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Zhengcheng Zhang
Lanzhou Jiaotong University

William Q. Meeker
Iowa State University, wqmeeker@iastate.edu

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Abstract

In this paper, we consider the residual lifetimes of surviving components of a failed coherent system with n independent and identically distributed components, given that before time t_1 ($t_1 > 0$), exactly r ($r < n$) components have failed and, at time t_2 ($t_2 > t_1$) the system just failed. Some aging properties and preservation results of the residual lives of the surviving components of such systems are obtained. Also some examples and applications are given.

Keywords

increasing failure rate, order statistics, pseudo signature, residual life, stochastic order

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On the Residual Lifetime of Surviving Components from a Failed Coherent Systems

Zhengcheng Zhang and William Q. Meeker

Abstract—In this paper, we consider the residual lifetimes of surviving components of a failed coherent system with n independent and identically distributed components, given that before time t_1 ($t_1 > 0$), exactly r ($r < n$) components have failed and, at time t_2 ($t_2 > t_1$) the system just failed. Some aging properties and preservation results of the residual lives of the surviving components of such systems are obtained. Also some examples and applications are given.

Index Terms—Order statistics, stochastic order, pseudo signature, IFR, residual life.

ACRONYMS

IFR increasing failure rate
DFR decreasing failure rate

NOTATIONS

X, Y random life lengths
 F, G distribution functions of X , and Y respectively
 f, g density functions of X , and Y respectively
 \bar{F}, \bar{G} reliability functions of X , and Y respectively
 \mathbf{X} vectors (X_1, \dots, X_n)
 τ structure functions of coherent systems
 \mathbf{s} signature vector of coherent systems
 $X_{r,n}$ the r -th order statistic of X_1, \dots, X_n
 \leq_{st} stochastic order
 \leq_{hr} hazard rate order
 \leq_{rh} reversed hazard rate order
 \leq_{lr} likelihood ratio order

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Z. Zhang is with the School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou, 730070, China (e-mail: zhzhcheng004@163.com).

William Q. Meeker is with the Department of Statistics, Iowa State University, Ames, Iowa, 50011-1210, USA (email: wqmeeker@iastate.edu).

I. INTRODUCTION AND PRELIMINARIES

Coherent systems are very important in reliability theory and survival analysis. A technical structure is said to be coherent if each of its components is relevant (i.e., the system would not contain any component whose functioning has no influence on whether or not the system works) and if its structure function is monotone (i.e., replacing a failed component by a working component can not cause a working system to fail). A physical system is usually a coherent system. A k -out-of- n ($k = 1, 2, \dots, n$) system, parallel-series and series-parallel systems are of important coherent systems. The k -out-of- n system works if and only if at least k components work. The cases of $k = n$ and $k = 1$ correspond to the usual series and parallel systems, respectively.

In order to investigate the performance of a coherent system and compare structures between two coherent systems, a very important concept, signature, is introduced. The signature has proven to be a useful proxy for a system's design, as it does not depend on the underlying distribution of component lifetimes. Let X_1, \dots, X_n be the lifetimes of n independent and identically distributed (i.i.d.) components of a coherent system, and let $X_{1,n}, \dots, X_{n,n}$ denote the corresponding order statistics. For a coherent system with n i.i.d. lifetimes X_1, \dots, X_n , the system lifetime can be represented as $T = \tau(X_1, \dots, X_n)$, where τ is a coherent life function, giving the failure time of the system as a function of the failure times of the n components. For example, if a parallel system has two components, then $\tau(X_1, X_2) = \max(X_1, X_2)$, where X_1 and X_2 are the components life lengths. For more details about coherent life functions, see Esary and Marshall [8], and Barlow and Proschan [5]. Samaniego [24] first defined the signature of a coherent

system as a probability vector $\mathbf{s} = (s_1, \dots, s_n)$ whose j th element is the probability that the system fails upon the failure of the j th component, that is,

$$s_j = \Pr(T = X_{j,n})$$

for $j = 1, 2, \dots, n$, and $\sum_{j=1}^n s_j = 1$. In Table 1, we give the signature of coherent systems with 3 i.i.d. components. Also Samaniego [24] and Kochar, Mukerjee and Samaniego [12] subsequently showed that the reliability function of a coherent system having n i.i.d. components can be expressed as a mixture of those of k -out-of- n systems. That is, for any $t > 0$,

$$\Pr(T > t) = \sum_{k=1}^n s_k \Pr(X_{k,n} > t). \quad (1)$$

It is shown by Navarro and Rychlik [21] that the representation (1) also holds when the lifetimes of components have an absolutely continuous exchangeable distribution (i.e., the joint distribution of X_1, X_2, \dots, X_n is invariant under permutation of the variables). For example, consider a coherent system with lifetime $T = \min(X_1, \max(X_2, X_3, X_4))$ has signature $\mathbf{s} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$. Then, for $t > 0$, its reliability function can be represented as

$$\begin{aligned} \Pr(T > t) &= \frac{1}{4} \Pr(X_{1,4} > t) \\ &\quad + \frac{1}{4} \Pr(X_{2,4} > t) + \frac{1}{2} \Pr(X_{3,4} > t). \end{aligned}$$

For more details about signatures, readers can refer to Boland [6], Navarro, Ruiz and Sandoval ([19], [20]), Navarro et al. [22], Samaniego ([24], [25]), and Zhang [31].

The residual lifetime of a coherent system is an important concept in reliability theory and survival analysis. Much has been written about the residual lifetime of coherent systems. For more details, readers can refer, for example, to Bairamov, Ahsanullah and Akhundov [2], Asadi and Bairamov [1], Navarro and Eryilmaz [17], Hu et al. [10], Khaledi and Shaked [11], Bairamov and Arnold [3], Li and Zhao [14], Navarro, Balakrishnan and Samaniego [16], Navarro and Hernandez [18], Gurler and Bairamov [9], Navarro and Shaked [23], Kochar, Mukerjee and Samaniego [12], Li and Zhang [13], Zhang and Li [28], Zhang and Yang [29], Eryilmaz [7], and Zhang [30].

Coherent systems of order n with one or more surviving components at the time of system failure have a signature of form

$$\mathbf{s} = (s_1, s_2, \dots, s_n), \quad s_n \neq 1. \quad (2)$$

Table 1: Signature vectors of some coherent systems.

System	$T = \phi(X_1, X_2, X_3)$	\mathbf{s}
1	$\min\{X_1, X_2, X_3\}$	$(1, 0, 0)$
2	$\min\{X_1, \max\{X_2, X_3\}\}$	$(\frac{1}{3}, \frac{2}{3}, 0)$
3	$X_{2,3}$ (2-out-of-3)	$(0, 1, 0)$
4	$\max\{X_1, \min\{X_2, X_3\}\}$	$(0, \frac{2}{3}, \frac{1}{3})$
5	$\max\{X_1, X_2, X_3\}$	$(0, 0, 1)$

For example, from Table 1, the signatures of Systems 1, 2, 3 and 4 (four of five systems with three i.i.d. components), have the form as that in equation (2). A coherent system with a signature of the form (2) has the property that the component with lifetime $X_{k,n}$ for $k > 1$ will have remaining life after the system has failed. Hence, after the failure of the system, the surviving components may be removed from the system and then can be used for other purposes. Therefore, the study of the reliability properties such as aging properties of such surviving components may be of interest to engineers and system designers.

In some applications, systems are continuously monitored and component failures are known as soon as they occur. In other applications, because of constraints or other reasons, systems are maybe inspected at some fixed time. For example, suppose that there is a planned inspection at time t_1 and it is known when the system fails at time $t_2 (> t_1)$, and at time t_1 , exactly r failed ($r < n$) components were noted.

In this paper we consider a coherent system (with lifetime T having n i.i.d. components) with a signature vector of form (2), and we are interested in the conditional residual life

$$(X_{k,n} - t_2 | X_{r,n} \leq t_1 < X_{r+1,n}, T = t_2) \quad (3)$$

for $0 \leq r < n$ (assume $X_{0,n} = 0$) and $n \geq k > r$. In fact the random variable in equation (3) is the residual life of a surviving component at time t_2 ($t_2 > t_1 > 0$) in the system, given that before time t_1 , exactly r ($r < n$) components have failed and the system failed at time t_2 .

In Section 2, first we obtain a mixture representation of the residual lifetime of surviving components in a

failed coherent system with i.i.d. components, given that before time t_1 ($t_1 > 0$), the number of failed components is exactly r ($r < n$), and that the system failed at time t_2 ($t_2 > t_1$). Secondly we present some aging properties and preservation results of the residual lives of the surviving components of a failed coherent system. In addition, we obtain some stochastic properties of the mixture distribution and the residual life of the surviving components of a failed k -out-of- n systems.

Throughout the paper, we assume that the coherent systems have a signature with the form in (2). Also we use the terms *increasing* and *decreasing* to mean nondecreasing and nonincreasing, respectively.

II. MAIN RESULTS

We consider a system with lifetime T having n original components with i.i.d. lifetimes X_1, \dots, X_n according to the continuous distribution function F . Suppose that the system has the signature of the form $\mathbf{s} = (s_1, \dots, s_n)$ ($s_n \neq 1$) and r is the number of failed components before time t_1 . For $0 \leq r < k \leq n$ and $0 < t_1 < t_2$, the reliability function of the conditional residual life ($X_{k,n} - t_2 | X_{r,n} \leq t_1 < X_{r+1,n}, T = t_2$) of the surviving components can be represented as follows:

$$\begin{aligned}
& \Pr(X_{k,n} - t_2 > x | X_{r,n} \leq t_1 < X_{r+1,n}, T = t_2) \\
&= \sum_{i=r+1}^n \Pr(X_{k,n} - t_2 > x, T = X_{i,n} | A_r(t_1), T = t_2) \\
&= \sum_{i=r+1}^n \Pr(X_{k,n} - t_2 > x | A_r(t_1), T = t_2, T = X_{i,n}) \\
&\quad \times \Pr(T = X_{i,n} | A_r(t_1), T = t_2) \\
&= \sum_{i=r+1}^n \Pr(X_{k,n} - t_2 > x | A_r(t_1), X_{i,n} = t_2) \\
&\quad \times \Pr(T = X_{i,n} | A_r(t_1), T = t_2) \\
&= \sum_{i=r+1}^n p_i(t_1, t_2) \Pr(X_{k,n} - t_2 > x | A_r(t_1), X_{i,n} = t_2),
\end{aligned} \tag{4}$$

where the event $A_r(t_1) = \{X_{r,n} \leq t_1 < X_{r+1,n}\}$, that r failures have been observed at the inspection time t_1 , and for $i = r + 1, \dots, n$, $p_i(t_1, t_2) = \Pr(T = X_{i,n} | A_r(t_1), T = t_2)$ such that $\sum_{i=r+1}^n p_i(t_1, t_2) = 1$. And $p_i(t_1, t_2)$ is the probability that the unit with lifetime $X_{i,n}$ ($i = r + 1, \dots, n$) would cause the system to fail given that

before time t_1 ($t_1 > 0$), exactly r ($r < n$) components have failed, and at time t_2 the system failed. Because, $p_i(t_1, t_2)$ depends on the signature s_i , the distribution F , also t_i ($i = 1, 2$), we call it a pseudo signature. For more information about other related signatures such as dynamic signatures, readers can see Balakrishnan and Asadi [4], Samaniego, Balakrishnan and Navarro [26], and Mahmoudi and Asadi [15]. The third equality holds because the events $\{T = X_{i,n}\}$ and $\{A_r(t_1), X_{i,n} = t_2\}$, $\{T = X_{i,n}\}$ and $\{X_{k,n} - t_2, A_r(t_1), X_{i,n} = t_2\}$ are independent, respectively. The vector

$$\mathbf{p}(t_1, t_2) = (0, \dots, 0, p_{r+1}(t_1, t_2), \dots, p_n(t_1, t_2)) \tag{5}$$

is called a pseudo signature vector.

Remark 1. It can be found that, from equation (4), that the residual life of surviving components in a failed coherent system can be represented as the mixture of the residual lives of the surviving components with lifetime $X_{k,n}$ in a $(n - i + 1)$ -out-of- n ($k > r$) system given that before time t_1 ($t_1 > 0$), exactly r ($r < n$) components have failed, and at time t_2 the system failed. The residual life of the surviving components in a failed coherent system can also be expressed as a mixture of the residual lifetimes of a $(n - k + 1)$ -out-of- n system given that i ($i < k$) components have failed before or at time t_2 , and the mixture weights are given by the pseudo signature vector $\mathbf{p}(t_1, t_2) = (0, \dots, 0, p_{r+1}(t_1, t_2), \dots, p_n(t_1, t_2))$.

For example, consider the system with lifetime $T = \min\{X_1, \max\{X_2, X_3, X_4\}\}$, whose signature is $\mathbf{s} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$, where X_1, X_2, X_3, X_4 are i.i.d. with survival function

$$\bar{F}(x) = \frac{1}{(x+1)^3},$$

for $x \geq 0$. Suppose that at time t_1 , exactly one component has failed and at time $t_2 > t_1$ the system failed. Now we consider the residual life of the fourth component that was surviving at the time that the system failed. By some computations, it can be shown that the coefficients in (4) (i.e., the pseudo signature) are given by

$$\begin{aligned}
p_2(t_1, t_2) &= \frac{1}{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 1}, \\
p_3(t_1, t_2) &= \frac{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 2}{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 1}.
\end{aligned}$$

Hence from (4), we have

$$\begin{aligned} & \Pr(X_{4,4} - t_2 > x \mid A_1(t_1), T = t_2) \\ &= \frac{1}{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 1} \Pr(X_{4,4} - t_2 > x \mid A_1(t_1), X_{2,4} = t_2) \\ &+ \frac{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 2}{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 1} \Pr(X_{4,4} - t_2 > x \mid A_1(t_1), X_{3,4} = t_2), \end{aligned}$$

where $A_1(t_1) = \{X_{1,4} \leq t_1 < X_{2,4}\}$.

In the following we assume that the pseudo signature vector of coherent systems takes the form in (5), implying that at time t_1 , exactly r components have failed while the system was still operating, and at time $t_2 (> t_1)$, the system failed. Some properties of the pseudo signature vector $\mathbf{p}(t_1, t_2)$ are given first. The following proposition shows that $p_k(t_1, t_2)$ can be represented as the function of the ratio $\bar{F}(t_1)/\bar{F}(t_2)$. Its proof can be found in the Appendix.

Proposition 1: For $0 \leq r < k \leq n$ and $0 < t_1 < t_2$,

$$p_k(t_1, t_2) = \frac{s_k(\phi(t_1, t_2) - 1)^k}{\sum_{j=r+1}^n s_j(\phi(t_1, t_2) - 1)^j}, \quad (6)$$

where $\phi(t_1, t_2) = \bar{F}(t_1)/\bar{F}(t_2)$.

Proposition 2: (a) Given a fixed t_1 , $p_{r+1}(t_1, t_2)$ is decreasing in t_2 , and $\lim_{t_2 \rightarrow \infty} p_{r+1}(t_1, t_2) = 0$; Given a fixed t_2 , $p_{r+1}(t_1, t_2)$ is increasing in t_1 ; (b) Given a fixed t_1 , $p_n(t_1, t_2)$ is increasing in t_2 , and $\lim_{t_2 \rightarrow \infty} p_n(t_1, t_2) = 1$; Given a fixed t_2 , $p_n(t_1, t_2)$ is decreasing in t_1 .

Proof. For $j = r + 1, \dots, n$, and $t_2 > t_1 > 0$,

$$\begin{aligned} p_j(t_1, t_2) &= \frac{s_j(\phi(t_1, t_2) - 1)^j}{\sum_{i=r+1}^n s_i(\phi(t_1, t_2) - 1)^i} \\ &= \frac{s_j}{\sum_{i=r+1}^n s_i(\phi(t_1, t_2) - 1)^{i-j}}. \end{aligned} \quad (7)$$

It is clear that $\phi(t_1, t_2) - 1$ is increasing in t_2 and hence $(\phi(t_1, t_2) - 1)^{i-j}$ is increasing in t_2 when $j = r + 1$. Thus $p_{r+1}(t_1, t_2)$ is decreasing in t_2 and $\lim_{t_2 \rightarrow \infty} p_{r+1}(t_1, t_2) = 0$. The proof of the remaining parts of (a) and (b) are similar and hence are omitted.

In order to properly study other stochastic behaviors of coherent systems, we review the following definitions of stochastic orders. Let X and Y denote random variables representing the lifetimes of two components, with respective distribution functions $F(x)$ and $G(x)$, and survival functions $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$. We

use $f(x)$ and $g(x)$ to denote the corresponding probability density functions of X and Y , respectively. Then X is said to be smaller than Y in the:

(a) usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x .

(b) hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{F}(x)/\bar{G}(x)$ is decreasing in x .

(c) reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $F(x)/G(x)$ is decreasing in x .

(d) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f(x)/g(x)$ is decreasing in the union of their supports.

For a more comprehensive discussions on properties and other details of these stochastic orderings, readers can refer to Shaked and Shanthikumar [27].

Proposition 3: For $0 < t_1 < t_2 < t_3$, suppose we have pseudo signature vector $\mathbf{p}(t_1, t_i) = (0, \dots, 0, p_{r+1}(t_1, t_i), \dots, p_n(t_1, t_i))$, $i = 2, 3$. Then $\mathbf{p}(t_1, t_2) \leq_{st} \mathbf{p}(t_1, t_3)$. That is, $\mathbf{p}(t_1, t_2)$ is stochastically increasing in time t_2 .

See its proof in the Appendix.

Proposition 4: Suppose that $\mathbf{s}^{(1)} = (s_1^{(1)}, \dots, s_n^{(1)})$ and $\mathbf{s}^{(2)} = (s_1^{(2)}, \dots, s_n^{(2)})$ are the signatures of two coherent systems, respectively. Let $\mathbf{p}^{(1)}(t_1, t_2) = (0, \dots, 0, p_{r+1}^{(1)}(t_1, t_2), \dots, p_n^{(1)}(t_1, t_2))$ and $\mathbf{p}^{(2)}(t_1, t_2) = (0, \dots, 0, p_{r+1}^{(2)}(t_1, t_2), \dots, p_n^{(2)}(t_1, t_2))$ be the corresponding pseudo signature vectors for $0 < t_1 < t_2$, where r is as defined in (3). If $\mathbf{s}^{(1)} \leq_{lr} \mathbf{s}^{(2)}$. Then $\mathbf{p}^{(1)}(t_1, t_2) \leq_{st} \mathbf{p}^{(2)}(t_1, t_2)$.

Proof. It is sufficient to show that, for all $i = r + 1, \dots, n$,

$$\begin{aligned} & \frac{\sum_{j=i}^n s_j^{(1)}(\phi(t_1, t_2) - 1)^j}{\sum_{k=r+1}^n s_k^{(1)}(\phi(t_1, t_2) - 1)^k} \\ & \leq \frac{\sum_{j=i}^n s_j^{(2)}(\phi(t_1, t_2) - 1)^j}{\sum_{k=r+1}^n s_k^{(2)}(\phi(t_1, t_2) - 1)^k}. \end{aligned} \quad (8)$$

The inequality in equation (8) is equivalent to

$$\begin{aligned} & \sum_{j=i}^n \sum_{k=r+1}^n (\phi(t_1, t_2) - 1)^j (\phi(t_1, t_2) - 1)^k \\ & \times (s_j^{(2)} s_k^{(1)} - s_j^{(1)} s_k^{(2)}) \geq 0. \end{aligned}$$

It is clear that, for $i = r + 1$, the left-hand summation above equals to zero. For $i = r + 2, \dots, n$, the inequality can be reduced to

$$\sum_{j=i}^n \sum_{k=r+1}^{i-1} (\phi(t_1, t_2) - 1)^{k+j} (s_j^{(2)} s_k^{(1)} - s_j^{(1)} s_k^{(2)}) \geq 0.$$

It is noted that $\phi(t_1, t_2) - 1$ is always nonnegative for $t_1 < t_2$. The condition $\mathbf{s}^{(1)} \leq_{lr} \mathbf{s}^{(2)}$ implies that the inequality above holds and hence the result follows.

The result above indicates that, for two systems having common i.i.d. components but having different structures (signatures), the stronger the signature vector of a system is in the likelihood ratio order, the stronger the corresponding pseudo signature vector of the system is in the usual stochastic order.

Proposition 5: For $i = r + 1, \dots, n$, and $n \geq k > r$, $(X_{k,n} - t_2 \mid A_r(t_1), X_{i,n} = t_2) \stackrel{d}{=} Z_{k-i, n-i}^{t_2}$, where $\stackrel{d}{=}$ means equality in distribution, and $Z_{k-i, n-i}^{t_2}$ is the $(k-i)$ th order statistic among $n-i$ random variables that are i.i.d. with the same distribution as $(X - t_2 \mid X > t_2)$.

Proof. Let h be a small positive real number, then, for any $x \geq 0$, and $0 < t_1 < t_2$,

$$\begin{aligned} & \Pr(X_{k,n} - t_2 > x \mid A_r(t_1), X_{i,n} = t_2) \\ &= \lim_{h \rightarrow 0} \frac{\Pr(X_{k,n} - t_2 > x, A_r(t_1), t_2 \leq X_{i,n} \leq t_2 + h)}{\Pr(A_r(t_1), t_2 \leq X_{i,n} \leq t_2 + h)} \\ &= \frac{\sum_{j=n-k+1}^{n-i} \binom{n-i}{j} \bar{F}^j(t_2 + x) (\bar{F}(t_2) - \bar{F}(t_2 + x))^{n-i-j}}{\bar{F}(t_2)^{n-i}} \\ &= \sum_{j=n-k+1}^{n-i} \binom{n-i}{j} [\bar{F}_{t_2}(x)]^j [1 - \bar{F}_{t_2}(x)]^{n-i-j} \\ &= \Pr(Z_{k-i, n-i}^{t_2} > x). \end{aligned}$$

where $\bar{F}_{t_2}(x) = \bar{F}(t_2 + x) / \bar{F}(t_2)$. Thus the result holds.

Remark 2. From the result in Proposition 5, it can also be found that $(X_{k,n} - t_2 \mid A_r(t_1), X_{i,n} = t_2) \stackrel{d}{=} (X_{k-i, n-i} - t_2 \mid X_{1, n-i} > t_2)$. Also, it is noted that the residual life of surviving components only depends on the recent time t_2 (system failure time), which reflects the Markov property of order statistics.

The following result follows from Theorem 1.C.31 of Shaked and Shantikumar [27].

Proposition 6: For $n \geq k$, $Z_{k-i, n-i}^{t_2} \leq_{lr} Z_{k-j, n-j}^{t_2}$ when $i > j$. That is, $(X_{k,n} - t_2 \mid A_r(t_1), X_{i,n} = t_2)$ is decreasing in i in the likelihood ratio order.

Remark 3. From Theorem 1.C.37 and Corollaries 1.C.38 and 1.C.39 of Shaked and Shantikumar [27], it is clear that, for $i \leq j$ and $m \geq n$,

$$Z_{k-j, m-j}^{t_2} \leq_{lr} Z_{k-i, n-i}^{t_2},$$

and hence, for $i \leq j$ and $m \geq n$, $(X_{k,m} - t_2 \mid A_r(t_1), X_{j,m} = t_2) \leq_{lr} (X_{k,n} - t_2 \mid A_r(t_1), X_{i,n} = t_2)$.

Recall that F is IFR (increasing failure rate) if its failure rate (hazard) function $h(t) = f(t) / \bar{F}(t)$ is increasing in $t \geq 0$, and F is DFR (decreasing failure rate) if $h(t)$ is decreasing in $t \geq 0$, given that F is absolutely continuous, where $f(t)$ is the density function of F . In the following, we show that if the components of the system are IFR then $\Pr(X_{k,n} - t_2 > x \mid A_r(t_1), T = t_2)$ is decreasing in t_2 , and if $\Pr(X_{k,n} - t_2 > x \mid A_r(t_1), T = t_2)$ is increasing in t_2 then the components of the system are DFR in $t \in [t_2, \infty)$. In order to get this result, we need the following lemma.

Lemma 7: For $0 \leq r < k \leq n$ and given a fixed $t_1 > 0$, if F is IFR, then $\Pr(X_{k,n} - t_2 > x \mid A_r(t_1), X_{i,n} = t_2)$ is decreasing in $t_2 (> t_1)$.

Proof. For any $x \geq 0$, and $0 < t_1 < t_2$,

$$\begin{aligned} & \Pr(X_{k,n} - t_2 > x \mid A_r(t_1), X_{i,n} = t_2) \\ &= \sum_{j=n-k+1}^{n-i} \binom{n-i}{j} [\bar{F}_{t_2}(x)]^j [1 - \bar{F}_{t_2}(x)]^{n-i-j} \\ &= \int_{1 - \frac{\bar{F}(t_2+x)}{\bar{F}(t_2)}}^1 (k-i) \binom{n-i}{k-i} u^{k-i-1} (1-u)^{n-k+i} du. \quad (9) \end{aligned}$$

Note that F is IFR if and only if $1 - \bar{F}(t_2 + x) / \bar{F}(t_2)$ is increasing in t_2 . Therefore the result follows immediately.

Proposition 8: (a) If F is IFR, then $\Pr(X_{k,n} - t_2 > x \mid A_r(t_1), T = t_2)$ is decreasing in t_2 for $0 \leq r < k \leq n$ and $0 < t_1 < t_2$; (b) If $\Pr(X_{k,n} - t_2 > x \mid A_r(t_1), T = t_2)$ is increasing in t_2 for $0 \leq r < k \leq n$ and $0 < t_1 < t_2$, then $F(t)$ is DFR in $t \in [t_2, \infty)$.

Its proof is in the Appendix.

In the following we give some sufficient conditions for stochastic orders of two systems having a set of i.i.d. components but with different structures. Their proofs can be found in the Appendix.

Proposition 9: Let $T_1 = \tau_1(X_1, \dots, X_n)$ and $T_2 = \tau_2(X_1, \dots, X_n)$ be the lifetimes of two coherent systems, both based on n components with i.i.d. lifetimes distributed according to the common continuous distribution F , having respective signatures $\mathbf{s}^{(1)} = (s_1^{(1)}, \dots, s_n^{(1)})$, and $\mathbf{s}^{(2)} = (s_1^{(2)}, \dots, s_n^{(2)})$. In addition, for all $t_2 > t_1$, their corresponding pseudo signature vectors are

$$\mathbf{p}^{(1)}(t_1, t_2) = (0, \dots, 0, p_{r+1}^{(1)}(t_1, t_2), \dots, p_n^{(1)}(t_1, t_2))$$

and

$$\mathbf{p}^{(2)}(t_1, t_2) = (0, \dots, 0, p_{r+1}^{(2)}(t_1, t_2), \dots, p_n^{(2)}(t_1, t_2)).$$

- (a) If $\mathbf{p}^{(1)}(t_1, t_2) \leq_{st} \mathbf{p}^{(2)}(t_1, t_2)$, then $(X_{k,n} - t_2 | A_r(t_1), T_1 = t_2) \geq_{st} (X_{k,n} - t_2 | A_r(t_1), T_2 = t_2)$;
 (b) If $\mathbf{p}^{(1)}(t_1, t_2) \leq_{hr} \mathbf{p}^{(2)}(t_1, t_2)$, then $(X_{k,n} - t_2 | A_r(t_1), T_1 = t_2) \geq_{rh} (X_{k,n} - t_2 | A_r(t_1), T_2 = t_2)$;
 (c) If $\mathbf{p}^{(1)}(t_1, t_2) \leq_{rh} \mathbf{p}^{(2)}(t_1, t_2)$, then $(X_{k,n} - t_2 | A_r(t_1), T_1 = t_2) \geq_{hr} (X_{k,n} - t_2 | A_r(t_1), T_2 = t_2)$;
 (d) If $\mathbf{p}^{(1)}(t_1, t_2) \leq_{lr} \mathbf{p}^{(2)}(t_1, t_2)$, then $(X_{k,n} - t_2 | A_r(t_1), T_1 = t_2) \geq_{lr} (X_{k,n} - t_2 | A_r(t_1), T_2 = t_2)$.

For example, consider the two coherent systems in Figure 1, the signature of the system in (a) on the top is $\mathbf{p} = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. The signature of the system in (b) on the bottom is $\mathbf{q} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$.

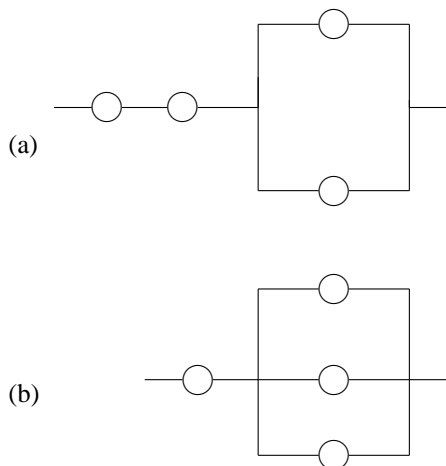


Figure 1. Two coherent systems (a) and (b) with likelihood ratio-ordered pseudo signature

Suppose that at time t_1 , the number of failed components is exactly 1, and at time t_2 ($t_2 > t_1$), the systems just failed. Following the methods outlined, it is easily to compute that the corresponding pseudo signature vectors are $\mathbf{p}(t_1, t_2) = (0, 1, 0, 0)$ and

$$\mathbf{q}(t_1, t_2) = \left(0, \frac{1}{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 1}, \frac{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 2}{2 \frac{(1+t_2)^3}{(1+t_1)^3} - 1}, 0\right),$$

respectively. Then it is easily shown that $\mathbf{p}(t_1, t_2) \leq_{lr} \mathbf{q}(t_1, t_2)$, hence $\mathbf{p}(t_1, t_2) \leq_{hr} (\leq_{rh})(\leq_{st}) \mathbf{q}(t_1, t_2)$. By the results of Proposition 9, the residual life of the last remaining component in the system (a) on the top is larger than that of the last remaining component in the system (b) on the bottom in the sense of usual stochastic order, (reversed) hazard rate order and likelihood ratio order.

The following proposition gives a sufficient condition for stochastic order of two systems having two different sets of i.i.d. components but with common structure.

Proposition 10: Let $T_1 = \tau_1(X_1, \dots, X_n)$ and $T_2 = \tau_2(Y_1, \dots, Y_n)$ be the lifetimes of two coherent systems having a same structure with a common signature $\mathbf{s} = (s_1, \dots, s_n)$. Assume that X_1, \dots, X_n are independent and identically distributed as F , Y_1, \dots, Y_n are independent and identically distributed as G . If $F \leq_{hr} G$, then for $t_1 < t_2$,

$$\Pr(X_{k,n} - t_2 > x | A_r^X(t_1), T_1 = t_2) \leq \Pr(Y_{k,n} - t_2 > x | A_r^Y(t_1), T_2 = t_2),$$

where $A_r^X(t_1) = \{X_{r,n} \leq t_1 < X_{r+1,n}\}$, $A_r^Y(t_1) = \{Y_{r,n} \leq t_1 < Y_{r+1,n}\}$.

The proof of this result is in the Appendix.

By the result above, for two systems with a common structure but with different components, if a system has stronger components in the sense of hazard rate order, then its remaining surviving components are better in the sense that they have stochastically longer general residual lifetimes, given that the system failed at time t_2 .

III. CONCLUSIONS

The paper provides some results about the residual lifetimes of surviving components of coherent systems with n independent and identically distributed components, given that before time t_1 ($t_1 > 0$), exactly r ($r < n$) components have failed and the system failed at time t_2 ($t_2 > t_1$). It is shown that the residual life of surviving components of a failed coherent system can be represented as the mixture of the residual lives of $X_{k,n}$ in a $(n-i+1)$ ($k = i+1, \dots, n$) out of n system that failed at time t_2 . By using the mixture representation, some aging properties and preservation results of the residual lives of the surviving components of failed systems are obtained. We present some independent stochastic properties of the residual life of $X_{k,n}$ in a $(n-i+1)$ out of n system and pseudo signature vectors. Some real-life examples and an explanation of main results are also given in order to help the reader to appreciate the proposed ideas and developed procedures and how they would be used in practical applications.

IV. ACKNOWLEDGMENTS

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V. APPENDIX

Proof of Proposition 1. Let $h_i > 0$ ($i = 1, 2$), then for $0 \leq r < k < n$ and $0 < t_1 < t_2$,

$$\begin{aligned}
p_k(t_1, t_2) &= \Pr(T = X_{k,n} | A_r(t_1), T = t_2) \\
&= \frac{\Pr(T = X_{k,n}, A_r(t_1), T = t_2)}{\Pr(A_r(t_1), T = t_2)} \\
&= \frac{\Pr(A_r(t_1), T = t_2 | T = X_{k,n}) \Pr(T = X_{k,n})}{\sum_{j=r+1}^n \Pr(A_r(t_1), T = t_2, T = X_{j,n})} \\
&= \frac{\Pr(A_r(t_1), T = t_2 | T = X_{k,n}) \Pr(T = X_{k,n})}{\sum_{j=r+1}^n \Pr(A_r(t_1), T = t_2 | T = X_{j,n}) \Pr(T = X_{j,n})} \\
&= \frac{s_k \lim_{h_1 \rightarrow 0} \Pr(A_r(t_1), t_2 \leq X_{k,n} \leq t_2 + h_1)}{\sum_{j=r+1}^n s_j \lim_{h_2 \rightarrow 0} \Pr(A_r(t_1), t_2 \leq X_{j,n} \leq t_2 + h_2)} \\
&= \frac{s_k \binom{n}{r} (\bar{F}(t_1) - \bar{F}(t_2))^{k-r-1} \bar{F}^{n-k}(t_2)}{\sum_{j=r+1}^n s_j \binom{n}{r} (\bar{F}(t_1) - \bar{F}(t_2))^{j-r-1} \bar{F}^{n-j}(t_2)} \\
&= \frac{s_k \left[\frac{\bar{F}(t_1) - \bar{F}(t_2)}{\bar{F}(t_2)} \right]^k}{\sum_{j=r+1}^n s_j \left[\frac{\bar{F}(t_1) - \bar{F}(t_2)}{\bar{F}(t_2)} \right]^j} \\
&= \frac{s_k (\phi(t_1, t_2) - 1)^k}{\sum_{j=r+1}^n s_j (\phi(t_1, t_2) - 1)^j},
\end{aligned}$$

where the fifth equality follows from the fact that the event $\{A_r(t_1), X_{k,n} = t_2\}$ and $\{T = X_{k,n}\}$ are independent.

Proof of Proposition 3. It is sufficient to show that for all $i = r + 1, \dots, n$,

$$\sum_{j=i}^n p_j(t_1, t_2) \leq \sum_{j=i}^n p_j(t_1, t_3).$$

That is,

$$\frac{\sum_{j=i}^n s_j (\phi(t_1, t_2) - 1)^j}{\sum_{k=r+1}^n s_k (\phi(t_1, t_2) - 1)^k} \leq \frac{\sum_{j=i}^n s_j (\phi(t_1, t_3) - 1)^j}{\sum_{k=r+1}^n s_k (\phi(t_1, t_3) - 1)^k},$$

which is equivalent to

$$\sum_{j=i}^n \sum_{k=r+1}^n s_j s_k [(\phi(t_1, t_3) - 1)^j (\phi(t_1, t_2) - 1)^k - (\phi(t_1, t_3) - 1)^k (\phi(t_1, t_2) - 1)^j] \geq 0.$$

It is clear that the equality above holds for $i = r + 1$. And for $i = r + 2, r + 3, \dots, n$, the left-hand side in the

inequality above can be represented as

$$\begin{aligned}
&\sum_{j=i}^n \sum_{k=r+1}^n s_j s_k [(\phi(t_1, t_3) - 1)^j (\phi(t_1, t_2) - 1)^k \\
&\quad - (\phi(t_1, t_3) - 1)^k (\phi(t_1, t_2) - 1)^j] \\
&= \sum_{j=i}^n \sum_{k=r+1}^{i-1} s_j s_k [(\phi(t_1, t_3) - 1)^j (\phi(t_1, t_2) - 1)^k \\
&\quad - (\phi(t_1, t_3) - 1)^k (\phi(t_1, t_2) - 1)^j] \\
&\quad + \sum_{j=i}^n \sum_{k=i}^n s_j s_k [(\phi(t_1, t_3) - 1)^j (\phi(t_1, t_2) - 1)^k \\
&\quad - (\phi(t_1, t_3) - 1)^k (\phi(t_1, t_2) - 1)^j] \\
&= \sum_{j=i}^n \sum_{k=r+1}^{i-1} s_j s_k [(\phi(t_1, t_3) - 1)^j (\phi(t_1, t_2) - 1)^k \\
&\quad - (\phi(t_1, t_3) - 1)^k (\phi(t_1, t_2) - 1)^j] \\
&= \sum_{j=i}^n \sum_{k=r+1}^{i-1} s_j s_k (\phi(t_1, t_3) - 1)^k (\phi(t_1, t_2) - 1)^k \\
&\quad \times [\phi(t_1, t_3) - 1)^{j-k} - (\phi(t_1, t_2) - 1)^{j-k}].
\end{aligned}$$

It is clear that the last term is nonnegative for $t_1 < t_2 < t_3$, and hence the result follows.

Proof of proposition 8. (a) For any $x \geq 0$, and $0 \leq t_1 < t_2$,

$$\begin{aligned}
&\frac{\partial}{\partial t_2} [\sum_{i=r+1}^n p_i(t_1, t_2) \\
&\quad \times \Pr(X_{k,n} - t_2 > x | A_r(t_1), X_{i,n} = t_2)] \\
&= \sum_{i=r+1}^n \frac{\partial p_i(t_1, t_2)}{\partial t_2} \Pr(X_{k,n} - t_2 > x | A_r(t_1), X_{i,n} = t_2) \\
&\quad + \sum_{i=r+1}^n p_i(t_1, t_2) \frac{\partial \Pr(X_{k,n} - t_2 > x | A_r(t_1), X_{i,n} = t_2)}{\partial t_2}.
\end{aligned}$$

From Lemma 7 above, that F is IFR means that $\frac{\partial \Pr(X_{k,n} - t_2 > x | A_r(t_1), X_{i,n} = t_2)}{\partial t_2}$ is negative.

Noting that

$$\begin{aligned}
\frac{\partial p_i(t_1, t_2)}{\partial t_2} &= \frac{\partial \frac{s_i (\phi(t_1, t_2) - 1)^i}{\sum_{j=r+1}^n s_j (\phi(t_1, t_2) - 1)^j}}{\partial t_2} \\
&= \frac{\frac{\partial \phi(t_1, t_2)}{\partial t_2} \sum_{j=r+1}^n s_i s_j (\phi(t_1, t_2) - 1)^{j+i-1} (i-j)}{(\sum_{j=r+1}^n s_j (\phi(t_1, t_2) - 1)^j)^2}.
\end{aligned}$$

Let $g(i) = \Pr(X_{k,n} - t_2 > x | A_r(t_1), X_{i,n} = t_2)$. Then, after some computations, we have

$$\begin{aligned}
&\sum_{i=r+1}^n \sum_{j=r+1}^n s_i s_j (\phi(t_1, t_2) - 1)^{j+i-1} (i-j) g(i) \\
&= \sum_{i=r+1}^n \sum_{j=r+1}^i s_i s_j (\phi(t_1, t_2) - 1)^{j+i-1} \\
&\quad \times [(i-j)(g(i) - g(j))].
\end{aligned}$$

By Proposition 6, $g(i)$ is decreasing in i . It holds that $(i-j)(g(i) - g(j)) \leq 0$. Thus

$$\sum_{i=r+1}^n \frac{\partial p_i(t_1, t_2)}{\partial t_2} g(i) \leq 0.$$

It follows that the proof of (a) is complete.

(b) It is noted that, from $\Pr(X_{k,n} - t_2 > x \mid A_r(t_1), T = t_2)$ is increasing in t_2 , for $t_1 < t_2 < t_3$ and any $x \geq 0$,

$$\sum_{i=r+1}^n p_i(t_1, t_2) [\Pr(X_{k,n} - t_3 > x \mid A_r(t_1), X_{i,n} = t_3) - \Pr(X_{k,n} - t_2 > x \mid A_r(t_1), X_{i,n} = t_2)] \geq 0.$$

That the inequality holds implies that there exists at least one i ($r+1 \leq i \leq n$) such that

$$\Pr(X_{k,n} - t_3 > x \mid A_r(t_1), X_{i,n} = t_3) - \Pr(X_{k,n} - t_2 > x \mid A_r(t_1), X_{i,n} = t_2) \geq 0,$$

which in turn implies from (9) that,

$$\frac{\bar{F}(t_2 + x)}{\bar{F}(t_2)} \leq \frac{\bar{F}(t_3 + x)}{\bar{F}(t_3)},$$

and hence F is DFR for $t \in [t_2, \infty)$.

Proof of Proposition 9. (a) By equation (4), for any $x \geq 0$,

$$\begin{aligned} & \Pr(X_{k,n} - t_2 > x \mid A_r(t_1), T_1 = t_2) \\ &= \sum_{i=r+1}^n p_i^{(1)}(t_1, t_2) \\ & \quad \times \Pr(X_{k,n} - t_2 > x \mid A_r(t_1), X_{i,n} = t_2) \\ &= \sum_{i=r+1}^n p_i^{(1)}(t_1, t_2) \Pr(Z_{k-i, n-i}^{t_2} > x) \\ &= \mathbb{E} \left[\Pr(Z_{k-I, n-I}^{t_2} > x) \right], \end{aligned}$$

where the random variable I is distributed as $\mathbf{p}^{(1)}(t_1, t_2)$. Also

$$\begin{aligned} & \Pr(X_{k,n} - t_2 > x \mid A_r(t_1), T_2 = t_2) \\ &= \mathbb{E}[\Pr(Z_{k-J, n-J}^{t_2} > x)], \end{aligned}$$

where the random variable J is distributed as $\mathbf{p}^{(2)}(t_1, t_2)$. By the Proposition 6, $\Pr(Z_{k-i, n-i}^{t_2} > x)$ is decreasing in i . Therefore the result of (a) follows from the equation (1.A.7) of page 4 in Shaked and Shantikumar [27].

By the conditions and Theorem 1.B.50. and Theorem 1.B.12. of Shaked and Shantikumar [27], the results (b) and (c) can be easily obtained respectively.

(d) Let $f_{T_1}(x)$, $f_{T_2}(x)$ be the density functions of $(X_{k,n} - t_2 \mid A_r(t_1), T_1 = t_2)$ and $(X_{k,n} - t_2 \mid A_r(t_1), T_2 = t_2)$, respectively. To prove the result we need only to show that

$$f_{T_1}(x)/f_{T_2}(x)$$

is increasing in $x \geq 0$. That is, we need to show that

$$\frac{\sum_{i=r+1}^n p_i^{(1)}(t_1, t_2) f_{Z_{k-i, n-i}^{t_2}}(x)}{\sum_{i=r+1}^n p_i^{(2)}(t_1, t_2) f_{Z_{k-i, n-i}^{t_2}}(x)} \text{ is increasing in } x \geq 0, \quad (10)$$

where $f_{Z_{k-i, n-i}^{t_2}}(x)$ denotes the density function of the random variable $Z_{k-i, n-i}^{t_2}$, i.e., $(X_{k,n} - t_2 \mid A_r(t_1), X_{i,n} = t_2)$. Then, for $x_1 \leq x_2$, equation (10) is equivalent to

$$\begin{aligned} & \frac{\sum_{i=r+1}^n p_i^{(1)}(t_1, t_2) f_{Z_{k-i, n-i}^{t_2}}(x_1)}{\sum_{i=r+1}^n p_i^{(2)}(t_1, t_2) f_{Z_{k-i, n-i}^{t_2}}(x_1)} \\ & \leq \frac{\sum_{i=r+1}^n p_i^{(1)}(t_1, t_2) f_{Z_{k-i, n-i}^{t_2}}(x_2)}{\sum_{i=r+1}^n p_i^{(2)}(t_1, t_2) f_{Z_{k-i, n-i}^{t_2}}(x_2)}. \quad (11) \end{aligned}$$

After some computations it can be shown that the inequality in (11) is equivalent to

$$\begin{aligned} & \sum_{i=r+1}^n \sum_{j=r+1}^i \left[p_i^{(1)}(t_1, t_2) p_j^{(2)}(t_1, t_2) \right. \\ & \quad \left. - p_i^{(2)}(t_1, t_2) p_j^{(1)}(t_1, t_2) \right] \\ & \quad \times \left[f_{Z_{k-i, n-i}^{t_2}}(x_2) f_{Z_{k-j, n-j}^{t_2}}(x_1) \right. \\ & \quad \left. - f_{Z_{k-i, n-i}^{t_2}}(x_1) f_{Z_{k-j, n-j}^{t_2}}(x_2) \right] \geq 0. \end{aligned}$$

The condition $\mathbf{p}^{(1)}(t_1, t_2) \leq_{lr} \mathbf{p}^{(2)}(t_1, t_2)$ implies that $p_i^{(1)}(t_1, t_2) p_j^{(2)}(t_1, t_2) - p_i^{(2)}(t_1, t_2) p_j^{(1)}(t_1, t_2) \leq 0$, and from Proposition 6 that, $f_{Z_{k-i, n-i}^{t_2}}(x_2) f_{Z_{k-j, n-j}^{t_2}}(x_1) - f_{Z_{k-i, n-i}^{t_2}}(x_1) f_{Z_{k-j, n-j}^{t_2}}(x_2) \leq 0$, for $x_1 \leq x_2$. It follows that the inequality in (11) holds and hence the proof is complete.

Proof of Proposition 10. For $i = r+1, \dots, n$, and $0 < t_1 < t_2$, let

$$p_i^X(t_1, t_2) = \frac{s_i \varphi_X^i(t_1, t_2)}{\sum_{j=r+1}^n s_j \varphi_X^j(t_1, t_2)},$$

and

$$p_i^Y(t_1, t_2) = \frac{s_i \varphi_Y^i(t_1, t_2)}{\sum_{j=r+1}^n s_j \varphi_Y^j(t_1, t_2)},$$

where $\varphi_X(t_1, t_2) = \bar{F}(t_1)/\bar{F}(t_2) - 1$ and $\varphi_Y(t_1, t_2) = \bar{G}(t_1)/\bar{G}(t_2) - 1$. By Proposition 5, for any $x \geq 0$,

$$\begin{aligned} & \Pr(X_{k,n} - t_2 > x | A_r^X(t_1), T_1 = t_2) \\ &= \sum_{i=r+1}^n p_i^X(t_1, t_2) \Pr(Z_{k-i, n-i}^{t_2, X} > x), \end{aligned}$$

and

$$\begin{aligned} & \Pr(Y_{k,n} - t_2 > x | A_r^Y(t_1), T_2 = t_2) \\ &= \sum_{i=r+1}^n p_i^Y(t_1, t_2) \Pr(Z_{k-i, n-i}^{t_2, Y} > x), \end{aligned}$$

where $Z_{k-i, n-i}^{t_2, X}$ is the $(k-i)$ th order statistic among $n-i$ i.i.d. random variables that are distributed as $(X - t_2 | X > t_2)$, $Z_{k-i, n-i}^{t_2, Y}$ is the $(k-i)$ th order statistic among $n-i$ random variables that are i.i.d. with the same distribution as $(Y - t_2 | Y > t_2)$.

To prove the result, we need only to show that

$$\begin{aligned} & \sum_{i=r+1}^n p_i^X(t_1, t_2) \Pr(Z_{k-i, n-i}^{t_2, X} > x) \\ & \leq \sum_{i=r+1}^n p_i^Y(t_1, t_2) \Pr(Z_{k-i, n-i}^{t_2, Y} > x). \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \frac{\sum_{i=r+1}^n s_i \varphi_X^i(t_1, t_2)}{\sum_{j=r+1}^n s_j \varphi_X^j(t_1, t_2)} \Pr(Z_{k-i, n-i}^{t_2, X} > x) \\ & \leq \frac{\sum_{i=r+1}^n s_i \varphi_Y^i(t_1, t_2)}{\sum_{j=r+1}^n s_j \varphi_Y^j(t_1, t_2)} \Pr(Z_{k-i, n-i}^{t_2, Y} > x). \end{aligned}$$

That is,

$$\begin{aligned} & \sum_{i=r+1}^n \sum_{j=r+1}^n s_i s_j \varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \Pr(Z_{k-i, n-i}^{t_2, X} > x) \\ & - \sum_{i=r+1}^n \sum_{j=r+1}^n s_i s_j \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \end{aligned}$$

is nonpositive, which is equivalent to

$$\begin{aligned} & \sum_{i=r+1}^n \sum_{j=r+1}^n s_i s_j \varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \\ & \times \left[\Pr(Z_{k-i, n-i}^{t_2, X} > x) - \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \right] \\ & + \sum_{i=r+1}^n \sum_{j=r+1}^n s_i s_j \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \end{aligned}$$

$$\times \left[\varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) - \varphi_Y^i(t_1, t_2) \varphi_X^j(t_1, t_2) \right] \quad (12)$$

is nonpositive. That $F \leq_{hr} G$ holds if and only if $\frac{\bar{F}(t_2+x)}{\bar{F}(t_2)} \leq \frac{\bar{G}(t_2+x)}{\bar{G}(t_2)}$, and hence from (9),

$$\Pr(Z_{k-i, n-i}^{t_2, X} > x) - \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \leq 0,$$

which in turn implies that the first term in equation (12) is nonpositive. Therefore in order to obtain the required result, it is sufficient to prove that the second term in equation (12) is nonpositive. Noting that

$$\begin{aligned} & \sum_{i=r+1}^n \sum_{j=r+1}^n s_i s_j \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \\ & - s_i s_j \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \\ & = \sum_{i=r+1}^n \sum_{j=i}^n s_i s_j \left[\varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \right. \\ & \quad \left. - \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \right] \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \\ & + \sum_{i=r+1}^n \sum_{j=r+1}^{i-1} s_i s_j \left[\varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \right. \\ & \quad \left. - \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \right] \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \\ & = \sum_{i=r+1}^n \sum_{j=i}^n s_i s_j \left[\varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \right. \\ & \quad \left. - \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \right] \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \\ & + \sum_{j=r+1}^n \sum_{i=j}^n s_i s_j \left[\varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \right. \\ & \quad \left. - \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \right] \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \\ & - \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \\ & = \sum_{i=r+1}^n \sum_{j=i}^n s_i s_j \left[\varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \right. \\ & \quad \left. - \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \right] \Pr(Z_{k-i, n-i}^{t_2, Y} > x) \\ & + \sum_{i=r+1}^n \sum_{j=i}^n s_i s_j \left[\varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \right. \\ & \quad \left. - \varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) \right] \Pr(Z_{k-j, n-j}^{t_2, Y} > x) \\ & = \sum_{i=r+1}^n \sum_{j=i}^n \left[\Pr(Z_{k-i, n-i}^{t_2, Y} > x) - \Pr(Z_{k-j, n-j}^{t_2, Y} > x) \right] \\ & \quad \times s_i s_j \left[\varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) - \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2) \right]. \end{aligned}$$

It follows from Proposition 6 that, $\Pr(Z_{k-i, n-i}^{t_2, Y} > x) - \Pr(Z_{k-j, n-j}^{t_2, Y} > x)$ is nonnegative for any $x \geq 0$ and $i < j$. Thus we need only to show that

$$\varphi_X^i(t_1, t_2) \varphi_Y^j(t_1, t_2) - \varphi_X^j(t_1, t_2) \varphi_Y^i(t_1, t_2)$$

is negative. This can be easily proven, from the condition $F \leq_{hr} G$. Thus the result follows directly.

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Zhengcheng Zhang is an Associate Professor at the School of Mathematics and Physics, Lanzhou Jiaotong University. He obtained his Ph.D. in 2008 at Lanzhou University. His research interests include system signatures, stochastic ordering, residual life, and mixtures of distributions. He has published twenty referred journal papers on applied probability and statistics.

William Q. Meeker is a Professor of Statistics and Distinguished Professor of Liberal Arts and Sciences at Iowa State University. He is a Fellow of the American Statistical Association, and of the American Society for

Quality. He is an elected member of the International Statistical Institute, and a past Editor of *Technometrics*. He is co-author of the books *Statistical Methods for Reliability Data* with Luis Escobar (1998), *Statistical Intervals: A Guide for Practitioners* with Gerald Hahn (1991), nine book chapters, and numerous publications in the engineering and statistical literature. He has won the American Society for Quality (ASQ) Youden prize five times, the ASQ Wilcoxon Prize three times, and the American Statistical Association Best Practical Application Award. He is also an ASQ Shewhart Medalist. He has consulted extensively on problems in reliability data analysis, reliability test planning, accelerated testing, nondestructive evaluation, and statistical computing.