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A solution method to a new class of inverse spectral problems

Kurugamega Clement Jayawardena
Iowa State University

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A solution method to a new class of inverse spectral problems

Jayawardena, Kurugamega Clement, Ph.D.
Iowa State University, 1992

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A solution method to a new class of inverse spectral problems

by

Kurugamega Clement Jayawardena

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

Let \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\phi_n(x)\}_{n=1}^{\infty} \) be the eigenvalues and eigenfunctions of the differential equation

\[-y'' + q(x)y = \lambda y \quad (1.1)\]

with the boundary conditions

\[y'(0) - hy(0) = 0 = y'(1) + H y(1). \quad (1.2)\]

If \( h = \infty \) then we interpret the boundary condition at \( x = 0 \) as \( y(0) = 0 \). Similarly, if \( H = \infty \) then the boundary condition at \( x = 1 \) is \( y(1) = 0 \). The inverse Sturm-Liouville problem consists of the reconstruction of the potential function \( q(x) \) from the knowledge of \( \{\lambda_n\}_{n=1}^{\infty} \) and another piece of data which can take different forms. In general the data available for the reconstruction of the potential \( q(x) \) is called spectral data. In the following we state the widely used forms of spectral data.

(a) Let \( \rho_n = \left\{ \frac{\|\phi_n(\cdot)\|_2^2}{\phi_n(0)^2} \right\} \) for \( h < \infty \)

One form of spectral data consists of two sequences \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\rho_n\}_{n=1}^{\infty} \). Here, we note that \( \rho_n \) is called a norming constant. Spectral data which consists of two sequences \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\rho_n\}_{n=1}^{\infty} \) is called spectral function data.

(b) Let \( \{\tilde{\lambda}_n\}_{n=1}^{\infty} \) be the eigenvalue sequence of (1.1) with respect to the boundary condition:

\[y'(0) - hy(0) = 0 = y'(1) + \tilde{H} y(1) \quad (\tilde{H} \neq \bar{H}) \quad (1.3)\]
So the two sequences \( \{\lambda_n\}_{n=1}^\infty \) and \( \{\hat{\lambda}_n\}_{n=1}^\infty \) gives another form of spectral data. This form of spectral data is called two spectrum data.

(c) In addition to the eigenvalue sequence \( \{\lambda_n\}_{n=1}^\infty \) if we know \textit{apriori} the function \( q(x) \) for \( x \in [\frac{1}{2}, 1] \) then we called this case as partially known case.

(d) If we know \textit{a priori} that \( q(x) \) is symmetric about \( x = \frac{1}{2}, \) i.e. \( q(1-x) = q(x), \) and the eigenvalue sequence \( \{\lambda_n\}_{n=1}^\infty \) corresponds to the differential equation (1.1) with respect to the boundary condition (1.2) with \( h = H \) then we called this the symmetric case.

(e) Another kind of spectral data is called end-point data. In this case, in addition to the sequence of eigenvalues \( \{\lambda_n\}_{n=1}^\infty \) we know another sequence \( \{\kappa_n\}_{n=1}^\infty \) given by:

\[
\kappa_n = \begin{cases} 
\ln \frac{\phi_n'(1)}{\phi_n(0)}, & \text{for } h \in \mathbb{R}, \ H = \infty \\
\ln \frac{\phi_n'(1)}{\phi_n(0)}, & \text{for } h = \infty, \ H = \infty \\
\ln \frac{|\phi_n(1)|}{|\phi_n(0)|}, & \text{for } h, H \in \mathbb{R}
\end{cases}
\]

One can classify the work in inverse spectral problems into three important categories. These categories are the work related to the existence of the potential \( q(x), \) the work related to the uniqueness of the potential \( q(x) \) and finally the work related to the actual reconstruction procedures. Recent work in this area is mainly concentrated in the third category mentioned above.

In the first part of the introduction, we discuss the existence and the uniqueness results. We begin this by stating the first result in the literature, which was stated by Ambarzumian [1].

Ambarzumian proved the following theorem: Let \( \{\lambda_n\}_{n=1}^\infty \) denote the eigenvalues of the operator (1.1)-(1.3) where \( h = H = 0, \) and \( q(x) \) is assumed to be a continuous function on the interval \([0,1].\) If \( \lambda_n = n^2 \pi^2, \) for \( n = 0, 1, 2, \ldots \) then \( q(x) = 0. \) Even though this result is correct the proof given by Ambarzumian is not correct. Later
several authors have proved this result. In [15] Levitan and Gasymov gave a proof using an extremal property of the first eigenvalue.

A very important step forward was taken by Borg [3] in 1946. Among other things he has proved that the potential \( q(x) \) can be uniquely determined from the spectral data consisting of two spectrum data as described in (b). His main result can be formulated as follows:

Let \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\tilde{\lambda}_n\}_{n=1}^{\infty} \) be two sequences of eigenvalues as in the case (b). Then \( q(x) \in C[0, 1] \) is uniquely determined from the eigenvalues \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\tilde{\lambda}_n\}_{n=1}^{\infty} \). In this paper Borg showed that, in general, one spectrum does not determine the unknown function \( q(x) \). One can easily see his claim. Consider the eigenvalue sequence of (1.1) with respect to the boundary condition \( y(0) = 0 = y(1) \) (i.e. when \( h = H = \infty \) in (1.2)). The eigenvalue sequence corresponding to this boundary condition is called Dirichlet eigenvalues. Let \( q^*(x) = q(1 - x) \). Now it is clear that the two functions \( q(x) \) and \( q^*(x) \) have the same Dirichlet eigenvalues. Therefore, one cannot reconstruct the potential function uniquely from only one sequence of eigenvalues. Hence, the result by Ambarzumian [1] which require only one spectrum is merely an extremal property of the first eigenvalue. In this paper Borg also showed that if \( q(x) \) is symmetric about \( x = \frac{1}{2} \), i.e., \( q(x) = q(1 - x) \) then the spectrum \( \{\lambda_n\}_{n=1}^{\infty} \) of (1.1) corresponding to the end conditions given by \( h = H = 0 \) or the eigenvalues \( \{\lambda_n\}_{n=1}^{\infty} \) corresponding to the end conditions given by \( h = H = \infty \) determines \( q(x) \) uniquely.

In [14] Levinson, proved that only one eigenvalue sequence is required to determine an even potential \( q(x) \) for \( x \in [0, 1] \) in the case of symmetric spectral data described in case (d).

In 1951 Gel'fand and Levitan [5], made a very significant contribution to this area by presenting a technique for the reconstruction of \( q(x) \in C(0.1) \). From the spectral function data described in case (a). In addition to this, they
have also proved a necessary and sufficient condition for the two sequences \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\rho_n\}_{n=1}^{\infty} \) to be the sequences of eigenvalues and norming constants of a Sturm-Liouville problem (1.1) and (1.2). We state their result in the following:

(1) For \( q \in C[0,1] \) if \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\rho_n\}_{n=1}^{\infty} \) represent the spectral function data for (1.1) and (1.2) then \( \lambda_n \) and \( \rho_n \) hold the following asymptotic formulae:

\[
\sqrt{\lambda_n} \approx n\pi + O\left(\frac{1}{n}\right)
\]

\[
\rho_n \approx \frac{1}{2} + O\left(\frac{1}{n}\right).
\]

(2) If

\[
\sqrt{\lambda_n} \approx n\pi + \frac{b_1}{n} + \frac{b_2}{n^2} + O\left(\frac{1}{n^3}\right)
\]

and

\[
\rho_n \approx \frac{1}{2} + \frac{a_1}{n^2} + O\left(\frac{1}{n^3}\right)
\]

where \( a_1, b_1, b_2 \) are constants, then there exists a \( q(x) \in C^1[0,1] \) which solves the inverse Sturm-Liouville problem with spectral function data \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\rho_n\}_{n=1}^{\infty} \).

In his paper Zhikov [27], has extended the necessary and sufficient condition described above by Gel’fand and Levitan, to the case when \( q(x) \) is a derivative of a function \( F(x) \) which is of bounded variation.

In 1951 Krein also published two papers [11],[12], where he has described a method to reconstruct the potential \( q(x) \) from the two sequences of eigenvalues \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\tilde{\lambda}_n\}_{n=1}^{\infty} \) described in case (b). In his paper [18] Marchenko has extended the Gel’fand and Levitan result, by showing that for \( q \in L^1(0,1) \) the two eigenvalue sequences \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\tilde{\lambda}_n\}_{n=1}^{\infty} \) described in (b), determine the potential \( q(x) \), \( h \), \( \bar{H} \) and \( \tilde{H} \) uniquely.

Now we describe the results relating to the other versions of spectral data. In 1976 Hochstadt and Lieberman [10], proved the following result:
Let \( \{\lambda_n\}_{n=1}^{\infty} \) be the spectrum of (1.1) subject to (1.2) whenever \( q \in L^1(0,1) \). Consider the second equation:

\[-y'' + \tilde{q} y = \lambda y \quad \text{ (1.4)}\]

where \( \tilde{q} \in L^1(0,1) \) and \( \tilde{q}(x) = q(x) \, \text{a.e. for } x \in (\frac{1}{2}, 1) \). If the spectrum of (1.4) subject to (1.2) is also \( \{\lambda_n\}_{n=1}^{\infty} \) then \( q(x) = \tilde{q}(x) \) almost everywhere in (0,1). In other words, only a single spectrum is required to determine the function \( q(x) \) for \( x \in [0, \frac{1}{2}] \), provided that the function \( q(x) \) is known for \( x \in [\frac{1}{2}, 1] \). In case of symmetric data it is only required to reconstruct the potential \( q(x) \) for \( x \in [0, \frac{1}{2}] \). Therefore, Levinson’s result is similar to the above described Hochstadt and Lieberman result.

Issacsou, McKean and Trubowitz have studied the inverse Sturm-Liouville problem with the end point data described in (e). They have proved that one can uniquely determined the potential \( q(x) \) from the two sequence \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\kappa_n\}_{n=1}^{\infty} \). Some of their work was published as a book by Pöschel and Trubowitz [23].

In 1986 McLaughlin [20], has written a review article on inverse spectral problems and in this paper she has discussed the results included in Zhikov [27]. Two other general references for this subject are the books by Levitan [16] and Gladwell [6].

At this point we change the direction of our discussion to the existing reconstruction methods. In their paper [4], Gel’fand and Levitan proved the following result. If the function

\[ a(x) = \sum_{n=1}^{\infty} \left[ \frac{1}{\rho_n} \frac{\cos \sqrt{\lambda_n} x}{\rho_n} - \frac{2}{\pi^2} \frac{\cos n \pi x}{n} \right] \]

has a continuous fourth derivative then there exists a kernel \( K(x,t) \), which is independent of \( \lambda \), such that the \( n \)th eigenfunction \( \phi(x,\lambda_n) \) of the boundary value problem for \( h, H \in \mathbb{R} \) can be written as

\[ \phi(x,\lambda) = \cos \sqrt{\lambda} x + \int_{t=0}^{x} K(x,t) \cos \sqrt{\lambda} t \, dt. \]
They have also established the relation between the potential \( q(x) \) and the Gel'fand and Levitan kernel \( K(x,t) \) of the integral equation, as

\[ q(x) = \frac{1}{2} \frac{dK(x,x)}{dx}. \]

Hence in order to reconstruct the potential \( q(x) \) we need to compute \( K(x,x) \).

Gel'fand and Levitan have shown that for fixed \( x \) the kernel \( K(x,t) \) solves the following second kind Fredholm integral equation:

\[ f(x,y) + \int_{t=0}^{x} f(y,t) K(x,t) \, dt + K(x,y) = 0. \]

Here the function \( f(x,y) \) can be obtain from the spectral data as

\[ f(x,y) = \frac{\partial^2 F}{\partial x \partial y}, \]

where

\[ F(x,y) = \sum_{n=1}^{\infty} \left( \frac{1}{\rho_n} \frac{\sin \sqrt{\lambda_n} x \sin \sqrt{\lambda_n} y}{\lambda_n} - \frac{1}{2} \frac{\sin n\pi x \sin n\pi y}{n^2} \right) + \frac{1}{\rho_0} \frac{\sin \sqrt{\lambda_0} x \sin \sqrt{\lambda_0} y}{\lambda_0} - xy. \]

In addition to a method for the reconstruction of \( q(x) \), this also provides a way to reconstruct the boundary conditions \( h \) and \( H \).

\[ h = K(0,0) \text{ and } H = \frac{\delta_n'(1)}{\phi_n(1)} \]

In order to solve the integral equation for \( K(x,t) \), we need to compute \( f(x,y) \).

It is clear from the formula for \( f(x,y) \) that \( f(x,y) \) is a difference of two divergent series. Therefore, the accurate computation of \( f(x,y) \) is an extremely difficult task.

So, this limits the accuracy of the computed \( q(x) \). Another disadvantage of this method is the following. In general one needs to find the values of the function \( q \) at the points \( x_k = \frac{k}{N}, \; k = 0, 1, \ldots, N \). In order to find these values, we have to
solve the integral equation for $K(x,t)$, $N$ different times. Each time we need to solve a linear system of order $k$, and this requires $O(k^3)$ operations. Therefore, the complete process requires $O(N^4)$ operations. Hence, this computational procedure is an expensive one. Other than these disadvantages, Gel'fand and Levitan have provided the first elementary method to solve the inverse spectral problem with spectral function data.

Hald has presented two algorithms [7] [8] to solve the inverse Sturm-Liouville problems. His algorithm in [7], based on the classical Rayleigh-Ritz method, was specialized for the reconstruction of the symmetric potential $q(x)$ from its Dirichlet spectrum. In the development of this algorithm, he expanded the potential function and the eigenfunctions as Fourier series and reduced the problem into a finite dimensional inverse eigenvalue problem by truncating these Fourier representations. He proved that the solution of the finite dimensional problem converges to the solution of the continuous problem, provided that the potential is sufficiently small in norm. The algorithm in [8] was based on a revised version of Hochstadt's [8] idea of reducing the problem to a system of nonlinear ordinary differential equations. He proved that the algorithm always provides a solution of the inverse problem for the case of sufficiently small symmetric potentials, and illustrates his method by some numerical examples.

In 1988 Sacks [25], contributed a time domain technique to reconstruct the unknown potential $q(x)$ from its Dirichlet spectral data. His key idea is to transform the frequency domain problem into a time domain problem by defining a map:

$$F : L^2(0,1) \rightarrow L^2(0,2)$$

given by

$$F(q)(t) = w_r(t)$$

where $w(x,t)$ solves the Goursat problem:

$$w_{tt} - w_{xx} + q(x)w = q(x), \ 0 < x < t < 2 - x \quad (1.9a)$$
\[ w(0,t) = 0, \quad 0 < t < 2 \quad (1.9b) \]
\[ w(t,t) = 0, \quad 0 < t < 1 \quad (1.9c) \]

and investigating the equation

\[ F(q) = g \quad (1.10) \]

where

\[ g(t) = -\delta'(t) - \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n} \rho_n} \sin \sqrt{\lambda_n} t. \quad (1.11) \]

He also showed that the function \( q(x) \) which solves the inverse spectral problem in turn solves the equation (1.10). In this paper Sacks proved that, if \( q \in L^\infty(0,1) \) and \( \|q\|_{L^\infty(0,1)} < M \), then the Quasi-Newton iterates:

\[ q_0 = 0 \]
\[ \tilde{q}_{n+1}(x) = q_n(x) + 2g(2x) - 2F(q_n)(2x) \]

and

\[ q_{n+1} = \begin{cases} \tilde{q}_{n+1}(x) & \text{if } |\tilde{q}_{n+1}(x)| \leq M \\ \pm M & \text{if } \pm \tilde{q}_{n+1}(x) > M \end{cases} \]

converges to \( q \) in \( L^2(0,1) \). McLaughlin [20], showed a way to compute an upper bound for \( |q(x)| \) in terms of spectral function data. Hence, \( M \) can be obtained using spectral data. In practice this numerical scheme converges very fast to the solution \( q \). In this paper he has also presented an algorithm based on the method of characteristics to solve the forward problem given by 1.9 (a-c). As we understand, his idea is very useful in solving inverse spectral problems.

In their recent work, Rundell and Sacks [24], have developed two numerical schemes to reconstruct the potential function \( q(x) \) for the inverse Sturm-Liouville problems with spectral data in the cases (a)-(e) described at the beginning of the chapter. In order to illustrate these methods they have considered the inverse
Sturm-Liouville problem with two spectrum data which corresponds to the boundary condition \( h = \infty \). In the following we describe one of their algorithms. From the Gel'fand and Levitan paper [5], if \( \phi(x, \lambda) \) satisfies

\[
\phi'' + (\lambda - q(x))\phi = 0
\]

\[
\phi(0, \lambda) = 0, \quad \phi'(0, \lambda) = 1
\]

then

\[
\phi(x, \lambda) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \int_{t=0}^{x} K(x, t) \frac{\sin \sqrt{\lambda}t}{\sqrt{\lambda}} dt
\]

where the function \( K(x, t) \) is defined on the triangle \( 0 \leq |t| \leq x \leq 1 \) and it is independent of \( \lambda \). Moreover \( K(x, t) \) satisfies the following Goursat problem:

\[
\frac{\partial K}{\partial t} - K_{xx} + q(x)K = 0
\]

\[
K(x, \pm x) = \pm \frac{1}{2} \int_{y=0}^{x} q(y) dy.
\]

They have described a method to compute \( K(1, t) \) and \( K_x(1, t) \) for \( t \in [-1, 1] \), using the spectral data. They showed that the solution \( q(x) \) of the inverse spectral problem satisfies the integral equation

\[
q(x) = G(x) - 2 \int_{y=x}^{1} q(y)K(y, 2x - y) dy
\]

where

\[
G(x) = 2[K_t(1, 2t - 1) + K_x(1, 2t - 1)].
\]

They have developed an iterative method to solve this integral equation and proved that this method converges to the solution of the inverse Sturm-Liouville problem. In this paper they have also developed a second method to solve inverse Sturm-Liouville problems. The solution methods in this thesis is based on a idea similar to their second method and it will be discussed in the third chapter of the thesis.
Recently Lowe, Pilant and Rundell [17] investigated the question of obtaining approximations to \( q(x) \) from a finite number of eigenvalues. Their solution method is based on the following idea:

For a given \( q(x) \) and eigenvalue sequence one can determine the eigenfunctions completely using left endpoint data. This leaves the boundary conditions imposed at the right endpoint as an additional information. Therefore, when \( q(x) \) is unknown one can use this additional information to reconstruct the potential \( q(x) \). They showed that their algorithm is always well defined and the associated iteration scheme converges for sufficiently small potentials. One of the advantages of this method is that it does not require the estimation of the mean of \( q \). In this algorithm it is require to compute the Frechet derivative and this requires one to solve \( N^2 \) (where \( N \) is the number of points at which \( q(x) \) is determined) Sturm-Liouville eigenvalue problems, and numerically this is an expensive proposition.

We will next state some existing well-posedness results in this area. The first result of this type seems to be invented by Borg. In his paper [3], Borg proved that if the potentials are symmetric then

\[
\| q - \hat{q} \|_2 \leq K(q) \sqrt{\sum_j |\lambda_j - \hat{\lambda}_j|},
\]

provided that the right hand side of this expression is small enough.

Under the assumption that the symmetric potentials have small \( L^2 \) norms, Barcilon [2] gave a well-posedness result similar to that of Borg.

Hochstadt [9] and Hald [8] had given \( L^\infty \) result which can be stated as follows:

\[
\| q - \hat{q} \|_{\infty} \leq K(h, \bar{h}, 1, \| q \|_1, \| \hat{q} \|_1) \sum_{j=1}^{\infty} |\lambda_j - \hat{\lambda}_j|.
\]

provided that \( q(x) \) is sufficiently small.

McLaughlin and Handelman [19] studied the problem of determining the effect on the potential \( q(x) \) and the boundary conditions \( h, H \) by altering the first \( n \)
eigenvalues and norming constants. In this paper they have developed a Gel'fand and Levitan type algorithm to find the eigenfunctions of the perturbed problem and they explicitly expressed the new potential in terms of the old potential and the eigenfunctions.

In 1988, Mclaughlin [22], proved a similar result for the general sturm-Liouville problem which gives an upper bound for the $L^2$ norm of the difference of the potentials in terms of eigenvalues and its norming constants as follows:

Let $q_1$ and $q_0$ be contained in a bounded set in $L^2(0,1)$, so that $\int_0^1 q_1 \, dt = 0 = \int_0^1 q_0 \, dt$. Let the associated spectral data be $\{ \lambda_n \}_{n=1}^{\infty}, \{ \rho_n \}_{n=1}^{\infty}$ and $\{ \lambda_0 \}_{n=1}^{\infty}, \{ \rho^0_n \}_{n=1}^{\infty}$. Then

$$\| q - q_0 \| \leq K \| (\lambda^1 \times \rho^1) - (\lambda^0 \times \rho^0) \|,$$

whenever $\| (\lambda^1 \times \rho^1) - (\lambda^0 \times \rho^0) \|$ is small enough.

1.2 The Statement of the Problems

In this thesis we will study the following new inverse problems.

Let $\{ \lambda_j \}_{j=1}^{\infty}$ be the complete eigenvalue sequence of the regular Sturm-Liouville problem (1.1-2) with $h = \infty$ and $H \in \mathbb{R}$. Let $\{ \mu_j \}_{j=0}^{\infty}$ be the complete eigenvalue sequence of the regular Sturm-Liouville problem (1.1-2) with $h = H \in \mathbb{R}$ and $\{ \nu_j \}_{j=1}^{\infty}$ denotes the complete eigenvalue sequence of the regular Sturm-Liouville problem with $h = H = \infty$.

(1) We assume that the potential function $q \in L^2(0,1)$ and $q$ is anti-symmetric about the mid-point $x = \frac{1}{2}$. Now the question is, can we reconstruct $q$ from the sequence $\{ \lambda_j \}_{j=1}^{\infty}$ or $\{ \mu_j \}_{j=0}^{\infty}$?

(2) We assume that the potential function $q \in L^2(0,1)$ and also that $q$ is symmetric about the mid-point $x = \frac{1}{2}$. We investigate the question of reconstructing $q$ from its eigenvalue sequence $\{ \lambda_j \}_{j=1}^{\infty}, \{ \mu_j \}_{j=0}^{\infty}$, or $\{ \nu_j \}_{j=1}^{\infty}$.

Here we note that only the case $h = H$ is discussed by Levinson [14], and problem (2) is a generalization of his result.
(3) Given, either a complete eigenvalue sequence \( \{\lambda_j\}_{j=1}^{\infty} \), \( \{\mu_j\}_{j=0}^{\infty} \) or \( \{\nu_j\}_{j=1}^{\infty} \), on which subsets of the unit interval \((0,1)\) does one needs to specify \( q(x) \) in order to determine \( q \) uniquely on \((0,1)\) ?

(4) Let \( \{\lambda_n\}_{n=0}^{\infty} \) be the eigenvalues of the differential equation

\[
- y'' + q y = \lambda y
\]

corresponding to the boundary conditions

\[
y'(0) - h y(0) = y'(1) + H y(1)
\]

and let \( \{\hat{\lambda}_n\}_{n=0}^{\infty} \) be the eigenvalues of the same differential equation subjected to the boundary conditions

\[
y'(0) - \hat{h} y(0) = y'(1) + \hat{H} y(1)
\]

where

\[
h \neq \hat{h} \text{ and } H \neq \hat{H}.
\]

Now the question is can we reconstruct \( q \) from the spectral data \( \{\lambda_n\}_{n=0}^{\infty} \) and \( \{\hat{\lambda}_n\}_{n=0}^{\infty} \).

Here we note that Borg [3] and others considered the problem with \( h = \hat{h} \). So, this is a more general problem.

In this thesis we will obtain some uniqueness results for these problems and also we will present constructive algorithms using time domain methods.
CHAPTER 2

REGULARITY PROPERTIES OF THE MAPS

2.1 An Energy Estimate

As we described in the last chapter, our goal in this thesis is to study some inverse spectral problems. In the next chapter we will discuss a solution method to these problems in the time domain. Solving these inverse problems is equivalent to finding an unknown function \( q \) in the non-linear equation \( F(q) = g \), where \( F(q) \) is given by a certain combination of the boundary values of the solution of a Goursat problem and its derivative. In order to study this non-linear equation via linearization, we need to know the differentiability of the map \( F \), which is the objective of this chapter.

We study the properties of the mapping \( F \), which maps the coefficient \( q \) of the linear hyperbolic characteristic boundary value problem

\[
\begin{align*}
  u_{tt} - u_{xx} + qu &= 0, \quad 0 \leq |t| \leq x \leq 1 \quad (2.1a) \\
  u(x, \pm x) &= h + \frac{1}{2} \int_{y=0}^{x} q(y) \, dy, \quad 0 \leq x \leq 1 \quad (2.1b)
\end{align*}
\]

or

\[
\begin{align*}
  u_{tt} - u_{xx} + qu &= 0, \quad 0 \leq |t| \leq 1 \quad (2.1c) \\
  u(x, \pm x) &= \pm \frac{1}{2} \int_{y=0}^{x} q(y) \, dy, \quad 0 \leq x \leq 1 \quad (2.1d)
\end{align*}
\]

to the boundary value of \( u_x(x, t) + H \, u(x, t) \) or \( u_{xt}(x, t) + H u_t(x, t) \)

\[
F : q \mapsto u_x(1, t) + H \, u(1, t) \quad \text{(or \( u_{xt}(1, t) + H u_t(1, t) \))}
\]

From well-known results about characteristic boundary value problems, see e.g. Courant and Hilbert [4] chapter 5, we know that the problem (2.1) is well-posed for smooth \( q \). Therefore, \( F \) is defined as a map : \( C^\infty[0,1] \rightarrow C^\infty[-1,1] \). We start
our study of the map $F$ when $q \in C^\infty[0,1]$ and extend this study to the case when $q \in L^2(0,1)$. Our main tool here is an energy estimate, therefore we first obtain such an estimate for the following generic problem.

\[
\begin{aligned}
\frac{\partial w}{\partial t} - w_{xx} + q_2 w &= h_1 w + h_2 q_1 u + h_5 v(x)p(x) \quad 0 \leq |t| \leq x \leq 1 \\
 w(x, \pm x) &= (h_4 \pm h_3) \int_0^x q_1 dy \quad 0 \leq x \leq 1
\end{aligned}
\]

(2.2a) \quad (2.2b)

In this chapter $D$ represents the set $\{(x,t) \mid 0 \leq |t| \leq x \leq 1\}$.

**Lemma 2.1.** Suppose $q_1, q_2 \in C^\infty[0,1], u \in C^\infty(D)$ and $h_1, h_2, h_3, h_4, h_5 \in \mathbb{R}$. We also suppose that $v, p \in L^2(0,1)$ and $|p| \leq C_0 \|q_1\|_{L^2(0,1)}$ for some constant $C_0$. Then the solution $w$ of (2.2) satisfies the $L^\infty$ estimate

\[
\|w\|_{L^\infty(D)} \leq C \|q_1\|_{L^2(0,1)}^2
\]

(2.3)

where $C$ is exponential/polynomial in $\|q_1\|_{L^2(0,1)}, \|q_2\|_{L^2(0,1)}, \|u\|_{L^\infty(D)}, \|v\|_{L^2(0,1)}, |h_1|, |h_2|, |h_3|, |h_4|$ and $|h_5|$.

**Proof.** Let $R$ be the characteristic rectangle bounded by the four characteristic lines $x-t = 0$, $x+t = 0$, $x-t = x_1-t_1$ and $x+t = x_1+t_1$. An application of Green's theorem, to the equation (2.2a) over $R$, yields

\[
\oint_R \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial t} dt = \iint_R \{(h_1 - q_2)w + h_2 q_1 u + h_5 vp\} dx dt.
\]

Using (2.2b) we rewrite this as

\[
\begin{aligned}
w(x_1, t_1) &= (h_4 - h_3) \int_0^{x_1-t_1} q_1(y) dy + (h_4 + h_3) \int_0^{x_1+t_1} q_1(y) dy \\
&\quad + \frac{1}{2} \iint_R \{(h_1 - q_2)w + h_2 q_1 u + h_5 vp\} dx dt.
\end{aligned}
\]
Therefore

\[ |w(x_1, t)| \leq (|h_3| + |h_4|) \left( \int_0^{x_1} |q_1(y)|dy + \int_0^{x_1} |q_2(y)|dy \right) + \frac{1}{2} \int_R \left( |h_1 - q_2|w| + |h_2||q_1||u| + |h_5||v||p| \right)dxdt \quad (2.4) \]

Rewriting the area-integral over \( R \) as

\[ \int_{x=0}^{x=x_1} \int_{t=\phi_1(x)}^{\phi_2(x)} \left( |h_1 - q_2||w| + |h_2||q_1||u| + |h_5||v||p| \right)dt\,dx \]

where \( |\phi_i(x)| \leq 1 \) for \( i = 1, 2 \), and defining, \( \alpha(x) = \sup_{t \in (-x,x)} |w(x,t)| \), the inequality (2.4) can be written as,

\[ |w(x_1, t_1)| \leq (|h_2||u|_{L^\infty(D)} + 2(|h_3| + |h_4|) + |h_5|C_0||v||_{L^2(0,1)})||q_1||_{L^2} \]

\[ + \int_0^{x_1} |h_1 - q_2|\alpha(x)dx. \]

Notice that the right hand side of the above inequality is independent of \( t_1 \). Therefore we get

\[ \alpha(x_1) \leq (|h_2||u|_{L^\infty(D)} + 2(|h_3| + |h_4|) + |h_5|C_0||v||_{L^2(0,1)})||q_1||_{L^2(0,1)} \]

\[ + \int_0^{x_1} |h_1 - q_2|\alpha(x)dx. \]

An application of Gronwall's inequality yields,

\[ \alpha(x_1) \leq C||q_1||_{L^2(0,1)} \]

where

\[ C = (|h_2||u|_{L^\infty(D)} + 2(|h_3| + |h_4|) + |h_5|C_0||v||_{L^2(0,1)}) \]

\[ (1 + (h_1 + ||q_2||_{L^2(0,1)}e^{(h_1 + ||q_2||_{L^2(0,1)})}). \]
Hence we get the desired result.

\[ \|w\|_{L^\infty(D)} \leq C\|q_1\|_{L^2(0,1)} \]

Our next lemma will establish an \(L^2\) estimate of \(w_x(1,t)\). We obtain this by estimating the energy \(E_w(x)\) of the equation (2.2). We define the energy \(E_w(x)\) by

\[ E_w(x) = \frac{1}{2} \int_{t=-x}^{x} (w_x^2 + w_t^2)(x,t)dt. \]

In order to obtain the energy form defined above, we integrate the divergence form of the equation over the trapezoidal region \(S\). Here, the region \(S\) is defined by \(S = \{(x,t) : x_0 \leq x \leq x_1, -x \leq t \leq x\}\).

**Lemma 2.2.** Under the assumptions made in lemma 2.1.

\[ E_w(x) \leq C_2\|q_1\|_{L^2(0,1)} \]

where \(C_2\) is exponential/polynomial in \(\|q_1\|_{L^2(0,1)}, \|q_2\|_{L^2(0,1)}, \|u\|_{L^\infty(D)}, \|r\|_{L^2(0,1)}, |h_1|, |h_2|, |h_3|, |h_4|\) and \(|h_5|\).

**Proof.** We multiply the equation (2.2a) by \(w_x\), in order to obtain its divergence form. So

\[ \frac{d}{dt}(w_tw_x) - \frac{d}{dx}\left(\frac{w_x^2 + w_t^2}{2}\right) = h_1ww_x + h_2q_1uw_x + h_5vpw_x - q_2ww_x. \]

Using the divergence theorem over \(S\) we get

\[ E_w(x_0) - E_w(x_1) + \frac{1}{2} \int_{x_0}^{x_1} (h_4 - h_3)^2q_1^2(y)dy + \frac{1}{2} \int_{x_0}^{x_1} (h_4 + h_3)^2q_1^2(y)dy 
= \int_{y=x_0}^{x_1} \int_{t=-x}^{x} (h_1w + h_2q_1u + h_5vp - q_2w)w_xdt dx. \]

So,

\[ E_w(x_0) - E_w(x_1) + (h_3^2 + h_1^2) \int_{x_0}^{x_1} q_1^2(y)dy = \int_{x=x_0}^{x_1} \int_{t=-x}^{x} (h_1ww_x + h_2q_1uw_x + h_5vpw_x - q_2ww_x)dt dx. \]
Setting \( x_0 = 0, \ x_1 = x \), rearranging the equation, using Cauchy’s inequality and finally using the \( L^\infty \) bound on \( w \) given in lemma 1, we get
\[
E_w(x) \leq \hat{C} \|q_1\|_{L^2(0,1)}^2 + \int_{x=0}^{x_1} E_w(x) dx,
\]
where \( \hat{C} = h_3^2 + h_4^2 + 4|h_1|^2C_1^2 + 4|h_2|^2\|u\|_{L^\infty(D)}^2 + 4|h_5|^2\|\varphi\|_{L^2(0,1)}^2 + 4\|q_2\|_{L^2(0,1)}^2 C_1^2 \). Here \( C_1 \) is the constant associated with lemma 2.1. By an application of Gronwall’s inequality, we get
\[
E_w(x) \leq C_2\|q_1\|_{L^2(0,1)}^2,
\]
where \( C_2 = \hat{C}(1 + \epsilon^2) \). In particular,
\[
\int_{-1}^{1} w_1^2(1,t) dt \leq C_2\|q_1\|_{L^2(0,1)}^2.
\] (2.0)
Therefore we get the desired estimate. \( \square \)

2.2 Existence of a Weak Solution to the Goursat Problem

Next, we establish the existence of a weak solution to the equation (2.1), for \( q \in L^2(0,1) \).

**Theorem 2.1.** (a). For \( q \in L^2(0,1) \) the solution \( u(x,t) \) of the equation 2.1 (a-h) exists. Moreover, \( u(x,t) \in H^1(D) \).

(b). For \( q \in L^2(0,1) \), the solution \( u(x,t) \) of 2.1(c-d) exists. Moreover, \( u(x,t) \in H^1(D) \).

**Proof.** (a). Since \( C^\infty[0,1] \) is dense in \( L^2(0,1) \) we have a sequence \( \{q_n\}_{n=1}^\infty \in C^\infty[0,1] \) such that \( q_n \rightarrow q \) in \( L^2(0,1) \). For every \( q_n \), the unique solution \( u_n(x,t) \) of (2.1) exists. Moreover this solution is \( C^\infty(D) \). Let
\[
w_n(x,t) = u_n(x,t) - \frac{1}{2}(\int_{y=0}^{e^{x_1}} q_n(y)dy + \int_{y=0}^{e^{x_1}} q_n(y)dy) \quad (2.7)
\]
Then \( w_n(x,t) \) satisfies the equations
\[
w_{ntt} - w_{nxx} + q_n w_n = -hq_n(x) - \frac{1}{2}q_n(x)(\int_{y=0}^{e^{x_1}} q_n(y)dy + \int_{y=0}^{e^{x_1}} q_n(y)dy)
\]
\( w_n(x,\pm x) = 0 \).
If \( \phi(x,t) \in C_0^\infty(D) \), then the weak form of the above partial differential equation is

\[
\int_{x=0}^{1} \int_{t=-x}^{x} \{ w_{nt} \phi_t - w_{nx} \phi_x - q_n w_{n} \phi \} dt \, dx = \\
- \int_{x=0}^{1} \int_{t=-x}^{x} h q_n(x) + \frac{1}{2} q_n(x) \left( \int_{y=0}^{x+y} q_n(y) dy + \int_{y=0}^{x+y} q_n(y) dy \phi(x,t) \right) dt \, dx.
\]

Next, we claim that \( \|w_n\|_{L^\infty(D)} \leq C q_n \|_{L^2(D)} \). In order to obtain this estimate we use lemma 2.1 with \( h_1 = -h, \ h_2 = -\frac{1}{2}, \ h_3 = 0, \ h_4 = 0, \ h_5 = 0, \ p(x) = 0, \ v(x) = 0, \ q_1 = q_n, \ q_2 = q_n \) and \( u = \int_{y=0}^{x+y} q_n(y) dy + \int_{y=0}^{x+y} q_n(y) dy \). It is clear that \( \|u\|_{L^\infty(D)} \leq 2 \|q_n\|_{L^2(0,1)} \). Therefore, by (2.3),

\[
\|w_n\|_{L^\infty(D)} \leq C q_n \|_{L^2(0,1)},
\]

where \( C = \|q_n\|_{L^2(0,1)} (1 + e^{h \|q_n\|_{L^2(0,1)}}) \).

By applying lemma 2.2 with the same parameters, we obtain that

\[
E_n(x) \leq C_2 \|q_n\|^2_{L^2},
\]

where \( C_2 = (h^2 + 2 \|q_n\|^2_{L^2})(1 + e^{(1 + \frac{\|q_n\|^2_{L^2}}{2})}) \). Using the above two estimates, we get

\[
\|w_n\|_{H^1(D)} \leq \sqrt{(C^2 + (h^2 + 2 \|q_n\|^2_{L^2})(1 + e^{(1 + \frac{\|q_n\|^2_{L^2}}{2})})} \|q_n\|^2_{L^2}.
\]

Since \( q_n \rightarrow q \) in \( L^2(0,1) \), we have \( (\|q\|_{L^2} + 1) \) as an upper bound for \( \|q_n\|_{L^2} \), for \( n \geq N \). Hence.

\[
\|w_n\|_{H^1(D)} \leq C(q).
\]

By weak compactness of \( H^1(D) \), there exists a subsequence \( \{w_{n_k}\} \) which converges weakly to \( w \) in \( H^1(D) \). Moreover, it can be shown that

\[
| \int_{x=0}^{1} \int_{t=-x}^{x} \{ -\frac{1}{2} q_n(x) \int_{y=0}^{x+y} q_n(y) dy + \frac{1}{2} q(x) \int_{y=0}^{x+y} q(y) dy \phi(x,t) \} dt \, dx |
\]

\[
\leq (\|q_n\|_{L^2} + \|q\|_{L^2}) \|q_n - q\|_{L^2} \max_{(x,t) \in D} |\phi(x,t)|
\]
and
\[
\left| \int_{x=0}^{1} \int_{t=-x}^{t} \left( -\frac{1}{2} q_n(x) \int_{y=0}^{x \frac{1}{2}} q_n(y) dy + \frac{1}{2} q(x) \int_{y=0}^{x \frac{1}{2}} q(y) dy \right) \phi(x,t) \, dx \, dt \right| \\
\leq (\|q_n\|_{L^2} + \|q\|_{L^2}) \|q_n - q\|_{L^2} \frac{\max}{(x,t) \in D} |\phi(x,t)|.
\]

Since \(q_n\) converges to \(q\), the right hand side of the inequality approaches 0. As a consequence, the left hand side also converges to 0. Hence
\[
\int_{x=0}^{1} \int_{t=-x}^{t} (w_t \phi_t - w_x \phi_x - qw \phi) \, dx \, dt = \\
\int_{x=0}^{1} \int_{t=-x}^{t} (-hv + \frac{1}{2} q(x) (\int_{y=0}^{x \frac{1}{2}} q(y) dy + \int_{y=0}^{x \frac{1}{2}} q(y) dy)) \phi(x,t) \, dx \, dt.
\]
This gives the existence of the weak solution \(w(x,t)\). The weak completeness of \(H^1\) gives us the desired regularity for \(w\). Now,
\[
u(x,t) = w(x,t) + h + \frac{1}{2} \left( \int_{y=0}^{x \frac{1}{2}} q(y) \, dy + \int_{y=0}^{x \frac{1}{2}} q(y) \, dy \right)
\]
satisfies the partial differential equation given by (2.1). From the above expression and the fact that \(w\) is an element \(\in H^1(D)\), we get the desired regularity for \(u(x,t)\).

(b) We replace (2.7) by
\[
w_n = u_n - \frac{1}{2} \int_{x \frac{1}{2}}^{x \frac{1}{2}} q_n(y) dy.
\]
The rest of the proof is similar to that of part (a). □

At this point we are ready to investigate the continuity and differentiability properties of the map defined by
\[
F_1(q)(t) = u_x(1,t) + Hu(1,t),
\]
where \(u(x,t)\) solves the characteristic boundary value problem 2.1(a-b). Here, we note that the \(u(x,t)\) which satisfies 2.1(a-b) and the \(u(x,t)\) which solves the following
characteristic boundary value problem

\[ u_{tt} - u_{xx} + q(x) u = -hq(x) \quad 0 \leq |t| \leq 1 \quad (2.9a) \]

\[ u(x, \pm x) = \frac{1}{2} \int_{y=0}^{x} q(y) dy \quad 0 \leq x \leq 1 \quad (2.9b) \]

differ only by a constant \( h \). Therefore, in the following analysis, we replace \( u(x,t) \) in equation 2.8 by the solution of 2.9(a-b).

**2.3 Continuity of the Map** \( F_2(q)(t) = u_x(1,t) + H u(1,t) \)

We begin our study by proving that map \( F_2(q) \) given by

\[ F_2(q)(t) = F_1(q)(t) - H h = u_x(1,t) + H u(1,t) \]

where \( u(x,t) \) solves (2.9), is well defined over smooth functions.

**Theorem 2.2.** The map \( F_2 : C^\infty[0,1] \rightarrow L^2(-1,1) \) given by

\[ F_2(q)(t) = u_x(1,t) + H u(1,t) \]

where \( u(x,t) \) solves (2.9), is well defined.

**Proof.** From, well known results about characteristic boundary value problems, the solution \( u(x,t) \) of 2.1(a-b) is smooth in \( \{(x,t) | 0 \leq |t| \leq x\} \). In particular \( u(1,t) \) and \( u_x(1,t) \) are continuous functions in \( t \) for all \( t \in [-1,1] \). Therefore

\[ u_x(1,t) + H u(1,t) \in L^2(-1,1). \]

and this proves the theorem. \( \Box \)

**Lemma 2.3.** Let \( q_i \in C^\infty[0,1] \), and let \( u_i(x,t) \) be the corresponding solution to (2.9), where \( i = 1, 2 \). Let \( v = u_1 - u_2 \). Then

\[ \|v\|_{L^\infty(D)} \leq C_1 \|q_1 - q_2\|_{L^2(0,1)} \]

and

\[ E_v(x) \leq C_2 \|q_1 - q_2\|^2_{L^2(0,1)}. \]
where the constant $C$ is exponential/polynomial in $\|q_1\|_{L^2(0,1)}$ and $\|q_1 - q_2\|_{L^2(0,1)}$.

**Proof.** Since each $u_i$ satisfies (2.9), $v(x, t)$ satisfies the characteristic boundary value problem

\[ v_{tt} - v_{xx} + q_1 v = -(h + u_2)(q_1 - q_2) \quad (2.10a) \]
\[ v(x, \pm x) = \frac{1}{2} \int_0^x (q_1 - q_2) dy \quad (2.10b) \]

Since $u_2$ appears in the inhomogeneous term of the equation, we need an $L^\infty$ estimate on $u_2$. We can obtain this estimate as a result of an application of lemma 2.1 to the equation (2.9). Doing so we get

\[ \|u_2\|_{L^\infty(D)} \leq C_0 \|q_2\|_{L^2(0,1)} \]

where $C_0 = (\frac{1}{2} + \|h\|)(1 + \|q_1\|_{L^2})$.

Another application of lemma 2.1 to (2.10) gives the estimate

\[ \|v\|_{L^\infty(D)} \leq C_1 \|q_1 - q_2\|_{L^2(0,1)} \quad (2.11) \]

where $C_1 = (\frac{1}{2} + \|h\| + C_0 \|q_2\|_{L^2})(1 + \|q_1\|_{L^2})$.

Now we have all of the necessary estimates in order to obtain the desired energy estimate. We apply lemma 2.2 to (2.10) and get

\[ E_v(x) \leq C_2 \|q_1 - q_2\|_{L^2(0,1)}^2 \]

for $0 \leq x \leq 1$ where $C_2 = (\frac{1}{2} + \|h\| + C_1 \|q_2\|_{L^2})(1 + \epsilon^2)$. \qed

**Theorem 2.3.** For $q_i \in C^\infty[0,1]$ in (2.9), $i=1,2$, there exists a constant $C$ which is exponential/polynomial in $\|q_1\|_{L^2}$ and $\|q_1 - q_2\|_{L^2}$ such that

\[ \|F_2(q_1) - F_2(q_2)\|_{L^2} \leq C \|q_1 - q_2\|_{L^2} \]
Proof. From lemma 2.3, we get $E_n(x) \leq C_1\|q_1 - q_2\|_{L^2(0,1)}^2$, for $0 \leq x \leq 1$. In particular, we get
\[
\frac{1}{2} \int_{-x}^{x} v^2(x,t)dt \leq C_1\|q_1 - q_2\|_{L^2}.
\]
By setting $x = 1$ we get
\[
\int_{-1}^{1} v^2(t)dt \leq 2C_1\|q_1 - q_2\|_{L^2}^2.
\]
(2.12)
Using the estimates (2.11) and (2.12) we get
\[
\|F_2(q_1) - F_2(q_2)\|_{L^2} \leq (C_1 + 2HC_1C_2 + H^2C_2^2)\|q_1 - q_2\|_{L^2}.
\]
Choose $C = C_1 + 2HC_1C_2 + H^2C_2^2$ to get the desired result. □

Corollary 2.1. The map $F_2$ extends to a locally Lipschitz map:
\[
F_2 : L^2(0,1) \rightarrow L^2(-1,1).
\]

2.4 Differentiability of the Map $F_2(q)(t) = u_x(1,t) + Hu(1,t)$

Now we have established the continuity of the map $F_2$ over $L^2(0,1)$ space. Our next task is to establish its differentiability. Here, we use the notion of Frechet differentiability. Next, we define the formal linearization of the map $F_2$. The formal linearization of the forward map $F_2$ at $q$ is given by
\[
DF_2(q)\delta q(t) = \delta u_x(1,t) + Hu(1,t)
\]
where $\delta u$ solves the partial differential equation
\[
\delta u_{tt} - \delta u_{xx} + q\delta u = -(h + u)\delta q
\]
(2.13a)
\[
\delta u(x, \pm x) = \frac{1}{2} \int_{y=0}^{x} \delta q(y)dy.
\]
(2.13b)
Note that the $u$ in equation (2.13a) is the solution of (2.9). We will show that $DF_2$ is the Frechet derivative of $F_2$ by proving the following theorem. We also note that the existence and uniqueness of the solution to 2.13a-b can be proven by following arguments similar to theorem 2.1.
Theorem 2.4. For \( q \in C^\infty[0, 1] \), there exists \( C \) such that, for \( \delta q \in C^\infty[0, 1] \),

\[
\|DF_2(q)\delta q - (F_2(q + \delta q) - F_2(q))\|_{L^2(-1, 1)} \leq C\|\delta q\|^2_{L^2(0, 1)}.
\]

Proof. Let \( \tilde{q} = q + \delta q \). We take \( u(x, t) \) and \( \tilde{u}(x, t) \) to be the solutions of (2.9) corresponding to the coefficients \( q \) and \( \tilde{q} \) respectively. We also take \( \delta u(x, t) \) to be the formal linearization which satisfies (2.13).

Define

\[
v(x, t) = \delta u(x, t) - (\tilde{u} - u)(x, t).
\]

In order to prove the differentiability of \( F \), we need to estimate \( \|v_x(1, t)\|_{L^2(-1, 1)} \).

We begin this by estimating the three quantities \( \|\tilde{u}\|_{L^\infty(D)}, \|u - \tilde{u}\|_{L^\infty(D)} \) and \( \|v\|_{L^\infty(D)} \).

Let \( u_1 = u - \tilde{u} \). Then \( u_1 \) and \( v \) satisfy the following equations

\[
u_1tt - u_1xx + qu_1 = \delta q(\tilde{u} + h) \tag{2.14a}
\]

\[
u_1(x, \pm x) = -\frac{1}{2}\int_{y=0}^{x} \delta q(y)dy \tag{2.14b}
\]

and

\[
vtt - vxx + qu = -\delta qu_1 \tag{2.15a}
\]

\[
v(x, \pm x) = 0 \tag{2.15b}
\]

respectively.

Also, \( \tilde{u}(x, t) \) satisfies

\[
\tilde{u}tt - \tilde{u}xx + \tilde{q}\tilde{u} = -h\tilde{q} \quad 0 \leq |t| \leq x \leq 1 \tag{2.16a}
\]

\[
\tilde{u}(x, \pm x) = \frac{1}{2}\int_{y=0}^{x} \tilde{q}(y)dy. \tag{2.16b}
\]

Now we apply lemma 2.1 to equation (2.16) and obtain the estimate

\[
\|\tilde{u}\|_{L^\infty(D)} \leq C_1\|\tilde{q}\|_{L^2}.
\]
To obtain an $L^\infty$ estimate on $u_1$ we again apply lemma 2.1 to 2.14(a-b) and obtain the estimate

$$\|u_1\|_{L^\infty(D)} \leq C_2 \|\delta q\|_{L^2(0,1)}$$

where $C_2 = (1 + C_1 \|q\|_{L^2(0,1)} + h)(1 + \epsilon(\|q\|_{L^2(0,1)}^2))$. Using these two estimates with the result of an application of lemma 2.1 to 2.15 we obtain

$$\|v\|_{L^\infty} \leq C_3 \|\delta q\|_{L^2(0,1)}$$

where $C_3 = \|u_1\|_{L^\infty(D)}(1 + \epsilon(1 + \|q\|_{L^2}))$. At this point we use the $L^\infty$ estimate of $u_1$ and obtain the following estimate

$$\|v\|_{L^\infty(D)} \leq \hat{C}_3 \|\delta q\|_{L^2(0,1)}^2$$  \hspace{1cm} (2.17)

Finally, we are in a position to obtain the desired $L^2$ estimate on $v_r(1,t)$. We apply the lemma (2.2) to (2.15) to obtain the estimate

$$E_r(x) \leq C_4 \|\delta q\|_{L^2}^2,$$

where $C_4 = \|u_1\|_{L^\infty(D)}^2\{1 + \|q\|_{L^2}^2(1 + \epsilon(\|q\|_{L^2(0,1)}^2))(1 + \epsilon^2)\}$ and $C_4 \leq \hat{C}_4 \|\delta q\|_{L^2}^2$. At this point we again incorporate the $L^\infty$ estimate of $u_1$ and obtain the following energy estimate

$$E_r(x) \leq \hat{C}_4 \|\delta q\|_{L^2(0,1)}^4 \quad 0 \leq x \leq 1.$$  

In particular,

$$E_r(1) \leq \hat{C}_4 \|\delta q\|_{L^2(0,1)}^4.$$  

Hence,

$$\|v_r(1,t)\|_{L^2(-1,1)} \leq \sqrt{2\hat{C}_4 \|\delta q\|_{L^2(0,1)}^2},$$  \hspace{1cm} (2.18)

So estimates (2.17) and (2.18) give us the following desired result.

$$\int_{-1}^1 (v_r(1,t) + Hv_r(1,t))^2 \, dt \leq C\|\delta q\|_{L^2(0,1)}^2 \quad \square$$

We would like to extend the differentiability of the map $F_2$ over to the $L^2$ space. We achieve this in two steps. First we extend $DF_2(q)$ as a bounded linear map from $L^2(0,1)$ to $L^2(-1,1)$ for $q \in C^\infty[0,1]$ by proving the following theorem.
Theorem 2.5. For $q \in C^\infty[0,1]$, $DF_2(q)$ extends to a bounded linear map from $L^2(0,1)$ to $L^2(-1,1)$.

Proof. Recall that $DF_2(q)\delta q = \delta u_x(1,t) + H\delta u(1,t)$ where $\delta u(x,t)$ satisfies (2.13). We first obtain an $L^\infty$ estimate of $\delta u$ as a result of an application of lemma 2.1 to the equation (2.13). So

$$\|\delta u\|_{L^\infty(D)} \leq C \|\delta q\|_{L^2(0,1)}$$

where $C = (\frac{1}{2} + \|u\|_{L^\infty(D)})(1 + \epsilon \|q\|_2)$.

In order to prove this theorem we need to show that energy $E_{\delta u}(x)$ is bounded above by $\|\delta q\|_2^2$.

Now we apply lemma (2.2) to equation (2.13) and obtain the estimate

$$E_{\delta u}(x) \leq C_2 \|\delta q\|_{L^2(0,1)}^2 \quad 0 \leq x \leq 1$$

where $C_2 = \{\frac{1}{2} + \|u\|_{L^\infty(D)}^2 + C_1^2 \|q\|_2^2\}(1 + \epsilon^2)$. In particular

$$E_{\delta u}(1) \leq C_2 \|\delta q\|_{L^2(0,1)}^2.$$

Hence

$$\int_{t=-1}^1 \delta u_x^2(1,t)dt \leq 2C_2 \|\delta q\|_2^2.$$

Therefore, we get

$$\|DF_2(q)\delta q\|_{L^2(-1,1)}^2 \leq (C^2 + 2C_2) \|\delta q\|_2^2.$$

So $DF_2(q)$ extends to a bounded linear map in $L^2(0,1)$. \qed

We need to extend this theorem for the case when $q \in L^2(0,1)$ by showing that the map $q \rightarrow DF_2(q)$ is continuous. We achieve this by proving the following theorem.
**Theorem 2.6.** The Frechet derivative $DF_2$ is a locally Lipschitz continuous map on $L^2(0,1)$, with values in $L(L^2(0,1), L^2(-1,1))$. where $DF_2$ extends to a locally Lipschitz map

$$L^2(0,1) \times L^2(0,1) \rightarrow L^2(-1,1).$$

**Proof.** Let $q$ and $\tilde{q} \in C^\infty[0,1]$ and let $u$ and $\tilde{u}$ be the corresponding solutions of (2.1). We take $\delta q \in C^\infty[0,1]$ and let $\delta u$ and $\tilde{\delta u}$ be the corresponding solutions of (2.13).

Let $\delta v = \delta u - \tilde{\delta u}$. We need to obtain an $L^2$ estimate of $\delta v(x, t)$, where $\delta v$ solves

$$\begin{align*}
\delta v_{tt} - \delta v_{xx} + q \delta v &= \delta q(\tilde{u} - u) - (q - \tilde{q})(\tilde{\delta u}) \\
\delta v(x, \pm x) &= 0.
\end{align*}
$$

(2.19a) (2.19b)

We first claim that

$$\|\delta v\|_{L^\infty(D)} \leq C \|q - \tilde{q}\|_{L^2(0,1)} \|\delta q\|_{L^2(0,1)}. \quad (2.20)$$

In order to prove the claim, we follow the steps outlined in lemma 2.1. We first apply the Green’s theorem to the equation (2.4) over the characteristic rectangle $R$, and then rearrange the result as we did in lemma 2.1. We arrive at

$$\alpha(x) \leq \|u - \tilde{u}\|_{L^\infty(D)} \|q\|_{L^2} + \|\tilde{\delta u}\|_{L^\infty(D)} \|q - \tilde{q}\|_{L^2} + \int_{x_0}^{x_1} |q(x)||\alpha(x)|dx,$$

where $\alpha(x) = \sup_{t \in (-x,x)} \|\delta v\|_{L^\infty(D)}$. An application of Gronwall’s inequality yields

$$\alpha(x) \leq (\|u - \tilde{u}\|_{L^\infty(D)} \|q\|_{L^2} + \|\tilde{\delta u}\|_{L^\infty(D)} \|q - \tilde{q}\|_{L^2})(1 + \|q\|_{L^2} \|\tilde{q}\|_{L^2}). \quad (2.21)$$

Note that $(u - \tilde{u})(x, t)$ satisfies the following partial differential equation:

$$(u - \tilde{u})_{tt} - (u - \tilde{u})_{xx} + q(u - \tilde{u}) = -(q - \tilde{q})\tilde{u}$$

$$(u - \tilde{u})(x, \pm x) = \frac{1}{2} \int_{y=0}^{x} (q - \tilde{q})(y)dy.$$
An application of lemma (2.1) on this equation gives
\[
\|u - \hat{u}\|_{L^\infty(D)} \leq C_1 \|q - \hat{q}\|_{L^2(0,1)}.
\] (2.22)

Another application of lemma (2.1) on the equation (2.13) gives
\[
\|\delta u\|_{L^\infty(D)} \leq C_2 \|\delta q\|_{L^2(0,1)}.
\]

Now the above two estimates and (2.20) result in
\[
\|\delta v\|_{L^\infty(D)} \leq \hat{C} \|q - \hat{q}\|_{L^2(0,1)} \|\delta q\|_{L^2(0,1)}.
\]

Next, we follow the steps outlined in lemma (2.2) in order to obtain a \(L^2\) estimate on \(\delta v_x(1,t)\). We first integrate the divergence form of equation (2.4) over the trapezoidal region \(S\) and rearrange the result as we did in lemma (2.2). We get
\[
E_{\delta v}(x_1) \leq \|\delta q\|^2_{L^2} \|q - \hat{q}\|^2_{L^2} (C \|q\|_{L^2} + \frac{3}{2}) + \int_{y=0}^{x_1} (|q| + 2) E_{\delta v}(y) dy.
\]

Applying Gronwall's inequality again yields
\[
E_{\delta v}(x_1) \leq C_3 \|\delta q\|^2_{L^2} \|q - \hat{q}\|^2_{L^2} \quad 0 \leq x_1 \leq 1.
\]

Therefore we get.
\[
\|\delta v_x(1,t)\|_{L^2(-1,1)} \leq \sqrt{2} C_3 \|\delta q\|_{L^2} \|q - \hat{q}\|_{L^2}.
\] (2.23)

Using (2.20) and (2.21) we get
\[
\|DF_2(q) - DF_2(\hat{q})\|_{L^2(-1,1)} \leq \hat{C} \|q - \hat{q}\|_{L^2(0,1)}.
\] (2.24)

Hence, \(DF_2(q)\) is a locally Lipschitz map on \(C^\infty[0,1]\) with values in \(L(L^2(0,1), L^2(-1,1))\), which is the desired result. \(\square\)
Corollary 2.2. The map $F_2(q)$ is continuously differentiable in $L^2(0,1)$.

At this point we can state a similar result for another useful map.

**Theorem 2.7.** The map $F_3: L^2(0,1) \rightarrow L^2(-1,1)$ given by

$$F_3(q) = u_x(1,t) + Hu(1,t)$$

where $u(x,t)$ solves the characteristic boundary value problem

$$u_t - u_{xx} + qu = 0 \quad 0 \leq |t| \leq x \leq 1$$

$$u(x, \pm x) = \pm \int_{y=0}^{x} q(y) \, dy \quad 0 \leq x \leq 1$$

is differentiable and its derivative $DF_3(q)$ extends to a locally Lipschitz map

$$L^2(0,1) \times L^2(0,1) \rightarrow L^2(-1,1).$$

**Proof.** The proof of this theorem is very similar to the case before. Hence we omit the proof.

\square

2.5 Continuity of the Map $F_4(q)(t) = u_{xt}(1,t) + Hu_t(1,t)$

Next, we study the continuity and differentiability properties of the map defined by

$$F_4(q)(t) = u_{xt}(1,t) + Hu_t(1,t) \quad (2.25)$$

where $u(x,t)$ solves the characteristic boundary value problem 2.1.

We begin this study by proving that the map $F_4(q)$ is well defined over the set of smooth functions.

**Theorem 2.8.** The map

$$F_4 : C^\infty[0,1] \rightarrow L^2(-1,1)$$
given by
\[ F_4(q)(t) = u_{xt}(1, t) + Hu_t(1, t), \]
where \( u(x, t) \) solves (2.1) is well defined.

**Proof.** From well known properties of the solutions to the characteristic boundary value problems, the solution \( u(x, t) \) of (2.1) is smooth in \( D \). In particular \( u_t(1, t) \) and \( u_{xt}(1, t) \) are continuous functions in \( t \) for all \( t \in [-1, 1] \). Hence,
\[ u_{xt}(1, t) + Hu_t(1, t) \in L^2(-1, 1), \]
and this proves the theorem. □

Our goal here is to establish the continuity of the map \( F_4 \). In order to achieve this we prove the following key estimate:

**Lemma 2.4.** Let \( O \) be the set of almost everywhere odd functions in \( L^2(0, 1) \). Suppose \( q_i \in O \cap C^\infty[0, 1] \) and \( u_i(x, t) \) is the corresponding solution to (2.1), where \( i = 1, 2 \). If \( v = u_1 - u_2 \) then
\[ \| v_{xt}(1, t) \|_{L^2(-1, 1)} \leq C \| q_1 - q_2 \|_{L^2(0, 1)}, \]
where the constant \( C \) is exponential/polynomial in \( \| q_1 \|_{L^2(0, 1)} \) and \( \| q_1 - q_2 \|_{L^2(0, 1)} \).

**Proof.** Since each \( u_i \) satisfies (2.1), \( v(x, t) \) satisfies the equation
\[
\begin{align*}
v_{tt} - v_{xx} + q_1 v &= -u_2(q_1 - q_2) \\
v(x, \pm x) &= \frac{1}{2} \int_0^x (q_1 - q_2)(y)dy.
\end{align*}
\]
let
\[ w(x, t) = v(x, t) - \frac{1}{2} \int_0^{\frac{x+t}{2}} (q_1 - q_2)dy + \int_0^{\frac{x-t}{2}} (q_1 - q_2)dy. \]
Then $w$ satisfies the following:

\begin{align}
    w_{tt} - w_{xx} + q_1 w &= f(x,t) \tag{2.30a} \\
    w(x, \pm x) &= 0 \tag{2.30b}
\end{align}

where

\begin{equation}
    f(x,t) = -\frac{1}{2} q_1(x) \left\{ \int_{0}^{\frac{x+t}{2}} (q_1 - q_2)(y)dy + \int_{\frac{x-t}{2}}^{0} (q_1 - q_2)dy \right\} - u_2(x,t)(q_1 - q_2)(x) \tag{2.31}
\end{equation}

Since $(q_1 - q_2)(x)$ is odd, we note that

\begin{equation}
    v_{xt}(1,t) = w_{xt}(1,t).
\end{equation}

An application of Green's theorem to the equation 2.30, over the characteristic rectangle $R$ results in the following expression for the solution $w(x,t)$ as in lemma 2.1:

\begin{equation}
    w(x,t) = \frac{1}{2} \left\{ \int_{y=0}^{\frac{x-t}{2}} \int_{s=-y}^{y} + \int_{y=\frac{x-t}{2}}^{x} \int_{s=y-x+t}^{y} \right. \\
    \left. + \int_{y=\frac{x+t}{2}}^{x} \int_{s=y-(x-t)}^{(x+t-y)} \right\} (q(y)w(y,s) + f(y,s)) \, dsdy.
\end{equation}
By differentiating this expression with respect to \( x \) we obtain the following:

\[
w_x(x,t) = \frac{1}{4} \int_{-\frac{x-t}{2}}^{\frac{x-t}{2}} \left( -q_1 \left( \frac{x-t}{2} \right) w \left( \frac{x-t}{2}, s \right) + f \left( \frac{x-t}{2}, s \right) \right) ds
+ \frac{1}{4} \int_{y=-\frac{x-t}{2}}^{\frac{x+t}{2}} \left\{ -q_1 \left( \frac{x-t}{2} \right) w(\frac{x-t}{2}, s) + f(\frac{x-t}{2}, s) \right\} ds
- \frac{1}{4} \int_{y=-\frac{x-t}{2}}^{\frac{x+t}{2}} \left\{ -q_1 \left( \frac{x-t}{2} \right) w(\frac{x-t}{2}, s) + f(\frac{x-t}{2}, s) \right\} ds
+ \frac{1}{2} \int_{y=-\frac{x-t}{2}}^{\frac{x+t}{2}} \left\{ -q_1(y) w(y, y-x+t) + f(y, y-x+t) \right\} dy
- \frac{1}{4} \int_{y=-\frac{x-t}{2}}^{\frac{x+t}{2}} \left\{ -q_1 \left( \frac{x-t}{2} \right) w(\frac{x-t}{2}, s) + f(\frac{x-t}{2}, s) \right\} ds
+ \frac{1}{2} \int_{y=-\frac{x-t}{2}}^{\frac{x+t}{2}} \left\{ -q_1(y) w(y, x+t-y) + f(y, x+t-y) \right\} dy
+ \frac{1}{2} \int_{y=-\frac{x-t}{2}}^{\frac{x+t}{2}} \left\{ -q_1(y) w(y, y-x+t) + f(y, y-x+t) \right\} dy
\]

Cancelling the first term with the third term and second term with fifth term results in,

\[
w_x(x,t) = \frac{1}{2} \int_{y=-\frac{x-t}{2}}^{\frac{x+t}{2}} \left\{ -q_1 w(y, y-x+t) + f(y, y-x+t) \right\} dy
+ \frac{1}{2} \int_{y=-\frac{x-t}{2}}^{\frac{x+t}{2}} \left\{ -q_1(y) w(y, x+t-y) + f(y, x+t-y) \right\} dy.
\]

Letting \( t = 1 \) in this expression, we get

\[
w_x(1,t) = \frac{1}{2} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} \left\{ -q_1(y) w(y, y-1+t) + f(y, y-1+t) \right\} dy
+ \frac{1}{2} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} \left\{ -q_1(y) w(y, 1+t-y) + f(y, 1+t-y) \right\} dy.
\]
By differentiating this expression with respect to $t$ we obtain the following:

$$w_{,t}(1, t) = \frac{1}{4}\{-q_{1}\frac{1-t}{2}w\frac{1-t}{2} + f(\frac{1-t}{2}, \frac{t-1}{2})\}$$

$$-\frac{1}{4}\{-q_{1}\frac{1+t}{2}w\frac{1+t}{2} + f(\frac{1+t}{2}, \frac{1+t}{2})\}$$

$$+ \frac{1}{2}\int_{y=\frac{1-t}{2}}^{1}\{-q_{1}(y)w_{t}(y, y - 1 + t) + f_{t}(y, y - 1 + t)\}dy$$

$$+ \frac{1}{2}\int_{y=\frac{1+t}{2}}^{1}\{-q_{1}(y)w_{t}(y, 1 + t - y) + f_{t}(y, 1 + t - y)\}dy$$

Using (2.30b) we get

$$w_{,t}(1, t) = \frac{1}{4}\{f(\frac{1-t}{2}, \frac{t-1}{2}) - f(\frac{1+t}{2}, \frac{1+t}{2})\}$$

$$+ \frac{1}{2}\int_{y=\frac{1-t}{2}}^{1}\{-q_{1}(y)w_{t}(y, y - 1 + t) + f_{t}(y, y - 1 + t)\}dy$$

$$+ \frac{1}{2}\int_{y=\frac{1+t}{2}}^{1}\{-q_{1}(y)w_{t}(y, 1 + t - y) + f_{t}(y, 1 + t - y)\}dy.$$

(2.33)

Now,

$$f(\frac{1-t}{2}, \frac{t-1}{2}) = -\frac{1}{2}\{\int_{0}^{\frac{1-t}{2}}(q_{1} - q_{2})dy\} - u_{2}(\frac{1-t}{2}, \frac{t-1}{2})(q_{1} - q_{2})(\frac{1-t}{2})$$

$$= -\frac{1}{2}q_{1}(\frac{1-t}{2})\int_{0}^{\frac{1-t}{2}}(q_{1} - q_{2})dy$$

$$- (h + \frac{1}{2}\int_{y=0}^{\frac{1-t}{2}}q_{2}(y)dy)(q_{1} - q_{2})(\frac{1-t}{2}).$$

Similarly,

$$f(\frac{1+t}{2}, \frac{1+t}{2}) = -\frac{1}{2}\{\int_{y=0}^{\frac{1+t}{2}}(q_{1} - q_{2})(y)dy\}$$

$$- (h + \frac{1}{2}\int_{y=0}^{\frac{1+t}{2}}q_{2}(y)dy)(q_{1} - q_{2})(\frac{1+t}{2}).$$
Therefore,

\[
f\left(\frac{1-t}{2}, \frac{t-1}{2}\right) - f\left(\frac{1+t}{2}, \frac{1+t}{2}\right) = -\frac{1}{2} q_1(\frac{1-t}{2}) \int_{y=0}^{\frac{1+t}{2}} (q_1 - q_2) dy \\
+ \frac{1}{2} q_1(\frac{1+t}{2}) \int_{y=0}^{\frac{1+t}{2}} (q_1 - q_2) dy \\
- \left\{ 2h + \frac{1}{2} \int_{y=0}^{\frac{1+t}{2}} q_2(y) dy \right\} \\
+ \frac{1}{2} \int_{y=0}^{\frac{1+t}{2}} q_2(y) dy \right\} (q_1 - q_2)(\frac{1-t}{2}).
\]

Using the Schwartz inequality, we get the following:

\[
\|f\left(\frac{1-t}{2}, \frac{t-1}{2}\right) - f\left(\frac{1+t}{2}, \frac{1+t}{2}\right)\|_{L^2(-1,1)} \leq C_1 \|q_1 - q_2\|_{L^2(0,1)} \quad (2.34)
\]

where \(C_1 = \sqrt{2\|q_1\|_{L^2}^2 + 2(2h + \|q_2\|_{L^2}^2)}\). It is clear that

\[
f_t(y, y - 1 + t) = -\frac{1}{4} \left\{ q_1(y)(q_1 - q_2)\left(\frac{2y - 1 + t}{2}\right) - q_1(y)(q_1 - q_2)\left(\frac{1-t}{2}\right) \right\} \\
\frac{1}{4} u_{2t}(y, y - 1 + t)(q_1 - q_2)(y)
\]

So,

\[
\| \int_{y=\frac{1-t}{2}}^{1} f_t(y, y - 1 + t) dy \|_{L^2(-1,1)}^2 \leq 3\left\{ \frac{1}{4} \|q_1\|_{L^2}^2 + \int_{t=-1}^{1} \int_{y=\frac{1-t}{2}}^{1} u_{2t}^2(y, y - 1 + t) dy dt \right\} \|q_1 - q_2\|_{L^2}^2.
\]

Since

\[
\int_{t=-1}^{1} \int_{x=0}^{x} \left\{ u_{2x}^2(x, t) + u_{2t}^2(x, t) \right\} dt \ dx = \\
\int_{t=-1}^{1} \int_{y=\frac{1-t}{2}}^{1} \left\{ u_{2x}^2(y, y - 1 + t) + u_{2t}^2(y, y - 1 + t) \right\} dy \ dt
\]
using lemma 2.2, we obtain that
\[
\int_{t=-1}^{1} \int_{y=\frac{1-t}{2}}^{1} u_{2}^{2}(y, y-1+t) dy dt \leq 2\hat{C}_{1}\|q_{2}\|_{L^{2}},
\]
where \( \hat{C}_{1} \) is a constant given by lemma 2.2. Hence we obtain that
\[
\| \int_{y=\frac{1-t}{2}}^{1} f_{1}(y, y-1+t) dy \|_{L^{2}(-1,1)} \leq C_{2}\|q_{1} - q_{2}\|_{L^{2}(0,1)},
\]
(2.35)
where \( C_{2} = \sqrt{3\left(\frac{\|q_{1}\|_{L^{2}}^{2}}{4} + 2C_{20}\|q_{2}\|_{L^{2}}^{2}\right)} \). Similarly,
\[
\| \int_{y=\frac{1-t}{2}}^{1} f_{2}(y, y-1+t - y) dy \|_{L^{2}(-1,1)} \leq C_{2}\|q_{1} - q_{2}\|_{L^{2}(0,1)}
\]
(2.36)
where \( C_{2} \) is the same as above. Next, we claim that
\[
\| \int_{y=\frac{1-t}{2}}^{1} q_{1}(y)w_{1}(y, y-1+t) dy \|_{L^{2}(-1,1)} \leq C_{3}\|q_{1} - q_{2}\|_{L^{2}(0,1)}.
\]
(2.37)
It's clear that
\[
\| \int_{y=\frac{1-t}{2}}^{1} q_{1}(y)w_{1}(y, y-1+t) dy \|_{L^{2}(-1,1)} \leq \|q_{1}\|_{L^{2}(0,1)}(\int_{t=-1}^{1} \int_{y=\frac{1-t}{2}}^{1} w_{1}(y, y-1+t)^{2} dy dt)^{\frac{1}{2}}.
\]
By integrating along the lines parallel to the \( x-t = 0 \) instead of the lines parallel
to the \( t \) axis we get
\[
\int_{x=0}^{1} \int_{t=-x}^{x} \left\{ w_{x}^{2}(x,t) + w_{t}^{2}(x,t) \right\} dt dx
\]
\[
= \int_{t=-1}^{1} \int_{y=\frac{1-t}{2}}^{1} \left\{ w_{x}^{2}(y, y-1+t) + w_{t}^{2}(y, y-1+t) \right\} dy dt.
\]
(2.38)
Now, an application of lemma 2.2 on 2.30 gives
\[
\int_{t=-1}^{1} \int_{y=\frac{1-t}{2}}^{1} w_{t}^{2}(y, y-1+t) dy dt \leq 2C_{30}\|q_{1} - q_{2}\|_{L^{2}}^{2}
\]
(2.39)
where $\hat{C}_2$ is the constant given by lemma 2.2. By taking $C_3 = \sqrt{2\hat{C}_2}\|q_1\|_{L^2}$, we prove the claim. Following through similar steps we obtain,

$$\| \int_{y=\frac{1-t}{2}}^{1} q_1(y)w_t(y, 1 + t - y) \|_{L^2(-1, 1)} \leq C_3\|q_1 - q_2\|_{L^2(0, 1)}. \quad (2.40)$$

An application of the Schwartz inequality with estimates (2.34), (2.35), (2.36), (2.37) and (2.39) finally gives,

$$\|w_{xt}(1, t)\|_{L^2(-1, 1)} \leq \hat{C}\|q_1 - q_2\|_{L^2(0, 1)}, \quad (2.41)$$

where $\hat{C} = \sqrt{5(C_1^2 + 2C_2^2 + 2C_3^2)}$. Since $(q_1 - q_2)(x)$ is odd,

$$v_{xt}(1, t) = w_{xt}(1, t).$$

Hence,

$$\|v_{xt}(1, t)\|_{L^2(-1, 1)} \leq \hat{C}\|q_1 - q_2\|_{L^2(0, 1)} \quad (2.42)$$

\[\square\]

**Theorem 2.9.** For $q_i \in O \cap C^\infty[0, 1]$ in 2.1. $i=1,2$, there exists a constant $C$ which is exponential/polynomial in $\|q_1\|_{L^2}$ and $\|q_1 - q_2\|_{L^2}$ such that

$$\|F_4(q_1) - F_4(q_2)\|_{L^2(-1, 1)} \leq C\|q_1 - q_2\|_{L^2(0, 1)}. \quad (2.43)$$

**Proof.** From lemma 2.3, we know that $E_v(x) \leq C_1\|q_1 - q_2\|_{L^2(0, 1)}^2$, for $0 \leq x \leq 1$. By setting $x = 1$ we get

$$\int_{-1}^{1} v_t^2(1, t)dt \leq 2C_1\|q_1 - q_2\|_{L^2(0, 1)}^2. \quad (2.43)$$

Using estimates (2.42) and (2.43) one can write

$$\|F_4(q_1) - F_4(q_2)\|_{L^2(-1, 1)} \leq \sqrt{2\hat{C}_2^2 + 2C_2^2}\|q_1 - q_2\|.$$

This proves the theorem. \[\square\]
Corollary 2.3. The map $F_4$ extends to a locally Lipschitz map as:

\[ F_4 : O \longrightarrow L^2(-1, 1). \]

2.6 Differentiability of the Map $F_4(q)(t) = u_{xt}(1, t) + Hu_t(1, t)$

We next establish the differentiability of the map $F_4$. So, we define the formal linearization of the forward map $F_4$ at $q$ is given by

\[ DF_4(q)\delta q(t) = \delta u_{xt}(1, t) + Hu_t(1, t) \quad (2.44) \]

where $\delta u$ solves the following boundary value problem:

\[ \begin{align*}
\delta u_{tt} - \delta u_{xx} + q \delta u &= -\delta qu \\
\delta u(x, \pm 1) &= \frac{1}{2} \int_{y=0}^{t} \delta q(y)dy.
\end{align*} \quad (2.45a,b) \]

Note that $u$ in equation (2.45a) is the solution of (2.26). We will show that $DF_4$ is the Frechet derivative of $F_4$, proving the following theorem.

**Theorem 2.10.** For $q \in O \cap C^\infty[0, 1]$, there exists $C$, such that for $\delta q \in O$.

\[ \|DF_4(q)\delta q - (F_4(q + \delta q) - F_4(q))\|_{L^2(-1, 1)} \leq C\|\delta q\|_{L^2}. \]

Proof. Let $\tilde{q} = q + \delta q$. We take $u(x, t)$ and $\tilde{u}(x, t)$ to be the solutions of (2.1) corresponding to the coefficients $q$ and $\tilde{q}$ respectively. $\delta u(x, t)$ corresponds to the formal linearization and satisfies (2.45). Define

\[ v(x, t) = \delta u(x, t) - \tilde{u} + u. \quad (2.46) \]

In order to prove the desired result, we need to estimate $\|v_{xt}(1, t)\|_{L^2(-1, 1)}$. We begin this by estimating $\|(\tilde{u} - u)_{xt}(1, t)\|_{L^2(-1, 1)}$. Let $u_1(x, t) = (\tilde{u} - u)(x, t)$. Now, $u_1$ and $v$ satisfy the following Goursat problem:

\[ u_{1tt} - u_{1xx} + qu_1 = -\delta q\tilde{u} \quad (2.47a) \]
\[ u_1(x, \pm x) = \frac{1}{2} \int_0^x \delta q(y) dy \] (2.47b)

and

\[ v_{tt} - v_{xx} + qv = -\delta u_1 \] (2.48a)

\[ v(x, \pm x) = 0 \] (2.48b)

respectively. Let \( f(x, t) = -\delta q(x)u_1(x, t) \). From 2.32, we know that

\[
v_{tt}(1, t) = \frac{1}{2} \left[ -\frac{1}{4} \int_0^{1+t} \delta q(y) dy - \delta q \left( \frac{1}{2} \right) \right] \int_0^{1+t} \delta q(y) dy - \delta q \left( \frac{1}{2} \right) \int_0^{1+t} \delta q(y) dy
- \int_{y=\frac{1-t}{2}}^1 q(y)v_t(y, y-1+t) dy - \int_{y=\frac{1-t}{2}}^1 \delta q(y)u_{1t}(y, y-1+t) dy
- \int_{y=\frac{1+t}{2}}^1 q(y)v_t(y, 1+t-y) dy - \int_{y=\frac{1+t}{2}}^1 \delta q(y)u_{1t}(y, 1+t-y) dy] \tag{2.49}
\]

First we claim that

\[
\|\delta q \left( \frac{1-t}{2} \right) \int_0^{1+t} \delta q(y) dy\|_{L^2(1,1)} \leq \sqrt{2} \|\delta q\|_{L^2}^2. \tag{2.50}
\]

An application of the Schwartz inequality results in

\[
\int_{t=-1}^1 \left( \delta q \left( \frac{1-t}{2} \right) \right) \int_0^{1+t} \delta q(y) dy^2 dt
\leq \|\delta q\|_{L^2}^2 \int_{-1}^1 \delta q^2 \left( \frac{1-t}{2} \right) dt
\leq 2 \|\delta q\|_{L^2(0,1)}^2.
\]

This proves the claim. Similarly, we get

\[
\|\delta q \left( \frac{1+t}{2} \right) \int_0^{1+t} \delta q(y) dy\|_{L^2(0,1)} \leq \sqrt{2} \|\delta q\|_{L^2}^2. \tag{2.51}
\]

Next, we show that

\[
\left\| \int_{y=\frac{1-t}{2}}^1 q(y)v_t(y, y-1+t) dy \right\|_{L^2} \leq C \|\delta q\|_{L^2}^2. \tag{2.52}
\]
In the proof of theorem 2.3, we showed that

\[
\int_{-\infty}^{x} (v_x^2 + v_t^2)(x,t)\,dt \leq \hat{C}_4 \|\delta q\|_{L^2}^4
\]

where \(\hat{C}_4\) depends exponential/polynomial in \(\|q\|_{L^2}\). So,

\[
\int_{x=0}^{1} \int_{t=-x}^{x} (v_x^2(x,t) + v_t^2(x,t))\,dt\,dx \leq \hat{C}_4 \|\delta q\|_{L^2(0,1)}^4.
\]

By integrating along the lines parallel to \(x-t = 0\) we can rewrite the above inequality as

\[
\int_{t=-1}^{1} \int_{y=-\frac{1-t}{2}}^{\frac{1-t}{2}} (v_x^2 + v_t^2)(y, y - 1 + t)\,dy\,dt \leq \hat{C}_4 \|\delta q\|_{L^2(0,1)}^4.
\]

(2.53a)

Also, by integrating along the lines parallel to \(x + t = 0\), we can rewrite the above inequality as

\[
\int_{t=-1}^{1} \int_{y=\frac{1-t}{2}}^{-\frac{1-t}{2}} (v_x^2 + v_t^2)(y, 1 + t - y)\,dy\,dt \leq \hat{C}_4 \|\delta q\|_{L^2(0,1)}^4.
\]

(2.53b)

In particular,

\[
\int_{t=-1}^{1} (\int_{y=\frac{1-t}{2}}^{-\frac{1-t}{2}} v_t^2(y, y - 1 + t)\,dy)\,dt \leq \hat{C}_4 \|\delta q\|_{L^2(0,1)}^4.
\]

(2.54)

Using the Schwartz inequality we get:

\[
\int_{t=-1}^{1} (\int_{y=\frac{1-t}{2}}^{-\frac{1-t}{2}} q(y)v_t(y, y - 1 + t)\,dy)^2\,dt
\]

\[
\leq \int_{t=-1}^{1} (\int_{y=\frac{1-t}{2}}^{-\frac{1-t}{2}} q^2(y)\,dy)(\int_{y=\frac{1-t}{2}}^{-\frac{1-t}{2}} v_t^2(y, y - 1 + t)\,dy)\,dt
\]

\[
\leq \hat{C}_4 \|q\|_{L^2}^2 \|\delta q\|_{L^2}^4.
\]

Choosing \(C = \sqrt{\hat{C}_4 \|q\|_{L^2}^2}\), we get the desired result. The energy estimate for (2.47) provides the following estimate:

\[
\int_{x=0}^{1} \int_{t=-x}^{x} (u_1^2 + u_2^2)(x,t)\,dt\,dx \leq C_1 \|\delta q\|_{L^2(0,1)}^2.
\]
Now as in (2.52) we get the following estimate:
\[
\int_{t=-1}^{1} \int_{y=-1}^{1} \left( u_{1}^{2} + u_{2}^{2} \right)(y, y - 1 + t)dydt \leq C_{1} \| \delta q \|_{L^{2}(0,1)}^{2}.
\]
By choosing \( C_{2} = \sqrt{2C_{1}} \), we get the following estimate:
\[
\left\| \int_{y=-1}^{1} \delta q(y)u_{1}(y, y - 1 + t)dy \right\|_{L^{2}(-1,1)} \leq C\| \delta q \|_{L^{2}(0,1)}^{2} \quad (2.55)
\]
where \( C_{2} = \sqrt{2C_{1}} \). Using the Schwartz inequality, (2.53b) and taking \( C_{3} = \sqrt{C_{3} \| q \|_{L^{2}(0,1)}} \) we obtain the following estimate:
\[
\left\| \int_{y=-1}^{1} q(y)v_{t}(y, 1 + t - y)dy \right\|_{L^{2}(-1,1)} \leq C_{3}\| \delta q \|_{L^{2}(0,1)}^{2} \quad (2.56)
\]
Using similar arguments as in (2.55) we obtain the following estimate:
\[
\left\| \int_{y=-1}^{1} \delta q(y)v_{t}(y, 1 + t - y)dy \right\|_{L^{2}(-1,1)} \leq C_{4}\| \delta q \|_{L^{2}(0,1)}^{2} \quad (2.57)
\]
Applying the Schwartz inequality on (2.49), and using estimates (2.50), (2.51), (2.55), (2.56) and (2.57) we get
\[
\| v_{xt}(1, t) \|_{L^{2}(-1,1)} \leq C_{5}\| \delta q \|_{L^{2}(0,1)}^{2} \quad (2.58)
\]
where \( C_{5} \) is exponential/polynomial in \( \| q \|_{L^{2}} \) and \( \| \delta q \|_{L^{2}} \).

From the proof of theorem 2.3,
\[
\| v_{t}(1, t) \|_{L^{2}(-1,1)} \leq C\| \delta q \|_{L^{2}(0,1)}^{2} \quad (2.59)
\]
Now estimates (2.58) and (2.59) together establish the final estimate
\[
\| v_{xt}(1, t) + H v_{t}(1, t) \|_{L^{2}(-1,1)} \leq \sqrt{2(C_{5}^{2} + C^{2})}\| \delta q \|_{L^{2}(0,1)}^{2}.
\]
Now choosing \( C = \sqrt{2(C_{5}^{2} + C^{2})} \) we get the desired result:
\[
\| DF_{4}(q) \delta q - (F_{4}(q + \delta q) - F_{4}(q)) \|_{L^{2}(-1,1)} \leq C\| \delta q \|_{L^{2}(0,1)}^{2}.
\]
\[\square\]

Next, we extend \( DF_{4}(q) \) as a bounded linear map from \( O \) to \( L^{2}(-1,1) \).
Theorem 2.11. For $q \in O \cap C^\infty[0,1]$, $DF_4(q)$ extends to a bounded linear map from $O$ to $L^2(-1,1)$.

Proof. Recall that $DF_4(q)\delta q(t) = \delta u_x(t,1,t) + H\delta u(t,1,t)$, where $\delta u(x,t)$ satisfies (2.45). From the proof of theorem 2.4,

$$\int_{t=-1}^{1} \delta u_t^2(1,t)dt \leq 2C_1\|\delta q\|^2_{L^2(0,1)}$$  \hfill (2.60)

where $C_1$ is polynomial in $\|\delta q\|_{L^2}$ and $\|q\|_{L^2}$. Replacing $(q_1 - q_2)$ by $\delta q$ in 2.28, we can obtain the following estimate:

$$\|\delta u_{x t}(1,t)\|_{L^2(-1,1)} \leq C_2\|\delta q\|_{L^2(0,1)}$$  \hfill (2.61)

where $C_2$ is exponential/polynomial in $\|q\|_{L^2}$ and $\|\delta q\|_{L^2}$. Using 2.60 and 2.61, we get

$$\|DF_4(q)\delta q\|^2_{L^2(-1,1)} \leq 2(4C_1^2 + C_2^2)\|\delta q\|^2_{L^2}.$$

Hence, $DF_4(q)$ extends to a bounded linear map in $L^2(0,1)$. \hfill $\Box$

We extend the results of theorem 2.11 to the case of $q \in O$ by showing that the map $q \mapsto DF_4(q)$ is continuous. So we prove the following theorem.

Theorem 2.12. The Frechet derivative $DF_4(q)$ extends to a locally Lipschitz map

$$O \times O \longrightarrow L^2(-1,1).$$

Proof. Let $q$ and $\tilde{q} \in O \cap C^\infty[0,1]$ and let $u$ and $\tilde{u}$ be the corresponding solutions of 2.1. We take $\delta q \in O \cap C^\infty[0,1]$ and let $\delta u$ and $\delta \tilde{u}$ be the corresponding solutions
of 2.45. Let \( \delta v = \delta u - \delta \tilde{u} \). Then \( \delta v \) solves 2.19. As in the previous theorem,

\[
\delta v_{xt}(1, t) = \frac{1}{2} \frac{1}{2} \delta \left( \frac{1 - t}{2} \right) \int_0^{\frac{1 - t}{2}} (q - \tilde{q}) dy + \frac{1}{2} (q - \tilde{q}) \left( \frac{1 - t}{2} \right) \int_0^{\frac{1 - t}{2}} \delta q(y) dy
\]

\[
- \frac{1}{2} \delta q (\frac{1 + t}{2}) \int_0^{\frac{1 + t}{2}} (q - \tilde{q}) dy - \frac{1}{2} (q - \tilde{q}) \left( \frac{1 + t}{2} \right) \int_0^{\frac{1 + t}{2}} \delta q(y) dy
\]

\[
+ \int_{\frac{1 - t}{2}}^{1} q(y) \delta v_t(y, y - 1 + t) dy \int_{y = \frac{1 - t}{2}}^{1} \delta q(y) (u - \tilde{u})_t(y, y - 1 + t) dy
\]

\[
+ \int_{\frac{1 - t}{2}}^{1} \delta u_t(y, y - 1 + t) (q - \tilde{q})(y) dy
\]

\[
+ \int_{y = \frac{1 - t}{2}}^{1} q(y) \delta v_t(y, 1 + t - y) dy \int_{y = \frac{1 - t}{2}}^{1} \delta q(y) (u - \tilde{u})_t(y, 1 + t - y) dy
\]

\[
+ \int_{y = \frac{1 - t}{2}}^{1} \delta u_t(y, 1 + t - y) (q - \tilde{q})(y) dy.
\]

(2.62)

From the proof of theorem 2.5, we get

\[
\int_{x = 0}^{1} \int_{t = -x}^{x} (\delta v^2_t(1, t) + \delta v^2_t(1, t)) dt dx \leq C_1 ||\delta q||^2_{L^2} ||q - \tilde{q}||^2_{L^2}.
\]

(2.63)

From the proof of lemma 2.3 we get

\[
\int_{x = 0}^{1} \int_{t = -x}^{x} ((u - \tilde{u})^2_t(1, t) + (u - \tilde{u})^2_t(1, t)) dt dx \leq C_2 ||q - \tilde{q}||^2_{L^2}.
\]

(2.64)

Using argument as in theorem 2.9, we obtain the following estimate:

\[
||\delta v_{xt}(1, t)||_{L^2(-1, 1)} \leq C_3 ||q - \tilde{q}||_{L^2} ||\delta q||_{L^2}.
\]

(2.65)

Combining two estimates (2.63) and (2.65),

\[
||\delta v_{xt}(1, t) + H \delta v_t(1, t)||_{L^2(-1, 1)} \leq C ||q - \tilde{q}||_{L^2(0, 1)} ||\delta q||_{L^2(0, 1)}.
\]

This estimate proves the desired result

\[
||DF_4(q) - DF_4(\tilde{q})||_{L^2(-1, 1)} \leq \tilde{C} ||q - \tilde{q}||_{L^2(0, 1)}
\]

So \( DF_4(q) \) is a locally Lipschitz map on \( O \) with values in \( L(L^2(0, 1), L^2(-1, 1)) \).

Corollary 2.4. The map \( F_4(q) \) given by (2.25) is continuously differentiable in \( O \).
CHAPTER 3

INVERSE SPECTRAL PROBLEMS

3.1 Asymptotics of the Eigenvalues

In this section we study the following inverse spectral problems.

Let \( \{\lambda_j\}_{j=1}^{\infty} \) be the complete eigenvalue sequence of the regular Sturm-Liouville problem (1.1a - b) with \( h = \infty \) and \( H \in \mathbb{R} \). Let \( \{\mu_j\}_{j=0}^{\infty} \) be the complete eigenvalue sequence of the regular Sturm-Liouville problem (1.1a - b) with \( h, H \in \mathbb{R} \) and let \( \{\nu_j\}_{j=1}^{\infty} \) denote the complete eigenvalue sequence of the regular Sturm-Liouville problem with \( h = H = \infty \).

(1) We assume that the potential function \( q \in L^2(0,1) \) and that \( q \) is antisymmetric about the mid-point \( x = \frac{1}{2} \). Now the question is, can we reconstruct \( q \) from the sequence \( \{\lambda_j\}_{j=1}^{\infty} \) or \( \{\mu_j\}_{j=0}^{\infty} \) ?

Since the Dirichlet spectrum is symmetric, \( \nu_j(q) = \nu_j(q^*) \) for \( j = 1, 2, 3, \ldots \). Here \( q^*(x) = q(1-x) \). Therefore, when \( q \) is odd, two odd functions \( q \) and \( q^* \) have the same Dirichlet spectrum, so we cannot reconstruct the odd \( q \) uniquely from this set of eigenvalues.

(2) We assume that the potential function \( q \in L^2(0,1) \) and also that \( q \) is symmetric about the mid-point \( x = \frac{1}{2} \). We investigate the question of reconstructing \( q \) from its eigenvalue sequence \( \{\lambda_j\}_{j=1}^{\infty}, \{\mu_j\}_{j=0}^{\infty} \) or \( \{\nu_j\}_{j=1}^{\infty} \).

Next, we study the following problem:

(3) Given, either a complete eigenvalue sequence \( \{\lambda_j\}_{j=1}^{\infty}, \{\mu_j\}_{j=0}^{\infty} \) or \( \{\nu_j\}_{j=1}^{\infty} \), on which subsets of the unit interval \((0,1)\) does one need to specify \( q(x) \), in order to determine \( q \) uniquely on \((0,1)\) ?

From the work of Gel'fand and Levitan [5], it is known that the asymptotic forms for the eigenvalue sequences \( \lambda_j, \mu_j \) and \( \nu_j \) are

\[
\lambda_j = (j - \frac{1}{2})^2 \pi^2 + 2H + \int_{s=0}^{1} q(s) ds + \alpha_j
\] (3.0a)
\[ \mu_j = j^2 \pi^2 + 2(h + H) + \int_{s=0}^{1} q(s) \, ds + \beta_j \]  

(3.0b)

\[ \nu_j = j^2 \pi^2 + \int_{0}^{1} q(s) \, ds + \gamma_j \]  

(3.0c)

where the sequences \( \{\alpha_j\}_{j=1}^{\infty} \), \( \{\beta_j\}_{j=0}^{\infty} \) and \( \{\gamma_j\}_{j=1}^{\infty} \) belong to \( l^2 \). The asymptotic behavior of \( \lambda_j \), \( \mu_j \), \( \nu_j \) provide the following completeness property.

**Lemma 3.1.** (Levinson [13], theorem 3).

The sequences \( \{\sin \sqrt{\lambda_j} t\} \), \( \{\cos \sqrt{\mu_j} t\} \) and \( \{\sin \sqrt{\nu_j} t\} \) are complete in \( L^2(0,1) \).

### 3.2 A Solution Method to Inverse Problems

Our solution method is based on the idea of solving a coefficient identification problem for a wave equation instead of the inverse spectral problem. Therefore, we translate our original frequency-domain problem to a time-domain problem using the celebrated work of Gel’fand and Levitan [5]. From their work, we know that for a given \( q \in L^2(0,1) \), there exist two kernels \( M(x,t;q) \) and \( L(x,t;q) \) defined on the characteristic triangle \( 0 \leq t \leq x \leq 1 \) which satisfy the following properties:

Suppose \( y_i(x,\lambda, q) \) satisfy

\[ -y_i'' + q(x)y_i = \lambda y_i \quad \text{for } i = 1,2 \]  

(3.1a)

\[ \begin{cases} 
    y_1(0,\lambda, q) = 1, y_1'(0,\lambda, q) = 0 \\
    y_2(0,\lambda, q) = 1, y_2'(0,\lambda, q) = 0 
\end{cases} \]  

(3.1b)

then

\[ \begin{cases} 
    y_1(x,\lambda, q) = \cos \sqrt{\lambda} x + \int_{0}^{x} M(x,t) \cos \sqrt{\lambda} t \, dt \\
    y_2(x,\lambda, q) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \int_{0}^{x} L(x,t) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \, dt 
\end{cases} \]  

(3.2a)

In case \( \lambda < 0 \), we understand \( \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \) and \( \cos \sqrt{\lambda} x \) are \( \frac{\sinh \sqrt{|\lambda|} x}{\sqrt{|\lambda|}} \) and \( \cosh \sqrt{|\lambda|} x \) respectively.

Let \( h \) be a real number. Then the function

\[ y(x,\lambda) = y_1(x,\lambda) + hy_2(x,\lambda) \]
satisfies the following differential equation:

\[-y'' + qy = \lambda y\]

and the boundary condition

\[y'(0) - hy(0) = 0.\]

For \(h = \infty\), by definition we take \(y(x, \lambda) = y_2(x, \lambda)\). Gel’fand and Levitan showed that there exists a kernel \(K(x, t; q, h)\) such that

\[y(x, \lambda) = \cos \sqrt{\lambda}x + \int_0^x K(x, t) \cos \sqrt{\lambda}t \, dt.\]  \hspace{1cm} (3.2b)

If \(h = 0\), it is clear that \(K = M\). Therefore, in the following discussion we need to use only the kernels \(K\) and \(L\).

One of the remarkable properties of the kernels \(K\) and \(L\) is their independence of \(\lambda\). When \(\lambda = \lambda_j\), \(y(x, \lambda_j)\) given by (3.2b) represents the \(j\)th eigenfunction of the Sturm-Liouville problem given by (1.1a-b). Moreover, \(K\) and \(L\) satisfy the characteristic boundary value problems

\[\begin{aligned}
K_{tt} - K_{xx} + q(x)K &= 0, \quad 0 \leq |t| \leq x \leq 1 \\
K(x, x) &= h + \frac{1}{2} \int_0^x q(y)dy, \quad 0 \leq x \leq 1 \\
K_t(x, 0) &= 0.
\end{aligned}\]  \hspace{1cm} (3.3a)

and

\[\begin{aligned}
L_{tt} - L_{xx} + q(x)L &= 0, \quad 0 \leq |t| \leq x \leq 1 \\
L(x, x) &= \frac{1}{2} \int_0^x q(y)dy, \quad 0 \leq x \leq 1 \\
L(x, 0) &= 0.
\end{aligned}\]  \hspace{1cm} (3.3b)

Note that the boundary conditions on the line \(t = 0\), result in the function \(K\) (L) being even (odd) in \(t\) for each fixed \(x\) respectively. Therefore, we extend the initial-characteristic boundary value problems given by (3.3) to characteristic boundary problems in \(0 \leq |t| \leq x \leq 1\) by introducing the following boundary conditions:

\[\begin{aligned}
K(x, -x) &= h + \frac{1}{2} \int_0^x q(y)dy, \quad 0 \leq x \leq 1 \\
L(x, -x) &= -\frac{1}{2} \int_0^x q(y)dy, \quad 0 \leq x \leq 1.
\end{aligned}\]  \hspace{1cm} (3.3c)
Next, we use spectral data to determine the boundary data \( \{K_x(1,t) + HK(1,t)\} \) or \( \{L_x(1,t) + HL(1,t)\} \) for \( 0 \leq t \leq 1 \). The \( j \)’th eigenfunction \( y_i(x, \lambda_j, q) \) for \( i = 1, 2 \) should satisfy the boundary condition at \( x = 1 \). Hence, differentiating both sides of (3.2) with respect to \( t \) and evaluating the result at \( \lambda = \lambda_j \) and \( x = 1 \), we obtain

\[
\begin{align*}
\int_0^1 (L_x(1,t) + HL(1,t)) \sin(\sqrt{\lambda_j}t) dt &= -(\sqrt{\lambda_j} \cos(\sqrt{\lambda_j}) + L(1,1) \sin(\sqrt{\lambda_j})) \\
\int_0^1 (K_x(1,t) + HK(1,t)) \cos(\sqrt{\mu_j}t) dt &= \sqrt{\mu_j} \sin(\sqrt{\mu_j}) - (H + K(1,1)) \cos(\sqrt{\mu_j}).
\end{align*}
\]

From the asymptotic behaviour of the eigenvalues in 3.0 (a-b) we may obtain \( \int_0^1 q(y) dy \) from either sequence. If \( \mu = \int_0^1 q(y) dy \), then we replace \( \lambda_k \) by \( \lambda_k = \lambda_k - \mu \), and compute the corresponding solution to the inverse problem. Now from 3.0 (a-b), the corresponding solution of the inverse problem has mean zero. So, without loss of generality we assume that \( \int_0^1 q(y) dy = 0 \).

Whenever the potential function has mean 0, we get

\[
\begin{align*}
K(1,1) &= h \quad \text{for } h \in \mathbb{R} \\
L(1,1) &= 0 \quad \text{for } h = \infty.
\end{align*}
\]

Therefore, we have the following integral equation:

\[
\begin{align*}
\int_0^1 (L_x(1,t) + HL(1,t)) \sin(\sqrt{\lambda_j}t) dt &= -\sqrt{\lambda_j} \cos(\sqrt{\lambda_j}) \\
\int_0^1 (K_x(1,t) + HK(1,t)) \cos(\sqrt{\mu_j}t) dt &= \sqrt{\mu_j} \sin(\sqrt{\mu_j}) - (H + h) \cos(\sqrt{\mu_j}).
\end{align*}
\]

(3.4)

Now, using the completeness result in lemma(3.1), we can recover the desired boundary data \( \{L_x(1,t) + HL(1,t)\} \) ( or \( \{K_x(1,t) + HK(1,t)\} \) ) uniquely.

Next, we form the inverse spectral problem as a time domain problem. For a given \( q \in L^2(0,1) \), maps \( F_1(q) \), \( F_2(q) \) and \( F_3(q) \) are given by

\[
\begin{align*}
F_1(q : h_1, h_2, h_3)(t) &= u_x(1,t) + Hu(1,t) \\
F_2(q : h_1, h_2, h_3)(t) &= u_{xt}(1,t) + Hu_t(1,t) \\
F_3(q : h_1, h_2, h_3)(t) &= u_t(1,t)
\end{align*}
\]

(3.5)

where \( u(x,t,q) \) satisfies

\[
u_{tt} - u_{xx} + q(x)u = 0 \quad 0 \leq |t| \leq x \leq 1
\]

(3.6a)
\[ u(x, \pm x) = h_1 + (\pm h_2 + h_3) \int_{y=0}^{x} q(y)dy \quad 0 \leq x \leq 1. \quad (3.6b) \]

Therefore, the solution \( q \) of the inverse spectral problem satisfies one of the equations

\[ F_1(q) = L(t,1) + HL(1,t), \text{for} \; h_1 = 0, \; h_2 = \frac{1}{2}, \; h_3 = 0 \quad (3.7a) \]

\[ F_2(q) = K_x(t,1) + HK(t,1), \text{for} \; h_1 = h, \; h_2 = 0, \; h_3 = \frac{1}{2} \quad (3.7b) \]

\[ F_3(q) = K_x(1,t) + HK(t,1), \text{for} \; h_1 = h, \; h_2 = 0, \; h_3 = \frac{1}{2} \quad (3.7c) \]

or

\[ F_3(q) = L_t(1,t) \text{ for } h_1 = 0, \; h_2 = \frac{1}{2}, \; h_3 = 0. \quad (3.7d) \]

If \( q \) is known then the problem (3.6 a-b) is well-posed. Hence for known \( q \), the boundary data \((L(x,1) + HL(1,t)) \) or \((K_x(1,t) + HK(t,1)) \) will lead to an overposed problem for the characteristic boundary value problem (3.6), and by an iterative method we can reconstruct \( q(x) \).

It is clear that the equation (3.7) is a non-linear equation. From chapter 2 we know that \( F_i \) for \( i = 1, 3 \) is a differentiable map from \( L^2(0,1) \) to \( L^2(-1,1) \) and \( F_2 \) is a differentiable map from \( O \) to \( L^2(-1,1) \). Here, we use a quasi-Newton method to solve the non-linear equations of the form \( F_i(q) = g \), for \( i=1,2,3 \). Our quasi-Newton iterates are defined as follows:

\[ q_0 = 0 \quad (3.8a) \]

\[ q_{n+1} = q_n - DF^{-1}(0)(F_i(q_n) - g), \; i = 1, 2, 3 \quad (3.8b) \]

In order to apply this method we should prove that \( DF_i(0) \), for \( i=1,2,3 \) is invertible.

In the following theorem we discuss uniqueness results for symmetric and anti-symmetric potentials whenever one set of eigenvalues are given. Let \( E \) be the set of \( a.e \) symmetric functions in \( L^2(0,1) \) and \( O \) be the set of \( a.e \) anti-symmetric functions in \( L^2(0,1) \). We will use the Inverse Function Theorem to prove the uniqueness of
the solution of the inverse problem for sufficiently small \( q \). The statement of the theorem is

**Inverse Function Theorem.** Let \( f : U \rightarrow C \) be a differentiable map from an open subset \( U \) of a Banach space \( B \) into a Banach space \( C \). Let \( a \in U \). If \( Df(a) \) is a linear isomorphism between \( B \) and \( C \), then \( f \) is a local isomorphism at \( a \).

**Theorem 3.1.** Let \( F_1, F_2 \) and \( F_3 \) be the maps given by 3.7a-d. Then,

(i) for the case when \( h_1 = 0, h_2 = \frac{1}{2} \) and \( h_3 = 0 \), there is a neighborhood of 0 in \( O \) where the map \( F_1 \) is one-one.

(ii a) for the case when \( h_1 = h, h_2 = 0, \) and \( h_3 = \frac{1}{2} \), there is a neighborhood of 0 in \( O \) where the map \( F_2 \) is one-one, provided that \( h \neq H \).

(ii b) for the case when \( h_1 = 0, h_2 = \frac{1}{2} \) and \( h_3 = 0 \), there is a neighborhood of 0 in \( E \) where the map \( F_2 \) is one-one, provided that \( H \neq 0 \).

(iii) for the case when \( h_1 = h, h_2 = 0 \) and \( h_3 = \frac{1}{2} \), there is a neighborhood of 0 in \( E \) where the map \( F_1 \) is one-one.

and

(iv) for the case when \( h_1 = 0, h_2 = \frac{1}{2} \) and \( h_3 = 0 \), there is a neighborhood of 0 in \( E \) where the map \( F_3 \) is one-one.

**Proof.** From theorem 2.6, the map \( F_1 \) is differentiable. We also know that

\[
DF_1(0)\delta q = \delta v_x(1, t) + H\delta v(1, t),
\]

where \( \delta v(x, t) \) satisfies the following:

\[
\delta v_{tt} - \delta v_{xx} = -h_1 \delta q
\]

\[
\delta v(x, \pm x) = (\pm h_2 + h_3) \int_{y=0}^{x} \delta q(y) dy.
\]
Using D’Alembert’s formula, we can write

\[
\delta v(x, t) = \frac{h_2}{2} \int_{y=x-t}^{x+t} \delta q\left(\frac{y}{2}\right) dy + \frac{h_1}{2} \left( \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} \delta q\left(\frac{y}{2}\right) dy + \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} \delta q(y) dy \right)
\]

\[
+ \frac{h_1}{2} \left(2 \int_{y=0}^{\frac{x-t}{2}} y \delta q(y) dy + \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} (x-t) \delta q(y) dy \right) + 2 \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} \delta q(y) dy. \tag{3.9a}
\]

From (3.9a) we get,

\[
(\delta v_x + H \delta v)(x, t) = \frac{h_2}{2} (\delta q(x + \frac{t}{2}) - \delta q(x - \frac{t}{2}))
\]

\[
+ \frac{h_3}{2} (\delta q(x + \frac{t}{2}) + \delta q(x - \frac{t}{2}))
\]

\[
+ h_1 \left( \frac{1}{2} \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} \delta q(y) dy + \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} \delta q(y) dy \right)
\]

\[
+ H \left( \frac{h_2}{2} \int_{x-t}^{x+t} \delta q\left(\frac{y}{2}\right) dy + \frac{h_3}{2} \left( \int_{y=0}^{\frac{x-t}{2}} \delta q\left(\frac{y}{2}\right) dy + \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} \delta q(y) dy \right) \right)
\]

\[
+ \int_{y=\frac{x-t}{2}}^{\frac{x-t}{2}} \delta q\left(\frac{y}{2}\right) dy + \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} (x-t) \delta q(y) dy + \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} \delta q(y) dy. \tag{3.9b}
\]

Case (i) After some simplifications of (3.9b), we get

\[
DF_1(0) \delta q(t) = \frac{1}{2} \delta q\left(\frac{1+t}{2}\right), \text{ for } \delta q \in O. \tag{3.10a}
\]

From this formula, it’s clear that \(DF_1(0)\) is one-one over odd functions. As a consequence, it has an inverse given by

\[
DF_1^{-1}(0) \delta g(x) = 2 \delta g(2x - 1). \tag{3.11}
\]

From (3.11), \(DF_1(0)\) has a bounded inverse. Therefore, \(DF_1(0)\) is a linear isomorphism over \(O\). Our intention here is to apply the inverse function theorem to obtain
the desired result. We take $B = O$ and $C = L^2(-1,1)$. From theorem 2.12, we know that $F_1$ is differentiable from $B$ to $C$. In particular, $F_1$ is differentiable from an open subset $U \subset B$ to $C$. The differentiability of $F_1$ along with 3.10b. gives us that $DF_1(0)$ is a linear isomorphism between $B$ and $C$. Therefore, by the inverse function theorem we conclude that the map $F_1$ is one-one over a neighborhood of 0 and this proves the case (i) of the theorem.

Case (ii a) From corollary 2.4 the map $F_2$ is differentiable. We also know that 

$$DF_2(0)\delta q(t) = \delta v_{xt}(1,t) + H\delta v_t(1,t),$$

where $\delta v(x,t)$ satisfies

$$\delta v_{tt} - \delta v_{xx} = -h\delta q$$

$$\delta v(x, \pm x) = \frac{1}{2} \int_0^x \delta q(y)dy.$$

From 3.9, we can write

$$\delta v(x,t) = \frac{1}{2} \left[ \int_0^{x/2} \delta q(y)dy + \int_0^{x/2} \delta q(y)dy \right]$$

$$- \frac{h}{2} \left[ \int_{y=0}^{\frac{x-t}{2}} y \delta q(y)dy + \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} (x-t) \delta q(y)dy + \int_{y=\frac{x-t}{2}}^{\frac{x+t}{2}} 2(x-y) \delta q(y)dy \right]$$

Using this we obtain

$$DF_2(0)\delta q(t) = \frac{h-H}{2} \delta q \left( \frac{1+t}{2} \right),$$

for $\delta q \in O$. (3.12a)

From this formula, it is clear that $DF_2(0)$ is one-one over odd functions, provided that $h \neq H$. As a consequence, it has an inverse given by,

$$DF_2^{-1}(0)\delta q(x) = \frac{2}{h-H} \delta q(2x-1)$$

(3.12b)

So, $DF_2^{-1}(0)$ is bounded. Following similar arguments as given in (i), and applying the inverse function theorem, we get the desired result.

Case (ii b) From corollary 2.4 the map $F_2$ is differentiable. We also know that 

$$DF_2(0)\delta q(t) = (\delta v_{xt} + H\delta v_t)(1,t),$$

where $\delta v(x,t)$ satisfies

$$\delta v_{tt} - \delta v_{xx} = 0$$
\[ \delta v(x, \pm x) = \pm \frac{1}{2} \int_{y=0}^{x} \delta q(y) \, dy. \]

Using (3.9a) we obtain,

\[ DF_2(0)\delta q(t) = \frac{H}{2} \delta q\left(\frac{1+t}{2}\right), \text{ for } \delta q \in E. \tag{3.12c} \]

From (3.12c), it is clear that \( DF_2(0) \) is one-one over even functions, provided that \( H \neq 0 \). As a consequence, it has an inverse given by,

\[ DF_2^{-1}(0)\delta g(x) = \frac{2}{H} \delta g(2x - 1). \tag{3.12d} \]

So, \( DF_2(0) \) has a bounded inverse over the subspace of even functions. Following similar arguments as given in (i) and applying the inverse function theorem, we get the desired result.

Case (iii) After some simplifications of (3.9b), we get

\[ DF_1(0)\delta q(t) = \frac{1}{2} \delta q\left(\frac{1+t}{2}\right) + H h\left( \int_{y=\frac{1+t}{2}}^{1} \left( 1 + t - 2y \right) \delta q(y) \, dy \right) \text{ for } \delta q \in E. \tag{3.13} \]

This is a Volterra integral equation of the second kind. So \( DF_1^{-1}(0) \) is bounded. Following similar arguments as given in case (i) and applying the inverse function theorem, we get the desired result.

Case (iv) From the proof of the corollary 2.4, it is clear that the map \( F_3 \) is differentiable. We also know that \( DF_3(0)\delta q(t) = \delta v_t(1,t) \), where \( \delta v(1,t) \) satisfies

\[ \delta v_{tt} - \delta v_{xx} = 0 \]

\[ \delta v(x, \pm x) = \frac{1}{2} \int_{0}^{x} \delta q(y) \, dy. \]

After some simplifications of (3.9), we get

\[ DF_3(0)\delta q(t) = \frac{1}{2} \delta q\left(\frac{1+t}{2}\right). \]
From this formula, it is clear that $DF_3(0)$ is one-one over even functions. As a consequence, it has an inverse given by,

$$DF_3(0)^{-1} \delta g(x) = 2 \delta g(2x - 1).$$

So, $DF_3^{-1}(0)$ is bounded. Following similar arguments as given in (i), and applying the inverse function theorem, we get the desired result. □

### 3.3 A Uniqueness Result for Odd and Even Potentials

Next, we prove a uniqueness theorem to the inverse problems (1) and (2) which were stated at the beginning of this section. This uniqueness result is a direct application of theorem 3.1.

Let us recall that:

(i) $\{\lambda_j\}_{j=1}^\infty$ is the complete eigenvalue sequence of the Sturm-Liouville problem 1.1a-b with $h = \infty$ and $H \in \mathbb{R}$,

(ii) $\{\mu_j\}_{j=0}^\infty$ is the complete eigenvalue sequence of the Sturm-Liouville problem 1.1a-b with $h, H \in \mathbb{R}$,

and

(iii) $\{\nu_j\}_{j=1}^\infty$ is the complete eigenvalue sequence of the Sturm-Liouville problem with $h = H = \infty$.

For the simplicity of the statements of the theorems, we define the following notations:

Let $\{\beta_j\}$ be any of the following eigenvalue sequences:

\[
\left\{ \begin{array}{ll}
(i) \text{ the spectrum } & \{\lambda_j\}_{j=1}^\infty \\
(ii) \text{ the spectrum } & \{\mu_j\}_{j=0}^\infty \text{ for } h \neq H
\end{array} \right. \quad (3.14a)
\]

Let $\{\gamma_j\}$ be any of the following eigenvalue sequences:

\[
\left\{ \begin{array}{ll}
(i) \text{the spectrum } & \{\lambda_j\}_{j=1}^\infty \text{ for } H \neq 0 \\
(ii) \text{the spectrum } & \{\mu_j\}_{j=0}^\infty \\
(iii) \text{the spectrum } & \{\nu_j\}_{j=1}^\infty
\end{array} \right. \quad (3.14b)
\]
Theorem 3.2. Let

\[ Ly = -y'' + qy, \]

where \( q \) is square integrable on \([0,1]\) and \( q(x) = -q(1-x) \) almost everywhere. Let \( \{\beta_j(q)\} \) be the spectrum of \( L \) defined by 3.14a. Consider now a second operator \( \tilde{L} \) defined by

\[ \tilde{L} = -y'' + \tilde{q}y, \]

where \( \tilde{q} \) satisfies the same conditions as \( q \). Let \( \{\tilde{\beta}_j(\tilde{q})\} \) be the spectrum of \( \tilde{L} \) corresponding to the same boundary conditions as \( \{\beta_j(q)\} \). If

\[ \beta_j(q) = \tilde{\beta}_j(\tilde{q}) \]

then

\[ q = \tilde{q} \]

provided that \( q \) is sufficiently small.

Proof. From 3.14a, it should be clear that the spectrum \( \{\beta_j\} \) can take two different forms. Therefore, we divide the proof into two cases.

Case (i) Here we consider the case when \( \beta_j(q) = \tilde{\beta}_j(\tilde{q}) \), for every \( j \) takes the following form:

\[ \lambda_j(q) = \tilde{\lambda}_j(\tilde{q}), \text{ for } j = 1, 2, ... \]

Therefore by equation 3.4, we get

\[ L_x(1, t; q) + HL(1, t; q) = L_x(1, t; \tilde{q}) + HL(1, t; \tilde{q}). \]

This together with equation 3.7a results in the following:

\[ F_1(q) = F_1(\tilde{q}). \]
At this point we see that the question of uniqueness of the inverse spectral problem is nothing but the question of injectivity of the map $F_1$. By theorem 3.1(i), we know that the map $F_1$ is one-one over a neighborhood of zero. Therefore,

$$q = \tilde{q}$$

provided that $q$ is sufficiently small.

Case (ii) Here we consider the case when $\beta_j(q) = \tilde{\beta}_j(\tilde{q})$, for every $j$, takes the following form:

$$\mu_j(q) = \tilde{\mu}_j(\tilde{q}), \quad j = 0, 1, 2, \ldots \text{for } h \neq H.$$ 

Therefore by equation 3.4, we get

$$K_{xI}(1, t; q) + HK_{I}(1, t; q) = K_{xI}(1, t; \tilde{q}) + \tilde{HK}_{I}(1, t; \tilde{q}).$$

This together with (3.7c) results in the following:

$$F_2(q) = F_2(\tilde{q}).$$

As in the case (i), the question of the uniqueness of the inverse spectral problem is nothing but the question of the injectivity of the map $F_2$. Therefore by theorem 3.1(iiia), $F_2$ is one-one over a neighborhood of zero. Hence,

$$q = \tilde{q},$$

provided that $q$ is sufficiently small. □

Next, we prove a uniqueness theorem to the inverse spectral problem (2), described at the beginning of this section.

**Theorem 3.3.** Let

$$L y = -y'' + qy.$$
where \( q \) is square integrable on \([0,1]\) and \( q(x) = q(1-x) \) almost everywhere. Let \( \{\gamma_j(q)\} \) be the spectrum of \( L \) defined by 3.14b. Consider now a second operator \( \tilde{L} \) defined by

\[
\tilde{L} = -y'' + \tilde{q}y,
\]

where \( \tilde{q} \) satisfies the same conditions as \( q \). Let \( \{\tilde{\gamma}_j(q)\} \) be the spectrum of \( \tilde{L} \) corresponding to the same boundary conditions as \( \{\gamma_j(q)\} \). If

\[
\gamma_j(q) = \tilde{\gamma}_j(\tilde{q}) \text{ for every } j
\]

then

\[
q = \tilde{q}
\]

provided that \( q \) is sufficiently small.

\textbf{Proof.} From 3.14b, the spectrum \( \{\gamma_j(q)\} \) can take three different forms. Therefore, we divide the proof into three different cases.

Case (i) Here we consider the case \( \gamma_j(q) = \tilde{\gamma}_j(\tilde{q}) \) for every \( j \) takes the following form:

\[
\lambda_j(q) = \tilde{\lambda}_j(\tilde{q}) \text{ for } j = 1, 2, \ldots \text{ whenever } H \neq 0
\]

Therefore by equation 3.4, we get

\[
L_{rt}(1,t;q) + HL_t(1,t;q) = L_{rt}(1,t;\tilde{q}) + HL_t(1,t;\tilde{q}).
\]

This together with equation 3.7c results in the following:

\[
F_2(q) = F_2(\tilde{q}).
\]

By theorem 3.1(ii b), we know that the map \( F_2 \) is one-one over a neighborhood of zero. Therefore,

\[
q = \tilde{q}
\]
provided that \( q \) is sufficiently small.

Case (ii) In this case \( \gamma_j(q) = \hat{\gamma}_j(q) \) for every \( j \) takes the following form:

\[
\mu_j(q) = \hat{\mu}_j(q) \quad \text{for} \quad j = 0,1,2,\ldots
\]

Therefore by equation 3.4, we get

\[
K_x(1,t; q) + HK(1,t; q) = K_x(1,t; \hat{q}) + HK(1,t; \hat{q}).
\]

This together with equation 3.7b results in the following:

\[
F_1(q) = F_1(\hat{q}).
\]

As in Case (i), this together with theorem 3.1(iii) proves the result.

Case (iii) In this case \( \gamma_j(q) = \hat{\gamma}_j(q) \) for every \( j \) takes the following form:

\[
\nu_j(q) = \hat{\nu}_j(q) \quad \text{for} \quad j = 1,2,\ldots
\]

Hence, by equation 3.4, we get

\[
K_t(1,t; q) = K_t(1,t; \hat{q})
\]

This together with equation 3.7d gives,

\[
F_3(q) = F_3(\hat{q})
\]

Following similar steps as in case (i), we prove the desired result. \( \square \)

3.4 A Uniqueness Result for Partially Known Potentials

Here, we study the following question: Given a spectrum, on which subsets of the unit interval \((0,1)\) does one need to specify \( q(x) \), in order to determine \( q \) uniquely? In the following, we answer this question provided that \( q \) is sufficiently small.
Theorem 3.4. Let $A$ be a measurable subset of $(0,1)$, with the following property: 
$x \in (0,1)$ implies either 

$x \in A$ and $1 - x \notin A$

or 

$1 - x \notin A$ and $x \notin A$.

For sufficiently small functions $q \in L^2(0,1)$, there is a neighborhood $N(0)$ such that for $\hat{q} \in N(0)$, if 

$q(x) = \hat{q}(x)$ a.e for $x \in A$

and 

$F_i(q) = F_i(\hat{q})$, where $F_i(q)$ is given by (3.7) for $i = 1, 2, 3$,

then 

$q(x) = \hat{q}(x)$ for $x \in (0,1)$.

Proof. Case (i) In this case we prove the theorem for $i = 1$.

So, we consider the map $F_1$ which takes the form 

$F_1(q)(t) = L_x(1,t;q) + H L(1,t;q)$.

For $x \in ([0,1] - A)$ we can write 

$q(x) - \hat{q}(x) = 2\left(\frac{q(x) - \hat{q}(x)}{2}\right) - 2\left(\frac{q(1-x) - \hat{q}(1-x)}{2}\right)$. (3.15)

So, we can write the difference $(q - \hat{q})$ in terms of the difference of odd parts of $q$ and $\hat{q}$ separately. At this point, we notice that as in theorem 3.1(i), $DF_1(0)q$ is the anti-symmetric part of the function $q$. Therefore, we write 

$q(x) - \hat{q}(x) = 2(DF_1(0)(q - \hat{q}))(x)$ for $x \in [0,1] - A$. (3.16)

From theorem 2.11, we get 

$\|DF_1(q) - DF_1(0)\| \leq C(\|q\|)\|q\|$. (3.17a)
We recall from theorem 2.11, that the constant $C$ is exponential/polynomial in $\|q\|$. Therefore, we can pick $q$ so that $4(e + C(\|q\|)\|q\|)^2 < 1$ for some $e > 0$. Since $F_i$ is differentiable at $q$, we can write

$$F_i(\hat{q}) = F_i(q) + DF_i(q)(\hat{q} - q) + \omega(\|\hat{q} - q\|).$$ \hspace{1cm} (3.17b)

We rewrite this as

$$\frac{\|F_i(\hat{q}) - F_i(q) - DF_i(q)(\hat{q} - q)\|}{\|\hat{q} - q\|} = \frac{\|\omega(\|\hat{q} - q\|)\|}{\|\hat{q} - q\|}. \hspace{1cm} (3.17c)$$

For the chosen $e > 0$, there exists a $\delta > 0$, so that $\|\omega(\|\hat{q} - q\|)\| \leq \epsilon$ whenever $\|\hat{q} - q\| \leq \delta$.

Let $F_i(\hat{q}) = F_i(q)$ for $\hat{q}$ such that $\|\hat{q} - q\| < \delta$. Using (3.17c)

$$\frac{\|DF_i(q)(\hat{q} - q)\|}{\|\hat{q} - q\|} \leq \epsilon \hspace{1cm} (3.17d)$$

and from (3.17a),

$$\|DF_i(0)(\hat{q} - q)\| \leq (e + C\|q\|)\|\hat{q} - q\|. \hspace{1cm} (3.17e)$$

As a result of (3.16), we get

$$\int_{[0,1] - A} |\hat{q}(x) - q(x)|^2 dx \leq 4\|DF_i(0)(\hat{q} - q)\|^2.$$ 

Finally (3.17e), gives the following :

$$(1 - 4(e + C\|q\|)^2)(\int_{[0,1] - A} |\hat{q}(x) - q(x)|^2 dx \leq 0. \hspace{1cm} (3.18)$$

Hence,

$$\hat{q}(x) = q(x) \text{ for } x \in ([0,1] - A).$$

Therefore, the map $F_i$ is one-one in $N_A(0)$.

Case (ii) In this case we prove the theorem for $i=2$. 

Here, we consider the case

\[ \hat{q}(x) = q(x) \text{ a.e for } x \in A \]

and

\[ F_2(\hat{q}) = F_2(q), \text{ where } F_2(q)(t) = K_{r_1}(1,t) + HK_1(1,t). \]

As in the first case, for \( x \in [0,1] - A \) we can write

\[ \hat{q}(x) - q(x) = \frac{1}{2}(\hat{q}(x) - q(x)) - \frac{1}{2}(\hat{q}(1-x) - q(1-x)) \]

As in theorem 3.1(ii), \( DF_2(0)q \) is a constant multiple of the odd part of the function. Therefore, we write

\[ \hat{q}(x) - q(x) = \frac{2}{h - H}(DF_2(0)(\hat{q} - q)(x)), \ h \neq H. \]

Now, using arguments similar to case 1, we get the desired result. In the following we extend this result for the case when \( h = H \).

Consider the map \( F_1 \) which takes the form

\[ F_1(q)(t) = K_r(1,t) + HK(1,t). \]

For \( x \in [0,1] - A \) we can write

\[ \hat{q}(x) - q(x) = \frac{1}{2}(\hat{q}(x) + \hat{q}(1-x)) - \frac{1}{2}(q(x) + q(1-x)). \]

As in theorem 3.1 (2), \( DF_1(0)(\hat{q}) \) approximates the even part of the function. Since \( \hat{q} - q \) is the difference of even parts of \( \hat{q} \) and \( q \) separately, we can approximate \( (\hat{q} - q)(x) \) by \( DF_1(0)\hat{q}(x) \) for \( x \in [0,1] - A \). Now, using arguments similar to case 1, we get the desired result.

Case (iii) In this case we prove the theorem for \( i=3 \).
In this case we consider the map $F_3$ which takes the form

$$F_3(q)(t) = L_t(1,t).$$

For $x \in [0,1] - \lambda$ we can write

$$\hat{q}(x) - q(x) = 2\hat{g}(x) + \hat{g}(1-x) - 2g(x) + g(1-x).$$

As in theorem 3.1 (3), $DF_3(0)q$ approximates the even part of the function. Now, using arguments similar to case 3, we get the desired result. □

Next, we prove a uniqueness theorem to the inverse problem (1), stated at the beginning of this section. This uniqueness result is a direct application of theorem 3.4. For the simplicity of the statement of the next theorem, we define the following notation:

Let $\{\delta_j\}$ be any of the following eigenvalue sequences:

$$\begin{align*}
&\text{(i) the spectrum } \{\lambda_j\}_{j=1}^{\infty} \\
&\text{(ii) the spectrum } \{\mu_j\}_{j=0}^{\infty} \\
&\text{(iii) the spectrum } \{\nu_j\}_{j=1}^{\infty}
\end{align*}$$

(3.19)

**Theorem 3.5.** Let

$$Ly = -y'' + qy,$$

where $q$ is square integrable on $[0,1]$. Let $\{\delta_j\}$ be the spectrum of $L$ defined by (3.19).

Consider a second operator

$$\hat{L}y = -y'' + \hat{q}y$$

where $\hat{q} \in L^2[0,1]$ and

$$\hat{q}(x) = q(x) \text{ for } x \in \lambda.$$

Here the set $\lambda$ is defined as in theorem 3.3. Let $\{\delta_j(\hat{q})\}$ be the spectrum of $\hat{L}$ corresponding to the boundary conditions as in $\{\delta_j(q)\}$. If

$$\delta_j(q) = \hat{\delta}_j(\hat{q}) \text{ for every } j,$$

$$\text{then } \delta_j(q) = \delta_j(q'').$$
then

\[ q(x) = \tilde{q}(x) \text{ a.e on } (0, 1). \]

provided that \( q \) is sufficiently small.

**Proof.** From 3.19, we know that the spectrum \( \{ \delta_j(q) \} \) can take three different forms. Therefore, we divide the proof into three different cases.

**Case(i)** Here we consider the case when \( \delta_j(q) = \tilde{\delta}_j(\tilde{q}) \), for every \( j \), takes the following form:

\[ \lambda_j(q) = \tilde{\lambda}_j(\tilde{q}), \text{ for } j = 1, 2, ... \]

Therefore by equation 3.4, we get

\[ L_x(1, t; q) + HL(1, t; q) = L_x(1, t; \tilde{q}) + HL(1, t; \tilde{q}). \]

This together with equation 3.7a results in the following:

\[ F_1(q) = F_1(\tilde{q}). \]

So, we have reduced our original problem to the problem of injectivity of the map \( F_1 \) given by (3.7a). By theorem (3.4), we know that the map \( F_1 \) is one-one over a neighborhood \( N_{\delta}(0) \). Hence,

\[ q = \tilde{q} \]

provided that \( \tilde{q} \in N_{\delta}(0) \). This proves the uniqueness.

In the other two cases, following similar arguments as above and applying theorem 3.4 we can prove the desired result. \( \Box \)

Note that \( A \) can be the set \( \cup_{i=0}^{k}(\frac{2i}{2k}, \frac{2i+1}{2k}) \). When \( A = [0, \frac{1}{2}] \), this result was proved by Hochstadt and Lieberman [9]. When \( A = [0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}] \), this question was asked by Pösche and Trubowitz in their book, [22]. One should note that the set \( A \) can not be an arbitrary measurable set of measure \( \frac{1}{2} \). In the following we argue
this point. Let \( A \) be the set \( \left\{ \frac{1}{4}, \frac{3}{4} \right\} \) and let \( \{ \nu_j \} \) be the Dirichlet spectrum. Given this data can one reconstruct \( q \) uniquely on \((0,1)\)? The answer to this question is no. Consider a function \( q \) such that

\[
\begin{cases}
q(x) = q(1-x) & \text{for } x \in A \\
q(x) \neq q(1-x) & \text{for } x \in [0,1] - A.
\end{cases}
\]

Let, \( q^*(x) = q(1-x) \). Using the symmetry of the Dirichlet spectrum, we know that \( \nu_j(q) = \nu_j(q^*) \) for \( j = 1, 2, \ldots \). Therefore, the two different functions \( q \) and \( q^* \) which agree on the set \( A \) have the same Dirichlet spectrum. So, one cannot reconstruct \( q \) uniquely in \((0,1)\) from this data.

We use the quasi-Newton method given by 3.8, to solve the non-linear equations of the form \( F_i(q) = g \) for \( i=1,2,3 \). In the following, we restate these iterates:

\[
q_0 = 0
\]

\[
q_{n+1} = q_n - DF_i^{-1}(0)(F_i(q_n) - g) \quad \text{for } i = 1, 2, 3
\]

For the inverse problems discussed in theorems 3.1 and 3.3, this scheme is well defined. In chapter 4, we will discuss its convergence properties.
CHAPTER 4
NUMERICAL EXAMPLES

4.1 A Numerical Method

In this chapter we will present the numerical results obtained by using the algorithm developed in chapter 3, for the following reconstruction problems:

1. Reconstruction of odd \( q \in L^2(0,1) \) from a finite number of pieces of spectral data namely \( \{\lambda_j\}_{j=1}^N \) or \( \{\nu_j\}_{j=0}^N \).
2. Reconstruction of even \( q \in L^2(0,1) \) from a finite number of pieces of spectral data, namely \( \{\lambda_j\}_{j=1}^N \), \( \{\mu_j\}_{j=0}^N \) or \( \{\nu_j\}_{j=1}^N \) and \( q \).
3. Reconstruction of \( q \in L^2(0,1) \) from a finite number of pieces of spectral data, namely \( \{\lambda_j\}_{j=1}^N \), \( \{\mu_j\}_{j=0}^N \) or \( \{\nu_j\}_{j=1}^N \) and the partial knowledge of \( q \).

Our reconstruction algorithm can be divided into the following three steps:

1. Computation of the mean (\( \mu \)) of \( q \) from the spectral data and modification of the data.
2. Computation of the boundary data \( (L_x(1,t) + HL(1,t)) \) or \( (K_x(1,t) + HK(1,t)) \) or \( (K_{xt}(1,t) + HK_t(1,t)) \) from eigenvalues.
3. Computation of \( q \) using:

\[
q_0 = 0 \\
q_{n+1} = q_n - DF_i^{-1}(0)(F_i(q_n) - g),
\]

where \( F_i \) is the forward map given by 3. for \( i = 1, 2, 3 \).

As we described in chapter 3, we need an estimate for the mean value \( \mu = \int_{y=0}^1 q(y)dy \). We cannot claim anything about \( \mu \) from a finite number of eigenvalues. However, we notice that the smoother \( q \) is, the faster the sequences \( \{a_j\} \), \( \{\beta_j\} \), and \( \{\gamma_j\} \) in (3.0a-c) will decay. Therefore, when \( q \) is well behaved, we expect that a good estimate for the value of \( \mu \) can be obtained from the following estimates:
\[ \mu = \lambda_N - (N - \frac{1}{2})^2 \pi^2 - 2H \]  
\[ \mu = \mu_N - N^2 \pi^2 - 2(h + H) \]  
\[ \mu = \nu_N - N^2 \pi^2 \]

In problem 1, it is clear that \( \mu = 0 \). Once we obtain the estimate for \( \mu \), we modify the eigenvalues by subtracting \( \mu \) from them. Therefore we will reconstruct \( q(x) - \mu \) and add the mean back.

From (3.4) we know that the boundary data \((L_x(1,t) + HL(1,t))\) or \((K_x(1,t) + HK(1,t))\) satisfies the following integral equation:

\[
\int_{t=0}^{1} (L_x(1,t) + HL(1,t)) \sin \sqrt{\lambda_j} t \, dt = -\sqrt{\lambda_j} \cos \sqrt{\lambda_j}, \quad j = 1..N \\
\int_{t=0}^{1} (K_x(1,t) + HK(1,t)) \cos \sqrt{\lambda_j} t \, dt = \sqrt{\mu_j} \sin \mu_j - (h + H) \cos \sqrt{\mu_j}, \quad j = 0..N.
\]

Let \( f(t) \) be the appropriate combination of boundary data. Then \( f \) satisfies the integral equation.

\[
\int_{t=0}^{1} f(t) \sin \sqrt{\lambda_j} t \, dt = b_j, \quad j = 1..N \quad (4.1a)
\]

or

\[
\int_{t=0}^{1} f(t) \cos \sqrt{\mu_j} t \, dt = d_j, \quad j = 0, 1, ... N. \quad (4.1b)
\]

In case \( f \) satisfies the \( N \) equations given by (4.1a), we approximate \( f \) using \( N \) basis functions. Whenever \( f \) satisfies the \((N + 1)\) equations given by (4.1b), we approximate \( f \) using \((N + 1)\) basis functions. In each of these two cases we expand \( f \) in terms of a set of basis functions denoted by \( \{u_k(x)\}_{k=1}^{N} \) or \( \{v_k(x)\}_{k=0}^{N} \) for \( x \in [0, 1] \). Hence, for \( f \) which satisfies (4.1a), we are looking for a function \( f(t) \) which takes the form:

\[
f(t) = \sum_{k=1}^{N} \alpha_k u_k(t). \quad (4.2a)
\]
Whenever \( f \) satisfies (4.2a), we need the function to be expressed as
\[
f(t) = \sum_{k=0}^{N} \beta_k v_k(t). \tag{4.2b}
\]

Now, we compute the coefficient \( \tilde{a} := (\alpha_1, \alpha_2, ..., \alpha_n) \) or \( \tilde{\beta} := (\beta_0, \beta_1, ..., \beta_N) \) from the system
\[
A \tilde{a} = \tilde{b} \tag{4.3a}
\]
or
\[
B \tilde{\beta} = \tilde{d} \tag{4.3b}
\]
where \( \tilde{b} := (b_1, b_2, ..., b_n) \) and \( A \) is the \( N \times N \) matrix with entries
\[
A_{jk} := \int_{t=0}^{1} u_k(t) \sin \sqrt{\lambda_j} t \ dt \tag{4.4a}
\]
or \( \tilde{d} := (d_0, d_1, ..., d_N) \) and \( B \) is the \((N + 1) \times (N + 1)\) matrix with entries
\[
B_{jk} := \int_{t=0}^{1} v_k(t) \cos \sqrt{\mu_j} t \ dt. \tag{4.4b}
\]

Next, we discuss some rules to pick the basis \( \{u_k(x)\}_{k=0}^{N} \) or \( \{v_k(x)\}_{k=0}^{N} \). We need these basis functions so that the matrix \( A \) (or \( B \)) is non-singular. Whenever these basis functions satisfy the boundary conditions satisfied by \( f(t) \), we observe better results. We discuss these in detail for the following cases.

(i) When \( h = \infty \) and \( H \in \mathbb{R} \).

It follows from (3.3b) and \( \mu = 0 \) that \( L(1,0) = 0, L_r(1,0) = 0, L(1,1) = 0 \). One can show that \( L_r(1,1) \neq 0 \) (unless \( q(1) = q(0) \)). Therefore, the basis \( \{ \sin(k - \frac{1}{2}) \pi t \}_{k=1}^{N} \) is suitable for the recovery of the boundary data \( L_r(1,t) + HL(1,t) \).

In the following, we argue that the matrix \( A \) obtained by replacing \( u_k(x) = \sin(k - \frac{1}{2}) \pi t \) in (4.4a) is non-singular. We know that \( \{ \sin(k - \frac{1}{2}) \pi t \}_{k=1}^{\infty} \) is complete and orthogonal in \( L^2(0,1) \). Consider the following sequence:
\[ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N, (N + \frac{1}{2})^2\pi^2, (N + \frac{3}{2})^2\pi^2, \ldots. \]

If \( \lambda_N < (N + \frac{1}{2})^2\pi^2 \), a theorem by Levinson [11] guarantees that
\[ \{ \sin(\sqrt{\lambda_k}t) \}_{k=1}^{N} \cup \{ \sin(k - \frac{1}{2})\pi t \}_{k=N+1}^{\infty} \]
is also complete in \( L^2(0,1) \). Now consider the change of basis matrix \( B \) from \( \{ \sin(\sqrt{\lambda_k}t) \}_{k=1}^{N} \cup \{ \sin(k - \frac{1}{2})\pi t \}_{k=N+1}^{\infty} \) to \( \{ \sin(k - \frac{1}{2})\pi t \}_{k=1}^{\infty} \). Clearly, \( B \) has the following block format:

\[
B = \begin{pmatrix}
A & 0 \\
C & I
\end{pmatrix}
\]

where (a) the matrix \( A \) is given by, \( A_{jk} = \int_{t=0}^{1} \sin(\sqrt{\lambda_j}t) \sin(k - \frac{1}{2})\pi t \, dt \) for \( j = 1, \ldots, N \) and \( k = 1, \ldots, N \).

(b) the matrix \( C \) is given by, \( C_{jk} = \int_{t=0}^{1} \sin(j - \frac{1}{2})\pi t \sin(\sqrt{\lambda_k}t) \, dt \) for \( j = N+1, \ldots, \infty \) and \( k = 1, \ldots, N \).

(c) the matrix \( I \) is the block identity form. Now \( B \) being a non singular matrix implies that \( A \) is non singular. In practice we observed that the matrix \( A \) is also well conditioned.

(ii) When \( h < \infty \) and \( H \in \mathbb{R} \).

From (3.3a) we know that \( K_t(1,0) = K(1,1) = h \). We also have that \( K_t(1,0) \neq 0 \) and \( K(1,1) \neq 0 \). So \( u_k(t) = \cos(k - \frac{1}{2})\pi t \) is a suitable choice for a basis in this case to recover the boundary data \( (K_t(1,t) + HK(1,t)) \). By an argument similar to case (i), we obtain that \( \{ \cos(\sqrt{\lambda_k}x) \}_{k=0}^{N} \cup \{ \cos(k - \frac{1}{2})\pi x \}_{k=N+1}^{\infty} \) is complete in \( L^2(0,1) \) provided that \( \lambda_k < (N + \frac{1}{2})^2\pi^2 \). By considering the change of basis matrix from \( \{ \cos(\sqrt{\lambda_k}x) \}_{k=0}^{N} \cup \{ \cos(k - \frac{1}{2})\pi x \}_{k=N+1}^{\infty} \) to \( \{ \cos(k - \frac{1}{2})\pi x \}_{k=0}^{\infty} \), we see that the matrix \( A \) given by (4.4b) is non-singular.

In case we need to recover the combination \( (K_t(1,t) + HK(1,t)) \), we make use the fact that this combination is the time derivative of the combination discussed before. The boundary data \( (K_t(1,t) + HK(1,t)) \) obtained this way is accurate enough for our computations.
Next, we discuss the computation of the forward map. From chapter 3, we know that, in order to compute the forward map, one needs to compute a certain combination of the quantities from \( v(1, t) \), \( v_t(1, t) \), \( v_{\tau t}(1, t) \) or \( v_{\tau \tau t}(1, t) \). Here, \( v(x, t) \) solves the boundary value problem of the wave equation given by:

\[
\begin{align*}
  v_{tt} - v_{xx} + qv &= 0 \\
  v(x, \pm x) &= h + \frac{1}{2} \int_0^x q(y) \, dy
\end{align*}
\]

or

\[
\begin{align*}
  v_{tt} - v_{xx} + qv &= 0 \\
  v(x, \pm x) &= \pm \frac{1}{2} \int_0^x q(y) \, dy.
\end{align*}
\]

In order to compute a discrete approximation to \( v(x, t) \), we represent \( v \) by its value at grid points \( (x, t) = (i\Delta x, j\Delta t) \), \( j = -i, i \). Let \( v^{i,j} = v(i\Delta x, j\Delta t) \).

\[
\begin{align*}
  &\text{do } i = 0, N \\
  &\quad v^{i,i} = h + \frac{1}{2} \int_{y=0}^{i\Delta x} q(y) \, dy. \\
  &\quad v^{i,-i} = h + \frac{1}{2} \int_{y=0}^{i\Delta x} q(y) \, dy. \\
  &\text{continue}
\end{align*}
\]

\[
\begin{align*}
  &\text{do } i = 0, (N - 1) \\
  &\quad v^{i+1,i} = v^{i,i-1} + \frac{1}{2} \Delta t q(i) \\
  &\quad v^{i+1,-i} = v^{i+1,i} \\
  &\text{continue}
\end{align*}
\]

\[
\begin{align*}
  &\text{do } i = 2, N \\
  &\text{do } j = 1, (i - 1) \\
  &\quad v^{i+1,j} = v^{i,j+1} + v^{i,j-1} + \Delta x^2 q(i) \\
  &\text{continue}
\end{align*}
\]

\[
\begin{align*}
  &\text{continue}
\end{align*}
\]

\[
\begin{align*}
  &\text{do } j = 0, (N - 1)
\end{align*}
\]
$$v_x(j) = \frac{v(N, j) - v(N-1, j)}{\Delta x}$$
continue
$$v_x(N) = 2v_x(N - 1) - v_x(N - 2)$$
do $j = 0, (N - 1)$
$$v_{xt}(j) = \frac{v_x(j + 1) - v_x(j - 1)}{2\Delta t}$$
continue
$$v_{xt}(N) = 2v_{xt}(N - 1) - v_{xt}(N - 2)$$
end.

### 4.2 Convergence of the Numerical Method

Next, we discuss the convergence properties of the algorithm developed in this chapter. Let us recall that the quasi-Newton iterates are given by:

$$q_0 = 0$$

$$q_{n+1} = q_n - DF^{-1}_i(F_i(q_n) - g)\text{ for } i = 1, 2, 3. \quad (4.7)$$

For the inverse problems discussed in theorems 3.1 and 3.3, this scheme is well defined. Next, we discuss its convergence properties.

**Theorem 4.1.** In each of theorem (3.1) and theorem (3.2) the quasi-Newton iterates are convergent to the solution and its rate of convergence is linear.

**Proof.** Proof of the convergence of the iterates to the solution is in the virtue of the proof of the Inverse function theorem. Next, we prove that the rate of convergence of the iteration scheme given by (4.7) is linear. We know that the iterates $q_n$ converges to the solution $q$ of $F_i(q) = g$. Let $\epsilon_n = q_n - q$. From (4.7) we get

$$\epsilon_{n+1} = DF^{-1}_i(0)DF_i(0)\epsilon_n - (F_i(q_n) - g) \quad (4.8)$$
Using the differentiability of $F_i$ we get

$$F_i(q_n) - g = DF_i(q_n)e_n + o(\|e_n\|)$$

Equations (4.8) and (4.9) together gives us

$$\|e_{n+1}\| \leq \|DF_i^{-1}(0)\|\{\|DF_i(0) - DF_i(q_n)\| + \frac{\|\omega(\|e_n\|)\|}{\|e_n\|}\}\|e_n\|$$

From theorems (2.5) and (2.11) there exist a constant $C_1(\|q_n\|)$ such that

$$\|(DF_i(q_n) - DF_i(q_0))(e_n)\| \leq C_1(\|q_n\|)\|q_n\||e_n||$$

Moreover,

$$\lim_{\|q_n\| \to 0} C_1(\|q_n\|) = 0.$$ 

By the definition of the Frechet differentiability, for $1 > \epsilon > 0$ there exists a $\delta > 0$ so that

$$\frac{\|\omega(\|e_n\|)\|}{\|e_n\|} \leq \epsilon \text{ whenever } \|e_n\| \leq \delta.$$ 

Using (4.10), (4.11) and (4.12)

$$\|e_{n+1}\| \leq \|e_n\||DF_i(0)^{-1}||C_1(\|q_n\|)||q_n|| + \epsilon$$

Let

$$C_2(\delta) = \max_{\|q_n\| \leq \delta} \|DF_i^{-1}(0)\|\|C_1(\|q_n\|)||q_n|| + \epsilon.$$ 

Here we note that $\lim_{\delta \to 0} C_2(\delta) = \epsilon$. So, we pick $\delta$ such that $C_2(\delta) < 1$. Hence there exists $\delta > 0$ such that

$$\|e_{n+1}\| \leq C_2(\delta)\|e_n\| \text{ with } C_2(\delta) < 1.$$ 

If $\|q\| \leq \delta$, then the quasi-Newton iterates \{q_n\} satisfy the inequality $\|q_n - q\| \leq \delta$, according with the inequality

$$\|e_{n+1}\| \leq C_2(\delta)\|e_n\|.$$
This proves that the quasi-Newton iteration scheme given by (4.7) converges linearly to the solution $q$.

4.3 Examples

In this section, we will display several numerical examples obtained by using the reconstruction algorithm described in the previous section. For each of these examples, we obtain eigenvalues by solving the forward problem using the FORTRAN software package SLEIGN [3].

We illustrate our computational results by reconstructing six functions $q_1(x)$, $q_2(x)$, $q_3(x)$, $q_4(x)$, $q_5(x)$ and $q_6(x)$ shown in the Figures 4.1 - 4.6 respectively. Note that $q_1$ represents a smooth even function, $q_2$ represents an even function with jump discontinuities, $q_3$ represents an odd function with a discontinuous first derivative, $q_4$ represents an odd function with jump discontinuities, $q_5$ represents a smooth function and $q_6$ represents a function with jump discontinuities. Figures 4.7 - 4.12 correspond to the reconstructions of these functions from a eigenvalue sequence of (1.1) with the boundary condition (1.2). In the following we list the parameters $h$ and $H$ which correspond to the each of the eigenvalue sequence, the number of eigenvalues ($N$) and the number of iterations ($K$). In Figure 4.7 $h = 0, H = 0.5, N = 4, K = 3$. in Figure 4.8 $h = 0.5, H = 0.8, N = 20, K = 10$. in Figure 4.9 $h = 0, H = 2.0, N = 15, K = 10$. in Figure 4.10 $h = 0.5, H = 4.0, N = 20, K = 10$. in Figure 4.11 $h = 0.1, H = 0.0, N = 7, K = 5$ and in Figure 4.12 $h = 0, H = 0.5, N = 20, K = 15$.

Note that the Figures 4.13 and 4.14 correspond to the reconstruction of a large even function from the eigenvalue sequence of (1.1) with respect to the boundary condition (1.2) with $h = 0, H = 0.9, N = 5, K = 10$ and to the reconstruction of a large odd function from the eigenvalue sequence of (1.1) with respect to the boundary condition (1.2) with $h = 1.0, H = 5.0, N = 15, K = 15$ respectively.
Figure 4.1 $q_1(x)$.

Figure 4.2 $q_2(x)$. 
Figure 4.3 $q_3(x)$.  

Figure 4.4 $q_4(x)$.  


Figure 4.5 $q_5(x)$.

Figure 4.6 $q_6(x)$. 

Figure 4.7 Reconstruction of $q_1(x)$.

Figure 4.8 Reconstruction of $q_2(x)$. 
Figure 4.9 Reconstruction of $q_3(x)$.

Figure 4.10 Reconstruction of $q_4(x)$.
Figure 4.11 Reconstruction of $q_5(x)$.

Figure 4.12 Reconstruction of $q_6(x)$. 
Figure 4.13 Reconstruction of a large even function.

Figure 4.14 Reconstruction of a large odd function.
CHAPTER 5

GENERAL TWO SPECTRUM PROBLEM

5.1 A Reconstruction Method for the Two Spectrum Case

In this chapter we study the following inverse spectral problem. Let \( \{\lambda_n\}_{n=0}^\infty \) be the eigenvalues of the differential equation

\[-y'' + q(x)y = \lambda y \quad (5.1)\]

with the boundary conditions

\[y'(0) - hy(0) = y'(1) + Hy(1). \quad (5.2)\]

Let \( \{\lambda_n\}_{n=0}^\infty \) be the eigenvalues of the differential equation (5.1) with the boundary conditions

\[y'(0) - \tilde{h}y(0) = y'(1) + \tilde{H}y(1). \quad (5.3)\]

Here we assume that \( h, \tilde{h}, H, \tilde{H} \in \mathbb{R}. \)

We assume that the potential function \( q(x) \in L^2(0,1) \) and investigate the question of reconstructing \( q \) from the eigenvalue sequences \( \{\lambda_n\}_{n=0}^\infty \) and \( \{\tilde{\lambda}_n\}_{n=0}^\infty \). From 3.0b it is clear that we may obtain \( \int_{y=0}^1 q(y)dy \) from either sequence. If \( \mu = \int_{y=0}^1 q(y)dy \) then we replace \( \lambda_n \) by \( \lambda_n - \mu \) and \( \tilde{\lambda}_n \) by \( \tilde{\lambda}_n - \mu \). Now we compute the corresponding solution to the inverse problem with the modified eigenvalue sequences. From the asymptotic forms 3.0a-b, the corresponding solution of the inverse problem has mean zero. So, as in chapter 3 without loss of generality \( \int_{y=0}^1 q(y)dy = 0 \).

Since \( h \in \mathbb{R} \) we know from chapter 3 that the \( j \)th eigenfunction of the Sturm-Liouville problem (5.1) and (5.2) is of the form

\[y_2(x, \lambda_j) = \cos \sqrt{\lambda_j}x + \int_{y=0}^x K(x,t) \cos \sqrt{\lambda_j}t dt \quad (5.4)\]
where $K(x,t)$ satisfies the characteristic boundary value problem

$$K_{tt} - K_{xx} + q(x)K = 0, \quad 0 \leq |t| \leq x \leq 1 \quad (5.5a)$$

$$K(x, \pm x) = h + \frac{1}{2} \int_0^x q(y)dy, \quad 0 \leq x \leq 1. \quad (5.5b)$$

Since $\hat{h} \in \mathbb{R}$ the $j$th eigenfunction of the Sturm-Liouville problem (5.1) and (5.3) is given by

$$y_2(x, \lambda_j) = \cos \sqrt{\lambda_j}x + \int_0^x \tilde{K}(x, t) \cos \sqrt{\lambda_j}t dt \quad (5.6)$$

where $\tilde{K}(x, t)$ satisfies the characteristic boundary value problem

$$\tilde{K}_{tt} - \tilde{K}_{xx} + q(x)\tilde{K} = 0, \quad 0 \leq |t| \leq x \leq 1 \quad (5.7a)$$

$$\tilde{K}(x, \pm x) = \hat{h} + \frac{1}{2} \int_0^x q(y)dy, \quad 0 \leq x \leq 1. \quad (5.7b)$$

Now, the $j$th eigenfunctions $y_2(x, \lambda_j)$ and $\tilde{y}_2(x, \lambda_j)$ should satisfy the boundary condition at $x = 1$. So, we get

$$\int_0^1 (K_x(1,t) + H\tilde{K}(1,t)) \cos \sqrt{\lambda_j}t dt = \sqrt{\lambda_j} \sin \sqrt{\lambda_j} - (H + h) \cos \sqrt{\lambda_j} \quad (5.8a)$$

$$\int_0^1 (\tilde{K}_x(1,t) + \hat{H}\tilde{K}(1,t)) \cos \sqrt{\lambda_j}t dt = \sqrt{\lambda_j} \sin \sqrt{\lambda_j} - (\hat{H} + \hat{h}) \cos \sqrt{\lambda_j} \quad (5.8b)$$

Using the completeness property of $\cos \sqrt{\lambda_j}t$ and $\cos \sqrt{\lambda_j}t$ given by lemma 3.1 we can recover the boundary data \{$K_x(1,t) + H\tilde{K}(1,t)$\} and \{$(\tilde{K}_x(1,t) + \hat{H}\tilde{K}(1,t))$\} uniquely.

Next, we form the inverse spectral problem as a time domain problem. For a given $q \in L^2(0,1)$ we define

$$\begin{cases} F_1(q)(t) = u_x(1,t) + Hu(1,t) \\ F_2(q)(t) = v_x(1,t) + \hat{H}v(1,t) \end{cases} \quad (5.9)$$

where $u(x,t; q), \ v(x,t; q)$ satisfy

$$u_{tt} - u_{xx} + q(x)u = 0, \quad 0 \leq |t| \leq x \leq 1 \quad (5.10a)$$
\[ u(x, \pm x) = h + \frac{1}{2} \int_0^x q(y)dy, \quad 0 \leq x \leq 1. \]  
(5.10b)

and

\[ v_{tt} - v_{xx} + q(x)v = 0, \quad 0 \leq |t| \leq x \leq 1 \]  
(5.11a)

\[ v(x, \pm x) = \tilde{h} + \frac{1}{2} \int_0^x q(y)dy, \quad 0 \leq x \leq 1 \]  
(5.11b)

respectively.

Now we introduce the following notation

\[ \bar{F}(q)(t) = \begin{pmatrix} F_1(q)(t) \\ F_2(q)(t) \end{pmatrix} \]  
(5.12)

We know that the solution \( q \) of the inverse spectral problem satisfy

\[ F_1(q)(t) = K_r(1,t) + HK(1,t) \]  
(5.13a)

and

\[ F_2(q)(t) = \tilde{K}_r(1,t) + \tilde{H}\tilde{K}(1,t). \]  
(5.13b)

Hence the solution \( q \) of the inverse spectral problem satisfies the non-linear system given by

\[ \bar{F}(q)(t) = \begin{pmatrix} K_r(1,t) + HK(1,t) \\ \tilde{K}_r(1,t) + \tilde{H}\tilde{K}(1,t) \end{pmatrix} \]  
(5.14)

**Theorem 5.1.** The map \( \bar{F} \) given by (5.12) is a locally Lipschitz map:

\[ \bar{F} : L^2(0,1) \rightarrow L^2(-1,1) \times L^2(-1,1). \]

Moreover, \( \bar{F} \) is Frechet differentiable in \( L^2(0,1) \) and the Frechet derivative \( D\bar{F} \) is a locally Lipschitz map:

\[ D\bar{F} : L^2(0,1) \times L^2(0,1) \rightarrow L^2(-1,1) \times L^2(-1,1). \]

**Proof.** Proof is clear from the corollary 2.1 and the theorem 2.6. \( \Box \)
We use the following quasi-Newton method to solve the system of non-linear equations given by 5.15:

\[ q_0 = 0 \]  
\[ q_{n+1} = q_n - D\bar{F}^{-1}(0)(\bar{F}(q_n) - \bar{g}), \]  

where

\[ \bar{g} = \left( \begin{array}{c} K_x(1,t) + HK(1,t) \\ \tilde{K}_x(1,t) + \tilde{H}\tilde{K}(1,t) \end{array} \right). \]

In order to apply the quasi-Newton method given by 5.16(a-b), we need to know the invertibility of the map \( D\bar{F}(0) \).

**Lemma 5.1.** Let \( \bar{F} \) be the function defined by (5.12). Then the inverse of the operator \( D\bar{F}(0) \) exists from \( L^2(-1,1) \) to \( L^2(0,1) \) provided that \( (h - H) \neq (\tilde{h} - \tilde{H}) \).

**Proof.** If \( D\bar{F}(0)q_1(t) = D\bar{F}(0)q_2(t) \) for \( q_1(x), q_2(x) \in L^2(0,1) \) then

\[ DF_1(0)q_1(t) = DF_1(0)q_2(t) \]  
\[ DF_2(0)q_1(t) = DF_2(0)q_2(t). \]

From (3.9b) we know that

\[ DF_1(0)q_1(t) = \frac{1}{2} q_1(x) \left( \frac{1 + t}{2} \right) + H h \int_{y=\frac{1}{2}}^{1} (1 + t - 2y) q_1(x) dy + (h - H) \int_{y=\frac{1}{2}}^{1} q_1(x) dy \]

and

\[ DF_2(0)q_1(t) = \frac{1}{2} q_1(x) \left( \frac{1 + t}{2} \right) + \tilde{H} \tilde{h} \int_{y=\frac{1}{2}}^{1} (1 + t - 2y) q_1(x) dy + (\tilde{h} - \tilde{H}) \int_{y=\frac{1}{2}}^{1} q_1(x) dy \]

where \( q_1(x) \), \( q_1(x) \) denotes the even and the odd parts of the function \( q_1(x) \) respectively. By taking \( q(x) = q_1(x) - q_2(x) \) we can rewrite (5.17a) and (5.17b) as

\[ \frac{1}{2} q(x) \left( \frac{1 + t}{2} \right) + H h \int_{y=\frac{1}{2}}^{1} (1 + t - 2y) q(x) dy + (h - H) \int_{y=\frac{1}{2}}^{1} q(x) dy = 0 \]
By subtracting (5.18b) from (5.18a) we get

\[(Hh - \tilde{H}h) \int_{y=\frac{1+t}{2}}^{1} (1 + t - 2y)q_c(y)dy + ((h - H) - (\tilde{h} - \tilde{H})) \int_{y=\frac{1+t}{2}}^{1} q_o(y)dy = 0\]

Since \((h - H) \neq (\tilde{h} - \tilde{H})\), differentiating the above equation with respect to \(t\) results

\[q_o\left(\frac{1+t}{2}\right) = 0\] for every \(t \in [0, 1]\).

Hence,

\[q_o(x) = 0\]

for \(x \in [\frac{1}{2}, 1]\). Substituting this back into the equation (5.18a) we get

\[\frac{1}{2} q_c\left(\frac{1+t}{2}\right) + Hh \int_{y=\frac{1+t}{2}}^{1} (1 + t - 2y)q_c(y)dy = 0\] for \(0 \leq t \leq 1\).

The solution to this integral equation is \(q_c(x) = 0\) for \(x \in [\frac{1}{2}, 1]\). Therefore \(DF(0)q_1 = DF(0)q_2\) implies that \(q_1 = q_2\). So, \(DF^{-1}(0)\) exists and this proves the lemma. \(\square\)

Here we note that if we are given the two spectrums correspond to the boundary conditions \(h = \infty, H = 0\) and \(\tilde{h} = \tilde{H}\) then we can prove that \(DF^{-1}(0)\) is not only exists but also bounded. In this case we have a uniqueness theorem similar to the cases in chapter 3 for sufficiently small potentials.

Next we describe our reconstruction algorithm:

1. Computation of mean \(\mu\) using spectral data and modify the spectral data.

2. Computation of the boundary data \((\tilde{K}_r(1,t) + HK(1,t))\) and \((\tilde{K}_r(1,t) + \tilde{H}_K(1,t))\) from the spectral data.
3. Computation of \( q \) using 5.15a-b.

Instead of computing \( q \) here we compute its even \((q_e)\) and odd \((q_o)\) parts. Let us introduce the following notation:

\[
\tilde{q} = \begin{pmatrix} q_e \\ q_o \end{pmatrix} \quad \text{and} \quad \tilde{q}_n = \begin{pmatrix} q_{n_e} \\ q_{n_o} \end{pmatrix}.
\]

Now the quasi-Newton method given by 5.16a-b take the following form:

\[
\tilde{q}_0 = 0
\]

\[
\tilde{q}_{n+1} = \tilde{q}_n - D\tilde{F}^{-1}(0)(\tilde{F}(q_n) - \tilde{g}).
\]

In order to compute \( D\tilde{F}^{-1}(0)(\tilde{F}(q_n) - \tilde{g}) \) we do the following:

If

\[
\delta q_n = D\tilde{F}^{-1}(0)(\tilde{F}(q_n) - \tilde{g})
\]

then

\[
D\tilde{F}(0)\delta q_n = (\tilde{F}(q_n) - \tilde{g}).
\]

Let \( \delta \tilde{g} = \begin{pmatrix} \delta g_1 \\ \delta g_2 \end{pmatrix} \).

Now the system given by

\[
\begin{pmatrix} DF_1(0) \\ DF_2(0) \end{pmatrix} \begin{pmatrix} \delta q_{n_e} \\ \delta q_{n_o} \end{pmatrix} = \begin{pmatrix} \delta g_1(t) \\ \delta g_2(t) \end{pmatrix}
\]

is equivalent to the following system of integral equations:

\[
\frac{1}{2} \delta q_{n_e}(\frac{1+t}{2}) + H_1 h_1 \int_{y=(\frac{1+t}{2})}^{1} (1+t-2y) \delta q_{n_e}(y) dy + (h_1 - H_1) \int_{y=\frac{1+t}{2}}^{1} \delta q_{n_o}(y) dy = \delta g_1(t)
\]

and

\[
\frac{1}{2} \delta q_{n_o}(\frac{1+t}{2}) + H_2 h_2 \int_{y=(\frac{1+t}{2})}^{1} (1+t-2y) \delta q_{n_e}(y) dy + (h_2 - H_2) \int_{y=\frac{1+t}{2}}^{1} \delta q_{n_o}(y) dy = \delta g_2(t)
\]

By solving this system for \( \begin{pmatrix} \delta q_{n_e} \\ \delta q_{n_o} \end{pmatrix} \) we compute the \( D\tilde{F}^{-1}(0)(\tilde{F}(q_n) - \tilde{g}) \).
5.2 Examples

In this section, we will display two numerical examples obtained by using the reconstruction algorithm described in the previous section. As in chapter 4, we obtain eigenvalues by solving the forward problem using the FORTRAN software package SLEIGN [3].

We illustrate our computational result by reconstructing two functions $q_1(x)$ and $q_2(x)$ shown in the Figures 5.1 and 5.2. Note that $q_1$ represents a function with a discontinuous first derivative and $q_2$ represents a function with a jump discontinuity. Let $N$, $\hat{N}$ be the number of eigenvalues from the first and the second sequence respectively. The number of iterations is denoted by $K$. In Figure 5.3 we illustrate the reconstruction of $q_1(x)$ from the eigenvalue sequences of (1.1) with respect to the boundary conditions (1.2) with $h = 1.0$, $H = 0.5$, $N = 7$ and $\hat{h} = 0.75, \hat{H} = 1.25, \hat{N} = 7$, $K = 5$. In Figure 5.4 we illustrate the reconstruction of $q_2(x)$ from the eigenvalue sequences of (1.1) with respect to the boundary conditions (1.2) with $h = 1.0$, $H = 0.4$, $N = 15$ and $\hat{h} = -0.5$, $\hat{H} = 0.9$, $\hat{N} = 15$, $K = 15$. In Figure 5.5 we illustrate the reconstruction of

$$q_3(x) = \begin{cases} 
4.35 & \text{for } x \leq \frac{1}{1024} \\
\frac{x^{3}}{4} & \text{for } \frac{1}{1024} < x \leq 1
\end{cases}$$

from the eigenvalue sequences of (1.1) with respect to the boundary conditions (1.2) with $h = 1.5$, $H = 1.0$, $N = 15$ and $\hat{h} = 0.75, \hat{H} = 1.25, \hat{N} = 15$, $K = 15$. 

Figure 5.1 $q_1(x)$.

Figure 5.2 $q_2(x)$. 
Figure 5.3 Reconstruction of $q_1(x)$.

Figure 5.4 Reconstruction of $q_2(x)$. 

Figure 5.5 Reconstruction of $q_3(x)$. 
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