1991

Application of the boundary element method to coupled fluid-structure interaction problems

Partha P. Goswami
Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Applied Mechanics Commons

Recommended Citation
https://lib.dr.iastate.edu/rtd/9643

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
Application of the boundary element method to coupled fluid-structure interaction problems

Goswami, Partha Pratim, Ph.D.

Iowa State University, 1991
Application of the boundary element method to coupled fluid-structure interaction problems

by

Partha P. Goswami

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Department: Aerospace Engineering and Engineering Mechanics
Major: Engineering Mechanics

Approved:
Signature was redacted for privacy.

In Charge of Major Work
Signature was redacted for privacy.

For the Major Department
Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa
1991

Copyright © Partha P. Goswami, 1991. All rights reserved.
# TABLE OF CONTENTS

ACKNOWLEDGEMENTS ................................................. xi

CHAPTER 1. THE INTRODUCTION .................................. 1
  Fluid-Solid Interaction ......................................... 1
  Numerical Solutions in Fluid-Structure Problems ................. 2
  BIE-BEM Approach in Wave Mechanics ........................... 4

CHAPTER 2. ACOUSTIC-ELASTIC INTERACTION IN SCATTERING BY A SUBMERGED FINITE ELASTIC BODY ........ 6
  Introduction .................................................. 6
  The Interaction Model .......................................... 7
  The B.I.E. Formulation ........................................ 10
    The boundary integral equations ............................ 11
  Computational Scheme ......................................... 16
    Numerical approximations in the BEM ....................... 17
    Numerical evaluation of integrals ......................... 19
    Integration for weakly singular kernels ................... 21
    System of equations ...................................... 23
  Solution Strategies .......................................... 24
    Ill-conditioning and scaling ............................... 25
CHAPTER 3. THE FICTIONAL EIGENFREQUENCY PROBLEM

Introduction............................................. 57
Numerical Solutions at Fictitious Eigenfrequencies............ 59
  Interior Dirichlet case.................................. 59
  Interior Neumann case.................................. 64
  Jones eigenfrequencies................................ 65

CHAPTER 4. ULTRASONIC TRANSMISSION THROUGH A CURVED INTERFACE

Introduction............................................. 67
Ultrasonic Transmission and NDE............................ 68
The BEM Model and the Problem Geometry...................... 69
  Integral formulation.................................... 70
A Gaussian Beam Model.................................. 73
Truncation.................................................. 76
The Boundary Element Solution............................. 77
An Exact Solution for a Planar Interface...................... 80
Integral representation for finite scatterers .................. 130
The representation integral for a half-space scatterer .......... 131
BIE Identities of Fundamental Solutions ....................... 133
Identities in the half-space ................................. 136

APPENDIX D. EXACT ANALYSIS OF SCATTERING BY ELASTIC SPHERE .................. 138

APPENDIX E. EIGENFREQUENCIES OF SPHERE: DIRICHLET PROBLEM .................. 143

APPENDIX F. THE REFLECTION COEFFICIENT $R(kr)$ ............. 147

APPENDIX G. NUMERICAL QUADRATURE .................. 151
LIST OF TABLES

Table 2.1: Pressure at Boundary Nodes ........................................ 35
Table 2.2: Normal Velocity at Boundary Nodes .......................... 35
Table 2.3: Material Properties .................................................. 52

Table 3.1: Interior Eigenfrequencies (Dirichlet) ......................... 60
Table 3.2: Condition Numbers At Different $ka$'s ....................... 61
Table 3.3: Interior Eigenfrequencies (Neumann) ......................... 64
Table 3.4: Eigenfrequencies: Free Torsional Oscillations of Sphere ........ 65

Table 4.1: Reciprocity Results For Curved Interface .................. 101
Table 4.2: CPU Time Distribution ............................................. 111
Table 4.3: Time Distribution: Interface Problem ......................... 111
LIST OF FIGURES

Figure 2.1: Fluid-solid interaction: Schematic model .......................... 9
Figure 2.2: Boundary elements and the connectivity ....................... 18
Figure 2.3: Mapping of curvilinear elements ................................. 20
Figure 2.4: Scheme for singular integrations ................................. 22
Figure 2.5: Acoustic-elastic interaction: Iterative solution ............... 29
Figure 2.6: Fluid(exterior)-solid(interior) domain and the far field ...... 32
Figure 2.7: Plane wave scattering by a sphere ............................... 34
Figure 2.8: Polar plot: Far field scattered pressure (brass sphere in glycerine, ka=1) .......................................................... 37
Figure 2.9: Polar plot: brass/glycerine ka=2 .................................. 38
Figure 2.10: Polar plot: sphere (aluminum sphere in water) ka=3 [solid line: Exact; Δ: BEM (105 node)] ................................. 39
Figure 2.11: Polar plot: sphere (lucite/water) ka=3 [solid line: Exact; •: BEM] ................................................................. 40
Figure 2.12: Polar plot: sphere (aluminum/water) ka=7 [solid line: Exact; •: BEM] ................................................................. 41
Figure 2.13: Polar plot: sphere (brass/glycerine) ka=7 [solid line: Exact; •: BEM] ................................................................. 42
Figure 2.14: Effect of mesh refinement (aluminum sphere in water) $ka=5$.
Line: Exact; •: BEM(144 Elem.); Δ: BEM(96 Elem.).

Figure 2.15: Effect of mesh refinement (aluminum sphere in water) $ka=6$.
Line: Exact; •: BEM(144 Elem.); Δ: BEM(96 Elem.).

Figure 2.16: BEM discretization (sphere) top: 96 Element; bottom: 144 Element.

Figure 2.17: Frequency distribution of far field (aluminum sphere in water).

Figure 2.18: BEM mesh (a) prolate spheroid (b) cylinder with hemispherical caps.

Figure 2.19: Scattering configuration of a prolate spheroid.

Figure 2.20: Scattered spectrum from aluminum spheroid. [top: Backscattering; bottom: Bistatic(90°)].

Figure 2.21: Scattered spectrum from capped cylinder [top: Backscattering; bottom: Bistatic (90°)].

Figure 2.22: Scattering patterns from spheroids of different impedances.

Figure 2.23: BEM meshes for spheres of four aspect ratios [from left: 1:1, 4:3, 2:1, 4:1].

Figure 2.24: Scattering from spheroids of four aspect ratios at $k=2$.

Figure 2.25: Scattering from 2:1 spheroid at different angles of incidence.

Figure 3.1: Normalized condition numbers of combined matrix (sphere).

Figure 3.2: Scattering in the vicinity of an eigenfrequency $ka = \pi$ (fundamental mode, first harmonic). Dotted Line: BEM at $ka - \delta$ ($\delta = 0.0005$).
ix

Figure 3.3:  $ka = 4.4934 \ (m=1, \ n=1) \ [\Delta: \ BEM \ at \ exact \ ka; \ ---: \ BEM \ at \ ka - \delta]$  

Figure 3.4:  $ka = 5.7625 \ (m=1, \ n=2)$  

Figure 3.5:  $ka = 6.9794 \ (m=1, \ n=3)$  

Figure 3.6:  First Jones eigenfrequency: Scattering from sphere  

Figure 4.1:  Curved fluid/solid interface: Schematic model  

Figure 4.2:  Curved interface: Mathematical model for boundary element analysis  

Figure 4.3:  Gaussian beam profile  

Figure 4.4:  Gaussian beam incident at an angle  

Figure 4.5:  Length parameters in the BEM model  

Figure 4.6:  Typical BEM discretization for a flat interface  

Figure 4.7:  Total surface pressure: Normal incidence $k = 1, \beta = 4$  

Figure 4.8:  Total surface pressure: Normal incidence $k = 3, \beta = 1.3$  

Figure 4.9:  Total surface pressure: oblique incidence (28.5°) $k = 1$  

Figure 4.10:  Verification in $X_1$-direction: Oblique incidence (Fig. 4.9)  

Figure 4.11:  Verification in $X_2$-direction: Oblique incidence (Fig. 4.9)  

Figure 4.12:  Normal incidence: 4 MHz ($k = 169.8, L/\lambda = 2, a = 0.3175cm$)  

Figure 4.13:  Oblique incidence ($8^\circ$): 4 MHz $R/a = 1.5, L/\lambda = 1.4$  

Figure 4.14:  Oblique incidence ($20^\circ$): 4 MHz $R/a = 1.5, L/\lambda = 1.4$  

Figure 4.15:  Oblique incidence ($28.5^\circ$): 4 MHz $R/a = 1.5, L/\lambda = 1.4$  

Figure 4.16:  BEM discretization of a concave interface  

Figure 4.17:  BEM mesh: Typical convex interface  

Figure 4.18:  Reciprocal work: Two different incident states
Figure 4.19: Boundary pressure: Concave interface $k = 2, R_C/a = 5$ . . . 102
Figure 4.20: Boundary pressure: Convex interface $k = 169, R_C/a = 5$ . . . 103
Figure 4.21: Normal incidence: Different surface curvatures (4 MHz.) . . . 105
Figure 4.22: Oblique incidence ($8^\circ$): Different curvatures .................. 105
Figure 4.23: Input beam profiles used in transmission study .................... 106
Figure 4.24: Ray representation of beam transmission: Reproduced from

[52] ..................................................... 106
Figure 4.25: Transmitted field along central axis: Normally incident beam
  on concave interface (4 MHz) .................................. 108
Figure 4.26: Radial variation of the transmitted field ............................... 109
Figure 4.27: 3D representation: Transmitted field at $Z=-0.5$ (Fig.56) . . . 110
Figure C.1: Interior and exterior domains ........................................... 134
Figure E.1: Spherical coordinate systems ............................................ 144
ACKNOWLEDGEMENTS

I would like to express my deep appreciation of my advisor Dr. Thomas Rudolph for his help, suggestions and patient guidance. Working with him was truly a pleasant experience. Thanks and gratitude are specially due to Dr. Frank Rizzo who initiated me into computational mechanics and has been very supportive throughout the research program. I am indebted to Dr. Ron Roberts for numerous helpful discussions on the 'interface' problem.

I take this opportunity to thank Drs. D. Fernandez-Baca, R. K. Hindman, E. Johnston and A. K. Mitra for serving on my P.O.S. committee and always being generous with their time.

I wish to thank the Office of Naval Research for providing partial support for the research under Contract Numbers N00014-86-K-0551 and N00014-89-K-0109. Thanks are also due to Dr. G. Everstine of the David Taylor Research Center for providing certain numerical comparison data. I gratefully acknowledge my association with Dr. David Shippy, whose organized approach to numerical research was a constant inspiration in managing piles of computer data.

Special thanks are due to Sreela, my significant other, for the crucial help with figures.
CHAPTER 1. THE INTRODUCTION

Fluid-Solid Interaction

The research described in this thesis deals with the scattering and transmission of acoustic waves through an acoustic-elastic interface while focus is primarily on numerical modelling and solutions using the boundary element method. Scattering of time-harmonic acoustic waves from a submerged elastic structure and subsequent transmission of elastic waves into the solid is a generic problem of interest to various disciplines, especially structural acoustics, geophysics and earthquake engineering. A solution strategy for this class of problems is useful in non-destructive evaluation (NDE) where such knowledge can contribute to a simulation scheme for ultrasonic scanning systems.

Interaction of a submerged elastic surface and an impinging oscillatory sound is a coupled problem and requires simultaneous solution of the dynamic or vibrational response of the solid structure and the interacting acoustic field. One field significantly influences the response of the other. A solid metal scatterer in vacuum behaves as a rigid body but responds differently when submerged in a fluid like water with reasonable impedance. The vibrational characteristics of a submerged oscillator is different than that in vacuo, e.g., submergence reduces its natural frequency. The early research in this area sought to predict scattering from simple shapes through
analytical methods, however, very few fluid-structure interface geometries are analytically tractable. A condition that makes an analytical treatment possible is that of separability, where the involved differential equations are separable in a coordinate system that describes the interface by a single coordinate. Typical examples are spheres requiring spherical coordinates, spheroids requiring a spheroidal geometry or an infinite, plane plate where a rectangular coordinate is appropriate.

Scattering and radiation solutions for spheres and cylinders were formulated by Faran [1] as early as 1951, presuming plane, harmonic incident waves. References [2] and [3], respectively, by Hickling and Doolittle et al., discuss various aspects of scattering by spheres and cylinders. Hickling obtained expressions describing the backscattered field and computed solutions for spheres of several materials. A detailed numerical evaluation of theoretical solutions describing monostatic scattering by elastic spheres was made by Rudgers [4] while a historical study of elastic structure under acoustic excitation is available in a review paper by Junger [5] and in the well known text by Junger and Feit [6].

Numerical Solutions in Fluid-Structure Problems

Some of the early numerical approaches for harmonic vibrations of structural and acoustic systems were based upon variational formulations. Gladwell et al. [7, 8] applied this method to study acoustic radiation from air-plate and air-membrane systems. There has since been diverse studies of the problem by coupled finite element analysis [9], finite difference method [10] as well as by a combined finite element-boundary element approach, wherein a finite element formulation of the solid is coupled with an integral equation model of the exterior acoustic domain.
The coupled treatment involves combination of two distinct domains as well as two different physical processes. Zienkiewicz [16] provides a detailed discussion of the numerical treatment of different class of coupled problems. Typically in this approach, equations representing each of the two processes in the respective sub-domains are reduced to matrix equations through a discretization procedure. The coupled algebraic equations are then solved together as a combined system or by a staggered iterative approach [16], where the acoustic and the structural part are solved separately, one after another, with appropriate interface conditions between.

The present work uses a three dimensional boundary integral formula exclusively for both the scatterer and the acoustic fluid and the subsequent solution is by the boundary element method (BEM) such that the discretization for the numerical solution is confined only to the scatterer surface. This feature of the boundary element approach is the principal attraction of the method.

The thesis is divided into two stages. The first part (Chapters 2 and 3) discusses the formulation and solution for scattering of plane waves by a finite and closed elastic body immersed in an infinite fluid. The second part (Chapter 4) deals with the transmission of an ultrasonic beam through an open curved interface. In both cases, the formalism has been verified through comparison with analytical solutions as well as solutions from other numerical schemes. A discussion in Chapter 3 examines the fictitious eigenfrequency problem for a finite scatterer in the context of the two-media interaction problem.
BIE-BEM Approach in Wave Mechanics

The boundary integral equation, or BIE, method starts from a surface integral representation of a boundary value problem and uses a numerical scheme - the boundary element method or BEM - to solve the equations by discretizing the boundary into surface portions or elements. By requiring only a surface discretization, the method thus reduces the dimension of the problem and consequently the number of unknowns required in the solution compared to domain methods. This is truly advantageous for most domain shapes except for slender bodies with high boundary to volume ratios. This method of analysis has been particularly successful in wave mechanics problems of acoustics and elastodynamics [17, 18, 19], where an infinite surface is involved. Even though the system matrices of the BEM involves lesser unknowns, they are neither banded nor symmetric, thus the BEM cannot use the many efficient numerical schemes that have been developed for banded or symmetric matrices. Also, the singular kernels of the BIE require sophisticated integration schemes. The various advantages and disadvantages of BEM solutions in computational acoustics is discussed in the reference by Shaw et al. [17]. In a coupled problem where matrices from two or more domains are combined, the resulting matrices from the other methods like the finite element method may also be unsymmetric unless special techniques are used to symmetrize them. Thus the usual advantages of a symmetric coefficient matrix in finite element analysis may also be lost in the coupled problem.

The present analysis combines an ideal fluid and an elastic solid with appropriate interface conditions; the fluid being inviscid and the solid is taken to be isotropic and linearly elastic. Wave propagation in an elastic solid is a vector phenomenon whereas the field in the ideal fluid is a scalar one since it does not support shear stresses.
One complexity in modeling fluid-solid interaction therefore lies in coupling a scalar field with a vector one. This thesis highlights the efficiency of BEM in capturing this coupling.

A FORTRAN code was developed, based on the integral equation approach and incorporates the new concepts in regularization of singularities in integral equations for both the scalar (acoustic) and the vector (elastodynamic) fields. The code is tailored to handle closed scatterers as well as open, curved interfaces. Once the acoustic field is solved at the interface, fields within both the fluid and the elastic domain can also be extracted through post-processing subroutines. Verification solutions are presented for different fluid and solid combinations. Results for closed scatterers cover a range of simple shapes like spheres, spheroids and cylinders with hemispherical caps. The axial symmetry of these shapes is, however, not exploited in the formulation and the numerical scheme is capable of handling completely general shapes. Results for open interfaces include concave, convex and flat planar surfaces.

Partial contents of this dissertation are already available in a journal article and several conference proceedings and technical reports. No reference has however been made to them in the text and all the details have been provided explicitly so that the dissertation may be read as a self-contained document without reference to other publications.
CHAPTER 2. ACOUSTIC-ELASTIC INTERACTION IN SCATTERING BY A SUBMERGED FINITE ELASTIC BODY

Introduction

This chapter develops the integral equation formulation for a class of time-harmonic coupled acoustic scattering problems where an incident plane wave impinges on the bounded elastic scatterer submerged in a fluid. An exact mathematical model is presented for the finite scatterer, where the surface integral equations are exclusively used to represent the acoustic-elastic interaction of the scattering process. The numerical procedure involves application of point collocation with quadratic isoparametric approximations that reduce the integral equations to a discrete set of complex linear algebraic equations.

The complexity involved in modelling the scattering of time-harmonic waves by the immersed body is two-fold. First, a full vector model of the wave propagation in a solid is required and second, the scalar field in the fluid must be coupled with the vector field of the solid. Use of the surface or boundary integral equations to describe both these fields and the subsequent numerical solution by boundary element method is shown to be an effective tool to solve such problems in the mid-range frequencies where asymptotic approximations do not work well. Validity of this BIE/BEM approach is established here through analytical and numerical verification for different
interacting fluids and elastic solids of different geometrical shapes and material properties. Verification examples are presented for plane wave scattering by spherical and non-spherical shapes over a wide range of impedances (brass, aluminum, lucite, solder, etc.) immersed in water or glycerine. Non-spherical shapes include axisymmetric bodies like spheroids and capped cylinders for which comparison solutions were obtained from other existing numerical methods. The BEM, however, is not limited by the axial symmetry of the scatterer and is capable of handling, in principle, completely general scatterer shapes.

The Interaction Model

The essence of the method is the coupling of the two sets of integral equations which represent, respectively, the elastodynamic response of the solid and the acoustic behavior of the fluid in the presence of an incident wave. The equations are coupled through continuity and equilibrium conditions at the scattering boundary. Within the assumptions of the formulation, the model is exact and captures all aspects of the interaction, e.g., diffraction, transmission, mode conversion, etc.

The acoustic and elastic response may respectively be characterized by the total acoustic pressure \( p \) and the elastic displacement \( u \). In the fluid, the total pressure \( p \) is the sum of the incident pressure \( p^I \) and the scattered pressure \( p^S \), i.e.,

\[
p = p^I + p^S
\]

where \( p^S \) is always outgoing, i.e., it satisfies the Sommerfeld radiation condition at infinity [Appendix B]. A schematic picture of the interacting domains is shown in
Figure 2.1, where $B_i$ is the (interior) solid scatterer with closed surface $\partial B$ and $B_e$ is the (exterior) acoustic medium carrying the incident beam $p^I$ of frequency $f$. A unit normal $n$ is defined outward from the fluid domain pointing into the solid and $q$ is the opposite normal pointing from solid into the fluid ($q=-n$).

The response of the elastic displacement $u(x,t)$ in $B_i$ is governed by the elastodynamic momentum balance equation without body forces or the Cauchy-Navier equations

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot u(x,t)) + \mu \nabla^2 u(x,t)$$

(2.2)

where $\rho$ is the density of the solid and $\lambda$ and $\mu$ are Lamé constants.

Propagation of the acoustic pressure wave $p(x,t)$ in the fluid is defined by the wave equation

$$\nabla^2 p(x,t) = \frac{1}{c^2} \frac{\partial^2 p(x,t)}{\partial t^2}$$

(2.3)

where $c$ represents the acoustic wave speed in the fluid. This study is limited to time-harmonic analysis with harmonic variation $e^{-i\omega t}$ assumed for all field variables, i.e., $u(x,t) = u(x)e^{-i\omega t}$ where $\omega$ is the circular frequency ($\omega = 2\pi f$) in radians per sec. For this steady-state problem, $p$, $p^I$ and $p^S$ all satisfy the Helmholtz wave equation (2.3) and the elastic displacement $u$ satisfies the Cauchy-Navier elastodynamic equation (2.2).

Under the time-harmonic assumption, equations 2.2 and 2.3 reduce to the elliptic differential equations

$$\left(\frac{k^2}{k_T^2} - 1\right) \nabla (\nabla \cdot u(x)) + \nabla^2 u(x) + k_T^2 u(x) = 0 \quad x \in B_i$$

(2.4)

and

$$\nabla^2 p(x) = 0 \quad x \in B_e$$

(2.5)
Figure 2.1: Fluid-solid interaction: Schematic model
where \( k_L \) and \( k_T \) are the longitudinal and shear wave numbers in the solid, \( k \) is the acoustic wave number of the fluid such that \( k = \omega/c, k_L = \omega/c_L, k_T = \omega/c_T \) and \( c, c_L, c_T \) denote the respective wave speeds with

\[
c_L = \sqrt{(\lambda + 2\mu)/\rho}; \quad c_T = \sqrt{\mu/\rho}
\]

At the fluid-solid interface \( \partial B \), the pressure \( p \) and displacement \( u \) are required to satisfy the interface equilibrium and compatibility conditions which may be written as

\[
t(x) = -p(x)n(x) \quad x \in \partial B \tag{2.6}
\]
\[
\frac{\partial p}{\partial n}(x) = \rho \omega^2 u(x).n(x) \quad x \in \partial B \tag{2.7}
\]

where \( t \) and \( \frac{\partial p}{\partial n} \) respectively denote the elastic traction and acoustic pressure gradient in the normal direction and \( \rho \) is the fluid density.

The B.I.E. Formulation

The starting point of a BIE formulation is to convert the differential equation and boundary conditions into an integral representation involving the field variable and a known solution (the fundamental solution or the free space Green's function). For the exterior fluid, this is accomplished by the use of Green's second theorem. If \( x_i \) and \( x_e \) respectively denote points inside and outside the scatterer, i.e., \( x_e \in B_e \) and \( x_i \in B_i \), the representation for the acoustic field at a point \( x_e \) may be written as an integral over the scatterer boundary [Appendix C], i.e.,

\[
p(x_e) = \int_{\partial B} \left[ G(r) \frac{\partial p}{\partial n}(y) - \frac{\partial G}{\partial n}(r)p(y) \right] ds(y) + p^I(x_e) \quad x_e \in B_e \tag{2.8}
\]
A similar representation for the elastodynamic PDE (Eqn. 2.4) valid inside the scatterer $B_i$ is based on the Betti's reciprocal relation [20] and is written as

$$u(x_i) = \int_{\partial B} [U(r)t(y) - T(r)u(y)] da(y) \quad x_i \in B_i$$

(2.9)

where $y$ is on the boundary and is called the field point. The point $x$ ($x_i$ or $x_e$) is referred to as the source point and $r(x,y)$ is the distance vector between points $x$ and $y$. The kernels $G$, $\frac{\partial G}{\partial n}$, $U$ and $T$ are the fundamental solutions or free space Green's functions characterizing responses for point disturbances in the fluid and solid [Appendix A]. In the numerical integration scheme of the BEM, $\partial B$ is typically divided into surface elements and integration over each element is performed by quadrature.

The boundary integral equations

By taking the respective domain points $x$ to the boundary $\partial B$ (i.e., $x_i(B_i) \rightarrow y(\partial B)$ and $x_e(B_e) \rightarrow y(\partial B)$ etc.), the integral representations lead to Fredholm integral equations of the second kind. The kernels $G$ and $U$ are $O(1/r)$ and $\frac{\partial G}{\partial n}$ and $T$ are $O(1/r^2)$ as $r \to 0$ and hence the limiting process results in singularities whenever $x \to y$. A regularization [21] is required to make them amenable to numerical integration.

A regularization for the elastodynamic integral is obtained as follows. For an interior point $x_i$ and

$$u_0 = u(x_i)$$

(2.10)

one can write the representation integral (Eqn. 2.9) as
\[ u_0 = \int_{\partial B} [U_t - T_u] \, ds(y) \]  \hspace{1cm} (2.11)

where

\[ u = u(y), \quad t = t(y), \quad \partial B = \partial B(y) \]  \hspace{1cm} (2.12)

The traction \( t \) and the Stoke's stress tensor \( T \) (Eqn. 2.11) are defined in terms of the normal \( n \) outward (Figure 2.1) from the interior \( B_i \). If instead the direction \( n \) is taken as a basis, the representation can be rewritten as

\[ u_0 = \int_{\partial B} [T_u - U_t] \, ds(y) \]  \hspace{1cm} (2.13)

since the traction \( t \) and the tensor \( T \) vary directly with respect to the normal \( n \).

Introducing now the elastostatic free space Green's traction tensor \( T^S \) [Appendix A], Eqn. 2.13 is rewritten as

\[
\begin{align*}
  u_0 &= \int_{\partial B} (T - T^S) u \, ds \quad + \quad \int_{\partial B} T^S (u - u_0) \, ds \\
  &\quad + u_0 \int_{\partial B} T^S \, ds \quad - \quad \int_{\partial B} U_t \, ds \\
  &\hspace{1cm} (2.14)
\end{align*}
\]

Now the singular part of the integral is essentially expressed as a difference involving the static and the dynamic kernels. It is noted that both the static and the dynamic Green's functions have the same order of singularity, i.e., \( T \rightarrow T^S + O(\frac{1}{r}) \) as \( r \rightarrow 0 \) \[22\]. Also, by identity C.21 [Appendix C],

\[ \int_{\partial B} [-T^S_q(x, y)] \, ds = I \]  \hspace{1cm} (2.15)

where the subscript indicates the normal basis of the tensor \( T^S \). Using this identity in Eqn. 2.14 one obtains,
\[ \int_{\partial B} (T - T^S)ud_s + \int_{\partial B} T^S(u - u_0)ds - \int_{\partial B} Ut ds = 0 \quad (2.16) \]

The resulting integrals are now regular since \((T - T^S) \rightarrow O(\frac{1}{r})\) and \((u - u_0) \rightarrow O(r)\) whenever \(r \rightarrow 0\) and thus \((T - T^S)u\) and \(T^S(u - u_0)\) are \(O(\frac{1}{r})\). The integrals \(\int_{\partial B} (T - T^S)ud_s\) and \(\int_{\partial B} T^S(u - u_0)ds\) are therefore only weakly singular.

For computational purposes, the surface \(\partial B\) is divided into a singular part \(\partial B_s\) where \(x - y\) and a non-singular part \(\partial B_n = \partial B - \partial B_s\). Typically, \(\partial B_s\) is a boundary element, one of whose nodes is the collocation point. The regularized form of the BIE (Eqn. 2.16) is then re-written as

\[ \int_{\partial B_n} Tu ds - \int_{\partial B_n} T^S u_0 ds + \int_{\partial B_s} (T - T^S) ud_s + \int_{\partial B_s} T^S (u - u_0) ds - \int_{\partial B} Ut ds = 0 \quad (2.17) \]

or,

\[ \left( - \int_{\partial B_n} T^S ds + \int_{\partial B_s} (T - T^S) ds \right) u_0 + \int_{\partial B_n} Tu ds + \int_{\partial B_s} (T - T^S) ud_s - \int_{\partial B} Ut ds = 0 \quad (2.18) \]

where, for computational purposes, the integral on \(\partial B_s\) is rearranged as

\[ \int_{\partial B_s} (T - T^S) ud_s + \int_{\partial B_s} T^S (u - u_0) ds = \int_{\partial B_s} T (u - u_0) ds + \int_{\partial B_s} (T - T^S) u_0 ds \quad (2.19) \]

The term in parenthesis (Eqn. 2.18) explicitly shows the diagonal coefficient of \(u_0\). The weakly singular integrals \(\int_{\partial B_s} (T - T^S) ud_s\) and \(\int_{\partial B_s} T^S (u - u_0) ds\) can be
numerically evaluated by the polar coordinate transformation method originally implemented by Rizzo, Shippy and Rezayat [23].

A similar regularization using the static Green's function $G^S$ (Appendix A: fundamental solution for $\nabla^2 p = 0$) in the acoustic representation (Eqn. 2.8) results in the acoustic BIE. Consider an exterior point $x_e$ where

$$p_o = p(x_e) \quad (2.20)$$

The representation integral with respect to the normal $n$ is

$$p_o = \int_{\partial B} \left[ G \frac{\partial p}{\partial n} - p \frac{\partial G}{\partial n} \right] ds + p^I \quad (2.21)$$

Introducing the static Green's function $G^S$ (Appendix A), one obtains the form

$$p_o + \int_{\partial B} \left( \frac{\partial G}{\partial n} - \frac{\partial G^S}{\partial n} \right) p ds + \int_{\partial B} \frac{\partial G^S}{\partial n} (p - p_o) ds + \int_{\partial B} G \frac{\partial p}{\partial n} ds = p^I \quad (2.22)$$

The term $(p_o \int_{\partial B} \frac{\partial G^S}{\partial n} ds)$ vanishes due to the identity C.25 [Appendix C], i.e.,

$$\int_{\partial B} \frac{\partial G^S}{\partial n} (x_e, y) ds = 0 \quad (2.23)$$

Dividing the boundary $\partial B$ as before into $\partial B_s$ and $\partial B_n$, Eqn. 2.22 is rearranged to

$$p_o + \int_{\partial B_n} \frac{\partial G}{\partial n} ds - \int_{\partial B_n} \frac{\partial G^S}{\partial n} p ds + \int_{\partial B_s} \left( \frac{\partial G}{\partial n} - \frac{\partial G^S}{\partial n} \right) p ds + \int_{\partial B_s} \frac{\partial G^S}{\partial n} (p - p_o) ds - \int_{\partial B} G \frac{\partial p}{\partial n} ds = p^I \quad (2.24)$$
The terms in the $\partial B_3$-integrals can be further regrouped by adding and subtracting $p_0 \int_{\partial B_3} \frac{\partial G^S}{\partial n} ds$ to get a form with an explicit diagonal coefficient of $p_0$ of the form

\[
\left[ 1 - \int_{\partial B_n} \frac{\partial G^S}{\partial n} ds + \int_{\partial B_s} \left( \frac{\partial G}{\partial n} - \frac{\partial G^S}{\partial n} \right) ds \right] p_0 + \int_{\partial B_n} \frac{\partial G}{\partial n} p ds + \int_{\partial B_s} \frac{\partial G}{\partial n} (p - p_0) ds - \int_{\partial B} G \frac{\partial p}{\partial n} dS = p^I (2.25)
\]

Here, it is noted that

\[
\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r} (r \cdot n)
\]

and

\[
\lim_{r \to 0} (r \cdot n) = O(r) \quad (2.26)
\]

Thus, even though $G$ and $G^S$ are weakly $O(1/r)$ singular and $\frac{\partial G}{\partial r}$ has a $O(1/r^2)$ singularity, $\lim_{r \to 0} \frac{\partial G}{\partial n}$ is $O(1/r)$ and hence the integral $\int_{\partial B} \frac{\partial G}{\partial n} ds$ will not be strongly singular, unlike $\int_{\partial B} Tuds$. Thus the regularization using the difference terms $(\frac{\partial G}{\partial n} - \frac{\partial G^S}{\partial n})$ and $(p - p_0)$ need not be implemented here. Therefore cancelling the terms $(p_0 \int_{\partial B_s} \frac{\partial G}{\partial n} ds)$ and $(-\int_{\partial B_s} \frac{\partial G}{\partial n} p_0)$ from Eqn. 2.25, the acoustic BIE finally reduces to

\[
\left[ 1 - \int_{\partial B_n} \frac{\partial G^S}{\partial n} ds - \int_{\partial B_s} \frac{\partial G^S}{\partial n} ds \right] p_0 + \int_{\partial B_n} \frac{\partial G}{\partial n} ds + \int_{\partial B_s} \frac{\partial G}{\partial n} ds - \int_{\partial B} G \frac{\partial p}{\partial n} dS = p^I (x) \quad (2.27)
\]
Hence all the terms in the acoustic BIE are thus devoid of any strong singularities and may be evaluated using the method [23] for weakly singular integrals.

The elastic and the acoustic integral equations (2.18 and 2.27) or BIE's as they are called, are commonly represented as

\[
C_f(x)p(x) = \int_{\partial B} \left[ G(r)\frac{\partial p}{\partial n}(y) - \frac{\partial G}{\partial n}(r)p(y) \right] ds(y) + p^f(x) \quad (2.29)
\]

The kernel \( G(r) \) and \( U(r) \) as mentioned before are of order \( 1/r \) and \( T \) and \( \frac{\partial G}{\partial n} \) have singularity of order \( 1/r^2 \). The integrals involving the strong singularities are usually interpreted as a Cauchy principal value (CPV) integrals [19, 23] and hence the integral notation \( \int \) is used above. The CPV integrals in this problem are calculated indirectly (by regularization) as discussed above. The coefficients \( C_f(x) \) and \( C_s(x) \) explicitly shown above depend exclusively on the local geometry of \( \partial B \) at \( x \). In particular \( C_f \) is \( 1/2 \) and \( C_s \) is \( 1/2 \) (\( I \) is the identity tensor) if \( \partial B \) is locally smooth, i.e., has a unique tangent at \( x \).

**Computational Scheme**

Equations 2.18 and 2.27 or 2.28 and 2.29 represent the uncoupled BIE's where the field variables pressure \( p \), displacement \( u \), traction \( t \) and \( \frac{\partial p}{\partial n} \) satisfy the interface continuity equations (2.6 and 2.7) and are a priori unknown. The determination of these quantities requires a simultaneous solution of the two integral equations, after incorporating the continuity conditions appropriately.
The BEM is a numerical procedure to solve integral equations of these types and is, in some regards, similar to the finite element method in that it involves a discretization, although in the BEM, only the surface requires discretization. Similar to the FEM, the unknown variables are approximated by shape functions on the elements of the surface $\partial B$ so that the continuum of the unknown functions is replaced by discrete nodal values. Then, to determine these nodal values, the BIE's are collocated at each node to provide a finite, algebraic set of equations approximating the solution of the integral equations. Each element integration in the BEM is evaluated relative to all collocation points or nodes and hence couples all nodes and not just its neighboring nodes as in the FEM. This results in a matrix of equations that is fully populated and, due to the asymmetry of $T$ and $\frac{\partial G}{\partial n}$, non-symmetric.

Numerical approximations in the BEM

The boundary integral equations are made amenable to approximation by dividing the boundary into surface elements (Figure 2.2) and approximating the field variables within each element (by shape functions). The evaluation of the discretized integrals reduce the integral equations into a system of linear algebraic equations, which, upon applying the boundary conditions, can be solved for the unknown variables. Figure 2.2 shows one quarter of a sphere divided into surface elements.

The numerical procedure in the present work follows the scheme implemented by Rizzo et al. [23] and Rezayat et al. [24] and uses quadratic isoparametric shape functions [Appendix G] to approximate both the unknowns and the boundary geometry. The boundary is discretized into curvilinear quadrilateral or triangular elements mapped to standard squares or equilateral triangles, respectively. The quadrilateral
Figure 2.2: Boundary elements and the connectivity
and the triangles have eight and six nodes, respectively, with the corner nodes numbered consecutively, followed by the mid-side nodes, in clockwise fashion (Figure 2.2). The triangles are considered as degenerate quadrilaterals, three of whose nodes (2nd, 3rd and 7th) have collapsed into one. The clockwise numbering convention sets the normal direction \( n \) for the element. The element mapping is shown in Figure 2.3.

If a boundary \( \partial B \) is divided into \( m \) elements and \( \xi_i \) denotes the local coordinates on each element, then a variable \( \phi \) over an element is represented as

\[
\phi(\xi) = \sum_{j=1}^{n} N_j(\xi) \phi_j
\]

where \( n \) is the total number of nodes on the element, \( \phi_j \) are the nodal values of \( \phi \) on the element and \( N_j(\xi) \) are the shape functions. Second order or quadratic shape functions of the quadrilaterals and triangles in terms of local coordinates \( \xi \) are listed in the Appendix G.

The surface geometry of each element is similarly approximated by a quadratic surface passing through the element nodes and the global cartesian coordinate \( x \) of any point on an element is represented as

\[
x = \sum_{j=1}^{n} N_j(\xi) x_j
\]

where \( x_j \) is the coordinate of the \( j \)-th node.

**Numerical evaluation of integrals**

The formulation discussed above involves integrals of the generic form

\[
I = \int_{\partial B} \phi(y) K(x, y) ds(y)
\]
Figure 2.3: Mapping of curvilinear elements
where \( \phi \) and \( K \) are the relevant field variables and the corresponding kernel. If \( \partial B \) is divided into \( m \) elements, each with \( n \) nodes and area \( \partial e \), the integrals is represented as the sum

\[
I = \sum_{i=1}^{m} \sum_{j=1}^{n} \phi_{ij} \left\{ \int_{\partial e} N_j(\xi) K(x, y(\xi)) J(\xi) \, ds(\xi) \right\} = \sum_{i=1}^{m} \sum_{j=1}^{n} \phi_{ij} \int_{\partial e} F(r(x, \xi)) \, ds(\xi)
\]

where, \( \phi_{ij} \) is the value of \( \phi \) at the \( j \)-th node of \( i \)-th element and \( J(\xi) \) is the Jacobian of coordinate transformation, i.e.,

\[
ds = J(\xi_1, \xi_2) d\xi_1 d\xi_2
\]

The integrals on each element can then be evaluated by Gaussian quadrature [Appendix G].

Integration for weakly singular kernels

If the function \( F(r(x, \xi)) \) in the integral above is weakly singular, i.e., \( O(1/r) \), the above integration has to be modified. Consider a quadrilateral (Figure 2.4), where the collocation point \( x \) is either a corner node or a mid-side node of the element. For a corner node, (say, the 4th node), the square is divided into two triangles diagonally with the distance \( r(x, \xi) \) expressed as \( r(\rho, \theta) \) in the local polar coordinates \( \rho \) and \( \theta \) as shown. The integral \( \int_{\partial e} F(x, \xi) d\xi_1 d\xi_2 \) over the element may then be expressed as integrals over the two sub-triangles

\[
I = \int_{-1}^{1} \int_{-1}^{1} F(x, \xi_1, \xi_2) d\xi_1 d\xi_2
\]

\[
= \sum_{\Delta=1,2} \int_{0}^{\pi/4} \left\{ \int_{0}^{2 \sec \theta} F(x, \rho, \theta) \rho d\rho \right\} d\theta
\]

\[
= \int_{0}^{\pi/4} \{ F(x, \theta) \} \, d\theta
\]

(2.30)
The element area $d\xi$ now becomes $\rho d\rho d\theta$ and the extra $\rho$ is $O(r)$ for sufficiently small $r$. It can be shown that the integrand under the transformation is non-singular and standard Legendre-Gauss quadrature is applicable on each triangle with minor modification. If the point $x$ is a mid side node of a quadrilateral, the mapped square needs to be divided into three triangles. If the collocation point $x$ is either a corner or a midside node of a triangular element, the triangle is divided into two sub-triangles (Figure 2.4) and the approach above is again used for each sub-triangle.

![Figure 2.4: Scheme for singular integrations](image-url)
System of equations

Following discretization of $\partial B$ (six and eight node bi-quadratic elements are used here) and subsequent nodal collocation and quadrature using isoparametric functions as detailed in refs. [23] and [24], Eqns. 2.28 and 2.29 become a finite system of algebraic equations. For a discretization involving $N$ nodes, Eqns. 2.28 and 2.29 may be written in discrete matrix form as

\[
[A] \{u\} + [B] \{t\} = \{0\} \tag{2.31}
\]

and

\[
[C] \left\{ \frac{\partial p}{\partial n} \right\} + [D] \{p\} = \{p^I\} \tag{2.32}
\]

where $[A],[B]$ are complex $3N \times 3N$ matrices and $[C],[D]$ are complex $N \times N$ matrices.

Using a discrete form of the boundary conditions (2.6) and (2.7), $\{t\}$ and $\left\{ \frac{\partial p}{\partial n} \right\}$ are eliminated from the Eqns. 2.31 and 2.32 to give a coupled system of equations in terms of $u$ and $p$ only, i.e.,

\[
[A] \{u\} + [B] \{-pn\} = \{0\} \tag{2.33}
\]

\[
[C] \left\{ (\rho\omega^2)n.u \right\} + [D] \{p\} = \{p^I\} \tag{2.34}
\]

or,

\[
[A] \{u\} + [B^*] \{p\} = \{0\} \tag{2.35}
\]

\[
[C^*] \{u\} + [D] \{p\} = \{p^I\} \tag{2.36}
\]

where, from Eqns. 2.6 and 2.7,

\[
[B] \{t\} = [B] \{-pn\} = [B^*] \{p\} \tag{2.37}
\]
and

\[
[C] \left\{ \frac{\partial p}{\partial n} \right\} = [C] \left\{ \rho \omega^2 u \cdot n \right\} = [C^*] \{ u \} \tag{2.38}
\]

Here, \([B^*]\) is a \(3N \times N\) matrix and \([C^*]\) is \(N \times 3N\).

\section*{Solution Strategies}

Equations 2.35 and 2.36 represent the \emph{coupled} system of equations which can be combined to give

\[
\begin{bmatrix}
A & B^* \\
C^* & D
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
p
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
p^T
\end{bmatrix} \tag{2.39}
\]

or,

\[
[K] \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ p^T \end{bmatrix} \tag{2.40}
\]

The combined system \([K]\) is a \(4N \times 4N\) matrix. In principle, one could evaluate \(\{u\}\) and \(\{p\}\) by reduction of the combined matrix \(K\). However, the matrix is normally ill-conditioned, unless properly scaled whenever realistic material parameters (such as steel in water) are used in the coefficient matrix. Such a combination is sometimes referred to as \emph{weakly coupled} as it represents the merger of two physically different sets of equations. The fact that the interface condition (Eqns. 2.6 and 2.7) provide coupling only in the normal direction may further contribute to this weak coupling (see discussion on fictitious eigenfrequencies in Chapter 3).
Ill-conditioning and scaling

The ill-conditioning of Eqn. 2.40 is artificially induced by the combination of the fluid and the solid equation within the same system. Order of magnitude estimates of the components $A, B^*, C^*$ and $D$ of the combined matrix may be related to the kernel functions $U, T, G$ and $\frac{\partial G}{\partial n}$ as follows:

$$A \simeq O(T); \quad B^* \simeq O(U)$$
$$C^* \simeq O(\rho\omega^2 G); \quad D \simeq O(\frac{\partial G}{\partial n})$$

Further examination of the kernel functions [Appendix A] shows that one can express the dominant orders of magnitude of the above quantities as

$$A \simeq O\left(\frac{\mu}{\rho\omega^2}\right); \quad B^* \simeq O\left(\frac{1}{\rho\omega^2}\right)$$
$$C^* \simeq O(\rho\omega^2); \quad D \simeq O(1)$$

where $\mu$ denotes the Lamé constant. In terms of the units used in the examples (Table 2.3), $O\left(\frac{\mu}{\rho\omega^2}\right)$ may be taken as $O(1)$ for intermediate frequencies. Thus the components $A, B^*, C^*$ and $D$ of a typical matrix may have the following distribution

$$O\left(\begin{bmatrix}A & B^* \\ C^* & D\end{bmatrix}\right) = \begin{bmatrix}O(1) & O\left(\frac{1}{\rho\omega^2}\right) \\ O(\rho\omega^2) & O(1)\end{bmatrix}$$

The existence of these large numbers $O(\rho\omega^2)$ (typically $10^{10} - 10^{14}$) and very small numbers $O\left(\frac{1}{\rho\omega^2}\right)$ makes the condition number of the matrix very high. The matrix $[K]$ may be scaled so that all the coefficients are of the same order of magnitude.
Since the stress tensor \( \tau \) and the displacement \( u \) are related as

\[
\tau_{ij} = \lambda u_{k,k} \delta_{ij} + \mu u_{i,j} + u_{j,i}
\]

i.e.,

\[
\tau \simeq O(\mu \nabla u)
\]

the order of magnitude of traction \( t_i = \tau_{ij}n_j \) is therefore

\[
t = O(\mu \nabla u) \simeq O(\mu \frac{u}{l})
\]

i.e.,

\[
u \simeq O(\frac{1}{\mu})
\]

where \( l \) is some length parameter, e.g., the largest distance between two nodes.

Thus for an input \( p^I = O(1) \),

one will have \( t = O(1) \)

and \( u = O(\frac{1}{\mu}) \)

Now if one uses a modified or scaled \( u \), denoted by \( u^S \), such that

\[
u^S = \frac{\mu u}{l}
\]

then,

\[
O(u^S) = 1
\]

The scaled (and coupled) equations then become

\[
[A] \left\{ \frac{\mu u}{l} \right\} + \left( \frac{\mu}{l} \right) [B^*] \{p\} = \{0\}
\]

\[
\left( \frac{1}{\mu} \right) [C^*] \left\{ \frac{\mu u}{l} \right\} + [D] \{p\} = \{p^I\}
\]

or,

\[
[A] \{u^S\} + \left( \frac{\mu}{l} \right) [B^*] \{p\} = \{0\}
\]

\[
\left( \frac{1}{\mu} \right) [C^*] \{u^S\} + [D] \{p\} = \{p^I\}
\]
Comparing Eqns. 2.43 and 2.49, it can be seen that the coefficients of $u^S$ and $p$ now have comparable orders of magnitude. The solution of the combined system therefore should incorporate such scaling and use at least double precision computations. The modified system for the scaled displacement with $u^S \approx O(1)$ is

$$
\begin{pmatrix}
    A & \mu B^* \\
    \frac{l}{\mu} C^* & D
\end{pmatrix}
\begin{pmatrix}
    u^s_1 \\
    u^s_2 \\
    u^s_3 \\
    p
\end{pmatrix}
= 
\begin{pmatrix}
    0 \\
    0 \\
    0 \\
    p^I
\end{pmatrix}
$$

(2.50)

The unscaled values of $u$ may be obtained from the displacement solutions of the scaled matrix. With the scaling described, it was found that the coupled system exhibits no instabilities or ill-conditioning during the simultaneous solution except at the eigenfrequencies to be discussed in the next chapter. One can, however, avoid a simultaneous solution of the combined matrix by using one of the following two approaches.

The first approach keeps the individual systems separate and substitutes the equivalent expression for the scalar unknown $p$ into the solid equation as follows. From Eqn. 2.36, the pressure $p$ can be determined in terms of $\{p^I\}$ and $\{u\}$ as

$$
[D] \{p\} = \{p^I\} - [C^*] \{u\}
$$

or,

$$
\{p\} = [D]^{-1} \left[ \{p^I\} - [C^*] \{u\} \right]
$$

(2.51)
Substituting this expression for \( \{p\} \) in Eqn. 2.35, one then has

\[
[A] \{u\} = -[B^*] \{p\} = -[B^*][D^{-1}] \{p^I\} + [B^*][D^{-1}][C^*] \{u\}
\]

i.e.,

\[
[A - B^*D^{-1}C^*] \{u\} = -[B^*D^{-1}] \{p^I\}
\]

(2.52)

From Eqn. 2.52, one can solve for \( \{u\} \) so that once it is known, the solutions for \( \{p\} \) is easily determined by Eqn. 2.51.

The second approach builds the elastic solution in an iterative way. The fluid equation (2.32) is first solved for the pressure by assuming a rigid scatterer, i.e., \( \frac{\partial p}{\partial n} = 0 \), or equivalently through Eqn. 2.7, \( u=0 \). Then the elastic equation (2.31) is solved as a Neumann problem, where applied tractions are obtained from the pressure resulting from the coupling condition Eqn. 2.6. The fluid equations are then solved again with the now known pressure gradients as obtained from the displacements \( \frac{\partial p}{\partial n} = \rho \omega^2 u.n \) by Eqn. 2.38. The process is repeated until some preset convergence criterion is reached. Figure 2.5 shows the iterative cycle of the process.

The first approach appears to be numerically less elegant as it involves actual matrix inversion and many matrix multiplications. Experience with the second (iterative) approach has shown that it generates reliable solutions at low frequencies. All results presented in this paper were obtained either by the iterative approach or the combined solution (Eqn. 2.50). Use of either solution scheme (i.e., Eqn. 2.50 or Eqns. 2.51-2.52 or iteration) appears to be a matter of choice, although at a given frequency or discretization, one may sometime prove to be more convenient than the other, e.g., a combined approach requires the solution of a \( 4N \times 4N \) system and can
Iterative Approach To Coupled Fluid/Solid Interaction

ELASTIC PROBLEM

ACOUSTIC PROBLEM

ASSUME RIGID SCATTERER \( \frac{dp}{dn} = 0 \)

ACOUSTIC RESPONSE

CALCULATE TRACTION \( t = -p n \)

APPLY TO SOLID

PRESSURE \( p \)

STRUCTURAL RESPONSE

APPLY TO FLUID

DISPLACEMENT \( u \)

Calculate \( \frac{dp}{dn} \) From \( u \)

Figure 2.5: Acoustic-elastic interaction: Iterative solution
be restrictive for small computers due to the higher storage requirements in the intermediate computations, whereas both the other approaches discussed above require the reduction of a maximum of a 3N×3N system. The iterative approach provides fast convergence for very hard scatterers (e.g., tungsten carbide) but requires more iterations for softer materials.

The bulk of the computation lies in constructing the coefficient matrix and obtaining the boundary solution. The field within the scatterer or anywhere in the acoustic fluid can then be easily computed using the known surface data through representation integrals like Eqns. 2.28 and 2.29, where the free term coefficients $C_s$ and $C_f$ will be 1 and 1, respectively, $x$ is the field point and all integrals are regular.

**Far Field Solutions**

Available solutions are mostly for far field [25, 26, 27, 28] scattered acoustic pressures only. Therefore, for comparison, the boundary solutions of the integral equations were used to further evaluate the scattered field in the exterior fluid using the integral representation. Data shown in the subsequent comparison are the far field values computed from the BEM surface solution through the acoustic integral representation

$$p^S(x_f) = \int_{\partial B} \left[ G(r) \frac{\partial p}{\partial n}(y) - \frac{\partial G}{\partial n}(r)p(y) \right] ds(y) \quad (2.53)$$

where, $p(y)$ and $\frac{\partial p}{\partial n}(y)$ are the boundary solutions on the scatterer surface $\partial B$, $x_f$ is a exterior field point in the far field and $r = r(x_f, y)$.

The far field solution can be normalized and expressed in a form independent
of the actual far field distance as follows. Expressing \( \frac{\partial G}{\partial n} \) in above representation as \\((\frac{\partial G}{\partial r})(\frac{\partial}{\partial n})\), one obtains

\[
4\pi p^S(x_f) = \int_{\partial B} \left[ \frac{\partial p}{\partial n} \left( \frac{e^{ikr}}{r} \right) - p \left( \frac{\partial r}{\partial n} \right) \left( \frac{ike^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right) \right] ds(y)
\]  

(2.54)

If \( x_f \) is located very far away from the scatterer (Figure 2.6), i.e., \( r \approx R - x, y, r \rightarrow \infty \) (Figure 2.6), one can use the approximations

\[
\frac{e^{ikr}}{r^2} \approx 0 \quad r \rightarrow \infty
\]

(2.55)

and

\[
\frac{e^{ikr}}{r} \approx \left( \frac{e^{ikR}}{R} \right) e^{-ik(x, y)}
\]

(2.56)

where \( R = |x_f| \) and \( x = x_f/R \). The far field representation now reduces to the form

\[
4\pi p^S(x_f) = \frac{e^{ikR}}{R} \int_{\partial B} \left( e^{-ikx, y} \left( \frac{\partial p}{\partial n} - ik \frac{\partial p}{\partial n} \right) \right) ds(y)
\]

(2.57)

or

\[
p^S = \frac{e^{ikR}}{R} A(x)
\]

(2.58)

where,

\[
A(x) = \int_{\partial B} \left( \frac{e^{-ikx, y}}{4\pi} \left( \frac{\partial p}{\partial n} - ik \frac{\partial p}{\partial n} \right) \right) ds(y)
\]

(2.59)

Thus \( A(x) \) is an expression for far field scattered pressure independent of the far field distance.
Figure 2.6: Fluid(exterior)-solid(interior) domain and the far field
Numerical Examples

Presented here are a series of problems to verify and exemplify the technique, progressing from geometrically simple problems with known analytical solution to more complex problems where comparisons are made with other numerical solutions. A few cases are presented at the end without comparison just to illustrate the ability of the numerical scheme to simulate data for a wide range of scattering configurations. The densities and wave speeds of the scatterer and host materials used in various examples are shown in Table 2.3. All examples presented are for a plane incident wave

\[ p^I(x) = p_0(x)e^{i\mathbf{k}\cdot\mathbf{x}} \]  
\[ \mathbf{k} = \left(\frac{\omega}{c}\right)\mathbf{e} = \mathbf{k}\mathbf{e} \]

where \( \mathbf{e} \) denotes the direction vector of the incident wave, although other more sophisticated incident wave models can be easily incorporated into the code (see e.g., [29]).

For calculations at points in the far field, the scattered pressure \( p^S \) is normalized with respect to the far field distance \( R \) and wave number \( k \) such that

\[ p_{\text{norm}} = \left(\frac{p^S}{p^I}\right)kR \]

**Boundary solution for a sphere**

As a first demonstration problem the boundary solution for a brass sphere submerged in glycerine is compared with a hybrid finite element (NASTRAN)/boundary element solution [30, 31]. The incident plane wave approaches along the Z-axis (Figure 2.7) and the solutions are compared at five nodes in the Y-Z plane at 45° interval.
Figure 2.7: Plane wave scattering by a sphere
Table 2.1: Pressure at Boundary Nodes

<table>
<thead>
<tr>
<th>Angle</th>
<th>BEM Mag.</th>
<th>tanθ</th>
<th>FEM-BEM Mag.</th>
<th>θ</th>
<th>tanθ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>1.306</td>
<td>4.627</td>
<td>1.304</td>
<td>77.8</td>
<td>4.625</td>
</tr>
<tr>
<td>45°</td>
<td>1.214</td>
<td>1.657</td>
<td>1.229</td>
<td>58.8</td>
<td>1.651</td>
</tr>
<tr>
<td>90°</td>
<td>0.983</td>
<td>0.147</td>
<td>0.962</td>
<td>8.3</td>
<td>0.146</td>
</tr>
<tr>
<td>135°</td>
<td>0.888</td>
<td>-1.532</td>
<td>0.898</td>
<td>303.1</td>
<td>-1.532</td>
</tr>
<tr>
<td>180°</td>
<td>0.973</td>
<td>-12.95</td>
<td>0.971</td>
<td>274.2</td>
<td>-13.00</td>
</tr>
</tbody>
</table>

Table 2.2: Normal Velocity at Boundary Nodes

<table>
<thead>
<tr>
<th>Angle</th>
<th>BEM Mag.</th>
<th>tanθ</th>
<th>FEM-BEM Mag.</th>
<th>tanθ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>7.236</td>
<td>-.151</td>
<td>7.256</td>
<td>-.151</td>
</tr>
<tr>
<td>45°</td>
<td>4.929</td>
<td>-.045</td>
<td>4.955</td>
<td>-.045</td>
</tr>
<tr>
<td>90°</td>
<td>0.907</td>
<td>-10.7</td>
<td>0.915</td>
<td>-11.0</td>
</tr>
<tr>
<td>135°</td>
<td>5.146</td>
<td>-.118</td>
<td>5.174</td>
<td>-.119</td>
</tr>
<tr>
<td>180°</td>
<td>7.360</td>
<td>-.011</td>
<td>7.384</td>
<td>-.014</td>
</tr>
</tbody>
</table>

The results for the pressure $p$ and the normal velocity $v_n = iωu_n$ are shown in Table 2.1 and Table 2.2 as magnitude-phase($θ$). They agree satisfactorily and lends confidence to the far field solutions that are presented in the following.

**Scattered far field for the sphere**

An analytical solution for the far-field pressure ([25, 32], also Appendix D) is available. With a host medium of either water or glycerine, computations were made (a) for an angular distribution of the scattered pressure in the far field for a fixed frequency and (b) for backscattered (pulse-echo type, 180°) and bistatic (pitch-catch type, 90°) echoes for a range of frequencies. The scatterer material is taken to be either aluminum, brass or lucite and the exact solution for the far field scattered pressure is expressible in series form with reflection coefficient $A_n$ for each term of the
series [25]. The analysis of Lin and Raptis [25], as was developed for a viscous fluid, is reduced to the inviscid case in Appendix D. The coefficients \( A_n \) are determined from the continuity conditions at the interface in terms of the spherical Bessel and Hankel functions. The basic form for the normalized scattered pressure \( p_{\text{norm}} \) at far-field (Appendix D) is

\[
\text{p}_{\text{norm}} = \frac{p_{kr}}{p_0} = \sum_{n=0}^{\infty} (2n + 1)A_n P_n \cos(\theta) \tag{2.63}
\]

where \( P_n \) are Legendre’s functions and \( \theta \) is the angle relative to incident direction.

Figures 2.8 and 2.9 compare the angular distribution of the far field (polar plot) for a brass sphere submerged in glycerine for \( ka = 1 \) and \( ka = 2 \) respectively. The data represent the amplitude of the normalized scattered pressure \( p_{\text{norm}} \) in different directions. Results at \( 0^\circ \) represents forward scattering and those at \( 180^\circ \) are the backscattered echo (Figure 2.7). Figures 2.10 and 2.11 present the polar plot for an aluminum and lucite sphere, respectively, in water at \( ka = 3 \). The scattered field from a lucite sphere is of particular significance as it is typically used in NDE experiments as a host material with embedded inclusions. Figures 2.12 and 2.13, respectively, present the field for an aluminum sphere in water and a brass sphere in glycerine at the higher frequency, \( ka = 7 \). Note the growth of the forward scattering lobe at higher \( ka \)'s which are accurately determined by the BEM calculations. Figures 2.14 and 2.15 compare the far field for an aluminum sphere in water for \( ka = 5 \) and \( ka = 6 \). Two different discretizations were used for these computations as shown in Figure 2.16. With the coarser mesh (96-element), the computations show some deterioration at the forward scattering zone but the error is decreased through use of the finer mesh (144-element model).
Figure 2.8: Polar plot: Far field scattered pressure (brass sphere in glycerine, $ka=1$)
Figure 2.9: Polar plot: brass/glycerine $ka=2$
Figure 2.10: Polar plot: sphere (aluminum sphere in water) $ka=3$ [solid line: Exact; Δ: BEM (105 node)]
Figure 2.11: Polar plot: sphere (lucite/water) $ka=3$ [solid line: Exact; •: BEM]
Figure 2.12: Polar plot: sphere (aluminum/water) $ka=7$ [solid line: Exact; •: BEM]
Figure 2.13: Polar plot: sphere (brass/glycerine) $ka=7$ [solid line: Exact; •: BEM]
Figure 2.14: Effect of mesh refinement (aluminum sphere in water) $ka=5$.
Line: Exact; •: BEM(144 Elem.); Δ: BEM(96 Elem.)
Figure 2.15: Effect of mesh refinement (aluminum sphere in water) $ka=6$.
Line: Exact; •: BEM(144 Elem.); Δ: BEM(96 Elem.)
Figure 2.16: BEM discretization (sphere) top: 96 Element; bottom: 144 Element
Figure 2.17 shows the spectrum (amplitude vs. frequency) of the backscattered and bistatic (90°) echoes for an aluminum sphere with the scattered amplitude plotted against non-dimensional frequency $ka$. For the frequency range considered, the BEM data are in consistent agreement with the far field values as calculated from the analytical (series) solution [Appendix D].

**Spheroidal and cylindrical scatterers**

Illustrative problems for nonspherical scatterers are given here for two geometries. The first is a prolate spheroid with an aspect ratio of 2:1 and the second is a cylinder with hemispherical end caps (Figure 2.18). Far field scattering amplitudes for the backscattered and bistatic (90°) are plotted against non-dimensional wave number $ka$, where, for the spheroid, $a$ is half the major diameter and for the capped cylinder, $a = h + r$; $h$ being half the length of cylinder and $r$ the cap radius. In all cases the incident wave is end-on, i.e., angle of incidence $\theta^I = 0^\circ$ as shown in the configuration for spheroid (Figure 2.19).

Figure 2.20 compares the BEM data with results determined by the T-matrix method [27, 28] and the hybrid finite element (NASTRAN)/boundary element solution [30, 31] for a 2:1 aluminum spheroid in water. Both the backscattered and bistatic (90°) spectrum are shown. The hybrid FEM/BEM scheme used a fine mesh (2000 interior nodes and 332 surface nodes) with bi-linear surface elements. BEM solutions were determined for two different meshes. Results for the 121-element model (155 nodes on half spheroid) agree well with the FEM/BEM data at low frequencies, but shows some differences for $ka > 2$. Use of a more uniform and finer mesh with 140 elements (229 nodes on half spheroid) improved the agreement dramatically.
Figure 2.17: Frequency distribution of far field (aluminum sphere in water)

Figure 2.18: BEM mesh (a) prolate spheroid (b) cylinder with hemispherical caps
Figure 2.19: Scattering configuration of a prolate spheroid
Figure 2.20: Scattered spectrum from aluminum spheroid. [top: Backscattering; bottom: Bistatic(90°)]
The T-matrix results, although reflecting the same trend as the above two, de­parts significantly in the numerical values. Reference [33] provides an informative discussion on the computational aspects of BIE and T-matrix method and gives some insight into these discrepancies. Figure 2.20 also shows the scattered field from a rigid spheroid. The significant differences with the scattered field from the elastic spheroid emphasizes the need for a coupled acoustic-elastic formulation for such analysis.

The backscattered spectrum for a capped aluminum cylinder (Figure 2.21) re­flects a trend similar to the spheroid and are compared only with T-Matrix data due to the unavailability of other comparable solutions. Agreement between the BEM results for the two different meshes (171 and 229 nodes on half cylinder) lends confi­dence to the BEM results for the frequency range shown in the figure.

More numerical experiments

The capability of the BEM code to generate new scattering data as function of material property (impedance), scatterer geometry (shape) and orientation (angle of incidence) is now exploited to illustrate the potential of the method as a tool for numerical simulation.

Figure 2.22 shows the far field BEM data for a spheroid of four solids (aluminum, lucite, tungsten-carbide and solder) for $ka=1.25$, aspect ratio 2:1 and direction of incidence along the axis of revolution ($\theta^I = 0^\circ$) of the spheroid (see Figure 2.19). The BEM solution appropriately captures the relative elasticity of the different inclu­sions and the magnitude of the backscattered amplitude clearly indicates the relative
Figure 2.21: Scattered spectrum from capped cylinder [top: Backscattering; bottom: Bistatic (90°)]
Table 2.3: Material Properties.

<table>
<thead>
<tr>
<th>Material</th>
<th>Density $\text{g/cm}^3$</th>
<th>$c_L$ cm/$\mu$s</th>
<th>$c_T$ cm/$\mu$s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>2.70</td>
<td>0.638</td>
<td>0.312</td>
</tr>
<tr>
<td>Brass</td>
<td>8.50</td>
<td>0.430</td>
<td>0.215</td>
</tr>
<tr>
<td>Lucite</td>
<td>1.18</td>
<td>0.272</td>
<td>0.134</td>
</tr>
<tr>
<td>Tungsten-Carbide</td>
<td>13.82</td>
<td>0.666</td>
<td>0.398</td>
</tr>
<tr>
<td>Solder</td>
<td>8.41</td>
<td>0.301</td>
<td>0.145</td>
</tr>
<tr>
<td>Water</td>
<td>1.00</td>
<td>0.149</td>
<td>-</td>
</tr>
<tr>
<td>Glycerine</td>
<td>1.26</td>
<td>0.191</td>
<td>-</td>
</tr>
</tbody>
</table>

Impedance of the scatterer.

The effect of the scatterer shape can be similarly studied by generating data first for a sphere and then for narrower spheroids of the same overall length but gradually decreasing aspect ratios (4:3, 2:1, and 4:1) as depicted in Figure 2.23. For a scatterer of aluminum and a direction of incidence again along the axis of revolution, Figure 2.24 shows the far field scattered pressure distribution at $ka = 2$. It is observed that the amplitude in all observation directions reduces sharply as the scatterer cross section, as viewed from the direction of incidence, diminishes.

The next illustration of Figure 2.25 compares the scattered field from a 2:1 aluminum spheroid for different angles of incidence. For angle of incidence $\theta^I = 0^\circ$ and $\theta^I = 90^\circ$, the direction of incidence coincides with an axis of symmetry of the scatterer. This symmetry is reflected in the scattered field pattern. For $\theta^I = 45^\circ$, scatterer orientation is asymmetric with respect to the direction of incidence and so is the far field pattern. The lobe in the forward shadow however clearly indicates the angle of incidence.
Figure 2.22: Scattering patterns from spheroids of different impedances

Figure 2.23: BEM meshes for spheres of four aspect ratios [from left: 1:1, 4:3, 2:1, 4:1]
Figure 2.24: Scattering from spheroids of four aspect ratios at k=2
Figure 2.25: Scattering from 2:1 spheroid at different angles of incidence
Discussion

In this chapter, a BIE/BEM formalism has been described to solve scattering from submerged, bounded elastic structures under time-harmonic acoustic loads and several examples with incident plane waves have been solved. The BIE formulation of the problem and their regularization for the numerical implementation is presented in detail and the various solution schemes have been discussed. The full space Green's functions or fundamental solutions of elastodynamic and acoustic wave equations used in the formulation are shown in Appendix A. The combined matrix generated by the numerical procedure is fully populated and was solved either directly or iteratively by solving the fluid and solid system, one after the other. Standard LINPACK [34] routines were used to solve the complex, linear system of equations. All other matrix manipulations were performed by IMSL routines. The numerical scheme based on the integral equation formulation is computationally intensive but the computational demand is largely offset by the need to model only the surface of the structure.

The coupled fluid/solid equations are written as a pressure-displacement formulation and are solved for the total boundary pressure and elastic displacement. Traction and the pressure gradients, whenever required, are extracted from the boundary continuity conditions and far field data are computed from the boundary solution through appropriate post-processors. Comparison with exact solutions and other numerical results shows that the coupled formulation and the solution scheme works well for various simple shapes, both on the boundary and in the far field. It has been shown that for a metallic scatterer, the scattering characteristics could be considerably different than the rigid behavior, and hence a coupled acoustic-elastic analysis may be crucial in many applications.
CHAPTER 3. THE FICTITIOUS EIGENFREQUENCY PROBLEM

Introduction

Integral equation methods for exterior scattering problems in acoustics are known to have uniqueness problems at certain discrete frequencies. These frequencies correspond to the characteristic (eigen) frequencies of the interior problem of a 'fictitious' body of the same shape and volume as the scatterer but containing the exterior fluid with appropriate boundary conditions. With the type of integral equations used here, non-uniqueness will be experienced only at the eigenfrequencies of the interior Dirichlet problem [35]. The difficulty is due to a breakdown in the integral equations for the scatterer at certain frequencies; not to a non-uniqueness of the physical problem [36]. In a discretized version of the integral equations, the problem manifests itself by ill-conditioning of the algebraic equations near the critical frequencies.

One way to remedy this problem is to overdetermine the algebraic system with supplementary equations as obtained from the Helmholtz representation integral for the fluid evaluated at points inside the scatterer (see Schenck [37]). This method of overdetermination is popularly known as CHIEF (Combined Helmholtz Integral Equation Formulation) [33, 37].

The coupled integral equation formulation used herein for scattering/transmission problems will suffer similar non-uniqueness difficulties as discussed by Martin [38].
The frequencies at which ill-conditioning or non-uniqueness is expected could be either of the cases (a), (b) and (c) as follows:

a. Frequencies that correspond to the eigensolutions for a fluid body of the same shape as the scatterer with Dirichlet boundary condition, i.e., the solutions of the problem

\[(\nabla^2 + k^2)p(x) = 0 \quad x \in B_i\]
\[p(x) = 0 \quad x \in \partial B\]

b. Eigen solutions for the same fluid body as above but with Neumann boundary condition, i.e., pressure field that satisfies

\[(\nabla^2 + k^2)p(x) = 0 \quad x \in B_i\]
\[\frac{\partial p}{\partial n}(x) = 0 \quad x \in \partial B\]

c. Frequencies that correspond to the free ‘torsional’ eigensolutions of the solid scatterer, i.e., solutions of the problem

\[
\left(\frac{k_L^2}{k_T^2} - 1\right)\nabla(\nabla \cdot u) + \nabla^2 u + k_T^2 u = 0 \quad x \in B_i
\]

or,

\[
\left(\frac{k_L^2}{k_T^2}\right)\nabla(\nabla \cdot u) - \left(\frac{1}{k_T^2}\right)\nabla \times (\nabla \times u) + u = 0 \quad x \in B_i
\]

with the boundary conditions

\[t(x) = 0\]
\[u \cdot n(x) = 0 \quad x \in \partial B\]
where $u_n = u \cdot n$ denotes the normal component of displacement. This last case is included for its special physical significance even though it is not exactly a 'fictitious frequency'. The continuity conditions of the fluid-solid interaction (Eqns. 2.6 and 2.7) are prescribed only in the normal ($n$) direction and since there is no interface coupling between the solid and the fluid for torsional displacement and stress, the fluid cannot share the motion of the solid as described by the above conditions. This intuitively may give rise to non-uniqueness in the solution and can be explained as a manifestation of the 'weak coupling' of the interaction model. This set of frequencies is informally referred to as Jones frequencies, after D. S. Jones, who emphasized their influence in this class of problems [39].

Without further theoretical details, examined below are the different sets of frequencies for a spherical scatterer for which the necessary eigenvalues are easily derived. The intent here is to examine the nature of our solution at or near those frequencies; not to suggest a remedy. However, the CHIEF method [37] as previously mentioned, if needed, could be incorporated into the solution procedure.

**Numerical Solutions at Fictitious Eigenfrequencies**

**Interior Dirichlet case**

For a spherical scatterer, non-dimensional wave numbers $ka$ corresponding to eigenfrequencies of interior Dirichlet problem ($p(a) = 0$) are zeroes of spherical Bessel function of first kind $j_n^m(ka) = 0$, where $a$ is the sphere radius, $n$ denotes the modes and $m$ denotes the harmonics. The solution by separation of variables is discussed in the Appendix E and the first three of these eigenfrequencies are shown in Table 3.1.
Table 3.1: Interior Eigenfrequencies (Dirichlet)

<table>
<thead>
<tr>
<th>Harmonic ( m )</th>
<th>( n = 0 )</th>
<th>( n = 1 )</th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.1416 (( \pi ))</td>
<td>4.4934</td>
<td>5.7635</td>
<td>6.988</td>
</tr>
<tr>
<td>2</td>
<td>6.2831 (2( \pi ))</td>
<td>7.7252</td>
<td>9.9095</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>9.4248 (3( \pi ))</td>
<td>10.9041</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is expected that at or near these frequencies, one could obtain spurious solutions as indicated by a sharp rise in condition number of the system matrix. Some of the values of the condition numbers for the matrix \([K]\) obtained for a 105 node sphere model (aluminum-water combination) are shown in the Table 3.2. A logarithmic plot of the condition numbers for a range of \( ka \)'s is shown in Figure 3.1 with the condition numbers normalized to the value at \( ka = 1 \) for visual simplicity. The numerical investigation was limited to \( ka < 8.0 \), as generation of Bessel functions (required for analytical solution [25] code) with higher argument was constrained by the machine constants of the computer being used (VAX 11/785). For this range of \( ka \)'s, spikes were encountered at the five values of \( ka = \pi, 4.4934, 5.7635, 2\pi \) and 6.988. Of these, the spikes for the fundamental mode (\( ka = \pi \) and 2\( \pi \)) seem to be the most prominent in the plot. It is to be noted that our BEM mesh only approximates the actual surface of the sphere and hence, with other inherent approximations in the procedure, one expects higher condition numbers to occur in the vicinity of the exact \( ka \)'s.

A close examination of the condition number and the BEM solutions near the eigenfrequencies was carried out. In the following, all the test frequencies (\( ka \)'s) used are correct up to four decimal places and the frequencies were examined at the exact
Table 3.2: Condition Numbers At Different $ka$'s

<table>
<thead>
<tr>
<th>$ka$</th>
<th>Condition numbers</th>
<th>Normalized values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>44</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>158</td>
</tr>
<tr>
<td>3</td>
<td>2.082</td>
<td>165</td>
</tr>
<tr>
<td>4</td>
<td>3.00</td>
<td>468</td>
</tr>
<tr>
<td>5</td>
<td>3.10</td>
<td>524</td>
</tr>
<tr>
<td>6</td>
<td>3.142</td>
<td>102810</td>
</tr>
<tr>
<td>7</td>
<td>3.3</td>
<td>625</td>
</tr>
<tr>
<td>8</td>
<td>3.342</td>
<td>814</td>
</tr>
<tr>
<td>9</td>
<td>4.00</td>
<td>1071</td>
</tr>
<tr>
<td>10</td>
<td>4.3</td>
<td>1426</td>
</tr>
<tr>
<td>11</td>
<td>4.49</td>
<td>12551</td>
</tr>
<tr>
<td>12</td>
<td>5.00</td>
<td>3334</td>
</tr>
<tr>
<td>13</td>
<td>5.265</td>
<td>6017</td>
</tr>
<tr>
<td>14</td>
<td>5.764</td>
<td>458348</td>
</tr>
<tr>
<td>15</td>
<td>5.94</td>
<td>4972</td>
</tr>
<tr>
<td>16</td>
<td>6.0</td>
<td>4438</td>
</tr>
<tr>
<td>17</td>
<td>6.28</td>
<td>920260</td>
</tr>
<tr>
<td>18</td>
<td>7.00</td>
<td>13420</td>
</tr>
</tbody>
</table>

$ka$'s as well as at $ka \pm \delta$, where $\delta = 0.0005$. Examination of the BEM solutions at the first harmonic ($m = 1$) of fundamental mode ($n = 0$) [$ka = \pi$] shows that no sensible solution is possible at the corresponding frequency for this mode. The numerical values simply 'blow up' at this $ka$. The result for $ka \pm \delta$ with $\delta = 0.0005$, though incorrect, produces numbers of the right order of magnitude (Figure 3.2) when using the 144-element mesh (Figure 2.16).

The results at the higher modes ($n > 0$) reveal that, with the given discretization (144-element), one can approach the exact $ka$ within $\delta = 0.0005$ for $n=1$ and $n=2$ and still get quite good estimates of the actual solution (Figures 3.3 and 3.4). Results at the next higher mode ($n=3$) (Figure 3.5) is however more erroneous, even at $\delta =$
Figure 3.1: Normalized condition numbers of combined matrix (sphere)

Figure 3.2: Scattering in the vicinity of an eigenfrequency $ka = \pi$ (fundamental mode, first harmonic). Dotted Line: BEM at $ka - \delta$ ($\delta = 0.0005$).
Figure 3.3: $ka = 4.4934 \ (m=1, \ n=1)$ [$\Delta$: BEM at exact $ka$; ---: BEM at $ka - \delta$]

Figure 3.4: $ka = 5.7625 \ (m=1, \ n=2)$
Table 3.3: Interior Eigenfrequencies (Neumann)

<table>
<thead>
<tr>
<th>Harmonic m</th>
<th>n = 0</th>
<th>n = 1</th>
<th>n = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.4934</td>
<td>2.0816</td>
<td>3.3421</td>
</tr>
<tr>
<td>2</td>
<td>7.7252</td>
<td>5.9404</td>
<td>7.2899</td>
</tr>
<tr>
<td>3</td>
<td>10.9041</td>
<td>9.2084</td>
<td></td>
</tr>
</tbody>
</table>

0.0005. A finer mesh should improve this result. Investigation at a higher harmonic (m=2) supports the same trend.

**Interior Neumann case**

The eigen solutions of the acoustic equations for a solid sphere with Neumann boundary condition \( \frac{\partial}{\partial n}(a) = 0 \) are the zeros of the first derivative of spherical Bessel functions of first kind \( j'(ka) = 0 \). A few of these are tabulated in Table 3.3 and a study of condition numbers and pressure solutions at the eigenfrequencies corresponding to Neumann boundary condition shows that BEM results at those \( ka \)'s exhibit no non-uniqueness, nor do the condition numbers show any spurious spike.

Therefore, it appears that for a spherical scatterer, the only fictitious eigenfrequencies at which the 'coupled' fluid-solid integral formulation is likely to break down are those corresponding to Dirichlet boundary condition \( j_n^m(ka) = 0 \). However, these fictitious eigenfrequencies are unlikely to produce significant pollution in an arbitrarily spaced frequency distribution of scattered amplitude since the results start deteriorating only within a very narrow band around the critical \( ka \)'s. This band gets narrower with the use of finer discretization. If this is true for a sphere, then it is likely to be true for any arbitrary scatterer.
Table 3.4: Eigenfrequencies: Free Torsional Oscillations of Sphere

<table>
<thead>
<tr>
<th>Harmonic $m$</th>
<th>$n = 0$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.763</td>
<td>9.095</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.501</td>
<td>7.136</td>
<td>10.51</td>
</tr>
<tr>
<td>3</td>
<td>3.865</td>
<td>8.444</td>
<td>11.88</td>
</tr>
<tr>
<td>4</td>
<td>5.095</td>
<td>9.712</td>
<td></td>
</tr>
</tbody>
</table>

**Jones eigenfrequencies**

The frequencies for the free, torsional ($u_n = 0$) oscillations for a vibrating sphere, in terms of non-dimensional shear wave number $k_T a$ or $(\frac{\omega}{c_T}) a$, may be obtained from the ref. [40]. They are also discussed in detail in the ref. [41] and are shown in the Table 3.4, in terms of $k_T a$, for different modes and harmonics. The value for the fundamental mode of first harmonic was not available. The BEM results at the second harmonic of the fundamental mode ($k_T a = 2.501$) are examined. The corresponding exterior (fluid) wave number for water/aluminum combination is $ka = 5.265$. Figure 3.6 compares the far field scattering solution at this $ka$ with the analytical solution. Here the numerical data appear acceptable and improve in accuracy with finer meshing of the scatterer geometry. Thus numerically, Jones frequencies pose no serious threat, at least for a spherical scatterer. For any other scatterer geometry with lesser axial symmetry, the problem should expectedly reduce even further.
Figure 3.5: $ka = 6.9794$ ($m=1, n=3$)

Figure 3.6: First Jones eigenfrequency: Scattering from sphere
CHAPTER 4. ULTRASONIC TRANSMISSION THROUGH A CURVED INTERFACE

Introduction

This chapter presents the specialization of integral equation method for modelling the transmission of an ultrasonic beam through a curved, open interface separating a fluid and a solid media. The work is motivated by the need for a numerical model for ultrasonic flaw detection in structural components with non-planar surface contours. The type of ultrasonic test commonly used in non-destructive evaluation (NDE) is that of immersion scanning where part of the incident ultrasonic pulse is scattered back from the interface and part of it transmits through the interface before it interacts with the subsurface defects.

The interface is assumed to be arbitrarily curved. In particular, behavior at a spherically concave and convex surface is studied for various beam widths at various angles of incidence from normal to the Rayleigh angle. A special case of the curved interface, namely the flat half-space, has been extensively treated by many authors for wave scattering problems in acoustics and elasticity [42, 43, 44]. Each of these problems required very specialized treatments. The integral formulation that follows is however extremely general, and is independent of the curvature and can solve a half-space or curved interface problem with equal ease without any special treatments.
Ultrasonic Transmission and NDE

Non-destructive evaluation is concerned with on-line monitoring of the integrity of the structural elements and identifying and discriminating between the inhomogeneities like inclusions, pores and cracks. Ultrasonic NDE techniques achieve this goal by evaluating the scattering characteristics of embedded inhomogeneities due to an incident ultrasonic beam impinging on the surface of the structure under consideration. The object under scrutiny is typically submerged in a liquid immersion bath so that the liquid can act as an effective couplant between the structure surface and the transmitting pulse from the transducer.

The immersion test for flaw detection can be segmented into following steps:

• Generation of appropriate electrical signals and conversion to acoustic wave by piezoelectric transducers.

• Propagation of the acoustic waves through a liquid bath or a couplant.

• Scattering from and transmission through liquid-solid interface.

• Scattering of the transmitted elastic wave (in the solid) from the flaw.

• Receiving these signals back by the transducer.

In this dissertation, only the interface transmission and scattering are examined. The incident pulse on striking the interface transmits into the solid as compressional and shear modes, and is scattered back into the fluid as a (compressional) pressure wave. When obliquely incident, the input beam generates surface waves at the interface. Such surface waves are coupled with the scattered wave-field propagating back
into the fluid. Particular attention is given to these non-normal incident waves that generate such surface waves.

Numerical modelling of transmission/scattering process can contribute to the construction of a reliable simulation scheme of the immersion test to help bypass expensive experimental procedures. Similar research to model the latter solid-flaw interaction (elastic wave scattering by flaws in solid matrix) through boundary integral equations has also been reported in the literature [45]. An appropriate combination of such schemes should lead to the ability to build probability of detection models for ultrasonic scanning when low or high frequency approximations are not applicable.

The BEM Model and the Problem Geometry

The model presented here is essentially an extension of the approach for the scattering by simple shapes developed in the Chapter 2. The details of the solution strategy that follows will show that both the geometrical configuration and the process of ultrasonic wave propagation has been simplified to accommodate our analytical and numerical approach.

The problem is defined, as before, in the frequency domain and all the field variables are assumed to have a harmonic time dependence ($e^{-i\omega t}$), where $\omega$ is the time-harmonic circular frequency in radians/sec. Time-domain results, whenever required, can be extracted via appropriate Fourier transforms. Alternately, one can also obtain appropriate representation integrals for the non-elliptic differential equations in time-domain analogous to Eqns. 2.28 and 2.29, increment the time in steps and perform at each step the BEM analysis similar to the steady-state case.
Consider the surface $ABCD$ of Figure 4.1 (a) to be the surface contour of a three-dimensional component under inspection. A two-dimensional schematic section of the concave part is represented by $AB$ of Figure 4.1 (b) that separates a fluid domain in $B^+$ and a solid domain in $B^-$. The fluid density as before is denoted by $\rho$ and the wave speed by $c$ while the solid is assumed to be isotropic, of density $\varrho$ and has longitudinal and shear wave speeds $c_L$ and $c_T$ respectively. Let $S$ be a source of a focussed ultrasonic beam (e.g., a transducer-lens assembly in pulse-echo mode) of half-width $a$ and is assumed to illuminate a circle of radius $a$ on the interface. The interface $AB$ presumably lies close enough to $S$ so that the beam remains well collimated when it strikes the surface.

The various theoretical models of the problem [46, 47, 48] take into consideration beam refraction, focussing and aberrations at non-normal incidence. The boundary element solution of the problem will essentially require (a) posing a two-domain boundary value problem for the ultrasonic incidence and scattering in the fluid (water) and the elastic wave transmission into the solid component ($B^-$) through the curved interface $S_H$, (b) simultaneous solution of equations representing the acoustic field and the transmitted elastodynamic field. The model automatically incorporates the beam aberration effects for any interface geometry.

**Integral formulation**

The total acoustic pressure $p$ of the coupling fluid and the displacement field $u$ in the solid are again the primary variables. The steady state wave mechanics of the domains $B^+$ and $B^-$ would then be represented by the familiar *Helmholtz* equation and *Cauchy-Navier* equation (Eqns. 2.4 and 2.5) respectively.
Figure 4.1: Curved fluid/solid interface: Schematic model
The boundary element model of the geometry retains the local curvature of $AB$, but extends the rest of the interface, now denoted by $S_H$, to infinity. In this idealized model, as shown in Figure 4.2, $S_H$ separates two infinite hemispherical fluid ($B^+$) and solid ($B^-$) domains with respective infinite boundaries $S_{\infty}^+$ and $S_{\infty}^-$. The semi-infinite interface $S_H$ is thus like a half-space but may have arbitrary curvature.

Introduction of the infinite boundaries $S_{\infty}^+$ and $S_{\infty}^-$ in this modified model makes the boundary element method ideally suited for this problem, provided that the incident pressure $p^I$ is a finite beam and the scattered pressure $p^S$ satisfies the Sommerfeld radiation condition [Appendix B] at upper infinite surface $S_{\infty}^+$. The transmitted elastic field in the lower half-space will be outgoing at the bottom infinite hemisphere and can also be assumed to satisfy the radiation condition at $S_{\infty}^-$. This requires that the following integral relations hold at the infinite boundaries $S_{\infty}^+$ and $S_{\infty}^-$ respectively.

\[
\int_{S_{\infty}^+} \left\{ G \frac{\partial p^S}{\partial n} - \frac{\partial G}{\partial n} p^S \right\} ds = 0 \quad (4.1)
\]
\[
\int_{S_{\infty}^-} \{ U(t) - T(u) \} ds = 0 \quad (4.2)
\]

where, $G$, $U$ and $T$ are the full-space Green's functions introduced in Chapter 2.

The fluid domain $B^+$ is bounded by $S_{\infty}^+$ and $S_H$ and the solid domain $B^-$ is bounded by $S_{\infty}^-$ and $S_H$. Using the radiation conditions (4.1) and (4.2), the formulation for the above geometry may be represented by integrals defined only on the semi-infinite interface $S_H$,

\[
u(x) = \int_{S_H} [U(r)t(y) - T(r)u(y)] ds(y) \quad x \in B^-, \quad y \in S_H \quad (4.3)
\]
\[
p(x) = \int_{S_H} \left[ G(r) \frac{\partial p}{\partial n}(y) - \frac{\partial G}{\partial n}(r)p(y) \right] ds(y) + p^I(x) \quad x \in B^+ \quad (4.4)
\]
The normal basis of the first integral is the unit vector $q$ pointing into $B^+$ and that for the second is the unit vector $n$ at $S_H$ pointing into the solid domain $B^-$ and $r(x,y)$, as before, is the distance vector between $x$ and $y$.

**A Gaussian Beam Model**

As a first beam model, a Gaussian function (Figure 4.3) was chosen to represent the incident beam at surface $S_H$. The incident excitation is thus bounded and the resulting surface insonification is finite. If a cartesian reference is fixed at the interface with origin at the center of the area illuminated by the beam, and vertical direction into the fluid $B^+$ as the positive $z$ or $x_3$ axis, then the acoustic pressure $p^I$, when normally incident into $B^-$ along negative $z$ direction, is assumed to be

$$p^I(x) = p_o \left[ e^{-\frac{x_1^2 + x_2^2}{4\beta}} \right] e^{-ikx_3} \quad x \in S_H$$  \hspace{1cm} (4.5)

where $\beta$ is a constant defining the width ($2\alpha$) of the beam and $k$ is the acoustic wave number. The amplitude is axially symmetric and decays rapidly with distance from the axis. This does not model a beam propagating in space but merely represents an assumed profile on reaching the interface and satisfies the requirement of the finiteness of the beam. For incidence in a non-normal direction, $p^I$ can be defined as

$$p^I(x) = p_o \left[ e^{-\frac{x_1^2 + x_2^2}{4\beta}} \right] e^{-ik \cdot x} \quad x \in S_H$$  \hspace{1cm} (4.6)

where $k$ is the wave vector defining the propagation direction of the incident beam. Figure 4.4 shows the three dimensional profile of a beam with angular incidence. For
Figure 4.2: Curved interface: Mathematical model for boundary element analysis.

Figure 4.3: Gaussian beam profile.
Figure 4.4: Gaussian beam incident at an angle
both these forms, an active scattering zone or effective area of insonification $\partial M$ may be estimated as an area whose radius $a$ is such that

$$p^I(a)/p^I(0) = \varepsilon \quad \varepsilon \ll 1$$

(4.7)

In subsequent computations, this ratio has been taken as $\varepsilon = 0.001$.

**Truncation**

Only a small part $\partial M$ of the semi-infinite interface $S_H$ is actually retained for the subsequent steps. The choice of a bounded Gaussian beam implies that the input beam $p^I$ effectively insonifies only a small segment $\partial M$ of the interface $S_H$ and the resultant field outside this segment can be considered insignificant, i.e.,

$$p^I(x) \approx 0 \quad x \in S_H - \partial M$$

(4.8)

This implies that

$$\int_{S'_H} \left[ G \frac{\partial p}{\partial n} - \frac{\partial G}{\partial n} p \right] ds \approx 0$$

(4.9)

where,

$$S'_H = \partial M + S'_H$$

Since the field of excitation vanishes outside $\partial M$, a similar truncation is also applied to the elastic representation, i.e.,

$$\int_{S'_H} [U_t - T_u] ds \approx 0$$

(4.10)

Intuition indicates that this approximation should hold, particularly, if one considers a small amount of material damping. Convergence of the solution is studied below.
for various sizes of $\partial M$.

**The Boundary Element Solution**

The boundary integral equations (BIE) for the problem are obtained by reformulating the integral representations of the solid and the fluid field in terms of the points on the interface $\partial M$ by taking the respective domain points $x$ to the boundary $\partial M$ (i.e., $x(B^+)-x(\partial M)$, $x(B^-)-x(\partial M)$ etc.). As discussed in the Chapter 2, this limiting process results in singular kernels whenever $x-y$ or as $r-0$ and needs to be *regularized* into a form amenable to the subsequent numerical solution. The regularization process is essentially same as in the finite scatterer problem with modifications for the new geometry and are summarized as follows:

Choose an interior point $x_s$ in the lower half space (solid or $B^-$) where $u_0 = u(x_s)$. The integral representation of this domain with respect to its normal $q$, with regularization is

$$ u_0 = \int_{S_H} U ds - \int_{S_H} (T - T^S) u ds - \int_{S_H} T^S (u - u_0) ds - u_0 \int_{S_H} T^S ds \quad (4.11) $$

where $T^S$, as before, is the static Green’s function or the Kelvin tensor [Appendix A]. For the half space $S_H$, one now uses the identity [Appendix C]

$$ \int_{S_H} T^S = -I/2 \quad (4.12) $$

to get

$$ (I/2)u_0 + \int_{S_H} (T - T^S) u ds + \int_{S_H} T^S (u - u_0) ds - \int_{S_H} U ds = 0 \quad (4.13) $$
Retaining the integrals only for the insonified area $\partial M$ (Eqn. 4.10), one obtains

$$\begin{align*}
(1/2)u_0 + \int_{\partial M}(T - T^S)uds + \int_{\partial M} T^S(u - u_0)ds - \int_{\partial M} Utds &= 0 \quad (4.14)
\end{align*}$$

The representation in this form reduces the problem to an integral equation devoid of any strong singularity even when both the source and the field points are on $\partial M$ (i.e., $x \in \partial M, y \in \partial M$), since $(T - T^S) \rightarrow O(\frac{1}{r^2})$ and $T(u - u_0) \rightarrow O(\frac{1}{r})$ as $r \rightarrow 0$.

During the integration process, the regularization is only carried out in the boundary element $\partial B_s$, one of whose nodes is a collocation point (i.e., $x \rightarrow y$ in $\partial B_s$). If $\partial B_n = \partial M - \partial B_s$ then the modified form for the integral is

$$\begin{align*}
(1/2)u_0 + \int_{\partial B_n} Tuds - \int_{\partial B_n} T^S u_0 ds + \int_{\partial B_s} (T - T^S)uds + \int_{\partial B_s} T^S(u - u_0)ds - \int_{\partial M} Utds &= 0 \quad (4.15)
\end{align*}$$

Further algebraic manipulation similar to the finite scatterer problem (Eqn. 2.19) results in an alternate form

$$\begin{align*}
\left[\frac{1}{2} - \int_{\partial B_n} T^S ds + \int_{\partial B_s} (T - T^S)ds\right] u_0 + \int_{\partial B_n} Tuds + \int_{\partial B_s} T(u - u_0)ds - \int_{S_H} Utds &= 0 \quad (4.16)
\end{align*}$$

Here the traction $t$ and the tensors $T$ and $T^S$ are defined in terms of the normal $q$.

The normal $n$ of the upper space $B^+$ is now taken as a reference direction so that the final form in terms of the normal $n$ is
The regularized form for the upper domain $B^+$ similarly becomes

\[
\left[ \frac{1}{2} + \int_{\partial B_n} T^S ds - \int_{\partial B_s} (T - T^S) ds \right] u_0
- \int_{\partial B_n} Tu ds - \int_{\partial B_s} T(u - u_0) ds + \int_{\partial M} Ut ds = 0
\] (4.17)

The regularized form for the upper domain $B^+$ similarly becomes

\[
\left[ \frac{1}{2} - \int_{\partial B_n} \frac{\partial G}{\partial n} ds + \int_{\partial B_s} \left( \frac{\partial G}{\partial n} - \frac{\partial G^S}{\partial n} \right) ds \right] p_o + \int_{\partial B_n} \frac{\partial G}{\partial n} ds
- \int_{\partial B_s} \frac{\partial G}{\partial n} (p - p_o) ds - \int_{\partial M} G \frac{\partial p}{\partial n} ds = p^I
\] (4.18)

where, to obtain this equation, the identity

\[
\int_{S_H} \frac{\partial G^S}{\partial n} ds = -1/2
\] (4.19)

has been used.

The BIE's (4.17) and (4.18) are valid for any arbitrarily curved interface as long as the surface geometry of $\partial M$ can be described by a unique normal $n$ at each point. One can therefore use the same formulation to solve problems for concave, convex, planar or any other surface profile by generating an appropriate model of the truncated interface. The two BIE's are coupled using the familiar continuity conditions at interface $\partial M$,

\[
t(x) = -p(x)n(x), \quad \frac{\partial p}{\partial n}(x) = \rho \omega^2 u(x).n(x) \quad x \in \partial M
\] (4.20)

The BIE's are then solved numerically by discretizing the interface $\partial M$ and using the boundary element method like the previous problem. The variables $u$ and
are approximated by their nodal values using isoparametric shape functions and the integral coefficients are evaluated by Gaussian quadrature. As before, one obtains a finite system of matrix algebraic equations of the form,

\[
[K] \begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  p
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  p^I
\end{bmatrix}
\]

where \(K\) is the complex matrix representing the fluid-solid interaction at the interface \(S_H\). The system is solved by standard Gaussian elimination using LINPACK routines [34].

An Exact Solution for a Planar Interface

Both the incident and scattered field due to an incident Gaussian beam \(p^I\) can be evaluated for the planar interface by Fourier transform techniques. The field due to the finite (Gaussian) beam can be evaluated as a superposition of plane waves, each identified by the propagation vector \(k = ke\ (k = \omega/c)\). Consider the transform pairs \(p(x_1,x_2)\) and \(\hat{p}(k_1,k_2)\) in spatial and k-domain.

\[
\hat{p}(k_1,k_2) = \int_{-\infty}^{\infty} p(x_1,x_2) \left[ e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3)} \right] dx_1 dx_2 \quad (4.22)
\]

and

\[
p(x_1,x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \hat{p}(k_1,k_2) \left[ e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)} \right] dk_1 dk_2 \quad (4.23)
\]

If the incident beam for all \(k\) is \(p^I(k) = p^I(k_1,k_2)\) then in the spatial domain,

\[
p^I(x_1,x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p^I(k_1,k_2) e^{i k \cdot x} dk_1 dk_2 \quad (4.24)
\]
where, \( k = (k_1, k_2, k_3) \) and \( k_3 = \sqrt{k_1^2 - k_2^2} \)

Here the root \( k_3 \) is chosen to ensure existence of the integral and propagation in the \( x_3 \) direction. Therefore for a given \( p^I(x) \), if one knows the corresponding \( p^I(k_1, k_2) \), then the total field \( p \) can be evaluated from the integral

\[
p(x_1, x_2) = p^I(x_1, x_2) + \left[ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(k_1, k_2) p^I(k_1, k_2) e^{ik \cdot x} dk_1 dk_2 \right] \tag{4.25}
\]

where \( R(k_1, k_2) \) represents the plane wave reflection coefficient. The theoretical derivation of the \( R(k) \) is discussed in the Appendix F.

Normal incidence

For a vertically incident Gaussian beam (Eqn. 4.5) with spatial representation

\[ p(x) = e^{-(x_1^2 + x_2^2)/4\beta} = e^{-r^2/4\beta} \]

on the interface \( (x_3 = 0) \), the transform, in terms of wave spectrum, is

\[
p^I(k_1, k_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x_1^2 - x_2^2/4\beta} e^{-ik \cdot x} dx_1 dx_2 \]

or,

\[
p^I(k) = 4\pi \beta e^{-(k_1^2 + k_2^2)\beta} \tag{4.26}
\]

In polar coordinates, if the transform pairs \( p(r, \theta) \) and \( p(k_r, \alpha) \) are such that

\[
x_1 = r(\cos \theta) \quad x_2 = r(\sin \theta) \\
k_1 = k_r(\cos \alpha) \quad k_2 = k_r(\sin \alpha)
\]
then,

\[ p(r, \theta) = \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} \hat{p}(k_r, \alpha) \left[ e^{i r k_r \cos \alpha \cos \theta + i r k_r \sin \alpha \sin \theta} \right] e^{i k_3 z k_r} dk_r d\alpha \]  

(4.28)

For vertical incidence, \( p \) and \( \hat{p} \) are independent of \( \theta \) and \( \alpha \) respectively, and hence

\[ p(r, \theta) = p(r) = \frac{1}{4\pi^2} \int_0^\infty \hat{p}(k_r) e^{i k_3 z} \left[ \int_0^{2\pi} e^{i r k_r \cos(\theta - \alpha)} d\alpha \right] k_r dk_r \]

\[ = \frac{1}{4\pi^2} \int_0^\infty \hat{p}(k_r) e^{i k_3 z} \left[ 2\pi J_0(r k_r) \right] k_r dk_r \]  

(4.29)

where, following Eqn. 4.1.6 in page 58 of ref. [49], it may be shown that

\[ \int_0^{2\pi} e^{i r k_r \cos(\theta - \alpha)} d\alpha = 2\pi J_0(r k_r) \]  

(4.30)

Hence, for the axially symmetric normally incident beam \( e^{-r^2/4\beta} \), the Fourier transform reduces to the familiar Hankel transform

\[ p(r) = \frac{1}{2\pi} \int_0^\infty \hat{p}(k_r) J_0(r k_r) e^{i k_3 z k_r} dk_r \]  

(4.31)

and inversely,

\[ \hat{p}(k_r) = 2\pi \int_0^\infty p(r) J_0(r k_r) e^{-i k_3 z r} dr \]  

(4.32)

At the interface \( z = 0 \), so that

\[ \hat{p}(k_r) = 2\pi \int_0^\infty p(r) J_0(r k_r) r dr \]

\[ = 2\pi \int_0^\infty e^{-r^2/4\beta} J_0(r k_r) r dr \]

\[ = 2\pi \left[ 2\beta e^{-k_r^2 \beta} \right] \]

\[ = 4\beta \pi e^{-\beta k_r^2} \]  

(4.33)
Non-normal incidence

For the special case of a non-normal beam incident in the X-Z plane at the Rayleigh angle, the Rayleigh wave number $k_R$ is approximately equal to the shear wave number $k_T$ and if $\theta$ is the angle of incidence, then

$$\theta \simeq \sin^{-1} \frac{k_T}{k}$$

and

$$p^I(x_1, x_2) = (e^{-\frac{ikR^2}{4\beta}})e^{ikRx_1} \simeq (e^{-\frac{ikR^2}{4\beta}})e^{ikTx_1}$$

$$p^I(k_x + k_2) = 4\pi\beta(e^{-\beta[(k_1 - k_T)^2 + k_2^2]})$$

and

$$p = p^I(x) + \frac{\beta}{\pi} \int \int R(k)(e^{-\beta[(k_1 - k_T)^2 + k_2^2]})e^{ik.x} dk$$

Length parameters in the interaction

The numerical prediction of the wave interaction is related to the various length parameters of the input and the BEM model. They are:

- $a =$ Transducer radius indicating the extent of the effective insonification of the interface.
- $R =$ Radial span of the discretized surface $\partial M$.
- $R_c =$ Radius of curvature of the interface.
- $\lambda =$ Wavelength of the input beam.
- $L =$ Characteristic size of an element of the surface mesh.
These length parameters are shown in the Figure 4.5 and a typical surface discretization of the modelled area $\partial M$ of radius $R$ is shown in Figure 4.6. The extent of the modelled area ($R$) and the characteristic mesh size $L$ affect the convergence of the BEM solutions significantly. The modelled area $\partial M$ should be at least equal to the illuminated area, i.e., $R$ should be at least equal to $a$, the transducer radius. Also the characteristic element size $L$ should be comparable to the wave length $\lambda$ of the incident beam. It has been observed that the accuracy of the amplitude is more sensitive to $R/a$ ratio, whereas the phase is more sensitive to $L/\lambda$ and in general reliable results may be expected if $R/a > 1.5$ and $\frac{L}{\lambda} \approx 1$.

The error of a solution, thus, may be influenced by lower $R/a$ and higher $L/\lambda$ ratios. The demand on the higher $R/a$ ratio in the boundary element model becomes more critical for non-normal incidence when surface waves are generated that propagate along the interface and radiate into the fluid. Various surface waves in a fluid-solid interface are discussed in detail in the ref. [50]. A leaky Rayleigh wave is strongly excited when the angle of incidence is near the Rayleigh angle [51, 52]. The BEM results for beams incident on flat interface at the Rayleigh angle were investigated. It was found that a considerably larger portion of the interface needs to be discretized to capture these surface wave phenomena. The modelling becomes more critical for a curved interface.

**Computational Results**

All numerical results presented here are for the total surface pressure at the fluid-solid interface. If the surface or boundary solutions can be shown to be accurate, validity of the scattered far-field pressure or the transmitted displacement field should
\[ \lambda = \frac{2\pi}{k} \]

Figure 4.5: Length parameters in the BEM model

Figure 4.6: Typical BEM discretization for a flat interface
follow. The BEM results show the real and imaginary components of total surface pressure and in all examples, aluminum and water are used as the solid and the fluid media whose material properties are taken from Table 2.3.

Normal incidence

The first example (Figure 4.7) involves a planar interface with a normally incident Gaussian beam and $k = 1$. Assuming the interface to be in the $x_1 - x_2$ plane and the beam incident along negative $z$ or $-x_3$ direction, the incident acoustic pressure reduces to the radially symmetric form

$$p^I(x) = p_o \left[ e^{-\frac{x_1^2 + x_2^2}{4\beta}} \right] = p_o \left[ e^{-\frac{r^2}{4\beta}} \right] \quad (4.38)$$

where the exponent $\beta$ controls the width $(2a)$ of the beam (Figure 4.3) and is related to the transducer diameter. For the example in Figure 4.7, the incident beam function uses $\beta = 4$. Here the total pressure $p$ on the surface obtained by the BEM is compared with the exact analysis by Fourier transform methods. Two boundary element solutions are computed from two different surface meshes of different $R/a$ ratios (1.1 and 1.9) but the same mesh size ($L = \lambda/2$). Accuracy is somewhat lost for the first model ($R/a = 1.1$), but the larger modelled area ($R/a = 1.9$) produces results in excellent agreement with the analytical solution. Figure 4.8 shows a similar calculation at $k = 3$. 
Figure 4.7: Total surface pressure: Normal incidence $k = 1, \beta = 4$
Figure 4.8: Total surface pressure: Normal incidence $k = 3, \beta = 1.3$
Oblique incidence

The next example involves oblique incidence on a planar interface. Typically, a beam incident at the Rayleigh angle will excite leaky surface waves and will affect the resultant field significantly. Existence of leaky Rayleigh waves offers a potential means of monitoring the characteristics of surface flaws close to the interface. Significant use is made of them in acoustic microscopy to locate subsurface flaws with incident focussed beams of very high frequency [53, 54]. Results that follow here are however limited to low to moderate frequencies. The presence of leaky Rayleigh waves, however, violates the assumption of the truncated surface model, since the propagating surface waves implies non-zero fields even outside the insonified area. A larger segment of the interface therefore needs to modelled for accurate solutions at oblique incidence. Figure 4.9 shows the boundary element simulation of the total surface pressure at \( k = 1 \) due to an incident beam with wave vector in the \( x_1 - x_3 \) plane striking the interface in the vicinity of Rayleigh angle (28.5°). The incident beam (Figure 4.4) has the following form

\[
p'(x) = p_0 \left[ e^{\frac{-x_1^2 + x_2^2}{4\beta}} e^{ik_T x_1}, \quad k_T \approx k_R \quad \theta_i = \sin^{-1}(\frac{k_T}{k}) \right] (4.39)
\]

where the shear wave number \( k_T \) was used, due to its proximity to the Rayleigh wave number \( k_R \). The incident beam uses \( \beta = 4 \), corresponding to a beam radius \( a \) of 11 cm., and an area with radius \( R = 50 \) cm. (≈4.5a) on the interface was discretized. Figures 4.10 and 4.11 compare the results in the \( x_1 \) and \( x_2 \) directions respectively with the exact solution. With this large \( R/a \) ratio (≈ 4.5), the BEM solution accurately discerns the surface disturbances in the \( x \)-direction. Note that
Figure 4.9: Total surface pressure: oblique incidence ($28.5^\circ$) $k = 1$
Figure 4.10: Verification in $X_1$-direction: Oblique incidence (Fig. 4.9)
Figure 4.11: Verification in $X_2$-direction: Oblique incidence (Fig. 4.9)
since the incident excitation is in the $x-z$ plane, the result in the $y$-direction (Figure 4.11) is relatively smooth.

**Higher frequencies**

For more realistic operating frequencies, a transducer model of 0.25 in diameter ($a = 1/8$ in) and 4 MHz. frequency ($k = 169, \lambda = 2\pi/k = 0.037 \text{cm}$.) was modelled. The radius of the illuminated area is taken to be $1/8$ in. or $0.318 \text{ cm}$. and the discretized circular area is of radius $R \approx 1.5a$ and the element size is $L = 0.074 \text{ cm} = 2\lambda$. A symmetry feature of the program allows one to actually model only half the circular area. The resulting mesh has 559 nodes and 256 triangular elements and Figure 4.12 shows the total pressure profile for the normally incident beam. Even with $L/\lambda = 2$, the BEM solution agrees satisfactorily with the analytical solution.

The non-normal incidence is however more demanding. Figures 4.13, 4.14 and 4.15 compare the total pressure at the same frequency ($k = 169$) but at increasing angular incidence ($8^\circ, 20^\circ$ and $28.5^\circ$). The modelled area radius is still $R = 1.5a$, but $L = 1.4\lambda$. The results are in good agreement, even at $20^\circ$ but show some disagreement near the Rayleigh angle ($28.5^\circ$). The mismatch agrees with intuition, since non-normal incidence, especially at the Rayleigh angle, generates surface waves propagating along the interface and demands a larger $R/a$ ratio for their representation. Therefore, for an adequate model, one should discretize a surface sufficiently larger than the beam radius.
Figure 4.12: Normal incidence: 4 MHz ($k = 169.8, L/\lambda = 2, a = 0.3175\text{cm}$)
Figure 4.13: Oblique incidence ($\theta^\circ$); 4 MHz $U/R = 1.5, L/\lambda = 1.4$
Figure 4.14: Oblique Incidence (20°): 4 M.H. $R/a = 1.3$, $L/a = 1.4$

- Imag(pressure)
- Real(pressure)

X-distance

BEM

Exact

96
Figure 4.15: Oblique incidence (28.5°): 4 MHz \( R/a = 1.5, L/\lambda = 1.4 \)
Fully curved interface

In this section, results for concave (Figure 4.16) and convex interfaces (Figure 4.17) are presented for normal as well as oblique incidence. For absence of experimental data for comparison, the results are validated by the reciprocity theorem of elasticity [55]. The theorem states that, in absence of any body force, if an elastic body is subjected separately to two systems of surface forces (tractions), then the work done by the first system (represented by the traction \( t^1 \)) in acting through the displacement \( u^2 \) of the second system is equal to the reciprocal work done by the traction \( t^2 \) of the second system in acting through the displacement \( u^1 \) due to the first system of forces, i.e.,

\[
\int_{\partial M} t_i^1 u_i^2 dS = \int_{\partial M} t_i^2 u_i^1 dS
\]  

(4.40)

where the superscripts 1 and 2 represent the two separate equilibrium states. Thus if \( p^1_1 \) and \( p^2_2 \) represent two incident beams striking the surface at different angles and possibly with different amplitudes (Figure 4.18) but the same frequency, and \( u^1 \) and \( u^2 \) represent the corresponding displacement solutions, then a necessary condition for the correctness for the solution is that the above equation should be satisfied. One can compare the two reciprocal work values by numerically evaluating the integrals \( \int t^1 . u^2 ds \) and \( \int t^2 . u^1 ds \) from the nodal solutions.

Table 4.1 compares the numerically integrated reciprocal work values based on the BEM nodal solutions of \( u \) and \( t \) for low and high frequencies at different angles of incidence. For all the cases, one observes that the BEM solutions reasonably satisfy the reciprocity relation (Eqn. 4.40). This, while not providing a complete validation of the method, at least lends confidence to it.
Figure 4.16: BEM discretization of a concave interface
Figure 4.17: BEM mesh: Typical convex interface

Figure 4.18: Reciprocal work: Two different incident states
Table 4.1: Reciprocity Results For Curved Interface

<table>
<thead>
<tr>
<th>$S_H$</th>
<th>$k$</th>
<th>States</th>
<th>$\mu \int t^1 \cdot u^2 dS$</th>
<th>$\mu \int t^2 \cdot u^1 dS$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>Real</td>
<td>Imag.</td>
</tr>
<tr>
<td>concave</td>
<td>2</td>
<td>0° 8°</td>
<td>13.05</td>
<td>2.64</td>
</tr>
<tr>
<td>concave</td>
<td>.2</td>
<td>8° 20°</td>
<td>3.55</td>
<td>0.73</td>
</tr>
<tr>
<td>Plane</td>
<td>169</td>
<td>0° 8°</td>
<td>$0.35 \times 10^{-6}$</td>
<td>$-.81 \times 10^{-4}$</td>
</tr>
<tr>
<td>convex</td>
<td>169</td>
<td>0° 14°</td>
<td>$0.14 \times 10^{-3}$</td>
<td>$0.67 \times 10^{-4}$</td>
</tr>
<tr>
<td>convex</td>
<td>169</td>
<td>0° 14°</td>
<td>$-.21 \times 10^{-4}$</td>
<td>$-.12 \times 10^{-4}$</td>
</tr>
<tr>
<td>concave</td>
<td>169</td>
<td>0° 8°</td>
<td>$-.84 \times 10^{-4}$</td>
<td>$0.42 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Figure 4.19 shows the BEM results for three angles of incidence at $k = 2$ of a concave interface whose radius of curvature $R_C = 5a$. Examination of the reciprocity results (Table 4.1) shows that while the values are in excellent agreement for the 0° and 8° states, those for the 8° and 20° states disagree somewhat in the imaginary part. This indicates a higher reliability of the results at 0° and 8° relative to 20°. The results for normal (0°) incidence does show the expected symmetry of the field (Figure 4.19). Also, significant surface wave phenomenon may be observed for 20° incidence. Similar features are also observed for results at higher frequency.

Figure 4.20 shows results at a higher frequency ($k = 169$ or $f = 4$ MHz.) for a convex surface ($R_C = 5a, a = 0.37$ cm.). As before, the results at 0° is symmetrical and those at higher angles show considerable structure. All the reciprocity results ($\theta_i = 0^0, 8^0$ and 14°) for $k = 169$ shown in Table 4.1 reasonably satisfy the reciprocity relations. The results beyond 14° produced increasingly erroneous reciprocal work values and were rejected. Figure 4.21 displays the numerical results at frequency of 4 MHz. for plane, concave and convex interface for normal incidence. Finally,
Figure 4.19: Boundary pressure: Concave interface $k = 2$, $R_c/a = 5$
Figure 4.20: Boundary pressure: Convex interface \( k = 169, R_c/a = 5 \)
Figure 4.22 is a result at 4 MHz. similar to Figure 4.21 but for a 8° incidence.

Transmitted field for the concave interface

The boundary solutions, once obtained, may be used to evaluate the transmitted field in the solid using the integral representation (Eqn. 4.5). Presented here are displacement profiles inside the solid domain when a normally incident ultrasonic beam penetrates a concave fluid-solid interface. Such studies could be useful in examining the focussing effect of the transmitted field. Examples that follow use incident beams whose initial profiles are (a) Gaussian \[ e^{-r^2/4\beta} \] or (b) more uniformly spread \[ e^{-(r/\beta)^2n} \] where \( n = 1, 2, 3... \) etc. Figure 4.23 shows the Gaussian and the uniform incident profiles \((n = 3)\) used in the examples. The geometrical ray representation of the transmission [52] for a concave interface is shown in Figure 4.24. The results which follow qualitatively agree with the 2-D solutions as discussed in the ref. [52].

From [52] the geometric focal length \( l_f \) of the transmitted longitudinal wave (for a cylindrical 2-D model) is given as

\[
l_f = \frac{-R_c(\cos \theta_l)}{1 - c_L(\cos \theta_l)/c(\cos \theta_l)} \tag{4.41}
\]

where \( \theta_i \) and \( \theta_l \) are the incident and the refracted (longitudinal) angle, \( c \) and \( c_L \) are the incident and the transmitted wave speeds and \( R_c \) is the radius of curvature. Since the input is normally incident, both \( \theta_i = \theta_l = 0^\circ \). For the interface model used here, the parameters are \( f = 4 \text{ MHz} \) (or, \( k = 169 \)), \( R_c = 5a \), \( a = 0.37 \text{ cm} \). The solid and the fluid domains are aluminum and water. Hence \( l_f = 0.56 \text{ cm} \). A plot of the
Figure 4.21: Normal incidence: Different surface curvatures (4 MHz.)

Figure 4.22: Oblique incidence (8°): Different curvatures
Figure 4.23: Input beam profiles used in transmission study

Figure 4.24: Ray representation of beam transmission: Reproduced from [52]
transmitted displacement amplitude (|\( u \)|) along the central axis in Figure 4.25 shows that the largest magnitude occurs approximately at a depth \( z = 0.5 \text{ cm.} \) and decays slowly with the increased depth. The analytical results for a cylindrical (2-D) concave interface [52] shows a similar variation. Radial variation of the amplitude intensities are next shown (Figure 4.26) at three different depths (0.2, 0.5, 0.8 cms.) The sharp rise of the amplitude at the central axis for \( z = 0.5 \text{ cm.} \) shows the focussing effect. Figure 4.27 shows the BEM results as a 3-D profile of the field at \( z = 0.5 \text{ cm.} \) depth.

**CPU time**

All the computations for smaller problems (52-420 degrees of freedom) were performed by the VAX 11/785 system while the larger problems (upto 2364 degrees of freedom) were solved by the HDS AS/9180 mainframe computer. A distribution of the CPU time required for various problems is shown in the Table 4.2. D.O.F or *degrees of freedom* indicates the problem sizes, e.g., a 559 node BEM model with 4 D.O.F per node \((u_1, u_2, u_3, p)\) will have a total of \( 4 \times 559 = 2236 \text{ D.O.F.} \) and will create a \( 2236 \times 2236 \) matrix. The *solid* and *fluid* part refers to the contribution required by the solid and the fluid BIE respectively. Examination of the time distribution shows that at the lower end (52 D.O.F for VAX, 780 for HDS), 73 and 94.5% of the total time was required for the numerical computations to form the system matrix, whereas the time for the solution was about 13 and 5%, respectively. The pattern changed significantly with increase in problem size and for larger problems, as is evident from the Table 4.2, solution time was much higher than the formulation time. Thus the efficiency of the code can be improved by using a more efficient solver.

Numerical quadrature consumes the largest time in the numerical procedure and
Figure 4.25: Transmitted field along central axis: Normally incident beam on concave interface (4 MHz)
Figure 4.26: Radial variation of the transmitted field
Figure 4.27: 3D representation: Transmitted field at $Z=-0.5$
Table 4.2: CPU Time Distribution

<table>
<thead>
<tr>
<th>Machine</th>
<th>D.O.F</th>
<th>Total (secs.)</th>
<th>Formulation (secs.)</th>
<th>Solution (secs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Solid</td>
<td>Fluid</td>
<td>Total</td>
</tr>
<tr>
<td>VAX</td>
<td>52</td>
<td>35.7</td>
<td>21</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>228</td>
<td>909.5</td>
<td>502</td>
<td>109</td>
</tr>
<tr>
<td></td>
<td>420</td>
<td>9954</td>
<td>2972</td>
<td>736</td>
</tr>
<tr>
<td>HDS</td>
<td>780</td>
<td>1133</td>
<td>838</td>
<td>233</td>
</tr>
<tr>
<td></td>
<td>2236</td>
<td>5537</td>
<td>1625</td>
<td>456</td>
</tr>
</tbody>
</table>

The program has an automatic scheme for increment of the order of the quadrature [23] as the frequency increases or more complicated geometry is encountered. Table 4.3 shows the increment in time requirement with increased frequency for a 1124 D.O.F half-space problem. A comparison between a concave and a half-space geometry at a fixed frequency \( k = 169 \) for a 2236 D.O.F problem also shows a similar increment in time.

Table 4.3: Time Distribution: Interface Problem

<table>
<thead>
<tr>
<th>D.O.F</th>
<th>k</th>
<th>Curvature</th>
<th>Time (Mins.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1124</td>
<td>100</td>
<td>Flat</td>
<td>13.0</td>
</tr>
<tr>
<td></td>
<td>127</td>
<td>Flat</td>
<td>17.0</td>
</tr>
<tr>
<td></td>
<td>169</td>
<td>Flat</td>
<td>30.4</td>
</tr>
<tr>
<td></td>
<td>254</td>
<td>Flat</td>
<td>41.3</td>
</tr>
<tr>
<td>2236</td>
<td>169</td>
<td>Concave</td>
<td>103.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Flat</td>
<td>68.0</td>
</tr>
</tbody>
</table>
Conclusions

A numerical formalism was presented to evaluate the scattered acoustic field and the transmitted displacement field due to a bounded ultrasonic beam striking a curved fluid-solid interface. It is assumed that the incident beam, on reaching the interface, has a Gaussian or uniform profile. A mathematical model was developed to approximate the fluid and the solid as infinite hemispherical domains around the interface. The integral equation formulation of the acoustic and the elastic field are simplified by using the radiation conditions at the infinite boundaries. The boundedness of the incident beam allowed one to truncate all but the illuminated portion of the interface. A discretized model of this truncated interface was then used to solve the boundary integral equations.

The solution scheme is shown to be independent of the surface curvature and was verified by comparison with an analytical solution for plane interface computed by Fourier transform integrals of the angular spectrum. The BEM solutions show extremely good agreement for normally incident beams. For incident angles near the Rayleigh angle, a leaky Rayleigh surface wave is excited and violates the assumption of the truncated model. The situation is examined and shows that a larger segment of the surface must be modelled to capture the propagating surface waves.

Solutions are also significantly affected by the mesh size of the discretized interface. The model was found to be reliable when the characteristic mesh size was comparable to the wave length of the incident wave. In the test problems, frequencies of the various input beams ranged from 24 kHz \((k = 1)\) to 4 MHz \((k = 169)\). While an analysis at the lower kHz range was not significant from an NDE point of view, the solution at higher frequencies was constrained by requirement for larger memory
storage of the system matrix, e.g., an incident beam of 4 Mhz frequency and 1/8 in. radius \((a = 1/8''')\) required an average element length of 0.02 in. or 0.052 cm. (using \(L = 1.4\lambda\)) and a discretized interface of 3/16 in. (1.5a) radius resulting in a 559 node model. This, with 4 degrees of freedom per node, produced a \(2236 \times 2236\) matrix. The matrix, being complex, required 80 Mbytes of storage in double precision. Improvement of the computational scheme, to circumvent this problem, is an area open to further research, e.g., use of functions other than polynomials could be explored for representation of the field variables. This could alleviate the pressure for increasing mesh refinement at higher frequencies.

Verification of the results for concave and convex interfaces was done indirectly. The boundary solutions were tested for satisfaction of reciprocity theorem. Solutions for angles of incidence above 14° increasingly violated the reciprocity theorem and were rejected.

Finally, the numerical capability was used to generate the transmitted displacement field into the solid and validated by qualitative comparison with existing 2-D solutions. The data, however, need to be verified with experimental results to establish validity on a quantitative basis.

Theoretical models for ultrasonic transmissions are available for several specialized cases [52, 48]. The present work provides a three-dimensional solution by the boundary element method, which is capable of solving completely general problems of scattering and transmission through curved fluid-solid interfaces at moderate frequencies. Such a numerical model could be extremely useful in generating a wide variety of near and far field data that can provide valuable insight into the mechanics of acoustic and elastic waves.


APPENDIX A. FUNDAMENTAL SOLUTIONS IN ELASTICITY
AND ACOUSTICS

In this appendix a short description of the fundamental solutions or free space Green's functions for the differential equations of elasticity and acoustics is provided.

**Green's Functions in Elasticity**

**Elastostatic Green’s tensor**

For a three dimensional space $E_3$ occupied by an elastic continuum, the elastostatic equilibrium requires satisfaction of

$$
\sigma_{ij}(x) + f_i = 0 \quad x \in E_3
$$

where $\sigma_{ij}$ and $f_i$ represent the cartesian components of stress tensor and the body force vector, respectively. If $u(x)$ and $t(x)$ are the displacement and traction vectors, $C_{ijkl}$ the material constants and $n$ the normal direction, then the stress components are related to the displacement gradients and the material constants through Hooke's law, i.e.,

$$
\sigma_{ij} = C_{ijkl}u_{k,l}
$$
with
\[ t_i = \sigma_{ij} n_j \]

For linear, isotropic materials, the material constants reduce to two, with
\[ C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \]
where \( \lambda \) and \( \mu \) are Lamé constants. Eqn. A.1, then, reduces to
\[ (\lambda + \mu)(u_{k,k},_j) + \mu \delta_{ik} u_{k,ll} = -f_i \]
or,
\[ L_{ik}(u_k) = -f_i \quad (A.3) \]

Now the fundamental solution \( U^S_{km}(x,y) \) of this operator \( L_{ik} \) is that particular solution that satisfies
\[ L_{ik} U^S_{km}(x,y) = -\delta_{im} \delta(x-y) \quad (A.4) \]
where \( \delta_{ik} \) is the Kronecker delta and \( \delta(x-y) \) is the Dirac delta function. The superscript \( S \) indicates the elastostatic differential operator. Physically, \( U^S_{km}(x,y) \) represents the \( k \)-th component of the displacement at the point \( x \) in an infinite continuum in response to a concentrated unit force acting in \( m \)-th direction at the point \( y \). That solution [23] is given by
\[ U^S_{ij}(r) = \frac{1}{16\pi \mu (1-\nu)} \left( \frac{1}{r} \right) \left[ (3-4\nu) \delta_{ij} + \frac{3}{2} \frac{\partial}{\partial r} \left( \frac{\delta_{ij} + \frac{3}{2} \frac{\partial n_j}{\partial r}}{1-2\nu} \right) + \frac{\partial}{\partial n_i} \right] \quad (A.5) \]
and the traction associated with \( U^S_{ij} \) as determined by Eqns. A.2 and Hooke’s law is,
\[ T^S_{ij}(r) = \frac{2\nu - 1}{8\pi (1-\nu)} \frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( \frac{\delta_{ij} + \frac{3}{2} \frac{\partial n_j}{\partial r}}{1-2\nu} \right) + \frac{\partial}{\partial n_i} \right] \quad (A.6) \]
where the comma indicates partial differentiation with respect to cartesian coordinates at \( y \), the normal derivative is taken at \( y \) and \( n_i \) are the components of \( n \). The quantities \( U^S_{ij} \) and \( T^S_{ij} \) are components of Kelvin tensors.
Elastodynamic Green’s tensor

The corresponding components \( U_{ij}(x,y) \) which form the fundamental solution of the time harmonic elastodynamic equation of motion and which denote the displacement component at position \( x \) in the \( i \)-th direction due to a unit time-harmonic point force in \( j \)-th direction applied at position \( y \), i.e., those which satisfy the equation

\[
(\lambda + \mu)U_{ij,ik}(x,y) + \mu U_{kj,il}(x,y) + \rho \omega^2 U_{kj}(x,y) = -\delta_{ij}\delta(x - y) \tag{A.7}
\]

are \([20, 23]\),

\[
U_{ij}(x,y) = \frac{1}{4\pi \rho \omega^2 r^3} \left[ \delta_{ij} \left\{ (k_2^2r)^2 e_2 + D \right\} + Cr_{i;},r,j \right] \tag{A.8}
\]

where \( \lambda \) and \( \mu \) are the elastic constants such that \((\lambda + 2\mu)/\rho = c_L^2\) and \(\mu/\rho = c_T^2\) and

\[
r = |x - y|
\]

\[
k_\alpha = \omega/c_\alpha, \quad \alpha = 1, 2
\]

\[
e_\alpha = e^{ik_\alpha r}
\]

\[
D = \Gamma_2 e_2 - \Gamma_1 e_1; \quad \Gamma_\alpha = -1 + ik_\alpha r
\]

\[
C = \Omega_2 e_2 - \Omega_1 e_1; \quad \Omega_\alpha = 3 - 3ik_\alpha r - k_\alpha^2 r^2.
\]

If \( \tau_{ijk} \) represents the stress tensor corresponding to \( U_{ij} \), then \( T_{ij}(x,y) \) represents the traction tensor with respect to the normal \( n_k \) at \( y \), such that, \( T_{ij} = \tau_{ijk} n_k \). The components \( T_{ij} \) of \( T \) (Stokes’ traction tensor) referred to a cartesian basis are

\[
T_{ij}(x,y) = \frac{1}{4\pi \rho \omega^2 r^4} \left\{ \lambda \beta_1 + \mu \beta_2 + 2\mu \beta_3 \right\} \tag{A.9}
\]

where
\[ \beta_1 = e_1 (k_1)^2 \Gamma_1 \nabla_{ij} n_i \]
\[ \beta_2 = e_2 (k_2)^2 \Gamma_2 (\delta_{ij} \frac{\partial r}{\partial n} + r_n i j) \]
\[ \beta_3 = \left[ C (\delta_{ij} \frac{\partial r}{\partial n} + r_n i j + r_n i j) + F r_n i j \frac{\partial r}{\partial n} \right] \]

and
\[ F = H_1 e_1 - H_2 e_2 \]
\[ H_\alpha = 15 - 15 i k_\alpha r - 6 k_\alpha^2 r^2 + i k_\alpha^3 r^3 \]
Helmholtz’s Equation and its Static Counterpart

The propagation of acoustic waves and certain special cases of electromagnetic waves is modelled by the wave equation

\[ \nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = f(x, t) \]  

(A.10)

where \( p \) is the acoustic pressure and \( c \) the wave speed. Assuming the time-harmonic motion (i.e., \( p = p(x)e^{i\omega t} \)) and absence of the forcing term \( f(x, t) \), Eqn. A.10 reduces to Helmholtz’s equation

\[ \nabla^2 p - \frac{\omega^2}{c^2} p = 0 \]

or

\[ (\nabla^2 + k^2)p = 0 \]  

(A.11)

where \( k = \frac{\omega}{c} \) is called the wave number.

The fundamental solution \( G(x, y) \) of the Helmholtz equation represents the pressure at position \( x \) due to a point source of dilatation at position \( y \), i.e., it is the solution which satisfies

\[ (\nabla^2 + k^2)G(x, y) = -\delta(r) \]  

(A.12)

This solution for \( G(x, y) \) can be derived by transform techniques (section 7.2 [56]) and the expressions for \( G(r) \) and its normal derivative are respectively

\[ G(r) = \frac{\exp(ikr)}{4\pi r} \]  

(A.13)

\[ \frac{\partial G}{\partial n} = \frac{\partial G}{\partial r} \left( \frac{\partial r}{\partial n} \right) \]

\[ = (-1 + ikr) \left[ \frac{\exp(ikr)}{4\pi r^2} \right] (r.n) \]  

(A.14)
Laplace’s equation

For the case $\omega = 0$, Helmholtz’s equation reduces to the Laplace’s equation, i.e.,

$$\nabla^2 p = 0$$  \hspace{1cm} (A.15)

and Laplace’s equation may be described as the static counterpart of the Helmholtz equation. The corresponding fundamental solution $G^S$ for the Laplace’s equation is the solution of

$$\nabla^2 G^S = -\delta(r)$$  \hspace{1cm} (A.16)

and is

$$G^S(r) = \frac{1}{4\pi r}$$  \hspace{1cm} (A.17)

Here again the superscript ‘$S$’ identifies the static solution.
APPENDIX B. THE SOMMERFELD RADIATION CONDITIONS

The Sommerfeld radiation conditions and their significance are discussed in this appendix in connection to the acoustic problem. The radiation conditions provide criteria for uniqueness of the solution to the acoustic wave equation for scattering in an unbounded domain by requiring that the scattered field always be outgoing. This is closely connected to the causality of the solution in the time-domain.

If \( p \) represents the acoustic field, \( k \) the wave number and \( r \) is the distance to any field point, then the condition [57] states that

\[
\lim_{r \to \infty} r (\frac{\partial p}{\partial r} - ikp) = 0 \tag{B.1}
\]

First the standard notation for orders of magnitude - \( O \) (big-O) and \( o \) (small-O) - is listed. If \( f(r) \) and \( g(r) \) are arbitrary functions and \( f(r)/g(r) \) is finite, i.e.,

\[
\lim_{r \to \infty} \frac{|f(r)|}{g(r)} < \infty \tag{B.2}
\]

then

\[
f(r) = O(g(r)) \quad r \to \infty \tag{B.3}
\]

and \( f \) is said to be of order \( g \), with the big-O connotation. Alternately if

\[
\lim_{r \to \infty} \frac{|f(r)|}{g(r)} = 0 \tag{B.4}
\]
then
\[ f(r) = o(g(r)) \quad r \to \infty \] (B.5)

and \( f \) is said to be of order \( g \), with the little-o connotation.

The Green's function with the time-harmonic term \( e^{-i\omega t} \) may be written
\[
G(r, k)e^{-i\omega t} = \left( \frac{e^{ikr}}{4\pi r} \right) \left[ e^{-i\omega t} \right] = \frac{e^{i\omega (r/c - t)}}{4\pi r} \tag{B.6}
\]

The surfaces of constant phase are thus spheres whose radii increase with time. The Fourier inversion of \( G(r, k) \) gives
\[
\int_{-\infty}^{\infty} G(r, k)e^{-i\omega t} dk = \int_{-\infty}^{\infty} \frac{e^{ik(r-ct)}}{4\pi r} dk
= \frac{1}{2r} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(r-ct)} dk \right]
= \frac{1}{2} \left[ \frac{\delta(r - ct)}{r} \right] \tag{B.7}
\]

which is non-zero only on this expanding sphere. Thus, in time \( G(r) = \frac{e^{ikr}}{4\pi r} \) represents a function propagating outwards to infinity.

Consider, now, the derivative of \( G(r) \),
\[
G_r(r) = \frac{\partial G}{\partial r} = ikG - \frac{e^{ikr}}{r^2}
\]
i.e.,
\[
G_r - ikG = -\frac{e^{ikr}}{r^2} \tag{B.8}
\]

Now,
\[
\lim_{r \to \infty} \left| \frac{e^{ikr}}{r^2} \right| = 0 \quad \lim_{r \to \infty} \left| \frac{e^{ikr}}{r} \right| = 0 \tag{B.9}
\]
\[
\lim_{r \to \infty} \left| \frac{e^{ikr}}{r} \right| = 0 \quad \lim_{r \to \infty} \left| e^{ikr} \right| < \infty \tag{B.10}
\]
i.e.,

\[ G_r - ikG = o\left(\frac{1}{r}\right) \quad r \to \infty \]  \hspace{1cm} (B.11)

\[ G = O\left(\frac{1}{r}\right) \quad r \to \infty \]  \hspace{1cm} (B.12)

The radiation conditions imply that for a unique solution of the Helmholtz equation in the unbounded domain \( S_\infty \), where the boundary data of the problem are prescribed over a finite domain \( S \), the solution at a point far removed from \( S \) should behave quantitatively as the Green's function \( G(r) \). Scattering at \( S_\infty \) from the domain \( S \) must, therefore, behave like a point source and the solution should represent a wave propagating outward from \( S \), i.e.,

- The solution \( p \) of the acoustic equation must decay at the rate \( \frac{1}{r} \) as \( r \to \infty \), i.e.,

\[ p(x) = O\left(\frac{1}{r}\right) \quad r \to \infty \]  \hspace{1cm} (B.13)

- The direction of propagation of \( p \) should be outgoing. This direction is characterized by \( r \)-th derivative of phase \( e^{ikr} \), i.e., the \( r \)-th derivative of solution should behave similar to that of \( G(r) \), i.e.,

\[ \frac{\partial^r p}{\partial r^r} - ikp = o\left(\frac{1}{r}\right) \quad r \to \infty \]  \hspace{1cm} (B.14)

The above two conditions comprise the Sommerfeld radiation conditions.
APPENDIX C. INTEGRAL REPRESENTATIONS AND IDENTITIES

The integral representation of the acoustic differential equation is developed in this appendix. Both full-space and half-space domains are considered and various identities of the BIE fundamental solutions in the full and half-space are discussed.

Reciprocal Integral Representation for Acoustic Equation

All BIE formalism starts from a reformulation of the differential equation form of the problem into a reciprocal integral form. The idea is illustrated here using the acoustic wave equation

\[(\nabla^2 + k^2)\psi = 0\]  \hspace{1cm} (C.1)

The known fundamental solution \(G\) of the above equation satisfies

\[(\nabla^2 + k^2)G = -\delta(x - y)\]  \hspace{1cm} (C.2)

Now, from Green's second identity, if \(\partial B\) and \(V\) are the boundary and the domain of the acoustic equations for \(G\) and \(p\) then Green's second theorem states that

\[
\int_V \left[ p \nabla^2 G - G \nabla^2 p \right] dV = \int_{\partial B} \left[ p \frac{\partial G}{\partial n} - G \frac{\partial p}{\partial n} \right] ds(y) \tag{C.3}
\]

But from (C.1) and (C.2), one has

\[
\int_V \left[ p \nabla^2 G - G \nabla^2 p \right] dV = \int_V -p(y)\delta(x - y)dV = -p(x) \tag{C.4}
\]
Combining Eqns. C.3 and C.4 one gets the representation

$$p(x) = \int_{\partial B} \left[ \frac{\partial p}{\partial n} G - p \frac{\partial G}{\partial n} \right] ds$$

The relation C.5 represents an equivalent integral representation of the differential equation C.1 in terms of the boundary values of $p$ and $\frac{\partial p}{\partial n}$.

**Integral representation for finite scatterers**

Consider the interior or scatterer domain $B_i$ and the surrounding unbounded exterior domain $B_e$ (Figure 2.1), where $p^I$ and $p^S$ are the incident and the scattered field ($B_i$ is bounded by surface $\partial B$). $B_e$ is bounded internally by $\partial B$ and externally by the infinite boundary $S_{\infty}$. The scattered field $p^S$ assumably satisfies the Sommerfeld radiation condition at infinity but $p^I$, in general, may not.

Both $p^S$ and $p^I$ satisfy the time harmonic wave equations

$$\left( \nabla^2 + k^2 \right) p^S(x) = 0 \quad x \in B_e \quad (C.6)$$

$$\left( \nabla^2 + k^2 \right) p^I(x) = 0 \quad x \in B_i + B_e \quad (C.7)$$

Let $n$ denote the normal directions outward from the exterior $B_e$ (Figure 2.1), and let $q$ be outward from $B_i$. Following Green's second identity, the integral representation of $p^S$ in the exterior domain $B_e$ can be written as

$$C p^S(x_0) = \int_{S_{\infty} + \partial B} \left[ p^S(y) \frac{\partial G}{\partial n}(r) - \frac{\partial p^S}{\partial n}(y) G(r) \right] ds(y) \quad y \in \partial B \quad (C.8)$$

where $C = 1$ for $x_0 \in B_e$ and $C = 0$ for $x_0 \in B_i$. Similarly, representation for $p^I$ in the interior domain $B_i$ is

$$C p^I(x_0) = \int_{\partial B} p^I(y) \frac{\partial G}{\partial q}(r) - \frac{\partial p^I}{\partial q}(y) G(r) ds(y) \quad y \in \partial B \quad (C.9)$$
where \( C = 1 \) for \( x_0 \in B_i \) and \( C = 0 \) for \( x_0 \in B_e \). The integral involving \( p^S \) over \( S_\infty \) vanishes when \( p^S \) satisfies the radiation condition i.e.,

\[
\int_{S_\infty} \left[ p^S(y) \frac{\partial G}{\partial n}(r) - \frac{\partial p^S}{\partial n}(y)G(r) \right] ds(y) = 0 \quad (C.10)
\]

Hence, for \( x \in B_e \), \( C = 1 \) and

\[
p^S(x) = \int_{\partial B} \left[ p^S(y) \frac{\partial G}{\partial n}(r) - \frac{\partial p^S}{\partial n}(y)G(r) \right] ds(y) \quad x \in B_e \quad (C.11)
\]

For the representation of \( p^I \) in \( B_i \), however, \( C = 0 \) whenever \( x \in B_e \) and there is no infinite boundary involved. Thus representation (C.9) then becomes

\[
0 = \int_{\partial B} p^I(y) \frac{\partial G}{\partial n}(r) - \frac{\partial p^I}{\partial n}(y)G(r) ds(y) \quad x \in B_e \quad (C.12)
\]

and, since \( q = -n \) one obtains

\[
0 = \int_{\partial B} \left[ p^I \frac{\partial G}{\partial n} - \frac{\partial p^I}{\partial n} G \right] ds \quad x \in B_e \quad (C.13)
\]

Then if \( p \) represents the total acoustic pressure such that \( p = p^I + p^S \), adding the two representations for \( x \in B_e \), [C.11 and C.13] provides the exterior representation

\[
p^S = \int_{\partial B} \left[ \frac{\partial G}{\partial n} - \frac{\partial p}{\partial n} G \right] ds
\]

\[
or, \quad p = \int_{\partial B} \left[ \frac{\partial G}{\partial n} - \frac{\partial p}{\partial n} G \right] ds + p^I \quad x \in B_e \quad (C.14)
\]

The representation integral for a half-space scatterer

The representation integral for the bounded domain (C.14) is readily adaptable to the case of a semi-infinite scatterer. Considering Figure 4.2 and the hemispherical sub-domains \( B^+ \) and \( B^- \) and the source \( p^I \) located in the upper domain \( B^+ \), the
lower domain $B^-$ may be considered the scatterer. Mathematically $p^I$ is oblivious to the scatterer so that by considering only the hemisphere $B^-$ in the lower half space below $S_H$, an interior representation of $p^I$ may be written for a point $x$ ($x \in B^-$).

With the domain of the representation integral denoted by $S_H + S_\infty$, one has

$$ p^I(x) = \int_{S_H} \left[ G \frac{\partial p^I}{\partial q} - \frac{\partial G}{\partial q} p^I \right] ds + \int_{S_\infty} \left[ G \frac{\partial p^I}{\partial q} - \frac{\partial G}{\partial q} p^I \right] ds \quad (C.15) $$

The incident beam propagates into the interface $S_H$ and is transmitted into $B^-$. Therefore it will be outwardly propagating at the infinite surface $S_\infty$ and the integral over $S_\infty$ will vanish due to the radiation conditions to give

$$ p^I(x) = \int_{S_H} \left[ G \frac{\partial p^I}{\partial q} - \frac{\partial G}{\partial q} p^I \right] ds \quad x \in B^- \quad (C.16) $$

This is valid for a point $x$ inside $B^-$. For a point outside $B^-$ in the upper domain,

$$ 0 = \int_{S_H} \left[ G \frac{\partial p^I}{\partial q} - \frac{\partial G}{\partial q} p^I \right] ds \quad x \in B^+ \quad (C.17) $$

For the field $p^S$ scattered from $S_H$ into $B^+$ outward at $S_\infty^+$,

$$ \int_{S_\infty^+} \left[ G \frac{\partial p^S}{\partial n} - \frac{\partial G}{\partial n} p^S \right] ds = 0 \quad (C.18) $$

Thus, for a point $x \in B^+$

$$ p^S(x) = \int_{S_H} \left\{ G \frac{\partial p^S}{\partial n} - \frac{\partial G}{\partial n} p^S \right\} ds + \int_{S_\infty^+} \left\{ G \frac{\partial p^S}{\partial n} - \frac{\partial G}{\partial n} p^S \right\} ds $$

$$ = \int_{S_H} \left\{ G \frac{\partial p^S}{\partial n} - \frac{\partial G}{\partial n} p^S \right\} ds \quad x \in B^+ \quad (C.19) $$

The integral on $S_\infty^+$ vanishes due to the radiation conditions so that by adding (C.17) and (C.19), one obtains the representation integral in terms of the total pressure $p$ and the incident beam $p^I$, i.e.,
\[ p(x) = \int_{S_H} \left[ G \frac{\partial p}{\partial n} - \frac{\partial G}{\partial n} p \right] + p^I(x) \quad x \in B^+ \] (C.20)

The representation integral for the half-space scatterer is thus of the same form as the bounded scatterer, except the integral is over the half-space interface.

**BIE Identities of Fundamental Solutions**

Consider a scatterer enclosing an interior domain \( B_i \) and surrounded by an exterior medium \( B_e \) with infinite boundary \( S_\infty \) (see Figure C.1) and let \( x_i \) and \( x_e \) be points in \( B_i \) and \( B_e \), respectively. Also if \( \partial B \) is the scatterer boundary, \( n \) the normal pointing into \( B_i \) and \( q \) the normal from \( B_i \) into \( B_e \) and if \( U^S \) and \( G^S \) are respectively the fundamental solutions of the elastostatic (vector) and the Laplace's (scalar) differential equation, and if \( T^S \) and \( \frac{\partial G^S}{\partial n} \) are the corresponding stress tensor and the normal gradient operator, then the following identities pertain:

\[ \int_{\partial B} -T^S_q(x_i, y) ds(y) = 1 \] (C.21)
\[ \int_{\partial B} T^S_n(x_e, y) ds(y) = 0 \] (C.22)
\[ \int_{S_\infty} -T^S_n(x_e, y) ds(y) = 1 \] (C.23)
\[ \int_{\partial B} -\frac{\partial G^S}{\partial q}(x_i, y) ds(y) = 1 \] (C.24)
\[ \int_{\partial B} \frac{\partial G^S}{\partial n}(x_e, y) ds(y) = 0 \] (C.25)
\[ \int_{S_\infty} -\frac{\partial G^S}{\partial n}(x_e, y) ds(y) = 1 \] (C.26)
Figure C.1: Interior and exterior domains
The subscripts \( q \) and \( n \) of the traction tensor \( T^S \) indicate the normal directions \( n \) and \( q \) in which they are defined. The first three vector identities are derived in the following. The scalar counterparts will analogously follow.

If upon the interior representation integral

\[
u(x_i) = \int_{\partial B} \left[ U^S t - T^S_q u \right] ds \tag{C.27}\]

one imposes a constant \( u \) (rigid body motion: \( u = \text{constant} \)), then the \( t \) vanishes and the representation becomes

\[
I = \int_{\partial B} \left[ -T^S_q(x_i, y) \right] ds \tag{C.28}
\]

Consider now a point \( x_e \) in \( B_e \) whose boundary is represented by \( \partial B + S_\infty \).

Using similar arguments as above, it can be shown show that

\[
I = \int_{\partial B + S_\infty} \left[ -T^S_n(x_e, y) \right] ds \tag{C.29}
\]

For a point \( x \) on the surface \( \partial B \) (Figure C.1) and the hemispheres \( \Gamma_i \) and \( \Gamma_e \) centered at \( x \) with a small radius \( \varepsilon \) intersecting \( B_e \) and \( B_i \) respectively.

Let

\[
\Delta = \text{Part of } \partial B \text{ intersecting the hemispheres}
\]

and,

\[
\partial S = \text{Reduced } \partial B = \partial B - \Delta
\]

If \( x \) is taken to be an interior point \( (x \in B_i) \) enclosed by the bump \( \Gamma_i \), then from (C.28)

\[
\int_{\partial S} \left[ -T^S_q \right] ds + \int_{\Gamma_i} \left[ -T^S_n \right] ds = I \tag{C.30}
\]
For an exterior point \( x \in B_e \) enclosed by the bump \( \Gamma_e \), the augmented inner boundary becomes \( \partial S + \Gamma_e \) and one obtains from (Eqn. C.29)

\[
\int_{\partial S} [-T^S_n] \, ds + \int_{\Gamma_e} [-T^S_n] \, ds + \int_{S_{\infty}} [-T^S_n] \, ds = I \tag{C.31}
\]

Since \( \Gamma_i \) and \( \Gamma_e \) together form a closed spherical bubble enclosing the point \( x \), both the normals \( n \) and \( q \) are outward normals with respect to the bubble. Therefore using Eqn. C.28,

\[
\int_{\Gamma_i} (-T^S_q) \, ds + \int_{\Gamma_e} (-T^S_n) \, ds = \int_{\Gamma_i + \Gamma_e} (-T^S) \, ds = I \tag{C.32}
\]

Also, since \( q = -n \),

\[
\int_{\partial S} (-T^S_q) \, ds + \int_{\partial S} (-T^S_n) \, ds = 0 \tag{C.33}
\]

Therefore, adding (C.30) and (C.31) and using (C.32) and (C.33)

\[
0 + I + \int_{S_{\infty}} (-T^S_n) \, ds = 2I
\]

Or,

\[
\int_{S_{\infty}} (-T^S_n) \, ds = I \tag{C.34}
\]

Now, from (C.29) and (C.34)

\[
\int_{\partial B} (-T^S_n) \, ds = 0 \tag{C.35}
\]

**Identities in the half-space**

For integration over a hemispherical surface, e.g., \( S^+_{\infty} \) or \( S^-_{\infty} \) as in Figure 4.2, where

\[
S^+_{\infty} = S^-_{\infty} = \frac{1}{2} S_{\infty}
\]
Consider now the upper domain $B^+$ as the exterior domain. Then from Eqn. C.29

\[ \int_{S_H + S_\infty^+} [-T_n^S] \, ds = I \]  \hspace{1cm} (C.37)

Similarly, considering $B^-$ as the interior domain one gets

\[ \int_{S_H + S_\infty^-} [-T_q^S] \, ds = I \]  \hspace{1cm} (C.38)

Combining the above

\[ \int_{S_H} T_q^S \, ds = -I/2 \]  \hspace{1cm} (C.39)
APPENDIX D. EXACT ANALYSIS OF SCATTERING BY ELASTIC SPHERE

The exact solution of the plane wave scattering by a solid elastic sphere is available in the ref. [25] for viscous fluid. Discussed in this appendix are the analysis and modifications for the inviscid fluid.

A viscous fluid, with kinematic viscosity $\nu$ and bulk viscosity $k_b$, will have a scalar velocity potential $\phi$ and a vector potential $\psi$ similar to the Helmholtz's resolution for a solid and will support shear stress. Following the discussion in [32], the incident compressional field in the fluid may be expressed as a series

$$\phi^I = \phi_0 \sum_{n=0}^{\infty} i^n(2n+1)j_n(K_1 r)P_n \cos(\theta)$$  \hspace{1cm} (D.1)

and the scattered longitudinal and viscous waves are

$$\phi^S = \phi_0 \sum_{n=0}^{\infty} i^n(2n+1)A_n h_n(K_1 r)P_n \cos(\theta)$$  \hspace{1cm} (D.2)

$$\psi^S = \phi_0 \sum_{n=0}^{\infty} i^n(2n+1)B_n h_n(K_2 r)P_n^1 \cos(\theta)$$  \hspace{1cm} (D.3)

where,

- $j_n$ = spherical Bessel function of first kind of order $n$
- $h_n$ = spherical Hankel function of first kind of order $n$
- $r$ = the field distance
- $P_n$ = Legendre function of order $n$
\( P_n^1 = \) associated Legendre function of order \( n \)

\( A_n \) and \( B_n \) are the reflection coefficients mentioned in the Chapter 2 to be determined. \( K_1 \) and \( K_2 \) are the two wave numbers in the viscous fluid corresponding to the longitudinal and the transverse field such that

\[
K_1 = \frac{\omega}{c} \left[ 1 + i \frac{\omega}{2c^2} \left( \frac{4}{3} \nu + \frac{k_b}{\rho_f} \right) \right]
\]

\[
K_2 = (1 + i) \sqrt{\frac{\omega}{2\nu}}
\]

For an inviscid fluid \( K_2 \) is non-existent and \( K_1 \) reduces to the wave number \( k = \frac{\omega}{c} \).

If \( \Phi \) and \( \Psi \) are similarly the transmitted potentials in the solid, then their series representations [25, 26] are

\[
\Phi = \Phi_0 \sum_{n=0}^{\infty} i^n (2n+1)a_n j_n(k_{\ell} r) P_n(\cos \theta)
\]

\[\text{(D.4)}\]

\[
\Psi = \Psi_0 \sum_{n=0}^{\infty} i^n (2n+1)b_n j_n(k_{T} r) P_n^1(\cos \theta)
\]

\[\text{(D.5)}\]

Here, \( a_n \) and \( b_n \) are the transmitted refraction coefficients to be determined together with reflection coefficients \( A_n \) and \( B_n \) from the continuity of traction and normal velocity at the fluid-sphere interface. The velocity components in the viscous fluid are

\[
v_r = -\frac{\partial \Phi}{\partial r} + \frac{\partial \psi}{r \partial \theta}
\]

\[
v_\theta = -\frac{\partial \Phi}{r \partial \theta} - \frac{\partial \psi}{\partial r}
\]

\[\text{(D.6)}\]

and those in the elastic solid are

\[
v_r = -i\omega \left( -\frac{\partial \Phi}{\partial r} + \frac{\partial \psi}{r \partial \theta} \right)
\]

\[
v_\theta = i\omega \left( \frac{\partial \Phi}{r \partial \theta} + \frac{\partial \psi}{\partial r} \right)
\]

\[\text{(D.7)}\]
The stress component in the fluid [26] are

\[ t_{rr} = -p_t + (k_b - 2\mu/3)\Delta + 2\mu d_{rr} \]
\[ t_{r\theta} = 2\mu d_{r\theta} \]  \hspace{1cm} (D.8)

and those in the elastic sphere are

\[ t_{rr} = \lambda e \Delta + 2\mu e_{err} \]
\[ t_{r\theta} = 2\mu e_{r\theta} \] \hspace{1cm} (D.9)

where \( p_t \) and \( d_{ij} \) \((i, j = r, \theta)\) are the thermodynamic pressure and the deformation-rate tensor of the fluid; \( e_{ij} \) \((i, j = r, \theta)\) is the deformation tensor of the elastic and \( \Delta \) is the dilatation. The coefficients \( \mu_e \) and \( \lambda_e \) are the Lame's constants of elasticity and \( \mu \) is the coefficient of viscosity.

Expressing the continuity of velocity and tractions (Eqns. 2.6 and 2.7) in terms of the potentials above, four linear algebraic equations are obtained in \( A_n, B_n, a_n \) and \( b_n \) as in Eqn. 23 of the ref. [25]. For the inviscid case, which is of interest here, the viscous shear in fluid is neglected, i.e., \( \mu = 0, \quad \nu = 0 \), and the coefficient \( B_n = 0 \) and \( K_1 \) reduce to \( k \). Defining now

\[ \xi = K_1 r; \quad \eta = K_2 r; \quad \dot{\xi} = k\xi r; \quad \dot{\eta} = k\eta r \]

and noting that

\[ \eta^2 = i\omega \rho r^2, \]

one of the four equations mentioned above drops out and the three equations to determine \( A_n, a_n, b_n \) are
where \( \nu_e \) is Poisson's ratio of the solid and

\[
\beta = \frac{\nu_e}{1 - \nu_e}
\]

For a time-harmonic acoustic problem in inviscid fluid, the acoustic pressure is related to the scalar potential \( \phi \) by

\[
p = -i\omega \rho \phi
\]  

(D.11)

Therefore, if \( p_0 \) is the amplitude of the incident wave, then from the Eqns. D.1 and D.2 one obtains

\[
p^I = p_0 \sum_{n=0}^{\infty} i^n (2n + 1) j_n(kr)P_n(cos\theta)
\]

(D.12)

\[
p^S = p_0 \sum_{n=0}^{\infty} i^n (2n + 1) A_n h_n(kr)P_n(cos\theta)
\]

(D.13)

For \( kr \rightarrow \infty \), \( h_n(kr) \approx \frac{1}{i^{n+1}} \left( \frac{e^{ikr}}{kr} \right) \) and Eqn. D.13 is

\[
p^S = \left[ p_0 \left( \frac{e^{ikr}}{i} \right) \right] \sum_{n=0}^{\infty} (2n + 1) A_n P_n(cos\theta)
\]

or,

\[
k_r \frac{p^S}{p_0} = \left( \frac{e^{ikr}}{i} \right) \sum_{n=0}^{\infty} (2n + 1) A_n P_n(cos\theta)
\]

(D.14)
i.e.,

\[ | \frac{p_{kr}}{p_0} | = \sum_{n=0}^{\infty} (2n+1) A_n P_n(\cos\theta) \] (D.15)

The quantity \( | \frac{p_{kr}}{p_0} | \) is defined as the normalized pressure or \( p_{norm} \) and is independent of the far field distance. The summation (D.15) can be determined for any orientation \( (\theta) \) of the far field point and usually converged within ten terms for \( ka<8 \).
APPENDIX E. EIGENFREQUENCIES OF SPHERE: DIRICHLET PROBLEM

The eigenfrequencies of elastodynamic vibration of a sphere is discussed in the reference by Eringen and Suhubi [40]. The analogous case is derived here for the scalar wave equation. The eigenfrequencies of a sphere for the acoustic wave equation are the zeroes of the solution for the Dirichlet boundary condition \( p(a) = 0 \), where \( a \) is the radius of the sphere and \( p \) is the pressure. They are evaluated using a spherical coordinate system as shown in the Figure E.1.

The Helmholtz's wave equation

\[
[\nabla^2 + k^2] p = 0 \tag{E.1}
\]

is written in spherical coordinates as

\[
\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial p}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial p}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 p}{\partial \phi^2} + k^2 \right] p = 0 \tag{E.2}
\]

or,

\[
\frac{\partial^2 p}{\partial r^2} + \frac{2 \partial p}{r \partial r} + \frac{\cot \theta \partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 p}{\partial \phi^2} + k^2 p = 0 \tag{E.3}
\]

Now,

\[
p(r, \theta, \phi) = R(r)F(\theta)G(\phi) \tag{E.4}
\]
so that by dividing the above equation by \( R(r)F(\theta)G(\phi) \), one obtains

\[
\frac{R''(r)}{R(r)} + \frac{2R'(r)}{rR(r)} + \frac{\cot \theta F'(\theta)}{r^2 F(\theta)} + \frac{1}{r^2} \frac{F''(\theta)}{F(\theta)} + \frac{1}{r^2 \sin^2 \theta} \frac{G''(\phi)}{G(\phi)} + k^2 = 0
\]  
(E.5)

Define \( f \) such that

\[
\frac{r^2 R''}{R} + \frac{2rR'}{R} + k^2 r^2 = f^2
\]

then, from (E.5)

\[
(sin^2 \theta)f^2 + \cos \theta \sin \theta \frac{F'}{F} + \sin^2 \theta \frac{F''}{F} + \frac{G''}{G} = 0
\]  
(E.6)

Also define \( g \) by

\[
(sin^2 \theta)f^2 + \cos \theta \sin \theta \frac{F'}{F} + \sin^2 \theta \frac{F''}{F} = g^2
\]
then

\( (f^2 - \frac{g^2}{\sin^2 \theta}) F + \frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta F') = 0 \)  \hfill (E.7)

From Eqns. E.5, E.6 and E.7, the three ODE's for \( R, F \) and \( G \) in terms of \( f \) and \( g \) are

\[
\begin{align*}
\frac{r^2 R''(r) + 2r R'(r) + (k^2 r^2 - f^2) R(r)}{r^2 + \frac{g^2}{\sin^2 \theta}} &= 0 \quad \text{(E.8)} \\
\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{dF}{d\theta}) + (f^2 - \frac{g^2}{\sin^2 \theta}) F &= 0 \quad \text{(E.9)} \\
g^2 + \frac{G''}{G} &= 0 \quad \text{(E.10)}
\end{align*}
\]

The solution for the last equation \( g^2 G + G'' = 0 \) is

\[ G(\phi) = A_1 e^{ig\phi} + A_2 e^{-ig\phi} \]  \hfill (E.11)

For a single valued solution for the whole sphere, \( G(\phi) \) must be periodic with period \( 2\pi \). Thus \( g \) must be an integer, say \( g = \pm 1, \pm 2, \pm 3, \ldots \).

Using the substitutions

\[
\begin{align*}
f^2 &= \nu(\nu + 1) \\
R(r) &= \frac{\Gamma(r)}{\sqrt{kr}}
\end{align*}
\]

Eqn. E.8 becomes

\[
\frac{r^2 d^2 \Gamma}{dr^2} + r \frac{d\Gamma}{dr} + [k^2 r^2 - (\nu + 1/2)^2] \Gamma(r) = 0 \]  \hfill (E.12)

The solution of the above equation ([58], pg. 179-180) is

\[ \Gamma(r) = B_1 J_{\nu + 1/2}(kr) + B_2 Y_{\nu + 1/2}(kr) \]  \hfill (E.13)
where $Y_{\nu+1/2}(kr)$ is unbounded as $r \to 0$. Therefore, for a bounded solution at $r = 0$, the second term must vanish. Therefore

$$\Gamma(r) = B_1 J_{\nu+1/2}(kr)$$

$$= B_1 \sqrt{\frac{2kr}{\pi}} j_{\nu}(kr)$$

$$= C(\sqrt{kr}) j_{\nu}(kr)$$

$$or, \quad R(r) = \frac{R}{\sqrt{kr}} = C j_{\nu}(kr) \quad (E.14)$$

where $\nu$ is an integer. To find solution for Eqn. E.9, $\cos \theta = \mu$ and $f^2 = \nu(\nu + 1)$ are substituted in Eqn. E.9

$$m^2 - 1 - \mu^2 F = 0 \quad (E.15)$$

The solution of this equation [58] will be a linear combination of the associated Legendre functions $P_{\nu}^m(\mu)$ and $Q_{\nu}^m(\mu)$, where $\nu$ is integer. The notation $\nu$ is now substituted by simply $n$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ etc. The function $Q(\mu)$ is singular at $\mu = \pm 1$, i.e. for $\theta = 0$ and $\pi$. Only $P(\mu)$ is retained in the solution and hence,

$$F(\theta) = P_{\nu}^m(\mu) = P_{\nu}^m(\cos \theta) \quad (E.16)$$

Therefore, the solution for pressure field is

$$p(r, \theta, \phi) = p_0 j_n(kr) P_{\nu}^m(\cos \theta)e^{-im\phi} \quad (E.17)$$

By applying the Dirichlet boundary condition $p(r = a) = 0$, one obtains

$$j_n^m(ka) = 0 \quad (E.18)$$

Hence, the required eigenfunctions are the zeroes of the above equation, where $n = 0, 1, 2\ldots$ are the modes and $m = 0, 1, 2\ldots$ are the harmonics of each mode.
APPENDIX F. THE REFLECTION COEFFICIENT $R(k_r)$

The reflection from a flat interface due to a plane incident wave has been treated extensively in [59] by Miklowitz (Chapter 3). In particular, the article 3.3.3 of the book treats the reflection and refraction at a fluid-solid interface. If $\theta$ is the angle of incidence and $\eta$ and $\beta$ denote the angles for longitudinal and shear vectors of the transmitted field in the solid, and if $A_R$ and $A_I$ are the reflected and incident amplitudes then the reflection coefficient is derived (3.117: [59]) as

$$R = \frac{A_R}{A_I}$$

$$= \frac{D_1 - D_2}{D_1 + D_2} \quad (F.1)$$

where

$$D_1 = \mu \cot \theta \left[ (\cot^2 \beta - 1) + 4 \cot \eta \cot \beta \right] \quad (F.2)$$

$$D_2 = \rho \left( \frac{c_4}{c_T} \right) \cot \eta \quad (F.3)$$

$$c_s = \frac{c}{\sin \theta} = \frac{c}{k_r/k} = \frac{ck}{k_r} \quad (F.4)$$

The expression for the reflection coefficient is now expressed in terms of the wave-spectrum $k$, which can be directly used in the Fourier transform to evaluate the spatial form of any wave form. If $e$ is the incident wave vector, then

$$k = ke = (\omega/c)e \quad (F.5)$$
If $k_T$ and $k_L$ are the wave numbers in solid, then the following may be defined

\[
\begin{align*}
  k_r &= k \sin \theta \\
  k_Z &= k \cos \theta = \sqrt{k^2 - k_T^2} \\
  k_{ZT} &= \sqrt{k_T^2 - k_r^2} \\
  k_{ZL} &= \sqrt{k_L^2 - k_r^2}
\end{align*}
\]

Now, using the Snell’s law,

\[
\frac{\sin \theta}{c} = \frac{\sin \eta}{c_L} = \frac{\sin \beta}{c_T}
\]

one obtains

\[
\begin{align*}
  \sin \eta &= (c_L/c) \sin \theta = (k/k_L) \sin \theta = k_r/k_L \\
  \cos \eta &= \sqrt{k_L^2 - k_r^2}/k_L = k_{ZL}/k_L \\
  \cot \eta &= k_{ZL}/k_r
\end{align*}
\]

i.e.,

\[
\begin{align*}
  \cot \beta &= k_{ZT}/k_r \\
  \cot \theta &= k_{Z}/k_r
\end{align*}
\]

and, similarly

Define, now

\[
b = \frac{\rho}{\epsilon}
\]

Then,

\[
D_1 = \mu \cot \theta \left[ (C \cot^2 \beta - 1)^2 + 4 \cot \eta \cot \beta \right]
\]

\[
= \mu \frac{k_Z}{k_r} \left[ (k_{ZT}/k_r)^2 - 1 \right]^2 + \frac{4k_{ZL}k_ZT}{k_T^2}
\]
and, now using

\[ \frac{kZ}{k_r} \left[ 1 - \frac{k_T^2}{k_r^2} - 2 \frac{k_T^2}{k_r^2} + 4 \frac{k_Z L k_Z T}{k_r^2} \right] \]

\[ = \frac{kZ}{k_r^2} \left[ k_T^2 + k_r^2 - 2 k_T^2 k_r^2 + 4 k_Z L k_Z T k_r^2 \right] \]

\[ = \frac{kZ}{k_r^2} \left[ (k_T^2 - k_r^2)^2 + k_r^4 - 2(k_T^2 - k_r^2)k_r^2 + 4 k_Z L k_Z T k_r^2 \right] \]

\[ = \frac{\mu}{k_T^2 k_r^2} \left( k_T^2 k_Z \right) \left[ (k_T^2 - 2k_r^2)^2 + 4 k_Z L k_Z T k_r^2 \right] \quad (F.15) \]

and,

\[ D_2 = \rho \left( \frac{c_T}{c} \right)^4 \cot \eta \]

\[ = b \rho \left( \frac{c_T}{c} \right)^4 \left( \frac{k_Z L}{k_r} \right) \]

\[ = b \left( \frac{\rho c^2}{c_T^2} \right) \left( \frac{k_Z L}{k_r^2} \right) \quad (F.16) \]

Now, using

\[ \frac{\rho c^2}{c_T^2} = \frac{\mu}{c_T^2} c^2 \]

\[ \frac{c^2}{c_T^2} = \frac{k_T^2}{k_r^2} \]

one obtains

\[ D_2 = \frac{\mu}{k_T^2 k_r^2} \left( \frac{b k_T^6 k_Z L}{b k_T^6 k_Z L} \right) \quad (F.17) \]

Combining \( D_1 \) and \( D_2 \), the reflection coefficient \( R(k) \) is

\[ R = \frac{D_1 - D_2}{D_1 + D_2} = 1 - 2 \frac{A}{B} \quad (F.18) \]

where,

\[ A = b k_T^6 k_Z L \]

\[ B = k_T^2 k_Z \left[ (k_T^2 - 2k_r^2)^2 + 4 k_Z L k_Z T k_r^2 \right] \quad (b k_T^6 k_Z L) \]
If the incident beam strikes the interface at some arbitrary angle such that

\[ k \cdot x = k_x \quad k \cdot y = k_y \]  \hspace{1cm} (F.19)

Then, the incident beams in spatial and k-domain are

\[ p^I(x_1, x_2) = \left( e^{\frac{-x_1^2 + x_2^2}{4\beta}} \right) e^{i k_x x_1 + i k_y x_2} \]  \hspace{1cm} (F.20)

\[ \hat{p^I}(k_x + k_2) = 4\pi\beta \left( e^{-\beta [(k_1 - k_x)^2 + (k_2 - k_y)^2]} \right) \]  \hspace{1cm} (F.21)

and the scattered field \( p^S(x) \) is given by the transform

\[ p^S = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(k_1, k_2) p^I(k_1, k_2) e^{ik \cdot x} dk_1 dk_2 \]  \hspace{1cm} (F.22)

where, the reflection coefficient \( R(k) \) is obtained from the expression for \( R(k_r) \) by substituting

\[ k_r^2 = k_1^2 + k_2^2 \]  \hspace{1cm} (F.23)

For the special case of a beam incident in the X-Z plane (which was discussed in Chapter 4) at the Rayleigh angle, the Rayleigh wave number \( k_R \) is approximately equal to the shear wave number \( k_T \) and if \( \theta \) is the angle of incidence then

\[ \theta \approx \sin^{-1} \frac{k_T}{k} \]  \hspace{1cm} (F.24)

and,

\[ p^I(x_1, x_2) = \left( e^{\frac{-x_1^2 + x_2^2}{4\beta}} \right) e^{i k_T x_1} \]  \hspace{1cm} (F.25)

\[ \hat{p^I}(k_x + k_2) = 4\pi\beta \left( e^{-\beta [(k_1 - k_T)^2 + k_2^2]} \right) \]  \hspace{1cm} (F.26)
APPENDIX G. NUMERICAL QUADRATURE

The quadratic shape functions used in numerical scheme are shown below for the standard quadrilateral and equilateral triangle. For an eight-noded square, the shape functions are

\[
\begin{align*}
N_1(\xi) &= (1/4)(\xi_1 + 1)(\xi_2 + 1)(\xi_1 + \xi_2 - 1) \\
N_2(\xi) &= (1/4)(\xi_1 - 1)(\xi_2 + 1)(\xi_1 - \xi_2 + 1) \\
N_3(\xi) &= (1/4)(1 - \xi_1)(\xi_2 - 1)(\xi_1 + \xi_2 + 1) \\
N_4(\xi) &= (1/4)(1 + \xi_1)(\xi_2 - 1)(\xi_2 - \xi_1 + 1) \\
N_5(\xi) &= (1/2)(\xi_1 + 1)(1 - \xi_2^2) \\
N_6(\xi) &= (1/2)(\xi_2 + 1)(1 - \xi_1^2) \\
N_7(\xi) &= (1/2)(\xi_1 - 1)(\xi_2^2 - 1) \\
N_8(\xi) &= (1/2)(1 - \xi_2)(1 - \xi_1^2) \\
\end{align*}
\]

and for a six-noded triangle, they are

\[
\begin{align*}
N_1(\xi) &= \xi_1(2\xi_1 - 1) \\
N_2(\xi) &= \xi_2(2\xi_2 - 1) \\
N_3(\xi) &= \xi_3(2\xi_3 - 1) \\
N_4(\xi) &= 4\xi_1\xi_3 \\
\end{align*}
\]
Numerical quadrature over each element is performed in terms of the local coordinate $\xi$ using Gaussian quadrature. If $F(\xi)$ is a function defined over an element area $\partial e$, the integrand $\int_{\partial e} F(\xi) d\xi_1 d\xi_2$ becomes

\[
I = \int_{\partial e} F(\xi) d\xi_1 d\xi_2
\]

\[
= \int_{-1}^{1} \int_{-1}^{1} F(\xi_1, \xi_2) d\xi_1 d\xi_2
\]

\[
= \sum_{\mu=1}^{n_1} \sum_{\nu=1}^{n_2} W_{\mu} W_{\nu} F(\xi_{\mu}, \xi_{\nu})
\]  

(G.3)

where $W_{\mu}$ and $W_{\nu}$ are the weight factors, $n_1$ and $n_2$ are number of Gauss points and $\xi_{\mu}$ and $\xi_{\nu}$ are the abscissas of Gauss points. The values of the abscissas and weights for quadrilateral and triangles can be found in refs. [60] and [61].