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Note on Nordhaus-Gaddum problems for
Colin de Verdière type parameters

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Abstract

We establish the bounds \( \frac{4}{3} \leq b_\nu \leq b_\xi \leq \sqrt{2} \), where \( b_\nu \) and \( b_\xi \) are the Nordhaus-Gaddum sum upper bound multipliers, i.e., \( \nu(G) + \nu(\overline{G}) \leq b_\nu |G| \) and \( \xi(G) + \xi(\overline{G}) \leq b_\xi |G| \) for all graphs \( G \), and \( \nu \) and \( \xi \) are Colin de Verdière type graph parameters. The Nordhaus-Gaddum sum lower bound for \( \nu \) and \( \xi \) is conjectured to be \( |G| - 2 \), and if these parameters are replaced by the maximum nullity \( M(G) \), this bound is called the Graph Complement Conjecture in the study of minimum rank/maximum nullity problems.

Keywords: Nordhaus-Gaddum; Colin de Verdière type parameter; Graph Complement Conjecture; maximum nullity; minimum rank; graph complement

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1 Introduction

Nordhaus-Gaddum problems have been studied for many different graph parameters, including chromatic number, independence number, domination number, Hadwiger number, etc. (see, for example, [6] and the references therein). In this note we discuss the Nordhaus-Gaddum sum upper bounds for the Colin de Verdière type parameters $\nu$, $\xi$, and $\mu$. The Graph Complement Conjecture in the study of minimum rank/maximum nullity problems is a conjectured lower bound for related Nordhaus-Gaddum problems.

All graphs in this paper are simple, undirected, and finite. The complement of a graph $G = (V,E)$ is the graph $\overline{G} = (V,\overline{E})$, where $\overline{E}$ consists of all possible edges between vertices in $V$ that are not in $E$. Let $G$ be a graph with vertices $\{1,\ldots,n\}$ and let $S_n$ denote the set of symmetric $n \times n$ real matrices. For $A = [a_{ij}] \in S_n$, the graph of $A$, denoted by $G(A)$, is the graph with vertices $\{1,\ldots,n\}$ and edges $\{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$. The set of symmetric matrices associated with $G$ is $S(G) = \{A \in S_n : G(A) = G\}$. The maximum nullity of $G$ is $M(G) = \max\{\text{null } A : A \in S(G)\}$, and the maximum positive semidefinite nullity of $G$ is $M_+(G) = \max\{\text{null } A : A \in S(G) \text{ and } A \text{ is positive semidefinite}\}$.

In [7] and [8], Colin de Verdière introduced the parameters $\mu(G)$ and $\nu(G)$, defined to be the maximum nullity among matrices $A \in S(G)$ that satisfy the Strong Arnold Hypothesis and additional conditions. In [4] the parameter $\xi(G)$ was defined to be the maximum nullity among matrices $A \in S(G)$ that satisfy the Strong Arnold Hypothesis. A real symmetric matrix $A$ satisfies the Strong Arnold Hypothesis if and only if there does not exist a nonzero real symmetric matrix $X$ satisfying $AX = 0$, $A \circ X = 0$, and $I \circ X = 0$, where $\circ$ denotes the Hadamard (entry-wise) product and $I$ is the identity matrix. The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally (see [13]). For $\nu$, the only additional condition (besides the Strong Arnold Hypothesis) is that the matrix must be positive semidefinite. For $\mu$, the additional conditions are that the matrix must be a generalized Laplacian (i.e., have nonpositive off-diagonal entries) and have exactly one negative eigenvalue. Clearly $\nu(G) \leq M_+(G) \leq M(G)$, $\nu(G) \leq \xi(G) \leq M(G)$, and $\mu(G) \leq \xi(G)$, and each of these inequalities can be strict (see [2]).

An important property of Colin de Verdière type parameters is minor monotonicity. The contraction of edge $e = uv$ of $G$ is obtained by identifying the vertices $u$ and $v$, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A minor of $G$ arises by performing a sequence of deletions of edges, deletions of isolated vertices, and/or contractions of edges. The notation $H \preceq G$ means that $H$ is a minor of $G$. A graph parameter $\beta$ is minor monotone if for any minor $H$ of $G$, $\beta(H) \leq \beta(G)$. In [7], [8], and [4] (respectively) it is shown that $\mu$, $\nu$, and $\xi$ are minor monotone.

For any graph $G$, the Hadwiger number $h(G)$ is the maximum order of a clique minor of $G$. It was shown in [7] and [8] that $\mu(K_n) = n - 1$ and $\nu(K_n) = n - 1$ (where $K_n$ denotes the complete graph on $n$ vertices), so by minor monotonicity $h(G) - 1 \leq \mu(G)$ and $h(G) - 1 \leq \nu(G)$.

Let $\kappa(G)$ denote the vertex connectivity of $G$, i.e., if $G$ is not complete, the smallest number $k$ such that there is a set of vertices $S$, with $|S| = k$, for which the graph obtained
by deleting the vertices in $S$ and all edges incident with a vertex in $S$, denoted by $G - S$, is disconnected (by convention, $\kappa(K_n) = n - 1$). It is proved in [15, 16] that $\kappa(G) \leq M_+(G)$ for every graph $G$. It was noted in [12] that the proof in [15] establishes $\kappa(G) \leq \nu(G)$ for all $G$. As defined in [2], the minor monotone ceiling of $\kappa$ is $[\kappa](G) = \max\{\kappa(H) : H \preceq G\}$. It follows from the definition that $h(G) - 1 \leq [\kappa](G)$, since the $K_{h(G)}$ minor of $G$ implies $\kappa(K_{h(G)}) \leq [\kappa](G)$, and $[\kappa](G) \leq \nu(G)$, since $\kappa(G) \leq \nu(G)$ and $\nu$ is minor monotone (see [2] for more detail).

A Nordhaus-Gaddum type result is a (sharp) lower or upper bound on the sum or product of a parameter of a graph and of its complement. The Graph Complement Conjecture for $\nu$ [3] is a Nordhaus-Gaddum sum lower bound.

**Conjecture 1.1 (GCC$_\nu$).** For any graph $G$,

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2. \tag{1}$$

It is not possible to raise the lower bound $|G| - 2$ since equality is attained for any tree that includes a $P_4$: For such a tree, it is shown in [1] that $M_+(T) = |T| - 3$. Since $M_+(T) = M_+(T) + M_+(\overline{T}) = |T| - 2$. It is shown in [17] that GCC$_\nu$ is true for graphs with tree-width at most 3, and thus for trees. Thus GCC$_\nu$ conjectures that $|G| - 2$ is a tight Nordhaus-Gaddum sum lower bound for $\nu$. This conjecture is studied in [3], where it is established for certain graphs. Various other forms of this conjecture have appeared, including: GCC$_+$, i.e., $M_+(G) + M_+(\overline{G}) \geq |G| - 2$, [3]; GCC, i.e., $M(G) + M(\overline{G}) \geq |G| - 2$, [5]; and GCC$_\xi$, i.e., $\xi(G) + \xi(\overline{G}) \geq |G| - 2$, [9]. Of course GCC$_\nu$ implies GCC+$\xi$ implies GCC, and GCC$_\nu$ implies GCC+$\xi$ implies GCC. The graph complement conjecture for $\mu$, i.e., $\mu(G) + \mu(\overline{G}) \geq |G| - 2$, appeared in [14].

Here we discuss values of the multiplier $b$ for a Nordhaus-Gaddum sum upper bound for the parameter $\beta$ where $\beta$ is one of $h, [\kappa], \nu, \xi$, or $\mu$. We denote by $b_\beta$ the least value of $b$ making

$$\beta(G) + \beta(\overline{G}) \leq b|G|$$

true for every graph of order at least two, and call $b_\beta$ the NG upper multiplier for $\beta$. Stiebitz [18] has shown that

$$h(G) + h(\overline{G}) \leq \frac{6}{5}|G|$$

and there exist graphs achieving $h(G) + h(\overline{G}) = \frac{6}{5}(1 - \varepsilon)|G|$ for arbitrarily small $\varepsilon$, so $b_\beta = \frac{6}{5}$. We establish bounds for $b_{[\kappa]}, b_{\nu},$ and $b_{\xi}$. Clearly $b_h \leq b_{[\kappa]} \leq b_{\nu} \leq b_{\xi}$, and $b_h \leq b_\mu \leq b_{\xi}$. In Section 2 we construct a family of graphs to show that $b_{[\kappa]} \geq \frac{4}{3}$. In Section 3 we show that $b_{\xi} \leq \sqrt{2}$. In Section 4 we summarize our conclusions. Note that the Nordhaus-Gaddum sum upper bound for the parameters $M$ and $M_+$ is not interesting because it is trivially $2|G| - 1$:

$$M(K_n) + M(\overline{K_n}) = M_+(K_n) + M_+(\overline{K_n}) = (n - 1) + n = 2|K_n| - 1.$$
2 Lower bound for NG upper multiplier for $\lceil \kappa \rceil$

In this section we construct a self-complementary graph $H(a)$ on $12a - 4$ vertices for $a \geq 2$, and show that $H(a)$ has a minor $\overline{H(a)}$ with $\delta(\overline{H(a)}) = \kappa(\overline{H(a)}) = 8a - 4$, where $\delta(G)$ denotes the minimum degree of a vertex of $G$. It is shown that this example implies that $b_{\lceil \kappa \rceil} \geq 4/3$.

**Example 2.1.** Construct the graph $H(a) = (V, E)$ as follows (see Figure 1): The $12a - 4$ vertices of $H(a)$ are partitioned into four sets $V_i, i = 1, 2, 3, 4$ of $r = 3a - 1$ vertices each. The sets $V_1$ and $V_2$ are the “sparse part” of $H(a)$, with $H(a)[V_i] = K_r, i = 1, 2$ (where $G[W]$ denotes the subgraph of $G$ induced by the subset $W$ of the vertices of $G$). The sets $V_3$ and $V_4$ are the “dense part” of $H(a)$, with $H(a)[V_i] = K_r, i = 3, 4$. Every edge between a vertex in $V_1$ and a vertex in $V_3$ is in the edge set $E$, and likewise for $V_2$ and $V_4$. There are no edges between $V_1$ and $V_4$, nor between $V_2$ and $V_3$. Regarding the edges between $V_1$ and $V_2$, number the vertices of $V_1$ as $u_{2i-1}, i = 1, \ldots, r$ and the vertices of $V_2$ as $u_{2i}, i = 1, \ldots, r$. Then vertex $u_s \in V_1$ is adjacent to the $a$ vertices $u_{s+j} \in V_2, j = 1, 3, \ldots, 2a - 3, 2a - 1$ (where for $k = 6a - 1, \ldots, 8a - 4$, $u_k$ is interpreted as $u_\ell$ with $\ell \equiv k \mod (6a - 2)$ and $1 \leq \ell \leq 2a - 2$). If $u \in V_1 \cup V_2$, then $\deg_{H(a)} u = r + a = 4a - 1$ (where $\deg_G w$ denotes the degree of $w$ in $G$). Regarding the edges between $V_3$ and $V_4$, number the vertices of $V_3$ as $v_{2i-1}, i = 1, \ldots, r$ and the vertices of $V_4$ as $v_{2i}, i = 1, \ldots, r$. Then vertex $v_s \in V_3$ is adjacent to all vertices $v_p \in V_4$ except for $p = s + j, j = 1, 3, \ldots, 2a - 3, 2a - 1$. If $v \in V_3 \cup V_4$, then $\deg_{H(a)} v = (r - 1) + r + (r - a) = 8a - 4$. It is clear from the construction that $\overline{H(a)} = H(a)$.

![Figure 1: Schematic diagram for the construction of $H(a)$](image)

Construct the minor $\overline{H(a)}$ by contracting the edges $u_{2i-1}u_{2i}, i = 1, \ldots, r$, and denote the set of these $r$ vertices by $V_{1,2}$. If $v \in V_3 \cup V_4$, then $\deg_{\overline{H(a)}} v = \deg_{H(a)} v = 8a - 4$. Note that each of the new vertices in $V_{1,2}$ has degree equal to $2((4a - 1) - 1) = 8a - 4$, so $\overline{H(a)}$ is $(8a - 4)$-regular. Furthermore, if $w \in V_{1,2}$, $w$ is adjacent to all $2r = 6a - 2$ vertices in $V_3 \cup V_4$ so $\overline{H(a)}[V_{1,2}]$ is $(2a - 2)$-regular. Since each vertex in $V_3 \cup V_4$ is adjacent to $r = 3a - 1$ vertices in $V_{1,2}$, $\overline{H(a)}[V_3 \cup V_4]$ is $(5a - 3)$-regular.
To establish that $\kappa(\hat{H}(a)) = \delta(\hat{H}(a))$, we use the property that for certain circulants $C$, $\kappa(C) = \delta(C)$, establish a method for computing $\kappa$, and examine parts of $\hat{H}(a)$ separately. For $1 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$, the consecutive circulant $\text{Circ}_n\{1, \ldots, t\}$ is the graph on the vertices $\{0, 1, \ldots, n-1\}$ with vertex $i$ adjacent to vertices $i + j$ and $i - j$ for $j = 1, \ldots, t$ (with arithmetic mod $n$). We will use the fact that for a consecutive circulant the vertex connectivity is equal to the (common) degree; Harary [11] gave the consecutive circulant as an example of a graph having maximum vertex connectivity $\frac{2m}{n}$ among graphs having $n$ vertices and $m$ edges (when $\frac{2m}{n}$ is an integer), and this result is now well known.

**Theorem 2.2.** [19, Theorem 4.1.5 (Harary)] For $1 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$,

$$\kappa(\text{Circ}_n\{1, \ldots, t\}) = \delta(\text{Circ}_n\{1, \ldots, t\}) = 2t.$$

**Theorem 2.3.** Let $G$ be a connected graph on $n$ vertices with $G \neq K_n$ and let $1 \leq t \leq n - 1 - \delta(G)$. Define

$$f(t) = \max\{s \in \mathbb{Z}^+ : K_{t, s} \text{ is a subgraph of } \overline{G}\},$$

where $K_{s,t}$ denotes the complete bipartite graph on $s$ and $t$ vertices. Then

$$\kappa(G) = \min\{n - (t + f(t)) : 1 \leq t \leq n - 1 - \delta(G)\}.$$

**Proof.** For every $t$ such that $1 \leq t \leq n - 1 - \delta(G)$, $\overline{G}$ contains a $K_{t,1}$ (by choosing a vertex $v$ of degree $\delta(G)$ as the partite set of 1 vertex, and $t$ of its non-neighbors as the other partite set), so $f(t)$ is defined.

Choose $t$ such that $1 \leq t \leq n - 1 - \delta(G)$. Let $U$ be a set of $t$ vertices and let $W$ be a set of $f(t)$ vertices such that $\overline{G}[U \cup W]$ contains a $K_{t,f(t)}$ subgraph. Then $G[U \cup W]$ is disconnected, so $\kappa(G) \leq |V \setminus (U \cup W)| = n - (t + f(t))$. Since this is true for every $t$ such that $1 \leq t \leq n - 1 - \delta(G)$,

$$\kappa(G) \leq \min\{n - (t + f(t)) : 1 \leq t \leq n - 1 - \delta(G)\}.$$

Choose a set $S$ such that $|S| = \kappa(G)$ and $G - S$ is disconnected. Let $U$ be the set of vertices of one connected component, let $t_0 = |U|$, and let $W = V \setminus (U \cup S)$; note $|W| = n - (t_0 + \kappa(G))$. Then $\overline{G}$ contains $K_{t_0,n-(t_0+\kappa(G))}$ with bipartition $U,W$. Thus $f(t_0) \geq n - (t_0 + \kappa(G))$, so

$$\kappa(G) \geq n - (t_0 + f(t_0)) \geq \min\{n - (t + f(t)) : 1 \leq t \leq n - 1 - \delta(G)\}. \tag*{\square}$$

We now return to establishing the properties of one of the graphs constructed in Example 2.1.

**Observation 2.4.** Let $G$ be a graph whose vertex set $V$ can be partitioned as $V = X \cup Y$ such that each vertex in $X$ is adjacent to each vertex in $Y$. Let $G_X = G[X]$ and $G_Y = G[Y]$. Then
\[ \bullet \, \delta(G) = \min\{\delta(G_X) + |Y|, \, \delta(G_Y) + |X|\}, \]
\[ \bullet \, \kappa(G) = \min\{\kappa(G_X) + |Y|, \, \kappa(G_Y) + |X|\}. \]

**Theorem 2.5.** For \( \widehat{H(a)} \) as in Example 2.1,
\[ \kappa(\widehat{H(a)}) = \delta(\widehat{H(a)}) = 8a - 4. \]

*Proof.* Let \( X = V_{1,2} \) and \( Y = V_3 \cup V_4 \). Then \( V(\widehat{H(a)}) = X \cup Y \) and every vertex in \( X \) is adjacent to every vertex in \( Y \). By Observation 2.4, if we show that \( \kappa(\widehat{H(a)}[V_{1,2}]) = \delta(\widehat{H(a)}[V_{1,2}]) \) and \( \kappa(\widehat{H(a)}[V_3 \cup V_4]) = \delta(\widehat{H(a)}[V_3 \cup V_4]) \), it follows that \( \kappa(\widehat{H(a)}) = \delta(\widehat{H(a)}) \).

Since \( \widehat{H(a)}[V_{1,2}] = \text{Circ}_r(1, \ldots, a - 1) \), we have \( \kappa(\widehat{H(a)}[V_{1,2}]) = \delta(\widehat{H(a)}[V_{1,2}]) \) by Theorem 2.2.

Recall that in \( \widehat{H(a)}[V_3 \cup V_4] = H(a)[V_3 \cup V_4] \), the vertices of \( V_3 \) are numbered as \( v_{2i-1}, \, i = 1, \ldots, r \) and the vertices of \( V_4 \) as \( v_{2i}, \, i = 1, \ldots, r \), and vertex \( v_s \in V_3 \) is adjacent to all vertices \( v_p \in V_4 \) except for \( p = s + j, \, j = 1, 3, \ldots, 2a - 3, 2a - 1 \) (where for \( p > 6a - 2, \, v_p \) means \( v_{p-(6a-2)} \)). Thus the vertex \( v_s \) in \( \widehat{H(a)}[V_3 \cup V_4] \) is adjacent to precisely the vertices \( v_p, \, p = s + j, \, j = 1, 3, \ldots, 2a - 3, 2a - 1 \). This is a bipartite regular graph with a great deal of symmetry, so in determining the maximum neighborhood intersection of a set of two vertices, no generality is lost by considering the vertices 1 and 1 + 2d (with \( d \leq \lceil \frac{3a-1}{2} \rceil \)). The size of the shared neighborhood is \( \max(a - d, 0) \). For \( 1 \leq t \leq a = 2r - 1 - (5a - 3) = |\widehat{H(a)}[V_3 \cup V_4]| - 1 = \delta(\widehat{H(a)}[V_3 \cup V_4]) \), the maximum neighborhood intersection of a set of \( t \) vertices happens when those vertices are consecutive in the same bipartition set, and it follows that in this case \( f(t) = a + 1 - t \). So for all \( t \in \{1, \ldots, a\} \), \( |\widehat{H(a)}[V_3 \cup V_4]| - (t + f(t)) = 5a - 3 \). Thus \( \kappa(\widehat{H(a)}[V_3 \cup V_4]) = 5a - 3 = \delta(\widehat{H(a)}[V_3 \cup V_4]). \)

**Corollary 2.6.** For the graph \( H(a) \) in Example 2.1,
\[ [\kappa](H(a)) \geq 8a - 4 \]
and
\[ [\kappa](H(a)) + [\kappa](\widehat{H(a)}) \geq \frac{4}{3}(1 - \frac{1}{6a - 2})|H(a)|. \]

Thus
\[ b_{[\kappa]} \geq \frac{4}{3}. \]

*Proof.* By Theorem 2.5, \([\kappa](H(a)) \geq \kappa(\widehat{H(a)}) = 8a - 4. \) Since \( H(a) \) is self-complementary, \([\kappa](\widehat{H(a)}) \geq 8a - 4. \) also, and thus \([\kappa](H(a)) + [\kappa](\widehat{H(a)}) \geq 16a - 8. \) The second statement can then be established by arithmetic. Since \( b_{[\kappa]} \geq \frac{4}{|H(a)|}([\kappa](H(a)) + [\kappa](\widehat{H(a)})) \), by taking the limit as \( a \to \infty \) we see that \( b_{[\kappa]} \geq \frac{4}{3}. \) \( \square \)
3 Upper bound for NG upper multiplier for $\xi$

In this section we show that the NG upper multiplier $b_\xi$ is at most $\sqrt{2}$.

**Theorem 3.1.** [10] Let $G = (V_G, E_G)$ be a connected graph. Then

$$|E_G| + a \geq \frac{\xi(G)(\xi(G) + 1)}{2}$$

where $a = 1$ if $G$ is bipartite and every optimal matrix for $\xi(G)$ has zero diagonal, and $a = 0$ otherwise.

Since $\xi(G)$ is the maximum of $\xi(G_i)$ taken over the connected components $G_i$ of $G$, the hypothesis that $G$ is connected is unnecessary.

**Corollary 3.2.** Let $G = (V_G, E_G)$ be a graph. Then

$$|E_G| + 1 \geq \frac{\xi(G)(\xi(G) + 1)}{2}.$$  

**Corollary 3.3.** Let $G = (V_G, E_G)$ be a graph with at least one edge. Then

$$\xi(G) \leq \sqrt{2|E_G|}.$$  

**Proof.** Algebraic manipulation of (3) gives $\xi(G) \leq \frac{1}{2}(-1 + \sqrt{8|E_G|} + 9)$. Further manipulation shows that the inequality $\frac{1}{2}(-1 + \sqrt{8|E_G|} + 9) \leq \sqrt{2|E_G|}$ is equivalent to $2 \leq |E_G|$, so (4) is true if $G$ has at least two edges. If $G$ has exactly one edge then $G$ has components $K_2$ and possibly some $K_1$'s, and thus $\xi(G) = 1 < \sqrt{2}$. 

Nordhaus-Gaddum bounds usually take one of two forms: additive or multiplicative. The form of inequality (4) suggests a third category of Nordhaus-Gaddum bound: Pythagorean.

**Corollary 3.4.** Let $G = (V_G, E_G)$ be a graph of order at least two. Then

$$\xi(G)^2 + \xi(G) \leq |G|^2 - |G|.$$  

**Proof.** Let $|G| = n \geq 2$. In the case where either $G$ has no edges or $\overline{G}$ has no edges, $\xi$ will take the value 1 for one of the two graphs and the value $n - 1$ for the other, in which case the result holds. In any other case inequality (4) applies both to $G$ and to $\overline{G}$, giving us two inequalities the sum of whose squares is

$$\xi(G)^2 + \xi(\overline{G}) \leq 2|E_G| + 2|E_{\overline{G}}| = |G|^2 - |G|.$$  

**Corollary 3.5.** Let $G = (V_G, E_G)$ be a graph of order at least two. Then

$$\xi(G) + \xi(G) \leq \sqrt{2|G|},$$

and thus $b_\xi \leq \sqrt{2}$. 

**Proof.** Let $|G| = n \geq 2$, and by Corollary 3.4 choose $x \geq \xi(G)$ and $y \geq \xi(\overline{G})$ such that $x$ and $y$ lie on the circle $x^2 + y^2 = n^2$. The maximum value of $x + y$ on this circle is $\sqrt{2}n$. 


4 Conclusions

In summary, we have established

\[ 1.333 < \frac{4}{3} \leq b_{[\kappa]} \leq b_{\nu} \leq b_{\xi} \leq \sqrt{2} < 1.415. \]

We have no evidence that the construction in Section 2 is tight, even for \( b_{[\kappa]} \). On the other hand, the inequality (2) with \( a = 0 \) is known to be tight for some small examples and for complete graphs (it is tight with \( a = 1 \) for \( K_{3,3} \)). For \( \nu \), since a diagonal entry for a vertex of degree at least one cannot be zero, \( a = 0 \) and the inequality (2) becomes \( |E_G| \geq \frac{\nu(G)(\nu(G)+1)}{2} \) for graphs with at least one edge; again this is tight for some small graphs and complete graphs. This leaves open the possibility that Corollaries 3.3 — 3.5 may be asymptotically tight.

**Question 4.1.** Given \( x \) and \( y \) positive with \( x^2 + y^2 = 1 \), does there exist an increasing sequence of graphs \( G_i \) on \( n_i \) vertices such that \( \nu(G_i)/n_i \) approaches \( x \) and \( \nu(G_i)/n_i \) approaches \( y \)? Or such that \( \xi(G_i)/n_i \) approaches \( x \) and \( \xi(G_i)/n_i \) approaches \( y \)?

The particular case of \( x = y = \frac{\sqrt{2}}{2} \) suggests the next question.

**Question 4.2.** Do \( b_{\nu} \) and \( b_{\xi} \) take the maximum possible value of \( \sqrt{2} \)?

On the other hand it seems more difficult to construct examples for \( b_{\mu} \), and the only bounds we know are those from \( h \) (due to Stiebitz [18]) and \( \xi \). i.e.,

\[ 1.2 = \frac{6}{5} = b_{h} \leq b_{\mu} \leq b_{\xi} \leq \sqrt{2} < 1.415. \]

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