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Bifurcation of periodic solutions of singularly perturbed delay differential equation

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Bifurcation of periodic solutions of singularly perturbed delay differential equation

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by

Abdullah Jamil Tamraz

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CHAPTER I. INTRODUCTION

Consider the singularly perturbed delay differential equation

(1.1) \[ \varepsilon \dot{x}(t) = -x(t) + f(x(t-1), \mu), \quad t \geq 0 \]

where \( \varepsilon > 0 \) is a small parameter and \( f \) is a nonlinear function depending on a parameter \( \mu \).

Recently, equations of the form (1.1) have occurred in physical applications. For example, the equation that describes an optically bistable device is

\[ \varepsilon \dot{x}(t) = -x(t) + \mu[1 - \sin x(t-1)]. \]

In the study of evolutionary biology, Wazewska-Czyzewska and Lasota [16] showed that the equation that describes the production of red blood cells is

\[ \varepsilon \dot{x}(t) = -x(t) + \varepsilon \mu x(t-1) \sin x(t-1). \]

Other examples can be found in Glass and Mackey [5] and Hoppensteadt [10].

Chow and Mallet-Paret [2] studied equation (1.1), they showed that Hopf bifurcation can occur for certain parametric values and they studied certain square wave solutions using perturbation methods. Mallet-Paret [12] studied equation (1.1) for a function \( f \) satisfying certain conditions with \( \mu \) fixed, he gave a result about the Hopf bifurcation and global continuation of periodic solutions. Ruiz-Claeyssen and Cockburn [15] investigated the delay differential equation
\[ \dot{x}(t) = g(x(t), x(t-1), a) \]

with \( a \) a real parameter and \( g \) a smooth real function with \( g(0, 0, a) = 0 \). Hopf bifurcation is assumed to occur at \( a = 0 \). They determined the Hopf bifurcation curves by studying the roots of the characteristic equation of the linearized problem.

If
\[
(1.2) \quad x(\theta) = \phi(\theta), \quad -1 \leq \theta \leq 0,
\]
then for \( \epsilon > 0 \), equation (1.1) determines a unique solution \( x(t) \) for all \( t \geq 0 \). This is found simply by integrating the equation
\[
(1.3) \quad \epsilon \dot{x}(t) = -x(t) + g(t), \quad 0 \leq t \leq 1,
\]
\[ x(0) = \phi(0) \]
where \( g(t) = f(x(t-1), \mu) \), to obtain \( x(t) \) for \( 0 \leq t \leq 1 \). Successive integrations determine \( x(t) \) for all \( t \geq 0 \). See the method of steps in Driver [4].

Setting \( \epsilon = 0 \) in equation (1.1), we get
\[ x(t) = f(x(t-1), \mu). \]
or
\[
(1.4) \quad x_{n+1} = f(x_n, \mu)
\]
where \( x_n = x(n) \).

The following theorem explains the relation between the singularly perturbed delay differential equation (1.1) and the reduced problem (1.4). We denote the composition of two functions \( f \circ g(x) = f(g(x)) \). The \( n \)-fold composition of \( f \) with itself is denoted by
\[ f^n(x) = f \circ f \circ \ldots \circ f(x). \]

**Theorem 1.5**

For fixed \( n \), let \( f_\mu(x) = f(x, \mu) \) be a \( C^m \) function. Let \( \delta > 0 \) and \( \varepsilon > 0 \) be such that for \( \theta \in [-1+\delta, 0] \), and for each positive integer \( n \),

\[ x(n+\theta) = f_\mu^n(\phi(\theta)) \text{ as } \varepsilon \to 0 \text{ uniformly in } \theta. \]

**Proof:**

From (1.3), we have for \( 0 \leq t \leq 1 \)

\[ \dot{x}(t) = \frac{1}{\varepsilon} x(t) + \frac{1}{\varepsilon} g(t) \]

\[ x(0) = \phi(0) \]

Using the variation of constants formula, we get

\[ x(t) = e^{-t/\varepsilon} x(0) + \frac{1}{\varepsilon} \int_0^t e^{(t-s)/\varepsilon} g(s) ds, \quad 0 \leq t \leq 1 \]

Therefore

\[ x(t+\theta) = e^{-(\theta+1)/\varepsilon} \phi(0) + \frac{1}{\varepsilon} \int_0^{\theta+1} e^{-(\theta+1-s)/\varepsilon} f_\mu(\phi(s-1)) ds \]

Write \( z = s-1 \) to get,
Using integration by parts we can show that

\[
\frac{1}{\epsilon} \int_{-1}^{\theta} e^{-\frac{\theta - z}{\epsilon}} f_\mu(\phi(z)) \, dz = f_\mu(\phi(\theta)) + O(\epsilon)
\]

Thus

\[
\frac{1}{\epsilon} \int_{-1}^{\theta} e^{-\frac{\theta - z}{\epsilon}} f_\mu(\phi(z)) \, dz \to f_\mu(\phi(\theta)) \text{ as } \epsilon \to 0 \text{ uniformly}
\]

So as \( \epsilon \to 0 \), we have for all \( \theta \in [-1+\delta, 0] \),

\[
\begin{align*}
\phi(l+\theta) &\to f_\mu(\phi(\theta)) \\
\phi(2+\theta) &\to f_\mu^2(\phi(\theta)) \\
\end{align*}
\]

Continue inductively, we get

\[
\phi(n+\theta) \to f_\mu^n(\phi(\theta))
\]

This establishes that, given any finite segment of an orbit \( \{\phi_n = f_\mu^n(\phi_0) | n \geq 0\} \) there is for \( \epsilon \) sufficiently small a solution of the singularly perturbed problem that "shadows" it. Also, as will be seen in Chapter IV, the delay differential equation exhibits much more complex solutions (period 2/3, 2/5,..., six-segment period-two solutions, etc.) so the relation between the delay differential equation and the reduced problem is not simple at all.
CHAPTER II. BIFURCATION OF FIXED POINTS

Let \( f: \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function, where \( \mathbb{R} \) denotes the real numbers.

Definition 2.1

The forward orbit of \( t \) is the set of points \( t, f(t), f^2(t), \ldots \) and is denoted by \( O^+(t) \). If \( f \) is a homeomorphism, we may define the full orbit of \( t \) \( O(t) \), as the set of points \( f^n(t) \) for \( n \in \mathbb{Z} \), and the backward orbit of \( t \), \( O^-(t) \), as the set of points \( t, f^{-1}(t), f^{-2}(t), \ldots \).

Definition 2.2

The point \( t \) is a fixed point of \( f \) if \( f(t) = t \). The point \( t \) is a periodic point of period \( n \) if \( f^n(t) = t \). The least positive \( n \) for which \( f^n(t) = t \) is called the prime period of \( t \). The set of all iterates of a periodic point form a periodic orbit.

Definition 2.3

A point \( t \) is a critical point of \( f \) if \( f'(t) = 0 \). The critical point is non-degenerate if \( f''(t) \neq 0 \). The critical point is degenerate if \( f''(t) = 0 \).

Definition 2.4

Let \( p \) be a periodic point of prime period \( n \). The point \( p \) is hyperbolic if \( |(f^n)'(p)| \neq 1 \).
Proposition 2.5

Let \( p \) be a hyperbolic fixed point with \( |f'(p)| < 1 \). Then there is an open interval \( U \) about \( p \) such that if \( t \in U \), then

\[
\lim_{n \to \infty} f^n(t) = p
\]

Definition 2.6

Let \( p \) be a hyperbolic periodic point of period \( n \) with \( |(f^n)'(p)| < 1 \). The point \( p \) is called an attracting periodic point or a sink.

Proposition 2.7

Let \( p \) be a hyperbolic fixed point with \( |f'(p)| > 1 \). Then there is an open interval \( U \) about \( p \) such that, if \( t \in U, t \neq p \), then there exists \( k > 0 \) such that \( f^k(t) \notin U \).

Definition 2.8

Let \( p \) be a hyperbolic periodic point of period \( n \) with \( |(f^n)'(p)| > 1 \). The point \( p \) is called a repelling periodic point or a source.

Proposition 2.9

Let \( 0 < \mu < 1 \). The function \( f_\mu(x) = -\mu x + x^3 \) has \( x = 0 \) as an attracting fixed point and \( x = \pm \sqrt{\mu} \) as repelling fixed points.

Proof:

The fixed points of \( f_\mu \) are \(-\sqrt{\mu} + 1, 0, \sqrt{\mu} + 1\).

\[
f'_\mu(x) = -\mu + 3x^2
\]

\[
f'_\mu(0) = -\mu
\]
Thus \( x=0 \) is attracting fixed point for \( 0<\mu<1 \).

\[
f'_\mu(\pm\sqrt{1+\mu}) = -\mu + 3(1+\mu) \\
= 2\mu + 3 > 0
\]

Hence the fixed points \( x = -\sqrt{1+\mu} \) and \( x = \sqrt{1+\mu} \) are repelling. See Figure 1.

\[
\]

**Remark**

Notice that \( x = 0 \) is a repelling fixed point for \( \mu>1 \). See Figure 2.

**Bifurcation of fixed points**

The only place where bifurcations of fixed points occur is near non-hyperbolic fixed and periodic points.

For proofs of the following three theorems, see Devaney [3].

**Theorem 2.10**

Let \( f_\mu \) be a one-parameter family of functions and suppose that \( f'_{\mu_0}(x_0) = x_0 \) and \( f''_{\mu_0}(x_0) \neq 1 \). Then there are intervals \( I \) about \( x_0 \) and \( N \) about \( \mu_0 \) and a smooth function \( p:N\rightarrow I \) such that \( p(\mu_0) = x_0 \) and \( f_\mu(p(\mu)) = p(\mu) \). Moreover, \( f_\mu \) has no other fixed points in \( I \).

**Theorem 2.11** (The saddle-node bifurcation)

Suppose that

1. \( f'_{\mu_0}(0) = 0 \)
2. \( f''_{\mu_0}(0) = 1 \)
3. \( f'''_{\mu_0}(0) \neq 0 \)
FIGURE 1. $x=0$ is stable for $0<\mu<1$
FIGURE 2. $x=0$ is unstable for $\mu > 1$
4. \((\delta f/\delta \mu)_{|\mu=\mu_0}(0) \neq 0\)

Then there exists an interval \(I\) about 0 and a smooth function \(p: I \rightarrow \mathbb{R}\) such that
\[ f_p(x)(x) = x. \]
Moreover, \(p'(0) = 0\) and \(p''(0) \neq 0\).

Remark

The bifurcation can be described graphically by a bifurcation diagram in which we plot the location of fixed (or periodic) points versus the parameter.

The signs of \(f''(0)\) and \((\delta f/\delta \mu)_{|\mu=\mu_0}(0)\) determine the "direction" of the bifurcation; if they have opposite signs, then the bifurcation diagram is as in Figure 3.

\[ \text{FIGURE 3. Bifurcation Diagram} \]
Theorem 2.12 (Period-doubling bifurcation)

Suppose that

1. $f_\mu(0) = 0$ for all $\mu$ in an interval about $\mu_0$.
2. $f'_\mu(0) = -1$
3. $(f_\mu)'(0) \neq 0$
4. $(\delta(f^2_\mu)/\delta\mu)|_{\mu=\mu_0}(0) \neq 0$

Then there is an interval $I$ about 0 and a function $p: I \rightarrow \mathbb{R}$ such that for all $x \in I$ different from 0,

$$f_p(x)(x) \neq x$$

but

$$f_p^2(x)(x) = x.$$ 

Proposition 2.13

1. The function $f_\mu(x) = -\mu x + x^3$ has two period-two points

$$x_1 = \sqrt{(\mu-1)} \quad \text{and} \quad x_2 = -\sqrt{(\mu-1)}$$

bifurcating from the fixed point $x=0$ as $\mu$ passes through 1.

2. These bifurcated period-two points are stable (attracting) for $1 < \mu < 2$.

Proof:

We can easily see that the conditions of theorem (2.12) are satisfied for the function

$$f_\mu(x) = -\mu x + x^3 \quad \text{and} \quad \mu_0 = 1.$$ 

So there are two points of period two bifurcate from the fixed point $x = 0$.

To find those two points, we solve
\[ f_\mu(x) = -x \]
\[-\mu x + x^3 = -x \]
\[ x(x^2 - \mu + 1) = 0 \]

Hence \( x \in \{0, -\sqrt{\mu-1}, \sqrt{\mu-1}\} \)

Therefore the period-two points that bifurcate from the fixed point \( x = 0 \) as \( \mu \) passes through 1 are
\[ x_1 = \sqrt{\mu-1} \text{ and } x_2 = -\sqrt{\mu-1}. \]

See Figures 4, 5, and 6.

\( x_1 \) and \( x_2 \) are the fixed points of \( f_\mu^2(x) \).

\[
\frac{d}{dx} (f_\mu^2(x)) = f_\mu' (f_\mu(x)) f_\mu'(x)
\]

\[
\frac{d}{dx} (f_\mu^2(x)) \bigg|_{x=\pm\sqrt{\mu-1}} = [-\mu + 3(\mu-1)]^2
\]

\[ = (2\mu-3)^2 \]

These points are stable for \((2\mu-3)^2 < 1\); equivalently
\[ |2\mu-3| < 1 \text{ or } 1 < \mu < 2 \]

Notice that for \( \mu_0 = 2 \),

\[
\frac{d}{dx} (f_\mu^2(x)) \bigg|_{x=\pm\sqrt{\mu_0-1}} = 1
\]

and the stability of the period-two points \( x_1 \) and \( x_2 \) changes as \( \mu \) passes through 2.
FIGURE 4. $f^2_\mu(x)$ for $\mu=0.9$
FIGURE 5. $f^2_{\mu}(x)$ for $\mu=1.0$
FIGURE 6. $f^2(x)$ for $\mu=1.1$
FIGURE 7. $I_\mu^2(x)$ for $\mu=1.9$
FIGURE 8. $f_\mu^2(x)$ for $\mu=2.0$
FIGURE 9. $f_\mu^2(x)$ for $\mu=2.1$
Proposition 2.14

1. The function \( f_\mu(x) = -\mu x + x^3 \) has two period-two points
   
   \[ a_1 = \frac{1}{\sqrt{2}} \left[ \mu + \sqrt{\mu^2 - 4} \right]^{1/2} \]
   
   \[ a_2 = \frac{1}{\sqrt{2}} \left[ \mu - \sqrt{\mu^2 - 4} \right]^{1/2} \]

   bifurcating from the period-two point \( \sqrt{\mu - 1} \), and two
   period-two points
   
   \[ a_3 = -\frac{1}{\sqrt{2}} \left[ \mu - \sqrt{\mu^2 - 4} \right]^{1/2} \]
   
   \[ a_4 = -\frac{1}{\sqrt{2}} \left[ \mu + \sqrt{\mu^2 - 4} \right]^{1/2} \]

   bifurcating from the period-two point \( -\sqrt{\mu - 1} \) as \( \mu \) passes through 2.

2. The bifurcated period-two points are stable for
   
   \[ 2 < \mu < \sqrt{5} \]

Proof:

To find all period-two points of \( f_\mu \), we solve the equation

\[ f_\mu^2(x) = x \]

\[ -\mu(-\mu x + x^3) + (-\mu x + x^3)^3 = x \]

\[ \mu^2 x - \mu x^3 + x^3 + x^3(x^2 - \mu) = x \]

Which implies

\[ x = 0 \text{ or } x^2(x^2 - \mu)^3 - \mu x^2 + \mu^2 - 1 = 0 \]

Put \( x^2 = y \) to get,

\[ y(y - \mu)^3 - \mu y + \mu^2 - 1 = 0 \]

\[ y(y^3 - 3\mu y^2 + 3\mu^2 y - \mu^3) - \mu y + \mu^2 - 1 = 0 \]

\[ y^4 - 3\mu y^3 + 3\mu^2 y^2 - (\mu^3 + \mu)y + \mu^2 - 1 = P(y) = 0 \]

We know that \( y = \mu + 1 \) and \( y = \mu - 1 \) are roots of \( P(y) \).
Thus we can write

\[ P(y) = (y - (\mu - 1))(y - (\mu + 1))(y^2 - \mu y + 1) \]

Hence the newly bifurcated fixed points of \( f_\mu^2(x) \) are the solutions of

\[ x^4 - \mu x^2 + 1 = 0 \]

which implies

\[ x^2 = (1/2)(\mu + \sqrt{\mu^2 - 4}) \text{ or } x^2 = (1/2)(\mu - \sqrt{\mu^2 - 4}) \]

Thus \( x = \pm(1/\sqrt{2})[\mu + \sqrt{\mu^2 - 4}]^{1/2} \)

or \( x = \pm(1/\sqrt{2})[\mu - \sqrt{\mu^2 - 4}]^{1/2} \)

These period-two points appear as \( \mu \) passes through 2.

See Figures 7, 8, and 9.

The bifurcation diagram is sketched in Figure 10.

Now we want to study the stability of the newly bifurcated period-two points; namely, \( a_1, a_2, a_3, \) and \( a_4 \)

\[
\frac{d}{dx} (f_\mu^2(x)) = f_\mu'(f_\mu(x)) f_\mu'(x)
\]

\[
= [-\mu + 3(f_\mu(x))^2][-\mu + 3x^2]
\]

\[
= [-\mu + 3(\mu^2 x^2 - 2\mu x + x^5)][-\mu + 3x^2]
\]

\[
= [-\mu + 3x^2(\mu^2 - 2\mu x^2 + x^4)][-\mu + 3x^2]
\]

\[
\frac{d}{dx} (f_\mu^2(x)) \bigg|_{x=a_1} = [-\mu + 3x^2(\mu^2 - 2\mu x^2 + \mu x^2 - 1)](-\mu + 3x^2)
\]

\[
= [-\mu + 3x^2(\mu^2 - \mu x^2 - 1)][-\mu + 3x^2]
\]

\[
= [-\mu + 3x^2(\mu^2 - 3\mu x^2 - 3x^2)][-\mu + 3x^2]
\]
FIGURE 10. Bifurcation Diagram

\[ \mu = x^2 + 1 \]

\[ = [-\mu + 3\mu^2x^2 - 3\mu^2x^2 + 3\mu - 3x^2][-\mu + 3x^2] \]

\[ = [2\mu - 3x^2][-\mu + 3x^2] \]

\[ = -2\mu^2 + 9\mu x^2 - 9x^4 \]

\[ = -2\mu^2 + 9\mu x^2 - 9\mu x^2 + 9 \]

\[ = 9 - 2\mu^2 \]

The period-two points \( a_i, i=1,2,3,4 \) are stable for \( |9 - 2\mu^2| < 1 \);
equivalently
\[-1 < 9 - 2\mu^2 < 1 \text{ or } 2 < \mu < \sqrt{5}. \]

Now
\[ f(a_1) = (1/\sqrt{2})[\mu + \sqrt{\mu^2 - 4}]^{1/2}[-\mu + (1/\sqrt{2})\mu + (1/2)\sqrt{\mu^2 - 4}] \]
\[ = -(1/\sqrt{2})[\mu + \sqrt{\mu^2 - 4}]^{1/2}[\mu - \sqrt{\mu^2 - 4}] \]
\[ = -(1/\sqrt{2})[(\mu + \sqrt{\mu^2 - 4})(\mu - \sqrt{\mu^2 - 4})]^{1/2}[\mu - \sqrt{\mu^2 - 4}]^{1/2} \]
\[ = -(1/\sqrt{2})[\mu^2 - (\mu^2 - 4)]^{1/2}[\mu - \sqrt{\mu^2 - 4}]^{1/2} \]
\[ = a_3 \]

Similarly, we can show that
\[ f(a_3) = a_1 \]
\[ f(a_2) = a_4 \]
\[ f(a_4) = a_1 \]

Thus \( a_1, a_3, a_1 \) and \( a_2, a_4, a_2 \).

Proposition 2.15

The function \( f_\mu(x) = -\mu x + x^3 \) has two period-4 points bifurcating from each \( a_i \), \( i=1,2,3,4 \). as \( \mu \) passes through \( \sqrt{5} \) where \( a_4 \) as in proposition 2.14.

Proof:

Write \( g_\mu(x) = f_\mu^2(x) \).

Thus \( a_i, i=1,2,3,4 \) are fixed points of \( g_\mu \).

for \( \mu = \sqrt{5} \),

\[ \left. \frac{d}{dx} \left( g_\mu(x) \right) \right|_{x=a_i} = -1 \]
with some computations we can see that conditions 3 and 4 of theorem 2.12 are satisfied. Hence $g_\mu$ has two period-two points bifurcating from each $a_i$, $i=1,2,3,4$. which implies that $f_\mu$ has two period-4 points bifurcating from each $a_i$, $i=1,2,3,4$. See Figures 11, 12, and 13.

Remark

$f_\mu$ undergoes a series of period-doubling bifurcations as $\mu$ increases which leads to chaotic behavior.
FIGURE 11. $f_\mu(x)$ for $\mu=2.1$
FIGURE 12. $f_{\mu}^6(x)$ for $\mu=\sqrt{5}$
FIGURE 13. $f_\mu(x)$ for $\mu=2.28$
CHAPTER III. THE NUMERICAL METHOD

We are considering the equation
\[ \dot{x}(t) = -x(t) + f_{\mu}(x(t-1)) \quad , \quad t \geq 0 \]
\[ x(\theta) = g(\theta) \quad , \quad -1 \leq \theta \leq 0 \]
or
\[ \dot{x}(t) = \frac{1}{c}x(t) + \frac{1}{e}f_{\mu}(x(t-1)) \quad , \quad t \geq 0 \]
\[ x(\theta) = g(\theta) \quad , \quad -1 \leq \theta \leq 0 \]
which can be written as
\[ (3.1) \quad \dot{x}(t) = F(x(t), f_{\mu}(x(t-1))) \quad , \quad t \geq 0 \]
\[ x(\theta) = g(\theta) \quad , \quad -1 \leq \theta \leq 0 \]

We now assume that \( F \) is Lipschitz continuous in both arguments, and \( f_{\mu} \) is Lipschitz continuous.

We can find the numerical solution at any point \( t \in [0,1] \) by using an ordinary differential equations solver. Suppose that we found the solution at the points \( 0 = t_0 < t_1 < \ldots < t_m < 1 \) and we want to find the numerical solution at \( t_j \) where \( 1 < t_j < 2 \). It is clear that we need to know \( x(t_{j-1}) \). Thus if \( t_{j-1} \) is not one of the points \( t_0, t_1, \ldots, t_m \) we need to use interpolation to approximate the value of \( x(t_{j-1}) \) and continue to find the numerical solution at \( t_j \).

To solve the problem numerically in the interval \([0,T]\), we use a onestep method for ordinary differential equations combined with Hermite quintic interpolation. The one step method used here is the diagonally implicit Runge-Kutta code DIRK. See Alexander [1]. At each
step from $t_i$ to $t_{i+1}=t_i+h_i$, $i=-N,-N+1,...,-1,0,1,2,...,k$, the code generates approximations to $x$ and $\dot{x}$ at $t_i$, $t_i+.5h_i$, and $t_i+h_i$, data for Hermite quintic interpolation. See Odoom [13]. We need a numerical approximation $\hat{f}_\mu$ of $f_\mu$. When the method steps from $t_j$ to $t_{j+1}$ we have available the history of $x$ and $\dot{x}$, so we use Hermite quintic interpolation to approximate $x(t_j-1)$ and then obtain $\hat{f}_\mu$. Assuming that $f_\mu$ is a known function of $t$, we get

$$x_{n+1} = x_n + (t_{n+1} - t_n) \phi(x_n, f_\mu)$$

where $\phi$ is the increment function for the method. For brevity of notation, we assume that the grid is equidistant with step $h$; the results can be modified for variable step size in exactly the same manner as for ordinary differential equations, see Henrici [8].

We apply the onestep method with interpolation $(\phi, \hat{f}_\mu)$ to (3.1) as follows

$$x_{n+1} = x_n + h \phi(x_n, \hat{f}_\mu) + p_n, \quad n \geq 0$$

$$x_j = g(t_j) + w_j, \quad j \leq 0$$

where $p_n$ are local perturbations arising in computations, and $w_j$ are errors in starting values. The following convergence result was given by Oppelstrup [14].

**Theorem 3.3** If

1. $\hat{f}_\mu$ is Lipschitz continuous
2. $\phi$ is consistent, hence convergent, for ordinary differential equations and, as $h \to 0$,
3. $w_j \to 0$
4. $p_n/h \to 0$
5. \[ \| f_\mu(x(t-1)) - \hat{f}_\mu(x(t-1)) \| \to 0 \]

Then the method \((\phi, f_\mu)\) converges.

**Proof:**

See Oppelstrup [14].

**Error of the method**

**Theorem 3.4**

Assume that the exact solution \(x\) and the function \(F\) are sufficiently differentiable. Then, if the local error of \(\phi\) is of order \(p+1\) and the interpolation error is of order \(q\), and \(q \geq p+1 \geq 2\),

\[ \| x(t_j) - x_j \| = h^p \varepsilon(t_j) + O(h^{p+1}) \]

where \(\varepsilon(t)\) satisfies

\[ \dot{\varepsilon} = F'\varepsilon + \psi(t), \quad t \geq 0 \]
\[ \varepsilon(s) = 0, \quad s < 0 \]

Here, \(F'\) is the Fréchet derivative of \(F\) along the exact solution \(x(t)\), \(\psi\) is the principal error function of \(\phi\). The initial error \(w_j\) and the local perturbations \(p_j\) were ignored.

**Proof:**

See Oppelstrup [14].
In the following, we will consider the delay differential equation

\[ e^x(t) = -x(t) + f(x(t-1), \mu), \quad t \geq 0 \]

with

\[ f(q, \mu) = -\mu q + q^3 \]

Proposition 4.3

Let \( 0 < \mu < 1 \).

\( x(t) = 0 \) is a uniformly asymptotically stable solution of the equation \( (4.1) \).

Proof:

The delay differential equation \( (4.1) \), linearized about the zero solution is

\[ e^x(t) = -x(t) - \mu x(t-1) \]

We shall look for nontrivial solutions of \( (4.4) \) of the form

\[ x(t) = \xi e^{\lambda t} \]

where \( \lambda \) is a complex number and \( \xi \) is a constant.

Equation \( (4.4) \) has a nontrivial solution of the indicated form if and only if

\[ e^{\xi \lambda} e^{\lambda t} = -\xi e^{\lambda t} - \mu \xi e^{\lambda(t-1)} \]

or

\[ \epsilon \lambda = -1 - \mu e^{-\lambda} \]

Write \( \lambda = a + i \beta \) to get
\[ e^{(a+ib)} = -1 - \mu e^{-a} \cos \beta - i \sin \beta \]
equating the real parts we get
\[ e^a = -1 - \mu e^{-a} \cos \beta \]
Thus
(4.6) \[ \mu e^{-a} \cos \beta = -1 - e^a \]
This equation cannot be satisfied for \( a \geq 0 \) because the absolute value of the left hand side of (4.6) is less than 1. Hence all the roots of the characteristic equation (4.5) have negative real parts. Thus \( x(t) = 0 \) is a uniformly asymptotically stable solution of the equation (4.1) with \( f(q,\mu) = -\mu q + q^3 \). See theorem 5.5 page 76 in Kolmanovskii and Nosov [11].

We know that as \( \mu \) passes through 1, the fixed point 0 for the reduced equation changes from an attracting fixed point to a repelling fixed point. We expect that the stability of the solution \( x(t) = 0 \) will change as \( \mu \) passes through 1.

For the Hopf bifurcation to occur, it is necessary that the characteristic equation (4.5) of the linearized delay differential equation have a pair of purely imaginary eigenvalues. We need the following theorem to study the roots of the characteristic equation (4.5).
Theorem 4.7

If $z = \pm i\omega$ is a purely imaginary root of the equation

$$(4.8) \quad (z + a)e^z + b = 0,$$

where $a \neq 0$, $b \neq 0$ are real, then there exist $k \in \mathbb{Z}$ such that

$$(4.9) \quad \omega = \omega_k(a) = k\pi - k\pi a^{-1} + k\pi a^{-2} - \frac{1}{3}[3k\pi - k^{3}\pi^{3}a^{-3} + O(a^{-4})]$$

and

$$(4.10) \quad b = b_k(a) = (-1)^{k+1} \left\{ a + \frac{1}{2}k^{2}\pi^{2}a^{-1} - k^{2}\pi^{2}a^{-2} + \frac{1}{8}k^{2}\pi^{2}[12 - k^{2}\pi^{2}]a^{-3} + \frac{1}{6}k^{2}\pi^{2}[5k^{2}\pi^{2} - 12]a^{-4} \right\} + O(a^{-5}).$$

In fact $\omega$ and $b + (-1)^{k}a$ are functions of $a$ analytic at $\infty$.

Proof:

Equation (4.8) can be written in the form

$$(z + a + b e^{-z} = 0)$$

Write $z = i\omega$ to get

$$(i\omega + a + b e^{-i\omega} = 0)$$

Or

$$(i\omega + a + b[\cos \omega - i \sin \omega] = 0)$$

Equate the real and imaginary parts to get

$$a + b \cos \omega = 0$$

$$\omega = b \sin \omega$$
Thus

\[
\begin{align*}
\cos \omega &= \frac{a}{b} \\
\sin \omega &= \frac{\omega}{b}
\end{align*}
\]  

(4.11)

We want to determine \( \omega \) and \( b \) as functions of \( a \) so that equations (4.11) are satisfied.

Now

\[
\tan \omega = \frac{\omega}{a}
\]

Or \( \tan \omega = -\rho \omega \) where \( \rho = \frac{1}{a} \)

This equation has for each \( k \in \mathbb{Z} \) a unique solution, call it \( \omega_k \), satisfying

\[
(k-\frac{1}{2})\pi < \omega_k < (k+\frac{1}{2})\pi,
\]

and these are all the solutions.

From equations (4.11), we get

\[
\cos^2 \omega_k + \sin^2 \omega_k = \frac{a^2}{b_k^2} + \frac{\omega_k^2}{b_k^2} = 1
\]

Thus

\[
b_k^2 = a^2 + \omega_k^2
\]

which implies that \( b_k = \pm \frac{1}{\rho}(1+\rho^2\omega_k^2)^{1/2} \)
The sign must be chosen in order to satisfy the constraints of (4.11), depending on the sign of \( p \) and the parity of \( k \). Thus

\[
b_k = (-1)^{k+1} \frac{1}{\rho} (1+\rho^2\omega_k^2)^{1/2}
\]

\[\text{Eqn (4.12)}\]

\[
b_k = (-1)^{k+1} \frac{1}{\rho} [1 + (1/2)\omega_k^2\rho^2 - (1/8)\omega_k^4\rho^4
\]

\[+ (1/16)\omega_k^6\rho^6 - ...]

Now \( \rho\omega = -\tan \omega \)

\[= \tan(k\pi - \omega_k)\]

\[= \tan x\]

where \( x = k\pi - \omega_k \), which implies that \( \omega_k = k\pi - x \)

Thus

\[\rho(k\pi - x) = \tan x\]

Then for every \( k \in \mathbb{Z} \), the equation

\[\rho = F_k(x) := \frac{\tan x}{k\pi - x}\]

determines \( x \) as analytic function of \( \rho \) for \( \rho = 1/a \) near 0.

\( F_k(x) \) is analytic at 0 and \( F_k(0) = 0 \).

Now

\[F_k'(x) = \frac{[k\pi - x]\sec^2 x + \tan x}{[k\pi - x]^2}\]

Therefore
\[
F_k(0) = \frac{k\pi}{[k\pi]^2}
\]

\[
= \frac{1}{k\pi} \neq 0
\]

Applying Lagrange-Burmann theorem, Henrici [7], we get

\[
x = F_k[-1](\rho) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Res}(F_k^{-n}) \rho^n
\]

Therefore

\[
\omega_k = k\pi - \sum_{n=1}^{\infty} \frac{1}{n} \text{Res}(F_k^{-n}) \rho^n
\]

Now \( F_k^{-n} = \left[\frac{k\pi - x}{\tan x}\right]^n \)

\[
= \left[\frac{k\pi - x}{x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots}\right]^n
\]

\[
= \frac{1}{n^n} \left[\frac{k\pi - x}{1 + \frac{x^2}{3} + \frac{2x^4}{15} + \ldots}\right]^n
\]

Write

\[
p = 1 + \frac{1}{3}x^2 + \frac{2}{15}x^4 + \ldots.
\]

Therefore
\[ \text{Res}(F_k^{-1}) = \text{Res} \frac{1}{x} \left[ \frac{k \pi - x}{p} \right] \]

\[ = \lim_{x \to 0} \frac{k \pi - x}{p} \]

\[ = k \pi \]

\[ \text{Res}(F_k^{-2}) = \text{Res} \frac{1}{x^2} \left[ \frac{k \pi - x}{p} \right]^2 \]

\[ = \frac{d}{dx} \left[ \frac{(k \pi - x)^2}{p^2} \right] \bigg|_{x=0} \]

\[ = \frac{-2p^2(k \pi - x) - 2(k \pi - x)^2p'}{p^4} \bigg|_{x=0} \]

\[ = -2k \pi \]

\[ \text{Res}(F_k^{-3}) = \text{Res} \frac{1}{x^3} \left[ \frac{k \pi - x}{p} \right]^3 \]

\[ = \frac{1}{2} \frac{d^2}{dx^2} \left[ \frac{(k \pi - x)^3}{p^3} \right] \bigg|_{x=0} \]

\[ = \frac{1}{2} \frac{d}{dx} \left\{ \frac{-3p^3(k \pi - x)^2 - 3p^2p'[k \pi - x]^3}{p^6} \right\} \bigg|_{x=0} \]

\[ = \frac{1}{2p^{12}} \left[ -3p^6 \{3p^2p'[k \pi - x]^2 - 2p^3[k \pi - x] \right. \]

\[ - 2pp' \left. \frac{2}{3} [k \pi - x]^3 + p^2p''[k \pi - x]^3 \right] \]
\[ -3p^2 p' [k_\pi - x]^2 + 3\{p^3 [k_\pi - x]^2 + p^2 p'[k_\pi - x]_0^3 6p^5 p' \} \bigg|_{x=0} \]

\[ = \frac{1}{2} [-3\{-2k_\pi + \frac{2}{3}k_\pi^3\}] \]

\[ = 3k_\pi - k^3_\pi^3 \]

Therefore

\[ (4.13) \quad \omega_k = k_\pi - k_\pi\rho + k_\pi\rho^2 - \]

\[ \frac{1}{3}[3k_\pi - k^3_\pi^3]\rho^3 + O(\rho^4) \]

Or

\[ \omega = \omega_k(a) = k_\pi - k_\pi a^{-1} + k_\pi a^{-2} - \]

\[ \frac{1}{3}[3k_\pi - k^3_\pi^3]a^{-3} + O(a^{-4}) \]

From (4.13) we get

\[ \omega_k^2 = k^2_\pi^2 - 2k^2_\pi^2\rho + 3k^2_\pi^2\rho^2 - \]

\[ 2[2k^2_\pi^2 - (1/3)k^4_\pi^4]\rho^3 + O(\rho^4) \]

and

\[ \omega_k^4 = k^4_\pi^4 - 4k^4_\pi^4\rho + O(\rho^2) \]

Substitute in equation (4.12) to get

\[ (4.14) \quad b_k = (-1)^{k+1}\{\rho^{-1} + \frac{1}{2}k^2_\pi^2\rho - k^2_\pi^2\rho^2 + \}

\[ \frac{1}{6}k^2_\pi^2[12 - k^2_\pi^2]\rho^3 + \]
\[
\frac{1}{6}k^2\pi^2[5k^2\pi^2 - 12]\rho^4 + O(\rho^5).
\]

Or
\[
b = b_k(a) = (-1)^{k+1} \left[ a + \frac{1}{2}k^2\pi^2a^{-1} - k^2\pi^2a^{-2} + \frac{1}{6}k^2\pi^2[12 - k^2\pi^2]a^{-3} + \frac{1}{6}k^2\pi^2[5k^2\pi^2 - 12]a^{-4}\right] + O(a^{-5}).
\]

To find when the characteristic equation (4.5) of the linearized delay differential equation has a pair of purely imaginary eigenvalues \( \pm iw \), we write equation (4.5) in the form

\[(4.15) \quad (\lambda + \frac{1}{\varepsilon})e^\lambda + \frac{\mu}{\varepsilon} = 0\]

and apply theorem (4.7) with

\[a = \frac{1}{\varepsilon} \quad \text{and} \quad b = \frac{\mu}{\varepsilon}\]

and considering the fact that both \( a \) and \( b \) are positive, we get for \( k=0,1,2,\ldots \)

\[(4.16) \quad \omega = (2k+1)\pi - (2k+1)\pi\varepsilon + (2k+1)\pi\varepsilon^2 - \frac{1}{3}[3(2k+1)\pi - (2k+1)^3\pi^3]\varepsilon^3 + O(\varepsilon^4)\]

and
(4.17) \[ \mu = 1 + \frac{1}{2}(2k+1)2\pi^2e^2 - (2k+1)2\pi^2e^3 + \]

\[ \frac{1}{9}(2k+1)^2\pi^2[12 - (2k+1)^2]e^4 + \]

\[ \frac{1}{6}(2k+1)^2\pi^2[5(2k+1)^2 - 12]e^5 + O(e^6). \]

For a given value of \( \epsilon \), equation (4.17) gives us the minimum value of \( \mu \) at which the solution \( x(t) = 0 \) of equation (4.1) loses its stability. We have the following theorem,

**Theorem 4.18**

The solution \( x(t) = 0 \) of (4.1) loses its stability along the curve \( \Gamma_k \) given by the equation (4.17), on which a pair of purely imaginary eigenvalues near \( \pm i(2k+1)\pi \) of the characteristic equation of the linearization of (4.1) at \( x(t) = 0 \) occur. Indeed, \( x(t) = 0 \) is exponentially stable for parameters to the left of \( \Gamma_k \), and unstable to the right, in a neighborhood of \( (\mu, \epsilon) = (1, 0) \).

Numerical experiments confirmed the existence of periodic solutions with period approximately \( 2\pi/\omega \) where \( \omega \) is given by (4.16) for \( k = 0, 1, 2, 3, \ldots \), that is periodic solutions with period approximately 2, 2/3, 2/5, \ldots were found.

Figures 18, 23, 28, respectively show solutions of periods approximately 2, 2/3, and 2/5.
In Figure 14, we illustrate the stability of the zero solution for parameters to the left of \( \Gamma_0 \) by considering initial function \( x = 1 \) with \( \mu = 0.7 \) and \( \epsilon = 0.01 \). It is seen that the solution converges rapidly to 0.

In Figure 15, we illustrate the instability of the zero solution for parameters to the right of \( \Gamma_0 \) by considering initial function \( x = 0.1 \) with \( \mu = 1.6 \), and \( \epsilon = 0.01 \), the solution converges rapidly to a square wave of period approximately two.

For small \( \epsilon \), as \( \mu \) passes \( \mu_{\text{min}} \) determined from equation (4.17), the bifurcating stable periodic solutions resemble sine waves and as we increase \( \mu \) they become square waves.

For \( \epsilon = 0.04 \), we can use equation (4.17) to find \( \mu_{\text{min}} \) at which the solution \( x = 0 \) loses its stability and bifurcates into a stable periodic orbit of period approximately two.

\[
\mu_{\text{min}} = 1.007264 \quad \text{for } \epsilon = 0.04
\]

In Figure 16, we take \( \mu = 1.004 < \mu_{\text{min}} \) with initial function \( x = 0.063246 \), the solution is decaying sine wave. Now fix \( \epsilon \) and take \( \mu = 1.04 > \mu_{\text{min}} \). The solution is a stable sine wave of period approximately two. Increasing \( \mu \) further to 1.14, we obtain in Figure 18 a stable square wave of period approximately two.

**Detailed look at bifurcations following \( \mu = 2 \)**

In Figure 19, we take \( \mu = 1.99 \) and \( \epsilon = 0.01 \), we obtain the stable symmetric square wave of period approximately two.
As $\mu$ increases through 2, the symmetric square wave loses its stability and bifurcates into two stable nonsymmetric square waves of period approximately two. See Figures 20 and 21.

Figure 22 shows that this does not occur precisely at $\mu = 2$. In this Figure we see that the symmetric square wave appears to be stable for $\mu = 2.0001 > 2$.

The nonsymmetric square wave loses its stability as $\mu$ increases through \sqrt{5} and bifurcates into two stable orbits of period approximately 4.

In Figure 23, by considering $\mu = 2.27 > \sqrt{5}$, $\epsilon = 0.01$ and initial function $x = 1.252$, we obtained a periodic orbit of period approximately 4.

The periodic solutions of period $2/3$, $2/5$, $2/7$, ... behave in the same way as the fundamental one of period two as the parameters $\mu$ and $\epsilon$ vary.

I. Sinusoidal behavior for "large" $\epsilon$.

This is illustrated by Figures 24 and 25, in Figure 23 we have a square wave of period approximately $2/3$, then we fixed $\mu$ and increased $\epsilon$ which results in obtaining the sine wave of period $2/3$ in Figure 25.

II. Splitting of periodic solutions with loss of stability as $\mu$ increases through 2.

This is illustrated by Figures 26 and 27.

III. Period doubling at $\mu = \sqrt{5}$. 

This is illustrated by Figure 28.

A Strange Periodic Solution

In Figures 30 and 31, $\mu = 2.1$ and $\epsilon = 0.01$. The Figures show a strange periodic orbit of period 2, it consists of three square waves of length $2/3$, each one is different from the other two.

It is worth noting that I tried unsuccessfully to obtain a similar orbit for the fundamental period-2 solution, i.e., an orbit of period 6 consists of three different square waves each has length 2.
FIGURE 14. Stability of the zero solution for $0 < \mu < 1$
FIGURE 15. Instability of the zero solution for $\mu > 1$
FIGURE 16. Decaying sine wave for $\mu < \mu_{\text{min}}$
FIGURE 17. Sine wave for $\mu > \mu_{\text{min}}$
FIGURE 18. Symmetric square wave of period 2 for $\mu > \mu_{\text{min}}$
FIGURE 19. Symmetric square wave of period 2 for $\mu=1.99$
FIGURE 20. Nonsymmetric square wave of period 2
FIGURE 21. Nonsymmetric square wave of period 2

MU=2.1, EP=.01
INIT. FUN. X=-1.170
FIGURE 22. The symmetric square wave stays for $\mu=2.0001$
FIGURE 23. Solution of period 4 for $\mu > \sqrt{5}$
FIGURE 24. Square wave of period 2/3

MU=1.1, EP=.01
FIGURE 25. Sine wave of period 2/3
FIGURE 26. Splitting of nonsym. square waves of period 2/3 for $\mu > 2$
FIGURE 27. Splitting of nonsym. square waves of period 2/3 for $\mu>2$
FIGURE 28. Period doubling for solution of period 2/3 and $\mu > \sqrt{5}$
FIGURE 29. Solution of period 2/5
Figure 30. Strange solution
Figure 31. Strange solution
We know that for $1<\mu<2$, the function $f_\mu(x) = -\mu x + x^3$ has a pair of period-two points bifurcating from the fixed point $x=0$ as $\mu$ passes through 1. These points are $\pm\sqrt{\mu-1} = \pm a$, with $f_\mu(a) = -a$ and $f_\mu(-a) = a$.

Now we will investigate the periodic solution of period near 2 for $\mu>1$, $\mu$ near 1.

From equation (4.16), we have for $k=0$,

$$\omega = \pi - \pi \epsilon + O(\epsilon^2)$$

which implies that the period $\frac{2\pi}{\omega} = 2 + O(\epsilon)$

We notice that the solution $x(t)$ of the delay differential equation (4.1) is near $a = f_\mu(-a)$ and $-a = f_\mu(a)$ on successive intervals of length $1+O(\epsilon)$ with an abrupt transition from $(-a)$ to $a$ on intervals of length $O(\epsilon)$. Assume that the period of $x(t)$ is $2 + 2 \epsilon + O(\epsilon^2)$, for some constant $r$. We want to find the boundary layer equation of the square wave $x(t)$ to describe the transition from $-a$ to $a$ or from $a$ to $-a$. As in Chow and Mallet-Paret [2], we introduce the following change of variables

$$\tau = -\frac{t}{\epsilon}$$

$$y(\tau) = x(t) \Rightarrow x(t+1+r\epsilon) = -y(\tau)$$
Substituting in the delay differential equation (4.1), using the periodicity of \( x(t) \) and taking formal limits as \( \epsilon \to 0 \), we obtain the differential equation

\[
(5.2) \quad y'(\tau) = y(\tau) + f_\mu(y(\tau - r))
\]
\[
y(\tau) \to a \text{ as } \tau \to \pm \infty
\]

Assume that

\[
(5.3) \quad r(\mu) = 1 + r_1\gamma + r_2\gamma^2 + \ldots \text{ where } \gamma = \sqrt{\mu - 1} \text{ is small}
\]

Let \( s = \frac{\tau \sqrt{r}}{r} \) be a new time variable and

\[
y(\tau) = \gamma w(s)
\]

\[
\frac{dy}{d\tau} = \frac{dy}{ds} \frac{ds}{d\tau} = \frac{\gamma^2}{r} \frac{dw}{ds}
\]

Notice that

\[
y(\tau - r) = \gamma w(\frac{2(\tau - r)}{r})
\]
\[
= \gamma w(s - \gamma)
\]

\[
f_\mu(y(\tau - r)) = f_\mu(\gamma w(s - \gamma))
\]
\[
= -\mu \gamma w(s - \gamma) + \gamma^3 w^3(s - \gamma)
\]

Substitute in equation (5.2) to get

\[
\frac{\gamma^2}{r} \frac{dw}{ds} = \gamma w(s) - \gamma(1 + \gamma^2)w(s - \gamma) + \gamma^3 w^3(s - \gamma)
\]
(5.4) \[ \gamma \frac{dw}{ds} = r(\mu)[w(s) - (1+\gamma^2)w(s-\gamma) + \gamma^2 w^3(s-\gamma)] \]

\[ w(s) \to \pm 1 + O(\gamma) \text{ as } s \to \pm \infty \]

Thus

\[ \gamma \frac{dw}{ds} = r(\gamma)[w(s) - (1+\gamma^2)w(s) - \gamma \frac{dw}{ds} + \frac{1}{2}\gamma^2 \frac{d^2w}{ds^2} - \ldots] \]

\[ + \gamma^2 \{w(s) - \gamma \frac{dw}{ds} + \frac{1}{2}\gamma^2 \frac{d^2w}{ds^2} - \ldots\}^3 \]

\[ \gamma \frac{dw}{ds} = (1 + r_1 \gamma + r_2 \gamma^2 + \ldots)[\{\gamma \frac{dw}{ds} - \frac{1}{2}\gamma^2 \frac{d^2w}{ds^2} + \frac{1}{6}\gamma^3 \frac{d^3w}{ds^3} - \ldots\] 

\[ \gamma^2 \{w(s) - \gamma \frac{dw}{ds} + \frac{1}{2}\gamma^2 \frac{d^2w}{ds^2} - \ldots\} \]

\[ + \gamma^2 \{w(s) - \gamma \frac{dw}{ds} + \frac{1}{2}\gamma^2 \frac{d^2w}{ds^2} - \ldots\}^3 + O(\gamma^3) \]

Write

\[ w(s) = w_0(s) + \gamma w_1(s) + \gamma^2 w_2(s) + \ldots \]

and equating the coefficients of \( \gamma^k \),

\( k = 0 \) and \( k = 1 \), coefficients agree.

\[ k = 2, \quad \frac{dw_1}{ds} = \frac{dw_1}{ds} - \frac{1}{2} \frac{d^2w_0}{ds^2} - w_0 + \gamma w_0 + r_1 \frac{dw_0}{ds} \]

or
with $w_0(s) \to \pm 1$ as $s \to \pm \infty$

$k = 3, \quad \frac{dw_2}{ds} = \frac{dw_2}{ds} - \frac{1}{2} \frac{d^2 w_1}{ds^2} + \frac{1}{6} \frac{d^3 w_0}{ds^3} - w_1 + \frac{dw_0}{ds}$

$+ 3w_0^2 w_1 - 3w_0 \frac{dw_0}{ds} + r_2 \frac{dw_0}{ds}$

or

(5.6) $\quad \frac{1}{2} \frac{d^2 w_1}{ds^2} - \frac{1}{6} \frac{d^3 w_0}{ds^3} + w_1 - \frac{dw_0}{ds} - 3w_0 w_1$

$+ (3w_0^2 - r_2) \frac{dw_0}{ds} = 0$

with $w_1 \to 0$ as $s \to \infty$

For the existence of a heteroclinic orbit for some value of $r$ joining the critical points $a$ and $-a$ at times $r = \pm \infty$, it is necessary that $r_1 = 0$. And in this case, equation (5.5) becomes

(5.7) $\quad \frac{1}{2} \frac{d^2 w_0}{ds^2} + w_0 - w_0^3 = 0$

Introducing $p = dw_0/ds = w_0'$, we get

$$\frac{d^2 w_0}{ds^2} = \frac{dp}{ds} = \frac{dp}{dw_0} \frac{dw_0}{ds} = \frac{dp}{p dw_0}$$
Substitute in (5.7) to get

\[
\frac{dp}{2dw_o} + w_o - w_o^3 = 0
\]

\[
dw_o = 2w_o^3 - 2w_o
\]

\[
\frac{1}{2}p^2 = \int (2w_o^3 - 2w_o)dw_o
\]

\[
= \frac{1}{2}w_o^4 - w_o^2 + C_1
\]

Now \( p \to 0 \) as \( s \to \infty \) which implies that \( C_1 = 1/2 \)

therefore \( p^2 = 1 - 2w_o^2 + w_o^4 \)

\[
dw_o = \sqrt{1 - 2w_o^2 + w_o^4}
\]

\[
= 1 - w_o^2
\]

Hence

\[
s = \int \frac{dw_o}{1 - w_o^2} = \tanh^{-1} w_o
\]

Which implies \( w_o = \tanh s \).

Thus the solution for equation (5.5) is

\[
w_o = \tanh s \text{ and } r_1 = 0.
\]

Now
\[ \frac{d\omega_0}{ds} = \text{sech}^2 s, \]

\[ \frac{d^2\omega_0}{ds^2} = -2\text{sech}^2 s \tanh s, \text{ and} \]

\[ \frac{d^3\omega_0}{ds^3} = -2[\text{sech}^4 s - 2\text{sech}^2 s \tanh^2 s]. \]

Substitute in equation (5.6) to get

\[ \frac{1}{2} \frac{d^2w_1}{ds^2} + w_1 - 3\tanh^2 s w_1 - \frac{1}{6}[-2\text{sech}^4 s + 4\text{sech}^2 s \tanh^2 s] \]

\[ = \text{sech}^2 s + 3\tanh^2 s \text{sech}^2 s - r_2 \text{sech}^2 s = 0 \]

The differential equation for \( w_1 \) becomes

\[ (5.8) \quad \frac{d^2w_1}{ds^2} + 2(1-3\tanh^2 s)w_1 = 4\text{sech}^4 s + (2r_2 - \frac{8}{3})\text{sech}^2 s \]

\[ , w_1 \xrightarrow{s \to \infty} 0 \]

Consider the homogeneous part

\[ (5.9) \quad \frac{d^2w_1}{ds^2} + 2(1-3\tanh^2 s)w_1 = 0 \]

Particular solution of the homogeneous part is

\[ w_1 = \text{sech}^2 s \]

which can be used as integrating factor for (5.8)
\[
\text{sech}^2 s \frac{d^2 w_1}{ds^2} + 2\text{sech}^2 s (1-3\tanh^2 s) w_1 = \\
4\text{sech}^6 s + (2r_2 - \frac{8}{3})\text{sech}^4 s
\]

The left hand side is exact.

\[
\frac{d}{ds}[\text{sech}^2 s \frac{dw_1}{ds} + 2\text{sech}^2 s \tanh s w_1] = \\
4\text{sech}^6 s + (2r_2 - \frac{8}{3})\text{sech}^4 s
\]

Therefore

\[
\text{sech}^2 s \frac{dw_1}{ds} + 2\text{sech}^2 s \tanh s w_1 = \\
4 \int \text{sech}^6 s ds + (2r_2 - \frac{8}{3}) \int \text{sech}^4 s ds
\]

Now

\[
\int \text{sech}^6 s ds = \frac{1}{5} \text{sech}^4 s \tanh s + \frac{4}{5} \int \text{sech}^4 s ds, \text{ and}
\]

\[
\int \text{sech}^4 s ds = \frac{1}{3} \text{sech}^2 s \tanh s + \frac{2}{3} \int \text{sech}^2 s ds
\]

\[
= (1/3)\text{sech}^2 s \tanh s + (2/3)\tanh s + C.
\]

which implies

\[
\int \text{sech}^6 s ds = (1/5)\text{sech}^4 s \tanh s + (4/15)\text{sech}^2 s \tanh s \\
+ (8/15)\tanh s + C_0.
\]

Therefore

\[
\frac{dw_1}{ds} + 2\tanh s w_1 = \frac{4}{5} \text{sech}^2 s \tanh s + (2r_2 - \frac{8}{3}) \frac{16}{9} \tanh s
\]
Multiplying both sides by an integrating factor 

\[ e^{2\tanh s} \, ds = \cosh^2 s, \]

we get 

\[
\frac{d}{ds}(\cosh^2 s \, w_1) = \frac{4}{5} \tanh s + \left(\frac{2}{3} \, r_2 + \frac{8}{45}\right) \sinh s \, \cosh s \\
+ \left(\frac{4}{3}^2 \, r_2 + \frac{16}{45}\right) \cosh^3 s \, \sinh s + C_1 \cosh^4 s
\]

Which implies 

\[
\cosh^2 s \, w_1 = \frac{4}{5} \ln \cosh s + \frac{1}{2} \left(\frac{2}{3} \, r_2 + \frac{8}{45}\right) \cosh^2 s + \frac{1}{4} \left(\frac{4}{3} \, r_2 + \frac{16}{45}\right) \cosh^4 s \\
+ C_1 \left(\frac{1}{4} \cosh^3 s \, \sinh s + \frac{3}{4} \left(\frac{s}{2} + \frac{1}{2} \sinh s \, \cosh s\right)\right) + C_2
\]

Notice that 

\[
\int \cosh^4 s \, ds = \left(\frac{1}{4}\right) \cosh^3 s \, \sinh s + \left(\frac{3}{4}\right) \cosh^2 s \, ds \\
= \left(\frac{1}{4}\right) \cosh^3 s \, \sinh s + \left(\frac{3}{8}\right) (s + \sinh s \, \cosh s) + C_4.
\]

Therefore 

\[
w_1 = \left(\frac{4}{5}\right) \sech^2 s \, \ln \cosh s + \left(\frac{1}{2}\right) \left(\frac{2}{3} \, r_2 + \frac{8}{45}\right) \cosh^2 s \\
+ \left(\frac{1}{4}\right) \left(\frac{4}{3} \, r_2 + \frac{16}{45}\right) \cosh^4 s \\
+ \left(\frac{C_1}{8}\right) \left[2 \cosh s \, \sinh s + 3(s + \tanh s)\right] + C_2 \sech^2 s
\]

Since \(\lim_{s \to \pm \infty} \sech^2 s \, \ln \cosh s = 0\) and \(w_1 \to 0\) as \(s \to \pm \infty\), then 

\[
C_1 = 0 \text{ and } (2/3) \, r_2 + (8/45) = 0 \Rightarrow r_2 = -4/15
\]
and the solution for equation (5.6) is

\[ w_1 = \frac{4}{5} \, \text{sech}^2 s \ln \cosh s + C_2 \, \text{sech}^2 s, \quad C_2 \text{ is constant} \]

and \( r_2 = \frac{4}{15} \)

Thus the equation (5.4) has an approximate solution

\[ w(s) = \tanh s + \sqrt{(\mu-1)((4/5)\text{sech}^2 s \ln \cosh s + C_2 \text{sech}^2 s + \ldots \right) \]

with \( r = 1 - (4/15)(\mu - 1) + \ldots \)

**Numerical evaluation of the period**

The period of the solution were evaluated numerically using the iterated inverse two-point Hermite interpolation, see Henrici [9]. We fixed \( \varepsilon = 0.01 \) and evaluated the period for different values of \( \mu, \)

\[ 1 < \mu < 2. \]

The results which are summarized in the next Table show clearly the dependence of the period on \( \mu. \)

**Wave form for \( 1 < \mu < 2 \)**

Figures 32 and 33 show monotone departure and monotone arrival in the boundary layer for \( \mu=1.1 \) while Figures 34 and 35 show oscillatory departure and monotone arrival in the boundary layer for \( \mu=1.99 \).

To get more informations about that, we need to find the characteristic equation of the equation (5.2). For this purpose we write

\[ y(\tau) = a + e^{\sigma \tau} \eta \]

and substitute in (5.2) to get
\[ e^{\sigma r} \eta = a + e^{\sigma r} \eta + f_\mu[a + e^{\sigma(r-r)} \eta] \]

Now,

\[ f_\mu[a + e^{\sigma(r-r)} \eta] = f_\mu(a) + e^{\sigma(r-r)} \eta f_\mu'(a) + \ldots \]

\[ = -a + e^{\sigma(r-r)} \eta(2\mu - 3) + \ldots \]

Approximating \( f_\mu[a + e^{\sigma(r-r)} \eta] \) by \(-a + e^{\sigma(r-r)} \eta(2\mu - 3)\),

we get the characteristic equation

\[ \sigma = 1 + (2\mu - 3)e^{-\sigma r} \]

Notice that we obtain the same equation if we use

\[ y(r) = -a + e^{\sigma r} \]

Let us study the real roots of this characteristic equation. We shall find that there are two cases.
Figure 32. Wave form for $\mu<1.5$
Figure 33. Wave form for $\mu<1.5$
Figure 34. Wave form for $\mu > 1.5$
Figure 35. Wave form for \( \mu > 1.5 \)
Case i. If \(-1<2\mu-3<0\) or \(1<\mu<1.5\), then the characteristic equation has two real solutions, one of them is positive and the other one is negative.

![Figure 36. 1 < \mu < 1.5](image)

Case ii. If \(0<2\mu-3<1\) or \(1.5<\mu<2\), then the characteristic equation has only one real root and it is positive.

**Theorem 5.10**

Consider the equation

\[
(5.11) \quad z = 1 + C e^{-\sqrt{r}},
\]

where \(C\) and \(r\) are constants, \(0<r\leq 1\).

If \(|C|<1\) then the equation (5.11) has a unique positive real root \(x_0\), and \(x_0\) is the only root with positive real part.
Proof:

Write \( z = x + iy \) to get

\[
Ce^{-r(x+iy)} = -1 + x + iy
\]

\[
Ce^{-FX}[\cos ry - i \sin ry] = -1 + x + iy
\]

Equating real and imaginary parts we get

\[
(5.12a) \quad Ce^{-FX} \cos ry = -1 + x
\]

\[
(5.12b) \quad Ce^{-FX} \sin ry = -y
\]

Let \( x > 0 \)

Thus there is no real \( y \neq 0 \) that satisfies equation (5.12b) with \( x > 0 \) because

\[
|\sin y| \leq |y|, \quad |C| < 1, \quad 0 < e^{-FX} < 1
\]

Thus the single real positive root is the only root with positive real part.

FIGURE 37. \( 1.5 < \mu < 2 \)
Remark

Squaring equations (5.12) and adding we get

\[(5.13) \quad C^2 e^{-2r\chi} = (x-1)^2 + \psi^2\]

Since \(A^2 < 1\), \(e^{-2r\chi} < 1\) for \(x > 0\) then equation (5.13) can be satisfied only for \(x \in (0, 2)\).

Theorem 5.14

If \(-1 < C < 0\), then equation (5.11) has a unique real negative solution \(x_o\). Moreover, if \(\alpha + \beta, \beta \neq 0\) is any solution of (5.11), then \(\alpha < x_o\) provided that \((1-x_o)r < 1\).

Proof:

Write \(z = \omega + x_o\) and substitute in the equation (5.11) to get

\[\omega + x_o = 1 + C e^{-(\omega + x_o)r}\]

Now \(x_o = 1 + C e^{-x_o r}\)

which implies that \(e^{-x_o r} = (x_o - 1)/C\)

Thus \(\omega + x_o = 1 + (x_o - 1) e^{-\omega r}\)

or \(\omega = 1 - x_o + (x_o - 1) e^{-\omega r}\)

which implies

\[(5.15) \quad \omega = (1-x_o)(1 - e^{-\omega r})\]

Write \(\xi = \frac{\omega}{1-x_o}\) to get

\[\xi = 1 - e^{-\xi(1-x_o)r}\]

or \(\xi - 1 + e^{-\xi(1-x_o)r} = 0\)

Let \(f(\xi) = \xi - 1\) and

\(g(\xi) = e^{-\xi(1-x_o)r}\)
\[ |g(\xi)| = e^{- (1-x_0) r} \text{Re}\, \xi \]
\[ |f(\xi)| = |\xi - 1| \]

For \((1-x_0)r < 1\), we can find \(\epsilon > 0\) such that
\[ |g(\xi)| < |f(\xi)| \] on any curve \(C_\epsilon\).

FIGURE 38.

By Rouche's theorem, the functions \(f\) and \(f+g\) have the same number of zeros inside \(C_\epsilon\). Since \(f(\xi) = 0\) only for \(\xi = 1\). Then \(f+g\) has \(\xi = 0\) as its only zero inside \(C_\epsilon\). For \(R\) arbitrarily large this implies that 
\((f+g)(\xi) = 0\) has solutions only in \(\text{Re}\, \xi < 0\) except for \(\xi = 0\).

Since \(\omega = (1-x_0)\xi\), \((1-x_0) > 1\) then all solutions of equation (5.15) with \(\text{Im}\, \omega \neq 0\) satisfy \(\text{Re}\, \omega < 0\).

Since \(\omega = z-x_0\) then all solutions of equation (5.11) with \(\text{Im}\, z \neq 0\) satisfy \(\text{Re}\, z < x_0\).
From the previous discussion, we conclude that

1. If $1<\mu<1.5$ then there exist $r>0$ such that the boundary layer equation has a monotone solution.

2. If $1.5<\mu<2$ then the boundary layer equation has no monotone solution. Moreover, there exists $r>0$ such that the boundary layer equation has a solution which oscillates about $a$ as $r \to \infty$

**Wave form for $2<\mu<\sqrt{5}$**

Recall the bifurcation diagram of Figure 10. To find the equation of the boundary layer equation of the nonsymmetric square wave we introduce the following change of variables

$$\tau = -t/\varepsilon$$

$$y_1(\tau) = x(t)$$

$$y_2(\tau) = x(t+1+r\varepsilon)$$

where the period of $x(t)$, as before, is assumed to be $2 + 2r\varepsilon + O(\varepsilon^2)$. Substituting in the delay differential equation (4.1), using the periodicity of $x(t)$ and taking formal limits as $\varepsilon \to 0$, we obtain the following differential equations

$$y_1'(\tau) = y_1(\tau) - f_\mu(y_2(\tau-\tau))$$

$$y_2'(\tau) = y_2(\tau) - f_\mu(y_1(\tau-\tau))$$

For the nonsymmetric square wave $(a_1, -a_2)$ the boundary conditions are

$$y_1(1) = a_1 \text{ and } y_2(T - \tau) = -a_2 \text{ as } \tau \to \infty$$

$$y_1(T - \tau) = -a_2 \text{ and } y_2(T - \tau) = a_1 \text{ as } \tau \to \infty$$
Equations (5.16) can be written in the form

\[ Y'(r) = Y(r) - F_\mu(Y(r-r)) \]

where

\[ Y(r) = \begin{bmatrix} Y_1(r) \\ Y_2(r) \end{bmatrix} \quad \text{and} \quad F_\mu(Y) = \begin{bmatrix} F_\mu(Y_2) \\ F_\mu(Y_1) \end{bmatrix} \]

write \( Y = Y_0 + \varepsilon e^{\sigma r} \eta \) where

\[ Y_0 = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix} \]

and substitute in equation (5.18) to get

\[ \varepsilon \sigma \eta = Y_0 + \varepsilon e^{\sigma r} \eta - F_\mu[(Y_0 + \varepsilon e^{\sigma(r-\Gamma)} \eta)] \]

Approximating \( F_\mu[(Y_0 + \varepsilon e^{\sigma(r-\Gamma)} \eta)] \) by \( F_\mu(Y_0) + \varepsilon F_\mu'(Y_0)e^{\sigma(r-\Gamma)} \eta \), we get

\[ \varepsilon \sigma \eta = Y_0 + \varepsilon e^{\sigma r} \eta - F_\mu(Y_0) - \varepsilon F_\mu'(Y_0)e^{\sigma(r-\Gamma)} \eta \]

Notice that

\[ F_\mu(Y_0) = \begin{bmatrix} f_\mu(-a_2) \\ f_\mu(a_1) \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix} = Y_0 \]

Therefore \( \sigma \eta = \eta - F_\mu'(Y_0)e^{\sigma r} \eta \)

or \( [(1-\sigma)I - e^{-\sigma r}F_\mu'(Y_0)] \eta = 0 \)

Now

\[ F_\mu'(Y_0) = \begin{bmatrix} 0 & f_\mu'(-a_2) \\ f_\mu'(a_1) & 0 \end{bmatrix} \]
Therefore

\[
\begin{bmatrix}
1 - \sigma & -e^{-\sigma r}f'_\mu(-a_2) \\
-e^{-\sigma r}f'_\mu(a_1) & 1 - \sigma
\end{bmatrix} \eta = 0
\]

The characteristic equation is

\[
\begin{vmatrix}
1 - \sigma & -e^{-\sigma r}f'_\mu(-a_2) \\
-e^{-\sigma r}f'_\mu(a_1) & 1 - \sigma
\end{vmatrix} = 0
\]

Which implies

\[(1-\sigma)^2 - e^{-2\sigma r}f'_\mu(-a_2)f'_\mu(a_1) = 0\]

Now \(f'_\mu(-a_2) = -\mu + 3a_2^2\)

\[= -\mu + (3/2)[\mu + \sqrt{\mu^2 - 4}]\]

\[= \frac{1}{2}\mu + \frac{3}{2}\sqrt{\mu^2 - 4}\]

and \(f'_\mu(a_1) = -\mu + 3a_1^2\)

\[= -\mu + (3/2)[\mu - \sqrt{\mu^2 - 4}]\]

\[= \frac{1}{2}\mu - \frac{3}{2}\sqrt{\mu^2 - 4}\]

Thus

\[f'_\mu(-a_2)f'_\mu(a_1) = (1/4)\mu^2 - (9/4)(\mu^2 - 4)\]

\[= -2\mu^2 + 9\]

Hence the characteristic equation is

\[(\sigma - 1)^2 = e^{-2\sigma r}(9 - 2\mu^2).\]
We have the following two cases for the real roots of this characteristic equations.

Case i. If $0 < -2\mu^2 < 1$ or $2 < \mu < 3/\sqrt{2}$, then the characteristic equation has three real solutions, two are positive and the third one is negative.

Case ii. If $-1 < -2\mu^2 < 0$ or $3/\sqrt{2} < \mu < 5$, then the characteristic equation has no real solutions.

\[
\begin{align*}
(x-1)^2 & \quad -2\mu \sqrt{x} \\
\frac{1}{\mu} (\mu - 2\mu^2) & \quad x
\end{align*}
\]

**FIGURE 39.** $2 < \mu < 3/\sqrt{2}$

Figures 41 and 42 show the wave form for $\mu = 2.05 < 3/\sqrt{2}$. And Figures 43 and 44 show the wave form for $\mu = 2.19 > 3/\sqrt{2}$. 
FIGURE 40. $3/2 < \mu < \sqrt{5}$
Figure 41. Wave form for $\mu<3/\sqrt{2}$
Figure 42. Wave form for $\mu<\sqrt{2}$
Figure 43. Wave form for $\mu > 3/\sqrt{2}$

MU=2.19, EP=.01
I. FUN. X=1.241419
Figure 44. Wave form for $\mu > \sqrt{3}/2$
REFERENCES


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