the average or most probable values of these properties in the resultant reduced ensemble. The a posteriori variances of these properties (i.e., the variances in the reduced ensemble) are used as a measure of significance or, equivalently, the "leverage" exerted by the scattering data on the properties of the scatterer.

Let us model the possible results of the scattering measurement (assumed in all cases to be longitudinal-to-longitudinal) by the stochastic expression:

\[
y_n = f_n(x) + \nu_n, \quad n = 1, \ldots, N
\]

where \( y_n \) is a possible measured value and \( \nu_n \) the measurement error. The function \( f_n(x) \) is given by

\[
f_n(x) = |A_2(\hat{e}_n^x, \hat{e}_n^y; x)|
\]

where \( A_2(\hat{e}_n^x, \hat{e}_n^y; x) \) is the coefficient of \( \omega^2 \) in the \( \omega \)-expansion of the longitudinal-to-longitudinal scattering amplitude \( A(\hat{e}_n^x, \hat{e}_n^y, \omega; x) \) as discussed in Section II. The unit vectors \( \hat{e}_n^x \) and \( \hat{e}_n^y \) define the directions of the incident and scattered longitudinal elastic waves. The subscript \( n \) added to these vectors denotes the configuration used in the \( n \)th measurement. The vector \( x \) represents the geometrical properties of the void. In the spheroidal case we assume that the semi-axis lengths are denoted by \( a, b \) and \( c \) and that the axis of symmetry is given by

\[
\hat{w} = \hat{e}_x (1 - \gamma_x^2 - \gamma_y^2)^{1/2}
\]
where \( \hat{e}_x, \hat{e}_y, \) and \( \hat{e}_z \) are the unit vectors in the \( x, y \) and \( z \) directions and where \( \gamma_x \) and \( \gamma_y \) are the direction cosines associated with the \( x \) and \( y \) directions as shown in Fig. 9. Thus the vector \( x \) is given by

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a \begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix}
\]

(4)

Fig. 9 Characterization of spheroidal geometry.

It is to be stressed that the Cartesian coordinates \((x,y,z)\) are defined in the laboratory frame of reference and have no necessary relation to the axis of symmetry of the spheroid. It is hoped that the state vector \( x \) and the Cartesian coordinate \( x \) will not be confused.

The definition of the stochastic model is completed by the specification of the a priori statistical properties of the state vector \( x \) and \( v_n \), and is characterized by the probability density (p.d.) \( P(x) \). The measurement errors \( v_n \) are assumed to the Gaussian random variables with properties

\[
E v_n = 0 \quad \text{and} \quad \text{cov}(v_n, v_{n'}) = \sigma_v^2 \delta_{nn'}
\]

where \( E \) is the averaging (or expectation) operator in the a priori sense.

Whatever is chosen for the criterion of performance of the estimation process, we must calculate the observationally conditioned p.d. of \( x \) given by

\[
P(x|y) = \frac{P(y|x)P(x)}{P(y)}
\]

(7)

where

\[
y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_N \end{pmatrix}
\]

(8)

and where

\[
P(y) = \int dx P(y|x)P(x)
\]

(9)

From the previous assumptions we obtain

\[
\log P(y|x) = - \frac{1}{2\sigma_v^2} \psi + \text{const.}
\]

(10)

where

\[
\psi = \sum_{n=1}^{N} (y_n - f_n(x))^2
\]

(11)

is the sum of squares of the deviations of the \( y_n \) from the functions \( f_n(x) \).

Let us consider the mean square criterion of optimality

\[
\epsilon = E(\hat{x} - x)^T W(\hat{x} - x)
\]

(12)

where \( \hat{x} = \hat{x}(y) \) is the estimator of \( x \) and where \( W \) is a symmetric positive-definite matrix of weighting factors. The minimum of \( \epsilon \) with respect to the functional form of \( \hat{x}(y) \) is given by a posteriori average

\[
\hat{x}(y) = E(x|y) = \int dx x P(x|y)
\]

(13)

where \( P(x|y) \) is given by (7). We will use the a posteriori covariance matrix defined by

\[
\text{Cov}(x|y) = E(x x^T|y) - E(x|y)E(x^T|y)
\]

(14)

as the measure of confidence or significance. This tells us how much the a priori p.d. \( P(x) \) is "narrowed down" by the experimental factor \( P(y|x) \) in (7). In other words, how much "leverage" the experimental data has on the scatterer parameters defined by \( x \). The measuring of a posteriori variances (or equivalently a posteriori standard
duration) is illustrated in Fig. 10 with a scalar state $a$.

In the explicit computations we made several approximations. The first was approximating the \textit{a posteriori} average by the \textit{a posteriori} mode, i.e.,

$$\hat{x} = E(x|y) = x_{\text{max}}$$

where $x_{\text{max}}$ is the value of $x$ for which $P(x|y)$ is a maximum. Alternatively, we could have used a different optimality criterion, in terms of which the mode is exact. The second approximation is the computation of the \textit{a posteriori} covariance by expanding the $P(x|y)$ defined by (11), in a power series about the point $x_{\text{max}}$ and ignoring terms higher than quadratic.

The first set of estimates were made with pitch-catch data as inputs. We considered both noiseless theoretical data and actual experimental data as summarized in Table II.

In Table IV we present estimates of $a$, $c$, $Y_x$, and $Y_y$ based upon the above experimental data. For the same of verification, we also present estimates based upon theoretical noiseless test data. The estimates based on actual experimental data compare surprisingly well with the exact values, even in spite of the effects of systematic error.

In Table V are given the \textit{a posteriori} standard deviations of the scatterer parameters (i.e., the square roots of the diagonal elements of $\text{Cov}(x|y)$ defined by (14)), appropriately normalized. In the case of the semi-axis lengths $a$ and $c$ we divide their respective standard deviations by their best estimates. In the case of the dimensionless direction cosines $Y_x$ and $Y_y$ such normalization seems to be unnecessary. In these computations, the r.m.s. experimental error is assumed to have the value $\sigma_{\text{e}} = 10^{-5}$ (corresponding to an approximate relative r.m.s. relative error of 10%). Since the experimental data, synthetic or actual, is confined to a single scattering plane (assumed to be the $xz$-plane in our coordinate system), the standard deviations of $Y_y$ (the cosine of the angle between $y$-axis and the symmetry axis of the spheroid) is omitted because the approximation involved in its computations is not valid.

It may appear inconsistent to present \textit{a posteriori} standard deviations based on noiseless theoretical test data. It must be pointed out that the standard deviations of $x$ are actually based upon the model (1) with the associated assumptions (5) and (6) giving the statistical nature of the experimental errors. The variance of the experimental errors is determined from an independent comparison of experiment with theory and not from the input data used in the estimation procedure.

We turn next to a consideration of estimates based upon pulse-echo data. Here we use the actual experimental data and noiseless theoretical test data summarized in Table III of the last section. In Table VI we present estimates of the scatterer parameters $a$, $c$, $Y_x$ and $Y_y$ for both kinds of input data. The agreement between the estimates based on actual experimental data and the exact parameter values is unbelievably good and must be regarded as partially accidental. But it is perhaps also due to the fact that it appears, as we will discuss later, that the pulse-echo data has considerably
In Table VII we give the normalized standard deviations (a posteriori) of the scatterer parameters in the present case of pulse-echo measurements. In the second and third columns are listed the values of these quantities based on actual experimental and on noiseless theoretical test data, respectively. In these computations, we assume that the r.m.s. experimental error has the value 

\[ \sigma_{\text{exp}} = 0.46 \times 10^{-5}, \]

a result based upon the deviations between experimental and theoretical results discussed in the last section. It is to be noted the relative standard deviation of the long semi-axis length \( c \) is, as usual, many times larger. The standard deviation of \( \gamma_x \) is about 10% which is still satisfactory for our purposes. The values of the standard deviation of \( \gamma_y \) have been omitted for the same reasons as before.

It is of fundamental interest to reconsider the normalized standard deviations, both in the present pulse-echo case and in the previous pitch-catch case, as measures of experimental "leverage" on the scatterer parameters. For the purpose of comparison, we have recomputed the normalized standard deviations in the pulse-echo with the same value of \( \sigma_{\text{exp}} \) as was used in the pitch-catch case, namely \( \sigma_{\text{exp}} = 1 \times 10^{-5} \). The results are tabulated in the fourth and fifth columns. Comparing the results for noiseless theoretical test data in both cases, the reader will note that the relative standard deviation of \( a \) is about 1/6 as large in the pulse-echo case as in the pitch-catch case, the relative standard deviation of \( c \) is about 1/9 as large and, finally, the standard deviation of \( \gamma_x \) is about 1/2 as large. Thus, the experimental leverage is markedly better in the pulse-echo case than in the pitch-case, particularly for the parameter \( c \). The number of data points is nearly the same in both cases.

It is of critical importance to consider the significance of the present results in the context of failure prediction or, almost equivalently, the making of an accept-vs-reject decision. Clearly, as discussed in Section I, there exists a more complex theoretical structure connecting our present results with the concerns of the final user of an NDE system. In any case, a significant first step in this direction is the estimation of certain quantities of significance in fracture mechanics. One such quantity is the stress intensity factor \( K_I \) measuring the tendency of a crack in, for example, a metal to propagate under the application of a mode I stress (i.e., a uniaxial stress oriented perpendicular to the plane of the crack).

To be sure, the spheroidal void, considered in the previous discussion, is hardly sufficiently degenerate to be regarded as a crack. However, based upon the discussion of Tetelman and McEvily,\textsuperscript{14,15} it appears that the concept can be meaningfully extended to the case of not-so-degenerate spheroidal voids. In any case the definition

\[ k_I = K_I / \sigma = (\pi a)^{1/2}, \]

where \( \sigma \) is the applied stress, will suffice for our present purposes.

In Table VIII we give both the best estimate \( k_I \) and the relative standard deviation \( \text{s.d.} k_I / K_I \) for both pitch-catch and pulse-echo input data. In our view, the significance of these estimates (as

\begin{table}
\centering
\caption{Estimates based on pulse-echo measurements.}
\begin{tabular}{|c|c|c|}
\hline
Parameter & Exact & Estimates (a posteriori) \\
\hline
\hline
\( a \) & 0.0400 & 0.03947 & 0.04000 \\
\hline
\( c \) & 0.0200 & 0.01999 & 0.02000 \\
\hline
\( \gamma_x \) & 0 & -1.24 x 10^{-5} & 2.3 x 10^{-6} \\
\hline
\hline
\end{tabular}
\end{table}

\begin{table}
\centering
\caption{Normalized standard deviations (a posteriori) based on pulse-echo measurements}
\begin{tabular}{|c|c|c|c|c|}
\hline
Quantity & P.E. Measured & Theoretical & P.E. Measured & Theoretical \\
\hline
\hline
\( k_I / \sigma \) & 0.0188 & 0.0186 & 0.0264 & 0.0260 \\
\hline
\( k_I / \sigma \) & 0.0008 & 0.0009 & 0.01099 & 0.01100 \\
\hline
\( \text{s.d.} \gamma_x \) & 0.1120 & 0.1007 & 0.2434 & 0.2295 \\
\hline
\hline
\end{tabular}
\end{table}

\begin{table}
\centering
\caption{Estimate of \( k_I \) and relative st. dev. (\( k_I / \sigma \) \( = k_I \ (\pi a)^{1/2} \))}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Quantity & Experimental Data (a) & Experimental Data (b) & Experimental Data (c) & Theoretical Test Data \\
\hline
\hline
\( k_I \) & 0.361 & 0.359 & 0.360 & 0.356 & 0.354 \\
\hline
\( \text{s.d.} k_I / K_I \) & 0.246 & 0.246 & 0.260 & 0.210 & 0.060 \\
\hline
\hline
\end{tabular}
\end{table}

Using Pitch-Catch Data:

Using Pulse-Echo Data:
measured by the relative standard deviation) is more than adequate for NDE purposes, particularly in the pulse-echo case.

\section*{Discussion}

Our accomplishments can be briefly summarized as follows:

1. Successful determination of $A_2$ ($A = A_2 + \ldots$) from scattering measurements.
2. Comparison of experiment with exact theory with highly satisfactory results.
4. Development of software for all modules involved in deducing fracture mechanics parameters from low frequency scattering measurements.

Even though these results have been obtained under simplifying constraints, they strongly suggest that this approach has promise for NDE.

The principal advantages of the present approach are listed below:

1. Exact scattering theory is available for ellipsoidal voids.
2. Low frequency scattering measurements are sensitive mainly to features that are important in fracture.
3. Low frequency scattering measurements are relatively insensitive to attenuation and spurious scattering in host medium.
4. In the inversion program the parameter $a$ (the long dimension of the spheroidal void) is estimated with good "leverage" and the fracture mechanics parameter $k_1$ is estimated with even better "leverage."
5. Good potential for the implementation of high speed automation.

Clearly, there remain a host of problems for future consideration. A few of these are:

1. Further improvement of the post-experiment data processing in the pitch-catch case.
2. Isolation of a particular scatterer from competing scatterers in taking the low frequency limit.
3. Extension of the analysis to include general ellipsoidal inclusions (voids are a special case).
4. Extension to the more general case of scatterers not having ellipsoidal geometry.
5. Transfer of algorithms to minicomputers suitable for field equipment.
6. Formulation of the theoretical structure extending from the outputs of the inverse scattering algorithms to the final accept-vs-reject decision.

\section*{References}