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Block Stanley Deompositions II. Greedy Algorithms, Applications and Open Problems

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BLOCK STANLEY DECOMPOSITIONS
II. GREEDY ALGORITHMS, APPLICATIONS, AND OPEN PROBLEMS

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ABSTRACT: Stanley decompositions are used in applied mathematics (dynamical systems) and $\mathfrak{sl}_2$ invariant theory as finite descriptions of the set of standard monomials of a monomial ideal. The block notation for Stanley decompositions has proved itself in this context as a shorter notation and one that is useful in formulating algorithms such as the "box product." Since the box product appears only in dynamical systems literature, we sketch its purpose and the role of block notation in this application. Then we present a greedy algorithm that produces incompressible block decompositions (called "organized") from the monomial ideal; these are desirable for their likely brevity. Several open problems are proposed. We also continue to simplify the statement of the Soleyman-Jahan condition for a Stanley decomposition to be prime (come from a prime filtration) and for a block decomposition to be subprime, and present a greedy algorithm to produce "stacked decompositions," which are subprime.

KEYWORDS AND PHRASES:

Geometry of monomial ideals
Simplest Stanley decompositions
Incompressible block decompositions
Algorithms
Organized decompositions
Stacked decompositions
Subalgebras
Hilbert bases
Algebraic relations
Classical invariant theory
Equivariants
Normal forms for dynamical systems
Prime filtrations
Soleyman-Jahan condition
Janet decompositions

1. **Introduction**

There are at least three reasons for taking an interest in Stanley decompositions:

1. **The Stanley conjecture that Stanley depth equals Hilbert depth.** Although this conjecture has been disproved in [8], it has has been suggested in [12] that there remain more subtle connections between the two notions of depth that deserve fuller investigation, so that this is not a time to disavow interest in the subject (although this question is not our concern here).
2. **The use of Stanley decompositions in the study of subalgebras**, either $A \subset K[\mathbf{x}] = K[x_1, \ldots, x_n]$ or, in some situations, $A \subset K[[\mathbf{x}]]$. The use of Stanley decompositions for this reason plays a role in invariant theory ([11], [23]) and normal forms for dynamical systems in the neighborhood of a rest point, both in the case of nilpotent linear part ([5],[6],[7],[16, Section 4.7],[15], [20], [17], [22, Chapter 12]) and semisimple linear part ([2], [16, pages 214-217]).
3. **Janet decompositions.** These are a special form of Stanley decompositions that were discovered prior to Gröbner bases, and are primarily used to study the solution of linear systems of ordinary or partial differential equations, via what are called Loewy decompositions. For partial differential equations, this application extends the algebras under consideration to differential algebras. Modern introductions to these topics are given in [1], [21] and [24].

The motivation for this paper and its predecessor ([19]) comes from item 2 above. Let $A \subset K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be a subalgebra of a polynomial algebra, $\mathcal{H}$ is a Hilbert basis for A, and let J be the (nonmonomial) ideal of relations among the elements of $\mathcal{H}$. Let I be the (monomial) ideal of leading terms (with respect to any Gröbner term order) of J. Then, as first shown by Sturmfels and White in [26], the standard monomials of I (monomials that are *not* in I) form a K-vector space basis for A; that is, each element of A can be written uniquely as a *standard polynomial* (a linear combination of standard monomials). A Stanley decomposition for K/I, or for A, provides a finite description of the set of standard monomials. The hardest part of such a calculation is the Gröbner basis work to get from J to I. We interpret this to mean that while $\mathcal{H}$ is insufficient information for unique expressions of elements of A (because of relations), the full ideal J of relations is too much information, and the exact right amount of information is given by a Stanley decomposition of A.

Karin Gatermann was a significant contributor to this "applied" use of Stanley decompositions through her book ([11]). In 2001, at a meeting of ISAAC in Berlin, the first author (JM) was present when Jan Sanders posed to Karin Gatermann the problem of finding the "simplest" Stanley decomposition in a given situation. (Of course this question is not limited to the "applied" use of Stanley decompositions to describe subalgebras.) The precise definition of "simplest" was not discussed, but it was clear from the context that the "size" of the Stanley decomposition was at the heart of the question, whether this "size" was measured by the number of characters in the symbol string, the number of Stanley spaces in the decomposition, or in some other way. This is the central question that we address in the present paper. Although Karin does not appear to have worked on this topic before her untimely death in 2005, we wish to dedicate this paper to her memory.

In [19] we made a start on Jan's question by defining *block decompositions*, in which a single rectangular block in Newton space may express several Stanley spaces, reducing, sometimes dramatically, the amount of space required to write a Stanley decomposition. The block notation focuses attention on the geometry of the set of standard monomials in Newton space. Disjoint unions of blocks correspond to direct sums of Stanley decompositions. Since blocks are given by matrices, they are easily handled by computers. Although at first sight a block decomposition is coarser than a Stanley decomposition, we interpret each block decomposition as a short notation for a specific Stanley decomposition easily obtained by Algorithm 2.2 in [19] (see Section 3 below). The question of "simplest Stanley decomposition" then passes to a question of simplest block decomposition, and for this we proposed the notion of incompressibility: a block decomposition is *incompressible* if it cannot be simplified further by combining some of its blocks to form larger blocks. An example in [19, §2] shows that incompressibility is a global property of a decomposition, and cannot be detected by examining the decomposition locally (for instance, two blocks at a time). Since incompressibility is difficult to detect in an arbitrary block decomposition, it is natural to seek an algorithm that produces incompressible block decompositions directly from I. The central result of the present paper is to give such an algorithm, which we call a *special organized decomposition* or $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-*organized decomposition*. A special organized decomposition is obtained by a "greedy algorithm" that first creates a (nonunique) "largest" block of standard monomials, and continues by choosing "largest" blocks that are disjoint from the blocks already created.

This paper falls loosely into three parts.

1. Sections 2 through 6 contain basic definitions, exploratory examples, and a discussion of the linkage between our papers on block decompositions and another series of papers (on "box products") by the first author. Some of this material is crucial for this paper, while other parts may be omitted without loss of continuity.
2. Sections 7 through 10 form the core of this paper; these contain the proof (Theorem 7.2) that *general organized decompositions* are incompressible, and present an (inefficient) procedure (Procedure 7.4) to produce general organized decompositions for the standard monomials of a monomial ideal, and the much more efficient, greedy algorithm (mentioned above) to produce special organized decompositions (Algorithm 10.2).
3. Finally, Sections 11-12 present a subordinate topic that the reader may omit. This continues the simplification of the Soleyman-Jahann condition for a block decomposition be prime (to come from a prime filtration), [25], that we began in [19], and gives a greedy algorithm for *stacked decompositions*, which are subprime.

Now we describe the individual sections briefly.

Section 2 is a brief historical introduction to the use of Stanley decompositions in the study of dynamical systems near a rest point, focusing on the recent idea of *box products*, first defined as such in [20] and [22, Chapter 12]), but having earlier roots. At about the same time that block notation was introduced in [19], one of us applied block notation to box products in [17]. We consider that [19] demonstrates the success of block notation in applications. We hope that the notion of box products will appeal to invariant theorists outside of the dynamical systems context. But this section may be omitted by a reader interested only in Stanley decompositions.

Section 3 reviews basic definitions from [19], improving some and adding others. Section 4 reviews and extends the notion of *elementary decomposition* from [19] and gives new lemmas that will be used later. These two sections are necessary for the rest of the paper.

Section 5 gives an easily pictured example of block decompositions in two dimensions. It gives six (general) organized decompositions for the standard monomials of the same ideal I, pointing out that two of these meet (at least) one of the requirement for being "special," while the other four do not. In doing so it motivates the work in Sections 8 and 9 below, for special decompositions. The example also shows that although *all* (general) organized decompositions are incompressible, and the special decompositions are the easiest of these to compute, the special decompositions do not necessarily minimize the number of Stanley spaces present in the block decomposition (another notion of simplicity for a Stanley decomposition). In this example, there is one organized decomposition, general but not special, that does minimize this number. Although useful for motivation, this section is not essential in the sequel.

Section 6 introduces *inner-minimal* block decompositions, in which the inner corners of the blocks are minimal in a specified sense. All decompositions created in this paper are inner-minimal, including all organized and stacked decompositions. Procedure 6.1 sketches a way to construct inner minimal decompositions. In this paper a **procedure** is a near-algorithm that falls short in some ways, such as being incompletely specified, or requiring choices, or not always terminating. The procedure in this section is not actually used, but serves as a framework around which later algorithms (that are truly algorithmic) are built. Example 6.5 illustrates difficulties that can arise in executing Procedure 6.1 and motivates a way to resolve one of them (collisions, that is, intersections, between blocks). Example 6.6 gives a counterexample (the "twisted cube") to a false conjecture about minimal inner corners that held us back for a while.

Section 7 defines (general) organized decompositions, which are both inner-minimal and outer-maximal (that is, the outer corners of the blocks satisfy a certain maximality condition). Theorem 7.2 proves that organized decompositions are incompressible, and Theorem 7.3 that each block in an organized decomposition is a union of elementary blocks. (This is essential for later algorithms.) Then it gives a procedure (which is almost algorithmic; it requires choices but always terminates) to produce (general) organized decompositions. This procedure is not efficient, and is still intended only as a step to the algorithms for special organized and stacked decompositions.

Sections 8 and 9 provide the techniques that allow true algorithms to be built for special organized decompositions. Specifically, Section 8 gives an algorithm for finding minimal inner corners for the blocks (a problem left over from Section 6), and 9 an algorithm for finding maximal outer corners for these blocks (a problem left over from Section 7).

Finally, Section 10 puts all of this together to give the algorithm for special organized decompositions, the principal result of this paper.

In [19] we simplified the Soleyman-Jahan condition for a Stanley decomposition to come from a prime filtration, adapted it a condition for block decompositions to come from what we called *subprime* filtrations, and proved that the gnomon block decomposition defined in that paper satisfies this condition. Section 11 gives yet other formulations of this condition. Section 12 gives a greedy algorithm, similar to the organized algorithm, that produces a type of block decomposition that we call *stacked*. These are always subprime, but not always compressed. (Sometimes these two conditions are incompatible.) The topics in these two sections are optional, and may be omitted.

As with [19], the authors are JM, a mathematician, and his son TM, a software engineer. The initial ideas of the greedy algorithm for organized decompositions, directional extension of blocks, lexicographic choice of inner corners, and the graph-theoretic criterion for subprimeness in Section 11, were due to TM. The idea for stacked decompositions, the notion of inner-minimal block construction, the use of the elementary decomposition to find minimal points, the proofs that organized decompositions are incompressible and stacked decompositions are subprime, and the final mathematical writing, were due to JM. Most of the examples and initial proof sketches (including many, not shown here, that guided the development) came from joint discussions.

For convenience we collect in one place the conventions governing the indices that will be introduced later.

1. n, r, and s are constants denoting, respectively, the dimension of Newton space $\mathbb{N}^n$, the unique minimal number of generators for the monomial ideal I, and the number of blocks in $\mathcal{D}$ (a block decomposition for the standard monomials of I).
2. h and i range from 1 to n and pick out a coordinate of a point or a face or edge of a block.
3. ℓ ranges from 1 to r and picks out a generator $\mathbf{m}^\ell$ of I.
4. j and k range from 1 to s and indicate blocks in a decomposition. Usually $j < k$.

Proofs are ended by open squares, remarks and examples by solid squares.

2. A digression on box products

As noted in the introduction, the box product idea has so far only appeared in the context of normal forms for dynamical systems near a rest point, in particular, a rest point with nilpotent nonlinear part. But the notion actually belongs to $\mathfrak{sl}_2$ invariant theory in greater generality. For easy cross-reference, this historical survey is presented as a sequence of remarks. The application of block notation to box products occurs at Remark 2.7 below, in connection with the paper [17], and demonstrates the utility of block notation in formulating algorithms for Stanley decompositions in the context of $\mathfrak{sl}_2$ representations. Some open questions are included in Remark 2.8.

2.1. **Remark. Overview of the box product.** Suppose representations of $\mathfrak{sl}_2$ are given on $K[\mathbf{x}] = K[x_1, \dots, x_n]$ and $K[\mathbf{y}] = K[y_1, \dots, y_m]$, where there is no overlap between the variables of $\mathbf{x}$ and $\mathbf{y}$. Let A_1 and A_2 be the subalgebras of seminvariants in $K[\mathbf{x}]$ and $K[\mathbf{y}]$ respectively, with associated Hilbert bases $\mathcal{H}_i$, ideals J_i of relations, I_i of leading terms of such relations (with respect to chosen term orders). (Covariants can be used in place of seminvariants, with only minor changes.) Classical invariant theory enables finding a Hilbert basis $\mathcal{H}$ for the seminvariants A of the tensor product of the two representations, using transvectants. But in order to find a Stanley decomposition$SD(A)$ for A, it is still necessary to do the Gröbner basis work to find J and pass from J to I. The goal of the box product (which is also built on transvectants) is to go directly from $SD(A_1)$ and $SD(A_2)$ to $SD(A)$; we write this (loosely) as $SD(A) = SD(A_1) \boxtimes SD(A_2)$, although additional information must be included in this notation to obtain a unique result for the box product. In other words, viewing Stanley decompositions as the "exact right amount of information" for representing elements of these subalgebras uniquely (see above), box products use this "right information" for the two inputs and the output, bypassing J and the Gröbner work. We have found two methods to do this, the *expansion method* and the *factoring method*; see below. ■

2.2. **Remark. The first stage.** Early work (prior to anything mentioned above) on invariants, equivariants, and normal forms for local dynamical systems (near a rest point) do not mention Stanley decompositions, but Stanley decompositions are nonetheless present implicitly. Typically one considered a nonlinear system of differential equations with rest point at the origin, written as $\dot{\mathbf{x}} = A\mathbf{x} + \cdots$, with a given linear term with matrix A (specified numerically) and arbitrary higher order terms. If $A = S$ is semisimple, the system is in normal form if the nonlinear terms are equivariant under the flow of the linear term; if $A = N$ is nilpotent, the system is in normal form if the nonlinear terms are equivariant under a *different* nilpotent linear linear flow, that of N^* (the conjugate transpose) in the *inner product normal form style* or under a linear flow determined via an $\mathfrak{sl}_2$ representation in the $\mathfrak{sl}_2$ *normal form style*; and if $A = S + N$ (a Jordan decomposition) the two approaches are combined. For each case, the systems that are in normal form can be characterized as containing only certain special nonlinear terms (thought of as terms that cannot be removed). This gives what is often called the *form of the normal form* for systems with a given (fixed) linear part. (To find the normal form of a specific system requires extra work to express the coefficients present in the "form of the normal form" as functions of the coefficients in the original system.) The statement of the *form of the normal form* is actually a (somewhat disguised) Stanley decomposition for the module of equivariants of the correct linear flow (for the style required) over the algebra of invariants of that same flow. See [16, Chapter 1] for easy examples done in this way. ■

2.3. **Remark. The second stage.** Once it was recognized (by Cushman and Sanders) that this was the case, it became possible, particularly in the nilpotent case using $\mathfrak{sl}_2$, that the required algebra of invariants and module of equivariants could be determined using ideas from classical invariant theory for representations of $\mathfrak{sl}_2$. (The *invariants* of the linear flow based on $\mathfrak{sl}_2$ are also *seminvariants* of the full $\mathfrak{sl}_2$ representation.) This marks the second stage in the development of these techniques. Key papers here are [5], [6], [7], [15, Section 4], and [13, Section 3]. Papers of the second stage typically make use of what we now call the "Cushman-Sanders test," which is a computation using aa two-variable Hilbert function (sometimes called the *table function*). A rigorous statement is given in [16, Lemma 4.7.9]: Given a subalgebra of the algebra of invariants, together with a Stanley decomposition of the subalgebra (from which the table function can be found), a "successful" Cushman-Sanders test proves that the subalgebra is in fact the full algebra of invariants and its Stanley decomposition is the correct one that we seek. An unsuccessful Cushman-Sanders test indicates that there are more invariants to be found (that are not in the given subalgebra), and gives indications about where (in terms of weight and degree) to look for them. Rigorous use of this test generally requires Gröbner basis calculations to find the leading terms of the relations among the Hilbert basis for the subalgebra, but Sanders does not illustrate these. Instead, he mostly used an informal version, in which the subalgebra is given and a set of known relations (obtained by inspection, and perhaps not complete) is also given. In this case a successful Cushman-Sanders test only shows that the subalgebra and (conjectured) Stanley decomposition are *probably* correct, on the grounds that it is unlikely that missing Hilbert basis elements and missing relations would exactly cancel to give a successful test. (It has not been determined whether such a cancellation is possible.) For a rigorous example, see [13, Section 3] for a difficult problem in which the Gröbner calculations are done explicitly, and the determination that the Cushman-Sanders test is successful relies on Zeilberger's algorithm. ■

2.4. **Remark. An infinite family of Stanley decompositions.** The problem solved by Malonza in Section 3 of his paper was to determine Stanley decompositions for the invariants of the entire family of nilpotent matrices having any number of two-by-two Jordan blocks (and no other Jordan blocks). Why is it possible to solve an infinite family with a finite amount of calculation? It is because the required Gröbner calculations fall into repetitive families, and as soon as enough Jordan blocks (of a given size) have been considered, all possible Gröbner calculations will already have been done. In the rest of Malonza's paper a Stanley decomposition for the equivariants (and thus the normal form) for the same family of dynamical systems is found using the boosting method in Remark 2.5 below. This problem had already been solved by a more complicated method ("covariants of special equivariants") in [7] and [6]. This method does not generalize well, and does not belong to the line of development leading to box products. See also Remark 2.8, item 3. ■

2.5. **Remark. The third stage: boosting; half way to the box product.** In [15, Section 5] I developed a technique for "boosting" a Stanley decomposition for the invariants of a nilpotent flow to an Stanley decomposition for the module of equivariants, without the extra work required by the method of covariants of special equivariants. This turned out later to be a special case of a box product. Malonza used this in [13, Section 4] to boost the Stanley decomposition for invariants of his problem to equivariants. ■

2.6. **Remark. The fourth stage: box products by the expansion method.** After the "boosting" procedure (from Stanley decompositions for invariants to Stanley decompositions for the module of equivariants) had been found (in [15]), Sanders pointed out to me that since boosting was based on tensor products, the method might be generalized to other situations involving tensor products of two $\mathfrak{sl}_2$ representations. As noted above in Remark 2.1, the box product of two Stanley decompositions for invariants of representations of $\mathfrak{sl}_2$ is a Stanley decomposition for their tensor product representation. In particular, the problem arising from a nilpotent matrix with two or more nilpotent Jordan blocks (of any size) falls into this category. We found that there were additional difficulties, not found in boosting, due to the fact that all of the representation spaces in the tensor product are infinite-dimensional. (In boosting, one of them was finite.) After working together on this, Sanders found an approach, the *expansion method*, that worked, although to me it seemed to work only formally, and to require justification. When I found a justification that satisfied me, we agreed that I would put my version in [20], and Sanders would put his in [22, Chapter 12], which we were writing at the same time. Each of these has valuable points that are not present in the other version. At this point the box product method absorbed the boosting method (which became unnecessary as a separate method). ■

2.7. **Remark. The fifth stage: box products by the factoring method, and the first application of block decompositions.** It turned out that there were some defects to the expansion method for box products, which did not lead to errors, but sometimes led to Stanley decompositions of an unexpected type: The monomials described by the Stanley decomposition were not always standard monomials of a monomial ideal (so these Stanley decompositions could not have arisen from an ideal of leading terms of relations in the usual way). Rather than having any advantages, these decompositions were also more complicated than necessary. Such an example arose in connection with the nilpotent matrix $N_{2,2,3}$ (where the subscripts indicate block sizes) in [20]), but we did not notice it at the time. I then formulated the *factoring method* for obtaining box products, [17]. The factoring method was better motivated than the expansion method, but more difficult to carry out; it amounted to propositional logic applied to integer inequalities–essentially, integer programming, but it did not seem to fit with any methods found in the integer programming literature. It was clear that this could, in principle, be made into an algorithm, since propositional logic is algorithmic (unlike first order logic), but the complexity was great enough that I had to handle each example separately with no uniform method. It happened that at the same time that I was working on [17] by myself, I was working with Theodore Murdock (TM) on [19], where TM had introduced the block notation as a more efficient way of writing Stanley decompositions, and one that could easily be handled by computers. Although very little from [19] beyond the definition of block decomposition was used in [17], the notation fully proved its worth in formulating the factoring method for box products algorithmically; it still seems to be nearly impossible to formulate some of the algorithms without this notation, because propositional logic without blocks has no natural connection to geometry in Newton space. (Although the algorithms in [17] have not yet been programmed, the examples in that paper were worked by hand, following the algorithms exactly.) This encourages us to think that block notation could be useful in other places where complicated calculations with Stanley decompositions might arise. ■

2.8. **Remark. Final remarks.**

1. We have adhered throughout this project to the use of classical methods that would be accessible to workers in dynamical systems, rather than modern methods using algebraic geometry, which obscure the concrete details in which we are interested.
2. It was observed in [20] that through the box product, we understand more about Stanley decompositions for high dimensional dynamical systems than we are able to use in studying their dynamics and its bifurcations. The applications of our results that we would hope for depend on a corresponding development in the techniques (such as blow-ups) that are used for this purpose. Therefore we do not expect any large quantity of citations of this work in the short term. It will be up to others to see what it is possible to do with these ideas.
3. Gachigua and Malonza ([10], [9]) have offered promising solutions to the infinite family of nilpotent matrices having any number of three-by-three Jordan blocks using box products by the expansion method. Their solutions satisfy the Cushman-Sanders test, but they do not completely explain the rules of formation for the diagrams that represent the solution. (These are not quite the same as the "maximal monotone path" diagrams that are used for this purpose in the two-by-two family.) So it seems fair to say that there are still open questions here, although the solution they have given is probably correct.
4. One of the original motivations for developing the factoring method was that it can be extended to covariants of the Lie algebra $\mathfrak{sl}_2^k$, while the expansion method cannot. This Lie algebra arises in several contexts: k commuting representations of SL_2; the covariants of certain multiforms in k pairs of variables; and the entangled states of k qubits in quantum computing, where the group is known as SLOCC (stochastic local operations with classical communication). See for instance [3]. In unpublished sketchy notes we have shown that an early form of the factoring method works in this context. (The expansion method does not work because it depends on the existence of certain least common multiples that do not exist in this situation.)
5. There also exists (in sketchy unpublished notes) a modification of the factoring method called the *method of tokens*. This applies whenever one or both of the given representations of $\mathfrak{sl}_2$ (on

$K[\mathbf{x}]$ or $K[\mathbf{y}]$) has more than one Hilbert basis element of the same weight. Each set of such Hilbert basis elements (even if they have different degrees) can be replaced by a single element called a *token* to create a simplified (but artificial) problem to be solved by the factoring method. It is then possible to "redeem" the token (replacing it by the original Hilbert basis elements) to obtain the solution of the actual problem. Crucial to this redemption process is Theorem 11.8 of [19]. The method of tokens has only been illustrated in specific examples, and we have not classified all the situations that can arise. To do so is an open problem.

6. We said in Remark 2.1 that a goal of this approach was to eliminate Gröbner calculations in connection with tensor product problems involving $\mathfrak{sl}_2$. One aspect of this goal has not yet been achieved, and remains an open problem: Although we can produce Stanley decompositions for the seminvariants of the tensor product representation, and we can prove that these are standard (in the sense that they describe standard monomials of a monomial ideal) we have not proved that this monomial ideal is the monomial ideal of leading terms of relations in $\mathcal{H}$ using Gröbner term orders. That is, the computation of a box product by the factoring method depends on a choice of *factoring order* for certain seminvariants that we call *primes*, while the computation by way of J depends on a choice of *Gröbner term orders* in the variables $\mathbf{x}$ and $\mathbf{y}$ (which do not appear at all in the box product calculation). It seems that it should be possible to match at least some of the possible factoring orders with Gröbner term orders, and show that these matched pairs lead to the same Stanley decompositions, but have made only slight and uncertain progress in that direction. We note that Sanders, in [22, Chapter 12], has given a method (which may well have been known classically) to obtain relations among transvectants that are also produced by Gröbner methods, but he has only illustrated his method rather than stated it precisely. Sanders' idea here would be our starting point in working on this problem, together with the notion of factoring order and the very important Theorem 11.8 of [17].
7. The work on box products in [17] does not use the idea of incompressibility of a block decomposition (the central concern of the present paper). It would be desirable to connect these further, perhaps by finding a box product algorithm that produces an incompressible output decomposition $SD(A)$ when the two inputs

$SD(A_1)$ and $SD(A_2)$ are incompressible. We have not worked on this question at all. ■

3. Definitions

Let K be a field of characteristic zero, and let $K[\mathbf{x}] = K[x_1, \dots, x_n]$. Monomials $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$ in $K[\mathbf{x}]$ will be represented by their exponent vectors $\mathbf{m} = (m_1, \dots, m_n)$ in the Newton space $\mathbb{N}^n$, where $\mathbb{N}$ is the set of nonnegative integers. For $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$, we say that **a divides b**, and write $\mathbf{a} \preceq \mathbf{b}$, if $\mathbf{x}^{\mathbf{a}}$ divides $\mathbf{x}^{\mathbf{b}}$, that is, if $a_i \leq b_i$ for $i = 1, \dots, n$. The statement $\mathbf{a} \prec \mathbf{b}$ means $\mathbf{a} \preceq \mathbf{b}$ and $a_i < b_i$ for some i. Bold Roman letters like $\mathbf{m}$, $\mathbf{a}$, $\mathbf{b}$, or $\mathbf{p}$ are always constants, denoting particular points in $\mathbb{N}^n$. For a variable point in $\mathbb{N}^n$ we use $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$. The **$i$-th direction** in $\mathbb{N}^n$ is the direction in which μ_i increases. A **coordinate hyperplane** in $\mathbb{N}^n$ is defined by $\mu_i = c$ for some constant c, and is perpendicular to the ith direction.

Let I be a monomial ideal in $K[\mathbf{x}]$ and let $G = \{\mathbf{m}^1, \dots, \mathbf{m}^r\}$ be the set of exponents of its unique minimal set of generators, so that

$$I = \langle \mathbf{x}^{\mathbf{m}^1}, \dots, \mathbf{x}^{\mathbf{m}^r} \rangle.$$

Here $\mathbf{m}^\ell = (m_1^\ell, \dots, m_n^\ell)$, with $\ell \in \{1, \dots, r\}$ being an index, not an exponent. We extend the notation $\langle\ \rangle$ to Newton space by writing

$$U = \langle G \rangle = \langle \mathbf{m}^1, \dots, \mathbf{m}^r \rangle = (\mathbf{m}^1 + \mathbb{N}^n) \cup \cdots \cup (\mathbf{m}^r + \mathbb{N}^n). \tag{3.1}$$

Then U is the set of exponents of monomials belonging to I, and it is an **upper set** in $\mathbb{N}^n$ under $\preceq$; that is, if $\mathbf{p} \in U$ and $\mathbf{p} \preceq \mathbf{q}$ then $\mathbf{q} \in U$. The complementary set

$$L = \mathbb{N}^n \setminus U \tag{3.2}$$

is the set of exponents of the **standard monomials** for I, that is, monomials that are **not** in I. This is a **lower set**; if $\mathbf{q} \in L$ and $\mathbf{p} \preceq \mathbf{q}$ then $\mathbf{p} \in L$. Throughout this paper, whenever one of the symbols G, U, or L is defined, all three are simultaneously defined and are related by (3.1) and (3.2). If $G = \emptyset$ then $U = \emptyset$ and $L = \mathbb{N}^n$. If G_1 and G_2 are two sets of generators and $G = G_1 \cup G_2$, then

$$U = U_1 \cup U_2 \quad \text{and} \quad L = L_1 \cap L_2. \tag{3.3}$$

An **interval** in $\mathbb{N}$ is denoted $[a, b]$ or $\begin{bmatrix} b \\ a \end{bmatrix}$, with $a, b \in \mathbb{N}^* = \mathbb{N} \cup \{\infty\}$; the interval is empty if $b < a$, or if $a = \infty$, and $[a, \infty]$ is understood as $[a, \infty)$, i.e. does not include ∞. A **block** $B \subseteq \mathbb{N}^n$ is a Cartesian product of n such intervals, represented by a $2 \times n$ matrix

$$(3.4) \qquad B = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} \times \cdots \times \begin{bmatrix} b_n \\ a_n \end{bmatrix}.$$

(The letter B denotes both the matrix and the set, according to context.) The rows $\mathbf{a}$ and $\mathbf{b}$ of B are elements of $N^{*n} = (\mathbb{N}^*)^n$; ∞ frequently occurs in $\mathbf{b}$, but rarely in $\mathbf{a}$, since this would make the set B empty. If the top and bottom entries in a column of B are equal, so that the interval represented by that column reduces to a point, the column may be replaced by a single entry on an intermediate level between the rows of the matrix. Two examples of this notation are

$$\begin{bmatrix} \infty & 3 & 7 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} \infty & & 7 \\ & 3 & \\ 0 & & 0 \end{bmatrix}$$

and the **singleton block** containing the point $\mathbf{a}$:

$$\{\mathbf{a}\} = \begin{bmatrix} \mathbf{a} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \end{bmatrix}.$$

3.1. **Remark.** It is sometimes useful to extend $\prec$ to N^{*n} in the obvious way (making ∞ larger than any finite integer). Then we may define U^* as the set of points in $\mathbb{N}^{*n}$ that are divisible by ($\succ$) an element of G, and $L^* = \mathbb{N}^{*n} \smallsetminus U^*$. This makes U^* and L^* into upper and lower sets in $\mathbb{N}^{*n}$. This usage is illustrated in Lemma 3.2 below. Any block $B \subseteq \mathbb{N}^n$ defines a possibly larger block $B^* \subseteq \mathbb{N}^{*n}$, which (for instance) contains its outer corner even if that corner lies at infinity. ∎

The **dimension** of a block B is the number of columns of the matrix B in which the top and bottom entries are unequal. (This is the dimension of the convex hull of B in $\mathbb{R}^n$.) The bottom row $\mathbf{a}$ of a block B is its **inner corner**, written $\mathbf{a} = \mathsf{IC}(B)$, and the top row $\mathbf{b} = \mathsf{OC}(B)$ will be called the **outer corner**, even though $\mathbf{b}$ is not an actual point of B if it has any infinite components. The intersection of two blocks is the block given by

$$(3.5) \qquad \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} \cap \begin{bmatrix} \mathbf{b}' \\ \mathbf{a}' \end{bmatrix} = \begin{bmatrix} \min\{\mathbf{b}, \mathbf{b}'\} \\ \max\{\mathbf{a}, \mathbf{a}'\} \end{bmatrix},$$

where the minimum and maximum are taken componentwise. The upper set generated by a single element $\mathbf{a} \in \mathbb{N}^n$ is a block given by

$$\langle \mathbf{a} \rangle = \begin{bmatrix} \infty \\ \mathbf{a} \end{bmatrix},$$

where $\boldsymbol{\infty} = (\infty, \ldots, \infty)$, and we have

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} \cap \langle \mathbf{a}' \rangle = \begin{bmatrix} \mathbf{b} \\ \max\{\mathbf{a}, \mathbf{a}'\} \end{bmatrix}. \tag{3.6}$$

3.2. **Lemma.** *A nonempty block B in $\mathbb{N}^n$ intersects an upper set U if and only if some element $\mathbf{m}^\ell$ of G divides its outer corner $\mathbf{b}$ ($\mathbf{m}^\ell \preceq \mathbf{b}$).*

Proof. Suppose first that $\mathbf{b}$ is a finite vector (i.e. does not have ∞ as an entry). Then, if any point $\mathbf{p} \in B$ belongs to U, so must $\mathbf{b}$, since $\mathbf{p} \preceq \mathbf{b}$ and U is an upper set. Therefore $\mathbf{b}$ is divisible by an element of G. If $\mathbf{b}$ is an infinite vector, the same argument applies using $\mathbf{b} \in U^*$ (Remark 3.1). In this case any finite points in U, obtained by replacing the infinities in $\mathbf{b}$ by large enough finite integers, will be divisible by the same element of G that divides $\mathbf{b}$. □

3.3. **Remark.** In [19] $\subset$ was used for subset ($\subseteq$). Here we distinguish these, so that $\subset$ means proper subset. In particular, for two blocks with the same inner corner $\mathbf{a}$ we have

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} \subset \begin{bmatrix} \mathbf{b}' \\ \mathbf{a} \end{bmatrix} \quad \text{if and only if } \mathbf{b} \prec \mathbf{b}'. \ \blacksquare$$

A block in which each b_i equals either a_i or ∞ is called a **Stanley block**. The span of a Stanley block (over K, in $K[\mathbf{x}]$) is a Stanley space in the usual sense; for instance

$$\operatorname{Span} \begin{bmatrix} \infty & & \infty \\ & 3 & \\ 2 & & 0 \end{bmatrix} = K[x_1, x_3] x_1^2 x_2^3;$$

the Stanley basis element is $\mathbf{x}^{\mathbf{a}} = \mathbf{x}^{(2,3,0)} = x_1^2 x_2^3$, and the coefficient ring $K[x_1, x_3]$ is generated by those x_i for which $b_i = \infty$. Any block can be written uniquely as the disjoint union of a minimal number of Stanley blocks, and the span of the original block is the direct sum of the associated Stanley spaces. See [19, Alg. 2.2].

Associated with any block B there are three important sets of "faces," the *inner faces, outer faces*, and *outer adjacent faces*; each face is itself a block. With B as in (3.4), the ith **inner face** (for $i = 1, \ldots, n$) is given by

$$\mathsf{IF}^i(B) = \begin{bmatrix} b_1 & \cdots & b_{i-1} & & b_{i+1} & \cdots & b_n \\ & & & a_i & & & \\ a_1 & \cdots & a_{i-1} & & a_{i+1} & \cdots & a_n \end{bmatrix}, \tag{3.7}$$

the ith **outer face** by

$$\mathsf{OF}^i(B) = \begin{bmatrix} b_1 & \cdots & b_{i-1} & & b_{i+1} & \cdots & b_n \\ & & & b_i & & & \\ a_1 & \cdots & a_{i-1} & & a_{i+1} & \cdots & a_n \end{bmatrix}, \tag{3.8}$$

and the ith **outer adjacent face** (which is not a subset of B) by

$$\mathsf{OAF}^i(B) = \begin{bmatrix} b_1 & \cdots & b_{i-1} & \multirow{2}{*}{$b_i + 1$} & b_{i+1} & \cdots & b_n \\ a_1 & \cdots & a_{i-1} & & a_{i+1} & \cdots & a_n \end{bmatrix}. \tag{3.9}$$

It follows from our definition of an interval that $\mathsf{OF}^i(B)$ and $\mathsf{OAF}^i(B)$ are empty if $b_i = \infty$, since this causes ∞ to occur in their bottom rows.

Let $h \in \{1, \ldots, n\}$. The hth **inner edge** $\mathsf{IE}^h(B)$ starts at $\mathbf{a}$ and runs along the block in direction h:

$$\mathsf{IE}^h(B) = \begin{bmatrix} a_1 & \ldots & a_{h-1} & \begin{matrix} b_h \\ a_h \end{matrix} & a_{h+1} & \ldots & a_n \end{bmatrix}. \tag{3.10}$$

The inner edges will be used only in Section 12. The following lemma is obvious from the definitions.

3.4. **Lemma.** *The edge $\mathsf{IE}^h(B)$ is the intersection of the inner faces $\mathsf{IF}^i(B)$ for $i \neq h$. The inner face $\mathsf{IF}^i(B)$ is the smallest block containing the edges $\mathsf{IE}^h(B)$ with $h \neq i$.*

Our main objects of study in this paper can be defined as follows, leaving some indicated details for later. Items 1 and 2 are defined for any $T \subseteq \mathbb{N}^n$. The rest are defined only for the case $T = L$ (a lower set).

1. A **partial block decomposition** of T is a finite ordered set $\mathcal{P}^k = (B^1, \ldots, B^k)$ of blocks in T that are pairwise disjoint.
2. A **block decomposition** $\mathcal{D}$ of T is a partial block decomposition $\mathcal{D} = \mathcal{P}^s = (B^1, \ldots, B^s)$ of T that is "total" in the sense that

   $$T = B^1 \sqcup \cdots \sqcup B^s$$

   Associated with $\mathcal{D}$ there is a sequence $\mathcal{P}^1, \ldots, \mathcal{P}^s$ of partial block decompositions $\mathcal{P}^k = (B^1, \ldots, B^k)$, as well as a filtration of T by subsets

   $$\emptyset = P^0 \subset P^1 \subset P^2 \subset \cdots \subset P^s = T.$$

   defined, for $k = 1, \ldots, s$, by

$$P^k = \bigsqcup \mathcal{P}^k = \bigsqcup (B^1, \ldots, B^k) = \bigsqcup_{j=1}^{k} B^j = B^1 \sqcup \cdots \sqcup B^k. \tag{3.11}$$

3. (For lower sets only.) An **inner-minimal** decomposition of a lower set L is a block decomposition of L in which each inner corner $\mathbf{a}^k$ of a block B^k is minimal (with respect to $\prec$) in the portion of L that has not been covered by previously constructed blocks. That is, $\mathbf{a}^k$ is minimal in $L \setminus P^{k-1}$. See Section 6 for details.

4. An **organized** decomposition (in the "general" sense) of L is an inner-minimal decomposition that also satisfies an outer-maximality condition stated in Section 7.
5. A **special organized** decomposition, or more specifically a **($\boldsymbol{\pi}, \boldsymbol{\sigma}$)-organized decomposition** of L is an organized decomposition in which the inner corners are determined by a lexicographic method described in Section 8, and the outer corners by a directional extension method described in Section 9. Here $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ are permutations of $(1, \ldots, n)$ which must be specified.
6. **Stacked decompositions** of lower sets are similar to organized decompositions but the outer-maximality conditions are stronger. See Section 12.

3.5. **Remark.** In [19], a block decomposition of L was defined to be the *equation* $L = B^1 \sqcup \cdots \sqcup B^s$ rather than the *ordered set* $\mathcal{D} = (B^1, \ldots, B^s)$. Thus a block decomposition was not a mathematical object with a name, but only an equation, perhaps with an equation number. This made it difficult to discuss several decompositions at once (for instance $\mathcal{D}_1, \ldots, \mathcal{D}_6$ in Section 5) below. ■

4. The unordered G-elementary decomposition of $\mathbb{N}^n$

Let G be a given set of generators for an upper set U, and let $L = \mathbb{N}^n \smallsetminus U$ as usual. For each fixed $\ell \in \{1, \ldots, r\}$, there exist n coordinate hyperplanes $\{\mu_i : \mu_i = m_i^\ell\}$ (one for each $i \in \{1, \ldots, n\}$) passing through the generator $\mathbf{m}^\ell \in G$. These hyperplanes, together with the n hyperplanes $\mu_i = 0$ passing through the origin, grid the entire Newton space $\mathbb{N}^n$ into "elementary blocks." But this rough definition fails to specify the elementary blocks exactly, since we must specify which faces belong to which blocks (to avoid intersections). So we begin again more formally. (See the beginning of Example 10.5 below.)

For $i = 1, \ldots, n$ let

$$S_i = \{0\} \cup \{m_i^\ell : \ell = 1, \ldots, r\} \subset \mathbb{N}. \tag{4.1}$$

Stated in words, S_i is the set of ith components of generators ℓ^m in G, with zero added in case it is not already present. For each i, the set S_i may be viewed as a set of points on the μ_i axis in $\mathbb{N}^n$, as long as these "points" are taken as integer *scalars* c and not integer vectors $c\mathbf{e}_i$. Although it may appear that S_i has either ℓ or $\ell + 1$ entries, this need not be the case, since several generators may have the same ith component. The set S_i can be ordered by the usual order $<$ for scalars, and if $c \in S_i$ we denote its **successor** under this ordering by $c^\sharp$. If c

is the last element of S_i, we put $c^\sharp = \infty$, but do not regard this as an element of S_i. Let $\mathbf{a} \in \mathbb{N}^n$ be a point such that $a_i \in S_i$ for $i = 1, \ldots, n$. Then the **G-elementary block** with inner corner $\mathbf{a}$ is the block

$$\begin{bmatrix} a_1^\sharp - 1 & \cdots & a_n^\sharp - 1 \\ a_1 & \cdots & a_n \end{bmatrix}. \tag{4.2}$$

Another way to say this is that a block $B = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix}$ is G-elementary if the following two conditions are satisfied:

1. Each component a_i of $\mathbf{a}$ belongs to S_i.
2. Each component b_i of $\mathbf{b}$ is such that $b_i + 1 = a_i^\sharp$. (If $b_i = \infty$, then $b_i + 1 = \infty$.)

It is obvious that each point of $\mathbb{N}^n$ belongs to one and only one elementary block; that is, the G-elementary blocks form what we will call the **unordered G-elementary block decomposition of** $\mathbb{N}^n$.

4.1. **Lemma.** *Any point in the ith inner face of any G-elementary block has its ith component in S_i.*

Proof. The ith inner face of (4.2) is

$$\begin{bmatrix} a_1^\sharp - 1 & \cdots & \multirow{2}{*}{a_i} & \cdots & a_n^\sharp - 1 \\ a_1 & \cdots & & \cdots & a_n \end{bmatrix}.$$

Any point of this set has ith component a_i, which (being the ith component of $\mathbf{a}$) must belong to S_i. □

4.2. **Lemma.** *Every G-elementary block $B = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$ belongs entirely to L or to U. Thus the sets L and U are (disjoint) unions of G-elementary blocks.*

Proof. Suppose that some G-elementary block B contains a point $\mathbf{p} \in L$ and a point $\mathbf{q} \in U$. Then some generator $\mathbf{m}^\ell \in G$ divides $\mathbf{q}$ (that is, $\mathbf{m}^\ell \preceq \mathbf{q}$). Therefore every component m_i^ℓ of this generator satisfies $m_i^\ell \leq q_i$. But no generator in G divides $\mathbf{p}$. In particular, $\mathbf{m}^\ell \not\preceq \mathbf{p}$. This means that for *some* value of i, $p_i < m_i^\ell$. For this choice of ℓ and i, then, we have $p_i < m_i^\ell \leq q_i$. So q_i is not the successor of p_i in S_i. But then B is not an elementary block, contradicting the supposition and proving the first statement in the Lemma. The second statement is an immediate consequence of the first. □

The **unordered G-elementary block decomposition of** L is the set of G-elementary blocks that lie in L, and similarly for U. This decomposition for L coincides with the one produced by Algorithm 3.1

of [19], except that step 6 of that algorithm imposes an order on the decomposition that was needed for a discussion on Section 5 of that paper.

4.3. **Example.** Consider $G = \{(3, 9), (7, 5)\}$, corresponding to equation (2.5) in [19]. An *ordered* G-elementary block decomposition for L, reproduced from [19], is shown in Figure 1 below. The *unordered* G-elementary block decomposition of L would delete the numbers showing the ordering. The unordered G-elementary decomposition of $\mathbb{N}^n$ would also add three G-elementary blocks lying in the upper right hand corner of the figure; for instance, the block to the right of block 5 in Figure 1 would be $\begin{bmatrix} \infty & 8 \\ 7 & 5 \end{bmatrix}$. The three extra blocks constitute the unordered G-elementary block decomposition of U. ■

The expression "disjoint union of G-elementary is actually redundant, since any two distinct G-elementary blocks (with the same G) are automatically disjoint.

4.4. **Corollary.** *A block* $B = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix}$ *is a union of (one or more) G-elementary blocks if and only if each component a_i of* $\mathbf{a}$ *belongs to S_i and each finite component b_i of* $\mathbf{b}$ *satisfies $b_i + 1 \in S_i$.*

Proof. Suppose that each a_i belongs to S_i and each finite b_i satisfies $b_i + 1 \in S_i$. The first of these is the same as item 1 in the definition of G-elementary block (above), but the second is weaker than item 2 (which requires $b_i + 1 = a_i^\sharp$). However, it is not much weaker; $b_i + 1$ must still come after a_i in the ordering $<$ on S_i. Therefore $b_i + 1$ must be one of the integers $a_i^\sharp$, $a_i^{\sharp\sharp}$, $a_i^{\sharp\sharp\sharp}$, ... Therefore, in any direction i, B may encompass more than one G-elementary block before reaching its (finite or infinite) boundary. Conversely, if B is both a block and a union of G-elementary blocks then $\mathbf{a}$ is the inner corner of the innermost of these elementary block, and $\mathbf{b}$ the outer corner of the outermost elementary block. The conditions on $\mathbf{a}$ and $\mathbf{b}$ specified in the corollary then follow from the definition of an elementary block. □

The next corollary is trivial but plays a crucial role later.

4.5. **Corollary.** *Let $Y \subset X \subset \mathbb{N}^n$ and suppose X and Y are unions of G-elementary blocks.. Then $X \setminus Y$ is also a union of G-elementary blocks. Any G-elementary block in X is entirely contained in either X or Y.*

In the sequel, when G is understood we often write "elementary block" to mean G-elementary block.

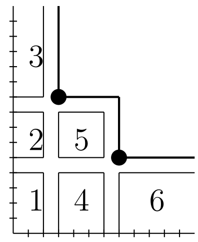

FIGURE 1. The elementary decomposition for Example 4.3 and Section 5.

5. An example in two dimensions

Let $I = \langle x^3y^9, x^7y^5\rangle$, so that $G = ((3,9),(7,5))$. Three block decompositions of L were given in [19], equations (2.6)-(2.8). The elementary decomposition, [19, (2.6)], is illustrated in Figure 1, taken from [19, §3].

As noted in [19, §3], elementary block decompositions are generally compressible, but when forming incompressible decompositions we "never need to consider blocks ... that are not disjoint unions of elementary blocks." We illustrate six ways (among others) of forming "organized" decompositions this way. The six fall into three groups, as follows.

1. Combine 1, 4, and 6 to form B^1.
   a. Combine 2 and 5 to form B^2 and use 3 for B^3.

$$\mathcal{D}_1 = \left(\begin{bmatrix} \infty & 4 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 8 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 2 & \infty \\ 0 & 9 \end{bmatrix} \right).$$

   b. Combine 2 and 3 to form B^2 and use 5 for B^3.

$$\mathcal{D}_2 = \left(\begin{bmatrix} \infty & 4 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & \infty \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 6 & 8 \\ 3 & 5 \end{bmatrix} \right).$$

   This is shown in Figure 4, Section 11.

2. Combine 1, 2, 4, and 5 to form B^1.
   a. Take 3 for B^2 and 6 for B^3.

$$\mathcal{D}_3 = \left(\begin{bmatrix} 6 & 8 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & \infty \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} \infty & 4 \\ 7 & 0 \end{bmatrix} \right).$$

   b. Take 6 for B^2 and 3 for B^3.

$$\mathcal{D}_4 = \left(\begin{bmatrix} 6 & 8 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \infty & 4 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 2 & \infty \\ 0 & 9 \end{bmatrix} \right).$$

3. Combine 1, 2, and 3 to form B^1.

a. Combine 4 and 5 to form B^2 and take 6 for B^3.

$$\mathcal{D}_5 = \left(\begin{bmatrix} 2 & \infty \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 8 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} \infty & 4 \\ 7 & 0 \end{bmatrix} \right).$$

b. Combine 4 and 6 to form B^2 and take 5 for B^3.

$$\mathcal{D}_6 = \left(\begin{bmatrix} 2 & \infty \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \infty & 4, \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 8 \\ 3 & 5 \end{bmatrix} \right).$$

Decompositions $\mathcal{D}_1$ and $\mathcal{D}_5$ share a property that the others do not have: The blocks can be build by *directional extension with a fixed order of directions*. To understand this, consider B^1 in $\mathcal{D}_1$. This block can be constructed by a sequence of extensions $B_0^1 \subset B_1^1 \subset B_2^1 = B^1$ as follows:

1. Form the singleton block $B_0^1 = \{(0,0)\}$ containing only the inner corner.
2. Slide the point $(0,0)$ to the right along the μ_1 axis as far as it can go without entering U. In this instance it can be slid to infinity, making B_1^1 equal to the μ_1 axis.
3. Now extend B_1^1 upward (in the μ_2 direction) by sliding the μ_1 axis upward until it collides with U at the point $(7,5)$, one of the generators of U. Backing off from this collision leaves the line $\mu_2 = 4$ as the top of the extended block B_2^1.
4. There are no more directions in which the block can be extended, so we are finished, and $B^1 = B_2^1$.

The same process, using the directions (1,2) in the same order, also works to construct B^2 and B^3 in $\mathcal{D}_1$. For $\mathcal{D}_5$, the directions must be used in the order $(2,1)$, and for $\mathcal{D}_3$ the method fails altogether for B^1 because sliding a point along an axis will never cause it to stop at $\mu_1 = 6$ or $\mu_2 = 4$. This makes $\mathcal{D}_3$ is "organized" but not " special organized."

Now we turn briefly to the topic of Stanley space minimization. Each of the decompositions $\mathcal{D}_1, \ldots, \mathcal{D}_6$ contains two unbounded blocks (with ∞ in the top row) and one bounded block. For instance, block B^3 in $\mathcal{D}_2$ is bounded and has 16 points, with inner corner $(3,5)$ corresponding to monomial $x_1^3 x_2^5$, and outer corner $(5,8)$ or $x_1^5 x_2^8$, and has 16 one-dimensional Stanley spaces:

$$\mathrm{Span}(B^3) = K x_1^3 x_2^5 \oplus \cdots \oplus K x_1^5 x_2^8.$$

But the unbounded block B^2 in $\mathcal{D}_2$ has only three Stanley spaces, each infinite-dimensional, namely

$$\mathrm{Span}(B^2) = K[x_2] x_2^5 \oplus K[x_2] x_1 x_2^5 \oplus K[x_2] x_1^2 x_2^5.$$

	B^1	B^2	B^3	total
$\mathcal{D}_1$	5	28	3	36
$\mathcal{D}_2$	5	3	16	24
$\mathcal{D}_3$	63	3	5	71
$\mathcal{D}_4$	63	5	3	71
$\mathcal{D}_5$	3	36	5	44
$\mathcal{D}_6$	3	5	16	24

TABLE 1. Stanley space count for $\mathcal{D}_1$ through $\mathcal{D}_6$.

Table 1 gives the number of Stanley spaces for each block of each decomposition. The unbounded blocks are more efficient (that is, require fewer Stanley spaces), since each Stanley space represents infinitely many points of the block. The most efficient block decompositions are $\mathcal{D}_2$ and $\mathcal{D}_6$, because these combine the bounded blocks labeled 1, 2, and 4 in Figure 1 with the unbounded blocks 3 and 6, leaving only the bounded block 5 with its 16 spaces (shown above as $\mathrm{Span}(B^3)$ of $\mathcal{D}_2$). These two decompositions, $\mathcal{D}_2$ and $\mathcal{D}_6$, are not built by directional extension, showing that the algorithm we develop for $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-organized decomposition cannot be expected to minimize the number of Stanley spaces.

6. INNER-MINIMAL BLOCK DECOMPOSITIONS OF LOWER SETS L

An *inner-minimal block decomposition* of a lower set L has been defined in item 3 of Section 3. To state this definition in greater detail, a **minimal point** of a subset $T \subseteq \mathbb{N}^n$ is a point $\mathbf{p} \in T$ such that no $\mathbf{q} \in T$ satisfies $\mathbf{q} \prec \mathbf{p}$ (or equivalently, divides $\mathbf{p}$). If T has more than one minimal point, these will be incomparable under $\prec$. Now let G, U, and L be as usual, and suppose that $\mathcal{D} = \mathcal{P}^s$ is a block decomposition of L. We say that $\mathcal{D}$ is an **inner-minimal** block decomposition of L if for each $k = 1, \ldots, s$, $\mathbf{a}^k$ is a minimal point of $L \smallsetminus P^{k-1}$.

Since P^{k-1} is the part of L that has already been covered by the blocks B^j with $j < k$, it is easy to apply this definition during the construction of $\mathcal{D}$. That is, suppose that $\mathcal{P}^{k-1}$ has been constructed, and we are ready to define $B^k = \begin{bmatrix} \mathbf{a}^k \\ \mathbf{b}^k \end{bmatrix}$. Let

$$\overline{P^{(k-1)}} = L \smallsetminus P^{k-1}$$

(this is the part of L that has not yet been covered by blocks), and let M^{k-1} be the set of minimal points of $\overline{P^{(k-1)}}$. In order that $\mathcal{D}$ be inner minimal, we must choose

$$\mathbf{a}^k \in M^{k-1},$$

and then choose $\mathbf{b}^k$ so that B^k does not intersect either P^{k-1} or U. This rule must be applied at each stage k in the construction. In particular, at the first stage ($k = 1$) we must have $P^{k-1} = P^0 = \emptyset$, $\overline{P^0} = L$, $M^{k-1} = \{\mathbf{0}\}$, and $\mathbf{a}^1 = \mathbf{0}$.

The following procedure is not fully specified (because methods are not obvious for some of the steps) and does not always terminate. Its purpose is to outline a strategy that will be followed (with modifications) in later procedures and algorithms for organized, special organized, and stacked decompositions.

6.1. **Procedure.** *To create an inner-minimal block decomposition of a lower set $L \subset \mathbb{N}^n$:*

1. *Put $P^0 = \emptyset$ and $M^0 = \{\mathbf{0}\}$.*
2. *For $k = 1, 2, \ldots,$ do the following:*
   a. *Choose a point $\mathbf{a}^k \in M^{k-1}$ to be the inner corner of the next block.*
   b. *Create B^k by choosing its outer corner $\mathbf{b}^k$ so that B^k does not intersect U or any previously constructed blocks.*
   c. *Set $\mathcal{P}^k = (B^1, \ldots, B^k)$ and $P^k = \bigsqcup \mathcal{P}^k$, as in (3.11).*
   d. *Find M^k.*
3. *Stop if $P^k = L$ and set $s = k$. (This may not terminate; for instance, it does not terminate if L is infinite and each $\mathbf{b}^k$ is chosen to equal $\mathbf{a}^k$, so that each block is a singleton.)*

For this procedure to become an algorithm, methods must be specified for steps 2a, 2b, and 2d. These will be provided later in specific situations. Note that item 2b implies that the procedure is (as yet) by no means "greedy," as promised in the introduction; there is nothing here to force each block to be (in some sense) "as large as possible." This will be done in different ways for organized and stacked decompositions, and doing so will cause the procedure to terminate.

We begin with an attempt to address 2d. It is easy to handle the first case, M^1.

6.2. **Lemma.** *Let B^1 be any block in $\mathbb{N}^n$ with inner corner $\mathbf{a}^1 = \mathbf{0}$. The set of minimal points of $\mathbb{N}^n \setminus B^1$ is the set of* ICOAFs *(inner corners of outer adjacent faces) of B^1.*

Proof. Any point of $\mathbb{N}^n$ that is not in an outer adjacent face of B^1 cannot be minimal because one of its components can be decreased by 1 without causing the point to enter B^1. Any point that lies in an outer adjacent face cannot be minimal unless it is minimal in the outer adjacent face; that is, it must be the inner corner of the outer adjacent face. The inner corner of any $\mathsf{OAF}^i(B^1)$ is minimal, because if its ith component is reduced by one, the point either enters B^1, while if any other component is reduced by one, that component becomes negative and the point leaves $\mathbb{N}^n$. □

6.3. **Example.** We give two slightly unusual examples of Lemma 6.2.

(i) A two-dimensional block in $L = \mathbb{N}^3$: The outer adjacent faces of $B^1 = \begin{bmatrix} 5 & 8 & \\ 0 & 0 & 0 \end{bmatrix}$ are $\begin{bmatrix} & 8 & \\ 6 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 5 & & \\ 0 & 9 & 0 \end{bmatrix}$, and $\begin{bmatrix} 5 & 8 & \\ 0 & 0 & 1 \end{bmatrix}$, so $M^1 = \{(6,0,0), (0,9,0), (0,0,1)\}$.

(ii) An unbounded block: For $B^1 = \begin{bmatrix} 2 & \infty & 4 \\ 0 & 0 & 0 \end{bmatrix}$ there are two nonempty OAFs, and $M^1 = \{(3,0,0), (0,0,5)\}$. ■

6.4. **Corollary.** *If L is a lower set in $\mathbb{N}^n$ and B^1 is a block in L with inner corner $\mathbf{a}^1 = \mathbf{0}$, M^1 is the set of* ICOAFs *of B^1 that lie in L.*

After the first block, things become more complicated. Example 6.5 below shows that ICOAFs still play a role in M^k, but an ICOAF that arises at one stage k may not become minimal until a later stage. This example also illustrates the possibility of "collisions" between blocks; prevention of such collisions is needed for an algorithmic version of the procedure.) Example 6.6 shows that there can exist minimal points that are not ICOAFs. (We had conjectured for a while that this could not happen.)

6.5. **Example.** Figure 2 illustrates the following partial block decomposition of $L = \mathbb{N}^2$.

$$\mathcal{P}^6 = \left(\begin{bmatrix} 3 & 11 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 5 \\ 8 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 10 & 13 \\ 4 & 9 \end{bmatrix}, \begin{bmatrix} 13 & 10 \\ 11 & 0 \end{bmatrix} \right).$$

The steps of the construction are as follows.

1. Initially $M^0 = \{(0,0)\}$. The first block B^1 covers this point, removing it from minimality, and replaces it with ICOAFs $(0,12)$ and $(4,0)$ to produce $M^1 = \{(0,12), (4,0)\}$.
2. Block B^2 removes $(0,4)$ from minimality and replaces it with ICOAFs $(4,3)$ and $(8,0)$. But B^2 does not remove $(0,12)$, which survives as a minimal point, so that $M^2 = \{(0,12), (4,3), (8,0)\}$.
3. Adding B^3 removes $(8,0)$ and adds its ICOAF $(11,0)$ to M^3, but the ICOAF $(8,6)$ is not a minimal point, because $(4,3) \prec (8,6)$. (We are not finished with $(8,6)$; it becomes a minimal point at the next stage). We have $M^3 = \{(0,12), (4,3), (11,0)\}$.

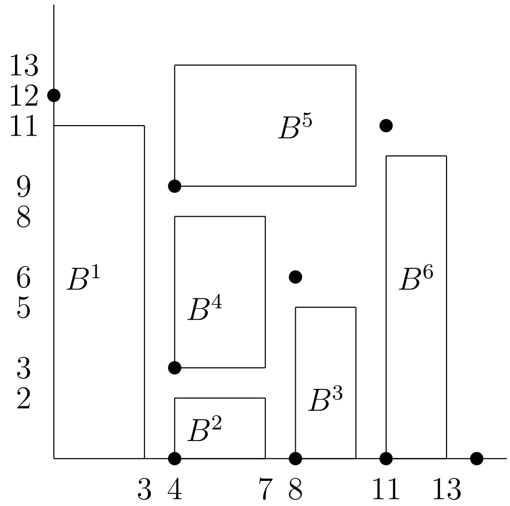

FIGURE 2. Diagram for Example 6.5

4. Adding B^4 removes $(4,3)$ and replaces it by $(8,6)$ and $(4,9)$.
5. Adding B^5 creates a situation in which, if a new block were to be built on $(8,6)$, it could not be taken so large as to intersect B^5. After B^6 is added, there is an additional constraint on the size of block that can be built on $(8,6)$. See Lemma 6.8 below. ■

6.6. **Example.** The **twisted cube** (Figure 3) is a compressible inner-minimal block decomposition of the cube $L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ given by

$$\mathcal{D} = \left(\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix}, \begin{bmatrix} & & 1 \\ 0 & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 0 & 1 \\ 0 & & \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right) \tag{6.1}$$
$$= \Big(\big\{(0,0,0)\big\}, \big\{(1,0,0),(1,1,0)\big\}, \big\{(0,1,0),(0,1,1)\big\}, \big\{(0,1,0),(1,0,1)\big\}, \big\{(1,1,1)\big\} \Big).$$

The generating set for U is $G = \{(2,0,0),(0,2,0),(0,0,2)\}$.

The first and last blocks in $\mathcal{D}$ are singletons, while the others contain two points each. At stage $\mathcal{P}^4$, every ICOAF is contained in a block, leaving no ICOAFs that could be minimal. Instead, the next inner corner is $\mathbf{a}^5 = (1,1,1)$, which is minimal even though it is not an ICOAF of any block in $\mathcal{P}^4$. ■

Motivated by Example 6.5, item 5, we now address the detection of block intersections when creating a block decomposition. At the kth stage, when $\mathcal{P}^{k-1}$ and $\mathbf{a}^k \in M^{k-1}$ are known, let $\mathbf{b}^k$ be a trial value for

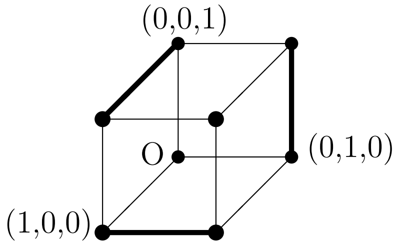

FIGURE 3. The twisted cube. Thick lines connect the two-point blocks. There are two singleton blocks, the origin O and $(1, 1, 1)$.

the outer corner of B^k. By (3.6),

$$B^j \cap B^k = \begin{bmatrix} \min\{\mathbf{b}^j, \mathbf{b}^k\} \\ \max\{\mathbf{a}^j, \mathbf{a}^k\} \end{bmatrix} = \begin{bmatrix} \min\{\mathbf{b}^j, \mathbf{b}^k\} \\ \mathbf{p}^{jk} \end{bmatrix}, \tag{6.2}$$

where

$$\mathbf{p}^{jk} = \max\{\mathbf{a}^j, \mathbf{a}^k\}. \tag{6.3}$$

Since the min and max operations are coordinate-wise, $B^j \cap B^k$ is nonempty if and only if $\mathbf{p}^{jk} \preceq \mathbf{b}^j$ and $\mathbf{p}^{jk} \preceq \mathbf{b}^k$. These two conditions are quite different in character; the first can be checked before $\mathbf{b}^k$ is chosen, but the second constitutes a condition on $\mathbf{b}^k$. Before choosing $\mathbf{b}^k$, we can determine the following set:

$$H^k = \{\mathbf{p}^{jk} : 1 \le j < k \quad \text{and} \quad \mathbf{p}^{jk} \preceq \mathbf{b}^j\}. \tag{6.4}$$

If $H^k = \emptyset$, it is not possible for any B^j with $j < k$ to intersect (collide with) B^k.

6.7. **Lemma.** *It is impossible to have $\mathbf{p}^{jk} \preceq \mathbf{b}^j$ if $j = 1$. Therefore*

$$H^k = \left\{\mathbf{p}^{jk} : j = 2, \ldots, k-1 \text{ and } \mathbf{p}^{jk} \preceq \mathbf{b}^j\right\}. \tag{6.5}$$

Proof. By definition, $\mathbf{p}^{1k} = \max\{\mathbf{a}^1, \mathbf{a}^k\} = \max\{\mathbf{0}, \mathbf{a}^k\} = \mathbf{a}^k$. We cannot have $\mathbf{a}^k \preceq \mathbf{b}^1$, since for $k > 1$, $\mathbf{a}^k$ must lie in $L \setminus B^1$. □

6.8. **Lemma.** *The proposed block B^k intersects a previous block if and only if one or more of the points of H^k divides $\mathbf{b}^k$ (that is, satisfies $\mathbf{p}^{jk} \preceq \mathbf{b}^k$).*

Proof. If B^k intersects a previous block, then it intersects some B^j with $j < k$. Then, as noted above, $\mathbf{p}^{jk} \preceq \mathbf{b}^j$ and $\mathbf{p}^{jk} \preceq \mathbf{b}^k$. The first of these says that $\mathbf{p}^{jk} \in H^k$, and the second says that this point divides $\mathbf{b}^k$. The converse is equally obvious. □

Because of the parallelism between Lemma 3.2 and Lemma 6.8, we call H^k the set of **temporary generators** at stage k. The (permanent) generators in G together with the temporary generators in H^k determine whether B^k is acceptable as the next block of the decomposition being constructed. (See the algorithms in later sections.)

7. General organized decompositions

The rough definition of organized decomposition given in item 4 of a list in Section 3 calls for an (unspecified) outer-maximality condition to be added. This condition is more complicated than the inner-minimality condition already stated; it is not enough to require that "$\mathbf{b}^k$ is maximal under $\preceq$ in $\overline{P^{k-1}}$." (See Remark 7.1.) We build up to it through a sequence of definitions.

1. The block $B' = \begin{bmatrix}\mathbf{b}' \\ \mathbf{a}'\end{bmatrix}$ in $\mathbb{N}^n$ **extends** the block $B = \begin{bmatrix}\mathbf{b} \\ \mathbf{a}\end{bmatrix}$ if both blocks have the same inner corner ($\mathbf{a} = \mathbf{a}'$) and $B \subset B'$. According to Remark 3.3, this is equivalent to
   i. $\mathbf{a} = \mathbf{a}'$.
   ii. $\mathbf{b} \prec \mathbf{b}'$.
2. Let G, U, and L be as usual. At the kth stage of construction of a block decomposition of L, suppose that $\mathcal{P}^{k-1}$, and the next inner corner $\mathbf{a}^k$, have been chosen. (For this definition only, $\mathbf{a}^k$ need not be inner-minimal.) Suppose that $B^k = \begin{bmatrix}\mathbf{b}^k \\ \mathbf{a}^k\end{bmatrix} \subset \overline{P^{k-1}} = L \setminus P^{k-1}$ is proposed as the next block to be constructed. We say that B^k is an **outer-maximal block** if any extension $(B^k)'$ of B^k must intersect either U or P^{k-1} (that is, either U or a previously constructed block). See Remark 7.1 below.
3. Let $\mathcal{D}$ be a block decomposition of L. Then $\mathcal{D}$ is an **outer-maximal decomposition** if each block B^k is an outer-maximal block at the stage when it is constructed.
4. Let $\mathcal{D}$ be a block decomposition of L. Then $\mathcal{D}$ is an **organized decomposition** if it is both inner-minimal (Section 6) and outer-maximal (item 2 of this list).

7.1. **Remark.** The outer-maximality condition for blocks, in item 2 above, is equivalent to "B^k is maximal under $\subseteq$ among all blocks contained in $\overline{P^{k-1}}$ having the same inner corner $\mathbf{a}^k$," and to "$\mathbf{b}^k$ is maximal under $\preceq$ among the set of all points $\mathbf{p}$ such that $\begin{bmatrix}\mathbf{p} \\ \mathbf{a}^k\end{bmatrix}$ is contained in $\overline{P^{k-1}}$, but is **not** equivalent to "$\mathbf{b}^k$ is maximal under $\preceq$ among all points $\mathbf{p}$ of $\overline{P^{k-1}}$." (Under the latter condition, $\begin{bmatrix}\mathbf{p}\\ \mathbf{a}^k\end{bmatrix}$ may contain a point $\mathbf{q} \neq \mathbf{p}$ with $\mathbf{q} \in P^{k-1}$.) ■

Recall from the introduction that a block decomposition is **compressible** ([19]) if the union of some subset of the blocks is itself a block; such a decomposition can be simplified by performing the union, so incompressible block decompositions are desirable. We now prove the fundamental theorem that justifies the consideration of organized decompositions.

7.2. **Theorem.** *Any organized decomposition $\mathcal{D}$ of a lower set L is incompressible.*

Proof. Suppose that $\mathcal{D}$ is both compressible and organized. Since it is compressible, there is an ordered subset $(B^{u_1}, \ldots, B^{u_v})$ of $\mathcal{D} = (B^1, \ldots, B^s)$, with $v > 1$ and $u_1 < \cdots < u_v$, whose (disjoint) union is a block B':

$$B^{u_1} \sqcup \cdots \sqcup B^{u_v} = B'.$$

Since $\mathcal{D}$ is organized, it is inner-minimal, and the inner corner $\mathbf{a}^{u_1}$ of block B^{u_1} is a minimal element of $\overline{P^{u_1-1}}$. Since $\mathbf{a}^{u_1}$ belongs to B^{u_1}, it belongs to B'. Since B' is a block, its inner corner $\mathbf{a}'$ is its only minimal point. Since $B' \subset \overline{P^{u_1-1}}$, a point that is not minimal in B' cannot be minimal in $\overline{P^{u_1-1}}$; therefore $\mathbf{a}' = \mathbf{a}^{u_1}$. Thus B' is a block that is larger than B^{u_1}, has the same inner corner, and is contained in $\overline{P^{u_1-1}}$. Therefore B^{u_1} is not outward maximal, contradicting the assumption that $\mathcal{D}$ is organized. □

The next theorem will allow us to show that organized decompositions can be constructed by a procedure that is almost algorithmic. Specifically, it enables us to provide methods to carry out steps 2a and 2b in Procedure 6.1, which proved problematic in Section 6.

7.3. **Theorem.** *Let $\mathcal{D} = \mathcal{P}^s$ be an organized decomposition of L. Then each block B^k in $\mathcal{D}$ is a union of G-elementary blocks of $\mathbb{N}^n$.*

Proof. The proof is by complete induction on k. So at the k stage we assume that each previous block B^j (for $j < k$) is a union of elementary blocks. For the first case ($k = 1$) there are no previous blocks, and this induction hypothesis is vacuously true. We will prove that B^k is a union of elementary blocks, by showing that it meets the conditions of Corollary 4.4, namely that $a_i^k \in S_i$ for each i, and $b_i^k + 1 \in S_i$ provided $b_i^k \neq \infty$.

The induction hypothesis implies that P^{k-1} is a union of elementary blocks. Lemma 4.2 states that L and U are unions of elementary blocks, and Lemma 4.5 then implies that $\overline{P^{k-1}} = L \setminus P^{k-1}$ is a union of elementary blocks. It follows easily that $P^{k-1} \cup U$ is a union of elementary blocks; we temporarily call this the set of "bad" points (because they are not allowed in B^k). The set $\overline{P^{k-1}}$ will be called the set of "good" points. Any elementary block consists entirely of good points or of bad points.

Since $\mathcal{D}$ is organized, B^k is inner-minimal in the set of good points. Its inner corner $\mathbf{a}^k$ belongs to a unique G-elementary block E in $\mathbb{N}^n$ that consists entirely of good points. Since $\mathbf{a}^k$ is minimal in $\overline{P^{k-1}}$, it must also be minimal in E. Therefore $\mathbf{a}^k$ is the inner corner of E. By the definition of G-elementary block, each component a_i^k belongs to S_i, as promised.

Again, since $\mathcal{D}$ is organized, B^k must be outer-maximal in the set of good points (in the strong sense defined above; see Remark 7.1). Let b_i^k be any component of $\mathbf{b}^k$ such that $b_i^k \neq \infty$; we will show that $b_i^k + 1 \in S_i$. Since $b_i^k \neq \infty$, the block

$$
\begin{aligned}
(7.1) \quad (B^k)' &= \begin{bmatrix} \mathbf{b}^k + \mathbf{e}_i \\ \mathbf{a}^k \end{bmatrix} \\
&= \begin{bmatrix} b_1^k & \cdots & b_i^k + 1 & \cdots & b_n^k \\ a_1^k & \cdots & a_i^k & \cdots & a_n^k \end{bmatrix} \\
&= \begin{bmatrix} b_1^k & \cdots & b_i^k & \cdots & b_n^k \\ a_1^k & \cdots & a_i^k & \cdots & a_n^k \end{bmatrix} \sqcup \begin{bmatrix} b_1^k & \cdots & & \cdots & b_n^k \\ & & b_i^k + 1 & & \\ a_1^k & \cdots & & \cdots & a_n^k \end{bmatrix} \\
&= B^k \sqcup \mathsf{OAF}^i(B^k)
\end{aligned}
$$

is an extension of B^k, and must contain a bad point $\mathbf{p}$ by the definition of outer-maximality. Let F be the unique elementary block that contains $\mathbf{p}$. Since B^k contains only good points, $\mathbf{p}$ must lie in $\mathsf{OAF}^i(B^k)$, in view of equation (7.1). This fact implies that $\mathbf{p} - \mathbf{e}_i \in B^k$, and this in turn implies that $\mathbf{p} \in \mathsf{IF}^i(F)$. We have seen (Lemma 4.1) that the ith component of any point in the ith inner face of an elementary block must lie in S_i; therefore $p_i \in S_i$. On the other hand, both $\mathbf{p}$ and $\mathbf{b}^k + \mathbf{e}_i$ belong to $\mathsf{OAF}^i(B^k)$. It follows from (3.9) that all points in the same ith outer adjacent face of any block have the same ith component. Therefore $b_i^k + 1 = p_i \in S_i$, as promised. In view of the strategy outlined at the beginning, this completes the proof. □

Next we state a procedure (that is algorithmic except that it calls for some choices, and always terminates) to produce organized block decompositions. This procedure is not at all efficient, and is stated primarily to show its possibility. (We will not actually use it in this paper.) In the next few sections it will be modified into an algorithm for "special organized decompositions" (a subset of general organized decompositions) that is much more efficient. The reader is invited to apply Procedure 7.4 to re-derive $\mathcal{D}_1$ through $\mathcal{D}_6$ in Section 5; the initial form of the dynamic lists $\mathcal{L}$ and $\mathcal{L}'$ needed in the procedure can be read off from Figure 1.

7.4. **Procedure.** *To create an organized block decomposition of a lower set $L \subset \mathbb{N}^n$:*

1. *Input the minimal set of generators of $U = \mathbb{N}^n \setminus L$.*
2. *Set $P^0 = \emptyset$ and $M^0 = \{\mathbf{0}\}$.*
3. *Find the elementary decomposition of L, either by Algorithm 3.1 of* [19] *or by Section 4 above.*
4. *Initialize a dynamic list $\mathcal{L}$ of the inner corners of elementary blocks of L, and a separate list $\mathcal{L}'$ of the outer corners. These lists will be updated during the execution of the algorithm (step 5e). (The order of elements in these lists does not matter at this time, but $\mathcal{L}$ will become an ordered list in later algorithms; $\mathcal{L}'$ will not be used again.)*
5. *For $k = 1, 2, \dots,$ do the following:*
   a. *Choose a minimal element, under $\preceq$, of the current version of $\mathcal{L}$, to be the inner corner $\mathbf{a}^k$ of the next block. Minimality can be determined by pairwise comparison of the elements of $\mathcal{L}$ (a finite process since $\mathcal{L}$ is a finite list).*
   b. *Determine the subset of points $\mathbf{b}$ in $\mathcal{L}'$ such that $\begin{bmatrix}\mathbf{b}\\ \mathbf{a}^k\end{bmatrix}$ does not intersect any previous blocks. (This can be done by finding H^k and using Lemma 6.8, or, much less efficiently, by computing all of the possible blocks $\begin{bmatrix}\mathbf{b}\\ \mathbf{a}^k\end{bmatrix}$ and finding the intersections of each of these with each of the previous blocks.) From this subset of the $\mathbf{b}$, choose one that is maximal under $\preceq$ to be the outer corner $\mathbf{b}^k$ of the next block.*
   c. *Set $B^k = \begin{bmatrix}\mathbf{b}^k\\ \mathbf{a}^k\end{bmatrix}$.*
   d. *Set $\mathcal{P}^k = (B^1, \dots, B^k)$ and $P^k = \bigsqcup \mathcal{P}^k$, as in (3.11).*
   e. *Remove any points from $\mathcal{L}$ and $\mathcal{L}'$ that are contained in B^k. (This will include $\mathbf{a}^k$ and $\mathbf{b}^k$.) Repeat step 5 with the updated lists until step 6 applies.*
6. *Stop when $P^k = L$ and set $s = k$. (This is guaranteed to occur, because there are finitely many elementary blocks in L.) Output $\mathcal{D} = \mathcal{P}^s$ as the desired decomposition of L.*

8. A lexicographic method for minimal points

In Procedure 7.4 for organized decompositions, two substeps (included in step 4a) are needed to select $\mathbf{a}^k$: the elements of the current list $\mathcal{L}$ must be compared with each other (under $\preceq$) to determine which are minimal, and then a choice (not specified algorithmically) must be made. In this section we give a way to do both steps at once, and to do them algorithmically. This method does not produce all possible organized decompositions.

Suppose that $\ll$ is a total order on $\mathbb{N}^n$ that meets two conditions:

1. $\ll$ refines (or extends) the partial order $\prec$; that is, if $\mathbf{a} \prec \mathbf{b}$ then $\mathbf{a} \ll \mathbf{b}$.
2. $\ll$ is a well-order; that is, every nonempty subset T of $\mathbb{N}^n$ has a (unique) least element under $\ll$.

Then the unique *minimum* element $\mathbf{p}$ (under $\ll$) of a nonempty subset $T \subset \mathbb{N}^n$ is also a *minimal* element of T under $\prec$. (If $\mathbf{p}$ were not minimal under $\prec$, there would exist $\mathbf{q} \prec \mathbf{p}$ in T, but then $\mathbf{q} \ll \mathbf{p}$ and $\mathbf{p}$ is not the minimum under $\ll$.) Thus, when $\mathcal{P}^{k-1}$ is known, $\mathbf{a}^k$ can be chosen as the minimum point of $\overline{P^{k-1}}$ under $\ll$ (by an algorithm stated below). It is obvious that the successive inner corners selected in this way satisfy

$$\mathbf{a}^1 \ll \mathbf{a}^2 \ll \cdots \ll \mathbf{a}^s.$$

To use this idea requires at least one total order that refines $\prec$. The familiar **lexicographic order** $<_{\mathrm{lex}}$ on $\mathbb{N}^n$ is defined as follows: Let i be the smallest integer such that $a_i \neq b_i$. Then $\mathbf{a} <_{lex} \mathbf{b}$ if and only if $a_i < b_i$. This is a well-order; see for instance [4, §2, Prop. 4].

8.1. **Lemma.** *The total order $<_{\mathrm{lex}}$ refines $\prec$.*

Proof. Suppose $\mathbf{a} \prec \mathbf{b}$. Then for all $i = 1, \ldots, n$, $a_i \leq b_i$, and for some i^*, $a_{i^*} < b_{i^*}$. Considering the first such i^* shows that $\mathbf{a} <_{lex} \mathbf{b}$, as required for a refinement. □

Now let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a permutation of $(1, \ldots, n)$. Define the total order $<_{\boldsymbol{\sigma}}$ on $\mathbb{N}^n$ to be lexicographic order using the coordinates $\boldsymbol{\mu}$ on $\mathbb{N}^n$ in the order $(\mu_{\sigma_1}, \ldots, \mu_{\sigma_n})$. That is,

$$\mathbf{a} <_{\boldsymbol{\sigma}} \mathbf{b} \text{ if and only if the entry with smallest } i \text{ such that } a_{\sigma_i} \neq b_{\sigma_i} \text{ satisfies } a_{\sigma_i} < b_{\sigma_i}.$$

This reduces to the standard lexicographic order if $\boldsymbol{\sigma} = \boldsymbol{\varepsilon} = (1, \ldots, n)$, the identity permutation. By the same reasoning as in Lemma 8.1, $<_{\boldsymbol{\sigma}}$ for any $\boldsymbol{\sigma}$ is a refinement of $\prec$.

8.2. **Remark.** In the case that $\boldsymbol{\sigma} = (n, n-1, \ldots, 2, 1)$, $<_{\boldsymbol{\sigma}}$ should not be confused with *reverse lexicographic order*, frequently used in Gröbner basis theory. ■

Now we modify Procedure 7.4 as follows: Before carrying out the algorithm, order the initial list $\mathcal{L}$ by one of the orders $<_{\boldsymbol{\sigma}}$. Whenever a new block is added during the construction, delete from this ordered list those points that are included in P^k; this leaves the *available* corners, which are still in order by $<_{\boldsymbol{\sigma}}$. When a new inner corner is required for the next block, choose the first one in the list. (It will automatically be deleted from $\mathcal{L}$ in step 4e.) This will be incorporated more formally in the algorithm for the $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-organized decomposition in Section 10.

9. Constructing blocks by directional extension

It is not feasible to use a lexicographic method to select the outer corners $\mathbf{b}^k$ in step 4b of Procedure 7.4. Instead, we use **directional extension**. To motivate this, consider first the following simplified problem (which does not involve any decomposition of L): A single block

$$B = \begin{bmatrix} b_1 & \cdots & b_h & \cdots & b_n \\ a_1 & \cdots & a_h & \cdots & a_n \end{bmatrix} \subset L$$

is given, and it is desired to extend this block as far as possible in direction h (leaving the inner corner fixed) without intersecting U. (There is no question here of intersecting "previous blocks.") Replacing the h-component b_h in $\mathbf{b}$ by a variable t produces a vector-valued function $\mathbf{b}_h(t) = (b_1, \ldots, b_{h-1}, t, b_{h+1}, \ldots b_n)$. (The subscript h on the boldface letter $\mathbf{b}$ does not indicate a component, but a new vector depending on t.) Consider the matrix-valued function

$$B_h(t) = \begin{bmatrix} \mathbf{b}_h(t) \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} b_1 & \cdots & t & \cdots & b_n \\ a_1 & \cdots & a_h & \cdots & a_n \end{bmatrix}.$$

Since $\mathbf{b} \in L$, no generator $\mathbf{m}^\ell \in G$ will divide $\mathbf{b}$, but there may be generators that divide $\mathbf{b}_h(t)$ if $t > b_h$ is large enough; this would indicate that $\mathbf{b}_h(t)$ has collided with U. Let $G_h \subseteq G$ be the set of such generators; this set may also be characterized as the set of generators that divide $\mathbf{b}_h(\infty)$, or as the set of generators that divide all components of $\mathbf{b}$ except the h-component (that is, $m_i^\ell \le b_i$ for $i \neq h$). Now define t^* as follows:

$$t^* = \begin{cases} \infty & \text{if} \quad G_h = \emptyset \\ \min \left\{ m_h^\ell : \mathbf{m}^\ell \in G_h \right\} - 1 & \text{otherwise.} \end{cases}$$

Then t^* is the largest value of t such that $\mathbf{b}_h(t)$ still belongs to L^* (see Remark 3.1), and $B_h(t^*)$ is the desired extension of B.

9.1. **Remark.** When we pass to the full (rather than simplified) problem, G^k will denote an enlarged set of (permanent and temporary) generators at the kth stage, and G_h^k will be obtained from G^k in the same way that G_h is obtained from G here. ■

9.2. **Example.** In $\mathbb{N}^2$, let $G = \{(5,0),(4,1),(2,6)\}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. To extend B in direction $h = 1$, form $\mathbf{b}_1(\infty) = (\infty, 1)$ by replacing the 1-component of $\mathbf{b} = (1,1)$ by ∞. Then observe that generators $(5,0)$ and $(4,1)$ divide $\mathbf{b}_1(\infty)$ while $(2,6)$ does not. The smallest 1-component of those that do is 4, so $t^* = 3$, and the desired extension is $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$. If we had chosen $h = 2$, none of the generators would have divided $\mathbf{b}_2(\infty) = (1, \infty)$, so t^* would equal ∞ and the extension would be $\begin{bmatrix} 1 & \infty \\ 0 & 0 \end{bmatrix}$. These results can be confirmed by drawing the upper set U and the block B, and observing the (separate) extensions in the 1 and 2 directions. ■

Now suppose that a partial organized block decomposition $\mathcal{P}^{k-1}$ has been constructed in a lower set L, and the next inner corner $\mathbf{a}^k$ has been selected, perhaps (but not necessarily) by the lexicographic method of Section 8. The next step is to construct a block B^k with inner corner $\mathbf{a}^k$ that is outer-maximal in $\overline{P^{k-1}}$. To apply directional extension, we extend the block successively in the directions $h = 1, \ldots, n$ (or a permutation $\boldsymbol{\pi}$ of these directions) to produce a sequence of blocks

$$\{\mathbf{a}^k\} = B_0^k \subseteq B_1^k \subseteq \cdots \subseteq B_n^k = B^k \tag{9.1}$$

beginning with the singleton block $\{\mathbf{a}^k\}$ and terminating with a maximal block B^k that does not intersect U or P^{k-1}. Each intermediate block B_h^k has inner corner $\mathbf{a}^k$ and outer corner $\mathbf{b}_h^k$, so that

$$B_h^k = \begin{bmatrix} \mathbf{b}_h^k \\ \mathbf{a}^k \end{bmatrix}.$$

As in the simplified problem above, the subscript h on the bold symbol $\mathbf{b}_h^k$ does not indicate a component of $\mathbf{b}^k$, but instead indicates a new vector, obtained by moving the previous outer corner $\mathbf{b}_{h-1}^k$ as far as possible in direction h, but now "as far as possible" means without causing the resulting block to intersect U or a previous block.

In order to avoid hitting P^{k-1} as well as U, we form the set H^k of **temporary generators** at stage k, defined by equation (6.5); according to Lemma 6.8, a proposed block B^k intersects a previous block if and only if its outer corner $\mathbf{b}^k$ is divisible by an element of H^k. This exactly parallels the fact that B^k intersects U if and only if $\mathbf{b}$ is divisible by a "permanent" generator from G. So we merely combine G with H^k to form $G^k = G \cup H^k$, and proceed as in the special case above. (For another way to view this, see Remark 9.4 below.) Note that in step 3 of the following algorithm we proceed from $\mathbf{b}^k_{h-1}$ to $\mathbf{b}^k_h$ through the intermediate steps of $\mathbf{b}^k_h(t)$ and $\mathbf{b}^k_h(t^*)$; it is only in these intermediate steps that $\mathbf{b}^k_h$ is a function that takes an argument (t or t^*). The algorithm is written for the identity permutation $\boldsymbol{\pi} = \boldsymbol{\varepsilon} = (1, \ldots, n)$ of the extension directions; for the general case see Remark 9.5.

9.3. **Algorithm.** *To create the unique outer-maximal extension B^k of the singleton block $\{\mathbf{a}^k\}$ in $\overline{P^{k-1}}$ with extension order $\boldsymbol{\pi} = \boldsymbol{\varepsilon} = (1, \ldots, n)$:*

1. *Initialize* $B^k_0 = \begin{bmatrix} \mathbf{a}^k \end{bmatrix} = \{\mathbf{a}^k\}$, *which implies* $\mathbf{b}^k_0 = \mathbf{a}^k$.
2. *Determine the set H^k of temporary generators $\mathbf{p}^{jk}$ at stage k according to (6.5), and create $G^k = G \cup H^k$.*
3. *For $h = 1, \ldots, n$, do the following:*
   a. *Replace the h-component of $\mathbf{b}^k_{h-1}$ by t and call the result $\mathbf{b}^k_h(t)$.*
   b. *Let G^k_h be the subset of G^k consisting of permanent and temporary generators that divide $\mathbf{b}^k_h(\infty)$. (See Remark 9.1.)*
   c. *Let*
   $$t^* = \begin{cases} \infty & \text{if} \quad G^k_h = \emptyset \\ \min\left\{m^\ell_h : \mathbf{m}^\ell \in G^k_h\right\} - 1 & \text{otherwise.} \end{cases}$$
   d. *Set* $\mathbf{b}^k_h = \mathbf{b}^k_h(t^*)$.
   e. *Repeat with the next h, unless $h = n$.*
4. *Set*
   $$B^k = \begin{bmatrix} \mathbf{b}^k_n \\ \mathbf{a}^k \end{bmatrix}$$
   and output this as the desired extension of $\{\mathbf{a}^k\}$.

9.4. **Remark.** Another way to explain the use of $G^k = G \cup H^k$ is to say that although P^k is not an upper set, $\langle H^k \rangle$ is, and Lemma 6.8 implies that B^k intersects P^k if and only if it intersects $\langle H^k \rangle$. By (3.3), $U \cup \langle H^k \rangle = \langle G \rangle \cup \langle H^k \rangle = \langle G \cup H^k \rangle = \langle G^k \rangle$. So B^k avoids $U \cup P^{k-1}$ if and only if it avoids $\langle G^k \rangle$. ■

9.5. **Remark.** To modify Algorithm 9.3 to handle a different extension order $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$, change line 3 to "For $i = 1, \ldots, n$, set $h = \pi_i$ and do the following:" The rest of the algorithm needs no changes. ■

The first example is easy. More examples appear in Section 10.

9.6. **Example.** Let U be the upper set in $\mathbb{N}^3$ defined by $G = \{(1,1,1)\}$, which is

$$U = \langle (1,1,1) \rangle = \begin{bmatrix} \infty & \infty & \infty \\ 1 & 1 & 1 \end{bmatrix}.$$

The associated lower set L consists of the three coordinate planes $\mu_1 = 0$, $\mu_2 = 0$, and $\mu_3 = 0$ in $\mathbb{N}^3$. We will construct the first block B^1 of an organized decomposition for L with extension order $\pi = (1,2,3)$, by using Algorithm 9.3 with $k = 1$. For step 1 we have $\mathbf{a}^1 = \mathbf{b}_0^1 = \mathbf{0}$. For step 2, Lemma 6.7 implies $H^1 = \emptyset$ because $k = 1$ implies $j = 1$.. For step 3 the first time (with $h = 1$), we have $\mathbf{b}_1^1(t) = (t, 0, 0)$. This can never be divisible by $(1,1,1)$, so $G_1^1 = \emptyset$, $t^* = \infty$, and $\mathbf{b}_0^1(t^*) = (\infty, 0, 0)$. For the second time (with $h = 2$), $\mathbf{b}_2^1(t) = (\infty, t, 0)$, $t^* = \infty$, and $\mathbf{b}_2^1(t^*) = (\infty, \infty, 0)$. For the third time, $\mathbf{b}_2^1(t) = (\infty, \infty, t)$. This is divisible by $(1,1,1)$ if $t \geq 1$, so $t^* = 0$ and $\mathbf{b}^1 = \mathbf{b}_3^1 = (\infty, \infty, 0)$. (Since G_3^1 contains only the one generator $(1,1,1)$, its third component is automatically the minimum called for in step 3c.) This implies that

$$B^1 = \begin{bmatrix} \infty & \infty & \\ 0 & 0 & 0 \end{bmatrix}.$$

That is, B^1 equals the "floor" of L, which is the $\mu_1\mu_2$ plane or the plane $\mu_3 = 0$. The reader should see that this same result is obvious geometrically without the algorithm: The singleton block can be extended to infinity in the 1-direction and then in the 2-direction without hitting U, but after that, cannot be pushed in the 3-direction. ■

10. $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-ORGANIZED DECOMPOSITIONS

Let $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ be two permutations of $(1, \ldots, n)$, which may or may not be equal. The $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-**organized decomposition** of L is the organized decomposition obtained by using the lexicographic method with order $<_{\boldsymbol{\sigma}}$ to determine each inner corner $\mathbf{a}^k$ and the directional extension method with extension order $\boldsymbol{\pi}$ to determine each block B^k. We now combine Sections 8 and 9 to state the algorithm for the unique $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-organized decomposition of L.

10.1. **Remark.** Since $\boldsymbol{\sigma}$ is used first (and only once) in the algorithm below, why do we put $\boldsymbol{\pi}$ first in $(\boldsymbol{\pi}, \boldsymbol{\sigma})$? In the first (and ultimately unsatisfactory) version of this research, $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ were used alternately, beginning with $\boldsymbol{\pi}$ (since we already knew $\mathbf{a}^1 = \mathbf{0}$ without needing $\boldsymbol{\sigma}$). When we changed the approach, the notation $(\boldsymbol{\pi}, \boldsymbol{\sigma})$ was so solidly established in all of our examples and communications between co-authors (often with $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ expressed numerically) that it was too late for us to change the notation without producing total confusion. ■

10.2. **Algorithm.** *To create the $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-organized block decomposition of a lower set $L \subset \mathbb{N}^n$:*

1. *Input the minimal set G of generators of $U = \mathbb{N}^n \setminus L$ and the permutations $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ of $\{1, \dots, n\}$.*
2. *Find the elementary decomposition of L, either by Algorithm 3.1 of* [19] *or by Section 4 above.*
3. *Initialize a dynamic list $\mathcal{L}$ of the inner corners of elementary blocks of L, ordered (from low to high) by the total order $<_\sigma$. This list will be updated in step 5d.*
4. *Set $P^0 = \emptyset$.*
5. *For $k = 1, 2, \dots$ do the following until $\mathcal{L}$ is empty:*
   a. *Let $\mathbf{a}^k$ be the first element of $\mathcal{L}$ (the list in its current form).*
   b. *Compute the $\boldsymbol{\pi}$-extension B^k of $\{\mathbf{a}^k\}$ in $\overline{P^{k-1}}$ by Algorithm 9.3, using Remark 9.5 if $\boldsymbol{\pi} \neq \boldsymbol{\varepsilon}$. (This step includes finding, and using, the temporary generators $\mathbf{p}^{jk}$ for $j = 2, \dots, k$.)*
   c. *Set $\mathcal{P}^k = (B^1, \dots, B^k)$ and $P^k = \bigsqcup \mathcal{P}^k$.*
   d. *Delete all elements of $\mathcal{L}$ that belong to P^k. (This always includes deleting at least $\mathbf{a}^k$).*
   e. *Repeat with the updated list $\mathcal{L}$ unless $\mathcal{L}$ is empty.*
6. *Set $s = k$ and output $\mathcal{D} = \mathcal{P}^s$ as the desired decomposition of L.*

The first example completes the calculation begun in Example 9.6. This example requires no temporary generators, so it does not illustrate the entire algorithm.

10.3. **Example.** We will construct the $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})$-organized decomposition of L for $G = \langle (1, 1, 1) \rangle$, showing that the result is

$$\mathcal{D} = \left(\begin{bmatrix} \infty & \infty & \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \infty & & \infty \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} & \infty & \infty \\ 0 & 1 & 1 \end{bmatrix} \right). \tag{10.1}$$

The initial list $\mathcal{L}$ of inner corners of the elementary decomposition of L, listed in lexicographic order, is

$$(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0).$$

The first block B^1, with $\mathbf{a}^1 = \mathbf{0}$, has been constructed in Example 9.6, and is the entire "floor" $\mu_3 = 0$, so removing the elements with $\mu_3 = 0$ from $\mathcal{L}$ leaves

$$(0, 0, 1), (0, 1, 1), (1, 0, 1).$$

Taking the first element of this updated list as the next inner corner gives $\mathbf{a}^2 = \mathbf{b}_0^2 = (0, 0, 1)$. By Lemma 6.7, there is no temporary generator $\mathbf{p}^{12}$ to include in the next calculation. (That is, B^2 cannot intersect B^1.) Then $\mathbf{b}_1^2(t) = (t, 0, 1)$, which cannot be divided by $(1, 1, 1)$ because of its second entry, so $t^* = \infty$, $\mathbf{b}_1^2(t^*) = (\infty, 0, 1)$, $\mathbf{b}_2^2(t) = (\infty, t, 1)$, $t^* = 0$ (since this is divisible by $(1, 1, 1)$ for $t \geq 1$), $\mathbf{b}_3^2(t) = (\infty, 0, t)$, $t^* = \infty$ (since this can never be divided by $(1, 1, 1)$), and $\mathbf{b}^2 = \mathbf{b}_3^2 = (\infty, 0, \infty)$. Therefore

$$B^2 = \begin{bmatrix} \mathbf{b}^2 \\ \mathbf{a}^2 \end{bmatrix} = \begin{bmatrix} \infty & 0 & \infty \\ 0 & 0 & 1 \end{bmatrix}.$$

Since this contains $(0, 0, 1)$ and $(1, 0, 1)$, we remove these from $\mathcal{L}$, which is now reduced to the single element $(0, 1, 1)$, which becomes $\mathbf{a}^3$. Again there are no temporary generators, since $\mathbf{p}^{23}$ does not divide $\mathbf{b}^1$ or $\mathbf{b}^2$. The reader can check that for B^3, $t^* = 0$ for the first time through the loop ($h = 1$) and $t^* = \infty$ for the next two times ($h = 2$ and 3). These results for the extension of the singleton blocks $\{\mathbf{a}^2\}$ and $\{\mathbf{a}^3\}$ can also be observed geometrically, as in Example 9.6. ∎

10.4. **Remark.** In example 10.3, the result does not depend on the choice of $\boldsymbol{\sigma}$ in $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$. It can be shown that $M^1 = \{(0, 0, 1)\}$, so any choice of $\boldsymbol{\sigma}$ must select $\mathbf{a}^2 = (0, 0, 1)$, and the construction of B^2 will be the same as above (since we are still using $\boldsymbol{\varepsilon}$-extension). Again, $M^2 = \{\mathbf{a}^3\} = \{(0, 1, 1)\}$, so $\mathbf{a}^3 = (0, 1, 1)$. ∎

The following example, again in $\mathbb{N}^3$, is still easy to picture geometrically but illustrates all of the ideas in the algorithm, including the use of temporary generators and the effect of changing $\boldsymbol{\sigma}$.

10.5. **Example.** The $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})$- and $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}')$-organized decompositions of L for $G = \{(1, 0, 9), (1, 5, 0), (7, 0, 0)\}$.

We first find the elementary inner corners for L (Section 4). This calls for sets S_i of ith components of generators, with repetitions deleted, ordered numerically, with an initial zero added to each ordered set if necessary; the result is

$$S_1 = 0, 1, 7$$
$$S_2 = 0, 5$$
$$S_3 = 0, 9.$$

There are twelve inner corners in $\mathbb{N}^3$, from which we drop seven that are in U. Arranging the remaining five in order by $<_{\boldsymbol{\varepsilon}}$ gives the initial list $\mathcal{L}$ of the algorithm as follows:

(10.2) $$(0, 0, 0), (0, 0, 9), (0, 5, 0), (0, 5, 9), (1, 0, 0).$$

Ordering instead by $<_{\varepsilon'}$ gives $\mathcal{L}$ as

$$(0,0,0),(1,0,0),(0,5,0),(0,0,9),(0,5,9). \tag{10.3}$$

For the $(\varepsilon, \varepsilon)$ decomposition, (10.2) is used for $\mathcal{L}$. The calculations for B^1 can be arranged in a table, listing as "relevant generator" the generator that divides $\mathbf{b}_i^1(t)$ if $t > t^*$:

		relevant generator	
$\mathbf{b}_1^1(t) = (t,0,0)$	$t^* = 6$	$(7,0,0)$	$\mathbf{b}_1^1(t^*) = (6,0,0)$
$\mathbf{b}_2^1(t) = (6,t,0)$	$t^* = 4$	$(1,5,0)$	$\mathbf{b}_2^1(t^*) = (6,4,0)$
$\mathbf{b}_3^1(t) = (6,4,t)$	$t^* = 8$	$(1,0,9)$	$\mathbf{b}_3^1(t^*) = (6,4,8)$.

Therefore

$$B^1 = B_3^1 = \begin{bmatrix} 6 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $(0,0,0)$ and $(1,0,0)$ are in B^1, these are deleted from (10.2), and $\mathbf{a}^2 = (0,0,9)$. By Lemma 6.7, $\mathbf{p}^{12}$ does not exist, and there are no temporary generators at this stage. The first line of the similar table for B^2 is $\mathbf{b}_1^2 = (t,0,9)$, $t^* = 0$, with relevant generator $(1,0,9)$; the next two lines each give $t^* = \infty$, and

$$B^2 = \begin{bmatrix} 0 & \infty & \infty \\ & 0 & 9 \end{bmatrix}.$$

Since $(0,0,9)$ and $(0,5,9)$ are in B^2, these are deleted from (10.2), and $\mathbf{a}^3 = (0,5,0)$. By Lemma 6.7, $\mathbf{p}^{1k}$ does not exist. However,

$$B^2 \cap \langle \mathbf{a}^3 \rangle = \begin{bmatrix} 0 & \infty & \infty \\ & 0 & 9 \end{bmatrix} \cap \begin{bmatrix} \infty & \infty & \infty \\ 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \infty & \infty \\ & 5 & 9 \end{bmatrix},$$

giving $\mathbf{p}^{23} = (0,5,9)$ as a temporary generator, so $H^2 = \{(0,5,9)\}$ and $G^2 = \{(1,0,9),(1,5,0),(7,0,0),(0,5,9)\}$. Then the table for B^3 is

		relevant generator	
$\mathbf{b}_1^3(t) = (t,5,0)$	$t^* = 0$	$(1,5,0)$	$\mathbf{b}_1^3(t^*) = (6,0,0)$
$\mathbf{b}_2^3(t) = (0,t,0)$	$t^* = \infty$	none	$\mathbf{b}_2^3(t^*) = (0,\infty,0)$
$\mathbf{b}_3^3(t) = (0,\infty,t)$	$t^* = 8$	$(0,5,9)$	$\mathbf{b}_3^3(t^*) = (0,\infty,8)$.

Therefore

$$B^3 = \begin{bmatrix} 0 & \infty & 8 \\ & 5 & 0 \end{bmatrix}.$$

Since $(0,5,0) \in B^3$, it is removed from $\mathcal{L}$, which is now empty, so the construction is finished. The reader should observe, in a three-dimensional plot, the collision of B^3 with B^2 that would occur if the temporary generator were not not used.

For the $(\varepsilon, \varepsilon')$-organized decomposition, $\mathcal{L}$ is given by (10.3). The computation of B^1 is the same as before, but $\mathbf{a}^2$ comes out as $(0,5,0)$

(which was formerly $\mathbf{a}^3$); the calculation is like the previous calculation of B^3, but without the temporary generator. The next inner corner is $\mathbf{a}^3 = (0,0,9)$, which was $\mathbf{a}^2$ before. The same temporary generator arises, $\mathbf{p}^{23} = (0,5,9)$, but it comes from a different intersection than before. The details are left to the reader; the final result is

$$(B^1, B^2, B^3) = \left(\begin{bmatrix} 6 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \infty & \infty \\ 0 & 5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & {4 \atop 0} & {\infty \atop 9} \end{bmatrix} \right).$$

In both cases B^2 and B^3 lie in the $\mu_2\mu_3$-plane and cover all of it that is not in B^1, but the region is partitioned differently. ■

The next example is four-dimensional and will be of interest in later sections.

10.6. **Example.** $G = \{(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,1)\}$.

We have computed three organized decompositions for this example, $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})$, $(\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}')$, and $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}')$. The last is the most interesting, as it resembles both the twisted cube (Example 6.6) and the McLagan-Smith example, and is not subprime. It does not seem that an *organized* decomposition of this type exists in less than four dimensions. First we put on record the three decompositions we found, then sketch the computation of the interesting one. The $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})$ decomposition is

$$\text{(10.4)} \quad \left(\begin{bmatrix} {\infty \atop 0} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & {\infty \atop 0} & 0 & {\infty \atop 1} \end{bmatrix}, \begin{bmatrix} 0 & {\infty \atop 0} & {\infty \atop 1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & {\infty \atop 1} & {\infty \atop 1} \end{bmatrix}, \begin{bmatrix} 0 & {\infty \atop 1} & 0 & 0 \end{bmatrix} \right).$$

The $(\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}')$ decomposition is

$$\text{(10.5)} \quad \left(\begin{bmatrix} 0 & 0 & {\infty \atop 0} & {\infty \atop 0} \end{bmatrix}, \begin{bmatrix} {\infty \atop 1} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & {\infty \atop 1} & 0 & {\infty \atop 0} \end{bmatrix}, \begin{bmatrix} 0 & {\infty \atop 1} & {\infty \atop 1} & 0 \end{bmatrix} \right).$$

The $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}')$ decomposition is

$$\text{(10.6)} \quad \left(\begin{bmatrix} {\infty \atop 0} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & {\infty \atop 1} & {\infty \atop 0} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & {\infty \atop 1} & {\infty \atop 0} \end{bmatrix}, \begin{bmatrix} 0 & {\infty \atop 0} & 0 & {\infty \atop 1} \end{bmatrix} \right).$$

The eight elementary inner corners in L that form the initial list $\mathcal{L}$ are listed in order by $<_{\boldsymbol{\varepsilon}'}$ in the following table. If $\mathbf{a}^k$ appears under one of these points, that point becomes the inner corner $\mathbf{a}^k$ in the $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}')$ decomposition; if B^k appears instead, the point is contained in B^k and does not become an inner corner.

(0,0,0,0) $\mathbf{a}^1$	(1,0,0,0) B^1	(0,1,0,0) $\mathbf{a}^2$	(0,0,1,0) $\mathbf{a}^3$
$(0,1,1,0)$ B^3	$(0,0,0,1)$ $\mathbf{a}^4$	$(0,1,0,1)$ B^4	$(0,0,1,1)$ B^3

For instance, B^1 is built on $(0, 0, 0, 0)$ by $\boldsymbol{\varepsilon}$-extension; it includes $(1, 0, 0, 0)$, which is deleted from $\mathcal{L}$, leaving $\mathbf{a}^2$ as $(0, 1, 0, 0)$, and so on across the table. After $\mathbf{a}^3$ has been found, the temporary generator $\mathbf{p}^{23} = (0, 1, 1, 0)$ arises and must be used in the computation of B^3, which goes as follows:

		relevant generator	
$\mathbf{b}_1^3(t) = (t, 0, 1, 0))$	$t^* = 0$	$\mathbf{m}^3 = (1, 0, 1, 0)$	$\mathbf{b}_1^3(t^*) = (0, 0, 1, 0)$
$\mathbf{b}_2^3(t) = (0.t.1, 0)$	$t^* = 0$	$\mathbf{p}^{23} = (0, 1, 1, 0)$	$\mathbf{b}_2^3(t^*) = (0, 0, 1, 0)$
$b_3^3(t) = (0, 0, t, 0)$	$t^* = \infty$	none	$\mathbf{b}_3^1(t^*) = (0, 0, \infty, 0)$
$b_4^3(t) = (0, 0, \infty, t)$	$t^* = \infty$	none	$\mathbf{b}_4^3 = (0, 0, \infty, \infty)$,

resulting in

$$B^3 = \begin{bmatrix} \mathbf{b}_4^3 \\ \mathbf{a}^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \begin{matrix} \infty \\ 1 \end{matrix} & \begin{matrix} \infty \\ 0 \end{matrix} \end{bmatrix}.$$

The remaining details are left to the reader. ■

11. Subprime decompositions

It is usual in the literature to speak of "Stanley decompositions that come from prime filtrations of $K[\mathbf{x}]$." In [19, §5] we called these "prime Stanley decompositions," discussed this idea in detail, extended it to "subprime block decompositions," and proved a version of Soleyman-Jahan's necessary and sufficient condition for a Stanley or block decomposition to be, respectively, prime or subprime. (Our formulation differs slightly from that of Soleyman-Jahan, since we work with *ordered* rather than unordered decompositions.) This result is restated in three ways in Theorem 11.1 below. Then we define a directed graph associated with any block decomposition, and re-state the Soleyman-Jahan condition once more in terms of this graph (Theorem 11.2). The graph makes it quite easy to determine whether a particular block decomposition is or is not subprime, and whether it can or cannot be re-ordered so as to become subprime. We use this to investigate the examples of organized decompositions obtained in previous sections, and see that some are subprime, some are not but can be re-ordered to be subprime, and some cannot be so re-ordered. The disjoint union of the outer adjacent faces of a block B is called its **outer adjacent set**:

$$\mathsf{OAS}(B) = \bigsqcup_{i=1}^{n} \mathsf{OAF}^i(B). \tag{11.1}$$

11.1. **Theorem** (Soleyman-Jahan). *An (ordered) block decomposition $\mathcal{D} = (B^1, \dots, B^s)$ of L is subprime if and only if any one of these equivalent conditions holds:*

1. *For each $k = 1, \dots, s$, the set*
$$U^k = (B^k \sqcup B^{k+1} \sqcup \cdots \sqcup B^s) \sqcup U$$
is an upper set.

2. *For each $k = 1, \dots, s$, $\mathsf{OAS}(B^k) \subset U^{k+1}$.*

3. *For each $k = 2, \dots, s-1$, the partial decomposition $P^k = \bigsqcup \mathcal{P}^k = B^1 \cup \cdots \cup B^k$ is a lower set.*

Proof. The first item is a restatement of [19, Thm. 5.2]. The second is a restatement of [19, Lm. 5.3]. For the third item, we have $P^{k-1} \sqcup U^k = \mathbb{N}^n$, so U^k is an upper set if and only if P^k is a lower set. □

Let $\mathcal{D} = (B^1, \dots, B^s)$ be any block decomposition of L. Create a directed graph Γ with s vertices labeled $1, \dots, s$, and an arrow (or directed edge) from j to k if $\mathsf{OAS}(B^j) \subset B^k$. (Here we make an exception to the rule that $j \le k$). The calculations needed to find Γ are illustrated in Example 11.4 below.

A **re-ordering** of $\mathcal{D}$ is a new decomposition $\mathcal{D}'$ obtained from $\mathcal{D}$ by permuting the blocks. (Here $\mathcal{D}$ is considered as the identity permutation of itself, so it counts as a re-ordering.) A **cycle** in Γ is a closed path or loop through a subset of the vertices, following the arrows. It is impossible for Γ to have a cycle containing only one or two vertices.

11.2. **Theorem.** *An (ordered) block decomposition $\mathcal{D}$ can be re-ordered so as to be subprime if and only if Γ does not contain a cycle. In this case the graph determines all possible re-orderings of $\mathcal{D}$ that are subprime.*

Proof. Observe first that if no arrows originate from a particular vertex j, then $\mathsf{OAS}(B^j)$ does not intersect any of the blocks B^k, and must be contained in U. In this case, $B^j \sqcup U$ is an upper set. On the other hand, if B^j does have an outward arrow, then $\mathsf{OAS}(B^j)$ intersects another block, is not contained in U, and $B^j \sqcup U$ is not an upper set.

If $B^j \sqcup U$ is an upper set, then $L \setminus B^j = L'$ is a lower set, and deleting B^j from $\mathcal{D}$ gives a block decomposition $\mathcal{D}'$ of L'. The graph G' of $\mathcal{D}'$ is obtained from G by deleting the vertex j and all arrows leading to it. If this can be continued until the graph is empty, a re-ordering $\mathcal{D}^*$ has been found that is subprime: the blocks of $\mathcal{D}$ should be permuted into $\mathcal{D}^*$ so that the blocks are added to U beginning with the last block

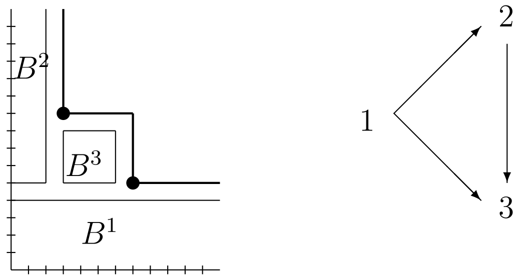

FIGURE 4. The decomposition $\mathcal{D}_2$ and its graph.

of $\mathcal{D}^*$ and ending with the first. If this can be done in several ways, several subprime re-orderings have been found. If a stage is reached in which every remaining vertex is the source of at least one outward arrow, no additional block can be added to the upper set, and $\mathcal{D}$ has no subprime re-ordering.

If the (original) graph G of $\mathcal{D}$ contains a cycle, every vertex in the cycle has an outward arrow, and no block in the cycle can be added to U (or to the growing upper set) before any other. Therefore $\mathcal{D}$ has no subprime re-ordering. Conversely, suppose there is no subprime re-ordering. Then there must come a time when (perhaps after some deletions) every vertex has at least one outward arrow. Since every outward arrow must point to some vertex, and there are finitely many vertices, there must be a cycle. This cycle has to have been present in the original graph G, since deleting vertices and arrows from G cannot produce a cycle. □

The information contained in G may also be presented in a table. The columns will be labeled $1, \ldots, s$ and the entries in the jth column are the values of k to which an arrow from j leads. If the jth column is empty, B^j can be added to U. To iterate, erase each occurrence of j in the table (both in the column headings and in the entries).

11.3. **Example.** The decomposition $\mathcal{D}_2$ in Section 5 is shown in the left-hand side of Figure 4. It can be seen that $\mathsf{OAF}^2(B^1)$ intersects both B^2 and B^3, while $\mathsf{OAF}^1(B^2)$ intersects B^3. Therefore the graph Γ for $\mathcal{D}_2$ is as shown in the right-hand side of Figure 3. The table is as follows.

1	2	3
2	3	
3		

The graph and the table both show that there are no cycles, and that B^3 is the only block that can be added to U first (giving an upper set).

After doing this, all occurrences of 3 in the table can be deleted, and B^2 can be added to the upper set. Then 2 can be removed, and B^1 can be added. Therefore $\mathcal{D}_2$ is subprime in the order that it is written, but not in any other order.

On the other hand, for $\mathcal{D}_3$, the graph is $3 \leftarrow 1 \rightarrow 2$, and we can add either B^2 or B^3 to U first, but must add both before adding B^1. Therefore the subprime re-orderings of $\mathcal{D}_3$ are either $\mathcal{D}_3$ itself (if B^3 is added first), or else $\mathcal{D}_4$ (which is already a re-ordering of $\mathcal{D}_3$). All of the decompositions $\mathcal{D}_1, \ldots, \mathcal{D}_6$ in Section 5 are subprime in one or more rearrangements; $\mathcal{D}_2$ and $\mathcal{D}_5$ are the only ones whose graph is a (noncyclic) triangle. ■

11.4. **Example.** The intersection table for the twisted cube (Example 6.6, Figure 3) is

1	2	3	4	5
2	4	2	3	
3	5	5	5	
4				

This table contains the cycle $2 \rightarrow 4 \rightarrow 3 \rightarrow 2$, so the twisted cube cannot be re-ordered to be subprime.

The calculation to establish the "2" column in the intersection table is as follows.

$$\begin{aligned} \mathsf{OAS}(B^2) &= \mathsf{OAF}^1(B^2) \cup \mathsf{OAF}^2(B^2) \cup \mathsf{OAF}^3(B^2) \\ &= \begin{bmatrix} 2 & \substack{1\\0} & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & \substack{1\\0} & 1 \end{bmatrix}. \end{aligned}$$

Since no point of the unit cube has a coordinate equal to 2, the first two blocks of $\mathsf{OAS}(B^2)$ cannot intersect any B^k. We have, by (3.5) and (3.9),

$$\begin{aligned} \mathsf{OAF}^3(B^2) \cap B^4 &= \begin{bmatrix} 1 & \substack{1\\0} & 1 \end{bmatrix} \cap \begin{bmatrix} \substack{1\\0} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \neq \emptyset, \\ \mathsf{OAF}^3(B^2) \cap B^5 &= \begin{bmatrix} 1 & \substack{1\\0} & 1 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \neq \emptyset. \end{aligned}$$

These calculations account for the presence of 4 and 5 in column 2 (or arrows $2 \rightarrow 4$ and $2 \rightarrow 5$ in the associated graph); the other intersections are empty. Column 4 is similar. Rather than compute the intersections, it is often quicker to show that an intersection is nonempty by exhibiting an element, or that it is empty by exhibiting a contradiction in the requirements for membership. ■

11.5. **Example.** The McLagan-Smith decomposition of the lower set L for $G = \{(1,1,1)\}$ is

$$\mathcal{D} = \left(\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \infty & \infty & 0 \\ 1 & 0 & \end{bmatrix}, \begin{bmatrix} 0 & \infty & \infty \\ & 1 & 0 \end{bmatrix}, \begin{bmatrix} \infty & 0 & \infty \\ 0 & & 1 \end{bmatrix} \right). \tag{11.2}$$

This was introduced in [14] as a Stanley decomposition that is not prime. It is inner-minimal but not organized, since its first block is not outer-maximal. Its intersection table is

1	2	3	4
2	4	2	3
3			
4			

which is the same as that of the first four blocks of the twisted cube, that is, the table in Example 11.4 with all occurrences of 5 deleted. To see this geometrically, expand the twisted cube decomposition (6.1) to infinity by replacing each 1 in the top row of each block by ∞. This gives a decomposition of $\mathbb{N}^3$, in which the last block is $B^5 = \begin{bmatrix} \infty & \infty & \infty \\ 1 & 1 & 1 \end{bmatrix}$, which is just the the upper set U for the McLagan-Smith example. Deleting B^5 yields (11.2). Since the cycle $2 \to 4 \to 3 \to 2$ in the twisted cube graph does not involve vertex 5, the same cycle exists in the McLagan-Smith graph, so (11.2) cannot be re-ordered to be subprime. ■

It does not seem possible to find an organized decomposition in 3 dimensions having the same geometry as the twisted cube and McLagan-Smith examples. Neither have we found a $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-organized decomposition in higher dimensions having that geometry and also satisfying $\boldsymbol{\pi} = \boldsymbol{\sigma}$. (We have not ruled this out, only failed to find one after a few tries in dimensions 4 and 5). The next example shows that an organized decomposition with this geometry does exist in dimension 4 with $\boldsymbol{\pi} \neq \boldsymbol{\sigma}$.

11.6. **Example.** The 4-dimensional $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}')$-organized decomposition (10.6) reduces to the McLagan-Smith decomposition (11.2) if the first column of each block is dropped. The table for (10.6) is exactly the same,

1	2	3	4
2	4	2	3
3			
4			

which again contains the cycle $2 \to 4 \to 3 \to 2$, and the decomposition cannot be re-ordered to be subprime. ■

12. Stacked decompositions

Let $\mathcal{D}$ be an inner-minimal block decomposition of a lower set $L \subset \mathbb{N}^n$. Let B^k and B^j be blocks in $\mathcal{D}$, and $i \in \{1, \ldots, n\}$. We say that B^k is **stacked on B^j in direction** i if

$$\mathsf{IF}^i(B^k) \subseteq \mathsf{OAF}^i(B^j). \tag{12.1}$$

(If the i direction is called "up," then the "bottom" of B^k rests on the "top" of B^j.)

12.1. **Lemma.** *If B^k is stacked on B^j in direction i, then $\mathbf{a}^k - \mathbf{e}_i \in B^j$, $a_i^k > 0$, and $j < k$.*

Proof. Since $\mathbf{a}^k \in \mathsf{IF}^i(B^k)$, the stacked hypothesis implies $\mathbf{a}^k \in \mathsf{OAF}^i(B^j)$. This implies $\mathbf{a}^k - \mathbf{e}_i \in B^j$, which implies $a_i^k > 0$. Since $\mathcal{D}$ is inner-minimal, $\mathbf{a}^k$ is minimal in $L \smallsetminus P^{k-1}$, and $\mathbf{a}^k - \mathbf{e}_i \prec \mathbf{a}^k$; it follows that $\mathbf{a}^k - \mathbf{e}_i \in P^{k-1}$. Of course $\mathbf{a}^k - \mathbf{e}_i$ can only belong to one block in $\mathcal{P}^{k-1}$, and this must be the block B^j. Therefore the block B^j that contains $\mathbf{a}^k - \mathbf{e}_i$ must satisfy $j < k$. □

If, for a particular i, $a_i^k = 0$, then B^k cannot be stacked on any block in direction i. In this case the "bottom" of B^k (in the direction i) is part of the "floor plane" $\mu_i = 0$. (Then B^k may be loosely regarded as "stacked on the floor," but actually its bottom points are part of the floor.)

At this point we may define a function $j = f(k, i)$ that will be very useful. Given an inner-minimal decomposition $\mathcal{D} = (B^1, \ldots, B^s)$ of a lower set L, let $k \in \{1, \ldots, s\}$ and let $i \in \{1, \ldots, n\}$ such that $\alpha_i^k \neq 0$. (Note that these conditions actually exclude $k = 1$, since $\mathbf{a}^1 = \mathbf{0}$ has no nonzero components). Then $\mathbf{a}^k - \mathbf{e}_i$ belongs to some unique block B^j with $j < k$, and we define $f(k, i)$ to be this value of j. Note that f is only defined for a restricted set of (k, i), and may be regarded as a "partial function" if desired. Now $B^{f(k,i)}$ is the only block on which B^k may (or may not) be stacked in direction i.

12.2. **Lemma.** *The the following three conditions are equivalent:*

1. *B^k is stacked on $B^{f(k,i)}$ in direction i;*
2. *for every $h \neq i$, $\mathsf{IE}^h(B^k) \subseteq \mathsf{OAF}^i(B^{f(k,i)})$;*
3. *$\mathbf{a}^k - \mathbf{e}_i \in B^{f(k,i)}$, and for every $h \neq i$, $b_h^k \leq b_h^{f(k,i)}$.*

Proof. The equivalence of items (1) and (2) is an immediate consequence of Lemma 3.4. If (2) holds, then $\mathsf{OAF}^i(B^j)$ is a block containing these inner edges, and must contain the smallest such block, which is $\mathsf{IF}^i(B^k)$. The converse is trivial.

For item (3), Lemma 12.1 shows that the condition $\mathbf{a}^k - \mathbf{e}_i \in B^j$, with $j = f(k,i)$, picks out the only block B^j on which B^k might be stacked in direction i. Since $\mathbf{a}^k \in B^k$ and $\mathbf{a}^k - \mathbf{e}_i \in B^j$, it follows that $\mathbf{a}^k - \mathbf{e}_i \in \mathsf{OF}^i(B^j)$. By (3.8),

$$\mathsf{OF}^i(B^j) = \begin{bmatrix} b_1^j & \cdots & b_{i-1}^j & \multirow{2}{*}{b_i^j} & b_{i+1}^j & \cdots & b_n^j \\ a_1^j & \cdots & a_{i-1}^j & & a_{i+1}^j & \cdots & a_n^j \end{bmatrix}.$$

Therefore the components of $\mathbf{a}^k$ satisfy

$$a_h^j \le a_h^k \le b_h^j \quad (h \ne i) \quad \text{and} \quad a_i^k - 1 = b_i^j. \tag{12.2}$$

Now rewrite item (2) using (3.9) and (3.10); we cannot write out every instance, but if $i = 1$ then the relevant values of h are those with $h > 1$, and the result of the rewriting is

$$\begin{bmatrix} a_1^k & a_2^k & \cdots & a_{h-1}^k & \begin{matrix} b_h^k \\ a_h^k \end{matrix} & a_{h+1}^k & \cdots & a_n^k \end{bmatrix} \subseteq$$
$$\begin{bmatrix} (b_1^j + 1) & \begin{matrix} b_2^j \\ a_2^j \end{matrix} & \begin{matrix} \cdots \\ \cdots \end{matrix} & \begin{matrix} b_{h-1}^j \\ a_{h-1}^j \end{matrix} & \begin{matrix} b_h^j \\ a_h^j \end{matrix} & \begin{matrix} b_{h+1}^j \\ a_{h+1}^j \end{matrix} & \begin{matrix} \cdots \\ \cdots \end{matrix} & \begin{matrix} b_n^j \\ a_n^j \end{matrix} \end{bmatrix}.$$

This inclusion is equivalent to the inclusion of each column of the first matrix (regarded as an interval) in the same column of the second matrix; for instance, $[a_h^k, b_h^k] \subseteq [a_h^j, b_h^j]$, which in turn says that $a_h^j \le a_h^k$ and $b_h^k \le b_h^j$. The first of these in proved in (12.2), while the second is not, but is assumed in item (3). The information in the remaining columns is contained in (12.2). Therefore (2) is equivalent to (3). □

Next we define what it means for an entire decomposition $\mathcal{D}$ to be stacked (rather than one particular block in $\mathcal{D}$ being stacked on another in a particular direction). We say that $\mathcal{D}$ is a **stacked decomposition** of lower set L if

1. $\mathcal{D}$ is an inner minimal block decomposition of L, and
2. for every $k \in \{1, \ldots, s\}$ and for every i such that $a_i^k > 0$, B^k is stacked in direction i on the block $B^{f(k,i)}$. As noted above, we may assume $k \ge 2$.

The following theorem "justifies" the definition of stacked decompositions (as Theorem 7.2 did for organized decompositions).

12.3. **Theorem.** *A stacked decomposition is subprime.*

Proof. Let $\mathcal{D} = \mathcal{P}^s$ be stacked. By Theorem 11.1, it is sufficient to prove that each P^k, $0 \le k \le s$, is a lower set. Suppose $\mathbf{q} \in P^k$ and $\mathbf{p} \prec \mathbf{q}$; we must prove $\mathbf{p} \in P^k$. To do so we consider a sequence $\mathbf{p}_1, \ldots, \mathbf{p}_v$

beginning with $\mathbf{p}_1 = \mathbf{q}$, ending with $\mathbf{p}_v = \mathbf{p}$, and nonincreasing (under division partial order $\prec$) by one unit at each step; that is,

$$\mathbf{q} = \mathbf{p}_1 \succ \mathbf{p}_2 \succ \cdots \succ \mathbf{p}_v = \mathbf{p},$$

with each $\mathbf{p}_{(u+1)} = \mathbf{p}_u - \mathbf{e}_d$ for some direction $d \in \{1, \ldots, n\}$ (where d may differ at each step). Such sequences always exist, usually in more than one way, and any of these can be used in the following argument. We then prove by induction that each $\mathbf{p}_u \in P^k$. Then $\mathbf{p} = \mathbf{p}_v \in P^k$, proving that P^k is a lower set.

The first step (that $\mathbf{p}_1 \in P^k$) is already known. So we assume $\mathbf{p}_u \in P^k$. So we consider $\mathbf{p}_{(u+1)}$. Since $\mathbf{p}_u \in P^k$, $\mathbf{p}_u$ is in some block B^t of $\mathcal{P}^k$. Since $\mathbf{p}_{(u+1)} = \mathbf{p}_u - \mathbf{e}_d$ for some d, we first ask whether this point remains in B^t or moves to a different block. There are three possible cases (considering all possible directions of motion d).

1. If $\mathbf{p}_u$ does not belong to any inner face of B^t, then $\mathbf{p}_{u+1}$ remains in B^t regardless of the direction of motion d. In this case $\mathbf{p}_{(u+1)}$ remains in P^k, and the induction step is already completed.
2. If $\mathbf{p}_u$ belongs to an inner face of B^t, then it may belong to one or more such inner faces. (For instance the inner corner of B^t belongs to all such inner faces.) If $\mathbf{p}_u \in \mathsf{IF}^i(B^t)$ for one or more directions i, but not for $i = d$ (the direction of motion), then passing to $\mathbf{p}_{(u+1)}$ merely moves the point within each of its inner faces, so again the point remains in B^t and the induction is completed as in case 1.
3. If $\mathbf{p}_u \in \mathsf{IF}^i(B^t)$ for one or more directions i, and one of these directions is the direction of motion d, then $\mathbf{p}_{(u+1)}$ is no longer in B^t, but has exited B^t through the inner face $\mathsf{IF}^d(B^t)$. Then the induction can no longer be completed as in cases 1 and 2. Instead we use (for the first time) the assumption that $\mathcal{D}$ is stacked, which implies that $\mathsf{IF}^d(B^t) \subseteq \mathsf{OAF}^d(B^{f(t,d)})$. So since $\mathbf{p}_u \in \mathsf{IF}^d(B^t)$ then $\mathbf{p}_u \in \mathsf{OAF}^d(B^{f(t,d)})$, and since $\mathbf{p}_{u+1} = \mathbf{p}_u - \mathbf{e}_d$ then $\mathbf{p}_{u+1} \in B^{f(t,d)}$. Since $f(t,d) < t$, by Lemma 12.1, and $t < k$, we again have $\mathbf{p}_{u+1} \in B^k$ and therefore in P^k, as desired.

This completes the induction for all cases. □

The next corollary shows that the issue of collisions with previous blocks does not arise in constructing stacked decompositions.

12.4. **Corollary.** *Let $\mathcal{D} = \mathcal{P}^s$ be a stacked decomposition. Then $H^k = \emptyset$ for each k. That is, there are no temporary generators of the type needed to prevent intersections with previous blocks.*

Proof. Suppose that H^k is nonempty. Then there is a point in P^{k-1} greater than $\mathbf{a}^k$. It follows from the proof of Theorem 12.3 that $\mathbf{a}^k \in P^{k-1}$ (because P^{k-1} is a lower set). But this contradicts the fact that $\mathbf{a}^k \in \overline{P^{k-1}}$. □

It follows from item 3 of Lemma 12.2 that a stacked decomposition must satisfy

$$b_h^k \leq b_h^{f(k,i)} \tag{12.3}$$

for each k and for each pair $i, h \in \{1, \ldots, n\}$ such that $a_i^k > 0$ and $h \neq i$. Now let

$$c_h^k = \min\{b_h^{f(k,i)} : a_i^k > 0 \quad \text{and} \quad i \neq h\}.$$

Then equation (12.3) is equivalent to $b_h^k < c_h^k + 1$ for each k and h. Next let

$$K^k = \{(c_h^k + 1)\mathbf{e}_h : h = 1, \ldots, n\}.$$

12.5. **Lemma.** *The inequalities (12.3) are equivalent to the statement that* $\mathbf{b}^k$ *does not belong to the upper set* $\langle K^k \rangle$ *generated by* K^k*.*

Proof. Both of these say that $\mathbf{b}^k$ is not divisible by any element of K^k. □

The important things to notice here are that both H^k and K^k work in the same way to restrict the size of B^k (namely, by requiring that $\mathbf{b}^k$ not be divisible by an element of H^k or K^k); and that K^k imposes stronger restrictions than H^k (since $H^k = \emptyset$ by Corollary 12.4, but K^k need not be empty).

Recall that in Section 7 we moved from *inner-minimal block decompositions* to *organized decompositions* by adding an outer-maximality condition, requiring that any extension $(B^k)'$ of B^k must intersect either P^{k-1} or U or both. We now make a similar move for stacked decompositions, but with a different definition of outer-maximality. In order to avoid confusion with the definition of "outer-maximal," we refer to the new condition instead as "maximally-stacked." A decomposition $\mathcal{D}$ of a lower set L is a **maximally-stacked** decomposition if no block B^k of $\mathcal{D}$ can be extended to a block $(B^k)'$ (having the same inner corner) without intersecting P^{k-1}, intersecting U, **or failing to be stacked** on $B^{f(k,h)}$ in **some** direction h with $a_d^k > 0$. (The logical negation of "every i" in the definition of stacked decomposition is "some i.") We can immediately drop the part about intersecting P^{k-1}, since this is impossible, according to Corollary 12.4.

12.6. **Lemma.** *If $\mathcal{D} = \mathcal{P}^s$ is a maximally-stacked block decomposition of L, each block of $\mathcal{D}$ is a (disjoint) union of blocks of the elementary block decomposition of L.*

Proof. The proof is similar to that of Theorem 7.3, except for the change in the maximality condition. So there is no change in the first part of the proof, showing that $a_i^k \in S_i$. It remains to prove that $b_i^k + 1 \in S_i$ whenever $b_i^k < \infty$. The proof is by complete induction, with induction hypothesis that B^j is a union of elementary blocks for all $j < k$.

Suppose $b_i^k < \infty$. Then, as before, the block $(B_i^k)'$ defined by (7.1) is an extension of B^k and, because $\mathcal{D}$ is maximally-stacked, it must either intersect U, or fail to be stacked on $B^{f(k,h)}$ for some direction $h \in \{1, \ldots, n\}$ of stacking.

If $(B^k)'$ intersects U, then its outer corner $\mathbf{b}^k + \mathbf{e}_i$ belongs to U because U is an upper set. Let F be the unique G-elementary block containing $\mathbf{b}^k + \mathbf{e}_i$. Then $F \subset U$, since every elementary block lies entirely in L or in U. Then $\mathbf{b}^k$, which is in L, does not belong to F, but $\mathbf{b}^k + \mathbf{e}_i$ does, so $\mathbf{b}^k + \mathbf{e}_i \in \mathsf{IF}^i(F)$. By Lemma 4.1 this implies that $b_i^k + 1 \in S_i$, as desired.

If, instead, $(B^k)'$ fails to be stacked on $B^{f(k,h)}$ for some h, then for that value of h, B^k is stacked on $B^{f(k,h)}$ but $(B^k)'$ is not. Fix h to be such a direction.

First, we write the condition that B^k is stacked on $B^{f(k,h)}$ in the form of item 3 of Lemma 12.2, after replacing some the variables. The stacking direction i is replaced by h (since h is the new stacking direction), and "for every $h \neq i$" is replaced by "for every $d \neq h$". Now item three of Lemma 12.2 reads

$$\mathbf{a}^k - \mathbf{e}_h \in B^{f(k,h)}, \text{ and for every } d \neq h,\ b_d^k \leq b_d^{f(k,h)}. \tag{12.4}$$

Second, $(B^k)' = \begin{bmatrix} \mathbf{b}^k + \mathbf{e}_i \\ \mathbf{a}^k \end{bmatrix}$ is *not* stacked on $B^{f(k,h)}$. Writing this condition in the form of item 3 of Lemma 12.2, using the variable in (12.4) and the correct logical negation, either $\mathbf{a}^k - \mathbf{e}_h \notin B^{f(k,h)}$ or there exists $d \neq h$ such that the d component of $\mathbf{b}^k + \mathbf{e}_i$ is greater than $b_d^{f(k,h)}$. But $\mathbf{a}^k - \mathbf{e}_h \in B^{f(k,h)}$, so there exists $d \neq h$ such that the d component of $\mathbf{b}^k + \mathbf{e}_i$ is greater than $b_d^{f(k,h)}$. We will now show that $d = i$, by contradiction. Suppose $d \neq i$. Then the dth component of $\mathbf{e}_i$ is zero, so

$$b_d^k > b_d^{f(k,h)} \text{ if } d \neq i \quad \text{(FALSE)}. \tag{12.5}$$

Equations 12.4 and 12.5 give a contradiction. Therefore $d = i$ (which also implies $i \neq h$). Now use once again the fact that the d component of $\mathbf{b}^k + \mathbf{e}_i$ is greater than $b_d^{f(k,h)}$ for $d \neq h$, but this time with $d = i \neq h$. The ith component of $\mathbf{e}_i$ is 1, so we no longer get a contradiction, but instead obtain

$$b_i^k + 1 > b_i^{f(k,h)}. \tag{12.6}$$

Subtracting 1 from the left-hand side of (12.6) gives $b_i^k \geq b_i^{f(k,h)}$, and combining this with (12.4) using $d = i$ gives

$$b_i^k = b_i^{f(k,h)}. \tag{12.7}$$

Since $f(k,h) < k$ the inductive hypothesis says that $B^{f(k,h)}$ is a union of elementary blocks, and then Corollary 4.4 implies that $b_i^{f(k,h)} + 1 \in S_i$. By (12.7), $b_i^k + 1 \in S_i$ as desired. This completes the induction. □

The $(\boldsymbol{\pi}, \boldsymbol{\sigma})$**-stacked decomposition** of L is defined to be the outer-maximal stacked decomposition of L having inner corners chosen by the lexicographic method using $<_{\boldsymbol{\sigma}}$, and blocks built by directional extension using the order $\boldsymbol{\pi}$, with H^k replaced by K^k. The algorithm for this is as follows.

12.7. **Algorithm.** *To create the* $(\boldsymbol{\pi}, \boldsymbol{\sigma})$*-stacked block decomposition of a lower set* $L \subset \mathbb{N}^n$*:*

1. *Input the minimal set* G *of generators of* $U = \mathbb{N}^n \setminus L$ *and the desired permutations* $\boldsymbol{\pi}$ *and* $\boldsymbol{\sigma}$ *of* $\{1, \ldots, n\}$*.*
2. *Find the elementary decomposition of* L*, either by Algorithm 3.1 of* [19] *or by Section 4 above.*
3. *Initialize a dynamic list* $\mathcal{L}$ *of the inner corners of elementary blocks of* L*, ordered (from low to high) by the total order* $<_\sigma$*. This list will be updated in step 5e.*
4. *Set* $P^0 = \emptyset$*.*
5. *For* $k = 1, 2, \ldots$ *do the following until* $\mathcal{L}$ *is empty:*
   a. *Let* $\mathbf{a}^k$ *be the first element of* $\mathcal{L}$ *(the list in its current form).*
   b. *Compute* $\mathbf{a}^k - \mathbf{e}_i$ *for each* i *such that* $a_i^k > 0$*, and determine the (already constructed) block* $B^{f(k,i)}$ *to which it belongs. From the outer corners* $\mathbf{b}^{f(k,i)}$ *of these blocks, determine* $c_h^k = \min\{b_h^{f(k,i)} : a_i^k > 0 \text{ and } i \neq h\}$ *for each* $h \in \{1, \ldots, n\}$*, and form the set* K^k *of points* $\overline{(c_h^k + 1)}\mathbf{e}_h$*.*
   c. *Compute the* $\boldsymbol{\pi}$*-extension* B^k *of* $\{\mathbf{a}^k\}$ *in* $\overline{P^{k-1}}$ *by Algorithm 9.3, with* H^k *replaced by* K^k*. (See Remark 9.5 if* $\boldsymbol{\pi} \neq \boldsymbol{\varepsilon}$*.)*
   d. *Set* $\mathcal{P}^k = (B^1, \ldots, B^k)$ *and* $P^k = \bigsqcup \mathcal{P}^k$*.*

e. *Delete all elements of $\mathcal{L}$ that belong to P^k. (This always includes deleting at least $\mathbf{a}^k$).*

f. *Repeat with the updated list $\mathcal{L}$ unless $\mathcal{L}$ is empty.*

6. *Set $s = k$ and output $\mathcal{D} = \mathcal{P}^s$ as the desired decomposition of L.*

12.8. **Example.** The $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})$-stacked decomposition of L for $G = \{(1,0,9),(1,5,0),(7,0,0)\}$. (Compare Example 10.5.)

The first block $B^1 = \begin{bmatrix} 6 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ and the second inner corner $\mathbf{a}^2 = (0,0,9)$ are the same as in the $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})$-organized case. But B^2 is different. The only positive entry in $\mathbf{a}^2$ is the third ($i = 3$). Then $\mathbf{a}^k - \mathbf{e}_i = \mathbf{a}^2 - \mathbf{e}_3 = (0,0,9) - (0,0,1) = (0,0,8)$, which belongs to B^1, so $f(2,3) = 1$, and this is the only value of the f function. Since $\mathbf{b}^1 = (6,4,8)$, and $b_h^{f(k,i)} = b_h^1$ for each $h \neq i$ (that is, for $h = 1,2$), we have $b_1^1 = 6$ and $b_2^1 = 4$. Since there is only one such number for each h, there is no minimum to take, and $c_1^2 = 6 + 1 = 7$, $c_2^2 = 5$. Therefore

$$K^2 = \{7\mathbf{e}_1, 5\mathbf{e}_2\} = \{(7,0,0),(0,5,0)\};$$

these are the points that block the extension of $\{\mathbf{a}^2\}$ in the directions 1 and 2. (This is geometrically completely obvious; $\mathbf{a}^2 = (0,0,9)$ rests on B^1, whereas $(7,0,9)$ and $(0,5,9)$ would overhang B^1. With a more complicated geometry, the algorithm would be necessary.) Taking G into account shows that $(1,0,9)$ blocks the extension of $\mathbf{a}^2$ in the 1 direction, so $t^* = 0$ as in the organized case, and $(7,0,0) \in K^2$ has no effect. But in direction 2, where $t^* = \infty$ for the organized decomposition, $(0,5,0)$ blocks the extension at $t^* = 4$. There is no obstacle in direction 3, so $B^2 = \begin{bmatrix} 0 & 4 & \infty \\ & 0 & 9 \end{bmatrix}$.

The full $(\boldsymbol{\pi}, \boldsymbol{\sigma})$-decomposition is given by

$$\mathcal{D} = \left(\begin{bmatrix} 6 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 & \infty \\ & 0 & 9 \end{bmatrix}, \begin{bmatrix} 0 & \infty & 9 \\ & 5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 8 & \infty \\ & 5 & 10 \end{bmatrix} \right).$$

Geometrically, this is clear; $\mathbf{a}^3 = (0,5,0)$ as in the organized case, and the extension in the 3-direction is stopped at 9 by the requirement that B^3 be stacked on B^1. The elementary inner corner $(0,5,9)$ in (10.2) is not contained in the first three blocks this time, and remains to form the inner corner of the fourth block. This stacked decomposition is not incompressible, as the organized decompositions are, because $B^3 \sqcup B^4$ is itself a block. ■

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