The Arrow-Lind Theorem in a Continuum Economy

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The Arrow-Lind Theorem in a Continuum Economy

Abstract
In their stimulating contribution to the theory of cost-benefit analysis, Arrow and Lind showed that "when the risks associated with a public investment are publicly borne, the total cost of risk bearing is insignificant and, therefore, the government should ignore uncertainty in evaluating public investments... This result is obtained not because the government is able to pool investments but because the government distributes the risk associated with any investment among a large number of people. It is the risk-spreading aspect of government investment that is essential to this result." Arrow and Lind proved their result from an asymptotic point of view, letting the number of taxpayers grow infinitely large.

Disciplines
Business Administration, Management, and Operations | Economic Theory | Income Distribution | Political Economy
The Arrow-Lind Theorem in a Continuum Economy

by

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No. 61
1. Introduction

In their stimulating contribution to the theory of cost-benefit analysis, Arrow and Lind showed that "when the risks associated with a public investment are publicly borne, the total cost of risk bearing is insignificant and, therefore, the government should ignore uncertainty in evaluating public investments... This result is obtained not because the government is able to pool investments but because the government distributes the risk associated with any investment among a large number of people. It is the risk-spreading aspect of government investment that is essential to this result." Arrow and Lind proved their result from an asymptotic point of view, letting the number of taxpayers grow infinitely large. Private national income grew large with the number of taxpayers, while the risk from government facing an individual taxpayer grew very small. Indeed, in the limit, the impact of the government investment relative to the private economy was negligible. With so many things happening at once, it was not altogether clear that the number of taxpayers alone was responsible for their striking result, although Arrow and Lind so argued. In an attempt to clarify the situation, an alternative approach, that of an economy with a continuum of agents, is adopted here. Such a model enables one to consider an economy with a fixed, very large number of agents, at the same time allowing the impact of risky government investment to vary. The major result for such a model is that the Arrow-Lind theorem holds as long as the government risk is very small relative to the economy. A counterexample shows that if the government risk is at all large, the Arrow-Lind theorem no longer holds, even though there are a great many agents over whom to spread the risk. These results decide the issue: only for small government risks is the total cost of risk bearing necessarily insignificant.
2. Model and Results

In what follows, we adhere as closely as possible to the definitions and notation of Arrow and Lind. Consider then an economy whose agents, called taxpayers, belong to the set $T = [0, 1]$. The assumption that the economy consists of many small agents is expressed by taking an individual agent to be an infinitesimal subset $dt$ of $T$. Since it is convenient to label agents by points $t$ in $T$, we will also consider $t$ as a typical point in the small set $dt$.

Each taxpayer $t$, $0 \leq t \leq 1$, has a twice differentiable and bounded utility function $U_t$ of income $Y(t)$. Since income $Y$ can be random, each taxpayer is assumed to maximize expected utility, $EU_t(Y(t))$. We require boundedness to avoid paradoxes of the St. Petersburg variety; we assume in view of differentiability that the risk aversion $r_t = -U_t''/U_t'$ is defined for all taxpayers and all values of income.

Taxpayer income $Y(t)$ is given by

(1) $Y(t) = A(t) + s(t)[\bar{B} + X]$

where $A(t)$ is private income, $\bar{B}$ is the mean government return on investment, is the random component of government return with mean zero and variance $\sigma_x^2$, and $s(t)$ is the distribution of taxes ($s(t)$ is continuous and non-negative).

Total private income is given by

$$A = \int_0^1 A(t) dt,$$

the tax distribution satisfies

$$1 = \int_0^1 s(t) dt,$$

and total expected government return $E \int_0^1 s(t)[\bar{B} + X] dt$ equals $\bar{B}$. 
The risk premium $\pi(t)$ for taxpayer $t$ satisfies by definition:

\[ (2) \quad EU_t(Y(t)) = EU_t(A(t) + s(t)B - \pi(t)). \]

Equation (2) expresses indifference on the part of taxpayer $t$ to bearing his share $s(t)X \, dt$ of the public risk or paying $\pi(t) \, dt$ out of his sure income and bearing no risk. For a risk averter, risk aversion and the risk premium are both positive. In this case, the risk premium represents a cost, namely the cost to taxpayer $t$ of bearing the risk of the government investment. The total cost of risk-bearing is then given by the integral

\[ \int_0^1 \pi(t) \, dt. \]

Finally, a small risk is a random variable whose variance is arbitrarily small.

We are now ready to state the following analogue of the Arrow-Lind theorem for a continuum economy:

**Theorem.** Let the economy have a continuum of agents. Then the total cost of bearing a small government risk is arbitrarily small.

**Proof.** Using Pratt's result for small risks, the risk premium for taxpayer $t$ satisfies

\[ (3) \quad \pi(t) = 1/2 \sigma_X^2 \tau_t(A(t) + s(t)B) + o(\sigma_X^2), \quad o(\sigma_X^2) \text{ being terms of order less than } \sigma_X^2. \]

Integrating over all the taxpayers, one finds the total cost of risk bearing

\[ (4) \quad \int_0^1 \pi(t) \, dt = 1/2 \sigma_X^2 \int_0^1 \tau_t(A(t) + s(t)B) \, dt + o(\sigma_X^2) \int_0^1 dt. \]

Both integrals on the right-hand side are obviously finite, so that for arbitrarily small variance the total cost of risk bearing is arbitrarily small.
From (4) the only other case where the total cost of risk-bearing vanishes is if \[ \int_0^1 r_t \, dt \] should vanish—this is that the agents of the economy on average are risk neutral. This is admittedly an extreme case. On the other hand, for fairly large risks (3) no longer applies. In this case we will show that even with an infinite number of agents, each bearing an infinitesimal share of the risk, the total cost of risk-bearing is not insignificant.

To this end, consider an economy of identical taxpayers:

\begin{align*}
U_t(Y(t)) &= -Y^{-1} \\
s(t) &= 1 \quad \text{for all } t \\
A(t) &= A > 1
\end{align*}

The government risk has positive mean \( \bar{B} \) and \( X = 1 \) with probability \( 1/2 \) and \( X = -1 \) with probability \( 1/2 \). Equation (2) in this case becomes

\[
\frac{1}{2} \left( \frac{-1}{A + \bar{B} + 1} \right) + \frac{1}{2} \left( \frac{-1}{A + \bar{B} - 1} \right) = \frac{-1}{A + \bar{B} - \pi(t)}
\]

Thus

\[
\pi(t) = \frac{1}{A + \bar{B}}, \text{ and the total cost of risk bearing}
\]

\[
\int_0^1 \pi(t) \, dt = 1/(A + \bar{B}) \text{ is positive.}
\]

Indeed, unless the variance of \( X \) decreases, there is no way for the total cost of risk bearing to vanish.
3. Conclusion

Arrow and Lind's argument for ignoring the uncertainty associated with public investment rested on a limit argument—applied to an economy with a large and increasing number of agents. At the same time, since each agent represented an addition to national income, the size of the government risk became small relative to the economy. It was a priori possible that one or both of these factors was responsible for the Arrow-Lind result, although Arrow and Lind argued exclusively in terms of the former. What we have seen is that, for a continuum economy, the decisive role is played by the latter. Benefit-cost analysis safely ignores the uncertainty associated with public investment only when that investment's risk is inherently small.
2. Arrow and Lind, p. 373.
3. Aumann and Shapley, Chapter VI, discuss the relationship between the two approaches and various interpretations of the latter.
4. In Arrow and Lind, private income is a random variable. In order to focus attention on public investment uncertainty, we have here suppressed this additional source of uncertainty.
5. Pratt, p. 124. Our $\pi(t)$ corresponds to Arrow and Lind's $k(n)$.
REFERENCES

