ELASTIC WAVE INVERSION TRANSFORMATION

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ABSTRACT

This paper presents results of a study on an inversion transformation $A$ for backscattering of ultrasonic waves from an obstacle embedded in a solid. For a rigid sphere the resulting function $A(t)$ is derived which is related to the cross-sectional area intercepted at any given time by a transverse plane moving across the obstacle in the same direction as the incident wave. The maximum value of $A(t)$ is the total backscattering cross section. From this study emerges a proposed inversion algorithm for cavities. The use of the inversion algorithm is demonstrated for "exact" theoretical data obtained from Opsal as well as experimental data collected on an ellipsoidal cavity and a double cavity. Associated with the inversion of the experimental data is an iterative technique which optimizes the construction of $A(t)$ and therefore the estimates for the size and shape of the scatterer.

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INTRODUCTION

An inversion transformation has been developed for the extraction of geometrical parameters of a scatterer embedded in a solid. The transformation is shown to be useful in the $1 < ka < 4$ range for the construction of an image. Examples given are a $400 \mu m \times 800 \mu m$ ellipsoidal void embedded in a Ti-alloy and a $800 \mu m$ spherical void with a small spherical perturbation also in Ti-alloy.

The inverse problem of extracting size and shape of an object from scattering data is important in many fields ranging from seismology, to underwater acoustics, to ultrasonic NDE, to radar. Although progress has been made towards solutions of the inverse problem for elastic waves, these efforts are still in the early stages with several approaches currently being evaluated. In a recent letter we proposed an approach based on a general inversion transformation $A(t)$ which when used on backscattering data produces a function $A(t)$.

For the special case of the Kirchhoff approximation, $A(t)$ was interpreted as the cross-sectional area intercepted by a transverse plane moving in the same direction as that of an incident wave. The maximum value of $A(t)$ was interpreted as the total cross section.

In another paper we briefly presented the application of the technique to several sets of scattering data with a resulting reconstruction of an image.

In this paper is presented in considerable detail the derivation of $A(t)$ for the rigid sphere, the inversion algorithm applied to theoretical data, the description of an iterative technique to be used on band limited experimental data and the inversion algorithm applied to experimental data.

THE INVERSION TRANSFORMATION - THEORY

The detailed discussion of the theoretical approach follows below but is presented here only in the form of a summary. The inversion transformation $A(t)$ is defined by

$$A(t) = J_A [P(k)]$$

such that

$$A(t) = \int_{-\infty}^{+\infty} [P(k)/k^2] \exp (ikct) dk$$
Here $P(k)$ is the pressure backscattered by a surface $S$ when illuminated by an incident plane wave with wave number $k$ and speed $c = 2z/t$, where $z$ is the position along the axis of propagation and $t$ is time. In general, $A(t)$ is difficult to derive explicitly but has been obtained for the case of acoustic waves scattered by a fixed, rigid sphere. However, much progress has been made in computing values for $P(k)$ for elastic waves scattered by obstacles of various shapes and it is possible to carry out $A$ on these theoretical data. In a later section we discuss the application to experimental data.

Kirchhoff Approximation

In the Kirchhoff approximation for a void, the pressure backscattered by a surface $S$ (see Fig. 1) when illuminated by an incident plane wave may be deduced from

$$P(k) = \frac{-iakB}{2\pi} \int_S \cos \theta \exp(-2ikz)da$$

where $k$ is the wave number, $\alpha$ is a phase factor depending on the choice of origin or coordinates, $B$ is a coefficient including the amplitude of the incident wave, $da$ is an elementary area of the scattering surface with $z$ as its position along $Oz$, the axis of wave propagation (see Fig. 1). $\theta$ is the angle wave incidence on the element $da$.

Fig. 1. Cross-sectional function as predicted by Kirchhoff approximation.
This double integral may be written as a simple integral along Oz:

\[ P(k) = \frac{-iakB}{2\pi} \int_{z_0}^{z_1} f(z) \exp(-2ikz) \, dz \]  

(2)

with \( f(z) \, dz \) being equal to the projection on to a plane perpendicular to Oz of the area of the scatterer between \( z \) and \( z + dz \); as shown in Fig. 1, \( z_0 \) and \( z_1 \) are the first and last positions of the illuminated portions of the surface along Oz. Let \( A_K(z) \) be a function such that \( f(z) = \frac{dA_K}{dz} \) with \( A_K(z_0) = 0 \). The Eq. (2) becomes

\[ P(k) = \frac{-iakB}{2\pi} \int_{-\infty}^{+\infty} \frac{dA_K}{dz} \exp(-2ikz) \, dz \]  

(3)

From integration by parts and assuming convergence at the limits:

\[ P(K) = \frac{k^2aB}{\pi} \int_{-\infty}^{+\infty} A_K(z) \exp(-2ikz) \, dz \]  

(4)

A convergence problem appears which seems inherent to the Kirchhoff interpretation of \( A(z) \): The term \( A_K(z)e^{-2ikz} \) appearing in the integrated term does not converge at \( z = +\infty \) far in the shadow region for which \( A_K(z) \) tends to the constant \( A \). This difficulty disappears when \( A(z) \) is defined as the result of the \( J_A \) transformation. This function tends to zero with \( z \) going to infinity.

It may be seen from (4) that \( A_K(z) \) may be obtained by taking the inverse Fourier transform of \( P(k)/k^2 \). Also given the relation between \( z \) and the time \( t \) in the backscattering geometry, \( z = ct/2 \) where \( c \) is the velocity of the wave, one can express \( A_K \) as a function of time.

\[ A_K(t) = \frac{1}{aB} \int_{-\infty}^{+\infty} \frac{P(k)}{k^2} \exp(ikct) \, dk \]  

(5)

\( A_K(t) \) may be interpreted as the cross-sectional area intercepted by a transverse plane moving at a velocity of one-half the speed of the wave in the same direction of the incident wave. In Fig. 1 is plotted schematically a typical cross section function \( A_K(t) \), as given by the Kirchhoff approximation. This function is increasing or is a constant. One obtains the length of the illuminated region along \( z \) by \( z_1 - z_0 = c(t_1 - t_0)/2 \). The maximum
value of $A_K(t)$ is the total cross section of the scatterer even for the case of elastic waves. In this more generalized context, $A_K$ is quite far removed from the Kirchhoff approximation and we demonstrate in a following section its usefulness for the inversion of elastic wave $P(k)$ derived from theoretical solutions of the scattering problem. 9, 10

Thus, within the context of the Kirchhoff approximation, the operation of calculating $A_K(t)$ constitutes an inversion in the sense that it provides a value for the length of the illuminated region of the scatterer along the direction of illumination. On the basis of these results, we define a general inversion transformation $J_A$, which, when used on any $P(k)$ of a scatterer produces a more general function $A(t)$. Now, $A(t)$ is no longer simply the cross sectional area, but will be demonstrated to provide valuable information about the geometrical parameters of the scatterer.

Fixed, Rigid Sphere

Given a sphere of radius $a$ assumed to be rigid and fixed, the velocity potential for backscattering of an acoustic wave is

$$P(k) = \frac{-i k a^2}{8 \pi R^2} e^{i k (-2R + a)} \int_{-1}^{1} \mu d\mu \sum_{n=0}^{\infty} \frac{e^{i k a \mu (2n + 1) P_n(\mu)}}{F_n(i k a)}$$

where $P_n$ is the Legendre polynomial and $F_n$ are functions related to spherical Bessel functions, $\mu = \cos \theta$ with $\theta$ being the angle between a point on the sphere and the direction of wave propagation, $k$ is the wave number, and $R$ is the distance of the transducer from the center of the sphere.

Applying the inversion transformation $J_A$, one obtains

$$A(t) = \frac{-a^2}{8 \pi R^2} \sum_{n=0}^{\infty} (2n + 1) \int_{-1}^{1} \mu P_n(\mu) d\mu \int_{-\infty}^{\infty} \frac{i e^{i k (ct - 2R + a + \mu a)}}{k F_n(i k a)} dk$$

The value of the integral is determined by the poles of the integrand. Let $x = (ct - 2R + a)/a$, then

for $n = 0$:

$$A_0 = \frac{-i a}{8 \pi R^2} \int_{-1}^{1} \mu d\mu \int_{-i \infty}^{+i \infty} \frac{e^{y(x - \mu)}}{y(1 + y)} dy$$

From the principle of causality

$$A_0 = 0 \quad \text{when} \quad x + \mu < 0$$
\[ \mu > 0: \]
\[ -1 < x < 1 \quad A_0(x) = \frac{a}{4R^2} \left( \frac{1 - x^2}{2} + 2e^{-(1+x)} + x - 1 \right) \]
\[ x > 1 \quad A_0(x) = \frac{a}{4R^2} [2e^{-(1+x)}] \]

For \( n = 1 \):
\[ F_1(y) = y + 2 + 2/y \quad \text{there are 2 poles for } x + \mu > 0. \]
\[ y_1 = -1 + i \]
\[ y_2 = -1 - i \]
\[ -1 < x < 1 \quad A_1(x) = \frac{-3a}{4R^2} - \frac{x^2}{2} + x = \frac{1}{2} + 2e^{-(1+x)} \cos (1 + x) \]
\[ x > 1 \quad A_1(x) = \frac{-3a}{4R^2} e^{-(1+x)} \cos (1 + x) \]

The functions \( A_n(x) \) are plotted in Fig. 2 individually for \( n = 0 \) and \( n = 1 \) as well as summed for \( n = 0 \) and \( n = 1 \). Figure 3 gives the sum of contributions from \( n = 0 \) to \( n = 5 \).

**INVERSION OF THEORETICAL DATA**

In Fig. 4 are plotted the functions \( A(t) \), as obtained from theoretical scattering amplitudes for scatterers in solids, i.e., a spherical void of radius \( a = 400 \mu m \); for a void in the shape of an oblate spheroid with aspect ratio \( a:b = 2:1 \) with its semi-minor axis \( b = 200 \mu m \) along the direction of incidence; for a void in the shape of a thin disc of radius \( a = 400 \mu m \) with axis along the direction of incidence. Also plotted is the function \( A_K(t) \) given by the Kirchhoff approximation for a spherical void. The origin of time is chosen at the centroid of the scatterer, for example for the sphere, the centroid corresponds to its center. For ellipsoidal shapes, the origin is also the position of the "equator" marking the border between the illuminated and shadow region. For the sphere, good agreement is observed between \( A_K(t) \) and \( A(t) \) in the domain of negative values of \( t \). The radius of the sphere and the semi-minor axis of the ellipsoid can be obtained by using the time difference \( \Delta t \) between the value where the function \( A(t) \) starts from zero and \( t = 0 \). Note that this difference is zero in the case of the flat disc.
Fig. 2. $A_0(x)$, $A_1(x)$, and $A_0(x) + A_1(x)$ calculated for back scattering of acoustic waves from rigid sphere.

Fig. 3. $\Sigma A_n(x)$ for $n = 0$ to 5 as calculated for rigid sphere.
Fig. 4. $A(t)$ obtained from exact theoretical data. Negative time corresponds to the illuminated region.

For positive values of $t$, $A_K(t)$ and $A(t)$ are in total disagreement. This is not surprising, since this domain is known to be dominated by waves emitted by the surface in the shadow region. Also the Kirchhoff approximation which assumes that the displacement is zero behind the scatterer; an assumption which is clearly invalid in the range of frequencies used here. The role of mode conversion from the incident longitudinal to other wave types (transverse, creeping) appears to be negligible for negative values of $t$. 
EXPERIMENTAL RESULTS

The experiments were performed on samples of Ti-alloy into which had been embedded scatterers of a variety of shapes with the aid of the diffusion bonding process already described elsewhere. The scattered waveforms were deconvolved of the transducer response with the use of waveforms reflected by a plane interface (metal-air) located at the same distance as the defect. In order to handle the typical band limitations of the experimental data, an iterative technique was developed to arrive at the experimental A(t).

Iterative Technique. In this technique, the first step is to force the phase of the scattered amplitude to be zero. The resulting A(t) gives a rough estimate of the dimension of the scatterer. This estimate is used to calculate the slope of the phase at high frequencies, taking advantage of the knowledge from theory that the phase for a spheroid varies linearly at high frequencies, with the slope given by the distance from the front face to the centroid. The calculation of A(t) is then performed again, with the waveform signal shifted in time to have the right slope for the phase at high frequencies but with a null still imposed in the unknown part at low frequencies. The function A(t) obtained is sufficiently stable with regard to this shift in time, that the result in the second step is usually not affected by a slight error in the evaluation of the position of the centroid. The operation is now repeated several times to refine A(t) using the criterion that A(t) has to be positive for t < 0. In the process the position of the centroid is also refined.

The comparison shown in Fig. 5 shows that the iterative technique is able to produce a fairly accurate A(t), even for band-limited data in the range 1 < ka < 4. Deviations for t > 0 could be due to the non-ideal shape of the void fabricated in the sample. It is to be noted that for negative values of t contributions from mode conversion from incident longitudinal to other wavetypes (transverse, creeping, etc.) are expected to be negligible but to play a key role for positive values of t. From a physical optics point of view, the part of the curve for t < 0 corresponds to the illuminated surface of the spheroid, whereas the part for t > 0 corresponds to the shadow regime.

When a wave is incident at an angle with respect to one of the axes of the spheroid, the process of calculating the dimensions becomes a little more complex. This is illustrated in Fig. 6a, where the line MP represents a wavefront just striking the spheroid at point M when incident at an angle θ, along the direction of the wavenormal OP. It is obvious that when A(t) is obtained, the dimension derived is OP which may be related to the geometrical parameters by
Fig. 5. A(t) void in Ti-alloy in the shape of an oblate spheroid with longitudinal waves incident along $\theta = 0$. 

Fig. 6. (a) Theoretically derived contour of spheroid. (b) Experimentally derived contour of spheroid.
\[ OP = b \ (1 + \beta^2 \tan^2 \theta)/(1 + \tan^2 \theta) \quad ; \quad \beta = a/b \quad (8) \]

The calculated points \( P \) for \( \theta \) ranging in 10° steps from 0 to 70° are shown in Fig. 6a and the calculated lengths \( OP \) in Table 1.

**Table 1. Comparison Between Theoretical and Experimental Determination of Geometrical Parameter \( OP \)**

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0°</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
<th>70°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( OP_{exp} ) (25%-85%) (( \mu )m)</td>
<td>198</td>
<td>194</td>
<td>205</td>
<td>261</td>
<td>294</td>
<td>355</td>
<td>370</td>
<td>359</td>
</tr>
<tr>
<td>( AVG \ OP_{exp} ) (( \mu )m)</td>
<td>208</td>
<td>207</td>
<td>221</td>
<td>251</td>
<td>303</td>
<td>355</td>
<td>371</td>
<td>359</td>
</tr>
<tr>
<td>( OP_{th} ) (( \mu )m)</td>
<td>200</td>
<td>209</td>
<td>232,5</td>
<td>265,6</td>
<td>299</td>
<td>332</td>
<td>360,6</td>
<td>382</td>
</tr>
</tbody>
</table>

\( \theta \) angle direction of incidence and minor axis of ellipse

\( OP_{th} \) distance calculated for normal from centroid to wave front tangent to ellipse

\( OP_{exp} \) (25%-85%) obtained from data using 25%-85% criterion on \( A(t) \)

\( AVG \ OP_{exp} \) average obtained from some data using three different criteria.

In the determination of \( OP \) from the experimental \( A(t) \) three criteria were used. One of these, illustrated in Fig. 1, is based on an empirical choice of 25% and 85% of the maximum value of \( A(t) \) for the start and end of the interval \( \Delta t \) and therefore \( OP \). Another was the interval between the 25% point and the centroid (\( t = 0 \)) obtained as a byproduct of the iterative technique. These two intervals correspond, respectively, to the time interval from the front face reflection to the onset of the shadow region on one hand and to the centroid on the other hand. These two intervals are the same only if the scatterer has inversion symmetry, as for example the oblate spheroid. For asymmetric scatterers the 25%-85% criterion provides additional information, since the location of the centroid may be far removed from the onset of the shadow region, as for example, for a cone with the tip pointed towards the incoming wave. The third criterion is based on the extraction of a feature in the scattered waveform usually identified as a creep wave signal. The delay of the creep wave which has travelled behind the scatterer with a known velocity gives the path length corresponding to one-half the perimeter. Table 1 compares the theoretical \( OP \) with the experimental \( OP \) based on the 25%-85% criterion alone and on the average of values obtained from all three criteria, labeled \( AVG \ OP_{exp} \).
Construction of Image. The values of Table 1 have been used to obtain an image which is compared with the actual contour in Fig. 6b. With the values $OP$ and the angles $\theta$ a family of tangent lines $MP$ are constructed which correspond, as described above to the wavefronts just striking the surface. The largest contour inscribed by all the tangent lines forms the desired image, which can be seen to be in good agreement with the actual contour. It is clear that additional experimental data for several different cuts through the oblate spheroid would allow the construction of a 3-D image.

In Fig. 7 is plotted $A(t)$ for a more complex scatterer without inversion symmetry. It consists of two overlapping spherical voids identified as "lumpy" sphere. As shown in the schematic, the result is shown for backscattering at an angle of 15° from the common axis of the two spheres. For $t < 0$, $A(t)$ clearly shows features identifiable with the presence of two overlapping voids. The features also allow estimates of their radii.

![Fig. 7. $A(t)$ experimentally obtained for spherical void with perturbation. $\theta = 75^\circ$.](image-url)
ACKNOWLEDGMENT

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REFERENCES

DISCUSSION

J.H. Rose (Ames Laboratory): I'd like to comment on the Kirchhoff derivation which becomes constant after you get past the shadow boundary. That arises because the Kirchhoff solution rises as $K$ instead of $K^2$ as it must at low frequencies. If you make the same derivation using the Born approximation similar to what John Richardson has talked about, then in fact you get it to come up and come back down; however, instead of being exactly right, it is symmetric, front to back.

B.R. Tittmann (Rockwell International Science Center): I see.

J. H. Rose: But that work has been derived, as in the Born. In fact, in some of the inversion algorithms that area of function is used to choose its zero time or find a centroid.