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Dynamic Duopoly Theory and Rational Expectations

Abstract
The determination of equilibrium prices and quantities in an oligopolistic market has been a troublesome problem for economic theory. The intrinsic nature of the problem is the interdependence of firms - the profit level of any firm depends upon not only aggregate demand and its own output level, but also on the output level of other firms. Thus, each firm, in choosing its own output level, needs to make some behavioral assumption — or conjecture - about how other firms will respond to these changes in output.

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Dynamic Duopoly Theory and Rational Expectations

by

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Abstract and Headnote:

This paper develops a dynamic model of duopoly behavior in order to investigate rational-expectations solutions to the duopoly problem. In the paper it is shown that there are multiple rational-expectations solutions, including the Cournot equilibrium, provided no adjustment costs are present. If adjustment costs occur, the Cournot solution cannot be a rational-expectations one, but multiple solutions still occur. Furthermore, the presence of adjustment costs provide a rationale as to why firms are not always in equilibrium and thus allow for a true dynamic solution to the duopoly problem. The analysis also treats the case of more than two firms.
Dynamic Duopoly Theory and Rational Expectations

The determination of equilibrium prices and quantities in an oligopolistic market has been a troublesome problem for economic theory. The intrinsic nature of the problem is the interdependence of firms - the profit level of any firm depends upon not only aggregate demand and its own output level, but also on the output level of other firms. Thus, each firm, in choosing its own output level, needs to make some behavioral assumption - or conjecture - about how other firms will respond to these changes in output.

Each of the various equilibrium concepts formulated for the duopoly problem differs in terms of the assumed conjectural variations, and each has been criticized on some ground. The Cournot model yields an equilibrium which is consistent in the sense that firms' conjectures about output levels of other firms are correct - in equilibrium. Yet, the Cournot model has been criticized, because the assumption of a zero conjectural variation is inconsistent with the firm's reaction function; in essence, the model is consistent in equilibrium but not out of equilibrium.

Recent models of duopoly behavior have focused attention on the role of the conjectural variation. Clearly, to each set of conjectures, there is a corresponding output/price equilibrium. Yet, the basic question seems to be - where did these conjectures come from? The emphasis on rational expectations in economic modelling seems to make it clear that a consistent framework requires that the conjectures assumed by firms about the behavior of other firms be consistent with the actual (or likely) behavior of those firms. Professor Timothy Bresnahan (1981), in a recent paper, presents a model in which he claims these conjectures are consistent - i.e., in which
the conjectures made by one firm are equal to the actual reaction of the other firm.

However, one difficulty in all models of duopoly behavior concerns the timing of decisions. If each firm must supply its output to the market at the same time, how can an unanticipated change in the output of one firm affect the current output level of other firms? Clearly, it is the anticipated level of output of other firms which is crucial in determining the current behavior of a firm. On the other hand, even if current output variations of one firm have no effect on the current output level of other firms, they may change the expectations these firms have concerning the future output level of that first firm. It would seem that what is needed is some explicit time framework in which these distinctions can be made clearer.

It is the purpose of this paper to develop a dynamic model of oligopoly behavior in which expectations are formed rationally. In the first section of this paper, we present a brief review of the approach used by Bresnahan and others. In the second section, we present a dynamic programming model in which one firm derives its optimal decision rule, given its expectations concerning the behavior of the other firms. The third section shows how the symmetric rational expectations solutions are derived, assuming there are no adjustment costs associated with output changes. One of the results that emerges is the multiplicity of rational expectations solutions; thus, rational expectations alone do not solve the basic duopoly problem of multiple solutions. It is also seen that the Cournot solution is one of the rational expectations solutions. In the fourth section we consider the role of adjustment costs. We see that multiple solutions still exist for this
case; however, the Cournot solution is no longer a rational expectations solution. Thus, the Cournot solution may be viewed as a singular one corresponding to a case in which output can be varied instantaneously; as such, there is never a reason to be out of equilibrium. The fifth section shows how the model can be extended to deal with more than two firms. Finally, we conclude with suggestions for future work.

I) Consistent Conjectures and Duopoly Equilibrium

As noted in the introduction, the duopoly equilibrium depends upon the conjectures made by each firm concerning the behavior of the other firm. Let us assume we have two producers of a homogeneous product, with given market demand. We will label the firms 1 and 2, and assume that each firm uses quantities as its choice variable. Following Bresnahan, let:

1. \[ P = P(Q); \quad Q = q_1 + q_2 \]
   where \( P(Q) \) is the inverse demand function, \( Q \) is total output, and \( q_i \) is the output of firm \( i \). Let each firm's cost function be given by \( C_i(q_i) \), so that firm \( i \)'s profits are:

2. \[ \pi_i = P(Q)q_i - C_i(q_i) \]

Firm \( i \), in choosing its (perceived) optimal output, \( q_i^* \), needs to make some assumption about the behavior of firm \( j \). From firm \( i \)'s perspective, assume that its beliefs about the behavior of firm \( j \) are summarized by the function:

3. \[ q_j^e = g_{ij}(q_i, \alpha_j), \quad i \neq j \]

In (3), \( q_j^e \) represents \( i \)'s belief about how firm \( j \) behaves; it need not represent the "truth"; \( q_i \) is the actual output of \( i \), and \( \alpha_j \) summarizes factors that \( i \) believes are external to its own behavior. For simplicity,
(3) can be specialized to:

\[ q^e_j = g_{ij}(q_i) + \alpha_j \]

where \( \alpha_j \) is firm j's "autonomous" output. Define:

\[ r_{ij}(q_i) = \left( \frac{\partial q^e_j}{\partial q_i} \right) = g_{ij}'(q_i) \]

Following Bresnahan, \( r_{ij} \) is i's conjectural variation. Using (3) or (5) in (2) yields:

\[ \pi^e_i(q_i, q^e_j, \alpha_j) = P(q_i + q^e_j)q_i - C_i(q_i) \]

where \( \pi^e_i \) represents the profit firm i expects to earn, given its output, \( q_i \), and its beliefs about the behavior of firm j. Optimizing (6) over \( q_i \) yields:

\[ \left( \frac{d\pi^e_i}{dq_i} \right) = P(0) + q_i P'(0) [1 + r_{ij}] - C_i'(q_i) = 0. \]

(7) serves to determine firm i's perceived optimal output, \( q^*_i \), given \( r_{ij} \) and \( q^e_j \). Note, however, that it is not methodologically legitimate to treat \( q^e_j \) as exogenous, since the derivation of (7) presupposes its endogeneity.

Thus, (7) implicitly determines \( q^*_i \), given the market demand function, firm i's cost curve, and firm i's beliefs about firm j's behavior. While superficially it seems to determine \( q^*_i \) as a function of \( r_{ij} \) and \( q^e_j \), it cannot consistently be so interpreted because of the endogeneity of \( q^e_j \). If we assume that \( g_{ij} \) is separable (as in (4)) and that \( r_{ij} \) is a constant, then we can write:

\[ q^*_i = h_i(\alpha_j, r_{ij}) \]

where \( \alpha_j \) represents the portion of j's output that i believes is autonomous to i's action, and \( r_{ij} \) is i's conjectural variation.
Similarly, from firm j's perspective, let its beliefs about the behavior of i be described by:

\[ q_i^e = g_{ji}(q_j, \alpha_i) = g_{ji}(q_j) + \alpha_i; \quad r_{ji} = g'_{ji}(q_j) \]

where we assume that the function is separable. Thus, j's conjectural variation is given by \( r_{ji} \) and, following Breshahan, may be assumed to be a constant. Hence, from j's perspective, profits are given by:

\[ \pi_j(q_j, q_i^e(q_j, \alpha_i)) = P(q_j + q_i^e)q_j - C_j(q_j) \]

Optimizing (10) over \( q_j \), given (9), yields:

\[ (d\pi_j^e/dq_j) = P(0) + q_j P'(0)(1 + r_{ji}) - C'_j(q_j) = 0 \]

As for (7), (11) determines \( q_j^* \) given \( r_{ji} \), \( q_i^e \); but since \( q_i^e \) is assumed to be endogenous, it cannot be said to determine a reaction function for firm j.

Clearly, (7) and (11) can be solved simultaneously to yield the equilibrium \((q_j^*, q_i^*)\), given the conjectural variations, as well as demand and cost parameters. And the values of \((\alpha_i, \alpha_j)\) can always be determined such that output conjectures are consistent - i.e., such that \( q_j^* = q_j^e = (r_{ij}q_i^* + \alpha_j) \) where \( q_j^* \) is j's actual output, and \( q_i^e \) is i's belief about j's output. For example, if we let:

\[ P = A - Bq; \quad C_i(q_i) = c_{i0} + c_{i1}q_i + \frac{c_{i2}}{2} q_i^2, \]

assuming, for simplicity, \( c_{i1} = c_{j1} = c_1 \), \( c_{i2} = c_{j2} = c_2 \), we find:

\[ q_j^* = (A - c_1)[B(1 + r_{ij}) + c_2]/\Delta; \quad x, k = i, j; \quad x \neq k. \]

\[ \Delta = B^2[3 + 2(r_{ij} + r_{ji}) + r_{ij}r_{ji}] + c_2^2 + Bc_2(4 + r_{ij} + r_{ji}) \]

\[ q_k = q^* - r_{jk}q_x^* = ((A - c_1)[B(1 - r_{ij}r_{ji}) + c_2][1 - r_{jk}]/\Delta); \quad x \neq k; \]
The problem arises, however, in defining what is meant by consistent - or rational - conjectures. As noted earlier, one cannot think of (7) as determining a reaction function for firm i - firm i's (subjective) optimal output does not depend on \( q_j \), since i thinks \( q_j \) is endogenous. Rather, \( q_i^* \) depends on \( q_j \) and \( r_{ij} \); how, then, can one derive from (7) a reaction function that is consistent with (9)? Furthermore, if i believes that it determines j's current output (since the term \( q_j \) presumably captures demand and cost conditions which are unchanging), and j believes it determines i's current output, how can they both be correct?

Furthermore, there seems to be a problem of timing. If both firms bring their output to the market simultaneously, then any unanticipated change in the output of firm j will have no (current) effect on i's output. On the other hand, if each changes its (announced) output ex ante, we are back to the same basic dilemma - how can each believe it determines the other's output, and how can both be correct?

I would suggest that a more fruitful way of looking at the problem would be in an explicit dynamic framework. Specifically, each firm may believe its output has no current effect on the output of the other firm but does change that other firm's future output as the latter firm revises its expectations concerning the future output level of the former firm. Thus, a dynamic framework can allow more explicitly for interactions between firms and can provide a basis for a rational expectations solution to the duopoly problem. We now turn to construct a simple dynamic model of the duopoly problem.
II) The Dynamic Framework

We assume that there are two firms, each producing an identical product. The firms are labelled 1 and 2, and their output at time $t$ is $q_1^t$, $q_2^t$ respectively. Total market demand is unchanging over time and is given by:

$$p^t = A - B(0^t); \quad 0^t = (q_1^t + q_2^t).$$

Total costs of firm $i$ at time $t$ depend on its output at time $t$ and, if adjustment costs are present, depend on its output at $(t-1)$. For simplicity, we assume the total cost function is quadratic:

$$c_i^t = c_{i0} + c_{i1}q_i^t + (c_{i2}/2)(q_i^t)^2 + (c_{i3}/2)(q_i^t - q_i^{t-1})^2; \quad i = 1,2$$

The term involving adjustment costs may be motivated by extra costs entailed (beyond simple factor payments) in changing labor employment or the capital stock. There is, of course, a large literature on the role of adjustment costs.

Each firm is assumed to choose its output (plan) in order to maximize the present discounted value of profits over its planning horizon. The essence of the duopoly problem is that the profits of firm $i$ depend on the behavior of firm $j$ and that firm $i$ may perceive its ability to influence firm $j$. However, in order to avoid the simultaneity problem in which each firm attempts to control the current output of the other firm (with its own current output level), we assume that the current output level of the competing firm is taken as exogenous. As discussed earlier, it is firm $i$'s beliefs about the output of firm $j$ – not the actual output of $j$ – that will influence $i$'s current decision. Instead, we assume that each firm believes it can, through its own current output decision, influence the future output
of the other firm. Specifically, from firm i's perspective, it believes j's output is governed by the following equation:\(^2\)

\[ q_j^t = \alpha_j + \phi_j q_j^{t-1} + \theta_j q_j^{t-1} \]

For the moment, (18) is strictly an ad hoc representation of i's beliefs; these beliefs may be correct or incorrect. Later, we shall deal with the issue of rational expectations.

At time t, firm i chooses \( q_i^t \) to maximize the present discounted value of profits over the planning horizon, which is given by:

\[ V_t(q_i^t, q_j^t, q_j^t) = (P^t q_i^t - C_i^t) + \delta V_{t+1}^*(q_j^{t+1}, q_i^t) \]

In (19), \( \delta \) is the discount rate, and \( V_{t+1}^* \) is the maximum present discounted value of profits, starting at (t+1). Note that \( V_{t+1}^* \) depends upon \( q_j^{t+1} \) since - at (t+1) - the jth firm's (then) current output is taken as exogenous; also, it will depend upon \( q_i^t \) if adjustment costs are present. Naturally, the form of \( V_{t+1}^* \) will also depend upon how (firm i believes) firm j's output (for \( \tau > t+1 \)) is determined - i.e., upon (18).

Due to the simple form of demand and cost functions, \( V_{t+1}^* \) will be quadratic. Assuming the planning horizon is infinite, the parameters of \( V_{t+1}^* \) will be time independent. Thus, let \( V_{t+1}^* \) be given by:

\[ V_{t+1}^* = s_0 + s_1 q_i^t + s_2 q_j^{t+1} - \left( s_3/2 \right) (q_i^t)^2 - s_4 (q_i^t)(q_j^{t+1}) - \left( s_5/2 \right) (q_j^{t+1})^2 \]

Substituting for \( P^t \), \( C_i^t \) and \( q_j^{t+1} \) from (16), (17) and (18) respectively, into (19) would give the objective function at t as a function of \( q_i^t, q_j^t \) and the parameters of the demand, cost, and firm j output determination equation. Optimizing (19) over \( q_i^t \), given \( q_j^t \) (and (16), (17), (18), and (20)) yields:
Using (18) and (20), and rearranging terms, yields:

\[(22) \quad (\partial V^t/\partial q_i^t) = h + c_{i1}q_i^{t-1} - Kq_j^t - Xq_i^t = 0, \text{ where}\]

\[(23) \quad h = (A - c_{i1}) + \delta(s_1 + \theta_j s_2 - \alpha_j(s_4 + \theta_j s_5))\]

\[(24) \quad K = [B + \delta\phi_j(s_4 + \theta_j s_5)]\]

\[(25) \quad X = [2B + c_{i2} + c_{i3} + \delta(s_3 + 2\theta_j s_4 + \theta_j^2 s_5)]\]

The SOC requires \(X > 0\)

Given the values of \(s\), and the other parameters, firm \(i\)'s optimal decision rule is given by:

\[(26) \quad (q_i^t)^t = (h/X) + (c_{i3}/X)q_i^{t-1} - (K/X)q_j^t\]

However, since \(q_j^t\) is taken by \(i\) to be exogenous and determined by (18), this can be rewritten as:

\[(27) \quad (q_i^t)^t = \{(h - \alpha_j K) + (c_{i3} - \theta_j K)q_i^{t-1} - (K\phi_j)q_j^{t-1}] / X\}

Note that the form of the decision rule for \(i\) is comparable to that postulated for \(j\). Call this optimal rule:

\[(27a) \quad (q_i^t)^t = \alpha_i q_i^{t-1} + \theta_i q_j^{t-1}\]

where \((\alpha_i, \phi_i, \theta_i)\) are determined from (27), and thus depend on all the parameters. The parameters \((s)\) can be recovered from (19) since, due to the infinite time horizon, \(V^*_t\) will have the same form as \(V^*_{t+1}\):

\[(28) \quad V^*_t(q_j^t, q_i^t) = \max\{V_t(q_i^t, q_i^{t-1}, q_j^t)\}\]

Thus, for example:

\[(29) \quad (\partial V^*_t/\partial q_j^t) = s_2 - s_4 q_i^{t-1} - s_5 q_j^t = (\partial V_t/\partial q_j^t) + (\partial V_t/\partial q_i^t)(\partial q_i^t/\partial q_j^t)\]
Since \( \frac{\partial V_t}{\partial q_j^t} = 0 \), we find:

\[
(30) \quad s_2 - s_4 q_1^{t-1} - s_5 q_j^t = -B(q_j^t) + \delta [s_2 - s_4 (q_j^t)^t - s_5 q_j^{t+1}] \phi_j
\]

where \( q_j^{t+1} \) is evaluated at (the optimal value) \( (q_j^t)^t \). Similarly:

\[
(31) \quad \frac{\partial V^*_t}{\partial q_j^t} = -s_5 = -\delta s_5 (\phi_j)^2 - [B + \delta \phi_j (s_4 + \delta s_5)] \frac{\partial (q_j^t)^t}{\partial q_j^t} = -\delta s_5 (\phi_j^2) + (K^2/X)
\]

Proceeding similarly for the other partials, but omitting the tedious calculations, yields:

\[
(32) \quad s_5(1 - \delta \phi_j^2) = -(K^2/X)
\]

\[
(33) \quad s_4 = (c_13 K/X)
\]

\[
(34) \quad s_3 = c_13 - (c_13^2/X)
\]

\[
(35) \quad s_2(1 - \delta \phi_j) = -\delta \phi_j c_3 s_5 - (K/X)
\]

\[
(36) \quad s_1 = (c_13 h/X)
\]

Note that \( s_1 = s_3 = s_4 = 0 \) if \( c_13 = 0 \); i.e., if no adjustment costs are present, the only state variable is \( q_j^t \).

Given \( (B, \delta, \phi_j, \theta_j, c_{12}, \text{ and } c_{13}) \), equations (32) - (34) can be solved simultaneously for \( s_3, s_4, s_5 \), and hence for \( (K, X) \). Thus, the interaction parameters of \( i \)'s reaction function \( \phi_i, \theta_i \) depend only on the discount rate, the interaction parameters of \( j \)'s reaction function, and the slopes of the demand and marginal cost schedules; they do not depend on the intercepts of these schedules. Finally, \( s_1, s_2, h \) and hence \( a_i^t \) can be recovered from (23), (27), (35) and (36), using the values of \( (s_3, s_4 \text{ and } s_5) \).

To summarize, given any arbitrary values for \( j \)'s reaction function of the form:

\[
(18) \quad q_j^t = a_j + \phi_j q_j^{t-1} + \theta_j q_1^{t-1},
\]
we have found i's optimal reaction function, contingent on its beliefs, in the form:

\[(27a) \quad (q_i^t) = a_i^t + \phi_i^t q_i^{t-1} + \theta_i^t q_j^{t-1}\]

where \((\phi_i^t, \theta_i^t)\) depend on \((\phi_j, \theta_j)\), and \(a_i^t\) depends on \((a_j, \phi_j, \theta_j)\); of course, they all depend on the parameters of the demand and i's cost schedules.

Proceeding similarly for j, we could assume j postulates i's reaction function to be of the form:

\[(37) \quad q_i^t = a_i^t + \phi_i^t q_i^{t-1} + \theta_i^t q_j^{t-1}\]

Following a similar procedure to that outlined above, we could derive j's optimal reaction function, which would be of the form:

\[(38) \quad (q_j^t) = a_j^t + \phi_j^t q_j^{t-1} + \theta_j^t q_i^{t-1}\]

where \(\phi_j^t, \theta_j^t\) depend upon \((\phi_i, \theta_i)\), and \(a_j^t\) depends upon \((\phi_i, \theta_i, \alpha_i)\); again, they all depend on the parameters of the (common) demand function and on the parameters of j's cost schedule.

By a rational expectations solution, we mean:

\[(39) \quad \phi_j^t(\phi_i, \theta_i) = \phi_j^t; \phi_i^t(\phi_j, \theta_j) = \phi_i^t\]
\[(40) \quad \theta_j^t(\phi_i, \theta_i) = \theta_j^t; \theta_i^t(\phi_j, \theta_j) = \theta_i^t\]
\[(41) \quad \alpha_j^t(\phi_i, \theta_i, \alpha_i) = \alpha_j^t; \alpha_i^t(\phi_j, \theta_j, \alpha_j) = \alpha_i^t\]

That is, a rational-expectations solution is one in which each firm uses the other's optimal reaction function in deriving its own reaction function. Hence, the beliefs by each firm about the behavior of the other firm are correct. We first turn to consider the case in which there are no adjustment costs.
III) Rational Expectations Solution without Adjustment Costs

In the analysis that follows, we assume the firms are symmetric - i.e., that they possess identical cost schedules; thus, we drop the firm subscript on the cost parameters. The fact that firms are symmetric does not imply that the rational expectations solution must be symmetric. Indeed, even if firms have identical cost curves, there will be an infinite set of rational-expectations solutions, yielding different reaction functions and output levels for the two firms. However, since these asymmetric solutions will result in different profit levels for the two firms, it is not clear why the firm with lower profits would "permit" such a solution. Thus, in the analysis that follows, we restrict ourselves to considering symmetric rational expectations solutions. Even for this case, multiple (but finite) solutions occur.

Assuming $c_3 = 0$ and that solutions are symmetric implies:

\[(42) \quad \phi_j = \phi_i = 0; \, \theta_j = \theta_i = 0; \, s_1 = s_3 = s_4 = 0\]

From (27) and (28):

\[(43) \quad \phi_i' = (-K/X)\theta_i'; \, \theta_i' = (-K/X)\phi_i'\]

Hence, the set of solutions is given by:

\[(44) \quad \phi_j' = \phi_j = \theta_i = \theta_i = 0, \text{ or} \]

\[(45) \quad (K^2/X^2) = 1; \, \phi^2 = \theta^2\]

Note that the Cournot solution is a rational expectations solution (corresponding to (44)). However, there are four other solutions (from (45)), in which firms perceive the interaction between them. These solutions for $(\phi, \theta)$ may be found using (24), (25), (32), (43), and (45).

The corresponding values of $\alpha$ can be found using (23), (27), (35), and the
solutions for $(\phi, \theta)$. These solutions are given below and are labelled (i) - (iv). Details are omitted to save space.

(46i) $K = X > 0; -\phi^i = \phi^i = \phi^* > 0; \phi^* = [(g - B)/2\delta g]^{1/2}$

(46ii) $K = X > 0; -\phi^{ii} = \phi^{ii} = -\phi^* < 0$

(46iii) $-K = X > 0; \phi^{iii} = \phi^{iii} = \phi^a > 0; \phi^a = [(g + B)/2\delta g]^{1/2}$

(46iv) $-K = X > 0; \phi^{iv} = \phi^{iv} = -\phi^a < 0$, where:

(47) $g \geq (2B + c_2^2) \geq 2B$.

The corresponding values for $a_j = a_j^i = a_j^i$ are given by:

(48i) $a^i = (A - c_1)(1 - \delta\phi^*)/[(g + B) - \delta\phi^*(g + 2B)]$

(48ii) $a^{ii} = (A - c_1)(1 + \delta\phi^*)/[(g + B) + \delta\phi^*(g + 2B)]$

(48iii) $a^{iii} = -(A - c_1)(1 - \delta\phi^a)/[\delta\phi^a g]$

(48iv) $a^{iv} = +(A - c_1)(1 + \delta\phi^a)/[\delta\phi^a g]$

The corresponding steady-state output levels $(q_j^i = q_j^i)$ are obtained by solving the reaction functions simultaneously. They are:

(49) $q^i = a^i > 0; q^{ii} = a^{ii} > 0$

(50) $q^{iii} = [a^{iii}/(1 - 2\phi^a)] > 0; q^{iv} = [a^{iv}/(1 + 2\phi^a)] > 0$

Making the appropriate substitutions for $\phi$ yields:

(51i) $q^i = [(A - c_1)[M + N]/(T)] > 0$

(51ii) $q^{ii} = [(A - c_1)[M - N]/(T)] > 0$

(52) $M \equiv (g^2 + gB)(2 - \delta) + 2\delta B^2$

(53) $N \equiv B[2\delta g(g - B)]^{1/2}$

(54) $T \equiv g^3(2 - \delta) + Bg^2(4 - 3\delta) + 2gB^2 + 4\delta B^3$

Similarly, for $\phi = \theta$,

(55i) $q^{iii} = [(A - c_1)(M^a - N^a)/(T^a)] > 0$
As is apparent, the equilibrium output levels depend upon the discount rate, $\delta$, the slopes of the demand and marginal cost schedules, and the difference between the intercepts of these schedules. Before discussing these values in more detail, let us first consider the stability of the system:

$$q_i^t = \alpha + \phi q_{i-1}^t + \theta q_{j-1}^t$$

The characteristic roots ($\lambda_1, \lambda_2$), and the corresponding characteristic vectors ($d_1, d_2$) are given by:

$$\lambda_1 = (\phi + \theta); \quad \lambda_2 = (\phi - \theta); \quad d_1 = (d_1, d_1); \quad d_2 = (d_2, -d_2)$$

For this case of no adjustment costs, one root is always zero, while the other always exceeds one in absolute value. Thus, the system is degenerate. While we expect a saddle-point solution to the rational-expectations model, in this case there is no stable branch because of the degeneracy. The only (symmetric) convergent solution is $q_i = q_j = q^*$ for all time. As we shall see, if adjustment costs are present, we do get a saddle-point solution.

Returning to the equilibrium output levels, consider first some reference solutions. Denote by $q_{pc}$, $q_c$ and $q_m$ the (individual firm) output solutions for the (static) competitive, Cournot, and joint-profit maximization monopoly models respectively. Then:
(61) \( q_{pc} = \left( A - c_1 \right) / g \) > \( q_c = \left( A - c_1 \right) / \left( g + B \right) \) > \( q_M = \left( A - c_1 \right) / \left( 2B + g \right) \)

In order to compare our solutions to the above requires specifying the value of \( \delta \); assuming \( \delta = 1 \), we find:

(62) \( q_{iii} < q_M < q_{ii} < q_c < q_{iv} < q_{pc} \leq q^i \)

In (62), \( q^i > q_{pc} \) unless \( c_2 = 0 \); for \( c_2 = 0 \), the solution \( q^i (\phi = -\theta > 0) \) corresponds to the Bertrand/Bresnahan solution. The corresponding steady-state profits can be ordered as:

(63) \( \pi_M > \pi_{ii} > \pi_c > \pi_{iii} = \pi_{iv} > \pi_{pc} \geq \pi^i \geq 0 \)

where equality between \( \pi_{pc}, \pi^i \) and \( 0 \) holds for \( c_2 = 0 \). Note, in the above, we are discussing only variable profits – i.e., we ignore fixed costs.

Furthermore, it is easily shown that as the discount rate increases (\( \delta \) decreases), each of the quantity solutions gets closer to the Cournot solution. In other words, decreases in \( \delta \) cause \( q_{ii} \) and \( q_{iii} \) to increase, while \( q_{iv} \) and \( q^i \) decrease. In the limit, as \( \delta \to 0 \), all solutions tend to the Cournot solution. By continuity, it follows that the stationary-state profits corresponding to \( q^i \) and \( q_{iv} \) increase as \( \delta \) decreases, since output is reduced towards the Cournot level. The profits corresponding to \( q_{ii} \) fall, since \( q_{ii} \) increases towards the Cournot level, whereas profits corresponding to \( q_{iii} \) first increase (as \( q_{iii} \) tends to the monopoly solution), then decrease.

If, somehow, the rational expectations solution were to be decided upon by which one yielded the highest profits, then this choice would depend upon the discount rate. Since \( q_M < q_{ii} < q_c \) for all \( 1 \geq \delta > 0 \), and \( (q^i, q_{iv}) > q_c \) for all \( \delta > 0 \), it follows that \( \pi_{ii} > \pi^i \) and \( \pi_{ii} > \pi_{iv} \) for all \( \delta \in (0, 1) \).
For $\delta = 1$, we have seen $q^{iii} < q_M < q^{ii}$, and $\pi^{ii} > \pi^{iii}$. However, as $\delta$ decreases, $\pi^{ii}$ falls, while $\pi^{iii}$ initially increases. In particular, since for some $\delta'$, $q^{iii} = q_M$, and for $\delta < \delta'$, $q_M < q^{iii} < q^{ii} < q_c$, it follows that there exists a $\delta^* > \delta'$ such that $\pi^{iii} > \pi^{ii}$ as $\delta \leq \delta^*$. Thus, the rational expectations solution which yields the highest profit level depends on the discount rate.

Finally, one could ask what (static) conjectures correspond to each of the rational expectations solution. First, note that for the dynamic reaction function with $c_3 = 0$:

(26) $q_1^t = (h/X) - (K/X)q^t$

Since $(K^2 = X^2)$, the absolute value of the contemporaneous conjectural variation is one. In particular, for $K = X$ ($q^i$ and $q^{ii}$), the conjecture is $-1$, whereas for $K = -X$ ($q^{iii}$ and $q^{iv}$), that conjecture is $1$. This, however, is not equivalent to the static notion of a conjectural variation, since each firm takes the other firm's contemporaneous output as given. The static conjectures corresponding to each of our equilibrium output levels can be found from the analysis of section I. Assuming symmetry, we find from equation (13):

(61) $q' = \{(A - c_1)/[g + B(1+r)]\}$

Obviously, for the Cournot solution, the corresponding conjectural variation is $0$. Note that the competitive solution corresponds to a conjecture of $(-1)$, even if marginal cost is not constant, while the joint profit maximizing solution corresponds to a conjecture of $+1$.

Since each of our four other rational expectations solutions depends upon $\delta$, the corresponding static conjectures will also depend upon $\delta$. Label these conjectures $(r^i, ..., r^{iv})$, corresponding to $(q^i, ..., q^{iv})$, as
defined earlier. Then:

(62) \( r^i = -[(\delta \phi)/(1 - \delta \phi^*)]; \quad r^{ii} = [(\delta \phi^*/(1 + \delta \phi^*)); \quad \phi^* = [(g-B)/(2g)]^{1/2} \)

(63) \( r^{iii} = [\delta \phi^a/(1-\delta \phi^a)]; \quad r^{iv} = -[\delta \phi^a/(1 + \delta \phi^a)]; \quad \phi^a = [(g+B)/(2g)]^{1/2} \)

In particular, for \( \delta = 1 \), we have

(64) \( r^i < -1 < r^{iv} < 0 < r^{ii} < 1 < r^{iii} \)

with \( r^i < -1 \) when \( c_2 > 0 \). Note that the output \( q^i \), and the corresponding conjecture, \( r^i \), correspond to the Bresnahan "consistent conjecture" equilibrium only for \( \delta = 1, c_2 = 0 \); for \( c_2 \neq 0 \), the conjecture \( r^i \) is less than minus one and differs from the Bresnahan conjecture.

To summarize the results of this section, we have found that:

(i) there are five rational-expectations solutions to the dynamic duopoly problem, assuming symmetric solutions. If symmetry is not assumed, there is an infinite set of solutions.

(ii) the Cournot solution is a rational-expectations solution when no adjustment costs are present.

(iii) the Bertrand/Bresnahan competitive solution emerges as a rational-expectations solution \( (q^i) \) when marginal cost is constant. However, this same solution (i.e., \( q^i \)) corresponds to output above the competitive level when the marginal cost curve is positively sloped. In some sense, then, this is a singular solution.

(iv) another rational expectations solution \( (q^{iv}) \) corresponds to output levels below the competitive level but above the Cournot level. Thus, the corresponding static conjecture, \( r^{iv} \in (0, -1) \).

For any discount rates, the Cournot solution, \( (q^i) \) and \( (q^{iv}) \) are
dominated by one of the other rational-expectation solutions, in terms of profits.

(v) of the remaining two rational-expectations solutions, \( q^{iv} \) corresponds to output levels between the Cournot and joint monopoly solution, while \( q^{vii} \) corresponds to an output level below the monopoly level when discount rates are small (\( \delta \) near 1). Which of these two solutions yields the higher profits depends upon the discount rate.

(vi) as firms become more myopic (\( \delta \) decreases), each solution tends towards the Cournot level. In the limit (\( \delta = 0 \)), the Cournot solution is the only rational-expectations solution.

In the next section, we reintroduce adjustment costs. Using these, we show the Cournot solution is not a rational-expectations solution. We also show that the rational expectations solution is a saddle-point, so that there is one stable trajectory which describes the dynamic output of firms outside of stationary equilibrium.

IV) The Role of Adjustment Costs

One of the most frequent criticisms aimed at duopoly models is that they use assumptions which are inconsistent with the actual behavior of firms. For example, the Cournot model is criticized because firms assume a zero conjectural variation, yet each firm does adjust to changes in the other firm's output level. The paper by Bresnahan, of course, is an attempt to rectify this problem by trying to define and determine consistent conjectures. The conceptual problem with all of these analyses, however, is that they are essentially static in nature. Since there is no intrinsic dynamic
characteristic of the model, why should firms ever be away from equilibrium output levels? Why don't they simply instantaneously adjust to the equilibrium level? And, if firms are always in equilibrium, what is meant by consistent conjectures?

In order to motivate why firms are ever out of equilibrium, and thus to meaningfully discuss consistent conjectures or dynamic behavior, it seems necessary to provide some mechanism which inhibits them from instantaneously jumping to the equilibrium level. This could be done by ad hoc restrictions on the level of output changes permitted in any period (treating output as a state variable), or by introducing adjustment costs. The precise nature of these costs is not crucial; the central point is that they provide a rationale for explaining why equilibrium (steady-state) output levels do not occur at all times.

In the last section we showed that, while we could derive "rational expectations" solutions, the only stable solution corresponded to stationary output levels. The reason is clear - treating output as a state variable was artificial, since it could be adjusted instantaneously and costlessly. However, with adjustment costs, we can show that while we get similar stationary-state results, we do find a true dynamic solution in which firms are not always in equilibrium.

Conceptually, the symmetric rational expectations solutions are found in precisely the same way for this case as they were in Section III. The difficulty that arises is that the solution entails solving higher-order polynomials, which cannot be solved explicitly. Thus, we need to resort to computer simulations to characterize these solutions. However, it can easily be shown that the Cournot case is no longer a rational-expectations
solution. Thus, one of the solutions is degenerate, and there are only four (symmetric) solutions for $c_3 \neq 0$.

Returning to the analysis of section II, recall that:

\[(c_4^t)^t = [(h - a_j^t) + (c_3 - \theta^j)q_j^{t-1} - (K\phi_j)q_j^{t-1}] / X\]

Assuming symmetry, this implies:

\[\phi = [(c_3 - \theta K)/X]; \theta = -(K\phi/X)\]

where $X > 0$ by the S.O.C. Thus, $\theta = 0$ implies $\phi > 0$ ($c_3 > 0$), but $\phi > 0$ implies $\theta = 0$ (since $K \neq 0$), a contradiction. Similarly, assuming $\phi = 0$ leads to a contradiction. Hence, both $\phi$ and $\theta$ must be nonzero, and consequently the Cournot solution cannot be a rational expectations solution.

The logic behind this is clear. For $c_3 > 0$, firm 1 knows that $q_2^t$ will depend on $q_2^{t-1}$; since, at $t$, firm 1 takes $q_2^t$ as exogenous, it follows immediately that, since $q_1^t$ depends on $q_2^t$, it must depend on $q_2^{t-1}$, as well as $q_1^{t-1}$. By symmetry, then, each firm's output at $t$ depends on the output level of both firms at $(t-1)$, and thus $(\phi, \theta) \neq 0$. Consequently, the Cournot solution is a singular one, corresponding to the case in which no real dynamic process is present. This mirrors the conventional criticism of the Cournot process out of equilibrium - i.e., if firms are not in steady-state equilibrium, the output adjustments are inconsistent, because they ignore the interaction among firms. The presence of adjustment costs provides a rationale for why firms do not instantaneously jump to the static equilibrium output levels, and thus these costs imply the Cournot solution cannot be a rational expectations one.

Using (65), the definitions of $K$ and $X$ ((24) and (25)), and (32)-(34) yield the equations that determine $\phi$ and $\theta$: 
Having solved for \( \phi, \theta \) the value of \( \alpha \), and the corresponding output levels, can be calculated using (23), (35), (36), and the symmetry assumption:

\[
\alpha'_{i} = \alpha'_{j} \rightarrow \alpha[X + K] = h
\]

Unfortunately, it is not possible to present an analytic solution except for the special case in which \( g = 2B \). For this case it can be shown that a set of solutions is given by:

\[
\phi = (1+\theta); c_{3}(3\theta^{2} + 2\theta^{3}) - B(2 + \theta)(1 + 2\theta) = 0.
\]

Thus, there are three solutions: \( \theta > 0 \), \( \theta \in (0, -1/2) \), \( \theta \in (-3/2, -2) \). These three solutions all yield the same equilibrium output level. Note, however, that the characteristic roots of the difference equation system are given by \((\phi-\theta), (\phi+\theta)\). Since \((\phi-\theta) = 1\), and \((\phi+\theta) > 1\) for \( \theta > 0 \), \((\phi+\theta) < -1\) for \( \phi \in (-3/2, -2) \), the only root which yields a stable saddle point branch is the root \( \theta \epsilon (0, -1/2) \). This root, of course, is the limiting solution for \( c_{3} = 0 \), corresponding to the Bertrand/Bresnahan solution \(-\theta = \phi = 1/2\), and output produced at the competitive level (given constant marginal costs).

Even for \( c_{3} \neq 0 \), the steady-state output level for this case is the competitive level, with price equal to marginal cost. However, firms need not start at this output level — there is one stable branch which converges.

The other three solutions for \((g = 2B)\) cannot be found analytically, nor can any solutions be found analytically when \( g > 2B \) (marginal cost is increasing). However, the corresponding solutions can readily be found by numerical techniques. Some illustrative solutions are presented in Table 1.
TABLE I: Solutions with Adjustment Costs*

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<tr>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$\phi^i$</th>
<th>$\theta^i$</th>
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*Note: we assume $B = 1$, $\delta = 1$ for simplicity. To obtain actual output levels, multiply $q^*$ by $(A-c_1)$. 
TABLE I - continued

<table>
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<th>$\theta^{iii}$</th>
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As is clear from the Table, for $c_3$ near zero, the adjustment costs do not substantially alter any of these four solutions. Thus, these rational expectations solutions are robust in the presence of adjustment costs; only the Cournot solution vanishes. Furthermore, the presence of adjustment costs, which provide a rationale for firms ever being out of equilibrium, also yield a rational expectations solution which is dynamic in nature.

From (60), we know there are two roots, $(\phi+\theta)$, and $(\phi-\theta)$, to the dynamic system. For $c_3 = 0$, one of these roots is always zero, whereas the other exceeds one in absolute value (except for the singular case of $c_2 = 0$ and the Bertrand solution). For $c_3 > 0$, however, one of these roots will be nonzero, and less than one in absolute value, whereas the other root still exceeds one in absolute value. In particular, when $\phi$ and $\theta$ are of the same sign, $|\phi+\theta| > 1$, but $|\phi-\theta| < 1$. Conversely, when $\phi$ and $\theta$ are of opposite sign, the root $(\phi+\theta)$ yields a stable trajectory.

Using the characteristic vector corresponding to each root, the stable saddle-point branch for each solution is readily found:

(69i) $q^{i_1(t)} = q^i + (q_1(0) - q^i)(\phi+\theta)^t; q^i_2(t) = q^{i_1(t)}; (\phi+\theta) > 0$

(69ii) $q^{ii_1(t)} = q^{ii} + (q_1(0) - q^{ii})(\phi+\theta)^t; q^{ii}_2(t) = q^{ii_1(t)}; (\phi+\theta) > 0$

(69iii) $q^{iii_1(t)} = q^{iii} + (q_1(0) - q^{iii})(\phi-\theta)^t; q^{iii}_2(t) = 2q^{iii} - q^{iii_1(t)}; (\phi-\theta) > 0$

(69iv) $q^{iv_1(t)} = q^{iv} + (q_1(0) - q^{iv})(\phi-\theta)^t; q^{iv}_2(t) = 2q^{iv} - q^{iv_1(t)}; (\phi-\theta) > 0$

As expected, convergence to equilibrium is monotonic. Of course, due to the saddlepoint solution, the initial values cannot be chosen arbitrarily; for cases (i) and (ii), the initial output of the two firms must be equal,
whereas for cases (iii) and (iv), their sum must be equal to the total steady-state output level.

To summarize the results of this section, we have found that if there are adjustment costs:

(i) dynamic rational-expectations solution can be found.
(ii) the Cournot solution cannot be a rational expectations solution.
(iii) the other four (symmetric) solutions are robust to the presence of these costs.
(iv) the resulting solutions are saddle-point solutions, and thus converge monotonically.
(v) however, the initial conditions cannot be chosen arbitrarily. This is, of course, a problem, but seems inevitable in a rational expectations framework.

Our final task is to see how our results are changed as the number of firms increases.

V) Extension to $N$ Firms

The preceding analysis has assumed there are only two firms. However, the analysis can readily be extended to deal with the case of $N$ firms, provided we maintain the assumption firms have identical cost functions and that the rational-expectations solutions are symmetric. For analytical simplicity, we further assume there is no discounting ($\delta=1$) and that there are no adjustment costs ($c^*=0$).

Using the approach of section II, assume there are $N$ firms, the output of each being denoted by $q_j(t)$. Define:
From the perspective of firm 1, we assume firm 1 believes all other firms have identical reaction functions which, it assumes, are given by:

\[ q_j(t) = e + \gamma q_j(t-1) + n_0 q_j(t-1); \quad j = 2, \ldots, N \]

Summing (71) over all \( j \), not equal to 1, yields:

\[ Q_1(t) = \alpha + \phi Q_1(t-1) + \theta q_1(t-1); \]

\[ \alpha = (N-1)e; \ \theta = (N-1)n; \ \phi = \gamma + (N-2)n \]

Equation (72) is of the same form as postulated in section II (equation 18); thus, using the same methodology, we can derive firm 1's optimal rule:

\[ q_1(t) = a + \gamma' q_1(t-1) + n' Q_1(t-1) \]

A symmetric rational-expectations solution can then be found, as was done earlier. Symmetry implies:

\[ n' = n = [\theta/(N-1)]; \ \gamma' = \gamma = \phi = [\theta(N-2)/(N-1)] \]

\[ a = c = [a/(N-1)] \]

As earlier, if there are no adjustment costs, the Cournot solution is a rational expectations solution. However, there are four other solutions, corresponding to the cases discussed earlier. These are given by:

\[ \theta^* = -\phi^*(N-1); \ (\phi^*)^2 = [(g-B)/NZ], \text{ where} \]

\[ Z = B N + c_2 > g, \ N > 2. \]

The corresponding output solutions for cases (i) and (ii) are:

\[ q^i = (A - c_1)(1 - \phi^*) / [(B + Z) - \phi^*(BN + Z)] \]

\[ q^{ii} = (A - c_1)(1 + \phi^*) / [(B + Z) + \phi^*(BN + Z)] \]

where \( \phi^* \) is defined as the positive root of \( (\phi^*)^2 \). The corresponding values of \( (\phi^a, \ \theta^a) \), and output, for cases (iii) and (iv) are:

\[ \phi^a = c^a = + \left[ \frac{(B+Z)}{Z} \left( \frac{N-1}{N} \right) \right]^{1/2} \]
As discussed in Sections II and III, we can calculate the static conjectural variations corresponding to each one of these solutions. Assuming N identical firms, define the conjectural variation, r, as:

\[ r = \frac{\partial q}{\partial q} \]

thus, \( r \) is the anticipated change in the output of all other firms due to a change in \( q \). If all firms use the same conjecture, and have identical costs, the equilibrium individual firm output is:

\[ q = \frac{A - c_1}{Z + B(1+r)} \]

Thus, the conjectures corresponding to each solution are given by (after some simplification):

\[ r^i = \frac{\phi^a(1 - N)/(1 - \phi^a)}{\phi^a(1 - N)/(1 + \phi^a)}; \quad r^{ii} = \frac{\phi^a(N - 1)/(1 + \phi^a)}{\phi^a(1 - N)/(1 + \phi^a)} \]

For \( N=2 \), these reduce to the same results given in Section III. As earlier, \( r^i = -1 \) if \( c_2 = 0 \); for \( c_2 > 0, \quad r^i < -1 \) (\( N > 1 \)). Also, \( r^{iii} > r^{ii} > r^{iv} \), \( r^{iii} > 1, \quad r^{iv} \in (0, -1) \), and \( r^{ii} > 0 \); for large \( N \), \( r^{ii} > 1 \), provided \( c_2 > 0 \).

As \( N \) grows large, it is clear that the output of each firm tends to zero. Total output, given by \( Nq \) is:

\[ \lim_{N \to \infty} [Nq^x] = \frac{(A - c_1)/B}{x = i, ii, iv} \]

\[ \lim_{N \to \infty} [Nq^{iii}] = 0 \]

Thus, for case iii (which corresponds to output levels below joint profit maximization), total output level actually falls, and tends to zero, as the
number of firms grows large. For the other three cases, total output expands, and price approaches $c_1$ asymptotically, the competitive equilibrium price if no fixed costs are present. Asymptotically, the conjectures are given by:

\[
\lim_{N \to \infty} r^i = -\frac{(g-B)}{B} \leq -1; \quad \lim_{N \to \infty} r^{ii} = \frac{(g-B)}{B} \geq 1,
\]

with strict inequality for $c_2 \neq 0$. Thus, as $N$ grows large, $r^i$ remains less than minus one, while $r^{ii}$ exceeds one, provided $c_2 > 0$. For the other two cases:

\[
\lim_{N \to \infty} r^{iii} = \infty; \quad \lim_{N \to \infty} r^{iv} = -(1/2)
\]

Note that even as the number of firms grows, the conjectures do not approach zero, the Cournot case.

If there are no fixed costs, then the competitive equilibrium is given by $P = c_1$; if marginal costs are constant ($c_2 = 0$), firm size – and hence the number of firms – is indeterminate. If $c_2 > 0$, then the competitive equilibrium entails a "large" number of firms, each producing an infinitessimal output.

For the rational expectations solutions, if the number of firms is determined endogenously – by the condition profits are zero – we get analogous results except for case (iii). If marginal costs are constant, case (i) – the Bertrand solution – always yields the competitive solution, regardless of the number of firms. The Cournot solutions, and cases (ii) and (iv) also yield the competitive solution asymptotically, as $N$ grows large. Even if $c_2 > 0$, these solutions all yield the competitive solution asymptotically. However, note that case (iii) does not converge to the competitive solution – instead, aggregate output shrinks towards zero.
If fixed costs are present, the situation is a bit more complex. Let
the representative firm's total cost curve be given by:

\[ \text{TC} = c_0 + c_1 q + \left( c_2 q^2 / 2 \right) \]

Then average total costs are minimized at:

\[ q^* = \left[ \frac{c_2}{c_1^2} \right]^{1/2}; \quad \text{Min}(\text{ATC}) = c_1 + \left( \frac{2c_0 c_2}{c_1^2} \right)^{1/2} = P^* \]

Thus, the competitive long-run price is given by \( P^* \), the output of each firm
by \( q^* \) and the number of firms, \( N^* \), by \( \left[ (A - B P^*) / q^* \right] \).

The zero-profit (free-entry) oligopoly solution depends on \( r \), the
conjecture. Given \( r \), and the number of firms, individual firm output is
given by (84). If the condition of zero profits is imposed on this
oligopoly equilibrium, the resulting solution is given by:

\[ q' = \left[ \frac{2c_0}{(c_2 + 2B(1+r))} \right]^{1/2}; \quad P' = c_1 + \left[ \frac{2^{1/2}(c_0 c_2 + B(1+r))/(c_0 c_2 + 2c_0 B(1+r))^{1/2}} \right] \]

For \( r = -1 \), this is the same solution as the competitive equilibrium.

Finally, note:

\[ q' \geq q^* \quad \text{as} \quad r \leq -1. \]

Returning to our rational-expectations solution, note that none of them
yields the competitive equilibrium (see Ruffin (1971) for similar results).
Case (i) implies \( r = -1 \) only if \( c_2 = 0 \); but for \( c_2 = 0 \), \( c_0 > 0 \), profits will
always be negative and hence this cannot be a viable rational expectations
solution. For the other four cases (including the Cournot case), \( r > -1 \).
If \( c_2 > 0 \), then \( r^i < -1 \), whereas the conjectures corresponding to the other
four cases exceed minus one. Thus, the zero-profit rational expectations
solution corresponding to case (i) corresponds to individual firms producing
output levels greater than those corresponding to efficient production,
whereas the other four cases correspond to the traditional Chamberlain
result of excess capacity. These results are summarized graphically in Figure I.

To summarize the results of this section, we have found that:

(i) the analysis of prior sections can readily be extended to more than two firms.

(ii) if no adjustment costs are present, there are five symmetric rational-expectations solutions.

(iii) if marginal costs are constant, the Bertrand solution \((r = -1)\) is a viable long-run solution only if there are no fixed costs (the same is true of a competitive equilibrium).

(iv) if no fixed costs are present, then all of the rational expectations solutions - except case (iii) - converge to the competitive equilibrium. For case (iii), aggregate output levels converge to zero.

(v) if fixed costs are present, and marginal cost is increasing, so that the firm cost curve is U-shaped, none of the rational expectations solutions converge to the competitive equilibrium.

(vi) with U-shaped cost curves, all of the rational-expectations solutions - except for case (i) - yield the standard monopolistic competition results of firms' having excess capacity. The excess capacity per firm is smallest for case (iv), whereas it is largest for case (iii).

(vii) finally, there is one rational-expectations solution - case (i) - in which firms produce above the optimal output level; thus, a zero-profit solution does not necessarily entail excess firm capacity. This result is similar to one of
Figure I: Rational Expectations Solutions with Free Entry*

*Notes: $q^*$ is the competitive solution

$q^C$ is the Cournot solution

$q^X$ corresponds to the other rational expectations solutions
Demsetz's (1959) solutions in which advertising costs are present.

VI) Summary and Conclusions

Most criticisms of standard duopoly models focus on the inconsistency between assumptions and actual behavior when firms are out of equilibrium. We have argued that this criticism is not a valid one, since the underlying models of the firm are static in nature, and thus there is no reason for the firm to ever be out of equilibrium. Consequently, we believe that it is necessary to specify a dynamic model of the firm in order to derive fully consistent rational-expectations solutions.

The model we have constructed in this paper is a relatively simple one. Its dynamic characteristics are derived from two assumptions: (i) the presence of output adjustment costs, and (ii) the assumed reaction function. Using this set-up, we have shown how (a multiplicity of) rational-expectations solutions can be derived. We have also seen that the Cournot solution is a rational-expectations solution, provided no adjustment costs are present. If adjustment costs occur, the model is truly dynamic in nature, and the Cournot solution ceases to be a rational-expectations one. However, there are four other symmetric rational expectations solutions, as well as an infinity of asymmetric solutions. Furthermore, it is possible to specify a true dynamic solution for each of these cases.

Of the four (non-Cournot) rational-expectations solutions, we have shown that one yields output levels above the competitive solution (when marginal cost is positively sloped), while a second yields output levels below the joint profit-maximization level. If there is free entry, we have
that the competitive equilibrium will not, in general, be a rational-expectations one. One surprising result that emerges is a monopolistic competitive solution in which firms produce at output levels above optimal capacity, the reverse of the usual Chamberlain-type result.

The model, and the postulated form of the reaction function, that we have used are quite simplistic. Moreover, we have ignored strategic behavior by firms, aimed at attempts to discover – or alter – the reaction functions of other firms. Extending the analysis in this direction would certainly be useful. The presence, however, of multiple solutions may indicate the inherent nonuniqueness of duopoly solutions. It would be interesting to see if other dynamic rational-expectations solutions yield the same problem of nonuniqueness.

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FOOTNOTES

1. If $r_{ij} = r_{ji} = -1$, and $c_{i2} = c_{j2} = 0$, the system is singular — total output is determinate, but individual firm output is indeterminate.

2. While we treat the parameters $(a_j, \phi_j, \theta_j)$ as known, little would be changed — due to the quadratic profit function — by assuming they are random, with known distributions.

3. Allowing the intercept terms on the marginal cost schedules to differ would be relatively trivial, since these do not affect the determination of the $\phi$'s and $\theta$'s. If the slope of the marginal cost curve differ across firms, the numerical analysis becomes quite tedious. However, there is no conceptual difficulty involved in assuming the firms are asymmetric.

4. If $\delta = 1$, $c_2 = 0$, then for $\phi = -\theta$, $|\phi| = 1/2$, and $|\lambda_2| = 1$.

5. Of course, for $\delta = 1$, the function $V_t^*$ is not convergent. However, overtaking principles can be used. Furthermore, it is clear the solution is continuous in $\delta$.

6. If $c_2 = 0$, there is no competitive solution for $c_0 > 0$, as the optimal solution entails having only one firm produce output.
References


