THE MATHEMATICAL ANALYSIS OF ELECTROMAGNETIC
FIELDS AROUND SURFACE CRACKS IN METALS

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INTRODUCTION

The work described in this paper arises from a program for the
detection and measurement of surface cracks in metals carried out
at University College London. The instrument which was developed
for the purpose, the Crack Microgauge, employs the acpd (alternating
current potential difference) method. An alternating electric current
at a frequency of 6 kHz is applied to the specimen, and the instrument
measures the voltage between the probe terminals which are applied to
the surface of the specimen. By examining the variation of the volt­
age readings with position on the surface and, in particular, the jump
in readings obtained when the probe crosses the crack, the crack can
be detected and features of its geometry deduced. The correlation
between instrument readings and information about the crack geometry
must be made by use of a theoretical model of the electromagnetic
field produced in the crack neighborhood. The authors have been prin­
cipally concerned in the study of this mathematical problem. In this
paper we have attempted to bring together in summary form the most
significant results arising from the studies on several different pro­
jects. While the emphasis of our work is on the mathematical analy­
sis, each example is motivated by practical use of the instrument in
different contexts, and we give in each case a comparison of our re­
sults with experimental data.

The basis of the analysis throughout is Maxwell's equations, 
which are

\[
\begin{align*}
\operatorname{curl} E &= \frac{\partial B}{\partial t}, \\
\operatorname{curl} H &= j + \frac{\partial D}{\partial t}, \\
\operatorname{div} D &= \rho, \\
\operatorname{div} B &= 0.
\end{align*}
\]
In addition we use the linear isotropic constitutive equations, $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{j} = \sigma \mathbf{E}$, where $\varepsilon, \mu, \sigma$ are taken to be constant. Experiments have been made on cracks on steels, aluminum, titanium and nickel alloys, all of which are good conductors in which displacement current is negligible, and the charge density $\rho = 0$ at the frequency of operation. It follows that the electric field $\mathbf{E}$ satisfies the equations

\begin{equation}
\text{div } \mathbf{E} = 0, \tag{5}
\end{equation}

\begin{equation}
\nabla^2 \mathbf{E} = k^2 \mathbf{E}, \tag{6}
\end{equation}

where $k^2 = \omega \mu \sigma$, and $\omega$ is the angular frequency.

The method of operation in the experiments is to establish a uniform current in the specimen, incident on the crack, and the object of the analysis is to establish the change in the electromagnetic field produced by the crack in its neighbourhood. If $L$ is a length scale typical of the crack dimensions, the dimensionless parameter $kL$ is of importance in determining the structure of the field. When $|kL| >> 1$ we have the classical skin effect in which current flow is confined to a thin skin over the surface of the specimen including the crack surface. When $|kL| << 1$ the skin depth becomes large compared with the crack dimensions, and in the neighbourhood of the crack the field is in effect a steady state field. Practical applications of the method extend to both these extremes, and we describe some results for thin and thick skin situations.

In the application of the a.c.p.d. method to cracks in steels, experimental results have been in good agreement with the simplest theoretical model. However, in nonferrous materials, in particular aluminum, serious discrepancies arose when the same formulae were used. The discrepancies were found to arise from the effect of magnetic induction in changing the probe readings. It can be seen that in nonferrous materials the scale of this effect is proportionately much greater than in steels. It arises either from the opening up of the crack or from the presence of an effective probe loop. Studies have been made on both of these effects. The adaptation of thin-skin theory to accommodate the first is briefly described here and good agreement between the modified theory and experiments on slots in aluminum alloys was obtained. Procedures developed from this work for characterizing probes by establishing their effective probe areas are described by Mirshekar-Syahkal (1982) and will not be discussed further here.

THIN-SKIN SOLUTIONS

Thin-skin solutions in which $|kL| >> 1$, are of importance in the testing of steel specimens. Cracks in large-scale steel structures have length dimensions $L$ typically of order 1 to 10 mm, whilst the
skin depth $\delta$ is of order 0.1mm. For the one-dimensional model of a crack with infinite aspect ratio shown in figure 1, the surface value of the incident field strength is conserved around the crack surface when the incident field is uniform. If probe readings $V_1$ and $V_2$ are taken in the positions ST and S'T', the crack depth $d$ is

$$d = \frac{1}{2} \Delta (V_2/V_1 - 1), \quad (7)$$

where $\Delta$ is the probe length, (Dover et al. 1980, 1981). Provided that the crack has a large aspect ratio, this one-dimensional interpretation of the instrument readings $V_1$ and $V_2$ gives good predictions of crack depth. It neglects the effects of order $\delta/d$ arising from the fields within a neighbourhood of length $\delta$ of the crack edges, but these effects have been calculated by Michael et al. (1982a) and have been found to be small in many circumstances. In a recent paper, Mirshekar-Syakhal et al. (1982) have also given a solution of this problem for arbitrary values of $\delta/d$ which shows that equation (7) gives a very good approximation to $d$ irrespective of the value of $\delta/d$, provided only that $\delta/\Delta$ is small. Current practice is to use probe lengths of 10mm so that in mild steels this condition is fulfilled.
Figure 2. Sketch of the geometry of a surface-breaking crack of finite aspect ratio.

Figure 3. Comparison of electrical and optical measurements of the centre-line depth in the growth of a semi-elliptical crack. ●, elliptical solution; ○, equation (7).
A serious limitation on the use of equation (7) is the neglect of the crack aspect ratio; figure 2 illustrates a more realistic crack geometry. In many cases the growth of fatigue cracks takes place in approximately semi-elliptical plan forms and the use of equation (7) to predict centre-line crack depth can give rise to serious underestimates if the crack aspect ratio is small. For example, figure 3 gives results for a fatigue crack growing as a semi-ellipse with aspect ratio approximately 3.65 and the results show that equation (7) gives a 30% underestimate of the true centre-line depth in this case. To overcome this defect an analysis of the global surface field distribution for cracks of finite aspect ratio was given by Dover et al. (1980, 1981) using an unfolded field algorithm. The resulting solution for a semi-elliptical crack was given by Collins et al. (1981). The revised centre-line depths, shown also in figure 3, are then in very good agreement with optical observations.

MAGNETIC INDUCTION EFFECTS

Another limitation to the use of equation (7) occurs when a crack opens out. When this happens probe readings taken across the crack are affected by an induced e.m.f. arising from the changing flux of the magnetic field through the crack cross section. The effect is in general difficult to calculate, but in the case of a two-dimensional slot, as illustrated in figure 4 it can easily be calculated when a uniform current is incident on the crack. In that case the $B$ field above the surface and in the slot is spatially constant. Allowing for this effect, and assuming an ideal probe which has no induction loop itself, it can be seen (Mirshekar-Syakhal et al. 1981) that equation (7) for thin skins is replaced by the equation

$$
\frac{V_2 - V_1}{V_1} = \frac{2d + \alpha d/2 \cos(\theta + \pi/2)}{\Delta \Delta \mu \delta \cos(\theta + \pi/4)}.
$$

The last term gives the additional induction effect, and when the slot is closed $\alpha = 0$ and we recover equation (7). The angle $\theta$ is an instrument-dependent phase angle which is a function of the instrument balance position, and $\delta$ is taken to be $\sqrt{2}/|k|$. The most significant parameter in the additional term is the relative permeability $\mu_r$ of the material. In steels $\mu_r$ is of order 500-1000. The induction effect is then unimportant. But in nonferrous metals, like aluminum and titanium where $\mu_r \approx 1$, the extra term is significant. To examine the effect experimentally measurements were made on a series of rectangular slots cut in aluminum and steel plates. Results are shown in Table 1. When equation (7) is used to predict the slot depth $d$ the values labelled $d_1$ in the table were obtained. These values are seen to be seriously in error for slots
in the aluminum alloys, but accurate for slots in steel. The value \( d_f \) calculated from equation (8) produces accurate values for the depths of slots in both materials. The values \( d_c \) give a further adjustment to the depth formula to allow for the edge effects calculated by Michael et al. (1982a) but in this context these effects are very small.
In the limiting case when $|k\ell| \ll 1$, the field is a static field, and we obtain Laplace's equation

$$\nabla^2 E = 0, \quad (9)$$

for the distribution around the crack at distances of order $\ell$. The solution of this problem for a uniform $E$ field incident on a two-dimensional normal crack is very well known in potential theory. It was shown by Mirshekar-Syahkal et al. (1982) that it leads to the limiting thick-skin formula

$$d = \frac{1}{4} \Delta (V_2/V_1 - 1)(V_1/V_2)^{1/2}. \quad (10)$$

Equations (7) and (10) give the two extremes of the solution for a normal crack of infinite aspect ratio. The general solution given by Mirshekar-Syahkal et al. (1982) spans the range of values of $|k\ell|$ between the limits 0 and $\infty$. In addition Michael and Collins (1982) have given perturbation solutions for the thick-skin limit correct to the order $|k\ell|^2$. A useful result of the general solution which is mentioned earlier is that provided only that the ratio $\delta/\Delta$ is small, the one-dimensional interpretation in equation (7) gives a very good approximation for $d$ for all values of $\delta/d$.

We turn finally to an interesting application of the thick-skin theory. Under contract from British Aerospace, the Crack Microgauge group tested titanium and Inconel bolts to be used in the pallet system of the Space Shuttle, and a method for detecting and measuring fatigue cracks in the bolt threads was required. The apparatus shown in figure 5 was constructed for this purpose. Bolts with diameters 12, 14 and 30 mm were tested, the thread depths were 1–2 mm and the fatigue cracks of large aspect ratio occurred in lines parallel to the threads with depths of order 0.1 mm. Alternating current at 6kHz was passed along the bolt through contacts on the bolt axis, and measurements were made of the voltage difference between adjacent thread crowns. The probe head which holds the voltage pick-up points is shaped as part of a nut so that rotation of the bolt causes the probe to travel along the thread. A displacement transducer attached to the probe head is used to measure the probe position relative to the bolt head. The transducer and probe output signals were recorded on an x–y plotter. Figure 6 shows some typical traces obtained. The main feature is that a peak in the probe reading occurs as the probe passes over the crack and in addition troughs are recorded at intervals of one complete turn away from the crack site. For a notch or crack located at the thread root, the troughs are symmetrical about the peak signal position (figure 6a), but cracks on the thread flanks give an asymmetric signal pattern (figure 6b). A detailed account of
Figure 5. General arrangement of experimental jig for tests on bolts.

Figure 6. (a) Symmetric signal trace for a centered notch at the root on a 28 mm titanium bolt. (b) Asymmetric trace for a naturally occurring crack on the flank of a 12 mm titanium bolt.
the analysis of this problem is given by Michael et al. (1982b). Here we give a brief outline of it. For titanium and Inconel, the skin depth at the operating frequency is 7-8 mm, which is several times greater than the thread depths, and much greater than the crack depth. A thick-skin analysis of the field is therefore appropriate. Also since the thread and crack depths are small compared with the bolt radius we have neglected the bolt curvature.

The mathematical problem may be posed in terms of a stream function $\psi(x,y)$, where $E_x = \partial \psi / \partial y$, $E_y = -\partial \psi / \partial x$. Figure 7a shows the boundary value problem to be solved for a normal crack $A_0 Q$ in a thread of saw-tooth profile $\ldots A_{-2} A_{-1} A_0 A_{+1} A_{+2} \ldots$. The probe, of length $\Delta$, samples the voltage between successive crowns $\ldots A_{-3}, A_{-1}, A_{+1}, A_{+3}, \ldots$. The problem is solved using the Schwarz-Christoffel conformal mapping

$$z = i^{-p} \int_0^w \tan^p(\pi w/\Delta)dw,$$

$$p = 1-2a/\pi$$

(11)

which transforms it into the normal-barrier problem in the $w$ plane, as shown in figure 7b.

![Figure 7](image-url)

Figure 7. (a) The boundary value problem in the $z$ plane for a normal crack at a thread root. (b) The conformal mapping to the $w$ plane with equation (11).
The solution for the complex potential $\Omega = \phi + i\psi$ is

$$\Omega = (w^2 + c^2)^{1/2},$$  \hfill (12)

and the relation between the lengths $c$ and $d$ in the $w$ and $z$ planes is

$$d/\Delta = \int_0^c \tanh^{\pi/\Delta} d\theta.$$  \hfill (13)

The difference in $\phi$ between two points is a measure of the voltage difference. The value of $\phi$ at the thread crown $A_{2n-1}$ is

$$\Delta\{(n-\frac{1}{2})^2 + (c/\Delta)^2\}^{1/2}.$$  

The voltage difference between consecutive thread crowns is thus $V_n = \phi_{n+1} - \phi_n$. We find that $V_\infty = \Delta$, and since the trace measures the difference between $V_n$ and $V$, we define

$$s_n = (V_n - V_\infty)/V_\infty$$

as a proportionate measure of the $n$th satellite signal strength. Figure 8 shows the ratios $s_n/s_0$, and the figure reproduces the pattern of satellite signals observed in figure 6a. We can calculate from the

![Figure 8](image)

Figure 8. Scale diagram of the theoretical satellite signal ratios for a centred normal crack.

theory the satellite signal ratios $s_n/s_0$ as a function of the peak signal strength $V_n/V_\infty$, as shown in figure 9 for $n = 1, 2, 3$. Figure 10 shows the relation between crack depth and $V_n/V_\infty$. These theoretical results are compared in figures 9 and 10 with experimental data obtained from tests on notches cut centrally at the thread root in Inconel bolts. Broad agreement between theory and experiment was obtained. Figure 10 shows in addition the thin-skin equation (7). As the notch depth increases it begins to penetrate through the skin, and as expected the data then begin to move towards the thin skin line. The analysis has been extended to describe other thread forms with truncated crowns and roots and the effect of such a truncation is shown in figure 10; it is seen to be slight. In this example the value given to the parameter $s$ produces a good approximation to the I.S.O. metric thread form.
Naturally-growing fatigue cracks may initiate at any position on the thread and in most cases appear on the flank near the change in section. Michael et al. (1982b) give a further extension of the mathematical analysis to describe this feature. It is found that crack positions can be deduced from the asymmetry of the satellite signals shown, for example, in figure 6 b. Figure 11 gives the theoretical relationships between crack depth $d_*/D$ and $(V_*/V_0-1)$ for a truncated profile with a centred crack ($f'=0$) and a crack at a position about 30% along the thread profile ($f'=0.3$). The data points on this figure are for natural fatigue cracks on 12mm titanium bolts and they fall within this band. The departure from this theoretical band and the increased scatter which occurs at larger crack depths are attributable to changes in the crack geometry which occur during its growth. The nature of the bolt material and the forging process are such that the material near the surface often has a greater resistance to fatigue cycling than that in the interior. Thus below a certain critical depth small fatigue cracks are found to be long and slender and their high aspect ratios are well described by the theoretical model. For deeper cracks beyond this critical depth the shape
Figure 10. Theory and experiment for the depths of centred notches.

Figure 11. Theory and experiment for off-centred fatigue cracks.
becomes distorted as the centre of the crack grows more rapidly and this results in a crack of bell-shaped form of relatively low aspect ratio. The two-dimensional theory outlined here is then no longer an adequate description of the field.

We conclude with references to related work which has been performed for application to eddy current instruments. Kahn et al. (1977) studied contributions from 90° corners and Kahn (1981) has also recently attacked the arbitrary skin-thickness problem for an inclined crack by numerical means. Auld et al. (1982) have taken into account induction effects produced by crack opening and in this meeting Muennemann et al. (1982) have considered developments of the unfolded field technique for the inversion of eddy current signals.

REFERENCES


DISCUSSION

C.M. Teller (Southwest Research Institute): You didn't mention the size of the slots that were used there. Could you give us some idea?

D.H. Michael (University College London): Yes, I should have mentioned it. The bolts were on 12 mm, 40 mm, and 30 mm diameters. The threads were typically the same size as the ones you mentioned, about 1 to 2 mm. The crack sizes ranged from about .1 mm up to 1 mm. One mm was a large crack size.

C.V. Dodd (Oak Ridge National Laboratory): Did you run into an end effect when you got near the end of the threads?

D.H. Michael: I don't think that was the problem; I don't know. We haven't looked at it. Certainly I think the data that we have are not subject to that.