It is known that the energy-release rates associated with translation, rotation, and self-similar expansion of cavities or cracks in solids are expressed by path-independent integrals $J$, $L$ and $M$, respectively. These integrals are of interest to NDE in that they can be used to characterize nondestructively defects in solids.

It is shown that these integrals for a crack may be evaluated by first considering an ellipse and then performing a limiting process. This obviates dealing with singularities at crack tips and holds promise for a more efficient numerical method in complicated cases, since modeling of singularities is always associated with difficulties and uncertainties.

INTRODUCTION

The strength of most brittle solids is determined by defects such as cracks. Thus one is interested in finding the stress distribution on the basis of the theory of elasticity. The linear theory leads to a so-called singular boundary value problem. The plane stress distribution near a sharp crack in an infinite body subjected to biaxial tension was first treated by Westergaard$^1$ and leads to singular (infinite) stresses at crack tips. Still, stress intensity factors, and energy release rates related to material toughness, can be calculated.

A path-independent (so-called $J$) integral was introduced by Rice$^2$ which provides alternate means in evaluated energy release rates and stress intensity factors. Other path-independent integrals,
later called L and M, were discovered by Günther and Knowles and Sternberg. All these path-independent integrals are useful in fracture mechanics in that they are related to energy release rates.

One purpose of this contribution is to show that the path-independent integrals leading to energy release rates for a crack can be evaluated by considering first the corresponding expression for an elliptic cavity and then passing to the limit of a vanishingly small semi-axis of the ellipse, i.e. a crack. This approach avoids completely dealing with a singular stress field. It is carried out for an infinite body.

Another purpose is to point out that this approach of evaluating energy release rates may be most useful in more complicated crack problems involving finite bodies. Numerical methods, such as finite elements, have then to be used and modeling of stress singularities is always associated with uncertainties. The method advanced here might circumvent these uncertainties and difficulties, since not a crack but an elliptic cavity would have to be modeled, retaining the value of the small semi-axis as a parameter and then taking the limit. These calculations are relegated to the future.

**PATH-INDEPENDENT INTEGRALS**

In a two-dimensional field referred to Cartesian coordinates $X_1$ and $X_2$, the $J$ integral is defined as

$$ J = \oint_C (W dX_2 - T_i u_i, d\ell) $$

where $C$ is a contour surrounding a crack tip (Fig. 1), $W$ is the strain energy density, $u_i$ is the displacement vector and $T_i$ is the traction vector defined as $\sigma_{ij} n_j$ where $n_j$ is the outward normal unit vector to the contour $C$.

![Figure 1. Path C for J integral](image-url)
In the presence of far-field homogeneous stresses $\sigma_{11}^A$, $\sigma_{12}^A$ and $\sigma_{22}^A$, $J$ is related to the stress intensity factors, in plane stress, as

$$J = \frac{K_1^2 + K_{II}^2}{E}$$  \hspace{1cm} (2)

where $K_1$ and $K_{II}$ are stress intensity factors for mode I and mode II, (mode I means crack opening and mode II is shearing parallel to the crack) respectively, which are defined as

$$K_1 = \sqrt{\pi a} \sigma_{22}^A \hspace{1cm} K_{II} = \sqrt{\pi a} \sigma_{12}^A$$  \hspace{1cm} (3)

and $E$ is Young's modulus and $2a$ is the crack length.

The integral is useful in fracture mechanics because it can be shown to be equal to the crack extension force $G$ obeying Irwin's relationship, i.e.

$$J = G = \frac{K_1^2 + K_{II}^2}{E}$$  \hspace{1cm} (4)

and it has been related to potential energy release rate associated with unit crack tip advancement, namely

$$J = -\frac{\partial U}{\partial a}$$  \hspace{1cm} (5)

where $U$ is the crack energy defined as

$$U = \frac{1}{2} \int_{-a}^{a} \sigma_{2j}^A \Delta u_j(X_1) \, dX_1$$  \hspace{1cm} (6)

Here $\Delta u_j(X_1)$ is the discontinuity in displacement across the crack and summation over $j$ is implied. Since $J$ is valid in nonlinear elastic solids and in elastoplastic solids, the relationship given in (4) still holds even in general yielding as long as no unloading occurs.

Actually the $J$ integral is the first component of the vector (2)

$$J_\kappa = \oint_C (W_{nk} - T_{ik} u_{i,k}) \, dl \hspace{1cm} i,k = 1,2$$  \hspace{1cm} (7)

where $n_k$ is the unit outward normal to $C$. If $J_2$ is evaluated for the infinitesimal contour enclosing the crack tip, the result gives, in plane stress

$$J_2 = -2K_1 K_{II}/E$$  \hspace{1cm} (8)

where $K_1$ and $K_{II}$ are given in (3). $J_2$ is a component normal
to the crack plane of the vector given by (7). But it is to be noted comparing (2) and (8) that $J_2$ is not independent of $J(J_1)$, as it should be since it is a component of a vector, in that if $J = 0$, $J_2$ is necessarily zero. It is also relevant to point out that $J_2$ is path-dependent in the same sense as $J$ is path-independent. The reason for this is that for $J$ (i.e. $J_1$) $T_k$ is zero along the crack faces; thus the second term of (1) is identically zero, while the first term vanishes because $dX_2$ is zero. But for $J_2$, the second term vanishes, whereas the first one does not. The usefulness of the $J_2$ integral and its relationship with the $L$ integral will be developed later.

The $M$ integral defined as

$$M = \oint_C (W_{i} n_i - T_{ik} u_k i X_i) \, dS$$

has been shown to be the rate of potential energy release for self-similar expansion of a cavity completely enclosed by the contour $C$. For a particular case of a crack undergoing a self-similar expansion (Fig. 2), such that there is a relative scale change $da/a$, the $M$ integral represents a measure of the energy release rate with respect to this change. A simple relationship can be established between $M$ and $J$, i.e.

$$M = 2aJ$$

which is particularly useful when it is more convenient to evaluate $M$ using a closed contour surrounding the whole crack rather than $J$ using a contour around a crack tip. Further, $M$ remains applicable in certain cases involving loading on the crack faces in which path-independence of $J$ would no longer hold.

The last path-independent integral to be discussed is the $L$ integral defined as

![Figure 2. Self-similar expansion of a crack](image-url)
ENERGY RELEASE RATES FOR VARIOUS DEFECTS

\[ L = \oint_C \epsilon_{3ij} (w X_{ji} n_i - T_i u_i - T_k u_k, i X_j) \, dS \]  

where \( \epsilon_{3ij} \) is the alternating tensor. When evaluating this integral around the whole crack, the result is obtained

\[ L = 2\sigma_{12}^A (\sigma_{22}^A + \sigma_{11}^A) \pi a^2 / E \]  

or

\[ L = -2aJ_2 + 2\sigma_{12}^A (\sigma_{11}^A - \sigma_{22}^A) \pi a^2 / E \]  

It can be observed from the above relationship that \( L \) cannot be expressed through \( J_2 \) completely. This is because of the contribution to \( L \) along the crack faces. Thus, \( L \) is not path-independent if the contour surrounds only part of the crack.

However, under a special stress field

\[ \sigma_{11}^A = \sigma_{22}^A \]  

\( L \) can be expressed in terms of \( J_2 \) as

\[ L = -2aJ_2 \]  

which is analogous to the relationship given by (10). \( L \) has been found to be the rotational energy release rate \(^4\) which can be interpreted as the measure of energy release rate of a crack per unit crack rotation. The relationship has the form

\[ L = - \frac{\partial U}{\partial \phi} \bigg|_{\phi=0} \]  

where \( U \) is defined in (6) and \( \phi \) is a small angle (Fig. 3) through which the crack rotates with respect to the applied stress field. The \( L \) integral may be useful in determining stress intensity factors of cracks subjected to mixed mode loading.

![Figure 3. Rotation of a crack by a small angle \( \phi \)](image)
The classical linear theory of elasticity for the crack problem with applied stresses $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{22}$ predicts a singular stress field and a singular displacement field near the crack tips which are of the form:

$$
\sigma_{ij}^m = K_m (2\pi r)^{-1/2} f_{ij}^m(\theta) \quad i,j = 1,2
$$

$$
u_i^m = (K_m/2E)(r/2\pi)^{1/2} f_i^m(\theta) \quad m = I,II
$$

where $K_m$ are the stress intensity factors defined in (3) and $f_{ij}^m(\theta)$ form a set of functions which are the same for all symmetrically loaded crack problems. For an infinite body path-independent integrals can be evaluated by integrating along a circular path of small radius $r$ around the crack tip making use of the above asymptotic expansion form. But for a finite body the energy release rates have to be calculated numerically. Thus, adequate modeling of the singularity is essential. This modeling causes difficulties in finite element calculations and numerous authors have tried many different methods to overcome this difficulty.

SUGGESTED ALTERNATE METHOD

The proposed alternate method of calculating path-independent integrals is based on the following ideas: given the fact that it is possible to obtain the stress and displacement fields around a crack as a limiting case of those around an elliptic cavity, it is proposed to calculate the integrals around an elliptic cavity first and then proceed to the limiting case of a crack (Fig. 4). Provided this method is valid, dealing with singularities is completely obviated.

Figure 4. Crack as a limiting case of an elliptical cavity
Table 1. Uniaxial Tension

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<th>J</th>
<th>M</th>
<th>L</th>
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<td>crack</td>
<td>0</td>
<td>$2\pi a^2 S^2 / E$</td>
<td>0</td>
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<tr>
<td>elliptical cavity</td>
<td>0</td>
<td>$\pi (2a^2 + ab) S^2 / E$</td>
<td>0</td>
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<tr>
<td>circular cavity</td>
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<td>$3\pi a^2 S^2 / E$</td>
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Table 2. Biaxial Tension

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<td>$2\pi a^2 S^2 / E$</td>
<td>0</td>
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<tr>
<td>elliptical cavity</td>
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<td>$2\pi (a^2 + b^2) S^2 / E$</td>
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</tr>
<tr>
<td>circular cavity</td>
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<td>$4\pi a^2 S^2 / E$</td>
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Table 3. Inplane Shear

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<tr>
<td>elliptical cavity</td>
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<td>$2\pi (a+b)^2 S^2 / E$</td>
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<td>circular cavity</td>
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<td>$8\pi a^2 S^2 / E$</td>
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To show the validity of the suggested method, the following calculations have been performed. The J, M and L integrals have been independently calculated for a crack, a circular cavity and an elliptical cavity, all embedded in an infinite body subjected to far field uniaxial tension, biaxial tension and inplane shear.

Figure 5. Infinite body subjected to a uniaxial tension $S$
(Fig. 5). The paths taken have been along the boundaries of each cavity, employing already existing plane stress elasticity solutions. The results are listed in Tables 1, 2 and 3.

For all three defects the $J$ and $L$ integrals were both zero. This is due to symmetry and the result of path chosen. Physically this means that the net material force ($J$) and the net material moment ($L$) acting on the cavity are zero. Table 1, for example, lists the $M$ integral calculated independently for 3 different cavities under uniaxial tension, and it is seen that $M$ for a crack, $2\pi a^2S^2/E$, can be deduced from that for an elliptical cavity, $\pi(2a^2+ab)s^2/E$, by taking the limit as $b \to 0$.

CONCLUDING REMARKS

It has been shown, for simple cases, that path-independent integrals for the plane crack in an infinite body can be derived by a limiting procedure applied to the relevant quantities calculated for an elliptical cavity.

The results obtained indicate the possibility of using this approach for more complicated three-dimensional stress fields, on one hand, and for finite bodies of complicated shape on the other.

It is intended to report the results of some of these problems in forthcoming papers.

ACKNOWLEDGMENT

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