A Mark Twainian View Of Consumer Surplus

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A Mark Twainian View Of Consumer Surplus

Abstract
Many economists have used CS or producer surplus (PS) to evaluate consequences of public policies. Although some economists question CS and PS formulations, the proponents of their use have carried the day. Almost every journal issue carries articles in which they are used. No one hears from the opponents any more—except for an occasional furtive whisper, "I don't believe it". For a refreshing exception, see Cochrane (1980). Cochrane’s objections to use of CS are mostly practical ones: that use of CS in public policy analysis is an effort to provide a scalar answer to a question that does not have a scalar answer, and that measures of CS do not provide public-policy makers information that they can use in choosing and defending policies. This paper complements Cochrane’s practical objections by presenting theoretical objections. The main thesis of this paper is that it is a good thing if Cochrane is right in asserting that CS is not used in making public policy, because the concept of CS is seriously flawed and our measures of CS are biased measures of a flawed concept.

Disciplines
Business Administration, Management, and Operations | Business Intelligence | Finance and Financial Management | Income Distribution
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George W. Ladd

January 1981

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A MARK TWAINIAN VIEW OF CONSUMER SURPLUS

George W. Ladd*

January 1981

The title reflects the fact that each time I read one of those marvelously sophisticated articles on consumer surplus (CS), the thought occurs to me that I must feel the same way about CS as Mark Twain felt about science. In Chapter 17 of *Life on the Mississippi* he wrote,

"In the space of one hundred and seventy-six years the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over one mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oolitic Silurian Period, just a million years ago next November, the Lower Mississippi River was upward of one million three hundred thousand miles long, and stuck out over the Gulf of Mexico like a fishing-rod. And by the same token any person can see that seven hundred and forty-two years from now the Lower Mississippi will be only a mile and three-quarters long, and Cairo and New Orleans will have joined their streets together, and be plodding comfortably along under a single mayor and a mutual board of aldermen. There is something fascinating about science. One gets such wholesome returns of conjecture out of such a trifling investment of fact."

It strikes me that "There is something fascinating about CS analysis. One gets such wholesome returns of conjecture out of such a trifling investment of fact."

Many economists have used CS or producer surplus (PS) to evaluate consequences of public policies. Although some economists question CS and PS formulations, the proponents of their use have carried the day. Almost every journal issue carries articles in which they are used. No one hears from the opponents any more—except for an occasional furtive

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*I am grateful to Roger Dahlgran, Marvin Hayenga, and John Miranowski for their helpful comments on an earlier draft of this paper.
whisper, "I don't believe it". For a refreshing exception, see Cochrane (1980). Cochrane's objections to use of CS are mostly practical ones: that use of CS in public policy analysis is an effort to provide a scalar answer to a question that does not have a scalar answer, and that measures of CS do not provide public-policy makers information that they can use in choosing and defending policies. This paper complements Cochrane's practical objections by presenting theoretical objections. The main thesis of this paper is that it is a good thing if Cochrane is right in asserting that CS is not used in making public policy, because the concept of CS is seriously flawed and our measures of CS are biased measures of a flawed concept.

SUMMARY

This paper is not going to review all the literature on and the evidence for and against use of CS. Currie, Murphy, and Schmitz (1971) present a useful summary and make it clear that they support the use of CS and PS. An excellent article by Harberger (1971) argues in favor of use of CS and PS in public-policy analysis.

The first two sections of this paper present excess utility (EU) and excess expenditure (EE) definitions, and the third discusses their equivalence. The fourth section considers empirical evidence on the model of consumer behavior in which CS is derived. It argues that the empirical evidence against the model is sufficiently strong that a person can reasonably conclude that the CS hypotheses should be rejected. The fifth section concerns definition and measurement of CS. This section argues that a satisfactory definition of CS requires that three conditions
be satisfied, and these three conditions cannot be satisfied simultaneously and consequently our standard definition of CS contains a serious flaw. This section concludes that the total area under a demand function overstates the amount spent by consumers under price discrimination. The next section concludes that our definition and measure of PS are subject to similar criticisms. In the seventh section it is argued that it is not possible to have nonzero PS and nonzero CS simultaneously. The eighth section discusses three value judgments that are made in measuring CS and using CS in public-policy analysis and suggests that many economists use measures of CS that violate their own value systems. The next section discusses standard errors of estimates of CS. The final section treats a methodological issue.

EXCESS UTILITY (EU) DEFINITION

The notation of Barten's (1977) review of studies on systems of consumer-demand equations will be used because his study will be cited again later in this paper. Assume that the consumer's preferences can be represented by the strictly quasi-concave monotone increasing utility function

\[ u = u(q) \]

where \( q \) is an \( n \)-element vector of quantities. The function \( u(q) \) is continuous and has continuous first- and second-order derivatives with respect to \( q \). The consumer maximizes \( u(q) \) subject to

\[ m = p'q \]

where \( m \) is a fixed scalar, interpreted either as income or as total
expenditure, and \( p \) is an \( n \)-element column vector of fixed prices. The first-order conditions are (2) and

\[
(3) \quad \frac{du}{dq} = \lambda p
\]

where \( \frac{du}{dq} \) is the \( n \)-element vector \( \frac{\partial u}{\partial q} \). The solution of (2) and (3) for \( q \) and \( \lambda \) in terms of \( p \) and \( m \) yields the demand equations

\[
(4) \quad q = D(p, m)
\]

Strict quasi-concavity and monotonicity ensure the existence of such a solution. If (4) has an inverse, each demand function can be expressed as

\[
p_i = d_i(q, m).
\]

A typical element of (3) is

\[
\frac{\partial u}{\partial q_i} = \lambda p_i
\]

If all other prices and incomes remain constant and marginal utility of money, \( \lambda \), remains constant, then

\[
(5) \quad u = \lambda \int_0^x p_i \, dq_i
\]

a measure of the total amount of utility obtained if good \( i \) is consumed in the amount \( x \). The excess of (6) over total expenditure on amount \( x \) is the consumer surplus, that is

\[
(6) \quad CS \equiv EU = \lambda \left( \int_0^x p_i \, dq_i - xp_{ix} \right)
\]

Equation (6) provides an EU definition of CS.

One objection to (5) and (6) is that they are based on the assumption of constant \( \lambda \). Harberger (1971, p. 788 fn. 2) provides one answer to this objection. If \( z \) is the value of a policy variable the change in utility that results from changing the policy variable from \( z_0 \) to \( z_1 \) is

\[
\Delta u = \int_{z_0}^{z_1} \sum_{i=1}^{m} u_i(z)(\frac{\partial q_i}{\partial z}) \, dz
\]
"This, being expressed in utils, is not invariant to a monotonic transformation. However, transforming utility into money continuously through the integration process, always at the marginal utility of money prevailing at that point," yields

\[
\Delta w = \int_{z_0}^{z_1} \sum \left( u_i(z) / \lambda(z) \right) (\partial q_i / \partial z) \, dz = \int_{z_0}^{z_1} \sum p_i(z) (\partial q_i / \partial z) \, dz.
\]

Expression (8) provides an EU definition of CS. Throughout the rest of this paper it will be assumed that \( \lambda \) is constant, except where otherwise specified.

**EXCESS EXPENDITURES (EE) DEFINITION**

CS can also be defined in terms of excess consumer expenditure (EE). Whereas the EU definition applies for an individual consumer, the EE definition can apply to an individual consumer or to a group of consumers.

Figure 1 is copied from Boulding (1946). He writes (p. 639) "The 'buyers' curve' \( b_1 \ldots b_n \), shows what quantities buyers are just willing to buy at various prices. Thus, at a price \( OB_1 \) there are buyers just willing to buy \( B_1 b_1 \); at a price \( ON_2 \), there are buyers just willing to buy an amount \( B_2 b_2 \); and so on. The total amount that will be bought at the price \( ON_2 \) is, of course, \( B_1 b_1 + B_2 b_2 \), or \( N_1 b_2 \), and, as the same principle applies all the way down the curve, the 'buyers' curve' is also the demand curve." On page 640, (italics mine) "Then the total buyers' surplus at the equilibrium price is measured by the area \( N_1 BP \ldots \). The buyers' surplus measures the difference between the total amount actually paid by the buyers (ONPM) and the total amount which they would have been willing to pay if perfect price discrimination could have been practiced—(i.e., if
Figure 1.
Source: Boulding (1946, p. 641).
each unit had been sold at the highest price that anyone was willing to pay for it)—which would be the area OB₁PM." On p. 642 "a fall in price may not only attract new buyers, but may also encourage each individual buyer to buy more. This fact is not excluded by Figure 1, where the buyers ... refer to quantities, not only individuals. Thus the quantity $b_2$, which would just be bought at the price ON₂, may represent an addition to the purchases of existing buyers; and the quantity $s_2$ likewise may represent an addition to the sales of existing sellers as well as the sales of new sellers." Boulding uses "buyers surplus" for what is commonly now called CS.

Also he writes on page 640 (italics mine), "The sellers' surplus measures the difference between what the sellers actually receive (ONPM) and the least sum for which the amount OM could be obtained under perfect price discrimination—i.e., if each quantity were to be paid for at a rate only just sufficient to induce the seller to part with it. This is the area OS₁PM." And the area $S₁NP$ measures sellers' surplus, or producers' surplus.

EQUIVALENCE OF THE DEFINITIONS

Currie, Murphy, and Schmitz (1971) assert that the EU and EE definitions are equivalent. In this paper it is accepted that they are equivalent (except possibly for the proportionately factor $\lambda$). Because EE is an amount of money and $\lambda$ is constant marginal utility of money, $EU = \lambda EE$ is marginal utility of the amount of money EE. Alternatively, one can accept Harberger's derivation of (7) and accept $EU = EE$. 
Braithwaite (1953, pp. 12-21) classified hypotheses into three levels. Level I hypotheses are the highest level hypotheses in a system. They have the greatest generality and serve only as premisses in the system. Level II hypotheses are less general. They are derived from level I hypotheses by logical arguments. They occur as conclusions but also serve as premises for level III hypotheses, which appear only as conclusions. A level II hypotheses is a general or universal statement that yields a great number of level III hypotheses, each of which is a specific instance of the universal statement. One level II hypothesis yields hypotheses IIIa, IIIb, IIIc, etc. Empirical contradiction of a level III hypothesis therefore implies rejection of the level II hypothesis from which it is derived. If, e.g., the level II hypothesis is "All A's are B's", an observation of one A that is not a B refutes the level II hypothesis. And refutation of a level II hypothesis implies refutation of the conjunction of level I hypotheses used in deriving the level II hypothesis.

In the previous section on EU definition, e.g., equation (1) is a level I hypothesis. Under the level I hypotheses of that section, the consumer demand functions satisfy four sets of constraints (let \( dq/dp \) be the \( n \times n \) matrix of derivatives of \( q \) with respect to \( p \)):

\[
\begin{align*}
(8) & \quad m \Delta q/\Delta m + p' dq/dp = 0: \text{ homogeneity} \\
(9) & \quad K = dq/dp + q' \Delta q/\Delta m = K': \text{ symmetry} \\
(10) & \quad y' Ky < 0 \text{ for all } y \neq 0: \text{ negativity} \\
(11) & \quad p' dq/dp + q' = 0; p' \Delta q/\Delta m = 1: \text{ additivity}
\end{align*}
\]
Each of these is a level II hypothesis; each is claimed to hold for all consumers of all consumer goods at all times. The statement "Expression (8) was true for consumers in the United States in 1929-72" is a level III hypothesis. What is expression (5)? Because it is a universal statement derived from level I hypotheses and level III hypotheses can be derived from it, it seems to be a level II hypothesis. But the level III hypotheses derived from it are unverifiable. We cannot, e.g., confront the statement "Expression (5) is true for George Ladd's 1980 purchases of Jack Daniels Whiskey" with empirical information. Evidence for or against (5) must be indirect evidence taken from the direct evidence bearing on other parts of the system.

Barten (1977) summarized the results from several studies that tested the constraints (8) through (11). Results are presented in Table 1, which is a copy of Barten's table (p. 46). The homogeneity, symmetry, negativity, and additivity restrictions were rejected in 4 out of 5, 6 out of 16, 0 out of 4, and 8 out of 11 tests, for a total of 18 rejections out of 36 tests. Half of the tests in Table 1 imply rejection of the conjunction of level I hypotheses. If one can show that the rejected level II hypotheses--(8), (9), and (11)--are due to a level I hypotheses which is not needed for the derivation of (5), the logical foundation of (5) is not rejected. But it has not been shown that (8), (9), and (11) are due to a hypothesis whose elimination leaves (5) unaffected. We are therefore justified in rejecting the conjunction of level I hypotheses that yields (5) and in concluding that this rejection eliminates the logical basis for the existence of CS, and consequently CS is no longer one of our analytical tools.
Table 1. Some Test Results of Constraints

<table>
<thead>
<tr>
<th>Source</th>
<th>Functional Form</th>
<th>Country</th>
<th>Period</th>
<th>No. of Groups</th>
<th>Homogeneity</th>
<th>Symmetry</th>
<th>Negativity</th>
<th>Additivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Court (1967)</td>
<td>Log-linear</td>
<td>New Zealand</td>
<td>1950/1960</td>
<td>3</td>
<td>-</td>
<td>R</td>
<td>P</td>
<td>-</td>
</tr>
<tr>
<td>5. Lluch (1971a, 1971b)</td>
<td>Log-linear &amp; Rotterdam</td>
<td>Spain</td>
<td>1958 and 1964</td>
<td>5</td>
<td>R</td>
<td>pc</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>8. Theil (1975a)</td>
<td>Rotterdam</td>
<td>Netherlands</td>
<td>1922/1939, 1949/1963</td>
<td>4</td>
<td>-</td>
<td>P</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>9. Christensen, et al. (1975)</td>
<td>Direct and Indirect Translog</td>
<td>U.S.A.</td>
<td>1929/1972</td>
<td>3</td>
<td>-</td>
<td>R</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

a R denotes rejection, usually at 95 percent level, P denotes no rejection, - denotes no result reported.
b Data refer to expenditure on three types of meat products.
c When testing homogeneity and symmetry together, rejection is reported.
d For Germany, only four groups.
e Refers to a joint test of homogeneity and symmetry.
Rather than eliminating CS from economics, we may choose to assert (3) and (5) as basic postulates that are self-evidently true. This is an arbitrary and unappealing procedure. Its general application would mean that a hypothesis would never be rejected. If it were contradicted by experience, we would make experience irrelevant to judging it by asserting it to be self-evidently true and immune to rejection.

Or one may assert that we should reject the rejections and retain the levels I and II hypotheses, including the CS hypothesis. What arguments can be presented in support of this position? One of the weakest is that half of the tests of (8) through (11) failed to reject them and a 50 percent success ratio is high enough to justify continued use of (5). Is that what you were taught in school? That so long as half of the statistical tests failed to reject a hypothesis, you should not reject it? Is that what you teach your students? Suppose that you were about to do something important, like advise a government agency on construction of a dam or bridge, or build a tree-house for your children. Would you be satisfied to base your design on a set of physical laws that lead to hypotheses that had been rejected half of the times that they were tested?

Any test of a set of level II hypotheses uses auxiliary hypotheses. In the articles summarized in Table 1 the auxiliary hypotheses concerned functional forms of the equations and appropriate estimation procedures. You may argue that the rejections of the level II hypotheses were not due to their falsity but to errors in the auxiliary hypotheses, that the level II hypotheses were rejected because incorrect functional forms or
methods of estimation were used. Don't put much faith in this argument because it can easily be turned on its head to produce the argument that the reason that level II hypotheses were not rejected was that incorrect functional forms or methods of estimation were used.

You might argue that the wrong critical levels were used in the tests and if correct probability levels had been used, few of the level III hypotheses would have been rejected. It is true that the criteria we use in determining rejection or acceptance can depend upon the use to be made of a hypothesis and consequently upon the consequences of Type I and Type II errors. But one can make the same kind of argument to support the conclusion that use of correct critical levels would have lead to rejection of more of the level III hypotheses. Braithwaite (1953, p. 172) writes

"If the use of the hypothesis to us is purely intellectual (e.g., in the way of organizing hypotheses at a lower level into a more comprehensive deductive system than that in which they could otherwise be arranged), we shall wish to reject the hypothesis unless it fits the observed facts pretty closely. That is to say, we shall wish to reject the hypothesis if there is strong evidence against it even if there is quite a moderate probability of our rejecting it when it is true.... But suppose that the hypothesis, if it were true, would be of the greatest practical use to us. It might, for example, be a hypothesis which, if it were true, would enable a treatment to be developed for a disease for which no other treatment was known. We should then not wish to reject the hypothesis unless the evidence against it was overwhelming; we should not wish to reject it unless the probability was minute that we should be rejecting it if it was true...."

In the same vein, Ackoff (1979, p. 102) has written

"Most, if not all, scientific inquiries involve either testing hypotheses or estimating the values of variables. Both of these procedures necessarily involve balancing two types of errors.... Choice of a significance level involves a value judgment by the scientist about the relative seriousness of these
two types of error ... statistical significance is not a property of data or a hypothesis, but is a consequence of an implicit or explicit value judgment applied to them.

This means that two persons can be expected to weight the two types of error differently when differences in their values are concerned. Confronted by the same theory and data, their different value systems may lead them to reach different conclusions on rejection of a hypothesis. Thus one can reasonably take the position, e.g., that "The evidence supporting the consumer demand model is strong enough that I will use that model for some things, but the evidence in support of it is not strong enough for me to use it in studies intended to guide public policy. The consequence of error in the latter studies is so serious that I want strong confirmation before I am justified in using the model in studies of public policy."

The discussion of empirical evidence is incomplete because it considers only the studies summarized by Barten and it ignores the many studies that have presented demand equations consistent with the hypothesis that consumer purchases are affected by prices and income. I am not overlooking those studies but I do not know how to weight their evidence. Frequently the reported demand equation that is consistent with the hypothesis is one equation selected from a set or series of estimated equations several of which are rejected because they have "wrong signs" or "unexpected results," or are, in other words, inconsistent with some hypothesis. One can use Clogg's marvelously descriptive phrase to describe the use made of the theory of consumer behavior in these studies. The consumer theory is used to tell the
investigator "what basket of variables ought to be looked at" (Clogg, p. 471). The basket of variables contains income, retail prices, and perhaps some socio-economic variables. The finding that a few of the variables can be selected from the basket and manipulated to produce significant coefficients is not strong evidence in favor of the level I hypotheses that yielded the level II basket of variables hypothesis. The common practice of reporting only the "best equations", incidentally, raises a question that is relevant to the present discussion: "Does our empirical work determine the hypotheses that will be rejected, or does our faith in hypotheses determine the empirical results that will be rejected?

Actually, at issue here is a fundamental methodological question that most writers on scientific methods and philosophy of science do not treat satisfactorily. According to them, observation and empirical work provide homogeneous evidence. Tests either confirm or disconfirm a hypothesis. If the first, the hypothesis is not rejected, if the latter it is rejected. What about heterogeneous evidence, when some tests disconfirm and other tests confirm? How should we then choose whether to reject or not? And what do we do if two independent tests confirm a hypothesis but their conjunction refutes the hypothesis? See Salmon (1973) for discussion of such situations. Salmon also writes (p. 83)

"Empirical scientists have been making observations and performing experiments in order to test sophisticated hypotheses ever since the rise of modern science in the 16th and 17th centuries. When it comes to drawing conclusions from the results of these observations and experiments, we are far from having a clear understanding of the kind of reasoning involved. We are now in a situation analogous to
that of mathematics during the millennia in which mathematical proof was used often and with good results while the logic behind it remained basically mysterious. Current work in confirmation theory and inductive logic is attempting to remedy the situation."

Until the work on confirmation theory that will remedy the situation is finished, I guess we will have to muddle through by relying on our scientific taste, intuition, and common sense.

Also at issue here is the meaning of "rejection." "Rejected" can mean "rejected as being empirically falsified". It can also mean "rejected as not being the best theory available". The two are not synonyms. One may reject a theory in the first meaning but refuse to reject it in the second meaning. One may also reject a theory in the second meaning but refuse to reject it in the first meaning.

In any event, the main point I wish to make is this: In our present state of knowledge, the person who objects to use of consumer surplus to measure gain or utility because he believes that it rests on unverified or rejected assumptions does have a respectable amount of empirical work to support his position.

CS DEFINITIONS CONTAIN INTERNAL CONTRADICTIONS²

The contention of this section is that (a) a satisfactory definition of CS requires that three conditions be met before a comparison is made between consumer expenditures at a fixed price and consumer expenditures under price discrimination, and (b) the three necessary conditions are also impossible conditions, no more than two of the three conditions can be satisfied. This contention is certainly anti-traditional, and may even
be secular heresy. In order to establish the assertion, it is useful to start with the very foundation of the concept by listing its basic components. Because it is accepted here that the EE and EU definitions are equivalent, establishing the existence of an internal contradiction in one establishes its existence in both. The EE definition will be used.

Elements of A Satisfactory Definition

The components of an EE definition are 13 in number:

\[ P_c = \text{constant price charged consumers when price discrimination is not practiced}, \]
\[ T_c = \text{length of time that } P_c \text{ is charged}, \]
\[ Q_c = \text{quantity purchased by consumers during } T_c \text{ at } P_c, \]
\[ E_c = \text{total consumer expenditures on } Q_c \text{ during } T_c, \]
\[ P_m = \text{minimum price charged during period of price discrimination}, \]
\[ P_M = \text{maximum price charged during period of price discrimination, } P_M > P_m, \]
\[ T_d = \text{length of time period during which price discrimination is practiced}, \]
\[ Q_d = \text{quantity purchased by consumers during } T_d, \]
\[ E_d = \text{total consumer expenditures on } Q_d \text{ during } T_d, \]
\[ R_p = \text{a relation between } P_c \text{ and } P_m, \]
\[ R_T = \text{a relation between } T_c \text{ and } T_d, \]
\[ R_Q = \text{a relation between } Q_c \text{ and } Q_d, \]
\[ R_E = \text{a relation between } E_c \text{ and } E_d. \]
Most measurements of CS are derived from annual demand functions and are referred to as "annual CS". Because I want to relate theory to empirical work, I will deal with annual CS. My readings of the uses made of CS and of theoretical treatments lead me to conclude that a satisfactory definition of annual CS specifies

(12) \( p_m = p_c \)

(13) \( t_c = t_d = 1 \) year

(14) \( q_d = q_c \)

and computes \( EE = E_d - E_c \).

Why "should" a satisfactory definition satisfy (12), (13), and (14)? Part of the answer comes from the way CS is used. If \( P_m > P_c = P_m \), CS measures gain in welfare from purchasing \( Q_c \) at \( P_c \) instead of purchasing \( Q_c \) at prices above and equal to \( P_c \). Suppose \( P_m > P_c > P_m \). Then CS measures consumer's gain in welfare from purchasing \( Q_c \) at \( P_c \) instead of purchasing \( Q_c \) at prices above and below \( P_c \). The argument for rejecting \( P_m > P_c \) is mainly a matter of convenience and neatness. If \( P_m > P_c \), then CS measures gain in welfare from buying at \( P_c \) instead of above \( P_c \), and one must specify the size of the difference \( P_m - P_c \), and the size of CS varies with this difference. (My criticisms will still be valid if we allow \( P_m > P_c \).)

If (12) and (14) are satisfied and (13) is not (say \( t_d > t_c = 1 \) year), CS measures gain in welfare from consuming quantity \( Q_c \) in one year instead of having to spread consumption of \( Q_c \) over, say, 18 months. If this is a meaningful measure of annual welfare, then a meaningful measure of annual GNP is obtained by adding one year of consumer expenditures, 15 months of
government expenditures, and 18 months of investment expenditures. Evidence that (14) is satisfied in present definitions is clearly seen in expressions (5) and (6). The range of integration is zero to x and the second term in (6) equals expenditures on quantity x at the constant price $p_{ix}$. Suppose (12) and (13) are satisfied and (14) is not (say $Q_d < Q_c$). CS then measures consumers gain in welfare from purchasing $Q_c$ in one year at $P_c$ instead of purchasing the smaller quantity $Q_d$ in one year at higher prices. It measures the utility of goods whose consumption is foregone if price discrimination is practiced. It measures

$$\int_0^x u_i dq_i - \int_0^y u_i dq_i$$

where $x > y$.

The standard treatments of CS seem to satisfy these conditions. They all assume (12). Some writers explicitly assume (13) and (14); some explicitly assume one and implicitly assume the other; but all writers do assume (13) and (14). And they conclude that

$$EE = E_d - E_c > 0.$$ 

The assumptions of these standard treatments are mutually inconsistent because if (12) is satisfied, (13) and (14) cannot both be satisfied. This follows intuitively from the existence of a negative slope in the demand function: If (12) and (13) are satisfied, then satisfaction of (14) requires that $Q_c$ be purchased in one year at price $P_c$ and that an equal amount be purchased in a year when prices are set at and above $P_c$. 
But if price is set above $P_c$ for part of the year and at $P_c$ for the rest of the year, consumer purchases will be smaller than if price is set at $P_c$ all year. Then (14) is not satisfied.

This section will establish four propositions

(P1) If (12) and (13) are satisfied, then $Q_d < Q_c$.

(P2) If (12) and (14) are satisfied, then $T_d > T_c$.

(P3) If (12) and (13) are satisfied, then the area under the annual demand function exceeds annual expenditures on $Q_d$.

(P4) If (12) and (13) are satisfied, $E_d > E_c$.

(P1) and (P2) are not independent; they are alternative ways of describing the same thing. Because (P2) can be derived from (P1), (P2) can be established anytime that (P1) is established. A consequence of (P3) is that the area under an annual demand function between two prices overstates the annual EE.

No Contradiction If Demand Functions Vertical

Perhaps the easiest way to approach the discussion is to consider a situation in which (12), (13), and (14) can be satisfied. In this situation, the demand functions have an odd property. Panels 2.1 through 2.4 in Figure 2 present annual demand functions for four groups of consumers and panel 2.5 presents total annual demand of the five groups. Price $p_i$ is the "entry price" for group $i$ of consumers, it is the highest price at which group $i$ buys a finite amount of product; and $q_i$ is the quantity purchased annually by group $i$ at and below $p_i$, and $Q_h = \sum_{i=1}^{h} q_i$. If a perfectly discriminating seller facing the demand functions in
Figure 2 charges groups 1, 2, 3, and 4 the prices $p_1, p_2, p_3,$ and $p_4$ respectively, for one year, total annual purchases equal $Q_4$ and total annual expenditures equal $\sum_{i=1}^{4} p_i q_i$, which equals the area bounded by the line connecting points F-G-H-O in panel 2.5. Total quantity sold to groups 1, 2, and 3 at prices $p_1, p_2,$ and $p_3$ is $Q_3$ and their total expenditures on $Q_3$ equal the area bounded by F-G-Q-O. The area F-G-I equals the EE of consumers in groups 1, 2, and 3 on $Q_3$ in one year of price discrimination over their expenditures on $Q_3$ in one year when $p_4$ is charged all year.

Demonstration of Contradiction

We expect a consumer to increase purchases as price falls below his entry price. The demand functions in Figure 2 violate this expectation. But the demand functions in the first three panels of Figure 3 do satisfy this expectation. The annual demands for four groups of consumers in the first four panels of Figure 3 are constructed so that their total—in panel 2.5—is the same as the total in Figure 2. In Figure 3, as in Figure 2, $p_i$ is the annual entry price for group $i$. Other symbols are defined as follows

$$q_{i1} = \text{quantity purchased yearly at } p_i \text{ by consumer group } i,$$

$$q_{ij} = \text{quantity purchased annually at } p_i \text{ by group } j \text{ for } i > j,$$

$$\Delta q_{ij} = q_{ij} - q_{i-1,j}.$$

Thus, for example, $q_{11}$ equals quantity purchased in one year at $p_1$ by consumers in group one; $q_{21}$ equals quantity purchased in one year at $p_2$ by consumers in group one; and $\Delta q_{21} = q_{21} - q_{11}$. Note that $q_{ij} = 0$ for $i < j$. 

so that $q_{01} = q_{12} = q_{23} = q_{34} = 0$, and that $\Delta q_{ii} = q_{ii}$. Assume that all consumers in a group have exactly the same annual demand functions. Also assume that there is no seasonal variation in demand so that quantity sold in fraction $f$ of a year at price $p_1$ equals $f$ multiplied by the quantity sold in one year at price $p_1$.

If the seller facing the demands in Figure 3 charges $p_4$ to each group of consumers for one year, their total quantity of purchases equals $Q_4 = \Sigma q_{4i}$ and their total annual expenditures equal the area bounded by I-G-H-O in panel 3.5. Now suppose that the seller discriminates in exactly the same way that the seller in Figure 2 discriminated: by charging groups 1, 2, 3, and 4 the prices $p_1$, $p_2$, $p_3$, and $p_4$ respectively for one year. Then total annual volume of sales to all four groups is $\Sigma_{i=1}^4 q_{ii} = Q_2 < Q_4$. Thus, proposition (P1) is established.

Figure 3 shows another noteworthy inequality. Total annual volume of sales to groups (1), (2), and (3) equals $\Sigma_{i=1}^3 q_{ii} = Q_2 < Q_3$. Total annual expenditures by groups 1, 2, and 3 are $\Sigma_{i=1}^3 p_i q_{ii}$ = area bounded by F-K-L-M-N-Q_2-O < area bounded by F-G-Q_3-I. Also, total annual expenditures by all four groups equal area bounded by F-K-L-M-N-P-Q_2-O < area bounded by F-G-H-O. Thus, the area under the annual demand function between zero and $p_1$ exceeds total expenditures during a year of price discrimination. This demonstrates (P3).

The argument can easily be extended to more than four prices and more than four groups of consumers. Let there be $n$ groups and suppose that $p_n$ is charged to every group all year long. Then total quantity sold during the year to group $j$ of consumers equals
(15) $tq_j(n) = q_{nj} = q_{jj} + \sum_{i\neq j} \Delta q_{ij} = \sum_{i>j} \Delta q_{ij}$

and total quantity sold to all consumers during the year equals

(16) $\sum_{j=1}^{n} tq_j(n) = \sum_{j=1}^{n} (q_{jj} + \sum_{i>j} \Delta q_{ij}) = \sum_{j=1}^{n} \Delta q_{jj} = \sum_{j=1}^{n} q_{nj}$

Now assume that for each $j = 1, 2, \ldots, n$ a price $p_j$ is charged all year to group $j$ of consumers and $p_j > p_n$ for $j < n$. Then total quantity sold to consumers in group $j$ during the year equals

(17) $tq_j(j) = q_{jj} = tq_j(n) - \sum_{i>j} \Delta q_{ij}$

and

(18) $tq_j(j) < tq_j(n)$.

Total quantity sold to all consumers during this year of inter-personal price discrimination equals

(19) $\sum_{j=1}^{n} tq_j(j) = \sum_{j} q_{jj}$

And, from (18)

(20) $\sum_{j=1}^{n} tq_j(j) < \sum_{j} tq_j(n)$

$\sum_{j} tq_j(j)$ equals $Q_d$ and $\sum_{j} tq_j(n)$ equals $Q_c$. Expressions (18) and (20) therefore establish proposition (PI).

Intra-personal price discrimination, in which a seller charges the same customer group different prices at different times during the year, is also possible. A complete analysis must consider both inter- and intra-personal price discrimination. Assume that the seller practices perfect inter- and intra-personal price discrimination. Because the
time period being considered is one year, it follows that each price that is charged a group must be charged to the group for less than a year. For \( i \geq j \) let

\[
f_{ij} = \text{fraction of year that } p_i \text{ is charged to group } j;
\]

\[
1 > f_{ij} > 0; \sum_{i \geq j} f_{ij} = 1
\]

The discriminating seller charges different prices to a group of consumers at different times. He may also charge different prices to different groups at the same time.

If \( p_j \) is charged to group \( j \) for one year, \( \Delta q_{jj} \) is the amount purchased by the group during the year. But when \( p_j \) is charged group \( j \) for only \( f_{jj} \) of a year, the amount purchased by the group during \( f_{jj} \) of a year equals \( f_{jj} \Delta q_{jj} \). If \( p_{j+1} \) is charged group \( j \) for \( f_{j+1,j} \) fraction of a year, total purchases by the group during this period equal \( f_{j+1,j} \Delta q_{j+1,j} \). After \( p_j \) has been charged for \( f_{jj} \) of a year and \( f_{j+1,j} \) has been charged for \( f_{j+1,j} \) of a year, total quantity purchased amounts to \( f_{jj} \Delta q_{jj} + f_{j+1,j} \Delta q_{j+1,j} \). During an entire year of intra-personal price discrimination, total purchases by group \( j \) amount to

\[
tq_{j}(j,n) = \Delta q_{jj} \sum_{i \geq j} f_{ij} + \Delta q_{j+1,j} \sum_{i \geq j+1} f_{ij} + \cdots + \Delta q_{nj} f_{nj}
\]

Let \( w_{ij} \) be the coefficient of \( \Delta q_{ij} \) in this expression. Then

\[
(21) \quad tq_{j}(j,n) = \sum_{i \geq j} \Delta q_{ij} w_{ij}
\]

Now \( w_{jj} = \sum_{i \geq j} f_{ij} = 1 \) but for \( i > j \) \( w_{ij} = \sum_{i \geq j} f_{ij} < 1 \). Consequently it follows from (15) and (21) that

\[
(22) \quad tq_{j}(j,n) < tq_{j}(n)
\]

and hence proposition (P1) is established for intra-personal price discrimination. When intra-personal discrimination is practiced all year
on all groups of consumers, total quantity sold during the year to all consumers equals

\[(23) \sum_{j=1}^{n} t_q(j,n) = \sum_{j} \sum_{i} w_{ij} \Delta q_{ij}\]

It follows from (22) that

\[(24) \sum_{j} t_q(j,n) < \sum_{j} t_q(n)\]

and (P1) is demonstrated for the case of combined inter- and intra-personal price discrimination.

**Area A Biased Measure of CS**

It is worth studying further the situation in which (12) and (13) are satisfied and \( Q_d < Q_c \) because it casts light on the meaning (or rather lack thereof) of the area under the annual demand function. If \( p_n \) is charged all groups all year long, quantity sold to group \( j \) of consumers is \( t_q(j,n) \) in (15) and total annual expenditures by group \( j \) equal

\[(25) \ t_e(j,n) = p_n t_q(j,n) = p_n \sum_{i} w_{ij} \Delta q_{ij}\]

If intra-personal price discrimination is practiced on group \( j \), their total annual volume is \( t_q(j,n) \) in (21) and their total annual expenditures equal

\[(26) \ t_e(j,n) = \sum_{i} p_i w_{ij} \Delta q_{ij}\]

The total area under the annual demand curve of group \( j \) equals

\[(27) \ t_a(j,n) = p_j q_{jj} + p_{j+1} \Delta q_{j+1,j} + \cdots + p_n \Delta q_{nj} = \sum_{i} q_{ij} p_i \Delta q_{ij}\]
Because \( w_{ij} < 1 \) for \( i > j \), it follows from (26) and (27) that

\[
(28) \quad t_{aj}(j,n) > t_{ej}(j,n)
\]

That is, the area under each group's annual demand function exceeds the total expenditures of that group during a year of intra-personal price discrimination. If inter- and intra-personal price discrimination is practiced on all groups of consumers for one year, their total quantity purchased is (23) and their total annual expenditures equal, from (26)

\[
(29) \quad \sum_{j=1}^{n} t_{ej}(j,n) = \sum_{j=1}^{n} \sum_{i \geq j} p_{i} w_{ij} \Delta q_{ij}
\]

The total area under the aggregate (over all groups) annual demand function equals, from (27)

\[
(30) \quad TA(1,n) = \sum_{j=1}^{n} \sum_{i \geq j} p_{i} \Delta q_{ij} = \sum_{j} t_{aj}(j,n)
\]

And it follows from (28) that

\[
(31) \quad TA(1,n) > \sum_{j} t_{ej}(j,n)
\]

Hence, the area under the aggregate annual demand function exceeds the total annual expenditures of all consumers during a year of inter- and intra-personal price discrimination. Expressions (28) and (31) establish (P3). From (25) and (29) because \( w_{ij} < 1 \) for \( i > j \) and \( p_{i} > p_{n} \) for \( i < n \) it follows that

\[
\sum_{j} t_{ej}(n) > \sum_{j} t_{ej}(j,n)
\]

and

\[
\sum_{j} \sum_{j} t_{ej}(n) > \sum_{j} \sum_{j} t_{ej}(j,n).
\]
That is, total annual expenditures under price discrimination may exceed equal, or fall short of total annual expenditures without price discrimination.

\[ E_d \geq E_c \]

This is (P4).

From the four propositions it follows that the area under the annual aggregate demand function and above \( p_n \) is a biased measure of a flawed concept. The size of the bias cannot be determined. Its size equals the difference between (29) and (30). This difference depends upon values of \( f_{ij} \) and these values are arbitrary (except for \( 1 > f_{ij} > 0 \) and \( \sum_{i,j} f_{ij} = 1 \)). The introduction of the \( f_{ij} \) into the analysis was not, however, an arbitrary step that can be dispensed with. Their introduction was necessary in order to use annual demand functions to determine purchases at various prices during one year of price discrimination.

Cochrane (1980, p. 508) wrote, "The concept of economic surplus is, for me, an illusion created by an alternative purchasing procedure that is nonoperational." This section finds that even if we could make the alternative purchasing procedure operational, it is an illusion to think that the perfectly discriminating seller can sell in one year of price discrimination the same amount that he can sell without price discrimination and it is an illusion to think that the discriminating seller can collect in one year of price discrimination an amount of money equal to the area under the annual demand function.

For convenience my argument has proceeded on the assumption that each \( p_i \) is an entry price for some group and each group's demand function
has a horizontal step at every \( p_i \) at and below its entry price. These conditions can easily be relaxed. If \( p_i \) is not an entry price for any group of consumers, but a horizontal step occurs at \( p_i \) in some group's demand functions, set \( \Delta q_{ij} = 0 \) for all \( i \) and \( \Delta q_{ij} > 0 \) for all \( j \) for which a step occurs at \( p_i \). If \( p_i \) is an entry price for group \( i \) and no step occurs at \( p_i \) for any other group, set \( \Delta q_{ij} = 0 \) for \( j < i \).

**Willingness to Pay: Time Dimension**

The EE definition, in Boulding e.g., is in terms of total amount consumers would be "willing to pay". But "willingness to pay" must be qualified to mean "willing to pay if perfect price discrimination were practiced." See this qualification in italics in the material quoted previously from Boulding (1946, p. 640). The results presented here satisfy this definition because the maximum amount that a perfectly discriminating seller could squeeze out of his customers was determined. "Willingness to pay" must also have a quantity and time dimension. The price that a consumer is willing to pay for a specified quantity depends upon the length of time that quantity is expected to last him. And the price that a consumer is willing to pay in a given length of time depends upon the quantity supplied. If a consumer is rationed to three pounds of pork, the price he is willing to pay is higher if the three pounds is his ration for three months than if it is his ration for a week. And the price he is willing to pay for pork in one week is less if his ration is 10 pounds of pork in one week than if it is three pounds.

Conventional treatments of CS ignore the time dimension or the quantity dimension or both. They write, e.g., of "the price a consumer
is willing to pay rather than go without." Does "going without" mean
going without for a day, or for a week, or for a month? Does it mean
go without all or go without part? If all, how much is all? If part,
how big a part of how big an all? Actually, I should prefer to dispense
entirely with the phrase "willing to pay" or "willingness". In some
peoples' minds the expression takes on almost a mystical meaning that
turns every consumer purchase from GM or A&P into a combination of
business transaction and charitable contribution. Instead of interpreting
a demand function as showing what a consumer is "willing to do", interpret
it as showing what he does under various circumstances. Remember that
the theory of consumer behavior does not define the vector $q$ in the
utility function and the budget constraint as "quantities consumers are
willing to buy". (Utility depends on what you get, not on what you are
willing to get. If it depended on the latter only, scarcity would be a
much less serious problem than it is.) Consequently, the solution and
the quantity variables in the demand functions are quantities purchased,
not "quantities consumers are willing to purchase."

Key to the Contradiction

The difference between the traditional results on CS and my anti-
traditional results originates in different interpretations of an annual
demand function. Your decision on which argument to accept will be
determined by your choice of interpretation of an annual demand function.

Let us use the group's annual demand function shown in panel 1 in
Figure 3. Among the messages this function conveys to me are
(32.1) If $p_1$ is charged one year, quantity $q_{11}$ is purchased that year by this group of consumers. (The phrase "by this group of consumers" will not be repeated but must be understood to be a part of each message conveyed.)

(32.2) If $p_2$ is charged for one year, $q_{21}$ is purchased that year.

(32.3) If $p_3$ is charged for a year, $q_{31}$ is purchased that year.

(32a) If $p_3$ is charged for a year, $q_{31}$ is purchased that year.

It is a property of a demand function that

(32.8) Exactly one of the statements (32.1), (32.2), ..., (32a) ... is true in any one year.

Statement (32.3) is equivalent to the statement

(32.3a) If $p_3$ is charged for a while, then $p_3$ is charged for a while longer, and finally $p_3$ is charged for the rest of the year, then quantity $q_{11} + \Delta q_{21} + \Delta q_{31}$ is purchased that year.

One message the demand function conveys to protagonists of CS, apparently is

(33) If $p_1$ is charged for a time, then $p_2$ is charged, then $p_3$ is charged, the total quantity purchased at these prices is $q_{11} + \Delta q_{21} + \Delta q_{31}$.

An annual demand function cannot provide this information because such a demand function provides information during a specified length of time, whereas (33) describes behavior during some unspecified length of time.
The length of the time period that is implied by discussions of annual CS is a year. Thus the CS protagonists obtain from the annual demand function the message

(34) If $p_1$ is charged fraction $f_{11}$ of a year, $p_2$ is charged for fraction $f_{21}$, and $p_3$ for fraction $f_{31}$ ($f_{11} + f_{21} + f_{31} = 1$),
then $q_{11} + \Delta q_{21} + \Delta q_{31}$ is sold that year (where $q_{11} + \Delta q_{21} + \Delta q_{31}$ = quantity sold in one year that $p_3$).

Now (34) is not equivalent to (32.3a) and it is not equivalent to the conjunction of (32.1), (32.2), and (32.3). It is a contradiction of these. It is a contradiction unless the demand function is multi-valued so that each quantity is taken in a year at two or more yearly prices.

My study of economists' discussions of the data and procedures that they use to estimate annual demand functions and study of their estimated demand functions—whether obtained from aggregate data or individual household data, whether from time series or cross-section data, whether from secondary data, from samples, or from experimental designs—convince me that these demand functions are intended to, and do, convey the kind of information in (32.1), (32.2), ..., (32.a), (32.3a). This is the kind of information the authors intend for them to provide because that kind is consistent with demand theory. I cannot imagine how to design an experiment that will provide a demand having interpretations like (34). Such a function may be conceptually possible and such a design may exist. But as yet no one has developed such a design or estimated such a function, nor determined how to use the function in analyses other than CS analyses. How can such a demand function be used to predict annual purchases at one price?
Looking at some other information that (32.1), (32.2), and (32.3) convey to me may clarify my reasons for rejecting (34). Set \( f_{21} = \Delta q_{21}/q_{21} \).

If price is set at \( p_2 \) for a year, \( q_{21} \) is purchased. If price is set at \( p_2 \) for \( f_{21} \) of a year, quantity \( f_{21} q_{21} = \Delta q_{21} \) is purchased. (In panel 3.1, \( \Delta q_{21}/q_{21} = 0.5 \). If price is set at \( p_2 \) for half a year, the quantity purchased equals 0.5 \( q_{21} = \Delta q_{21} \).) Set \( f_{31} = \Delta q_{31}/q_{31} \). If price is set at \( p_3 \) for \( f_{31} \) fraction of a year, quantity purchased equals \( \Delta q_{31} \) at \( p_3 \). Therefore

(35) If price is set at \( p_1 \) for one year, then reduced to \( p_2 \) for \( f_{21} \) fraction of a year, then reduced to \( p_3 \) for \( f_{31} \) of a year, during the period of \((1 + f_{21} + f_{31})\) years the quantity purchased equals \( q_{11} + \Delta q_{21} + \Delta q_{31} \).

Note that (32.3a) and (35) simply restate proposition (P3) and are inconsistent with (34).

Let "orex" mean "or" in the exclusive sense of "one or another but not all" so that in a list of statements connected by "orex", exactly one is true. Then my argument can be briefly summarized: I maintain that the information conveyed by a demand function is

(35.1) If \( p_1 \) is charged, \( q_1 \) is purchased, orex

(35.2) If \( p_2 \) is charged, \( q_1 + \Delta q_{21} = q_{21} \) is purchased, orex

(35.3) If \( p_3 \) is charged, \( q_1 + \Delta q_{21} + \Delta q_{31} \) (= \( q_{31} \)) is purchased, orex

(35.4) If \( p_1 \) is charged, \( q_1 + \sum_{i>1} \Delta q_{i1} \) is
purchased,

orex

CS analyses interpret a demand function as telling something quite different. They interpret a demand function as saying:

(36.1) If $p_1$ is charged, $q_1$ is purchased,

and

(36.2) If $p_2$ is charged after $p_1$, $\Delta q_{21}$ additional is purchased,

and

\vdots

(36.1) If $p_1$ is charged next, $\Delta q_{21}$ additional is purchased,

and

\vdots

Your acceptance of the criticisms embodied in propositions (P1) through (P4) depends upon your acceptance of statements (35.1) as the proper interpretation of a demand function. If, however, you accept statements (36.1) as the proper interpretation, please answer this question for me. Assume the function in the first panel of Figure 3 is an annual demand function. If quantity $q_{41}$ is to be sold in one year and prices $p_1$, $p_2$, $p_3$, and $p_4$ are to be charged, for how long must each price be charged? Acceptance of statements (36.1) seems to interpret the annual demand function as a relation between quantity sold during a year and the minimum (and last) price charged that year.
Stepped Demand Function

Is it possible that the anti-traditional nature of my results is due to incorrect use of finite price and quantity changes along a stepped demand function instead of infinitesimal changes along a correct smooth demand function? The assumption that a stepped demand function is incorrect and a smooth demand function is correct is a wrong assumption. Smoothness of demand functions is not something that the world imposes on us; we impose it on the world for our convenience. A demand function is a stepped function because price is not a continuous variable. Price differences commonly amount to at least one cent. What about 3 cans for $1.00 versus 3 for 99¢. Even here the per-can price difference is finite: 1/3 cent. Small, certainly; but finite. The argument in this paper did not specify the differences between successive prices. We can assume them to be a quarter or even a tenth of a cent: small, but finite differences. Because price is not a continuous variable, a correct analysis can be carried out with a stepped demand function and finite differences.

This argument amounts to asserting that the integrals in (5) and (7) are approximations to the area under a stepped function. Whereas in integral calculus we learn that the area under a smooth function is accurately measured by an integral and can be approximated by the area under a stepped function; the argument here is the reverse of that. The integrals in (5) and (7) are approximations to the area under a stepped function.

If discontinuous price is not enough to convince you, observe that quantities are also discontinuous variables for most products. Milk,
beer, pop are purchased by the can or bottle or keg, not by the drop. 
Corn flakes are bought by the box, not by the flake, etc. Even 
products that come from the farm in continuous units are measured in 
discontinuous units. A lot of hogs may weigh 22,149.143+ε pounds, but 
that is not the way weight is reported. Number of bushels of grain is 
a continuous variable but weights and volumes of grain are measured in 
finite units.

Suppose that, for convenience, one chose to work with a smooth 
annual demand function. Then consider how (5) or (7) is applied. 
From the annual demand function; we compute $q_i$, $q_{i-1}$, and $dq_i$, where 
$q_i =$ annual volume of consumer purchases at $p_i$ 
$\frac{dq_i}{dq_{i-1}} = q_i - q_{i-1-\varepsilon} =$ excess of annual sales at price $p_i$ 
over annual sales at price $p_i - \varepsilon$.

If we are computing the amount of CS enjoyed by a consumer during one 
year then each of the various prices must be charged for less than one 
year. If each price is charged for a "month," we should not use values 
of $dq_i$ in the integral but should use, say, $dv_i$ where 
$v_i =$ volume of purchases during one month at price $p_i$ 
$dv_i = v_i = v_{i-\varepsilon}$ 
$p_i \ dv_i =$ money value of purchases in one month

Clearly $v_i$ and $dv_i$ cannot be read directly from an annual demand function. 
If the year has $m$ months, then $v_i = q_i/m$ and $dv_i = dq_i/m$. To obtain annual 
CS, we should use the monthly demands and evaluate 
$$\int_0^x p_i \ dv_i.$$
There remains the question: "What should be the value of x? Should it be the quantity purchased during 12 months of constant price or the smaller quantity purchased during a year of price discrimination?"

PS Definitions Also Internally Contradictory

Propositions (P1), (P2), and (P4), and a proposition analogous to (P3) can be derived for PS by redefinition of symbols used to study CS. $P_c$, $P_m$, and $P_M$ now represent prices paid to suppliers; $P_m$ and $P_M$ are maximum and minimum prices paid to suppliers during price discrimination. $Q_c$ and $Q_d$ represent quantities supplied, and $T_c$ and $T_d$ represent lengths of time periods that prices are received by sellers. $E_c$ and $E_d$ now represent amounts of money received by suppliers. A satisfactory definition satisfies (12), (13), and (14). Let

$p_i = \text{entry price of group } i \text{ of suppliers, i.e., the lowest price at which group } i \text{ supplies a finite amount;}$

$p_1 < p_2 < \cdots < p_n$

$q_{ij} = \text{quantity supplied annually at } p_i \text{ by group } j$

$f_{ij} = \text{fraction of year that } p_i \text{ is paid to group } j.$

Then the preceding analysis establishes (P1), (P2), (P4) and (P5).

(P5) If (12) and (13) are satisfied, then the area under the annual supply function exceeds annual receipts from $Q_d$. A consequence of (P5) is that the area above the annual supply function and below $p_n$ understates annual excess revenues.

Analysis of PS probably should allow seasonal variation in supply. This can be done, by making the notation and the algebra more complicated without affecting the conclusions. Simply introduce seasonal supply
functions and add a seasonal subscript. Then \( f_{ij} \) equals proportion of season \( s \) that \( p_i \) is paid to producers in group \( j \) and \( q_{ijs} \) equals the quantity supplied at \( p_i \) by group \( j \) in season \( s \).

**NONZERO PS OR NONZERO CS, BUT NOT BOTH**

The argument of this section supports the argument of the previous section—that areas or integrals obtained from annual demand and annual supply functions do not measure annual CS and PS—and also concludes that excess buyer receipts and excess consumer expenditures cannot exist simultaneously.

Notice what was done in the discussion of Figures 1 and 3. To derive the existence of CS it was assumed that (a1) sellers are price-making perfect discriminators and (b1) buyers are price takers. To derive the existence of PS, it was assumed that (a2) sellers are price takers and (b2) buyers are price making discriminators. Notice that assumptions (a1) and (a2) are contradictory, as are (b1) and (b2). In any one year in which (a1) and (b1) are satisfied, conditions (a2) and (b2) cannot be, and vice versa. It is impossible to satisfy the conditions that justify EE or EU and simultaneously to satisfy the conditions that justify the existence of excess receipts. We can have one or the other set of conditions satisfied in any one year, but cannot have both sets satisfied simultaneously. It is not legitimate to have either the buyer or the seller be simultaneously a price maker and a price taker. It is wicked methodology to assume contradictory things in the same argument.
It may be argued that (al), (a2), (bl), and (b2) are wrong and that the phrase "are willing to" should be inserted into each one between "sellers are" (or "buyers are") and "price." The modified (al) and (a2) are still contradictory, as are the modified (bl) and (b2). Can a consumer simultaneously be willing to act as a price-taking buyer and as a price discriminating buyer? A sensible consumer will be willing to behave as one or the other—whichever one he perceives as offering him the greatest advantage—but not in both ways. Likewise, a supplier will choose the role that offers him the greatest advantage and will not be willing to act in the other role. Therefore, under the "willing to be" assumption, as under the initial assumption, nonzero PS and nonzero CS cannot be extracted simultaneously.

VALUE JUDGMENTS

Income Distribution

Harberger (1971, p. 785) wrote, "When evaluating the net benefits or costs of a given action (project, program, or policy), the costs and benefits accruing to each member of the relevant group (e.g., a nation) should normally be added without regard to the individuals to whom they accrue." This is a value judgment, that sounds rather neutral and probably would be accepted by many. But some who accept this value judgment would probably change their minds if they learned that its application produces the same result as application of the value judgment that "the preferences of a wealthy consumer ought to have a greater influence on public policy than the preferences of a poor consumer."
This section argues that our present measures of CS do in fact contain this latter value judgment. It also argues that every measure of CS contains a value judgment as to whose preferences should carry more weight and whose should carry less, and consequently it is impossible to have a value-free measure of CS.

Assume each individual's demand function to be \( q = a_0 + a_1 y + a_2 p \) where \( q, p, \) and \( y \) are the individual's level of purchases, price paid, and income and \( a_1 > 0 \) and \( a_2 < 0 \). Let \( Z \) be the price at which \( q = 0 \). Then \( Z = -\frac{(a_0 + a_1 y)}{a_2} \).

Let \( \bar{p} < Z \) and CS = area under the demand function and above \( \bar{p} \). Then
\[
CS = (Z - \bar{p}) \frac{q}{2}
\]

(37) \[
CS = -\left(\frac{a_0 + a_1 y}{a_2} - \bar{p}\right) \left(\frac{a_0 + a_1 y + a_2 \bar{p}}{2}\right)
\]

Now consider a simple two-person economy consisting of a low-income consumer for whom \( y = m \) and a high income consumer for whom \( y = 3m \). The demand functions for these two consumers are then
\[
q = a_0 + a_1 m + a_2 p
\]
and
\[
q = a_0 + 3a_1 m + a_2 p.
\]
Suppose that we have also aggregated the market-data for this two-person economy and estimated a per capita demand curve. It will be
\[
q = a_0 + 2a_1 m + a_2 p.
\]
Now let CSL and CSH be consumer surplus
for the low and high income consumers and CSA be the per capita consumer surplus obtained from the per capita demand function. And let \( Z_L \) and \( Z_H \) be values of \( Z \) for the low and high income consumers, and let \( Z_A \) be the value of \( Z \) computed from the per capita demand function. Then

\[
\begin{align*}
Z_L &= - \left( a_0 + a_1 m \right) / a_2 \\
Z_A &= - \left( a_0 + 2a_1 m \right) / a_2 \\
Z_H &= - \left( a_0 + 3a_1 m \right) / a_2 \\
Z_H &> Z_A > Z_L
\end{align*}
\]

At any price above \( Z_L \) and below \( Z_A \), CSL = 0, CSA > 0, and CSH > 0; the low income consumer enjoys no consumer surplus but the high income consumer does and according to the per-capita demand curve, the "typical" consumer enjoys a consumer surplus. At any price above \( Z_L \) and below \( Z_A \) each person’s weight in the CSA is determined by his income. A rich consumer receives a positive weight and a poor consumer a zero weight. This same weighting pattern—more consideration given to a rich than to a poor consumer—also holds for prices below \( Z_L \). If we use (37) to compute CSL and CSH and subtract, we find

\[
(38) \quad \text{CSH} - \text{CSL} = - 4a_1 m \left( \tilde{q}_L + a_1 m / 2a_2 \right) > 0
\]

where \( \tilde{q}_L \) = value of \( q \) for a low income consumer at \( \tilde{p} \). If CSH and CSL are computed separately and added together, more than half of the economy's total CS comes from the richest half of the consumers because CSH > CSL.

Suppose we use the per capita demand equation to compute total CS as 2CSA. It can be shown that

\[
(39) \quad \text{CSH} + \text{CSL} + a_1^2 m^2 / a_2 = \text{CSA}
\]
Average CS computed from the two individual demand curves exceeds CS computed from the average demand curve by the amount $\frac{a_1^2}{2a_2}$. Total CS estimated as $2CS_A$ falls short of total CS by $\frac{a_1^2}{a_2} < 0$. Suppose that all of this difference is subtracted from $CS_H$. Then

$$CS_H + \frac{a_1^2}{a_2} = CS_L - \frac{2a_1^2m(2q_1^L + a_1^m)}{2a_2}$$

or

(40) \hspace{1cm} CS_H + \frac{a_1^2}{a_2} > CS_L

And again it turns out that more than half of the estimated $CS_A$ is contributed by the wealthy half of the consumers.

Whether you compute total CS as $CS_H + CS_L$ or as $2CS_A$, the high-income consumer carries more weight in your computation than the low-income consumer carries. In any study of a commodity having a positive income elasticity, these two measures of total CS assign greater weight to a high-income than to a low-income consumer. If you use either of these measures of CS in a study of a commodity having a positive income elasticity of demand, you are making the value judgment that the preferences of a wealthy consumer should count for more than the preferences of a poor consumer. My perceptions of economists' values lead me to believe that some economists have used measures of CS that violate their own systems of values. If the product under study has a negative income elasticity, of course, use of $2CS_A$ or of $CS_H + CS_L$ assigns higher weight to preferences of a low-income consumer. And if the product has a zero income elasticity, use of these measures carries the value judgment that a consumer with a large value of $a$ (whoever he may be) deserves greater
consideration than a consumer having a small value of a (whoever he may be).

The analysis can be carried farther to show that use of 2CSA or of CSH + CSL on a commodity having a positive income elasticity also carries with it the value judgment that as the nation's per capita income rises, the weight given to high income consumers ought to rise faster than the weight given to low income consumers. Suppose that each consumer's net income were 100\% percent higher so that the low and high incomes were \( m (1+\lambda) \) and \( 3m (1+\lambda) \), \( \lambda > 0 \). According to (38), where \( dm = \lambda m \),

\[
\frac{d(CSH - CSL)}{dm} = \frac{\partial (CSH - CSL)}{\partial m} \lambda m
\]

\[
= -\lambda \left( \frac{8q^2 m^2}{2a_2} \right) + \lambda (CSH - CSL) > 0
\]

And according to (40)

\[
\frac{d(CSH + a_1^2 m^2/a_2 - CSL)}{dm} = \frac{\partial (CSH + a_1^2 m^2/a_2)}{\partial m} - \frac{\partial CSL}{\partial m} = -2\lambda a_1 m \frac{\bar{q}_H}{a_2} > 0
\]

where \( \bar{q}_H \) = value of \( q \) at \( \bar{p} \) for a high-income consumer.

And what if we avoid these value judgments by using only low income consumer's demand functions to compute total CS as 2CSL? Then we are making the value judgments that every consumer ought to receive the same weight and the measure for every consumer should equal the CS of the low-income consumer. There is no value-free way to measure consumer surplus for a public policy decision. Every measure contains its own value judgment.

If economists are going to make value judgments, they should make
them consciously and not unconsciously. I would prefer, of course, that
they use my values. And mine would assign higher weight to low- than to
high-income consumers. If he won't use mine, perhaps he ought to use
those of the people expected to use the study or of the people financing
the study or of society in general. Whatever ones he uses, he ought to
describe them in the report of the research.

Harberger (1971) pleaded that the economics profession accept three
basic postulates "as providing a conventional framework for applied
welfare economics", i.e., a framework for use of producer and consumer
surplus. He argued (p. 795) that "the postulates can readily be used to
define a set of policies that characterizes a full optimum." Now,
optimum means no more than "optimum according to the criterion used."
Because every measure of CS involves a value judgment, Harberger's "full
optimum" policy is no more than optimum according to the value judgments
incorporated into the measure of CS. Different measures of CS, based
on different value judgments, would lead to different optima.

Do our measures of PS carry the implicit value judgment that each
producer's influence on public policy should be in proportion to his
ownership of specialized resources?

Well Informed Consumer

The use of CS to evaluate public policies carries with it another
value judgment: every consumer is the best judge of his own welfare,
and public policy ought to be based on that precept. I question whether
that value is fully accepted in any culture. It is not universally
accepted in North America where there are a number of laws that contradict
it, for example, laws limiting prescription drugs; laws prohibiting use of some drugs such as marijuana, cocaine, heroin; laws prohibiting prostitution; laws and public campaigns aimed at discouraging cigarette smoking. It is not even accepted by all U.S. economists.

Some economists welcomed Lancaster's approach to consumer demand (1966, 1971) because it provided an intellectual justification for their criticisms of other people's purchase patterns as irrational or inefficient. Scitovsky (1966, p. 47) wrote

"I should just like to say how much I welcome Professor Lancaster's new approach to the theory of consumer's demand. There is nothing more frustrating than to watch, and watch silently, the stupid way in which some people squander their money and get little to show for it.... At last I can look down my nose, without a pang of professional conscience, upon the sorry mess some people make of the noble art of spending money. Now I can respect the poor sucker's sovereignty and still criticize him for his inefficiency in catering to his own sovereign tastes."

The economists who agree with Scitovsky certainly do not accept the value judgment that every consumer is the best judge of his own welfare.

The model's underlying ideal of a well-informed consuming public is certainly not a universally held ideal among businessmen or among public employees, as Lindblom (1977, pp. 219, 220) points out.

"Leading corporations succeed in persuading consumers to buy automobiles with risk of carbon monoxide discharges into the interior of the passenger compartment; pesticides harmful to those who use them, children's toys with lead paint, cosmetics and other drugstore products that do not perform as claimed, foodstuffs with dangerous additives, all kinds of goods with hidden or deliberately misstated credit charges, costly life insurance policies that do not provide savings that customers think they are arranging for, flammable rugs and fabrics, and development houses that quickly require expensive repairs."
For many years, Consumers Union has been trying to compel the U.S. government to release the results of product tests carried out by U.S. purchasing agents in the course of buying products for government use. Its failure reveals the degree to which business rights to protect profitability—again, the privileged position of business—take precedence over consumers' rights to be informed about their market choices."

Businessmen and public employees are part of society and their values do, and should, affect public policy.

STANDARD ERRORS OF CS

Economists generally report their measures of CS as parameters rather than as what they actually are, estimates. I hazard a guess that many standard errors of estimated CS would be so large relative to estimated CS that economists would be reluctant to base policy recommendations on their estimated values of CS. For the linear demand equation,

\[ q = a_0 + a_1 y + a_2 p \]

the variance of CS can be estimated as

\[ \text{Var} (CS) = D' \Sigma D \]

where

\[ D = (\partial CS/\partial a_0, \partial CS/\partial a_1, \partial CS/\partial a_2) \]

\[ \partial CS/\partial a_0 = - \bar{q}/a_2 \]

\[ \partial CS/\partial a_1 = - y \bar{q}/a_2 \]

\[ \partial CS/\partial a_2 = - p \bar{q}/a_2 + \bar{q}^2/8a_2 \]

\[ \Sigma = \begin{pmatrix} s_{00} & s_{01} & s_{02} \\ s_{01} & s_{11} & s_{12} \\ s_{02} & s_{12} & s_{22} \end{pmatrix} \]

\[ s_{ii} = \text{variance of } a_i \text{ for } i = 0, 1, 2 \]

\[ s_{ij} = \text{covariance of } a_i \text{ and } a_j \]
ONE LINE OF DEFENSE FOR PS OR CS OR ANY ABSTRACT ARGUMENT, IS "THIS IS AN EXTREMELY COMPLEX TOPIC, AND ITS CRITICS DON'T UNDERSTAND IT. IF YOU COULD UNDERSTAND IT YOU WOULD BELIEVE IT." THIS WAS USED, E.G., BY HAHN (1973) TO DEFEND GENERAL EQUILIBRIUM THEORY. HE ACCUSED CRITICS OF GENERAL EQUILIBRIUM THEORY OF INABILITY "TO UNDERSTAND WHAT THE BEST MINDS IN THEIR SUBJECT ARE SAYING" AND WROTE "THE VULGARIZATIONS OF GENERAL EQUILIBRIUM WHICH ARE THE SUBSTANCE OF MOST TEXT-BOOKS OF ECONOMICS ARE BOTH SCIENTIFICALLY AND POLITICALLY HARMFUL." ONE THING THAT CAN BE SAID ON BEHALF OF A DEFENSE THAT ACCUSES CRITICS OF IGNORANCE OR INCOMPETENCE IS THAT IT HAS A LONG--THOUGH NOT, IT SEEMS TO ME, HONORABLE--TRADITION BEHIND IT. HUTCHISON (1977, PP. 162, 170) POINTS OUT THAT JAMES MILL USED THE SAME ARGUMENT IN 1836. AND IT WAS USED AT LEAST TWO MILLENNIA BEFORE THAT. IN PSALMS 92:6 APPEARS, "A BRUTISH MAN KNOWETH NOT; NEITHER DOTH A FOOL UNDERSTAND THIS."

THE MEANING OF A STATEMENT LIKE HAHN'S CLEARLY IS DETERMINED BY THE DEFINITION OF "BEST MINDS." (IT IS CLEAR THAT PEOPLE WRITING TEXT-BOOKS DID NOT SATISFY HAHN'S DEFINITION.) IT IS USUALLY CLEAR FROM THE PAPER THAT THE PAPER'S AUTHOR MEANS BY "BEST MINDS" THE "MINDS OF THOSE WHO BELIEVE THE ARGUMENT." ONES WHO DON'T BELIEVE THE ARGUMENT DON'T HAVE THE BEST MINDS. EVEN IF THEIR DEFINITION IS RELAXED TO PERMIT "BEST MINDS" TO MEAN "ANY MINDS WHO UNDERSTAND THIS ABSTRACT THEORY" WHETHER THEY BELIEVE OR NOT, WE DON'T HAVE TO ACCEPT THE AUTHOR'S DEFINITION. AND EVEN IF WE ACCEPT THIS DEFINITION WE STILL DON'T NEED TO EQUATE BEST MINDS WITH BEST SCIENTISTS. ONE MAY BELIEVE AS HEIMANN...
and Schumpeter *inter alia* do that a good scientist should be in close touch with reality. In discussing Ricardo's economics, Heimann (1945, p. 92) wrote "But we have since [Ricardo's time] come to recognize that good judgment may be more important than logical acumen." In evaluating Ricardo, Schumpeter (1954, p. 473) concludes that a famous Ricardian theory of profits "is an excellent theory that can never be refuted and lacks nothing save sense." In his evaluation of the dispute between Malthus and Ricardo, he wrote (p. 483) that Malthus "was throughout in the most unenviable position an economist can be in, namely, in the position of having to defend plain sense against another man's futile but clever pirouettes."

It's really rather amusing when you stop to think of it.

Psychologists use the word "psychotic" to describe people who are badly out of touch with reality. Hahn (and others) use the term "best minds in the profession" to describe economists who are badly out of touch with economic reality.

And, of course, Hahn's line of defense overlooks some rather obvious things. A person may be able to identify nonsense without understanding it. It may happen that occasionally the defense, "If you understood it you would believe it" is true. It may also happen that an honest response to the charge is, "I don't intend to waste my time trying to understand it. If you really understood the assumptions and realized how badly they violate reality, you would realize that its a waste of time to learn it." Another thing Hahn's attitude overlooks is that it may be impious, but it is not bad judgment, to refuse to worship in a cathedral built upon beach sand.
This paper certainly didn't end up as I expected it to. The findings in the sections on CS Definitions Contain Internal Contradictions, PS Definitions Internally Contradictory, and Nonzero PS or Nonzero CS But Not Both came as surprises to me.

These findings are embarrassing because in earlier reports my colleagues and I presented annual measures of CS. Ladd and Updegraff (1969) studied some dairy pricing policies and presented measures of compensating variation. Ladd and Fuleihan (1970) presented measures of compensating and equivalent variation for meats, beer, and dairy products.
FOOTNOTES

1/ I use "methodology" in the dictionary sense, as meaning "science of method, branch of logic dealing with principles of procedure" and not in the modern depreciated sense of "method or technique". By my definition, least squares and two stage least squares, e.g., are two methods and not two methodologies.

2/ I can hardly believe the conclusions reached in this and the next two sections. I've gone over the argument on many days in many ways and find no logical fault with it. The question that bothers me is "If the conclusions are correct, how come no one found them before?" If this section seems long and tedious, I ask your indulgence. The shocking, anti-traditional, nature of the results makes a thorough discussion desirable in order that a person may clearly understand my argument and results and their differences from the standard results on CS.
REFERENCES


