2016

Carathéodory’s Theorem and moduli of local connectivity

Timothy H. McNicholl
Iowa State University, mcnichol@iastate.edu

Follow this and additional works at: http://lib.dr.iastate.edu/math_pubs
Part of the Analysis Commons, and the Geometry and Topology Commons

The complete bibliographic information for this item can be found at http://lib.dr.iastate.edu/math_pubs/109. For information on how to cite this item, please visit http://lib.dr.iastate.edu/howtocite.html.
CARATHÉODORY’S THEOREM AND MODULI OF LOCAL CONNECTIVITY

TIMOTHY H. MCNICHOLL

Abstract. We give a constructive proof of the Carathéodory Theorem by means of the concept of a modulus of local connectivity and the extremal distance of the separating curves of an annulus.

1. Introduction

The goal of this paper is to give a new proof of the Carathéodory Theorem which states that if $D$ is a Jordan domain, and if $\phi$ is a conformal map of $D$ onto the unit disk, then $\phi$ extends to a homeomorphism of $D$ with the closed unit disk (see e.g. [4] and [5]). This proof has a feature which appears to be new in that for each $\zeta \in \partial D$ it explicitly constructs a $\delta$ for each $\epsilon$ when proving the existence of $\lim_{z \to \zeta} \phi(z)$. Furthermore, a closed form expression for $\delta$ in terms of $\epsilon$ and $\zeta$ is obtained. Such expressions are potentially useful when estimating error in numerical computations. The proof also makes two seemingly new connections. First, we construct $\delta$ from $\epsilon$ by means of a modulus of local connectivity for the boundary of $D$. Roughly speaking, this is a function that predicts how close two boundary points must be in order to connect them with a small arc that is included in the boundary. Second, the proof constructs an upper bound on $|\phi(z) - \phi(\zeta)|$ from the extremal distance of the separating curves of an annulus.

The paper is organized as follows. Section 2 covers background material. Section 3 states the main ideas of the proof. Sections 4 and 5 deal with topological preliminaries. Our estimates are proven in Section 6 and Section 7 completes the proof.

2. Background

Let $\mathbb{N}$ denote the set of non-negative integers.

When $A$ is an annulus with inner radius $r$ and outer radius $R$, let

$$\lambda(A) = \frac{2\pi}{\log(R/r)}.$$ 

$\lambda(A)$ is the extremal length of the family of separating curves of $A$; see e.g. [3].

When $X$, $Y$, and $Z$ are subsets of the plane, we say that $X$ separates $Y$ from $Z$ if $Y$ and $Z$ are included in distinct connected components of $\mathbb{C} - X$. In the case where $Y = \{p\}$, we say that $X$ separates $p$ from $Z$. In the case where $Y = \{p\}$ and $Z = \{q\}$ we say that $X$ separates $p$ from $q$.

A topological space is locally connected if it has a basis of open connected sets. By the Hahn-Mazurkiewicz Theorem, every curve is locally connected; see e.g.

1991 Mathematics Subject Classification. 30.

Key words and phrases. Complex analysis, conformal mapping.
Section 3-5 of [6]. Suppose $X$ is a compact and connected metric space. Then, $X$ is locally connected if and only if it is uniformly locally arcwise connected. This means that for every $\epsilon > 0$, there is a $\delta > 0$ so that whenever $p, q \in X$ and $0 < d(p, q) < \delta$, $X$ includes an arc from $p$ to $q$ whose diameter is smaller than $\epsilon$ (although its length may be infinite); again, see Section 3-5 of [6]. Accordingly, $f$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $p, q \in X$ and $0 < d(p, q) < \delta$, $X$ includes an arc from $p$ to $q$ whose diameter is smaller than $2^{-k}$. Thus, a metric space is uniformly locally arcwise connected if and only if it has a modulus of local connectivity. Note that if $f$ is a modulus of local connectivity, then $\lim_{k \to \infty} f(k) = \infty$. In addition, if a metric space has a modulus of local connectivity, then it has a modulus of local connectivity that is increasing.

Moduli of local connectivity originated in the adaptation of local connectivity properties to the setting of theoretical computer science in [1] and [2]. Computational connections between moduli of local connectivity and boundary extensions of conformal maps are made in [7]. Here, we attempt to show that this notion may be useful in more traditional mathematical settings.

3. OUTLINE OF THE PROOF

We first observe the following which is proven in Section 4.

**Theorem 3.1.** If $\zeta_0$ is a boundary point of a simply connected Jordan domain $D$, then for every $r > 0$, $\zeta_0$ is a boundary point of exactly one connected component of $D_r(\zeta_0) \cap D$.

Suppose $\zeta_0$ is a boundary point of a simply connected Jordan domain $D$. In light of Theorem 3.1, when $r > 0$ we let $C(D; \zeta_0, r)$ denote the connected component of $D_r(\zeta_0) \cap D$ whose boundary contains $\zeta_0$. Suppose $\phi$ is a conformal map of $D$ onto the unit disk. The fundamental strategy of the proof is to bound the diameter of $\phi[C(D; \zeta_0, r)]$. To do so, we first construct an upper bound on the diameter of $\phi[C]$ where $C$ is a connected component of $D_r(\zeta) \cap D$ for some point $\zeta$ in the complement of $D$. Namely, in Section 6 we prove the following.

**Theorem 3.2.** Let $\phi$ be a conformal map of a domain $D$ onto the unit disk. Suppose $\mathcal{A}$ is an annulus so that $\mathcal{A}$ separates its center from $\phi[D_r(0)]$ where $r \geq \sqrt{\pi\lambda(\mathcal{A})}$. Suppose $C$ is a connected component of the points of $D$ that are interior to the inner circle of $\mathcal{A}$. Then, the diameter of $\phi[C]$ is at most $\sqrt{l^2 + 4\pi\lambda(\mathcal{A})}$ where $l = 1 + \sqrt{r^2 - \pi\lambda(\mathcal{A})}$.

Note that Theorem 3.2 applies to non-Jordan domains.

With Theorem 3.2 in hand, some basic calculations, which we perform in Section 6, lead us to the following.

**Theorem 3.3.** Suppose $\phi$ is a conformal map of a Jordan domain $D$ onto the unit disk. Let $\zeta_0$ be a boundary point of $D$, and let $\epsilon > 0$. Then, the diameter of $\phi[C(D; \zeta_0, r_0)]$ is smaller than $\epsilon$ whenever $r_0$ is a positive number that is smaller than

$$
\sup_{0 < l < \epsilon} \left( \exp \left( \frac{8\pi^2}{l^2 - \epsilon^2} \right) \min \left\{ |\zeta_0 - \phi^{-1}(w)| : |w| \leq \sqrt{(1-l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\} \right).
$$


When $0 < \epsilon < 1$ and $l = \frac{\epsilon}{2}$,
\[
\frac{7}{16} < (1 - l)^2 + \frac{\epsilon^2 - l^2}{4} < 1.
\]
Thus, (3.1) is positive when $0 < \epsilon < 1$. In other words, for all sufficiently small $\epsilon > 0$, there is a positive number $r_0$ that is smaller than (3.1).

So, suppose $\phi$ is a conformal map of a Jordan domain $D$ onto the unit disk. We use Theorem 3.3 to form an extension of $\phi$ to $\overline{D}$ as follows. Let $\zeta_0$ be a boundary point of $D$. Note that $C(D; \zeta_0, r') \subseteq C(D; \zeta_0, r)$ when $0 < r' < r$. It follows from Theorem 3.3 that there is exactly one point in
\[
\bigcap_{r > 0} \phi(C(D; \zeta_0, r)).
\]
We define this point to be $\phi(\zeta_0)$.

Our next goal is to show that this extension of $\phi$ is continuous. That is, $\lim_{z \to \zeta} \phi(z) = \phi(\zeta)$ whenever $\zeta$ is a boundary point of $D$. This is accomplished by showing that $z \in C(D; \zeta, r)$ whenever $z \in D$ is sufficiently close to $\zeta$. This is where we begin to use moduli of local connectivity. Namely, in Section 4 we prove the following.

**Theorem 3.4.** Suppose $g$ is a modulus of local connectivity for a Jordan curve $\sigma$. Suppose $D$ is an open disk whose boundary separates two points of $\sigma$. Suppose $z_0$ and $\zeta_0$ are points so that $\zeta_0 \in \sigma \cap D$, $z_0 \in D - \sigma$, and $|z_0 - \zeta_0| < 2^{-g(k)}$ where $2^{-k + 2^{-g(k)}} \leq \max\{d(\zeta_0, \partial D), d(z_0, \partial D)\}$. Then, $\zeta_0$ is a boundary point of the connected component of $z_0$ in $D - \sigma$.

Theorem 3.4 was previously proven by means of the Carathéodory Theorem in [8]. We give another proof here with a few extra topological steps so as to avoid circular reasoning.

We then obtain the following form of the Carathéodory Theorem from Theorems 3.3 and 3.4.

**Theorem 3.5.** Suppose $\phi$ is a conformal map of a Jordan domain $D$ onto the unit disk. Let $\zeta_0$ be a boundary point of $D$. Then, $\lim_{z \to \zeta_0} \phi(z) = \phi(\zeta_0)$. Furthermore, if $g$ is a modulus of local connectivity for the boundary of $D$, then for each $\epsilon > 0$, $|\phi(z_0) - \phi(\zeta_0)| < \epsilon$ whenever $z_0$ is a point in $D$ so that $|z_0 - \zeta_0| < 2^{-g(k)}$ and $k$ is a non-negative integer so that $2^{-k + 2^{-g(k)}}$ is smaller than (3.1). Finally, the extension of $\phi$ to $\overline{D}$ is a homeomorphism of $\overline{D}$ with the closed unit disk.

The proof of Theorem 3.5 is given in Section 7.

Suppose $\phi, D, g, \zeta_0$ are as in Theorem 3.5. Without loss of generality suppose $g$ is increasing. Thus $2^{-k} + 2^{-g(k)} \leq 2^{-k+1}$. Let $0 < \epsilon < 1$. We define a positive number $\delta(\zeta_0, \epsilon)$ so that $|\phi(z) - \phi(\zeta_0)| < \epsilon$ whenever $|z - z_0| < \delta(\zeta_0, \epsilon)$. Let:
\[
k(\zeta_0, \epsilon) = 2 - \sup_{0 < l < \epsilon} \left( \frac{8\pi^2}{l^2 - \epsilon^2} + \min \left\{ \log |\zeta_0 - \phi^{-1}(w)| : |w| \leq \sqrt{(1 - l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\} \right)\]
\[
\delta(\zeta_0, \epsilon) = 2^{-k(\zeta_0, \epsilon)} + 2^{-g(k(\zeta_0, \epsilon))}.
\]
(Here, $[x]$ denotes the largest integer that is not larger than $x$.) Thus, by Theorem 3.5, $|\phi(z) - \phi(\zeta_0)| < \epsilon$ whenever $z \in D$ and $|z - \zeta_0| < \delta(\zeta_0, \epsilon)$. 

4. Proofs of Theorems 3.1 and 3.4

Theorem 3.4 is used to prove Theorem 3.1. The proof of Theorem 3.4 is based on the following lemma and theorem.

**Lemma 4.1.** Let $D$ be a Jordan domain. Let $\alpha$ be a crosscut of $D$, and let $\gamma_1, \gamma_2$ be the subarcs of the boundary of $D$ that join the endpoints of $\alpha$. Then, the interior of $\gamma_1 \cup \alpha$ is one side of $\alpha$, and the interior of $\gamma_2 \cup \alpha$ is the other side of $\alpha$.

**Proof.** Let $U_j$ denote the interior of $\alpha \cup \gamma_j$. Choose a point $p$ in $\alpha \cap D$. There is a positive number $\delta$ so that $D_\delta(p) \subseteq D$. Since $p$ is a boundary point of $U_j$, $U_j \cap D_\delta(p)$ is non-empty. So, let $q_j \in U_j \cap D_\delta(p)$, and let $D_j$ be the side of $\alpha$ that contains $q_j$.

We show that $U_j = D_j$: $U_j$ is a connected subset of $D - \alpha$ that contains a point of $D_j$ (namely $q_j$). So, $U_j \subseteq D_j$. On the other hand, $D_j$ is a connected subset of $\mathbb{C} - (\gamma_j \cup \alpha)$ that contains a point of $U_j$. So, $D_j \subseteq U_j$.

$D_1 \neq D_2$ since $\partial D_1 \neq \partial D_2$. Thus, $U_1$ and $U_2$ are the two sides of $\alpha$. $\square$

**Theorem 4.2.** Let $D$ be an open disk, and let $\sigma$ be a Jordan curve. Suppose the boundary of $D$ separates two points of $\sigma$. Let $C$ be a connected component of $D - \sigma$. Then, $C$ is the interior of a Jordan curve. Furthermore, if $p$ is a boundary point of $C$ that also lies in $D$, then $p$ lies on $\sigma$ and the boundary of $C$ includes the connected component of $p$ in $D \cap \sigma$.

**Proof.** Since $C \neq D$, the boundary of $C$ contains a point of $\sigma$; let $p$ denote such a point.

Since the boundary of $D$ separates two points of $\sigma$, if $G$ is a connected component of $D \cap \sigma$, then $\overline{G}$ is a crosscut of $D$.

Let $E$ denote the connected component of $p$ in $\sigma \cap D$. Since $C$ is a connected subset of $D - E$, there is a side of $E$ that includes $C$; let $E^-$ denote this side, and let $E^+$ denote the other side. By Lemma 4.1, each of these sides is a Jordan domain. Again, since the boundary of $D$ separates two points of $\sigma$, if $G$ is a connected component of $\sigma \cap E^-$, then $\overline{G}$ is a crosscut of $E^-$.

We aim to show that the boundary of $C$ is a Jordan curve which includes $E$. To this end, we construct an arc $F$ so that $E \cup F$ is a Jordan curve whose interior is $C$. $F$ will be a union of subarcs of $\sigma$ and connected subsets of the boundary of $D$. To define these subarcs of $\sigma$, we define a partial ordering of the connected components of $\sigma \cap E^-$. Namely, when $G_1, G_2$ are connected components of $\sigma \cap E^-$, write $G_1 \prec G_2$ if $G_2$ is between $G_1$ and $E$; that is if $E$ and $G_1$ lie in opposite sides of $\overline{G_2}$.

Since $\sigma$ is locally connected, it follows that there is no increasing chain $G_1 \prec G_2 \prec G_3 \prec \ldots$. It then follows that if $G_1$ is a connected component of $\sigma \cap E^-$, then there is a $\prec$-maximal component of $\sigma \cap E^-$, $G$, so that $G_1 \preceq G$.

We now define $F$. Let $F' = \partial E^+ \cap \partial D$. Thus, $E \cup F' = \partial E^-$. Let $\mathcal{M}$ denote the set of all $\preceq$-maximal components of $\sigma \cap E^-$. For each $G \in \mathcal{M}$, let $\lambda_G$ be the subarc of $F'$ that joins the endpoints of $\overline{G}$. Let $F$ be formed by removing each $\lambda_G$ from $F'$ and replacing it with $\overline{G}$.

Thus, $F$ is an arc that joins the endpoints of $E$ and that contains no other points of $E$. Let $J = E \cup F$. Then, $J$ is a Jordan curve. We show that $C$ is the interior of $J$. Note that since $J \subseteq \overline{E^+}$, $E^-$ includes the interior of $J$. 

When \( G \in \mathcal{M} \), let \( G^+ \) be the side of \( \overline{G} \) that includes \( E \) (when \( \overline{G} \) is viewed as a crosscut of \( D \) rather than \( E^- \)), and let \( G^- \) denote the other side. The rest of the proof revolves around the following four claims.

1. For each \( G \in \mathcal{M} \), the exterior of \( J \) includes \( G^- \).
2. The interior of \( I \) includes \( \bigcap_{G \in \mathcal{M}} G^+ \cap E^- \).
3. For each \( G \in \mathcal{M} \), \( G^+ \) includes \( C \).
4. The interior of \( J \) contains no point of \( \sigma \).

Claims (2) and (3) together imply that the interior of \( J \) includes \( C \). Claim (1) will be used to prove (4). Claim (4) shows that the interior of \( J \) is included in a connected component of \( D - \sigma \) which then must be \( C \).

We begin by proving (1). Let \( p' \in G^- \). Let \( z_0 \in \mathbb{C} - \overline{D} \). Thus, \( z_0 \) is exterior to \( J \) since \( J \subseteq \overline{D} \). We construct an arc from \( p' \) to \( z_0 \) that contains no point of \( J \). Let \( q \in \lambda G - \overline{G} \). By Lemma 4.1, \( G^- \) is the interior of \( G \cup \lambda G \). So, there is an arc \( \sigma_1 \) from \( p' \) to \( q \) so that \( \sigma' \cap \partial G^- = \{ q \} \). There is an arc \( \sigma_2 \) from \( q \) to \( z_0 \) so that \( \sigma_2 \cap \partial \overline{D} = \{ q \} \). Thus, \( \sigma_1 \cup \sigma_2 \) is an arc from \( p' \) to \( z_0 \) that contains no point of \( J \). Thus, \( p' \) is exterior to \( J \) for every \( p' \in G^- \).

We now prove (2). Suppose \( p_0 \in E^- \) belongs to \( G^+ \) for every \( G \in \mathcal{M} \). By way of contradiction, suppose \( p_0 \) is exterior to \( J \). Again, let \( z_0 \in \mathbb{C} - \overline{D} \). Thus, the exterior of \( J \) includes an arc from \( p_0 \) to \( z_0 \); let \( \alpha \) denote such an arc. By examination of cases, \( \alpha \) cannot cross the boundary of \( D \) at any boundary point of \( E^- \). So, it must do so at a boundary point of \( E^+ \). But, this entails that \( \alpha \) crosses \( E \) which it does not since \( J \) includes \( E \). This is a contradiction, and so \( p_0 \) is interior to \( J \).

Next, we prove (3). Let \( G \in \mathcal{M} \). Since \( \sigma \) is locally connected, and since \( p \in E \), there is a positive number \( \delta \) so that \( D_\delta(p) \) contains no point of any connected component of \( \sigma \cap E^- \). However, this disk must contain a point of \( C, p' \). So, \([p', p] \cap E^- \) contains a point of \( E \) but no point of \( G \). Hence, \( p' \in G^+ \). Since \( C \) is a connected subset of \( D - G, C \subseteq G^+ \).

Finally, we prove (4). By way of contradiction, suppose \( p' \) is a point on \( \sigma \) that is interior to \( J \). As noted above, \( E^- \) includes the interior of \( J \). So, \( p' \in \sigma \cap E^- \). Let \( G_1 \) be the connected component of \( p' \in \sigma \cap E^- \). Let \( G \) be a \( \leq \)-maximal component of \( \sigma \cap E^- \) so that \( G_1 \subseteq G \). Since \( p' \) is interior to \( J \), and since \( J \) includes \( G, p' \notin G \). So, \( G_1 \prec G \). This means that \( G_1 \subseteq G^- \). By (1), \( p' \) is exterior to \( J \), a contradiction. So, the interior of \( J \) contains no point of \( \sigma \).

By the remarks after (4), \( C \) is the interior of \( J \) and the proof is complete.

**Proof of Theorem 3.4.** Let \( C \) be the connected component of \( z_0 \) in \( D - \sigma \). Let \( l = [z_0, \zeta_0] \). Let \( z_1 \) be the point in \( l \cap \sigma \) that is closest to \( z_0 \). Thus, \( z_1 \in \partial C \). Since \(|z_1 - \zeta_0| < 2^{-g(k)} \), \( \sigma \) contains an arc from \( z_1 \) to \( \zeta_0 \) whose diameter is smaller than \( 2^{-k} \); call this arc \( \sigma_1 \).

We claim that \( D \) includes \( \sigma_1 \). For, let \( q \in \sigma_1 \). It follows that

\[
\max\{|q - z_0|, |q - \zeta_0|\} < 2^{-k} + 2^{-g(k)}.
\]

Since \( 2^{-k} + 2^{-g(k)} \leq \max\{d(\zeta_0, \partial D), d(z_0, \partial D)\} \), it follows that \( q \in D \).

Since \( \sigma_1 \subseteq D, \zeta_0 \) belongs to the connected component of \( z_1 \) in \( D \cap \sigma \). So, by Theorem 4.2, \( \zeta_0 \) is a boundary point of \( C \) since \( z_1 \) is.

**Proof of Theorem 3.1.** Without loss of generality, suppose \( D_r(\zeta_0) \) does not include \( D \). Let \( J \) denote the boundary of \( D \). It follows that \( \partial D_r(\zeta_0) \) separates two points of \( J \).
It follows from Theorem 3.4 that $\zeta_0$ is a boundary point of at least one connected component of $D_r(\zeta_0) - J$. We now show it is a boundary point of exactly two such components. Let $E$ be the connected component of $\zeta_0$ in $D_r(\zeta_0) \cap J$. Thus, as noted in the proof of Theorem 4.2, $E$ is a crosscut of $D_r(\zeta_0)$. If $C$ is a connected component of $D_r(\zeta_0) - J$, and if $\zeta_0$ is a boundary point of $C$, then exactly one side of $E$ includes $C$. By the proof of Theorem 3.1, if $C$ is a connected component of $D_r(\zeta_0) - J$, then the side of $E$ that includes $C$ completely determines the boundary of $C$. Thus, $\zeta_0$ is a boundary point of exactly two connected components of $D - J$; one for each side of $E$.

So, let $C_1, C_2$ denote the two connected components of $D_r(\zeta_0) - J$ whose boundaries contain $\zeta_0$. Each of these components is a connected subset of $C - J$. So each is either included in the interior of $J$ or in the exterior of $J$. Since there are points of the interior and exterior of $J$ that are arbitrarily close to $\zeta_0$, it follows from Theorem 3.4 that one of these components is included in the interior of $J$ and one is included in the exterior of $J$. Suppose $C_1$ is included in the interior of $J$; that is, $D \supseteq C_1$.

Let $p \in C_1$, and let $U$ be the connected component of $p$ in $D \cap D_r(\zeta_0)$. We show that $U = C_1$. Since $C_1$ is a connected subset of $D \cap D_r(\zeta_0)$ that contains $p$, $C_1 \subseteq U$. Since $U$ is a connected subset of $D_r(\zeta_0) - J$ that contains $p$, $U \subseteq C_1$. This completes the proof of the theorem.

5. Preliminaries to Proof of Theorem 3.2: Polar Separations

**Definition 5.1.** Let $A$ be an annulus, and let $\Omega$ be an open subset of $A$. A polar separation of the boundary of $\Omega$ is a pair of disjoint sets $(E, F)$ so that whenever $C$ is an intermediate circle of $A$, there is a connected component of $C \cap \Omega$ whose boundary contains a point of $E$ and a point of $F$.

Our goal in this section is to prove the following.

**Theorem 5.2.** Let $A$ be an annulus, and let $D$ be a simply connected Jordan domain. Suppose that $A$ separates two boundary points of $D$, and let $\gamma_1$ and $\gamma_2$ be the subarcs of the boundary of $D$ that join these points. Then, $(\gamma_1 \cap A, \gamma_2 \cap A)$ is a polar separation of the boundary of $D \cap A$.

Our proof of Theorem 5.2 is based on the following lemma.

**Lemma 5.3.** Let $C$ be a circle, and let $D$ be a simply connected Jordan domain. Suppose $C$ separates two boundary points of $D$. Then, there is a connected component of $C \cap D$ whose boundary hits both subarcs of the boundary of $D$ that join these two boundary points of $D$.

**Proof.** Let $p$ be a boundary point of $D$ that is exterior to $C$, and let $q$ be a boundary point of $D$ that is interior to $C$.

Let $\gamma_1, \gamma_2$ denote the subarcs of the boundary of $D$ that join $p$ and $q$. Let $\alpha$ be a crosscut of $D$ so that $\alpha \cap C$ consists of a single point; label this point $p'$. Let $D_j$ denote the interior of $\alpha \cup \gamma_j$. By Lemma 4.1, $D_1$ and $D_2$ are the sides of $\alpha$.

Now, for each $j \in \{1, 2\}$, we construct a point $q_j$ in $C \cap D_j$ so that $p'$ is a boundary point of the connected component of $q_j$ in $C \cap D_j$. Since $D$ is open, there is a positive number $\delta$ so that $D_2(p') \subseteq D$. Let $C' = C \cap D_2(p')$. Thus, $C'$ is a subarc of $C$. Let $q \in C' - \{p'\}$. Then, $q \notin \alpha$ since $C \cap \alpha = \{p'\}$. So, $q \in D_1 \cup D_2$. Without loss of generality, suppose $q \in D_1$. Relabel $q$ as $q_1$. Let $q_2$ be a point of $C'$
so that $p'$ is between $q_1$ and $q_2$ on $C'$. Again, $q_2 \in D_1 \cup D_2$. Since $D_1$ is the interior of a Jordan curve, and since the subarc of $C'$ from $q_1$ to $q_2$ crosses the boundary of $D_1$ exactly once, $q_2 \notin D_1$. So, $q_2 \in D_2$.

Let $E_j$ denote the connected component of $q_j$ in $C \cap D_j$. By construction, $p'$ is a boundary point of $E_j$. So, the other endpoint of $E_j$ must be in $\gamma_j$ since $C \cap \alpha = \{p'\}$. Set $E = E_1 \cup E_2$. Thus, $E$ is a connected component of $C \cap D$. One endpoint of $E$ belongs to $\gamma_1$, and the other belongs to $\gamma_2$. This proves the lemma.

**Proof of Theorem 5.2.** By assumption, $A$ separates two boundary points of $D$. One of these points is interior to the inner circle of $A$, and the other is exterior to the outer circle of $A$. Let $p$ denote a point that is exterior to the outer circle of $A$, and let $q$ denote a point that is interior to the inner circle of $A$.

Let $C$ be an intermediate circle of $A$. Then, $p$ is exterior to $C$ and $q$ is interior to $C$. So, by Lemma 5.3, there is a connected component of $C \cap D$ so that one of its endpoints lies on $\gamma_1$ and the other lies on $\gamma_2$. Thus, $(\gamma_1 \cap A, \gamma_2 \cap A)$ is a polar separation of the boundary of $D \cap A$.

6. **Proof of Theorems 3.2 and 3.3**

When $X, Y \subseteq \mathbb{C}$, let $d_{inf}(X, Y)$ denote the infimum of $|z - w|$ as $z$ ranges over all points of $X$ and $w$ ranges over all points of $Y$.

The proof of the following is essentially the same as the proof of Lemma 4.1 of [7] which is a standard length-area argument.

**Lemma 6.1.** Let $A$ be an annulus, and let $\Omega$ be an open subset of $A$. Suppose $(E, F)$ is a polar separation of the boundary of $\Omega$. Then,

$$\lambda(A) \geq \sup_{\phi} \frac{d_{inf}(\phi[E], \phi[F])^2}{\text{Area}(\phi[\Omega])}$$

where $\phi$ ranges over all maps that are conformal on a neighborhood of $\overline{\Omega}$.

**Proof of Theorem 3.2.** Note that $r < 1$ since $C$ is non-empty.

We begin by constructing a rectangle $R$ as follows. Let $z_0$ be any point of $\phi[C]$. Choose $m, l_0$ so that $l_0 > l$, $m > \sqrt{\pi \lambda(A)}$, and $(1 - l_0)^2 + m^2 < (1 - l)^2 + \pi \lambda(A)$. Since $r^2 = (1 - l)^2 + \pi \lambda(A)$, $z$ is exterior to the outer circle of $A$ whenever $|\phi(z)| \leq \sqrt{(1 - l_0)^2 + m^2}$. Let:

$$\nu_1 = \frac{z_0}{|z_0|} (1 - l_0 + mi)$$
$$\nu_2 = \frac{z_0}{|z_0|} (1 - l_0 - mi)$$

Thus, the radius $[0, z_0/|z_0|]$ is a perpendicular bisector of the line segment $[\nu_1, \nu_2]$. The midpoint of $[\nu_1, \nu_2]$ is $(1 - l_0)z_0/|z_0|$, and the length of $[\nu_1, \nu_2]$ is $2m$. Let:

$$\nu_3 = \frac{z_0}{|z_0|} (1 + mi)$$
$$\nu_4 = \frac{z_0}{|z_0|} (1 - mi)$$

Thus, the line segment $[\nu_3, \nu_4]$ is perpendicular to the radius $[0, z_0/|z_0|]$. Furthermore, the length of this segment is $2m$ and its midpoint is $z_0/|z_0|$.
Let $R$ be the open rectangle whose vertices are $\nu_1$, $\nu_2$, $\nu_3$, and $\nu_4$. That is, $R$ is the interior of $[\nu_1, \nu_3] \cup [\nu_3, \nu_4] \cup [\nu_4, \nu_2] \cup [\nu_2, \nu_1]$.

Note that the diameter of $R$ is $\sqrt{l_0^2 + 4m^2}$. Also, the diameter of $R$ approaches $\sqrt{l^2 + 4\pi \lambda(A)}$ as $(l_0, m) \to (l, \sqrt{\pi \lambda(A)})$. It thus suffices to show that $\phi[C] \subseteq R$.

We claim that it suffices to show that $\phi[C]$ contains no boundary point of $R$. For, since $\phi^{-1}(z_0)$ is interior to the outer circle of $A$, the modulus of $z_0$ is larger than $\sqrt{(1 - l_0)^2 + m^2}$ which is larger than $l - l_0$. This implies that $z_0 \in R$. Since $R$ contains at least one point of $\phi[C]$, namely $z_0$, and since $\phi[C]$ is connected, it suffices to show that $\phi[C]$ contains no boundary point of $R$.

Since $[\nu_3, \nu_4]$ contains no point of the unit disk, it contains no point of $\phi[C]$. By construction, $|\nu_1| = |\nu_2| = \sqrt{(1 - l_0)^2 + m^2}$. Thus, $|z| \leq \sqrt{(1 - l_0)^2 + m^2}$ whenever $z \in [\nu_1, \nu_2]$. It follows from what has been observed about $l_0$ and $m$ that $[\nu_1, \nu_2]$ contains no point of $\phi[C]$. So, it suffices to show that $[\nu_1, \nu_3] \cup [\nu_4, \nu_2]$ contains no point of $\phi[C]$.

Let us begin by showing that $[\nu_1, \nu_3]$ contains no point of $\phi[C]$. By way of contradiction, suppose otherwise. In order to obtain a contradiction, we construct a Jordan curve $J$ so that $A$ separates two points of $J$ as follows. Let $z_1$ be a point of $\phi[C]$ that belongs to $[\nu_1, \nu_3]$. Thus, by what has just been observed, $z_1 \neq \nu_1$. Let $\sigma_0$ be the pre-image of $\phi$ on $[\nu_1, 0]$. Let $\sigma_1'$ be the pre-image of $\phi$ on $[\nu_1, z_1]$. Let $\sigma_2'$ be the pre-image of $\phi$ on $[0, \nu_3]$. Since $C$ is connected, it includes an arc from $\phi^{-1}(z_1)$ to $\phi^{-1}(z_0)$; label this arc $\sigma_2'$. Let $w_1$ be the first point on $\sigma_1'$ that belongs to $\sigma_2'$. Let $w_2$ be the first point on $\sigma_2'$ that belongs to $\sigma_2'$. Let $\sigma_1$ be the subarc of $\sigma_1'$ from $\phi^{-1}(v_1)$ to $w_1$, and let $\sigma_3$ be the subarc of $\sigma_2'$ from $w_2$ to $\phi^{-1}(0)$. Let $\sigma_2$ be the subarc of $\sigma_2'$ from $w_1$ to $w_2$. Let $J = \sigma_0 \cup \sigma_1 \cup \sigma_2 \cup \sigma_3$. Thus, $J$ is a Jordan curve. By construction, $A$ separates two points of $J$.

Let $D'$ denote the interior of $J$. Let $\Omega = D' \cap A$. Let $E = \sigma_0 \cap A$, and let $F = \sigma_3 \cap A$. We claim that $(E, F)$ is a polar separation of the boundary of $\Omega$. For, let $p = \phi^{-1}(v_1)$, and let $q = w_1$ (where $w_1$ is as in the construction of $J$). Thus, $p$ is exterior to the outer circle of $A$. Since $q \in C$, $q$ is interior to the inner circle of $A$. Let $\gamma_1 = \sigma_1$, and let $\gamma_2 = \sigma_2 \cup \sigma_3 \cup \sigma_0$. Therefore, $\gamma_1, \gamma_2$ are the subarcs of the boundary of $D'$ that join $p$ and $q$. So, by Theorem 5.2, $(\gamma_1 \cap A, \gamma_2 \cap A)$ is a polar separation of the boundary of $\Omega$. Since $\sigma_0$ is the pre-image of $\phi$ on $[v_1, 0]$, $\sigma_0$ contains no point of $A$. Since $\sigma_2 \subseteq C$, $\sigma_2$ contains no point of $A$. Thus, $E = \gamma_1 \cap A$, and $F = \gamma_2 \cap A$. Hence, $(E, F)$ is a polar separation of the boundary of $\Omega$.

By construction, $d_{\inf}(\phi[E], \phi[F]) = m$. So, by Lemma 6.1, the area of $\phi[\Omega]$ is at least as large as

$$m^2 \lambda(A)^{-1} > \pi.$$  

This is impossible since the unit disk includes $\phi[\Omega]$. Thus, $[\nu_1, \nu_3]$ contains no point of $\phi[C]$.

By similar reasoning, $[\nu_4, \nu_2]$ contains no point of $\phi[C]$. Thus, $\phi[C] \subseteq R$, and the theorem is proven.  

Proof of Theorem 3.3. Suppose $r_0$ is a positive number that is smaller than (3.1). We begin by defining an annulus $A$ as follows. Choose $l$ so that $0 < l < \epsilon$ and so that

$$r_0 < \exp(\frac{8\pi^2}{l^2 - \epsilon^2}) \min \left\{ |\zeta_0 - \phi^{-1}(w)| : |w| \leq \sqrt{(1 - l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\}.$$
There is a positive number \( r_1 \) so that
\[
    r_1 < \min \left\{ |\zeta_0 - \phi^{-1}(w)| : |w| \leq \sqrt{(1-l)^2 + \frac{1}{4}(\epsilon^2 - l^2)} \right\}
\]
and so that
\[
    r_0 < \exp \left( \frac{8\pi^2}{l^2 - \epsilon^2} \right) r_1.
\]
Since \( l < \epsilon \), \( r_0 < r_1 \). So, define \( \mathcal{A} \) to be the annulus whose center is \( \zeta_0 \), whose outer radius is \( r_1 \), and whose inner radius is \( r_0 \).

We now show that the diameter of \( \phi[C(D; \zeta_0, r_0)] \) is smaller than \( \epsilon \). First, note that \( \pi \lambda(\mathcal{A}) < (\epsilon^2 - l^2)/4 \). Set \( r = \sqrt{(l-1)^2 + \pi \lambda(\mathcal{A})} \). Then, \( \mathcal{A} \), \( r \), and \( l \) satisfy the hypotheses of Theorem 3.2. By Theorem 3.2, the diameter of \( \phi[C(D; \zeta_0, r_0)] \) is at most
\[
    \sqrt{l^2 + 4\pi \lambda(\mathcal{A})}.
\]
We have
\[
    l^2 + 4\pi \lambda(\mathcal{A}) = l^2 + \frac{8\pi^2}{\log(r_1/r_0)} < l^2 + \epsilon^2 - l^2 = \epsilon^2.
\]
Thus, the diameter of \( \phi[C(D; \zeta_0, r_0)] \) is smaller than \( \epsilon \). \( \square \)

7. Proof of the Carathéodory Theorem

We now conclude with the proof of Theorem 3.5. Set \( r_0 = 2^{-k} + 2^{-g(k)} \). By Theorem 3.4, \( \zeta_0 \in C(D; \zeta_0, r_0) \). By Theorem 3.3, \( |\phi(\zeta_0) - \phi(\zeta_1)| < \epsilon \). Thus, \( \lim_{z \to \zeta_0} \phi(z) = \phi(\zeta_0) \).

We now show that the extension of \( \phi \) is injective. It suffices to show that \( \phi(\zeta_0) \neq \phi(\zeta_1) \) whenever \( \zeta_0 \) and \( \zeta_1 \) are distinct boundary points of \( D \). By way of contradiction, suppose \( \phi(\zeta_0) = \phi(\zeta_1) \). Let \( p = \phi(\zeta_0) \).

We construct a Jordan curve \( \sigma \) as follows. Let \( \alpha \) be a crosscut of \( D \) that joins \( \zeta_0 \) and \( \zeta_1 \). Thus, \( \phi[\alpha] \) is a Jordan curve that contains no unimodular point other than \( p \). Let \( \sigma = \phi[\alpha] \).

We now construct an annulus \( \mathcal{A} \) that separates two points of \( \sigma \). Choose a positive number \( R \) so that \( R < \max\{|z-p| : z \in \sigma\} \). Choose another positive number \( r \) so that \( r < R \). Let \( \mathcal{A} \) be the annulus whose center is \( p \), whose inner radius is \( r \), and whose outer radius is \( R \). By the choice of \( R \), there is a point \( q \in \sigma \) that is exterior to the outer circle of \( \mathcal{A} \). Let \( \gamma_1 \) and \( \gamma_2 \) be the subarcs of \( \sigma \) that join \( p \) and \( q \). Let \( E = \gamma_1 \cap \mathcal{A} \), and let \( F = \gamma_2 \cap \mathcal{A} \). Finally, let \( \Omega = \mathcal{A} \cap \mathbb{D} \) (where \( \mathbb{D} \) is the unit disk). Then, by Theorem 5.2, \( (E, F) \) is a polar separation of the boundary of \( \Omega \). Now, as \( r \to 0^+ \), \( \lambda(\mathcal{A}) \to 0 \). However, by the choice of \( R \), \( d_{\text{inf}}(E, F) \) is bounded away from 0 as \( r \to 0^+ \). Thus, by Lemma 6.1, \( \text{Area}(\phi^{-1}[\Omega]) \to \infty \) as \( r \to 0^+ \). Since \( \phi^{-1}[\Omega] \subseteq D \), this is a contradiction. Thus, \( \phi(\zeta_0) \neq \phi(\zeta_1) \).

Finally, we show that this extension of \( \phi \) is surjective. Let \( \zeta \) be a point on the unit circle. It follows from the Balzano-Weierstrauss Theorem that there is a boundary point of \( D \), \( \zeta_1 \), so that \( \zeta_1 \in \{\phi^{-1}(r\zeta) : 0 < r < 1\} \). Thus, \( \phi(\zeta_1) = \zeta \) by the continuity of \( \phi \).
References


Department of Mathematics, Iowa State University, Ames, Iowa 50011
E-mail address: mcnichol@iastate.edu