Economic Considerations in Measurement of Economic Value of Genotypes

George W. Ladd
Iowa State University

Follow this and additional works at: http://lib.dr.iastate.edu/econ_las_staffpapers

Part of the Agribusiness Commons, Economic History Commons, and the Economic Theory Commons

Recommended Citation
http://lib.dr.iastate.edu/econ_las_staffpapers/109

This Report is brought to you for free and open access by the Economics at Iowa State University Digital Repository. It has been accepted for inclusion in Economic Staff Paper Series by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
Economic Considerations in Measurement of Economic Value of Genotypes

Abstract
The questions, "What's a better animal. (or seed) worth?" and "What makes one animal (or seed) worth more than another?" are faced regularly by animal and plant breeders when they select animals or seeds to use in breed improvement programs. Answering the question is complicated by two facts. (a) A strain of animals or seeds that is superior in some of the traits it will pass on to its offspring may be only average or inferior in other traits. (b) Because some pairs of traits have negative genetic correlations, breeding to improve one trait may degrade another trait. The breeder needs to make trade-offs. How much can he afford to degrade one trait while improving another?

Disciplines
Agribusiness | Economic History | Economic Theory
Economic Considerations in Measurement of Economic Value of Genotypes

George W. Ladd

No. 74

May 1978

(Revision of Staff Paper No. 51)
<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Selection index theory*</td>
<td>2</td>
</tr>
<tr>
<td>Some common methods of measuring economic values*</td>
<td>5</td>
</tr>
<tr>
<td>Neoclassical firm theory†</td>
<td>7</td>
</tr>
<tr>
<td>Production function</td>
<td>8</td>
</tr>
<tr>
<td>Profit</td>
<td>12</td>
</tr>
<tr>
<td>Constrained maximization</td>
<td>14</td>
</tr>
<tr>
<td>First-order conditions</td>
<td>14</td>
</tr>
<tr>
<td>Second-order conditions</td>
<td>15</td>
</tr>
<tr>
<td>Economic interpretations</td>
<td>15</td>
</tr>
<tr>
<td>Application of economic theory to animal breeding</td>
<td>19</td>
</tr>
<tr>
<td>Intuitive explanation</td>
<td>19</td>
</tr>
<tr>
<td>Analysis</td>
<td>22</td>
</tr>
<tr>
<td>Interpretation</td>
<td>27</td>
</tr>
<tr>
<td>Use of procedure</td>
<td>28</td>
</tr>
<tr>
<td>Effects of variations in prices†</td>
<td>29</td>
</tr>
<tr>
<td>Application of procedure to M-H-W data</td>
<td>33</td>
</tr>
<tr>
<td>An alternative</td>
<td>35</td>
</tr>
<tr>
<td>Difference between neoclassical and activity analysis views of production†</td>
<td>36</td>
</tr>
<tr>
<td>Linear programming statement of theory of multi-product firm†</td>
<td>40</td>
</tr>
<tr>
<td>Numerical example</td>
<td>40</td>
</tr>
<tr>
<td>General statement</td>
<td>44</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Solution</td>
<td>47</td>
</tr>
<tr>
<td>Solution to numerical example</td>
<td>50</td>
</tr>
<tr>
<td>Justification for ignoring fixed costs</td>
<td>52</td>
</tr>
<tr>
<td>Use of linear programming to find economic values</td>
<td>53</td>
</tr>
<tr>
<td>Procedure</td>
<td>53</td>
</tr>
<tr>
<td>Application to hogs</td>
<td>56</td>
</tr>
<tr>
<td>Farm I</td>
<td>56</td>
</tr>
<tr>
<td>Farm IA</td>
<td>64</td>
</tr>
<tr>
<td>Farm II</td>
<td>64</td>
</tr>
<tr>
<td>Results</td>
<td>64</td>
</tr>
<tr>
<td>Using economic values determined by linear programming*</td>
<td>69</td>
</tr>
<tr>
<td>Implications of neoclassical and linear programming models</td>
<td>71</td>
</tr>
<tr>
<td>Variation in output prices</td>
<td>71</td>
</tr>
<tr>
<td>Variations in prices of variable inputs</td>
<td>72</td>
</tr>
<tr>
<td>Variation in livestock traits</td>
<td>73</td>
</tr>
<tr>
<td>Optimum or efficient operation</td>
<td>77</td>
</tr>
<tr>
<td>Comparison of different methods of measuring economic values</td>
<td>79</td>
</tr>
<tr>
<td>References</td>
<td>85</td>
</tr>
</tbody>
</table>

(An asterisk (*) by the title of a section identifies a section that contains material that is familiar to animal breeders, but not to economists. A dagger (†) identifies a section that contains material that is familiar to economists, but not to animal breeders.)
ECONOMIC CONSIDERATIONS IN MEASUREMENT
OF ECONOMIC VALUES OF GENOTYPES

The questions, "What's a better animal (or seed) worth?" and "What makes one animal (or seed) worth more than another?" are faced regularly by animal and plant breeders when they select animals or seeds to use in breed improvement programs. Answering the question is complicated by two facts. (a) A strain of animals or seeds that is superior in some of the traits it will pass on to its offspring may be only average or inferior in other traits. (b) Because some pairs of traits have negative genetic correlations, breeding to improve one trait may degrade another trait. The breeder needs to make trade-offs. How much can he afford to degrade one trait while improving another?

Forty years ago, Smith (1936-37) introduced the selection index into plant breeding as a device for selecting plants for seed improvement programs. Thirty-five years ago, Hazel (1943) introduced it into animal breeding. To compute a selection index, a breeder needs measures of economic values of traits. Economic value of a trait has been defined as "the amount by which net profit may be expected in increase for each unit of improvement in that trait." (Hazel 1943). Although breeders have been measuring and using economic values of traits for some 35 years, they have yet to receive help from economists in developing economically sound methods for measurement. Economists do have insights into firm behavior that have not been incorporated into these measures.

1/Parts of this manuscript are taken from Craig Gibson's M.S. thesis (1976). The entire manuscript reflects the influence, encouragement, and help of P. Jeffrey Berger, Lauren L. Christian, A. E. Freeman, and Richard L. Willham.
This paper reports on an interdisciplinary effort that involved animal breeders and economists in development of procedures for measuring economic values, and presents two procedures and an application of one of them.

**SELECTION INDEX THEORY**

Living things pass onto their offspring only their inherited characteristics. Thus breeders are interested in knowing these. But an animal's inherited characteristics are not observable. Observable characteristics are the result of heredity and environment. This is usually expressed as

\[ P_i = G_i + E_i \]

where

- \( P_i \) = animal's phenotypic (observable) value of \( i \)-th trait = value of \( i \)-th phenotype
- \( G_i \) = animal's genotypic (unobservable) value of \( i \)-th trait = value of \( i \)-th genotype
- \( E_i \) = environmental effect on animal's \( i \)-th trait

and where

\( G_i \) and \( E_i \) are assumed to be independently distributed, and the mean value of \( E_i \) over all animals is zero. The traits of offspring are determined by the parents' genotypes. In selecting individuals for breeding programs, therefore, the breeder is interested in genotypes. But he can only observe phenotypes. He must therefore find some method of linking the observed phenotypes with the unobserved genotypes.

---

2/ Seminal papers on this topic are by Smith (1936-37) and Hazel (1943). The best single reference on the topic is Henderson (1963). Arboleda, Harris and Nordskog (1976a, 1976b) provide a convenient summary and an application.
The purpose of breeding is to select parents with superior genotypes. Breeders define an aggregate genotype or aggregate breeding value, $H$, as a weighted sum of individual genotypes

$$H = \sum_{i=1}^{m} a_i G_i = G'A$$

where $a_i$ is economic value (or weight) of the $i$-th trait.

In Hazel's 1943 description of the economic weights, he wrote, "The relative economic value for each trait depends upon the amount by which profit may be expected to increase for each unit of improvement in that trait. Good approximations to relative economic values often can be obtained from long-time price averages and cost-of-production figures."

Hazel, in his description of the selection index, never explicitly defined relative economic value but instead related it to influencing factors. In his application of the selection index, Hazel, like Smith (1936-37), used the idea that the economic weights for each of the characters should be ratios in terms of a single character.

Hazel (1956) explicitly defined economic values and showed examples of the derivation of economic values for some characters in each of beef cattle, swine, and sheep. Hazel (1956) wrote, "The economic values are of primary importance. These should reflect the net profit which will result to the livestock enterprise for one unit of change in the particular trait, but should not include the profit which might result from improvement in an associated trait." From this, it must be said that the economic value for a character should reflect the net profit expected to accrue to the livestock enterprise as the result of one unit of change in that trait alone. It should not include any net profit that will accrue to the livestock enterprise as the result of a change in correlated traits that may change as the initial trait changes, thereby causing net profit to accrue
to the livestock enterprise indirectly.

H is unobservable because the $G_i$ are unobservable and therefore the value of H cannot be determined directly. The value of H can be estimated by defining a selection index I that is linear function of observable traits and that is related to H

$$I = \sum_{j=1}^{n} b_j P_j = P'\beta$$

where $b_j$ is a weight whose value is to be determined. It is desired that I be a good predictor or estimator of H. Consequently, two plausible criteria for determining values of the $b_j$ are: minimizing $E(I-H)^2$ where E means expectation, and maximizing the correlation between I and H. Both criteria yield the same value of $\beta$.

$$\beta = (P'P)^{-1} G'G A$$

where

$P'P = n \times n$ matrix of phenotypic variances and covariances

$G'G = n \times m$ matrix of genotypic variances and covariances of n traits

in I with m traits in H

Phenotypic variance is the sum of genotypic variance and environmental variance. Phenotypic covariance is the sum of genotypic covariance and environmental covariance.

If a breeder knows A, he can use his information on $P'P$ and $G'G$ to determine $\beta$ from (2). Then $\beta$ and phenotypic values of traits of different potential breeding stock can be used to determine I from (1). Animals (or plants) with the highest values of the index can then be selected for breeding programs.

Economists cannot contribute anything to the determination of $P'P$ or $G'G$. They can, however, make a contribution to the measurement of the $a_i$, the economic values.
Three methods that are used to measure economic values of traits are: "budgeting", "gross-revenue", and multiple regression. Hazel (1956) exemplified the derivation of economic values for beef cattle, swine, and sheep. Of the following three examples from his 1956 mimeographed paper the first and last example illustrate the "gross-revenue" method, and the second illustrates the "budgeting method":

... The economic value of slaughter grade can be computed by the range in price between very good and very poor animals at slaughter, divided by the range in score for good and poor animals. This value should be multiplied by average selling weight. For example, if we score very good animals 9 and they sell for .20 per pound, and very poor animals 1 and they sell for .16 and average sale weight is 1000, the economic value is \[ \frac{0.20 - 0.16}{9-1} \times 1000 = \frac{0.04}{8} \times 1000 = 5.00 \.

One of Hazel's examples associated with swine is as follows:

... The value of growth rate is a function of labor cost, insurance, maintenance of equipment, etc. Figuring $.03 per pig per day for labor, $.002 per day for insurance, and $.003 per day for maintenance of equipment, we have $.035 per pig per day. Pigs which gained 1.6 lbs. per day instead of 1.5 would get to market 8 days sooner. On this basis, growth rate is worth 8 x $.035 = $.28 for each 1/10 lb. gain per day, or $2.80 per lb. per day ....

One of Hazel's examples associated with sheep is as follows:

... The values of a single lamb at weaning is about $11.25, while the value of twins is about $18.20. Thus, the economic value of number of lambs born is $6.95. Perhaps no additional credit should be given for triplets as mortality among them is very high ....
Comparing Hazel's 1956 examples of deriving economic values to the examples shown by Smith in 1936 and by Hazel in 1943, it can be seen that the concept of how the economic values must be represented changed substantially. The earlier work on economic values expressed the economic values as ratios. As a result of the ratio idea, the term "relative economic value" was used for economic weights used in the selection index.

During the 1950's and 1960's, the definition of the economic value of traits selected for use in the selection index became accepted as "the amount by which profit may be expected to increase for each unit of improvement in that trait".

The budgeting approach uses simple relationships of costs of inputs incurred in breeding and managing an animal and prices received in marketing an animal or its product to estimate the economic value of a trait. By budgeting the costs and revenues of the animal and then finding the change in the costs of inputs incurred and/or price received due to a change in the trait, the net change of costs and revenues which reflect the change in profit due to a direct change in a trait can be found.

Another method that has been used in deriving economic weights is the multiple regression technique. The general problem to which the multiple regression analysis is applied is to determine the extent to which income can be predicted from different combinations of traits or performance variables. Nordskog (1960) applied multiple regression to records from 21 random sample egg laying tests for 1957. His dependent variable was income over feed costs. The independent variables were
number of eggs per chick started, number of eggs per pullet per hen-day, percentage of eggs over 24 ounces per dozen, body weight at end of test, and percentage mortality.

**NEOClassical Firm Theory**

This section of the paper reviews some standard economic theory of a profit-maximizing business firm. The next section discusses the possible application of this theory to the problem of defining and measuring aggregate genotype and net economic values of genotypic traits.

This paper deals with a producer who has a set of fixed resources. He owns, for example, a certain number of plows, discs, planters; he has a fixed number of buildings of various characteristics and sizes; he owns a given number of tillable acres and a given number of non-tillable acres; etc.

These resources (or inputs) are referred to as "fixed inputs (or resources)" to denote the fact that the total available amount of each one is fixed. The firm faces a fixed upper limit on the amount of each one that it can use. The producer uses these fixed resources in combination with variable resources (variable inputs) to produce one or more products (outputs) for sale. These resources are "variable" because the producer can acquire as many or few units of each variable resource as he desires. He buys each unit of a variable resource at a constant price. He can also sell as much or as little of each product as he desires at a constant price. Briefly, the producer's problem can be stated as: Utilize the fixed resources to make as much money as possible. More fully, his problem can be stated as: Determine the amount of each variable resource (variable input) to combine with the fixed resources, and determine the amounts of products to produce, to maximize his profit.
Production Function

A fundamental concept in economic analysis of profit-maximizing firms is the "production function." A production function is a description of the state of technology; it relates the quantities of fixed and variable inputs used to the quantities of outputs produced. It is assumed that the production function is continuous and has continuous derivatives up to at least the second order.

For our purposes, it will be assumed that, either: (a) the firm produces one output, or (b) it produces several outputs, but the production function for each product is independent of the production functions for other products. Each assumption permits us to analyze each output separately. Let the subscript zero (0) identify the firm's output, let the subscript i identify the firm's i-th variable input, and let $q_0$ and $q_i$ (i $\geq$ 1) equal the quantity of output produced and the quantity of variable input i purchased and used. A simple illustration of a production function with one variable input is presented in Figure 1. This firm has a set of fixed resources symbolized by K. The curve labeled $q_0 = f(q_1, K)$ shows the maximum quantities of output attainable by combining K with various quantities of input one.

Suppose the firm combines $q_{1A}$ of input one with fixed resources K. The maximum output it is technically possible to obtain is $q_{0A}$ at point A. It is not technically possible, for example, to obtain $q_{0A}'$ at point A'. If the firm uses $q_{1B}$, it can obtain $q_{0B}$ of output at point B. It cannot obtain $q_{0B}'$ at point B'. Points above the curve $q_0 = f(q_1, K)$ are technologically impossible to obtain. Points on and below the curve are technologically possible. Hence, we can write $q_0 \leq f(q_1, K)$. Now consider point A'. A firm operating at this point is technically inefficient. It is only obtaining an output of $q_{0A}'$ while using enough resources to permit it to produce $q_{0A}$ of output. Likewise a firm operating at point B' is technically inefficient. It is using enough resources to
produce $q_{OB}$ of output, but is only producing $q'_{0B}$. If the firm is technologically efficient, its production function is $q_0 = f(q_1, K)$. This paper will assume that the firm under analysis is technologically efficient; and the multi-input production function will be written

$$q_0 = f(q_1, q_2, \ldots, q_n, K)$$

Some production functions that have been reported in the literature are presented here. The first production function relates milk production to feed use, cow characteristics and environmental conditions (Heady, et. al., 1964). These data used in estimation of this production function came from a 12-week experiment.

$$M = 248.42 + 1.8358G + 1.4117H - 0.00505G^2 - 0.00109H^2 - 0.00352GH$$
$$- 0.00557GT + 0.00069WG - 0.00015HA + 0.0749A + 1.0060F + 3.1619J$$
$$- 5.4269K + 0.3694W + 0.09091T^2 - 0.00398F^2 + 15.3569K^{\frac{3}{2}} - 27.0461W^{\frac{1}{2}}$$
$$- 0.00164AT - 0.00023AF + 0.00065WF - 0.00187WJ + 0.00164KA$$
$$+ 0.03865KT - 0.02967KF - 0.03864JT - 0.01454JF$$

Variables are defined as:

- $H$ = alfalfa hay, measured as pounds consumed by a cow during 1 week.
- $G$ = grain, measured as pounds consumed by a cow during 1 week.
- $M$ = milk, measured as pounds of 4-percent FCM (fat corrected milk) produced by a cow during 1 week.
- $T$ = stage of lactation, measured as the ordinal number of the week, with $T = 1$ for the first experimental week.
- $A$ = index of ability, measured as total 4-percent FCM produced during the 50-day preliminary period.
- $K$ = coefficient of inbreeding, measured in percentage (cows with unrelated parents for many past generations have an inbreeding percentage of zero).
- $W$ = body weight, measured in pounds at the beginning of the experimental period.
F = outside temperature, measured weekly to correspond with weekly input-output data and computed as the arithmetic mean of daily high temperature readings, in degrees Fahrenheit, as recorded at the Iowa State University Agronomy Farm. High temperatures were used, since evidence indicates that feed consumption is reduced during severely high temperatures.

J = index of maturity, measured in months from time of birth but with an upper value of J = 66 for mature cows. The maturity index is truncated at 66 months, because Holstein population studies indicate that cows mature at about that age, with milk production approaching a plateau or a mathematical limit.

The next two production functions relate to beef production; the first describes the relation when cattle were fed stilbesterol, and the second, when cattle were not fed stilbesterol (Heady et al 1963).

\[
G = 0.11637150C + 0.02316051F - 0.0000049955C^2 - 0.0000007455F^2 + 0.0000000374CF - 1.2236046H
\]

\[
G = 0.14971812C + 0.02128774F - 0.0000122612C^2 - 0.0000007455F^2 - 0.0000037907CF - 2.2005042H
\]

Variables were defined as:

G = pounds of beef gain.
C = pounds of corn.
F = pounds of soilage (freshly cut alfalfa).
H = deviations of the average maximum temperature of each observation interval from the mean maximum temperature for the over-all feeding period.

The next two equations relate to raising hogs in dry lot from 34 to 200 pounds. The first is double-logarithmic; the second is quadratic. Both were
estimated using the same set of data (Heady and Dillon 1961).

\[ Y = 1.36c^{0.63}p^{0.20} \]

\[ Y = 2.03 + 0.32c + 0.46p - 0.00013c^2 - 0.00092p^2 - 0.00011cp \]

The variables are defined as

- \( Y \) = gain in pounds per pig after weaning.
- \( C \) = pounds of corn fed.
- \( P \) = pounds of soybean meal fed after weaning.

Previously presented production functions are not adequate to allow for our interest in animal improvement. For a given number of milk cows and given amounts of various feeds fed to the cows, the amount of milk produced will depend upon the inherited characteristics of the milk cows. The amount of meat produced by feeding a specified amount of feeds to hogs, or cattle, will depend upon the genetic traits of the livestock fed. Let \( G_h \) represent the genotypic value of the h-th trait of the livestock fed by the producer. Then the fact that the output of meat or other livestock products is affected by productive inputs used and by inherited characteristics is represented in the production function

\[ q_0 = f(q_1, q_2, \ldots, q_n, G_1, G_2, \ldots, G_m, K) \]

Initially it will be assumed that the values of \( G_h \) are fixed; the producer cannot vary them. Their values have been determined by previous decisions of the producer. Later this assumption will be relaxed, and values of \( G_h \) will be allowed to vary. It will be at this point of the paper, when effects of variations in \( G_h \) on producers are analyzed, that economic analysis and animal breeding analysis tie together.

**Profit**

It was stated earlier that the firm's objective is assumed to be profit maximization: maximization of the excess of total revenue over the sum of variable costs (costs of variable inputs) and fixed costs (costs of fixed inputs). Letting \( s \) represent the firm's selling price for one unit of output, the
firm's total revenue is $s_0 q_0$. The firm's fixed cost can be represented by CK; this cost does not vary, it is the constant amount CK regardless of the levels of $q_0, q_1, \ldots, q_n$. To define variable costs, we need to distinguish among three kinds of variable inputs appearing in the production function.

(a) Variable inputs in this class do not require the use of any other variable inputs.

(b) Variable inputs in this class are not used unless certain other variable inputs are used, and these other variable inputs are included in the firm's production function.

(c) Variable inputs in this class cannot be used unless certain other variable inputs are used, and these other variable inputs are not included in the firm's production function.

Suppose that the first variable input included in the production function is corn silage, that electric motors are used to feed corn silage and that the quantity of electricity used in feeding silage does not appear in the production function. Then corn silage is in class (c) of the variable inputs. A fourth class of variable inputs is

(d) These variable inputs do not appear in the production function. The level of use of each of these is proportional to $q_0$. An example of this kind is the number of hours of milking machine use.

For variable inputs in classes (a) and (b) define

$$p_i = \text{price paid per unit of variable input.}$$

For variable inputs in class (c) define

$$p_i = \text{price paid per unit of variable input plus cost, per unit of this input used, of the variable inputs not included in the production function, that are required for use of this variable input.}$$

For variable inputs in class (d) define

$$v_0 = \text{cost, per unit of output, of variable inputs not appearing in the production function.}$$
Then define
\[ p_0 = s_0 - v_0. \]
For convenience, throughout the rest of this paper, \( p_0 \) will be referred to as output or product price and \( p_i \) will be referred to as price of \( i \)-th variable input.

The firm's profit can now be written as
\[ (2) \pi = p_0 q_0 - \sum_{i=1}^{n} p_i q_i - CK. \]
The values of \( p_0, p_1, \ldots, p_n \) are assumed to be constants whose values are beyond the influence of the firm. This means that the price received for output and prices paid for inputs are not affected by the producer's levels of output or use of inputs.

Constrained Maximization

To maximize profit, the producer selects values of \( q_0, q_1, \ldots, q_n \) that maximize \( (2) \). He is not, however, completely free to vary \( q_0 \) and the \( q_i \) \((i \geq 1)\). He is restricted by his production function. To incorporate this restriction into \( (2) \) and to obtain a profit expression that will be easier to work with, substitute \( (1) \) into \( (2) \) to obtain
\[ (3) \pi = p_0 f(q_1, q_2, \ldots, q_n, c_1, c_2, \ldots, c_m) - \sum_{i=1}^{n} p_i q_i - CK \]
Profit is maximized by selecting values of \( q_1, q_2, \ldots, q_n \) that maximize \( (3) \).

First-Order Conditions

First-order conditions for determining values of \( q_1, q_2, \ldots, q_n \) that maximize \( \pi \) are obtained by equating to zero the partial derivatives of \( (3) \) with respect to the \( q_i \). These first-order conditions are (letting \( f = f(q_1, q_2, \ldots, q_n, c_1, c_2, \ldots, c_m, k) \) and \( f_i = \partial f / \partial q_i \))
\[
\begin{align*}
(4.1) \quad & \partial \pi / \partial q_1 = 0 = p_0 f_1 - p_1 \\
(4.2) \quad & \partial \pi / \partial q_2 = 0 = p_0 f_2 - p_2 \\
& \vdots \\
(4.n) \quad & \partial \pi / \partial q_n = 0 = p_0 f_n - p_n
\end{align*}
\]
Expressions (4.1) through (4.n) constitute a (possibly nonlinear) system of \( n \) equations in the \( n \) unknowns \( q_1, q_2, \ldots, q_n \) and the \( n+m+l \) constants or parameters \( p_0, p_1, \ldots, p_n, c_1, \ldots, c_m \). A solution to this system may or may not exist. If a solution does exist, it may or may not represent a profit-maximizing solution. (It may, e.g., represent a profit-minimizing solution.) The existence and characteristics of a solution depend upon second-order partial derivatives.

Select some point in \( n \)-space whose coordinates are \( q_1, q_2, \ldots, q_n \) and consider its properties if it is to represent a profit-maximizing solution. Let \( f^* \) (=\( q_0^* \)) be the value of \( f \) evaluated at the point \( q_1^*, \ldots, q_n^* \). Let \( f_{ii}^* \) be the value of \( f_i \) evaluated at this same point and let \( f_{ii}^* = \frac{\partial^2 f}{\partial q_i^2} \) and \( f_{ij}^* = \frac{\partial^2 f}{\partial q_i \partial q_j} \) evaluated at this same point. From (4.1) through (4.n) it is clear that we must have

\[
(5.1) \quad p_0 f_i^* = p_i; \quad i = 1, 2, \ldots, n
\]

**Second-Order Conditions**

To assure that a profit-maximizing solution exists it is necessary to consider the Hessian matrix \( H \) of second-order partial derivatives of \( \pi \).

\[
(6) \quad H = p_0 \begin{pmatrix} f_{11}^* & f_{12}^* & \cdots & f_{1n}^* \\ f_{21}^* & f_{22}^* & \cdots & f_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}^* & f_{n2}^* & \cdots & f_{nn}^* \end{pmatrix}
\]

If \( H \) is a negative definite matrix, the point \( q_i^* \) (\( i = 1, 2, \ldots, n \)) is a solution to (4.1) through (4.n) and it is a profit-maximizing solution. In the rest of this paper it will be assumed that \( H \) is negative definite. Note that \( H \) is a symmetric matrix: \( f_{ij}^* = f_{ji}^* \).

**Economic Interpretations**

\( f_i^* \) is the marginal physical product of input \( i \). It represents the change in level of output resulting from a one unit change in the amount of input.
i used (when all inputs are used at their profit-maximizing levels). $p_0 f_i^*$ is the marginal revenue product of the $i$-th input. It equals the change in revenue that results from a one unit change in the amount of input $i$ used. From (5),

(7) $p_0 f_i^* = p_i$

It will help to understand (7) if we first perform some algebraic manipulations using the second-order conditions. Because $H$ is negative definite, every element on its main diagonal is negative: $f_{ii}^* < 0$ for all $i$. Let $f_1^*$ and $f_{ii}^*$ be the values of $f_i^*$ evaluated at $q_1 < q_1^*$ and $q_{ii} > q_1^*$. Reducing $q_i$ from $q_i^*$ to $q_1^*$ raises the value of $f_i^*$ from $f_1^*$ to $f_1$. Raising the value of $q_i$ from $q_i^*$ to $q_1^*$ reduces the value of $f_i^*$ from $f_1^*$ to $f_{ii}^*$.

(8) If $q_{ii} < q_i^*$, then $f_{ii} < f_i^*$ and $p_0 f_i^* > p_0 f_{ii}^*$.

Hence $p_0 f_i^* > p_i$.

(9) If $q_{ii} > q_i^*$, then $f_{ii}^* > f_i^*$ and $p_0 f_i^* < p_0 f_{ii}^*$.

Hence $p_0 f_i^* < p_i$.

The argument presented mathematically in (7), (8) and (9) is presented graphically in Figure 2. Figure 2 deals with a production function having a single input. The line labelled TC identifies the total (fixed + variable) cost of using various amounts of input. The distance between points 0 and CK is fixed cost. The line labelled TR shows total revenue resulting from use of various levels of $q_1$. Because $TR = p_0 q_0$, the shape of TR is much like the shape of a production function (see Figure 1). The distance between TR and TC measures profit. At the point $q_i^*$ the distance between TR and TC is a maximum. The slope of TC is $\partial p_1 q_1 / \partial q_1 = p_1$, the slope of TR is $\partial p_0 f = p_0 f_1$. At the
point $q_1^*$, $p_0 f_i^* = p_1$, the slopes of the TR and TC curves are equal. At point $q_1^*$, $\frac{\partial TC}{\partial q_1} = p_1$ but $\frac{\partial TR}{\partial q_1} = p_0 f_i^* p_1$. The slope of TR exceeds the slope of TC, and increasing $q_1$ above $q_1^*$ increases TR by more than it increases TC and hence, increases profit. At point $q_1^*$, $\frac{\partial TC}{\partial q_1} = p_1$ but $\frac{\partial TR}{\partial q_1} = p_0 f_i^* p_1$. The slope of TR is less than the slope of TC and increasing $q_1$ above $q_1^*$ increases TR by less than it increases TC, and hence reduces profit.

So far we have dealt with varying the level of one input at a time. What happens if the use of all inputs is varied? From equation (1), changing the amounts of all variable inputs used by the amounts $dq_1, dq_2, \ldots, dq_n$ changes the level of output by $dq_0 = \Sigma f_i dq_i$. Changes in levels of inputs used from the profit-maximizing levels result in changing $q_0$ by the amount

$$dq_0 = \Sigma f_i^* dq_i$$

Letting $C_T = \text{total cost} = \Sigma p_i q_i + CK$, the change in total cost resulting from the variations in the $q_i$ are

$$dC_T = \Sigma p_i dq_i$$

Dividing yields

$$(10) \frac{\partial C_T}{\partial q_0} = \frac{\Sigma p_i dq_i}{\Sigma f_i dq_i}$$

$\partial C_T / \partial q_0$ is commonly called marginal cost. It is the change in total cost that results when output is varied by one unit. If the firm is maximizing profit, marginal cost is $\partial C_T^*/\partial q_0$ and according to (5) the numerator of (10) can be written $\Sigma p_i f_i^* dq_i$ and the denominator is $\Sigma f_i^* dq_i$. Then

$$(11) \frac{\partial C_T^*/\partial q_0} = p_0 \frac{\Sigma f_i^* dq_i}{\Sigma f_i^* dq_i} = p_0$$
By using (8) and (9) some things can be learned about the nature of marginal cost.

If all \( q_i < q_i^* \), then \( \frac{\partial C_T}{\partial q_0} = \sum p_i dq_i / \sum f_i dq_i < p_0 \sum f_i dq_i / \sum f_i dq_i = p_0 \).

Hence \( \frac{\partial C_T}{\partial q_0} = \sum p_i dq_i / \sum f_i dq_i < p_0 \sum f_i dq_i / \sum f_i dq_i = p_0 \).

Thus

(12) If all \( q_i < q_i^* \), then \( \frac{\partial C_T}{\partial q_0} < p_0 \).

If all \( q_i > q_i^* \), then \( \frac{\partial C_T}{\partial q_0} > p_0 \).

Thus

(13) If all \( q_i > q_i^* \), then \( \frac{\partial C_T}{\partial q_0} > p_0 \).

If the firm is operating at the point of maximum profit, the marginal cost of output equals product price. If the firm is operating at \( q_0 < q_0^* \), increasing output would increase total cost by less than it increased revenue and would therefore increase profit. If the firm is operating at \( q_0 > q_0^* \), reducing output would reduce total revenue by less than it reduced total cost, and would increase profit.

APPLICATION OF ECONOMIC THEORY TO ANIMAL BREEDING

Intuitive Explanation

Now, suppose \( p_0, p_1, \ldots, p_n, G_1, G_2, \ldots, G_m \) and \( K \) are fixed and the farmer is maximizing profit. Let \( q_0^*, q_1^*, \ldots, q_n^* \) be the values of the variables that maximize his profit subject to the production function: the equilibrium values. Then \( \pi^* = p_0 q_0^* - \sum p_i q_i^* \) is the maximum level of profit. Now suppose
the genotypic value of one trait, say the h-th, varies by $\partial G_h$. This affects the production function and, in turn, affects the values of output and inputs that maximize profit. Assume the producer is adjusted to the new production function and is again maximizing profit. Then $\partial q_0^*/\partial G_h$, $\partial q_1^*/\partial G_h$, ..., $\partial q_n^*/\partial G_h$ represent the changes in the levels of $q_0$, $q_1$, ..., $q_n$ per unit change in $G_h$ between the old equilibrium level and the new equilibrium point. The change in the firm's profit between the old and new equilibrium point is

$$\frac{\partial \pi^*}{\partial G_h} = p_0 \frac{\partial q_0^*}{\partial G_h} - \sum_i p_i \frac{\partial q_i^*}{\partial G_h}$$

if variation in $G_h$ does not affect $p_0$. Variation in $G_h$ may, however, affect $p_0$, for example, it may result in a leaner hog for which the farmer receives a higher price. Then

$$\frac{\partial \pi^*}{\partial G_h} = p_0 \frac{\partial q_0^*}{\partial G_h} + q_0^* \frac{\partial p_0}{\partial G_h} - \sum_i p_i \frac{\partial q_i^*}{\partial G_h}$$

The first analysis to follow will assume that $\frac{\partial p_0}{\partial G_h} = 0$.

To investigate the value of animal improvement programs to individual producers, we investigate $\frac{\partial q_0^*}{\partial G_h}$ and $\frac{\partial q_i^*}{\partial G_h}$ for $i = 1, 2, \ldots, n$, and then investigate $\frac{\partial \pi^*}{\partial G_h}$. For notational convenience, asterisks will not be used as superscripts on the variables. But it must be kept in mind throughout the rest of this paper that $\frac{\partial q_i}{\partial G_h}$ really represents $\frac{\partial q_i^*}{\partial G_h}$ and that $\frac{\partial \pi}{\partial G_h}$ really represents $\frac{\partial \pi^*}{\partial G_h}$; $\frac{\partial q_i}{\partial G_h}$ represents the change in the level of $q_i$ that is consistent with maximum profit before and after the change in $G_h$.

To illustrate the mathematical procedure that will be used, consider the two equations (16) and (17) containing the two unknown or dependent variables $y_1$ and $y_2$ and the two known or independent variables $x_1$ and $x_2$.

$$a_1 y_1 + a_2 y_2 + a_3 y_1 y_2 + c_1 x_1 + c_2 x_2 = 0 = F_1$$

$$b_1 y_2^2 + b_2 y_2 + b_3 y_1 y_2^3 + e_1 x_1^2 + e_2 x_2 = 0 = F_2$$

We are concerned with two issues. (1) Given $x_1 = x_{10}$ and $x_2 = x_{20}$, does this system have a solution? (2) If it has a solution, what is the behavior of
y_1 and y_2 in the neighborhood of the solution as x_{10} and x_{20} vary? The fact that these equations are nonlinear in y_1 and y_2 creates some problem in answering these two questions. It is known, however, that an equation that is non-linear in the variables becomes linear in the differentials of the variables if the total differential of the equation is determined. Taking the total differential of \( F_1 \) yields
\[
dF_1 = \left( \frac{\partial F_1}{\partial y_1} \right) dy_1 + \left( \frac{\partial F_1}{\partial y_2} \right) dy_2 + \left( \frac{\partial F_1}{\partial x_1} \right) dx_1 + \left( \frac{\partial F_1}{\partial x_2} \right) dx_2 = 0
\]
Taking the total differential of \( F_2 \) yields
\[
dF_2 = \left( \frac{\partial F_2}{\partial y_1} \right) dy_1 + \left( \frac{\partial F_2}{\partial y_2} \right) dy_2 + \left( \frac{\partial F_2}{\partial x_1} \right) dx_1 + \left( \frac{\partial F_2}{\partial x_2} \right) dx_2 = 0
\]
Because \( x_1 \) and \( x_2 \) are independent variables, they can be varied independently of each other. Suppose \( dx_2 = 0 \) but \( dx_1 \neq 0 \). Dividing \( dF_1 \) and \( dF_2 \) by \( dx_1 \) yields partial derivatives \( \frac{\partial F_1}{\partial x_1} \), \( \frac{\partial F_2}{\partial x_1} \), and \( \frac{\partial y_1}{\partial x_1} \).
\[
(18) \quad \frac{dF_1}{dx_1} = \left( \frac{\partial F_1}{\partial y_1} \right) \frac{dy_1}{dx_1} + \left( \frac{\partial F_1}{\partial y_2} \right) \frac{dy_2}{dx_1} + \left( \frac{\partial F_1}{\partial x_1} \right) = 0
\]
\[
(19) \quad \frac{dF_2}{dx_1} = \left( \frac{\partial F_2}{\partial y_1} \right) \frac{dy_1}{dx_1} + \left( \frac{\partial F_2}{\partial y_2} \right) \frac{dy_2}{dx_1} + \left( \frac{\partial F_2}{\partial x_1} \right) = 0
\]
In vector-matrix notation, these can be written
\[
\begin{pmatrix}
\frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\
\frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2}
\end{pmatrix}
\begin{pmatrix}
\frac{dy_1}{dx_1} \\
\frac{dy_2}{dx_1}
\end{pmatrix}
= \begin{pmatrix}
-\frac{\partial F_1}{\partial x_1} \\
-\frac{\partial F_2}{\partial x_1}
\end{pmatrix}
\]
or, still more compactly, as
\[
B \frac{dy}{dx_1} = -\frac{\partial F}{\partial x_1}
\]
The system (18) and (19) has a solution if \( B \) is non-singular, that is, if \( \det B \neq 0 \). (The determinant of \( B \) is referred to as the Jacobian determinant of (16) and (17).) If \( \det B \neq 0 \), the solution to (18) and (19) is \( \frac{dy}{dx_1} = -B^{-1} \frac{\partial F}{\partial x_1} \). The solution can also be expressed by use of Cramer's rule. Use of Cramer's rule yields the solution for \( \frac{dy}{dx_1} \):
\[
\frac{dy}{dx_1} = \begin{vmatrix}
-\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial y_2} \\
-\frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial y_2}
\end{vmatrix}
= \begin{vmatrix}
\frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\
\frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2}
\end{vmatrix}
\]
If \( \det B \neq 0 \), the original system (16) and (17) also has a solution.

In equations (16) and (17):

\[
\begin{align*}
\frac{dF_1}{dx_1} &= (a_1 + a_3y_2^2)\frac{\partial y_1}{\partial x_1} + (2a_2y_2 + a_3y_1)\frac{\partial y_2}{\partial x_1} + c_1 = 0 \\
\frac{dF_2}{dx_1} &= (2b_1y_1 + b_3y_2^3)\frac{\partial y_1}{\partial x_1} + (b_2 + 3b_3y_1y_2^2)\frac{\partial y_2}{\partial x_1} + 2c_1x_1 
\end{align*}
\]

Analysis

To determine the effect of one unit of change in the value of \( G_h \) on the levels of inputs and output and on profit, it is convenient first to rewrite (4.1) through (4.n) as (20.1) through (20.n) in order to use the procedure used on (16) and (17).

\[
\begin{align*}
(20.1) \quad F_1 &= p_0^1 - p_1 = 0 \\
(20.2) \quad F_2 &= p_0^2 - p_2 = 0 \\
& \quad \vdots \\
(20.n) \quad F_n &= p_0^n - p_n = 0 
\end{align*}
\]

Variation in \( G_h \) may affect the optimum levels of inputs. Taking the total differential of (20.e) with respect to variations in \( G_h \) and the \( q_j \) yields

\[
dF = \sum_{j=1}^{n} \left( \frac{\partial F}{\partial q_j} \right) dq_j + \left( \frac{\partial F}{\partial G_h} \right) dG_h = 0
\]

Dividing through by \( dG_h \) yields the following expression in partial derivatives

\[
(21) \quad \frac{dF}{dG_h} = \sum_j \left( \frac{\partial F}{\partial q_j} \right) \left( \frac{\partial q_j}{\partial G_h} \right) + \frac{\partial F}{\partial G_h} = 0
\]
(21) can be written

\[ \frac{dF}{dG} = p_0 \sum_j f_{ej} \left( \frac{\partial a_j}{\partial G} \right) + p_0 f_e G = 0 \]

Let \( \frac{\partial a}{\partial G} = a G \). Applying (22) to (20.1) through (20.n) yields

\[ p_0 \sum_j f_{1j} a_j + p_0 f_{1G} = 0 \]
\[ p_0 \sum_j f_{2j} a_j + p_0 f_{2G} = 0 \]
\[ \vdots \]
\[ p_0 \sum j a_j + p_0 f_{nG} = 0 \]

In vector-matrix notation these equations can be written

\[
\begin{pmatrix}
  f_{11} & f_{12} & \cdots & f_{1n} \\
  f_{21} & f_{22} & \cdots & f_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{n1} & f_{n2} & \cdots & f_{nn}
\end{pmatrix}
\begin{pmatrix}
  a_{G1} \\
  a_{G2} \\
  \vdots \\
  a_{Gn}
\end{pmatrix}
\begin{pmatrix}
  f_{1G} \\
  f_{2G} \\
  \vdots \\
  f_{nG}
\end{pmatrix}
\]

or as, say

\[(23) \quad H \frac{dQ}{dG} = -F_h \]

Because \( H \) is negative definite, \( H^{-1} \) exists and the solution to (23) is

\[(24) \quad \frac{dQ}{dG} = -H^{-1} F_h \]

(24) shows how the profit-maximizing levels of the various inputs change as
$G_h$ changes. The optimum level of output also changes. Its change is determined by the values of $\partial q_i / \partial G_h$ and by the nature of the production function.

$$\frac{\partial q_0}{\partial G_h} = \sum_{j=1}^{n} \frac{\partial q_j}{\partial G_h} + \frac{\partial f}{\partial G_h}$$

Define $F' = (f_1, f_2, \ldots, f_n)$ and $\partial f / \partial G_h = f_{G_h}$. Then $\frac{\partial q_0}{\partial G_h} = F' \frac{dQ}{dG_h} + F_{G_h}$

Finally, the effect on maximum profit of varying $G_h$ is

$$\frac{\partial \pi}{\partial G_h} = \frac{\partial q_0}{\partial G_h} p_0 - \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial G_h}$$

Define $P' = (p_1, p_2, \ldots, p_n)$. Then

$$\frac{\partial \pi}{\partial G_h} = p_0 F' \frac{dQ}{dG_h} + p_0 f_{G_h} - P' \frac{dQ}{dG_h}$$

Because $P' = p_0 F'$,

$$\frac{\partial \pi}{\partial G_h} = p_0 f_{G_h}$$

Thus, the analysis leads to a fairly simple result. If the firm is maximizing profit with existing levels of traits, and one trait changes and the firm adopts the improved variety of input and continues to maximize profit, the change in profit equals output price multiplied by the partial derivative of the production function with respect to the changed trait.

The result, however, is not quite as simple as it seems. From what was written earlier under Intuitive Explanation, it is seen that $f_{G_h}$ must be evaluated at the point $q_1^*, \ldots, q_n^*$. The first-order conditions must be solved to obtain the optimum levels of the inputs and $f_{G_h}$ must be evaluated at the point of optimum levels of inputs. To emphasize this point, rewrite
(25) as

\( \frac{\partial \pi^*}{\partial G_h} = p_0 f_{G_h} \)

The effect on maximum profit of varying \( G_h \) by the amount \( dC_h \) is

\[(27) \ d\pi^* = \left( \frac{\partial \pi^*}{\partial C_h} \right) dC_h = p_0 f_{C_h} dC_h \]

Generally a breeding program affects more than one trait. The effect on maximum profit of simultaneous variation in more than one trait is

\[(28) \ d\pi^* = \sum \left( \frac{\partial \pi^*}{\partial G_h} \right) dG_h = \sum p_0 f_{C_h} dG_h \]

Up to this point it has been assumed that \( \frac{\partial p_0}{\partial G_h} = 0 \). Now suppose that variation in the \( h \)-th trait affects the quality of the firm’s product, and hence affects \( p_0 \); \( \frac{\partial p_0}{\partial G_h} \neq 0 \). To obtain (23), \( G_h \) and \( q_j \ (j \geq 1) \) varied. Now, \( p_0 \) is also varied. Define the vector \( F_h \) as

\[
F_{h0} = (p_0 f_{C_h} + f_h \frac{\partial p_0}{\partial G_h}, \ldots, p_0 f_{C_n} + f_n \frac{\partial p_0}{\partial G_n})
\]

The results of taking the total differentials can now be expressed as

\[(29) \ H dQ/dG_h = - F_h \]

The solution for \( dQ/dG_h \) is (30) instead of (24).

\[(30) \ dQ/dG_h = - H^{-1} F_h \]

The effect of \( dG_h \) on optimum level of output is

\[
\frac{\partial q_0}{\partial G_h} = F' dQ/dG_h + f_{G_h}
\]

where \( dQ/dG_h \) comes from (30). The effect of varying \( G_h \) on the maximum
level of profit is now

\[ \frac{\partial \pi}{\partial q_h} = p_0 \frac{\partial f}{\partial q_h} + q_0 \frac{\partial p_0}{\partial q_h} \]  

(compare this with (26)).

Now suppose that the price the firm received depends upon the amount it produces \( \frac{\partial p_0}{\partial q_0} \neq 0 \), but that \( \frac{\partial p_0}{\partial q_h} = 0 \). The first-order conditions now are

\[ \frac{\partial \pi}{\partial q_e} = 0 = p_0 + f \left( \frac{\partial p_0}{\partial q_0} \right) f_e - p_e = F_e \]

Expression (22) becomes

\[ \frac{dF_e}{dG_h} = \sum_j \left[ \left( p_0 + f \frac{\partial p_0}{\partial q_0} \right) f_e f_j + 2 f_e f_j \frac{\partial p_0}{\partial q_0} \right] \frac{\partial q_j}{\partial G_h} \]

\[ + \left( p_0 + f \frac{\partial p_0}{\partial q_0} \right) f_e G_h + 2 f_e f_{G_h} \frac{\partial p_0}{\partial q_0} \]

In expression (23), the element in row e and column j of H is

\[ \left( p_0 + f \frac{\partial p_0}{\partial q_0} \right) f_e f_j + 2 f_e f_j \frac{\partial p_0}{\partial q_0} \]

and the e-th element of F_h is

\[ \left( p_0 + f \frac{\partial p_0}{\partial q_0} \right) f_e G_h + 2 f_e f_{G_h} \frac{\partial p_0}{\partial q_0} \]

and (25) becomes

\[ \frac{\partial \pi}{\partial G_h} = \left( p_0 + q_0 \frac{\partial p_0}{\partial q_0} \right) f_{G_h} \]

Before proceeding to the interpretation of \( \frac{\partial \pi}{\partial G_h} \) and \( \Delta \pi \), the determination of \( f_{G_h} \) and \( f_{1G_h} \) will be discussed. Suppose the production function is quadratic, say
Suppose the production function is double-logarithmic

\[ q_0 = a_0 + \sum_{i=1}^{n} a_i q_i + \sum_{i=1}^{n} \alpha_{ij} q_i q_j + \sum_{i=1}^{n} \beta_{ij} G_i G_j + \sum_{i=1}^{n} \gamma_{ij} G_i G_j \]

Then

\[ f_i = \alpha_i + 2\alpha_{ii} q_i + \sum_{j \neq i} \alpha_{ij} q_j + \gamma_{ij} G_j \]

\[ f_{C_h} = \beta_h + 2\beta_{hh} G_h + \sum_{j \neq h} \beta_{ij} G_j + \gamma_{ih} q_i \]

\[ f_{C_h} = \gamma_{ih} \]

Suppose the production function is double-logarithmic

\[ q_0 = a_0 q_1 q_2 \ldots q_n G_1 G_2 \ldots G_m \]

Then

\[ f_i = \alpha_i q_0 / q_i \]

\[ f_{C_h} = \beta_h q_0 / C_h \]

\[ f_{C_h} = \beta_h f_i / C_h \]

**Interpretation**

The term \( \partial \pi / \partial G_h \) is the net monetary value to the firm of a one unit change in the h-th genotypic value; net monetary value because it measures effect on firm's net profit. And \( dG_h \) is the change in the h-th genotypic value. Then \( (\partial \pi / \partial G_h) dG_h \) measures the net monetary value to the firm of changing \( G_h \) by the
Suppose \( dG_1 \) is change in back fat probe in tenths of an inch. Then the unit of measurement of \( \frac{\partial \pi}{\partial G_1} \) is dollars of profit per tenth of an inch of backfat, and \( \left(\frac{\partial \pi}{\partial G_1}\right)dG_1 \) equals the change in profit resulting from changing backfat probe by \( dG_1 \) tenths of an inch. And, suppose \( dG_2 \) is change in average daily gain of hogs, measured in pounds. Then \( \frac{\partial \pi}{\partial G_2} \) is the change in profit per one pound change in average daily gain, and \( \left(\frac{\partial \pi}{\partial G_2}\right)dG_2 \) equals the change in profit resulting from changing average daily gain by \( dG_2 \) pounds. Combining the effects of \( dG_1 \) and \( dG_2 \) yields

\[
d\pi = \left(\frac{\partial \pi}{\partial G_1}\right)dG_1 + \left(\frac{\partial \pi}{\partial G_2}\right)dG_2
\]

which is the change in profit resulting from changing backfat by \( dG_1 \) tenths of an inch and changing average daily gain by \( dG_2 \) pounds. This interpretation suggests that \( d\pi \) might appropriately be termed "differential aggregate genotype." To notationally dramatize its relation to the aggregate genotype \( H = \sum a_i G_i \), call it \( H(d\pi) \)

\[
(35) \quad H(d\pi) = \sum_h \left(\frac{\partial \pi}{\partial G_h}\right)dG_h
\]

Both \( H \) and \( H(d\pi) \) contain measures of net economic value of genotypic traits. Whereas \( H \) utilizes an intuitive definition of net economic value, \( H(d\pi) \) provides a rigorous definition of net economic value. Another difference is that \( H \) focusses on levels of genotypic values \( (G_i) \) whereas \( H(d\pi) \) focusses on changes in genotypic values \( (dG_h) \).

Use of Procedure

Use of the procedure developed here requires the following steps if variation in \( G_h \) does not affect \( \pi_0 \):

(a) Estimate a production function -- equation (1) -- that relates output to amounts of variable inputs and values of traits.
(b) Specify product price \( (p_0) \), prices of variable inputs \( (p_1, p_2, \ldots, p_n) \), and levels of traits \( G_1, G_2, \ldots, G_m \).

(c) Differentiate the profit function -- equation (3) -- with respect to \( q_1, q_2, \ldots, q_n \) to obtain the first-order conditions (4.1) through (4.n).

(d) Solve the equations (4.1) through (4.n) for the values of \( q_1^*, q_2^*, \ldots, q_n^* \). These values of the \( q_j^* \) satisfy expression (5.1).

(e) Express the production function in terms of the \( q_j^* \) and the specified levels of traits: \( q_0^* = f^* (q_1^*, q_2^*, \ldots, q_n^*, G_1, G_2, \ldots, G_m) \).

(f) From the production function in step (e) obtain \( f_{G_h}^* = \partial f^*/\partial G_h \).

(g) Apply equation (26): \( \partial \pi^*/\partial G_h = p_0 f_{G_h}^* \).

If variation in \( G_h \) affects \( p_0 \), three additional steps are required:

(h) Obtain \( \partial p_0/\partial G_h \)

(i) Multiply \( q_0^* \) from (e) by \( \partial p_0/\partial G_h \)

(j) Add results from steps (g) and (i) to obtain expression (31).

**EFFECTS OF VARIATIONS IN PRICES**

The neoclassical firm model can be used to gain some additional insights into firm behavior. In deriving \( G_h \) it was assumed that \( G_h \) changed but prices did not. The same procedure used to derive (23) can be used to study effects of variations in prices. In deriving (23) it was assumed that \( G_h \) changed but prices did not. Now suppose \( p_j \) varies but other prices and all traits remain constant. The differential equation to be considered now is (36) instead of (22).

\[
(36) \quad \frac{df}{dp_j} = \sum_i \left( \frac{\partial f}{\partial q_i} \frac{\partial q_i}{\partial p_j} \right) \left( \frac{\partial q_i}{\partial p_j} \right) + \left( \frac{\partial f}{\partial p_j} \right)
\]

The results of applying (36) to (20.1) through (20.n) can be expressed as

\[
(37) \quad \frac{dQ}{dp_j} = e_j
\]
where \( \frac{dQ}{dp_j} \) = \( \frac{\partial q_1/\partial p_j, \partial q_2/\partial p_j, \ldots, \partial q_n/\partial p_j} \) and \( e_j \) is the \( j \)-th unit vector. It has a one in the \( j \)-th position and zeros elsewhere. Letting \( H^{ij} \) be the element in the \( i \)-th row and \( j \)-th column \( H^{-1} \), the solutions to (34) can be expressed \( \frac{\partial q_i}{\partial p_j} = H^{ij} \). Now \( H^{ij} \) may be positive, negative or zero.

\[
(38) \ \frac{\partial q_i}{\partial p_j} = H^{ij}\geq 0.
\]

Because \( H \) is negative definite, \( H^{-1} \) is negative definite. Therefore, every element on the main diagonal of \( H \) is negative and

\[
(39) \ \frac{\partial q_i}{\partial p_i} = H^{ii} < 0.
\]

A variation in \( p_i \) changes the profit-maximizing value of \( q_i \) in the opposite direction.

Suppose the selling price \( p_0 \) varies. The results from differentiating (20.1) through (20.n) can be expressed

\[
(40) \ H \frac{dQ}{dp_0} = -F
\]

The solution to (40) is

\[
(41) \ \frac{dQ}{dp_0} = -H^{-1}F
\]

A typical element of (41) is

\[
(42) \ \frac{\partial q_j}{\partial p_0} = -h^j F
\]

where \( h^j \) is the \( j \)-th row of \( H^{-1} \). The effect of variation in \( p_0 \) on level of output is

\[
(43) \ \frac{\partial q_0}{\partial p_0} = F' \frac{dQ}{dp_0} = -F' H^{-1} F > 0
\]

Because \( H^{-1} \) is negative definite \( F'H^{-1}F < 0 \) and \( \frac{\partial q_0}{\partial p_0} \) is positive. An increase or decrease in selling price raises or lowers, respectively, the level of output.

From (5.1) and (43) it is possible to infer something about the signs of
(42). (5.1) shows that each \( f_i^* \) equals the ratio of two (positive) prices. Therefore, \( f_i^* > 0 \) for all \( i \) and all elements of \( F \) are positive; \( P > 0 \). Because \( F' > 0 \) and \( F' \frac{dQ}{dp_0} > 0 \), some (or all) elements of \( dQ/dp_0 \) are positive. Hence

\[
\frac{\partial q_j}{\partial p_0} > 0 \text{ for some } j \neq 0.
\]

Variation in product price affects levels of some inputs.

Variation in the \( p_j \) affects levels of inputs, and hence also level of output. \( \frac{\partial q_0}{\partial p_j} = F' \frac{dQ}{dp_j} \). From (31) this can be expressed

\( \frac{\partial q_0}{\partial p_j} = F' \frac{dQ}{dp_j} = F' H^{-1} e_j \). Let \( H^j = j \)-th column of \( H^{-1} \). Then

\[ \frac{\partial q_0}{\partial p_j} = F' H^j. \]

Because \( H \) is symmetric the elements of \( H^j \) are the same as the elements \( h^j \) and \( H^j = h^j \). Therefore \( \frac{\partial q_0}{\partial p_j} = F' h^j = h^j F \). Hence from (42) \( \frac{\partial q_0}{\partial p_j} = - \frac{\partial q_j}{\partial p_0} \) and from (44) we have

\[
(45) \quad \frac{\partial q_0}{\partial p_j} < 0 \text{ for some } j.
\]

For some (and possibly for all) variable inputs, an increase in price leads to a reduction in output.

The economic value of a trait is also affected by price changes. Suppose product price changes but other prices remain constant. Now \( \partial (\partial \pi/\partial G_h)/\partial p_0 \) must be evaluated. From (25) this can be written

\[
\partial (\partial \pi/\partial G_h)/\partial p_0 = p_0 \frac{\partial f_G}{\partial p_0} + \frac{f_G}{\partial p_0}
\]

And

\[
p_0 \frac{\partial f_G}{\partial p_0} = p_0 \sum_j \left( \frac{\partial f_G}{\partial p_j} \right) \frac{\partial q_j}{\partial p_0}
\]

\[ p_0 \frac{\partial f_G}{\partial p_j} = p_0 f_j^G. \text{ Hence }
\]
The effect of an increase in product price is to increase profit. It may increase or reduce economic value of a trait.

The effect of variation in an input price \( p_j \) is

\[
\frac{\partial \pi}{\partial p_j} = p_0 \frac{\partial f}{\partial p_j}.
\]

This can be expressed as

\[
\frac{\partial \pi}{\partial p_j} = \frac{\partial f}{\partial p_j} = \frac{\partial f}{\partial p_j} = \frac{\partial f}{\partial p_j} > 0
\]

Example: Quadratic Production

Function: One Variable Input

This section will present a simple example of application of neoclassical firm theory to a firm having one variable input. Equations are numbered to correspond to equations in the previous two sections.

Thus (1X) is an example of expression (1), (4X) is an example of (4), etc.

The production function and the profit function for this example are

\[
\begin{align*}
(1X) \quad q_0 &= \alpha_0 + \alpha_1 q_1 + \alpha_2 q_1^2 = f(q_1) \\
(3X) \quad \pi &= p_0 (\alpha_0 + \alpha_1 q_1 + \alpha_2 q_1^2) - p_1 q_1
\end{align*}
\]

The first-order condition is

\[
(4X) \quad \frac{\partial \pi}{\partial q_1} = p_0 (\alpha_1 + 2\alpha_2 q_1) - p_1 = 0
\]

To obtain \( \frac{\partial \pi}{\partial q}, \) we can go directly to expression (23). In this example, (23) becomes

\[
(23X) \quad p_0 \frac{\partial f}{\partial q_1} = - p_0 f_{1G_h}
\]
APPLICATION OF PROCEDURE TO M-H-W DATA

The procedure outlined in the section on Application of Economic Theory to Animal Breeding is here applied to a production function estimated by Melton, Heady, and Willham (M-H-W) (1977). The data used to estimate the production function were obtained from steers in a cross breeding experiment.

(a) Define \( q_0 \) = kg of retail product
\( q_1 \) = number of days on feed
\( G_1 \) = kg of average daily gain\(^3\)/

\( G_2 \) = kg of weaning weight

The production function is

\[
q_0 = f(q_1, G_1, G_2) = -14.515 - 19.558 G_1 \\
+ .486 G_2 + .108 q_1 + 20.909 G_1^2 \\
- .000224 G_2^2 - .00141 q_1^2
\]

\(^3\)/On page 9, line 15, M-H-W define \( G_1 \) as gr of average daily gain. I think this is a typing error. The arithmetic on page 10 of the M-H-W ms does not come out right for me unless average daily gain is measured in kg.
\[+.279 \ q_1 \ G_1 - .000095 \ q_1 \ G_2 -.111 \ G_1 \ G_2\]

(b) Let \(p_0 = \$2/\text{kg}\)
\[p_1 = \$0.50/\text{day}\]
\[G_1 = .785 \ \text{kg}^{1/}\]
\[G_2 = 193.8 \ \text{kg}\]

(c) \[\partial q_0 / \partial q_1 = f_1 = .108 - .00282 \ Q_1 + .279 (.785)\]
\[= .000095 (193.8)\]
\[\partial \pi^* / \partial q_1 = 2 f_1 -.5\]

(d) .000564 \(q_1 = 2[.108 + .219015 - .018411] - .5\)
\[q_1^* = .117208/.000564 = 207.8 \ \text{days}\]
\[= 208 \ \text{days}\]

(e) In \(f(q_1^*, G_1, G_2)\), set \(G_1 = .785, G_2 = 193.8, q_1^* = 208\)

(f) \[\partial f^* / \partial G_1 = - 19.558 + 41.818 (.785)\]
\[+ .279 (208) - .111 (193.8) = 49.78933\]

(g) \(2(49.78933) = \$99.57/\text{kg} = \partial \pi^* / \partial G_1\)

(h) \[\partial f^* / \partial G_2 = .486 - .000448 (193.8)\]
\[-.000095 (208) -.111 (.785) = .292283\]

(i) \[\partial \pi^* / \partial G_2 = 2(.292283) = \$ .58/\text{kg}\]

\(^{1/}\) See footnote 3.
AN ALTERNATIVE

Melton, Heady, and Willham (1977) have presented another method of obtaining $\pi/\pi^h$. They assume prices for traits are known; let $w_j$ be the price per unit of the $j$-th trait. They write the profit function as

$$\pi = p_0 f(q_1, q_2, \ldots, q_n, G_1, G_2, \ldots, G_m) - \sum p_i q_i - \sum w_j G_j - CK$$

They assume that the variables the firm can manipulate in order to maximize profit are $q_1, q_2, \ldots, q_n$, and also $G_1, G_2, \ldots, G_m$. Their first order conditions are (4.1) to (4.n) and (4.n+1) to (4.n+m)

$$\frac{\partial \pi}{\partial G_1} = p_0 f_{G_1} - w_1 = 0$$

$$\vdots$$

$$\frac{\partial \pi}{\partial G_m} = p_0 f_{G_m} - w_m = 0$$

Solving (4.n+m) for $w_h$ yields

$$w_h = p_0 f_{G_h}$$

The right-hand-side of (49) is exactly the right-hand-side of (25).

In the previous procedure, the parameters in the initial profit-maximization problem are $p_0, p_1, p_2, \ldots, p_n$ and $G_1, G_2, \ldots, G_m$. The $n$ first-order conditions are used to obtain $q_1, q_2, \ldots, q_n$. From these $q_0$ and $\pi$ are determined. Then $\partial \pi/\partial G_h$ is determined. The M-H-W procedure contains the same initial parameters: $p_0, p_1, p_2, \ldots, p_n$ and $G_1, G_2, \ldots, G_m$. The $n+m$ first order conditions are solved for $q_1, q_2, \ldots, q_n$ and $w_1, w_2, \ldots, w_m$.

Although it turns out that $w_h = \partial \pi/\partial G_h$ = economic value of $h$-th trait, this equality cannot be proved in the M-H-W model. The reason is clearly shown in expressions (4.n+1) through (4.n+m). According to these, $\partial \pi/\partial G_h = 0$. But according to (25) $\partial \pi/\partial G_h > 0$. 
DIFFERENCE BETWEEN NEOCLASSICAL AND ACTIVITY ANALYSIS VIEWS OF PRODUCTION

It was pointed out in the earlier section on Production Function that the production function is continuous and has continuous 1st and 2nd order derivatives. One implication of this continuity is: for a fixed level of output, ratios among inputs are infinitely variable. Consider a simple case of two variable inputs, $q_1$ and $q_2$. Suppose the level of output, $q_0'$, is fixed at $q_0'$ and let $q_1'$ and $q_2'$ be the levels of $q_1$ and $q_2$ used to produce $q_0'$. The ratio $q_1'/q_2'$ can take on infinite number of values, as can the ratios $q_1'/q_0'$ and $q_2'/q_0'$. This is illustrated in figure 2. The constant output curve $q_0'q_0$ shows different combinations of $q_1$ and $q_2$ that can be used to produce level of output $q_0'$. The constant output curve $q_0''q_0''$ shows different combinations of $q_1$ and $q_2$ that can produce the level of output $q_0''$ (where $q_0'' > q_0'$).

In linear programming, on the other hand, ratios among inputs can have only a finite number of values. A fundamental concept in linear programming is an "activity". This represents a way of producing something. Within a given activity, the ratio between inputs is fixed and the ratio of each input to output is also fixed. Increasing the use of each input by the percent $p$, increases the output from that activity by $p$ percent. Figure 3 represents a situation in which output can be produced by any one or a combination of 4 activities. The slope of the ray for each activity equals the ratio of $q_2'/q_1'$ for that activity. The points B, C, D and E show the combinations of levels of $q_1$ and $q_2$ that produce the level of output $q_0'$ by activities 1, 2, 3 and 4. Activity 4 has a lower ratio of $q_2'/q_1'$ than do the other activities. The line ABCDE is a constant output curve. It shows various combinations of levels of $q_1$ and $q_2$ that produce $q_0'$ of output. Points between two activities are achieved by using both activities. For example, point X between D and E
FIGURE 2

Constant-output Curves

$q_0'$

$q_0''$

$q_0$
is achieved by producing some output by activity 3 and some by activity 4. Each constant output curve consists of straight line segments and corners, each corner occurring on a ray identifying an activity. This contrasts with Figure 2, where each constant output curve is a smooth (continuous) curve possessing continuous derivatives.

A second difference between neoclassical production function and activity analysis is this: Increasing the levels of all inputs used in an activity by the same percent, say P percent, results in a P percent increase in output. A production function may possess this property, but need not. A production function that possesses this property is said to exhibit "constant returns to scale."

A third difference is in the assumptions concerning the use of fixed resources. The neoclassical model implicitly assumes that all fixed inputs are fully employed. In activity analysis of the firm, this assumption is not made. Instead, the amount of each fixed resource to be used is determined by the analysis, not predetermined before the start of the analysis.

A fourth difference between the production function and activity analysis approaches, and the one that is most important for empirical work is this. The previous discussion of production function related to a single product firm. Conceptually, it is simple to use the concept of multi-product production function to handle the analysis of a firm producing and selling more than one product. Multi-product production functions have been used very little in empirical work because of the big data collection and statistical analysis problems encountered in trying to obtain reliable estimates of them. Data collection and analysis problems encountered in estimating activities are much less formidable. Consequently, we have a substantial body of information on production activities.
The activity analysis view of production is the one adopted in linear programming. The next section discusses use of linear programming to study multi-product firms.

LINEAR PROGRAMMING STATEMENT OF THEORY OF MULTI-PRODUCT FIRM

This section presents a numerical example of a linear program, its solution, and a general application of linear programming to profit maximization.

This discussion is a condensation of part of Craig Gibson's thesis (1976). The mathematical theory of linear programming is discussed in many books, among them Dantzig (1963), Hadley (1962) and Gass (1964). Heady and Candler (1958) and Beneke and Winterboer (1973) discuss applications of linear programming to agriculture.

Numerical Example

In this example it is assumed that a firm feeds cattle to slaughter weight and then markets the cattle. The firm has three alternative activities which it may use to finish cattle to slaughter weight. The first activity is to buy 450 pound feeder calves, feed them a high roughage ration, and then sell them for slaughter at 1,050 pounds. The second activity is to buy 450 pound feeder calves, feed them a high grain ration, and then sell them for slaughter at 1,100 pounds. The third activity is to buy 650 pound yearling steers, feed them a medium roughage-medium grain ration, and then sell them for slaughter at 1,100 pounds.

The firm has a set of fixed inputs available for use in feeding the cattle. The firm has 11,000 bushels of corn, 900 tons of silage, 300 tons of hay, and 1600 hours of labor. The feed inputs are fixed in availability because they equal the amounts of feeds the firm has produced and the firm is unwilling to buy any of these feeds. The time input is fixed in availability because
it is the maximum amount of time the firm feels it can allot to the processes of finishing the cattle to slaughter weight. Table 1 shows the amount of each of these fixed inputs needed to produce one fed animal ready for market by each activity. To buy a 450 pound feeder calf, feed it a high roughage ration, and market it at 1,050 pounds (activity 1) requires 40 bushels of corn, 3.25 tons of silage, 0.11 ton of hay and 6.0 hours of labor.

The firm also requires a set of variable inputs for use in feeding the cattle. The firm requires such things as supplement, veterinary services and medicine, power and fuel, and other miscellaneous variable inputs. These inputs are variable because the firm can buy whatever amounts of these it needs: there is not an upper limit on the amount of each the firm can use. These purchased inputs are also available at a constant price. Prices of the purchased inputs and selling price for fed cattle are as follows:

Cattle prices

| Purchasing choice 450# calves | $44.50/cwt. |
| Purchasing choice 650# yearlings | $40.50/cwt. |
| Marketing choice 1050# steers | $35.00/cwt. |
| Marketing choice 1100# steers | $36.00/cwt. |

Supplement costs are:

| for high roughage ration | $24.00/600# gain |
| for high grain ration | $28.44/600# gain |
| for medium roughage-medium grain ration | $14.40/350# gain |

Veterinary services and medical costs are:

| for steer on high roughage ration | $9.40/steer |
| for steer on high grain ration | $12.50/steer |
| for yearling steer | $4.50/steer |

Power and fuel costs are:

| for steer on high roughage ration | $9.50/steer |
| for steer on high grain ration | $12.00/steer |
| for yearling steer | $7.22/steer |
TABLE 1

Production and Revenue Alternatives Facing Cattle Feeder

<table>
<thead>
<tr>
<th>Fixed input</th>
<th>Stock of input</th>
<th>Activity 1 a/</th>
<th>Activity 2 b/</th>
<th>Activity 3 c/</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corn (bu.)</td>
<td>11,000</td>
<td>40</td>
<td>50</td>
<td>35</td>
</tr>
<tr>
<td>Silage (tons)</td>
<td>900</td>
<td>3.25</td>
<td>0.72</td>
<td>2.0</td>
</tr>
<tr>
<td>Hay (tons)</td>
<td>300</td>
<td>0.11</td>
<td>0.25</td>
<td>0.3</td>
</tr>
<tr>
<td>Labor (hrs.)</td>
<td>1,600</td>
<td>6.0</td>
<td>6.4</td>
<td>5.0</td>
</tr>
<tr>
<td>Net revenue per unit of output d/</td>
<td></td>
<td>$123.35</td>
<td>$112.66</td>
<td>$105.88</td>
</tr>
</tbody>
</table>

\( a/ \) 450 lb. feeder calves raised to 1,050 lb. on high-roughage ration.

\( b/ \) 450 lb. feeder calves raised to 1,050 lb. on high-grain ration.

\( c/ \) 650 lb. yearling steers raised to 1,100 lb. on medium roughage, medium grain ration.

\( d/ \) Selling price per animal minus total variable costs per animal.
Miscellaneous costs are:

- for steer on high roughage ration — $1.00/steer
- for steer on high grain ration — $1.50/steer
- for yearling steer — $0.75/steer

The bottom row of table 1 shows for each activity the excess of selling price over total variable cost of buying, raising and marketing one animal.

For activity number 1 this is computed as follows:

Selling the steer at 1050# at $35.00/cwt. = $367.50 minus the variable input costs:

- supplement 24.00
- veterinary and medical 9.40
- power and fuel 9.50
- miscellaneous 1.00
- feeder calf (which weighs 450# and is purchased at $44.50/cwt.) 200.25

$244.15

gives the net revenue for activity #1 $123.35

Let \( x_1 \), \( x_2 \) and \( x_3 \) represent the numbers of animals raised by activities 1, 2 and 3 respectively. For now, we will ignore fixed costs. (Justification for this will be presented later.) If the cattle feeder wants to maximize his net revenue (excess of gross revenue over total variable costs), he must determine values of \( x_1 \), \( x_2 \) and \( x_3 \) to maximize

\[
123.35x_1 + 112.66x_2 + 105.88x_3
\]

In determining these values of \( x_1 \), \( x_2 \) and \( x_3 \), the feeder cannot use more of any fixed resource than is available. The total amount of corn used in activity 1 is 40\( x_1 \). The total amount of corn used in all three activities is 40\( x_1 \) + 50\( x_2 \) + 35\( x_3 \). The total amount used cannot exceed 11,000 bushels.
Consequently, \( x_1, x_2 \) and \( x_3 \) must satisfy

\[
(41.1) \quad 40x_1 + 50x_2 + 35x_3 \leq 11,000
\]

Likewise, the feeder cannot use more silage, hay or labor than is available. These restrictions on his behavior can be expressed

\[
(41.2) \quad 3.25x_1 + 0.72x_2 + 2.0x_3 \leq 900
\]

\[
(41.3) \quad 0.11x_1 + 0.25x_2 + 0.3x_3 \leq 300
\]

\[
(41.4) \quad 6.0x_1 + 6.4x_2 + 5.0x_3 \leq 1,600
\]

The feeder's net revenue maximization problem is to determine levels of \( x_1, x_2 \) and \( x_3 \) to maximize expression (40) subject to (41.1), (41.2), (41.3) and (41.4). One other condition must also be met. It does not make sense to speak of producing a negative number of animals. The values of \( x_1, x_2 \) and \( x_3 \) must satisfy the non-negativity conditions

\[
(42) \quad x_1, x_2, x_3 \geq 0
\]

The problem (40) through (42) is a "linear program". Before presenting a solution to this problem, a general linear-programming statement of profit-maximization will be considered.

**General Statement**

A fundamental concept in linear programming is the "activity." Activity means a way of producing something by a firm (or farm). (A firm being any technical unit in which output is produced.) Thus, if a farm produced market hogs by two different techniques, these two different techniques would be considered to represent two different activities. Activities are the alternative ways in which to produce different types of output, or, in some cases the same output.

A fundamental concept in linear programming, as in the neoclassical model of the firm, is the concept of "inputs." An "input" may be defined as
any good or service which contributes to the production of an output. A firm will normally use many different inputs for the production of an output. It is possible that some of the inputs used in one firm may be outputs of other firms.

Inputs are classified as "fixed" or "variable" with respect to their availability in the production of outputs. A "fixed input" is an input that is necessary for the production of output, but whose quantity available for the production of output is limited or "fixed." A "variable input" is defined as an input that is necessary for the production of output, but whose quantity available for the production of output is unlimited or "variable."

As a result of classifying inputs as "fixed" or "variable," total costs can be classified as "fixed" or "variable." Total cost is defined as the cost of production which results from using fixed and variable inputs in the production of output. "Fixed" cost is defined as the cost of fixed inputs. "Variable" cost is defined as the cost of variable inputs.

Another fundamental concept in linear programming is the concept of the "objective function." The "objective function," sometimes called the criterion function, defines the goal or objective of the linear program. It is the objective function which is optimized when solving the linear programming problem.

It is possible to optimize the objective function by either maximization or minimization, depending upon the objective. Maximization of the objective function is often used when the objective function expresses the returns of various "activities" of the linear programming problem and when the objective is to maximize profits. Minimization of the objective function is often used when the objective function expresses the costs of various "activities" of the linear programming problem and when the objective is to minimize costs.

By using these concepts, linear programming can be used to develop an
economic theory of a competitive profit-maximizing firm. The firm has a set of fixed inputs available for use. The firm owns, for example, a certain number of machines; the firm has available a certain number of buildings; the firm has available certain amounts of natural resources, etc. The firm uses these fixed inputs together with variable inputs to produce one or more different types of output. The firm purchases each unit of variable input it needs at a constant price. The firm sells each unit of output also at a constant price. Thus, the firm faces the problem of determining the amount of variable inputs to purchase and combine with its fixed inputs, while also determining the quantities of outputs to produce, in order to maximize its profit.

To measure net revenues, define the symbols

\[ p_j = \text{price received for one unit of output produced by the j-th activity} \]

\[ r_k = \text{purchase price of the k-th variable input} \]

\[ q_{kj} = \text{quantity used of the k-th variable input in the production of one unit of output by the j-th activity} \]

\[ c_j = \text{net revenue received by producing and selling one unit of output by the j-th activity} \]

Then

\[ c_j = p_j - \sum_k r_k q_{kj} \quad (43) \]

\( \sum_k q_{kj} \) is the average variable cost for activity j. Average variable cost is variable cost per unit of output. The firm's net revenue, Z, then can be expressed as

\[ Z = \sum_{j=1}^{n} c_j x_j \quad (44) \]

if the firm has n different activities. To express the restrictions on the use of fixed inputs, define
\( a_{ij} \) = amount of \( i \)-th fixed input used to produce one unit of output by \( j \)-th activity

\( a_{i0} \) = amount of \( i \)-th fixed input available to the firm

If the firm has \( m \) fixed inputs, the restrictions on the firm's behavior are

\[
\begin{align*}
\sum_{j=1}^{n} a_{ij} x_j &\leq a_{i0} \\
\sum_{j=2}^{n} a_{2j} x_j &\leq a_{20} \\
& \vdots \\
\sum_{j=1}^{m} a_{mj} x_j &\leq a_{m0}
\end{align*}
\]

(46) all \( x_j \geq 0 \)

The firm's net revenue maximization problem is to maximize (44) subject to (45) and (46).

Solution

The first step in solving the linear program is to convert the inequalities to equalities by adding a nonnegative slack variable to each constraint. A slack variable contributes nothing to the value of the objective function. The \( i \)-th constraint now becomes

\[
\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = a_{i0}
\]

The variables \( x_1, x_2, \ldots, x_n \) are called real variables, to differentiate them from the slack variables. The linear program can now be written as

\[
\begin{align*}
\text{Maximize} \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} s_{n+i} = Z \\
\text{subject to} \quad & (a_{11} \ x_1 + a_{12} \ x_2 + \ldots + a_{1n} \ x_n) + (1 \ s_{n+1}) = (1)
\end{align*}
\]
Define \( A_j \) to be vector multiplying \( x_j \) in (48)
\[
A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}
\]
and \( e_i \) to be the \( i \)-th unit vector; \( e_i \) has a unit in the \( i \)-th place and zeroes elsewhere. \( e_i \) multiplies \( x_{n+i} \) in (48). Then (48) can be written
\[
(48a) \quad \sum_j A_j x_j + \sum_i e_i x_{n+i} = A_0
\]

A feasible solution is a solution to (45) or (48), that is, a set of nonnegative variables that satisfy the inequalities in (45) or the equalities in (48). (Any feasible solution to (45) is also a feasible solution to (48), and vice versa.)

An optimal feasible solution is a feasible solution that maximizes the value of \( Z \). (48) is a system of \( m \) linear equations. A basic feasible solution is a feasible solution that contains \( m \) variables, and the vectors that are multiplied by these \( m \) variables in (48) are linearly independent, and all other variables are zero. For example, suppose \( x_1 = x_{10} > 0, x_2 = x_{20} > 0, \ldots, x_m = x_{m0} > 0, x_{m+1} = x_{m+2} = \ldots = x_{m+n} = 0 \) and \( A_1, A_2, \ldots, A_m \) are linearly independent. This is a basic feasible solution because the vectors \( A_1, A_2, \ldots, A_m \) form a basis in \( m \)-space. The matrix \( (A_1, A_2, \ldots, A_m) \) is nonsingular: it has an inverse. A fundamental theorem tells us: if a linear program has an optimal feasible solution, it has a basic optimal feasible solution. Thus
we need only investigate basic feasible solutions to the linear program.

Assume \( B \) is an optimal feasible basis. \( B \) is an \( m \times m \) matrix. The \( m \) columns of \( B \) are made up of some of the \( A_j \) and some of the \( c_i \) in (48a). Let \( X_0 \) be the vector of values of the basic variables. Then

\[
(49) \quad BX_0 = A_0, \quad X_0 = B^{-1}A_0 = (x_{j0}) \geq 0
\]

Let \( c_B' \) be a vector of weights, from the objective function (47), of the basic variables. \( c_B' \) consists of \( c_j \) for basic real variables and zeroes for basic slack variables. Then the maximum value of the objective function is

\[
Z_0 = c_B' X_0 = c_B' B^{-1} A_0
\]

For each real variable \( x_j (j = 1, 2, \ldots, n) \) define

\[
(50) \quad c_j - z_j = c_j - c_B' B^{-1} A_j
\]

For each slack variable \( x_{n+i} (i = 1, 2, \ldots, m) \) define

\[
(51) \quad c_{n+i} - z_{n+i} = 0 - c_B' B^{-1} e_i
\]

The \( c_j - z_j \) and \( c_{n+i} - z_{n+i} \) are sometimes referred to as criterion elements.

If \( B \) is an optimal feasible basis, then \( c_j - z_j \leq 0 \) for all \( j \) and \( c_{n+i} - z_{n+i} \leq 0 \) for all \( i \). If \( x_t (t = 1, 2, \ldots, n+m) \) is a basic variable, \( x_t \geq 0 \) and

\[
(52) \quad c_t - z_t = 0.
\]

The criterion elements for nonbasic variables provide useful information.

Letting \( \Sigma \) denote summation over all variables in the basis \( B \), the maximum value of the objective function can be written as

\[
Z_0 = \sum_{i \in B} c_i x_i 0
\]

All variables not in the basis have a value of zero. Now what happens to the value of the objective function if some nonbasic variable, say \( x_n \), is forced into the solution? In order to maintain a feasible solution, the basic variables must change in value. Hence the total change in the objective
function for a unit change in \( x_h \) is

\[
\frac{dZ_0}{dx_h} = \sum_{i \in B} \frac{\partial Z_0}{\partial x_i} \frac{\partial x_i}{\partial x_h}
\]

And it can be shown that

\[
(52) \quad \frac{dZ_0}{dx_h} = c_h - z_h \leq 0
\]

Thus the criterion element for a nonbasic variable shows the reduction in the objective function that would result from forcing the nonbasic variable into the solution at a value of one. The criterion elements for slack variables not in the basic solution provide a second piece of information. Note from (48) that the \( i \)-th slack variable appears with a non-zero coefficient only in the \( i \)-th constraint. Thus \( x_{n+i} \) corresponds to \( a_{10} \). Suppose \( a_{10} \) decreases by a small amount. What is the effect on the maximum value of the objective function? The answer is

\[
(53) \quad \frac{\partial Z_0}{\partial a_{10}} = c_{n+i} - z_{n+i} \leq 0
\]

Thus \( c_{n+i} - z_{n+i} \) measures the amount by which net revenue would decline if the firm had one less unit of the \( i \)-th fixed resource.

**SOLUTION TO NUMERICAL EXAMPLE**

After adding slack variables to the constraints (41.1) through (41.4) of the numerical problem presented earlier, the constraints become:

- (corn) \( 40x_1 + 50x_2 + 35x_3 + x_4 = 11,000 \)
- (silage) \( 3.25x_1 + 0.72x_2 + 2.0x_3 + x_5 = 900 \)
- (hay) \( 0.11x_1 + 0.25x_2 + 0.3x_3 + x_6 = 300 \)
- (labor) \( 6.0x_1 + 6.4x_2 + 5.0x_3 + x_7 = 1,600 \)

The objective function becomes

\[
123.35x_1 + 112.66x_2 + 105.88x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7
\]
The values of the basic variables in the basic optimal feasible solution obtained from expression (49) are:

\[ x_{10} = 100 \]
\[ x_{30} = 200 \]
\[ x_{50} = 175 \]
\[ x_{60} = 229 \]

The values of 100 and 200 for \( x_{10} \) and \( x_{30} \) mean that 100 and 200 steers are produced by activities 1 and 3, respectively. \( x_5 \) is the slack variable in the silage constraint. Its value of 175(\( x_{50} \)) means that 175 of the 900 tons of silage are not used. The value of 229 for \( x_6 (x_{60} = 229) \) means that 229 of the 300 tons of hay are not used. The value of the objective function is \( Z_0 = 33,511 \). The maximum net income the firm can earn is $33,511. It can do this by raising 100 animals by activity 1 and 200 by activity 3.

The values of criterion elements for real variables, from expression (50) are:

\[ c_1 - z_1 = 0 \]
\[ c_2 - z_2 = -32.50 \]
\[ c_3 - z_3 = 0 \]

The values of the criterion elements for slack variables, from expression (51), are:

\[ c_4 - z_4 = -1.85 \]
\[ c_5 - z_5 = 0 \]
\[ c_6 - z_6 = 0 \]
\[ c_7 - z_7 = -8.21 \]

The interpretation of \( c_2 - z_2 = -32.50 \) is, as shown in expression (52), that if one steer were produced by activity 2, net income would fall from $33,511.
by the amount $32.50. According to (53), the interpretation of $c_4 - z_4 = -1.85$ is, that the firm's net income would have been $1.85 less than $33,511 if it had owned one bushel less of corn. If the amount of available labor had been one hour less than it was, the firm's net income would have been $8.21 less.

In this example, the feeder was assumed to have 11,000 bushels of corn available. If the feeder had only 9,000 bushels of corn available, his optimum feeding program would be much different. If the amount of corn available is reduced from 11,000 to 9,000 and the other parameters of the linear program are unchanged, the optimum feeding program is to feed 225 steers under activity 1 and no steers under activities 2 or 3. The resulting maximum net income is $27,753.75.

If the costs for veterinary services and medicine for a steer fed under activity 1 were $10 rather than $9.40, the net income per unit of activity 1 would be $0.60 less: $122.75 rather than $123.35. Changing $c_1$ in the original problem to $122.75 results in still a third optimum feeding program: feed 314 steers under activity 3 for a net income of $33,246.32.

These three examples show that the optimum feeding program is affected by the amounts of fixed resources that are available and by net revenues per unit of the various activities.

**JUSTIFICATION FOR IGNORING FIXED COSTS**

We need to justify our failure to consider fixed costs. The firm's total revenue is $r_{p,j} x_j$. The firm's total variable costs are $\sum_{j} \sum_{k} r_{k,j} x_j$. The linear program maximizes their difference, $Z$. Total fixed costs are the costs of the fixed inputs. Thus, letting $F = \text{total fixed costs}$, we may write $F = \sum_{i} f_{i} a_{i0}$ where $f_{i} = \text{fixed costs per unit of the } i\text{-th fixed input}$. Because the $a_{i0}$ are fixed, and prices of fixed inputs are constant, $F$ is a
constant. Let profit be \( \pi = Z - F \). The linear program selects values of \( x_j \) to maximize \( Z \). Because \( F \) is a constant (specifically, its value is independent of all \( x_j \)) it follows that the values of \( x_j \) that maximize \( Z \) are the values that maximize \( \pi \). After \( Z_0 \) is determined, maximum profit can be determined from \( \pi_0 = Z_0 - F \).

USE OF LINEAR PROGRAMMING TO FIND ECONOMIC VALUES

What would happen to the net income of the cattle feeder in the numerical example if average daily gain of steers were to be increased by 0.1 pound? Fewer days would be required to bring animals to market weight. This would reduce the labor required per animal in each activity. The values of 6.0, 6.4, and 5.0 hours per animal in the labor constraint would decline, perhaps to 5.8, 6.2, and 4.9 hours. Reducing the number of days animals are fed would also reduce power and fuel costs per unit for each activity, say from \$9.50, \$12.00 and \$7.22 to \$9.20, \$11.60, and \$7.12. This change increases the net revenue per unit of output from each activity. If we change the net revenues in the objective function and change the coefficients of the labor constraint in the original problem, and then solve the new problem, the difference between the maximum values of the two objective functions provides a measure of the effect on maximum net income of the change in average daily gain. This simple example presents the idea underlying the procedure to be presented in this section and to be applied in the next section.

Procedure

Our definition of economic value of a trait is "the amount by which maximum profit may be expected to increase for each unit of improvement in that trait in each animal, other traits remaining constant."

Expression (49) shows that the optimum level of output from the \( j \)-th activity, \( x_{j0} \), depends upon elements of \( B^{-1} \) and \( A_0 \). Elements of \( B^{-1} \), of course
depend upon elements of B. This dependence of \( x_j \) upon the parameters of the linear program can be expressed as

\[
x_j = g_j(a_{11}, a_{12}, \ldots, a_{21}, a_{22}, \ldots, a_{mn}, a_{10}, a_{20}, \ldots, a_{m0}, c_1, c_2, \ldots, c_n)
\]

The value of \( Z_0 \) depends upon \( x_j \) (or \( g_j \)) and upon the \( c_j \). Therefore, the dependence of \( Z_0 \) upon the parameters of the linear program can be expressed as

\[
(54) \quad Z_0 = Z(a_{11}, a_{12}, \ldots, a_{21}, a_{22}, \ldots, a_{mn}, a_{10}, a_{20}, \ldots, a_{m0}, c_1, c_2, \ldots, c_n)
\]

And the dependence of \( \pi_0 \) upon the parameters of the linear program can be expressed as

\[
\pi_0 = Z(a_{11}, a_{12}, \ldots, a_{mn}, a_{10}, a_{20}, \ldots, a_{m0}, c_1, c_2, \ldots, c_n) - F
\]

Some parameters of the linear program of the farm are functions of traits. These parameters are shown in the following equations as functions of the \( h \)-th trait,

\[
(55) \quad a_{ij} = \Theta_{ij}(G_h)
\]

\[
p_j = \Theta_j(G_h)
\]

\[
q_{kj} = \rho_{kj}(G_h)
\]

It is clear that before the linear program can be written, it is necessary to specify the traits of the livestock. If feed efficiency and average daily gain, for example, are not known, it is not possible to determine how much feed or how many days will be needed to raise an animal. If the number of days is not known, it is not possible to determine hours of labor used.

Suppose that a basic optimal feasible solution to the profit-maximizing linear program has been determined, and it is desired to determine economic value of the \( h \)-th trait. Let \( G_h \) be the value assigned to that trait in the original program, and suppose it changes by the amount \( dG_h \). We ask, "What
is the effect of this variation on $\pi_0$, i.e., what is the value of $d\pi_0/dG_h$?"

This change does not affect the amounts of any of the fixed resources available. It does not, e.g., increase the number of acres in the farm or the total building space. From (54) and (55), it follows that $d\pi_0/dG_h$ can be written as

$$
d\pi_0/dG_h = dZ_0/dG_h = \sum_j (\partial Z_0/\partial c_j)(\partial c_j/\partial G_h) + \sum_i \sum_j (\partial Z_0/\partial a_{ij})(\partial a_{ij}/\partial G_h)
$$

This shows that economic value can be computed without paying any attention to the fixed cost of an activity. If the change in $G_h$ does not cause the optimal feasible basis to change, it can be shown that this expression can be written as

$$
(56) \quad dZ_0/dG_h = \sum_{j=1}^{n} (\partial c_j/\partial G_h) + \sum_{i=1}^{m} \sum_{j=1}^{n} \left( c_{n+i} - z_{n+i} \right) x_{j} (\partial a_{ij}/\partial G_h)
$$

To obtain a measure compatible with the definition of economic value, (56) must be divided by the number of animals experiencing the genetic change $dG_h$. Let $N_{h0}$ be the number of animals in the basic optimal feasible solution that experience the change. Thus, if $G_h$ is backfat in hogs, $N_{h0}$ is the number of slaughter hogs marketed. A computable form for economic value of the $h$-th trait is, then,

$$
(57) \quad EV_h = (dZ_0/dG_h) / N_{h0}
$$

where $dZ_0/dG_h$ comes from (56).

If the change in $G_h$ does cause the optimal feasible basis to change, $EV_h$ can be computed from

$$
(58) \quad EV_h = (Z'_0 - Z_0) / (N'_{h0} + N_{h0}) / 2
$$

where $Z'_0$ is the maximum net revenue in the new linear program obtained from the original linear program by changing values of $c_j$ and $a_{ij}$ to conform with the new value of the $h$-th trait. In the new linear program, the objective function weights are $c_j + (\partial c_j/\partial G_h)dG_h$ and the new values at the input-output
coefficients in the constraints are \( a_{ij} + (\partial a_{ij}/\partial G_h) dG_h \). \( N'_{nh0} \) is the number of animals affected by the change in \( G_h \) in the new linear program.

A profit-maximization linear program of a livestock farm provides the basic data needed for computation of economic values. The only additional data needed are specifications of \( dG_h \) and values of \( \partial c_j/\partial G_h \) and \( \partial a_{ij}/\partial G_h \).

Application To Hogs

The procedure outlined in the previous section was used to measure economic values (EVs) of three heritable characteristics in swine: backfat (BF), feed efficiency (FE), and average daily gain (ADG). Results for three production conditions are summarized here. For further discussion see Gibson (1976) and Ladd and Gibson (1978).

The analysis covered the 22 month period from November 1, 1972 through August 1974. This allowed for two complete cycles of breeding, gestation, feeding, and marketing. Prices of outputs and variable inputs used in the analysis were monthly Iowa prices during this period.

Farm I

Farm I may farrow its own pigs, buy feeder pigs, or do both. Farrowing times are May, August, November, and February. Feeder pigs can be purchased in June, September, December, or March. Pigs can be fed to weights of 180, 200, 220, 240, or 260 pounds. The farm has a total of 40 different activities for producing market hogs. Pigs from any one of the 4 farrowings can be fed to any one of 5 market weights. Each combination of farrowing month and finished weight is one activity. The firm has 20 activities (4 x 5) that involve producing pigs it farrows. It has 20 more activities that involve feeding purchased feeder pigs to market weight. Females that farrow can be purchased or raised on the farm.

The farm's fixed resources are central farrowing facilities, growing-
finishing units, monthly family labor, and number of boars. The farm has a central farrowing house that is fully insulated and environmentally controlled, and that has a 25 sow capacity. It has partial confinement growing-finishing units sufficient to house 250 head of 220 pound market hogs during the summer, and has two boars.

Tables 2 through 5 summarize the amounts of fixed resources available to this swine farm. The 31 numbers in these tables are the values of the $a_{ij}$s in the constraints in expression (45). Each $a_{ij}$ in the constraints measures the amount of one fixed resource used to produce one market hog by one of the 40 activities defined in the preceding paragraph. Before the values of the $a_{ij}$ could be determined, it was necessary to select the values of the traits of the hogs to be grown. For example, to determine labor requirements for each activity, average daily gain needs to be specified so that the number of days that hogs will be on hand can be determined. Characteristics of hogs are summarized in Table 6.

The firm's purchased inputs are: all feed and feed additives, veterinary and medical expenses, fuel and power, feeder pigs and breeding stock purchased, and transportation of animals purchased or sold. Input-output coefficients were based on experience of typical mid-west swine operations and recommendations of the Iowa Agriculture Experiment Station and Cooperative Extension Service. Table 7 shows the computation of variable cost per unit of output for one activity. Average variable cost for activity $j$ is $\sum_{k} q_{kj}$ in expression (43).
### TABLE 2

Monthly fixed labor inputs during the 22 month period:

**Farm I**

<table>
<thead>
<tr>
<th>Row number</th>
<th>Month</th>
<th>Available hours</th>
<th>Row number</th>
<th>Month</th>
<th>Available hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>November 1972</td>
<td>160</td>
<td>12</td>
<td>October 1973</td>
<td>160</td>
</tr>
<tr>
<td>2</td>
<td>December 1972</td>
<td>196</td>
<td>13</td>
<td>November 1973</td>
<td>160</td>
</tr>
<tr>
<td>3</td>
<td>January 1973</td>
<td>216</td>
<td>14</td>
<td>December 1973</td>
<td>196</td>
</tr>
<tr>
<td>4</td>
<td>February 1973</td>
<td>192</td>
<td>15</td>
<td>January 1974</td>
<td>216</td>
</tr>
<tr>
<td>5</td>
<td>March 1973</td>
<td>198</td>
<td>16</td>
<td>February 1974</td>
<td>192</td>
</tr>
<tr>
<td>6</td>
<td>April 1973</td>
<td>160</td>
<td>17</td>
<td>March 1974</td>
<td>198</td>
</tr>
<tr>
<td>7</td>
<td>May 1973</td>
<td>160</td>
<td>18</td>
<td>April 1974</td>
<td>160</td>
</tr>
<tr>
<td>8</td>
<td>June 1973</td>
<td>160</td>
<td>19</td>
<td>May 1974</td>
<td>160</td>
</tr>
<tr>
<td>9</td>
<td>July 1973</td>
<td>216</td>
<td>20</td>
<td>June 1974</td>
<td>160</td>
</tr>
<tr>
<td>10</td>
<td>August 1973</td>
<td>208</td>
<td>21</td>
<td>July 1974</td>
<td>216</td>
</tr>
<tr>
<td>11</td>
<td>September 1973</td>
<td>168</td>
<td>22</td>
<td>August 1974</td>
<td>208</td>
</tr>
<tr>
<td>Farrowing number</td>
<td>Farrowing month</td>
<td>Number of sows</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>------------------</td>
<td>--------------------</td>
<td>----------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>May 1973</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>August 1973</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>November 1973</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>February 1974</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 4

Finishing capacity for market hogs: Farm I

<table>
<thead>
<tr>
<th>Confinement building</th>
<th>Market hog group</th>
<th>Number of square feed available</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>May 1973</td>
<td>3250</td>
</tr>
<tr>
<td>2</td>
<td>August 1973</td>
<td>3250</td>
</tr>
<tr>
<td>1</td>
<td>November 1973</td>
<td>3250</td>
</tr>
<tr>
<td>2</td>
<td>February 1974</td>
<td>3250</td>
</tr>
</tbody>
</table>
TABLE 5

Number of boars available: Farm I

<table>
<thead>
<tr>
<th>Boars</th>
<th>Number of boars</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>
TABLE 6

Assumed Phenotypic Measures of Market Hogs:
Herd Averages On Farms I, IA and II

<table>
<thead>
<tr>
<th>Trait</th>
<th>Weights (kg and lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>81.6 kg.</td>
</tr>
<tr>
<td>Backfat (cm)</td>
<td>3.30</td>
</tr>
<tr>
<td>(in)</td>
<td>1.3</td>
</tr>
<tr>
<td>Feed efficiency</td>
<td>3.4143</td>
</tr>
<tr>
<td>(kg. feed/kg. gain)</td>
<td>.6916</td>
</tr>
<tr>
<td>Avg. daily gain</td>
<td>1.5246</td>
</tr>
<tr>
<td>(lb. gain/day)</td>
<td></td>
</tr>
</tbody>
</table>

Table 7
Finding Average Variable Cost ($r_k q_{kj}$) for Activity 27 (j=27) in Farm I:
Feeding May Farrowed Pigs From 40 to 180 Pounds

<table>
<thead>
<tr>
<th>Variable Input</th>
<th>FE$^a$</th>
<th>TG</th>
<th>VFI $^b$</th>
<th>q$^c_{k27}$</th>
<th>r$^c_k$</th>
<th>$\Sigma r_k q_{kj}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Veterinary and medicine</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$1.90$</td>
</tr>
<tr>
<td>Power and fuel</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$1.00$</td>
</tr>
<tr>
<td>Miscellaneous</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$.20</td>
</tr>
<tr>
<td>Corn</td>
<td>3.4143</td>
<td>140 lbs.</td>
<td>29.182 bu.</td>
<td>6.9744 bu.</td>
<td>$2.20$</td>
<td>15.34</td>
</tr>
<tr>
<td>Soybean oilmeal</td>
<td>3.4143</td>
<td>140 lbs.</td>
<td>305.81 lbs.</td>
<td>73.0885 lbs.</td>
<td>.12</td>
<td>8.77</td>
</tr>
<tr>
<td>Limestone</td>
<td>3.4143</td>
<td>140 lbs.</td>
<td>15 lbs.</td>
<td>3.585 lbs.</td>
<td>.02</td>
<td>.07</td>
</tr>
<tr>
<td>Dicalcium phosphate</td>
<td>3.4143</td>
<td>140 lbs.</td>
<td>23 lbs.</td>
<td>5.497 lbs.</td>
<td>.10</td>
<td>.55</td>
</tr>
<tr>
<td>Salt</td>
<td>3.4143</td>
<td>140 lbs.</td>
<td>10 lbs.</td>
<td>2.39 lbs.</td>
<td>.025</td>
<td>.06</td>
</tr>
<tr>
<td>Trace mineral premix</td>
<td>3.4143</td>
<td>140 lbs.</td>
<td>2 lbs.</td>
<td>.478 lbs.</td>
<td>.10</td>
<td>.05</td>
</tr>
<tr>
<td>Vitamin premix</td>
<td>3.4143</td>
<td>140 lbs.</td>
<td>10 lbs.</td>
<td>2.39 lbs.</td>
<td>.60</td>
<td>1.43</td>
</tr>
<tr>
<td>Tylosin</td>
<td>3.4143</td>
<td>140 lbs.</td>
<td>20 gm.</td>
<td>4.78 gm.</td>
<td>.12</td>
<td>.57</td>
</tr>
<tr>
<td>Total (Σ)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$29.94$</td>
</tr>
</tbody>
</table>

$^a$ From Table 6.

$^b$ Variable feed intake. Basic 14% protein finishing ration from Life Cycle Swine Nutrition; Iowa State Agriculture and Home Economics Experiment Station, Pm-489 Rev. August 1974.

$^c$ Actual Iowa average prices during study period.
Farm IA

The only difference between Farms I and IA is that Farm IA had 140 hours of family labor available monthly in October, November, April, May and June, whereas Farm I had 160 hours.

Farm II

The differences between Farms I and II are these. Farm I can farrow four times a year and has two boars. Farm II can farrow twice a year and has one boar. Farm I has a central farrowing house. Farm II uses a pasture farrowing system.

Results

The term "initial solution" for each farm refers to the solution to the initial linear program. Farm I. Table 8 presents part of the initial optimal feasible solution: It shows amount of market hogs produced. This solution also called for purchasing feeder pigs and for marketing gilts that did not conceive, cull gilts and sows.

To determine EV of BF, FB was assumed to increase by .381 cm. (.15 in.) and to decrease by .381 cm. (.15 in.) For convenience these will be referred to as $1\sigma$ and $-1\sigma$ changes because 0.15 inch is approximately 1 standard deviation.

A change in BF does not affect any of the $a_{ij}$ and $q_{kj}$. That is, a change in BF alone does not affect the amount of any fixed input or of any variable input required to produce a kg of live hog. A change in BF affects only the $p_j$. Setting $dC_h = .381$ cm., computing resulting changes in $3c_j/3C_h$ and evaluating (56) yielded an EV for a $1\sigma$ change in BF of $.95 per hog. This is presented in the top row of table 10.

FE is measured as (kg of feed/kg of gain). According to its definition, EV of a trait is measured by varying only that trait; all other traits are held constant. To permit varying FE by itself and ADG by itself, to change FE we
Table 8
Levels of Marketings of Slaughter Hogs in Basic Optimal Feasible Solutions

<table>
<thead>
<tr>
<th>Farm</th>
<th>Activity</th>
<th>Amount marketed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(100 lb.)</td>
</tr>
<tr>
<td>I</td>
<td>Market May farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>220 pound (99.8 kg.) hogs</td>
<td>590</td>
</tr>
<tr>
<td></td>
<td>Market August farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>260 pound (117.9 kg.) hogs</td>
<td>697</td>
</tr>
<tr>
<td></td>
<td>Market November farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200 pound (99.8 kg.) hogs</td>
<td>387</td>
</tr>
<tr>
<td></td>
<td>Market February farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>260 pound (117.9 kg.) hogs</td>
<td>598</td>
</tr>
<tr>
<td>IA</td>
<td>Market May farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>220 pound (99.8 kg.) hogs</td>
<td>590</td>
</tr>
<tr>
<td></td>
<td>Market August farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>260 pound (117.9 kg.) hogs</td>
<td>603</td>
</tr>
<tr>
<td></td>
<td>Market November farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200 pound (90.7 kg.) hogs</td>
<td>360</td>
</tr>
<tr>
<td></td>
<td>Market February farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>260 pound (117.9 kg.) hogs</td>
<td>598</td>
</tr>
<tr>
<td>II</td>
<td>Market April farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200 pound (90.7 kg.) hogs</td>
<td>169</td>
</tr>
<tr>
<td></td>
<td>Market April farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>260 pound (117.9 kg.) hogs</td>
<td>238</td>
</tr>
<tr>
<td></td>
<td>Market October farrowed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200 pound (90.7 kg.) hogs</td>
<td>387</td>
</tr>
</tbody>
</table>
### Table 9
Computing Change in Average Variable Cost and Per Unit Net Revenue
For Activity 27 as Feed Efficiency Rises by 0.15 lb. of Feed Per lb. of Gain<sup>a/</sup>

<table>
<thead>
<tr>
<th>Feed input</th>
<th>Change of $q_{k27}$</th>
<th>$r_k$</th>
<th>Change in $\sum r_k q_{k27}$</th>
<th>Change of $c_{27}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corn</td>
<td>+0.3064 bu.</td>
<td>$2.20$</td>
<td></td>
<td>$+.67$</td>
</tr>
<tr>
<td>Soybean oilmeal</td>
<td>+3.2110 lbs.</td>
<td>.12</td>
<td></td>
<td>+.39</td>
</tr>
<tr>
<td>Limestone</td>
<td>+0.1575 lbs.</td>
<td>.02</td>
<td></td>
<td>+.01</td>
</tr>
<tr>
<td>Dicalcium phosphate</td>
<td>+0.2415 lbs.</td>
<td>.10</td>
<td></td>
<td>+.02</td>
</tr>
<tr>
<td>Salt</td>
<td>+0.1050 lbs.</td>
<td>.025</td>
<td></td>
<td>+.00</td>
</tr>
<tr>
<td>Trace mineral premix</td>
<td>+0.0210 lbs.</td>
<td>.10</td>
<td></td>
<td>+.00</td>
</tr>
<tr>
<td>Vitamin previx</td>
<td>+0.1050 lbs.</td>
<td>.60</td>
<td></td>
<td>+.06</td>
</tr>
<tr>
<td>Tylosin</td>
<td>+0.2100 gm.</td>
<td>.12</td>
<td></td>
<td>+.03</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td>$+1.18$</td>
</tr>
</tbody>
</table>

<sup>a/</sup>Compare with Table 7, showing computation of $\sum r_k q_{k27}$.
changed kg of feed fed, and to change ADG we varied number of days animals are fed. In both cases we kept total gain (and possible market weights) unchanged.

The only parameters of the linear program whose values are changed by the change in FE are the \( q_{kj} \) that measure quantities of feed needed to produce 1 kg. (or lb.) of market hogs. The \( p_j \) are unchanged because BF if unchanged. Weights at which hogs can be marketed are assumed to be unchanged from the initial problem. Therefore, quantities of the fixed resources (labor, space and boar services) required to produce 1 kg. of market hog are not changed. This means that none of the \( a_{ij} \) is changed by a change in FE. Table 9 presents the computation of the effect of raising FE by \( \sigma \) on average variable cost and net revenue per unit of output for activity 27.

FE was assumed to change by \( .15 \) and \( -.15 \) kg. of feed per kg. of gain. For convenience these will be referred to as \( \sigma \) and \( -\sigma \) changes in FE because .15 kg. of feed per kg. of gain is approximately 1 standard deviation. Setting \( dG_h = .15 \) kg. of feed per kg. of gain, determining the appropriate values of \( \partial q_{kj}/\partial G_h \) and \( \partial c_j/\partial G_h \), and substituting these values into (56) yielded an EV of \(-$1.44\) per market hog, as shown in table 10.

ADG is defined as an animal's kg. of gain per day fed. Kilograms of gain per day fed were assumed to change by \( .068 \) and \( -.068 \) kg. per day (\( .15 \) and \( -.15 \) lb. per day) for each market animal. For convenience these will be referred to as \( \sigma \) and \( -\sigma \) changes because .068 change in kg. of gain per day is approximately 1 standard deviation. Changing ADG by changing number of days animals were fed lead to changes in two sets of \( a_{ij} \) coefficients (those for labor and finishing space) and in values of some \( c_j \). Reducing the number of days that hogs are fed reduces the amount of labor needed to care for the hogs. A hog needs more space in the summer than in the fall and spring and needs more space in the fall and spring than in the winter. Because of this seasonal variation in
space requirements and because farrowing dates are assumed fixed, shortening the number of days needed to raise an animal to a fixed market weight affects the finishing space needed per hog. Reducing the number of days hogs are fed reduces power and fuel costs and hence affects the \( c_j \).

Setting \( d_{c_h} = .068 \) kg. of gain per day fed, determining appropriate values of \( \partial a_{ij} / \partial c_h \) and \( \partial c_{ij} / \partial c_h \) and substituting these values into (56) yielded an EV of $.09 per market hog, as shown in table 10. Other values are also shown in this table.

The optimum farm plan under the -.068 change in ADG was different from the optimum farm plan in the initial solution. Changing ADG by -.068 reduced the number of May farrowed 220 pound hogs that were produced by three-fourths (from 590 to 153 cwt.) and increased marketings of May farrowed 260 pound hogs from zero to 475 cwt.

**Farm IA.** The initial optimal feasible solution for Farm IA was much like that for Farm I. EVs were computed in the same way for Farm IA as for Farm I. EVs for changes in BF and FE were the same in the two models. Whereas EVs of \( \lambda \) and \(-\lambda\) changes in ADG were $.09 and -.21 for Farm I, they were $.94 and -$.1.02 for Farm IA. Changes in ADG brought about by changing the number of days hogs are on feed (and consequently changing labor needs) has a greater affect on the net income of the farm with the smaller labor supply.

**Farm II.** The initial optimal feasible solution for Farm II differed substantially from that for Farm I: it called for marketing only hogs of 81.6 and 90.6 kg. The economic values derived from Farm II are smaller in absolute value than those derived from Farm I.

The rank orders of EVs are the same in Farms I and II. For changes of \( \lambda \), for example, EV is largest for ADG, and smallest for FE in both Farms I and II. Although the rank orders are the same for both farms in each column, the ratios among the EVs are quite different. In the \( \lambda \) column, for example,
the ratio of BF EV to ADG EV is -16 for Farm I but -6.9 for Farm II; the ratio of FE EV to ADG EV is -10.6 for Farm I but -4.6 for Farm II. Within each farm, the ratios among the EVs also vary between the 1o and -1o columns.

Some of the genetic changes caused changes in the weights of hogs marketed. The initial solution for Farm II called for marketing 81.6 and 90.7 kg. market hogs. In the solutions for 10 decreases in BF and FE, 81.6, 90.7 and 117.9 kg. hogs were sold.

**USING ECONOMIC VALUES DETERMINED BY LINEAR PROGRAMMING**

Suppose that a swine breeder employed by a public agency believed that Farm I was typical of \( \alpha_1 \) proportion of the hog farms in his state, Farm IA was typical of \( \alpha_2 \) proportion, and Farm II was typical of \( \alpha_3 \) proportion \((\alpha_1 + \alpha_2 + \alpha_3 = 1)\). And suppose he further believed that prices during the study period were typical of future prices. Then he would proceed as follows to select hogs for breeding programs:

(a) For each combination of farm and trait, average the absolute values of the EVs in table 3,

(b) Express aggregate genotype, equation (1) as

\[
H = [-0.955\alpha_1 - 0.955\alpha_2 - 0.79\alpha_3] \text{BF} + [-1.44\alpha_1 - 1.44\alpha_2 - 1.155\alpha_3] \text{FE}
\]

\[
+ [0.15\alpha_1 + 0.98\alpha_2 + 0.09\alpha_3] \text{ADG}
\]

and use the coefficients in brackets as elements of the vector \( \mathbf{A} \),

(c) Use this \( \mathbf{A} \) vector and his knowledge of \( \mathbf{P}'\mathbf{P} \) and \( \mathbf{G}'\mathbf{G} \) to determine \( \mathbf{E} \) from (2),

(d) Substitute this \( \mathbf{E} \) into (1) along with phenotypic measures of traits of available potential breeding stock and

(e) Select animals with the highest values of the index for his swine improvement program.
<table>
<thead>
<tr>
<th>Farm</th>
<th>Trait</th>
<th>$dG_h$ = genetic change in trait</th>
</tr>
</thead>
<tbody>
<tr>
<td>I, IA</td>
<td>Backfat</td>
<td>$-0.95$, $0.96$</td>
</tr>
<tr>
<td>I, IA</td>
<td>Feed efficiency</td>
<td>$-1.44$, $1.44$</td>
</tr>
<tr>
<td>I</td>
<td>Average daily gain</td>
<td>$0.09$, $-0.21_{a/}$</td>
</tr>
<tr>
<td>IA</td>
<td>Average daily gain</td>
<td>$0.94$, $-1.02$</td>
</tr>
<tr>
<td>II</td>
<td>Backfat</td>
<td>$-0.77$, $0.81_{a/}$</td>
</tr>
<tr>
<td>II</td>
<td>Feed efficiency</td>
<td>$-1.12$, $1.19_{a/}$</td>
</tr>
<tr>
<td>II</td>
<td>Average daily gain</td>
<td>$0.08$, $-0.10$</td>
</tr>
</tbody>
</table>

$\text{a/}^\text{These values were obtained from equation (58). Others were obtained from (57).}$
IMPLICATIONS OF NEOCLASSICAL AND LINEAR PROGRAMMING MODELS

Five different methods that can be used to estimate EVs of traits have been presented: (a) the methods of budgeting, gross-revenue, and multiple regression that have been used by animal scientists, (b) neoclassical firm analysis, and (c) linear programming. The next section of the paper will present some comparisons among these methods. This section will present some insights into firm behavior that are provided by economic analyses. These insights are derived from the neoclassical model of a single product firm presented previously, a neoclassical model of a multi-product firm, and from the linear programming model of a firm. These insights will be used in the next section.

Variation In Output Prices

1. A change in the price of an output tends to have the effect of causing the amount of that output produced to change in the same direction as price changes. For demonstration of this, see expression (36) and the numerical example of a linear program of a beef feeding operation. In that example, reducing the net revenue per unit of output of activity 1 from $123.35 to $122.75 by increasing average variable cost for activity 1 by $0.60 reduced the number of cattle fed under activity 1 from 100 to zero. The same effect

The phrase "a change in a parameter tends to have the effect of" or "tends to" is used frequently in this section, and it has a specific meaning. It means "a change in a parameter has the effect of" or "a change in a parameter does" for a firm whose production conditions are best described by a continuous production function having continuous derivatives. It means "a change in the parameter has the effect of" or "a change in a parameter does" for a firm whose production conditions are best described by activities, as in linear programming, provided the change in the parameter is large enough.
would have been obtained by leaving average variable cost unchanged and reducing the selling price of an animal fed according to activity 1 by $0.60.

2. A change in the selling price of one output of a multi-product firm tends to affect the levels of outputs of other products. It may reduce the levels of outputs of some products and increase the levels of outputs of others. In the numerical example of a beef feeding operation; reducing price of an animal produced by activity 1 by $0.60 caused the number of animals produced by activity 3 to rise from 200 to 314.

3. A change in the price of an output tends to cause a change in the same direction in the quantities of at least some variable inputs in its production. See expression (37).

4. A change in the price of an output tends to cause a change in the amounts of some variable inputs used per unit of output. This was illustrated by reducing \( p_1 \) by $0.60 per animal in the beef feeding linear program. This change in \( p_1 \) reduced the optimum number of animals raised by activity 1 (450 pound feeder calves raised on a high roughage ration) from 100 to zero and increased the optimum number of animals raised by activity 3 (650 pound yearlings raised on a medium roughage, medium grain ration) from 200 to 314.

Variations in Prices of Variable Inputs

5. A change in the price of a variable input tends to reduce the level of output of products using that input. See expression (38). The beef feeding example showed that reducing \( c_j \) by increasing the average variable cost of activity \( j \) tends to reduce the optimum level of that activity. An increase in average variable cost of one activity might be caused by an increase in price of a variable input that is not used in any other activity.

6. A change in the price of a variable input tends to reduce the amount of that input used. See expression (39) and the discussion in the preceding paragraph.
7. An increase in the price of a variable input also tends to reduce the amount of that input that is used in each unit of product and to increase the amounts of some other inputs that are used for each unit of output. In other words, an increase in the price of a variable input tends to affect what a producer does (how much he produces of each product) and also how he does it (the amount of each input he uses to produce one unit of output).

Variation in Livestock Traits

8. A change in one trait of one class of livestock tends to have a number of direct and indirect effects on the firm's maximum profit. (a) It can affect the price received for that class of livestock. (b) It can affect the optimum level of output of that class of livestock. (c) It can also affect the optimum levels of output of other products. (d) It can affect the amounts of variable inputs used per unit of output of the affected class of livestock. (e) It can affect the amounts of variable inputs used per unit of output of other products.

Some of these effects are demonstrated in expression (28). Others are observed in the previous application of linear programming to obtain economic values.

Suppose that a corn-hog-beef farmer produces most of the corn he uses for feed. He may buy additional corn, or he may sell corn. Assume ADG for hogs rises. Then the labor needed per hog falls. The farmer has a number of options open to him for using the labor that is not needed for producing the number of hogs he formerly produced. It might be that it would be profitable for him to use this labor to produce more corn and also to greatly increase his hog production. Suppose he cannot do this because he has little additional space that he can use for increasing hog production. The farmer can now use some of the labor saved to produce more corn and use some of the labor saved
and use some of the additional corn to produce a few more hogs and use some of the labor and corn to feed more cattle. He may also produce his hogs by using more home-grown corn and less purchased corn per hog. He can now change the ration (activity) used in feeding cattle to include more corn and less roughage and less supplement.

9. A change in a trait that does not affect the production function or input-output coefficients but does affect the selling price of output tends to affect optimum levels of inputs and output. If a change in a trait $dG_h$ does not affect the production function then $f_{G_h} = f_{IG_h} = 0$ (in the notation of (23) through (28)). Then the effects of varying $dG_h$ can be obtained from the following expression

$$
\begin{pmatrix}
0 & -1 & F' \\
-1 & 0 & 0 \\
F & 0 & p_0 H
\end{pmatrix}
\begin{pmatrix}
\partial q_0 / \partial G_h \\
\partial q_0 / \partial G_h \\
\vdots \\
\partial q_n / \partial G_h
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
-\partial p_0 / \partial G_h \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

Note how closely this expression resembles (35). By making use of this resemblance, it can be seen from (36), then, that

$$\partial q_0 / \partial G_h = - (\partial p_0 / \partial G_h) F' H^{-1} F/p_0 = (\partial q_0 / \partial p_0) (\partial p_0 / \partial G_h) > 0$$

And also, from the argument following (36) that

$$\partial q_j / \partial G_h = - (\partial p_0 / \partial G_h) H_j^{-1} F/p_0 = (\partial q_j / \partial p_0) (\partial p_0 / \partial G_h) < 0$$

Hence from (37), it follows that

$$\partial q_j / \partial G_h > 0 \text{ for some } j \neq 0$$

Then, the partial derivative of $\pi$ with respect to $G_h$ becomes

$$\partial \pi / \partial G_h = q_0 (\partial p_0 / \partial G_h) + p_0 (\partial q_0 / \partial p_0) (\partial p_0 / \partial G_h) - \sum_{i=1}^{n} p_i (\partial q_i / \partial p_0) (\partial p_0 / \partial G_h)$$
10. A change in technology affecting one product (or activity) affects EVs of traits of other products. Consider a swine farm that is like Farm I studied earlier except it grows all the corn it will feed instead of buying it. If the per acre yield of the corn it plants is increased genetically or if a new, bigger tractor permits it to do its field work more quickly, then EVs of BF, FE and ADG may be different from those presented earlier for Farm I. Or, refer to expression (28) and suppose that the firm replaces some old machinery with some new machinery. This change in machinery may affect the production function and cause the values of $f_i$, $f_{ij}$, $f_h$ and $f_{ih}$ to be different. The result is that the value of (28) is different because the values of $D_{h0}$ and $D'_{hi}$ are different, and the value of (27) is different because the values of $D_{h0}$ and $D_{hi}$ are different.

11. EV of a trait is affected by prices of outputs and variable inputs. Two effects of input and output prices can be seen in (27) and (28). (a) The values of $p_0$ and $p_i$ appear explicitly on the right-hand side of (28). (b) From expression (5), it follows that $p_0 = \lambda$. From expression (6), then, $f_i = p_i/\lambda = p_i/p_0$. It therefore follows that $D$, $D_{h0}$, $D_{hi}$, $D'_{h0}$ and $D'_{hi}$ are all functions of input and output prices.

12. To compute EVs for traits on a multi-product farm, average fixed costs should be ignored.

The last paragraph preceding the section entitled Use of Linear Programming To Find Economic Values demonstrated that it is permissible to ignore fixed costs in a linear programming analysis of a profit-maximizing firm. A slight extension of that argument shows why it is desirable to do so. As shown in that discussion, a firm's profit is

$$\pi = Z - F = \sum_{j} c_j x_j - F$$
where total fixed cost \( F \) is a fixed constant. Suppose fixed costs are allocated among activities and let \( F_j \) be the total amount of fixed cost allocated to the \( j \)-th activity. Then \( F_j/x_j \) is the average fixed cost of the \( j \)-th activity. And the firm's profit can be expressed

\[
\pi = \sum (c_j - F_j/x_j)x_j
\]

The firm's profit maximizing problem then is to find the nonnegative values of \( x_j \) that maximize this expression. But the value of each \( c_j - F_j/x_j \) must be determined before the profit-maximizing analysis is carried out. If, however, we can specify each \( F_j \) and \( x_j \) (and therefore \( F_j/x_j \)) before carrying out the analysis, the analysis is not necessary because the solution is already specified. It consists of the values of \( x_j \) used in computing the \( F_j/x_j \). If the analysis is carried out anyway, the values of \( x_j \) obtained in the solution may not be (and in most applications will not be) the same as the values used to determine \( F_j/x_j \). Hence, the solution of the problem will be inconsistent with the data of the problem.

To remove this inconsistency we could carry out an iterative procedure. Specify \( F_j \) and \( x_j \) and solve the problem. Adjust values of \( F_j \) and \( x_j \) and solve again. Continue adjusting values of \( F_j \) and \( x_j \) and solving until the values of \( x_j \) used in computing average fixed cost equal the solution values of \( x_j \).

Three comments on this procedure are relevant. (a) It can require a great deal of work to determine values of the \( F_j \). (b) A number of linear programs (or other numerical analyses) must be solved. (c) The final values of the \( x_j \)s will be the same as those determined by ignoring fixed costs.

13. A change in a trait of livestock may change the optimum grades or weights of livestock to be produced or may change the optimum activities for producing animals of a given grade and weight.
Optimum or Efficient Operation

Consideration of the preceding implications shows that the terms "optimum" or "efficient" (as in "optimum farm plan or "efficient feeding program") have specific and highly restricted meaning. (The discussion will be in terms of the linear programming model, but the same conclusions can be drawn from neoclassical models.)

14. Firstly, an optimum farm plan or efficient program is conditional upon the constraints specified. Reducing the amount of corn available to the cattle feeder in the earlier numerical example changed the profit-maximizing solution. Changing the available amounts of family labor between Farm I and Farm II reduced the numbers of August farrowed and November farrowed pigs marketed.

If two farms have the same stocks of fixed resources, but the operators differ in the effectiveness with which they use the resources, the basic optimal feasible solutions may be different for the two farms. For example, values of $a_{1j}$ and $a_{2j}$ might be small and values of $a_{3j}$ and $a_{4j}$ large for one farmer, whereas the reverse is true for a neighbor.

15. Secondly, an optimum or efficient program is conditional upon the prices of outputs and variable inputs. Increasing the selling price of one product or reducing the average variable cost of producing the product makes it profitable to produce more of that product and produce less of other products.

16. An optimum or efficient plan is conditional upon the activities or products considered. The swine farms analyzed to compute EVs had the option of feeding animals to 180, 200, 220, 240 or 260 pounds. Profit maximizing solutions for Farms I and II called for marketing some hogs at 260 pounds and some at other weights. If the operators of these farms did not want to raise
260 pound hogs, the activities of raising 260 pound hogs would be removed from the linear program and the profit maximizing solution would be different. If the swine enterprises were part of a diversified farming operation that involved beef cattle, the profit-maximizing solution in a time of high fed cattle prices and low hog prices would involve much cattle feeding and little hog production.

17. An optimum farm program or plan is affected by the objectives of the manager. The neoclassical and linear programming models presented earlier assumed his objective to be profit-maximization. In the cattle feeding example, the objective was to maximize net revenue, which was

\[ 123.35x_1 + 112.66x_2 + 105.88x_3 \]

One constraint was a labor constraint. It was

\[ 6.0x_1 + 6.4x_2 + 5.0x_3 \leq 1,600 \]

A farm operator does not have to be a profit-maximizer. His desire might be to earn a satisfactory level of income with a minimum amount of work. Let \( S \) represent the specified amount of income that he wants to earn. Then his objective would be to minimize

\[ 6.0x_1 + 6.4x_2 + 5.0x_3 \]

The values of \( x_1 \), \( x_2 \) and \( x_3 \) would still have to satisfy the constraints (41.1), (41.2) and (41.3) of the original problem. They would also have to satisfy the constraint

\[ 123.35x_1 + 112.66x_2 + 105.88x_3 \geq S \]

What has happened here is that one constraint and the objective function have been interchanged. The basic optimal feasible solution for this farm operation might well be quite different from the basic optimal feasible solution to the original problem.

18. Finally, an optimum plan can call for raising animals to several
different weights, or grades, or for raising different lots of animals by
different rations. Profit maximizing solutions for all three swine farms
called for marketing hogs at two or more different weights. The original
cattle feeding numerical example called for feeding some animals from 450
to 1,050 pounds on a high-roughage ration and for feeding some animals from
650 to 1,100 pounds on a medium roughage, medium grain ration.

The basic thrust of these 5 implications is this. To speak of "the
optimum way" or "the efficient way" implies that one is discussing "the (only)
optimum way or "the (only) efficient way." To speak in this manner is not
informative unless one also specifies: (a) the constraints included in his
analysis, (b) the activities included, (c) prices, and (d) the objective
function. It can actually be misleading to speak in this way because "the
optimum way" can, in reality, consist of several alternatives, as several
weights of hogs.

COMPARISON OF DIFFERENT METHODS OF MEASURING ECONOMIC VALUES

The neoclassical model of the firm will not be considered here. Although
this model provides valuable insight into firm behavior, it is of limited use-
fulness for empirical analysis: generally much less useful than the linear
programming model.

One obvious feature of linear programming is the large amount of inform-
ation needed to compute the parameters of the linear program (the $c_j, a_{ij}$
and $a_{10}$) and the changes in these parameters resulting from genetic change
(the $\partial c_j / \partial G_h$ and $\partial a_{ij} / \partial G_h$). A number of "tricks" are available to simplify
the organization of information for the linear program; see Beneke and Winter-
boer (1973) for discussion of these. The multiple regression also requires
collection of a substantial body of data.

A major strength of linear programming is its ability to include more
than one production activity for growing and finishing animals and to allow
the procedure to determine which combination of these activities provides
the maximum profit. On Farms I, IA and II feeder pigs were purchased, or
farrowed on the farm, or both, and hogs could be raised to any one or all of
five different weights. In all solutions, profit maximization required that
more than one weight of hogs be marketed.

This feature of our model relates to a problem Harris (1970, p. 862)
raised when he wrote, "If a constant slaughter weight and/or age is considered, it should probably be the optimum weight and/or age from economic considerations for the genetic group under consideration." These results show that "the (unique) optimum weight and/or age" may not exist. The optimum farm plan may call for several weights or ages or grades of animals to be marketed. The solutions for Farms I and II also showed that the farrowing times and selling weights that maximize profit depend upon the genetic composition of the animals. This is consistent with Harris' position.

The linear programming method is consistent with each of the 16 implications in the two preceding sections. This is obviously true, because linear programming was used in deriving these implications.

The budgeting and gross revenue methods have two important advantages: simplicity and modest data requirements. They also have some limitations.

When the budgeting method is used, fixed costs should be ignored (see implication 12). Including fixed costs will make additional work but have no other consequence in only one circumstance: when all animals affected by genetic change are raised to the same grade and weight by the same activity (see implication 18), when the genetic change affects neither the activity used nor the number of animals raised, (see implications 9 and 13), and when levels of output of other products and activities used to produce them are not indirectly
affected (see implication 8) by the genetic change.

Implications 13 through 18 are relevant to the selection of the activity or production method to be budgeted. The usual budgeting procedure is to select one activity and budget costs and revenue for it alone. According to implication 13, the genetic change may cause a producer to select a different activity. According to 14 through 18, one producer may use more than one grade and weight. And the combination of activities selected can be expected to vary among producers and over time.

Average variable cost of a product or an activity is a function of input prices and input-output coefficients \( (q_{kj} \text{ in the linear programming model}). This follows from implications 4 and 7 and from the method of constructing average variable cost in the linear program. This implies that to budget variable cost for an activity, one should budget inputs used \( (q_{kj}) \) and input prices \( (r_k) \) separately and multiply them together to obtain variable cost. By doing this, budgeted costs can accurately be adjusted to reflect changes in input prices or changes in input-output coefficients.

Suppose \( V_j \) represents average variable cost for activity \( j \). Then \( V_j = \sum_{k} r_k q_{kj} \). Assume every \( r_k \) and \( q_{kj} \) is known at one point in time. Then \( V_j \) can easily be computed for this same point in time. Suppose that at a later date various \( r_k \)s have changed by different amounts or some \( q_{kj} \)s have changed. The new value of \( V_j \) can easily be computed by using new values of the \( r_k \)s and \( q_{kj} \)s. If, however, the values of the \( r_k \)s and \( q_{kj} \)s were not known at the initial time, and all that was known was \( V_j \), it will be difficult to accurately determine the new value of \( V_j \).

According to implication 11, \( EV \) is a function of input and output prices. Constructing \( V_j \) by determining the \( r_k \)s and \( q_{kj} \)s individually takes account of the impact of input prices on the activity selected. But it does not take
account of the indirect impacts that result from changes in average variable costs of other activities brought about by these same input price changes.

The budgeting method, as pointed out earlier, assumes that the genetic change will not affect the optimum method of production or activity. It assumes that the only affect of the genetic change is to change the price received for the product. Even if the first assumption is true, the second can be false. This is shown by expression (59) in implication 9. The usual application of budgeting assumes that the values of $\partial q_0/\partial p_0$ and $\partial q_1/\partial p_0$ in that expression are zero.

In summary, application of the budgeting method is limited by implications 4, 7, 8, 9, 11, 12, 13, and 14 through 18. Whether it does or does not lead to error to ignore any one of these implications in a particular application is a question that can only be answered by empirical investigation. For example, it might be that a firm uses two different activities to produce two different grades and weights of animals, but that assuming that the firm uses only one of these activities (or uses an average of the two) leads to negligible error in the results obtained by budgeting.

Implication 10 is also a limitation. EV of one livestock trait may change over time because of changes in cropping technology or genetic changes in other livestock.

The question of the relevance of these implications for the gross revenue approach can be quickly covered. The same implications that apply to and limit the usefulness of the budgeting approach also apply to and limit the usefulness of the gross revenue approach: implications 4, 7, 8, 9, 10, 11, 12, 13 and 14 through 18. The gross revenue approach is a special case of the budgeting approach: the former assumes that the only effect of a genetic change is to affect the price received for the commodity. Even if the genetic
change has no effect on optimum combination of activities (i.e., has no indirect effects on other activities or products), it still can have indirect effects on profit through its effect on level of output and on combination of inputs used to produce the livestock affected by the genetic change. This is shown by expression (59) in implication 9. The gross revenue method assumes that values of $\frac{\partial q_0}{\partial p_0}$ and $\frac{\partial q_i}{\partial p_0}$ in that expression are all zero.

Equation (27) expresses the effect on maximum profit of simultaneous variation of genetic values of several traits. It is reproduced here for convenience:

$$ (27) \frac{d\pi}{dG} = \sum (p_0D_{h0}/D - \sum p_iD_{hi}/D)dG_h $$

Use of multiple regression involves estimation of the $\beta_h$ in the regression equation

$$ \pi = \sum \beta_h p_h $$

from which is obtained

$$ (60) \frac{d\pi}{dP} = \sum \beta_h dP_h $$

The statistical model assumes each $\beta_h$ is a constant. But, if $\beta_h$ is an estimate of $EV_h$, $\beta_h$ is an estimate of $(p_0D_{h0}/D - \sum p_iD_{hi}/D)$, and $\beta_h$ is not an estimate of a constant but of a variable whose value depends upon prices. See especially implication 11. Because values of $D$, $D_{h0}$, and $D_{hi}$ are also dependent upon the production function (or choices of activities), $\beta_h$ also depends upon the method of production used. And this method depends upon many factors: see implications 4, 7, 13 and 14 through 18.

Now for a purely statistical comment. Note that (27) and (60) do not contain the same differentials. The first contains $dG_h$, the second $dP_h$. But in general $dG_h$ need not equal $dP_h$ and $G_h$ need not equal $P_h$. For the moment, ignore the variability of $(p_0D_{h0}/D - \sum p_iP_{hi}/P)$ and assume this is constant and
equals $\alpha_h$. Then, because $\alpha_h$ is intended to equal $\partial^2 \eta/\partial \theta_h$, the equation whose coefficients we want to know is

$$\pi = \sum \alpha_h g_h$$

But because genotypic values are not known the equation estimated is

$$(61) \pi = \sum \beta_h p_h$$

But $p_h = g_h - e_h$, so

$$\pi = \sum \beta_h (g_h - e_h) = \sum \beta_h g_h - \sum \beta_h e_h$$

The independent variables ($p_h$) contain errors of measurement, $e_h$. Least-squares estimates of $\beta_h$ are therefore biased and inconsistent estimates of $\alpha_h$. Various statisticians have investigated the question of development of consistent measures of coefficients when data contain errors of measurement.

Warren, White and Fuller (1974) recently presented a method that could be applied to contain consistent estimates of the $\alpha_h$ when environmental variances or covariances are known. In applying their procedure, these variances and covariances become the variances and covariances of the errors-in-variables or measurement errors.
REFERENCES


