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Application of displacement and traction boundary integral equations for fracture mechanics analysis

Seungwon Youn
Iowa State University

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Application of displacement and traction boundary integral equations for fracture mechanics analysis

Youn, Seungwon, Ph.D.

Iowa State University, 1993

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Application of displacement and traction boundary integral equations for fracture mechanics analysis

by

Seungwon Youn

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Signature was redacted for privacy.

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CHAPTER 1. INTRODUCTION

The boundary element method (BEM) is now well established and is in widespread use in many application areas [1]. This method is a numerical procedure for the solution of singular integral equations or the so-called boundary integral equations (BIE's) that are typically used to formulate boundary value problems. Jaswon [2] first used the direct formulation in the solution of potential problems and this method was extended to elastostatics by Rizzo [3] and Cruse [4] in the 1960's.

The method is referred to as a "boundary element method" [5] in that numerical approximations are required only on the boundary of a domain. The BIE formulation contains only integrals over the boundary so the BEM for the solution of such equations requires only the numerical integration of data on the boundary. There are three different approaches [6] in the formulation of the BIE's. These are Betti's reciprocal work theorem, the weighted residual method and the superposition method due to linear operator theory. The integrals are, however, singular in that the integrands become singular at the point on the integration surface or path.

Compared with the well-known finite element method and the finite difference method, the BEM has some notable advantages. The primary advantage of the BEM is that the degrees of freedom of the system of algebraic equations are comparatively small since the BEM requires discretization only on the boundary rather than throughout the volume. Meshes can easily be generated and design modifications do not require a com-
plete remeshing. Meshing only the surface is a big advantage especially for three-dimensional problems. Also, the BEM is a particularly powerful tool in problems where high accuracy is desired such as in stress concentration problems and where the domain extends to infinity [1]. One can find various applications of the BEM in the literature [7, 8]. Other applications of the BIE's are in potential theory [9-12], elastic waves and acoustics [13-19], thermoelasticity [20], heat conduction [21], elastostatics [4, 22, 23], composite materials [24-26], elastodynamics [27] and fracture mechanics [27-45].

Most elastostatic problems can be solved using only the displacement BIE's which are determined by the limit forms of the Somigliana's identity or the displacement representation. However, in crack problems, when the upper and lower crack surfaces lie in the same location, the collocation point on one crack surface occupies the same spatial coordinates as the opposite crack surface. In this case, the displacement BIE's are degenerated. Two collocation points in the same plane generate the same equation and leads to a singular coefficient matrix for the system. For this reason, closed crack problems cannot be solved by the use of the displacement BIE's alone when the body is modelled with a single region.

Background

To overcome the singularity problem in overlapping crack surfaces, several special techniques have been developed. These include the displacement discontinuity method [31], the specialized Green's function method [32], the multidomain method [33] and the dual integral equation method [6]. Cruse [7, 32] solved two-dimensional problem by approximating a crack by an open notched and by using a specializing a Green's function so
that it satisfies a traction free boundary condition on the crack surface. Cruse [34] and Smith and Aliabadi [35] used the multidomain method, where the body is artificially divided through the crack, to solve their two- and three-dimensional problems. Ang [36] solved an arc crack problem in terms of an integral taken only around the exterior boundary of the body under the assumption that the crack is stress free. The drawback of a multidomain method is non-uniqueness of the artificial boundary and a larger system of equations than needed. Jia et al. [42] have used displacement and traction singular shape functions for the crack surfaces together with a combination of Cartesian and polar coordinate transformation to solve a penny-shaped crack in an infinite body and in a finite cylinder by a multidomain method without the traction BIE's.

To solve a general mixed-mode crack without the special techniques mentioned above, one needs another integral equation in addition to the displacement BIE to ensure a unique solution [32]. The traction BIE's, which are formed by linear combinations of derivatives of the displacement BIE's through Hooke's law, have been shown to provide the additional equations. Guidera and Lardner [43] have first used the traction BIE's explicitly to solve the problem of a penny-shaped crack subjected to the arbitrary tractions in an infinite medium. The traction and the displacement BIE's are herein referred to as dual boundary integral equations and are both required for the solution of crack problems.

The fundamental solutions of the displacement BIE's contain both weak and strong (Cauchy) singularities and can be integrated in the sense of improper integrals and as Cauchy principal value integrals. However, the fundamental solution kernels associated with the traction BIE's have both strong (Cauchy) and stronger, termed hypersingular singularities, due to the differentiation process. The hypersingular kernels in the traction BIE's cause some difficulties in the numerical implementation. Polch et al. [44, 45] used a
finite element interpolation scheme for the unknown displacement discontinuities and its derivatives on the crack surface to analyze a circular and elliptical buried crack. By taking the test function with the displacement gradients instead of the displacement field for the equilibrium equation, Okada et al. [46] derived a weak vector form, which has a lower order singularity than the usual traction BIE's. They solved two-dimensional open-crack problems using discontinuous corner elements.

To reduce the singularity order of the hypersingular integrals, Cruse and Vanburen [30] subtracted the first term of a Taylor expansion from the density (displacement) function. They also derived some integral identities for the stress tensor at points on the surface by considering a rigid body translation. Similarly, Jeng and Wexler [47] and Aliabadi et al. [48, 49] subtracted the first term of a Taylor expansion. In either case, the added back terms are still hypersingular and remain to be computed. Brandao [50] and Kaya and Erdogan [51] subtracted the first two terms of a Taylor expansion to evaluate one-dimensional hypersingular integrals in the Hadamard sense. For multi-dimensional problems, Krishnasamy et al. [17] also subtracted the first two terms of a Taylor expansion to regularize the hypersingular integrals. The added back terms were converted from surface integrals to the line integrals by using Stokes' theorem.

Recently, many researchers have contributed to the use of the hypersingular BIE's [52-67]. Some important identities pertaining to the fundamental solutions were established by Rudolphi et al. [52-56] and have been used to avoid the stronger integrals. The regularization process using a Taylor expansion and the identities pertaining to the fundamental solutions to get regularized forms of the traction and the displacement BIE's has become a popular method. The added back terms which appear during the regularization process are replaced by weakly singular integrals through the pertinent identities. These
identities can be used to avoid one hypersingular integral, but in crack problems there are always two such integrals to regularize, one on each crack surface. As a part of what they called the modal solution method (which is the same as the above mentioned identities), Lutz et al. [64] have constructed tent-like closure surfaces and used the integral identities on both crack surfaces. Portela et al. [65] and Mi and Aliabadi [66] solved two- and three-dimensional closed crack problems by the dual boundary integral equations, respectively. After the regularization with the key terms of a Taylor expansion, they have integrated the finite part integrals, the stronger singularities, in the traction BIE’s on a vanishingly small part of the boundary near the limit point following the technique Guiggiani et al. [67].

**Present Work**

The main object of this thesis is concerned with a solution for a general mixed-mode crack problem in a linearly elastic isotropic medium by use of the dual boundary integral equation approach, together with the auxiliary surface approach of Lutz et al. [64]. The computational algorithm for the solution of general mixed-mode crack problems in bounded or unbounded domains, and two- or three-dimensions has been developed.

By presuming the density functions are sufficiently continuous at the singular point, the displacement and the traction BIE’s are regularized by subtracting and adding back the key terms of the Taylor expansion to the density function. The regularized BIE’s have, at most, weakly singular integrals.

The added back integrals contain the hypersingular kernels on both crack surfaces. To integrate these added back singular integrals, the integration is carried out over
smoothly curved auxiliary surfaces that replace the actual singular surface. The introduced auxiliary surfaces always provide non-singular integration paths and result modified forms of identities for uncracked bodies.

This thesis is organized as follows.

Chapter 2 provides the notation, the basic equations and the fundamental solutions for elastostatics. The fundamental solutions are given without detailed derivation.

Chapter 3 gives the mathematical foundation of the dual boundary integral equations. The displacement representation is derived by Betti's reciprocal work theorem. By taking the derivatives of the displacement representation and by contracting on the elastic moduli as required by Hooke's law, the traction representation is formed. After derivation of the integral identities from the rigid body translation and the constant strain fields in elastostatics, several modified identities are developed for the auxiliary surface idea. Then, the displacement and the traction representations are regularized by the Taylor expansion and the relevant integral identities. During the regularization process, the whole integration range is divided into two, that is, a singular part and the remainder, for convenience. By taking the limit of the source point from the interior to the surface, the displacement and the traction BIE's are formed from the representation equations.

Chapter 4 describes the development of all possible computational forms for both the displacement and the traction BIE's in elastostatics. The appropriate integral identities and the auxiliary surfaces are used during the development process. Special attention is given to the crack surfaces. The developed computational forms all have similar features.

Chapter 5 describes the numerical integration methods. The weak singularities are removed by coordinate transformation, that is, polar coordinates in three-dimensional problems. To isolate the stronger singularities, the actual surfaces are replaced by the
smoothly curved auxiliary surfaces which provide the detoured, non-singular integration paths. Convergence tests for different number of subdivisions or sub-elements on the auxiliary surfaces are performed.

Chapter 6 describes the numerical implementation for the BIE's. Element types and the discretization process are explained. The focus here is on the strongly singular integrals and the normalization of the linear equations. The usual Gaussian quadrature technique is used for the numerical integration. The stress intensity factor calculations and the data interfacing with I-DEAS and AutoCAD are studied.

Chapter 7 gives numerical examples by the developed BEM program and a translator BEMUNV. A number of typical two- and three-dimensional crack models, a crack in an unbounded region, an embedded crack, an edge crack, a single crack and a multiple crack, are analyzed. The last problem in three-dimensional examples is a practical edge crack model with a mixed-mode stress field. With the results, all deformed shapes are given. Crack tip stress intensity factors are determined from the nodal displacements on the crack surfaces. Comparison to known analytic solutions is made whenever possible.

Chapter 8 gives the conclusions and recommendations for future work.

Finally, the surface gradients on three-dimensional surface and all shape functions used in the program development are provided in the appendixes. The derivations of the shape functions are not difficult but takes time. Appended shape functions for continuous, discontinuous and partially discontinuous elements in two- and three-dimensional problems may help others working with similar problems.
CHAPTER 2. THE FUNDAMENTAL SOLUTION TENSORS FOR ELASTOSTATICS

This chapter introduces the notation, the basic equations of linear elastostatics \[68\] and the fundamental solution tensors pertinent to elastostatics as required for the boundary integral formulation. The fundamental solution tensors, due to Lord Kelvin, correspond to the basic field variables, that is, displacements and tractions, due to a unit force at a point in an infinite domain with the same material properties as the problem to be solved. Throughout the thesis, index notation and the summation convention are to be used in all equations, except where noted.

Domains and Notation

Shown in Figure 2.1 is the typical domain for an elastic body, the first an interior region and the second an exterior region. Some definitions for the basic symbols used throughout the following development are as follows:

- \( V \) interior domain or region
- \( E \) exterior domain or region
- \( S \) surface or boundary
- \( \vec{x} \) field point position vector or coordinates
- \( \vec{\xi} \) source point position vector or coordinates
(a) Interior domain: external surface (outward normal)

(b) Exterior domain: internal surface (inward normal)

Figure 2.1: Domains, notation and surface normal directions
\[ \bar{r} \] position vector from \( \xi \) to \( \bar{r} \)

\[ \bar{n} \] outward normal at a point on the surface \( S \)

When the domain \( V \) is bounded externally as in Figure 2.1(a) the domain is called an interior domain, whereas when bounded externally as shown in Figure 2.1(b) it is called an exterior domain. In either case the unit normal \( \bar{n} \) on the surface is conventionally taken as outward from the material.

### The Basic Equations of Linear Elastostatics

In the linear theory of elasticity, the governing equations assumably satisfy the linear elastic stress-strain law. In this thesis, the material is also assumed to be isotropic. An isotropic material is characterized by Hooke's stiffness tensor \( E_{ijkl} \), which contains only two independent properties called Lame's constants, denoted by \( \lambda \) and the modulus of elasticity in shear or the modulus of rigidity \( \mu \). In terms of these constants, Hooke's stiffness tensor \( E_{ijkl} \) can be written

\[
E_{ijkl} = \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda \delta_{ij} \delta_{kl}
\]  

(2.1)

where \( \delta_{ij} \) is the Kronecker delta. Also, the tensor \( E_{ijkl} \) has the symmetry property that

\[
E_{ijkl} = E_{klij}
\]

(2.2)

which is a consequence of the positive definiteness of the strain energy density function. Lame's constants can also be expressed in terms of two alternates and commonly used constants, that is, Young's modulus \( E \) and Poisson's ration \( \nu \). These relationships are
The complete system of equation of linear elasticity for homogeneous isotropic solids includes equilibrium equations (with the body force distribution \( f_i \))

\[ \sigma_{ij,j} + f_i = 0 \] (2.5)

stress-strain law (generalized Hooke's law)

\[ \sigma_{ij} = E_{ijkl} \varepsilon_{kl} = \lambda \varepsilon_{mm} \delta_{ij} + 2 \mu \varepsilon_{ij} \] (2.6)

and strain-displacement relations (Cauchy strain tensor)

\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \] (2.7)

The displacement field \( u_i \) in equation (2.7) should have a single-valued continuous field for a simply connected region. From this condition, the compatibility relations follow.

\[ \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ij,ki} - \varepsilon_{ki,ij} = 0 \] (2.8)

Traction or stress vector components are obtained from the stresses \( \sigma_{ij} \), or vice versa, at a point by

\[ t_i(\xi) = \sigma_{ij}(\xi) n_j(\xi) \] (2.9)

where \( n_j \) are the unit vector components in an arbitrary direction.

By employing right side of equation (2.6) and equation (2.7) in the equilibrium equation (2.5), one gets the well-known Navier's equations of elasticity.
\[
\left(\frac{1}{1-2\nu}\right)u_{i,jj} + u_{i,jj} + \frac{f_i^j}{\mu} = 0 \tag{2.10}
\]

which have only the displacement fields as the dependent variables.

The Fundamental Solution for Elastostatics

The fundamental solution for elastostatics corresponds to the displacements which satisfy Navier's equations (2.10) with a unit force at a point \(\xi\) in an infinite medium (Figure 2.2). The fundamental solution displacements are denoted by \(U_{ij}(\xi,\bar{\xi})\) and represent the displacements in the \(j\) direction at a point \(\bar{x}\) due to a unit point force acting in the \(i\) direction at \(\bar{\xi}\). The traction components associated with these displacements are denoted by \(T_{ij}(\bar{x},\bar{\xi})\) and are obtained through equations (2.6) and (2.9). Thus one has

\[
T_{ij}(\bar{x},\bar{\xi}) = \mathbf{E}_{iklm} U_{lj,m} n_k(\bar{\xi}) \tag{2.11}
\]

Figure 2.2: An actual region \(V\) in an infinite region \(V_\infty\)
One can find the following fundamental solutions from many available references [1, 5, 34, 38].

For two dimensional plane strain conditions \((i,j,k = 1,2)\), one has

\[
U_{ij}(\bar{x}, \bar{\xi}) = \frac{-1}{8\pi\mu(1-\nu)} \left[ \delta_{ij}(3-4\nu) \ln r - r_i r_j \right] \tag{2.12}
\]

\[
T_{ji}(\bar{x}, \bar{\xi}) = \frac{-1}{4\pi(1-\nu) r} \left[ \left( (1-2\nu) \delta_{ij} + 2r_i r_j \right) r_k n_k + (1-2\nu)(r_j n_i - r_i n_j) \right] \tag{2.13}
\]

\[
T_{ji,k}(\bar{x}, \bar{\xi}) = \frac{-1}{4\pi(1-\nu) r^2} \left[ 2(\delta_{jk} r_i + \delta_{ik} r_j) r_m n_m \right. \\
+ \left. -2\left( (1-2\nu) \delta_{ij} r_k + 4r_i r_j r_k \right) r_m n_m + \left\{ 2r_j r_i + (1-2\nu) \delta_{ij} \right\} n_k \right. \\
+ \left. (1-2\nu)(2r_i r_k - \delta_{ik}) n_j + (1-2\nu)(\delta_{jk} - 2r_j r_k) n_j \right] \tag{2.14}
\]

and for three dimensional problems \((i,j,k = 1,3)\)

\[
U_{ij}(\bar{x}, \bar{\xi}) = \frac{1}{16\pi\mu(1-\nu) r} \left( (3-4\nu) \delta_{ij} + r_i r_j \right) \tag{2.15}
\]

\[
T_{ji}(\bar{x}, \bar{\xi}) = \frac{-1}{8\pi(1-\nu) r^2} \left[ \left( (1-2\nu) \delta_{ij} + 3r_i r_j \right) r_k n_k + (1-2\nu)(r_j n_i - r_i n_j) \right] \tag{2.16}
\]

\[
T_{ji,k}(\bar{x}, \bar{\xi}) = \frac{-1}{8\pi(1-\nu) r^3} \left[ 3(\delta_{jk} r_i + \delta_{ik} r_j) r_m n_m \right. \\
+ \left. -3\left( (1-2\nu) \delta_{ij} r_k + 5r_i r_j r_k \right) r_m n_m + \left\{ 3r_j r_i + (1-2\nu) \delta_{ij} \right\} n_k \right. \\
+ \left. (1-2\nu)(3r_i r_k - \delta_{ik}) n_j + (1-2\nu)(\delta_{jk} - 3r_j r_k) n_j \right] \tag{2.17}
\]

where \(r = |\bar{x} - \bar{\xi}|\) is the distance between \(\bar{x}\) and \(\bar{\xi}\) and \(r_i = \frac{x_i - \xi_i}{r}\).
CHAPTER 3. THE BOUNDARY INTEGRAL EQUATIONS

This chapter begins with the definition of more notation used in the development of boundary integral equations for elastostatics. The boundary integral equations can be derived through three different approaches [6]. These are Betti's reciprocal work theorem, the weighted residual method and the superposition method due to linear operator theory. As a starting point, the well-known Somigliana's identity or the displacement representation is derived from the governing equation in elasticity and Betti's reciprocal work theorem with the known fundamental solutions. The stresses at internal and on surface points follow. The traction representation is formed from the displacement representation by contracting the material property on the derivatives of the displacements.

By imposing the rigid body translation and the constant strain field solutions onto the displacement or the traction representation, some important integral identities pertaining to the fundamental solutions are formed [30, 52-56]. Then, a discussion about the characteristics of the fundamental solutions and the order of the singularities is made.

To isolate the strongly singular integrals, the displacement and the traction representations are regularized by subtracting and adding back the key terms of a Taylor expansion to the density (displacement) functions and the relevant integral identities. In the regularization process, it is convenient to divide whole integration range into two parts, the singular part $S_0$ and the remainder $S'$. Through the regularization process, one gets the regu-
larized displacement and the traction representations that contain at most weakly singular integrals.

The displacement and the traction boundary integral equations are determined through the limiting process for the relevant regularized representations.

**Geometric Notation**

Figures 3.1 and 3.2 show the geometric notation and the vector components used. The descriptions of the variables are as follows.

- $S'$, $S_o$: surfaces, $S'$ for a regular and $S_o$ for a singular part
- $S_1$: surface which the natural boundary condition, traction, is prescribed
- $S_2$: surface which the essential boundary condition, displacement, is prescribed
- $\Gamma_o$: artificial auxiliary boundary or surface
- $\bar{x}$, $\bar{X}$: field point vector components
- $\bar{\xi}$: source point vector components in the domain
- $\bar{z}$: source point vector components on the surface
- $\bar{r}$: distance vector components from the source point to the field point

Vector components on the surface are represented by

- $\bar{e}$, $\hat{e}$: basis vector components
- $\bar{n}$: normal vector components at the field point
- $\bar{s}_m$: tangential vector components at the field point
- $\bar{v}$: normal vector components at the source point
- $\bar{\zeta}_m$: tangential vector components at the source point

where $m = 1$ for two-dimensional and $m = 1, 2$ for three-dimensional surfaces.
Figure 3.1: Geometric notation and the vector components in a 2-D domain

Figure 3.2: Geometric notation and the vector components in a 3-D domain
Reciprocal Work Theorem

First, consider two different equilibrium states. Superscript \(^{(1)}\) denotes the first state and \(^{(2)}\) denotes the second state. For each state, the equilibrium equations are

\[
\sigma_{ij}^{(1)} + f_i^{(1)} = 0 \\
\sigma_{ij}^{(2)} + f_i^{(2)} = 0
\]  
(3.1)  
(3.2)

Stress-strain products for two different states are

\[
\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = E_{ijkl} \varepsilon_{kl}^{(1)} \varepsilon_{ij}^{(2)} = E_{ijkl} \varepsilon_{ij}^{(1)} \varepsilon_{ij}^{(2)} \\
\sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} = E_{ijkl} \varepsilon_{kl}^{(2)} \varepsilon_{ij}^{(1)} = E_{ijkl} \varepsilon_{ij}^{(1)} \varepsilon_{ij}^{(2)}
\]  
(3.3)  
(3.4)

Equations (3.3) and (3.4) are identical due to the symmetry property of an isotropic material. The strains and the stresses are also symmetric, that is,

\[
E_{ijkl} = E_{klij}  \\
\varepsilon_{ij} = \varepsilon_{ji}  \\
\sigma_{ij} = \sigma_{ji}
\]  
(3.5)  
(3.6)  
(3.7)

For identical body force conditions \(f_i^{(1)} = f_i^{(2)}\), one can write

\[
\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = \sigma_{ij}^{(1)} \frac{1}{2} (u_i^{(2)} + u_j^{(2)}) \\
= \sigma_{ij}^{(1)} u_i^{(2)}
\]  
(3.8)

and

\[
\sigma_{ij}^{(1)} u_i^{(2)} = [\sigma_{ij}^{(1)} u_i^{(2)}]_j = \sigma_{ij}^{(1)} u_i^{(2)}
\]  
(3.9)

Then, equations (3.1) and (3.2) can be written as
Noticing that \( \sigma_{ij,j} = -f_i \), equation (3.10) may be written as

\[
\left[ \sigma_{ij}^{(1)} u_i^{(2)} \right]_j + f_i^{(1)} u_i^{(2)} = \left[ \sigma_{ij}^{(2)} u_i^{(1)} \right]_j + f_i^{(2)} u_i^{(1)}
\] (3.11)

By integrating equation (3.10) over the entire volume, converting the derivative terms by the divergence theorem and finally applying the Cauchy's stress formula on the surface integrals, one gets

\[
\int_S t_i^{(1)} u_i^{(2)} dS + \int_V f_i^{(1)} u_i^{(2)} dV = \int_S t_i^{(2)} u_i^{(1)} dS + \int_V f_i^{(2)} u_i^{(1)} dV
\] (3.12)

This equation (3.12) is the vector form of Betti's reciprocal work theorem.

The Displacement Representation Formulation

If the first state is identified with the fundamental state, one can substitute the fundamental solution \( u_i^{(1)} \to U_{ij} \) and its derivative \( t_i^{(1)} \to T_{ij} \) into equation (3.12). Then Betti's theorem, with the condition \( f_i^{(1)} = 0 \) (\( \forall r \neq 0 \)) and without the superscript \(^{(2)}\), yields

\[
\int_S T_{ij}(\vec{x},\vec{\xi}) u_j(\vec{x}) dS(\vec{x}) = \int_S U_{ij}(\vec{x},\vec{\xi}) t_j(\vec{x}) dS(\vec{x}) + \int_V U_{ij}(\vec{X},\vec{\xi}) f_j(\vec{X}) dV(\vec{X})
\] (3.13)

As shown in Figure 3.3, now consider the two different situations in which the source point is at an interior point of the domain and then at an exterior point, respectively. Then consider equation (3.13) term by term as \( \varepsilon \to 0 \) (\( S_{\varepsilon} \to 0 \) and \( (S - S_{\varepsilon}) \to S \)). The left side becomes
\[ \int_{S} T_{ij}(\vec{x}, \vec{\zeta}) u_j(\vec{x}) dS(\vec{x}) = \lim_{\varepsilon \to 0} \int_{S-S_{\varepsilon}} T_{ij}(\vec{x}, \vec{\zeta}) u_j(\vec{x}) dS + \lim_{\varepsilon \to 0} \int_{S_{\varepsilon}} T_{ij}(\vec{x}, \vec{\zeta}) u_j(\vec{x}) dS \quad (3.14) \]

and

\[ \lim_{\varepsilon \to 0} \int_{S_{\varepsilon}} T_{ij}(\vec{x}, \vec{\zeta}) u_j(\vec{x}) dS = \lim_{\varepsilon \to 0} \int_{S_{\varepsilon}} T_{ij}(\vec{x}, \vec{\zeta}) [u_j(\vec{\zeta}) + \vec{\nabla} u_j(\vec{\zeta}) \cdot \vec{\phi} + \cdots ] d\theta \]

\[ = \lim_{\varepsilon \to 0} \left[ \delta_{ij} u_j(\vec{\zeta}) + O(\varepsilon) \right] \quad (3.15) \]

\[ = \begin{cases} u_j(\vec{\zeta}) & (\forall \vec{\zeta} \in V) \\
0 & (\forall \vec{\zeta} \in E) \end{cases} \]

The first term on the right side of equation (3.13) may be written as

\[ \int_{S} U_{ij}(\vec{x}, \vec{\zeta}) t_j(\vec{x}) dS(\vec{x}) = \lim_{\varepsilon \to 0} \int_{S-S_{\varepsilon}} U_{ij}(\vec{x}, \vec{\zeta}) t_j(\vec{x}) dS + \lim_{\varepsilon \to 0} \int_{S_{\varepsilon}} U_{ij}(\vec{x}, \vec{\zeta}) t_j(\vec{x}) dS \quad (3.16) \]

and

\[ \lim_{\varepsilon \to 0} \int_{S_{\varepsilon}} U_{ij}(\vec{x}, \vec{\zeta}) t_j(\vec{x}) dS = 0 \quad (\forall \vec{x}, \vec{\zeta} \in V \text{ or } \forall \vec{x}, \vec{\zeta} \in E) \quad (3.17) \]
For the body force term, by taking the limit as $\varepsilon \to 0$, $V_e \to 0$ and $(V - V_e) \to V$, one has

$$
\int_U U_{ij}(\bar{X}, \bar{\zeta}) f_i(\bar{X}) dV = \lim_{\varepsilon \to 0} \int_U U_{ij}(\bar{X}, \bar{\zeta}) f_i(\bar{X}) dV + \lim_{\varepsilon \to 0} \int_{V_e} U_{ij}(\bar{X}, \bar{\zeta}) f_i(\bar{X}) dV \quad (3.18)
$$

with

$$
\lim_{\varepsilon \to 0} \int_{V_e} U_{ij}(\bar{X}, \bar{\zeta}) f_i(\bar{X}) dV = 0 \quad (\forall \bar{X}, \bar{\zeta} \in V \text{ or } \forall \bar{X}, \bar{\zeta} \in E) \quad (3.19)
$$

By substituting the above equations into equation (3.13), one gets

$$
u_i(\bar{\zeta}) + \int_S T_{ij}(\bar{x}, \bar{\zeta}) u_j(\bar{x}) dS = \int_S U_{ij}(\bar{x}, \bar{\zeta}) t_j(\bar{x}) dS + \int_U U_{ij}(\bar{X}, \bar{\zeta}) f_i(\bar{X}) dV \quad (3.20)
$$

for the interior problem ($\forall \bar{x} \in S$ and $\forall \bar{\zeta}, \bar{X} \in S \cup V$) and

$$
\int_S T_{ij}(\bar{x}, \bar{\zeta}) u_j(\bar{x}) dS = \int_S U_{ij}(\bar{x}, \bar{\zeta}) t_j(\bar{x}) dS + \int_U U_{ij}(\bar{X}, \bar{\zeta}) f_i(\bar{X}) dV \quad (3.21)
$$

for the exterior problem ($\forall \bar{x} \in S$ and $\forall \bar{\zeta}, \bar{X} \in S \cup E$).

Equation (3.20) is known as Somigliana's identity or the interior representation for $u_i(\bar{\zeta})$ and is valid for any point in terms of the boundary values $u_j$ and $t_j$. The interior representation does not predict $u_i(\bar{\zeta})$ is zero when $\bar{\zeta}$ is outside the domain, since it does not apply there. Equation (3.21) applies outside but it does not say that $u_i(\bar{\zeta})$ is zero at outside either. The displacement components at any point in the field and on the surface can be obtained from the boundary values $u_j$ and $t_j$, the body forces $f_j$ and the fundamental solutions. Boundary conditions may be imposed anywhere on the surface as

$$
t_j = \bar{t}_j \quad \text{on } S_1 \quad (3.22)
$$

$$
u_j = \bar{u}_j \quad \text{on } S_2 \quad (3.23)
$$
The Displacement Boundary Integral Equation

Somigliana's identity (3.20) is valid for points in the domain $V$, including the boundary $S$, when $u_i$ and $t_i$ are known at every boundary point. However, as the source point is taken to the boundary the integrals become singular. To get the boundary integral equation from equation (3.20), one lets the source point $\tilde{\xi}$ go to a typical surface point $\tilde{z}$ through a limit process. To avoid singular integration on the surface, one can argument the boundary by an outward hemisphere as shown in Figure 3.4. With the boundary partitioned as $S = S' - S_e + S'_e$, equation (3.20) can be written as

$$u_i(\tilde{\xi}) = \int_{S' - S_e + S'_e} U_{ij}(\tilde{x}, \tilde{\xi}) t_j(\tilde{x}) dS(\tilde{x}) - \int_{S' - S_e + S'_e} t_{ij}(\tilde{x}, \tilde{\xi}) u_j(\tilde{x}) dS(\tilde{x})$$

$$+ \int_{V} U_{ij}(\tilde{X}, \tilde{\xi}) f_j(\tilde{X}) dV(\tilde{X})$$

(3.24)

Figure 3.4: Boundary surface $S'_e$ for the surface point $\tilde{z}$ inclusion
By taking the limit $\varepsilon \to 0$, $S^\varepsilon_\i$ tends to $S_c$ and the displacement components at the point on the surface are obtained as

$$c_{ij}(\vec{z})u_i(\vec{z}) = \int_S U_{ij}(\vec{x},\vec{z})t_j(\vec{x})dS(\vec{x}) - \int_S T_{ij}(\vec{x},\vec{z})u_j(\vec{x})dS(\vec{x})$$

$$+ \int_V U_{ij}(\vec{X},\vec{z})f_j(\vec{X})dV(\vec{X})$$

$$\quad \forall \vec{x}, \vec{z} \in S$$

$$\forall \vec{X} \in S \cup V$$

(3.25)

where

$$c_{ij}(\vec{z}) = \lim_{\varepsilon \to 0} \int_{S^\varepsilon_\i} T_{ij}(\vec{x},\vec{z})dS(\vec{x}) + \delta_{ij}$$

(3.26)

Now, the integrals on $S$ must be interpreted in the sense of Cauchy principal value integrals.

**Multiple Regions**

For large problems or if the body consists of different homogeneous linear elastic materials, it may be convenient or necessary to partition the domain into two or more regions by introducing artificial internal boundaries (Figure 3.5).

![Figure 3.5: Multiple regions](image-url)
On an internal boundary $S^{ij}$ as shown in Figure 3.5, the equilibrium and displacement compatibility conditions require that [69]

$$\bar{u}^{ij} = \bar{u}^{ji} \quad (3.27)$$

$$\bar{r}^{ij} = -\bar{r}^{ji} \quad (3.28)$$

With a partitioned domain, the fully populated matrices will be transformed into banded or block banded matrices. If the domain consists of regions with different material properties, this method of internal boundaries is necessary so that equation (3.25) can be applied to each homogeneous and isotropic region.

**Stresses at Internal Points**

To calculate the stresses at internal points from the interior representation, one needs to differentiate equation (3.20) with respect to the coordinates $\xi_k$. By contracting the material property $E_{pqik}$ on the derivatives, one obtains the stress equations at any internal point. First, by taking the derivatives of the interior representation, one has

$$\frac{\partial u_i(\vec{\xi})}{\partial \xi_k} = \int_S \frac{\partial U_{ij}(\vec{x}, \vec{\xi})}{\partial \xi_k} t_j(\vec{x}) dS(\vec{x}) - \int_S \frac{\partial T_{ij}(\vec{x}, \vec{\xi})}{\partial \xi_k} u_j(\vec{x}) dS(\vec{x})$$

$$+ \int_V \frac{\partial U_{ij}(\vec{X}, \vec{\xi})}{\partial \xi_k} f_j(\vec{X}) dV(\vec{X}) \quad \forall \vec{x} \in S, \forall \vec{\xi}, \vec{X} \in S \cup V \quad (3.29)$$

With Hooke's law, in the regularized form, the stresses at the interior point $\vec{\xi}$ are given by

$$\sigma_{pq}(\vec{\xi}) = E_{pqik} \frac{\partial u_i(\vec{\xi})}{\partial \xi_k}$$

$$\quad (3.30)$$
By substituting equation (3.29) into (3.30), one obtains the following stress formula.

\[
\sigma_{pq}(\bar{\xi}) = \int_S E_{pqik} \frac{\partial U_{ij}(\bar{x}, \bar{\xi})}{\partial \xi_k} t_f(\bar{x}) dS(\bar{x}) - \int_S E_{pqik} \frac{\partial T_{ij}(\bar{x}, \bar{\xi})}{\partial \xi_k} u_f(\bar{x}) dS(\bar{x})
\]

\[
+ \int_V E_{pqik} \frac{\partial U_{ij}(\bar{X}, \bar{\xi})}{\partial \xi_k} f_f(\bar{X}) dV(\bar{X})
\]

(3.31)

Note that the fundamental solution kernels in equation (3.31) have higher order singularities than in the displacement representation due to the differentiation process. The stress and the displacement components are determined from the internal equation solution. So, when the internal point is near the boundary, one needs special integration schemes to obtain accurate results because the fundamental solutions contain higher singularities. To get a more compact form of equation (3.31), two new variables are defined as

\[
\Sigma_{pq} = E_{pqik} \frac{\partial U_{ij}(\bar{x}, \bar{\xi})}{\partial \xi_k}
\]

(3.32)

\[
\Theta_{pq} = E_{pqik} \frac{\partial T_{ij}(\bar{x}, \bar{\xi})}{\partial \xi_k}
\]

(3.33)

With these new variables, one can calculate the stresses at any point in the interior and on the boundary from

\[
\sigma_{pq}(\bar{\xi}) = \int_S \Sigma_{pq} t_f(\bar{x}) dS(\bar{x}) - \int_S \Theta_{pq} u_f(\bar{x}) dS(\bar{x}) + \int_V \Sigma_{pq} f_f(\bar{X}) dV(\bar{X})
\]

(3.34)
Stresses on the Surface

One can calculate the unspecified displacements and the tractions on the boundary from the displacement BIE's (3.25). Once the displacements and the tractions at the points on the surface are known, it is then possible to compute the stress and the strain tensors on the surface by computing them in a local coordinate system on the surface and then transforming them to the global system. The transformation can be accomplished by the transformation law

\[ \hat{e}_i = a_{ij} \hat{e}_j \quad (3.35) \]

where \( a_{ij} \) denote the direction cosines of the local system \( (\hat{e}_1, \hat{e}_2, \hat{e}_3) \) with respect to the global system \( (\bar{e}_1, \bar{e}_2, \bar{e}_3) \). The hat in the vector notation indicates the local coordinate system. Similarly, the boundary displacement components can be transformed as

\[ \hat{u}_i = a_{ij} u_j \quad (3.36) \]

The normal stress components at the surface follow directly from the calculated traction components, that is,

\[ \hat{\sigma}_{k3} = \hat{\sigma}_{3k} = \hat{\sigma}_k \quad (3.37) \]

To calculate the remaining three stress components, one needs the three strain components which, from equation (2.7), are

\[ \hat{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial \hat{u}_j}{\partial \hat{x}_i} + \frac{\partial \hat{u}_i}{\partial \hat{x}_j} \right) \quad (i,j = 1,2) \quad (3.38) \]

Here, one should notice that all tangent components of the displacements and the strains are in the surface coordinates (see Appendix A for the surface gradients calculation). The three strain components are given by
\[
\hat{e}_{33} = \frac{1}{1 - \nu} \left[ \left( \frac{1 - 2\mu}{2\mu} \right) \hat{\sigma}_{33} - \nu (\hat{e}_{11} + \hat{e}_{22}) \right]
\]  
(3.39)

\[
\hat{e}_{13} = \hat{e}_{31} = \frac{1}{2\mu} \hat{\sigma}_{13}
\]  
(3.40)

\[
\hat{e}_{23} = \hat{e}_{32} = \frac{1}{2\mu} \hat{\sigma}_{23}
\]  
(3.41)

From equations (2.6) and (3.38), one can get the other three stress components by

\[
\hat{\sigma}_{12} = \hat{\sigma}_{21} = 2\mu \hat{e}_{12}
\]  
(3.42)

\[
\hat{\sigma}_{11} = \frac{1}{1 - \nu} \left[ \nu \hat{\sigma}_{33} + 2\mu (\hat{e}_{11} + \nu \hat{e}_{22}) \right]
\]  
(3.43)

\[
\hat{\sigma}_{22} = \frac{1}{1 - \nu} \left[ \nu \hat{\sigma}_{33} + 2\mu (\hat{e}_{22} + \nu \hat{e}_{11}) \right]
\]  
(3.44)

Using the transformation law, all components of the strain and stress tensors in the global system on the surface are then obtained from

\[
\sigma_{ij} = a_{ki} a_{lj} \hat{\sigma}_{kl}
\]  
(3.45)

\[
\varepsilon_{ij} = a_{ki} a_{lj} \hat{e}_{kl}
\]  
(3.46)

**The Traction Representation Formulation**

The traction representation is the derivative form of the displacement representation. This gives one an additional independent integral representation to solve many engineering problems together with the displacement representation. The traction representation has higher singular kernels compared to the displacement representation due to the differen-
tiation process in the stress-displacement relation. The traction representation is useful in solving the crack problems of fracture mechanics.

The Cauchy's stress formula for the traction components at any point $\tilde{\xi}$ on a plane with normal $\bar{v}$ is given by

$$ t_p(\tilde{\xi}) = \sigma_{pq}(\tilde{\xi}) v_q(\tilde{\xi}) $$

Equation (3.47) may be written with the material properties through the stress-strain law as

$$ t_p(\tilde{\xi}) = E_{pqik} v_q(\tilde{\xi}) \frac{\partial u_i(\tilde{\xi})}{\partial \bar{v}_k} $$

By contracting the material properties $E_{pqik} v_q(\tilde{\xi})$ on the displacement representation gradients (3.29) and applying the stress-strain law with the notations (3.32) and (3.33), one has following traction representation.

$$ t_p(\tilde{\xi}) = \int_S \sum_{j,pq} v_q(\tilde{\xi}) t_j(\bar{x}) dS(\bar{x}) - \int_S \Theta_{jpq} v_q(\tilde{\xi}) u_j(\bar{x}) dS(\bar{x}) $$

$$ + \int_V \sum_{j,pq} v_q(\tilde{\xi}) f_j(\bar{X}) dV(\bar{X}) $$

By defining the following two variables

$$ U^*_{pj} = \sum_{j,pq} v_q(\tilde{\xi}) $$

$$ T^*_{pj} = \Theta_{jpq} v_q(\tilde{\xi}) $$

one gets the final form of the traction representation from equation (3.49) as
where $U_{pi}^*$ contains the derivatives of $U_{ij}$ and $T_{pj}^*$ contains the derivatives of $T_{ij}$ with respect to $\xi_k$, together with the elastic constants and the normal vector components $\tilde{v}$. One can see that the traction representation has a form similar to the interior representation of equation (3.20).

**Identities Pertaining to the Fundamental Solutions**

Since the interior representation (3.20) and the exterior representation (3.21) hold for all equilibrated states on any arbitrary region, one can use it for known or self-consistent fields that are in equilibrium. The following three identities can be established by considering the constant displacement and the constant strain field for the fundamental solutions [56]. First, consider the constant displacement or the rigid body translation field $u_i(\tilde{x}) = a_i$. Since there are no stresses in the rigid body translation, there exist no traction. By ignoring the body force term, the interior representation (3.20) are written as

$$a_i = -\int_S T_{ij}(\tilde{x}, \tilde{\xi}) \alpha_j \ dS \quad (3.53)$$

or

$$\left[ \delta_{ij} + \int_S T_{ij}(\tilde{x}, \tilde{\xi}) \ dS \right] a_j = 0 \quad (3.54)$$
From equation (3.54), the first kind identity for the interior domain is obtained as

$$\int_{S} T_{ij}(\bar{x}, \bar{\xi}) \, dS = -\delta_{ij} \quad (\bar{\xi} \in S \cup V) \tag{3.55}$$

Similarly, the first kind identity for the exterior domain is obtained as equation (3.57).

$$\int_{S} T_{ij}(\bar{x}, \bar{\xi}) \, a_j \, dS = 0 \tag{3.56}$$

$$\int_{S} T_{ij}(\bar{x}, \bar{\xi}) \, dS = 0 \quad (\bar{\xi} \in S \cup E) \tag{3.57}$$

From the first kind identities, one can obtain several modified identities by introducing an auxiliary boundary to the domain (see Figure 3.1 in the two-dimensional domain case). With the partitioned integration ranges $S = S' + S_o$ or $S = \Gamma_o + S_o$, the first kind identities may be written as

$$\int_{S'} T_{ij}(\bar{x}, \bar{\xi}) \, dS = \begin{cases} -\delta_{ij} & (\bar{\xi} \in S \cup V) \\ 0 & (\bar{\xi} \in S \cup E) \end{cases} \tag{3.58}$$

$$\int_{\Gamma_o + S_o} T_{ij}(\bar{x}, \bar{\xi}) \, dS = \begin{cases} -\delta_{ij} & (\bar{\xi} \in S \cup V) \\ 0 & (\bar{\xi} \in S \cup E) \end{cases} \tag{3.59}$$

By subtracting equation (3.58) from (3.59) for the interior or the exterior domain respectively, one gets a unified identity for the first kind identities, both for the interior and the exterior domains, in the form

$$\int_{S'} T_{ij}(\bar{x}, \bar{\xi}) \, dS = \int_{\Gamma_o} T_{ij}(\bar{x}, \bar{\xi}) \, dS' \quad (\bar{\xi} \in S \cup V \cup E) \tag{3.60}$$

With the exterior identity, one also obtains two modified identities from the basic form as
\[ \int_{S'} T_{ij}(\vec{x}, \vec{\xi}) \, dS = - \int_{S_o} T_{ij}(\vec{x}, \vec{\xi}) \, dS \quad (\vec{\xi} \in S \cup E) \] (3.61)

and

\[ \int_{\Gamma_o} T_{ij}(\vec{x}, \vec{\xi}) \, dS = - \int_{S_o} T_{ij}(\vec{x}, \vec{\xi}) \, dS \quad (\vec{\xi} \in S \cup E) \] (3.62)

The basic and the modified forms of the first kind identities for the interior and the exterior domains will be used to formulate the computational forms of the boundary integral equations for the crack problems.

Now, consider linear displacement fields given by

\[ u_j(\vec{x}) = a_i + b_{ij}(x_j - \xi_j) = a_i + b_{ij}r_j \] (3.63)

where \( a_i \) and \( b_{ij} \) are constants and \( r_j = x_j - \xi_j \) are the distance vector components. The derivative of equation (3.63) yields

\[ u_{i,k}(\vec{x}) = b_{ij} \delta_{jk} = b_{ik} \] (3.64)

By putting the above strain fields into the stress-strain relations, one gets

\[ \sigma_{pq}(\vec{x}) = E_{pqik} u_{i,k}(\vec{x}) = E_{pqik} b_{ik} \] (3.65)

and from equation (3.48), one has

\[ t_p(\vec{x}) = E_{pqik} n_q(\vec{x}) b_{ik} \] (3.66)

Then, one obtains the following traction fields by substituting equations (3.63) and (3.66) into the traction representation (3.52) while ignoring the body force term \( f_j(\vec{X}) = 0 \).

\[ E_{pqik} v_q(\vec{\xi}) b_{ik} = \int_{S} U_{pj}^* E_{jqik} n_q(\vec{x}) b_{ik} \, dS - \int_{S} T_{pj}^*[a_j + b_{jk}r_k] \, dS \] (3.67)

Equation (3.67) provides two more identities as follows.
First, from the rigid body translation fields \( \alpha_j \neq 0 \) and \( b_{jk} = 0 \), one has the second kind identity for both the interior and the exterior domains

\[
\int_{S} T_{ij}^* (\bar{x}, \bar{\xi}) \, dS = 0 \quad (\forall \, \bar{\xi} \in \mathcal{S} \cup \mathcal{V} \cup \mathcal{E}) \tag{3.68}
\]

By introducing the auxiliary surface, one can derive two more modified forms of the identities from the second kind identity as

\[
\int_{S} T_{ij}^* (\bar{x}, \bar{\xi}) \, dS = \int_{\Gamma_0} T_{ij}^* (\bar{x}, \bar{\xi}) \, dS \quad (\forall \, \bar{\xi} \in \mathcal{S} \cup \mathcal{V} \cup \mathcal{E}) \tag{3.69}
\]

and

\[
\int_{\Gamma_0} T_{ij}^* (\bar{x}, \bar{\xi}) \, dS = -\int_{S_o} T_{ij}^* (\bar{x}, \bar{\xi}) \, dS \quad (\forall \, \bar{\xi} \in \mathcal{S} \cup \mathcal{V} \cup \mathcal{E}) \tag{3.70}
\]

Next, from the constant strain fields \( \alpha_j = 0 \) and \( b_{jk} \neq 0 \), one can obtain the third kind identity. From the knowledge of the tensor operation, the dummy index \( j \) may be changed to \( i \) in \( T_{pj}^* b_{jk} \) so that

\[
T_{pj}^* b_{jk} = T_{pj}^* \delta_{ji} \delta_{ij} b_{jk} = T_{pi}^* b_{ik} \tag{3.71}
\]

By substituting equation (3.71) into equation (3.67), one has

\[
\int_{S} T_{pi}^* r_k \, dS = \int_{S} U_{pj}^* E_{jqik} n_q (\bar{x}) \, dS - \delta_{pj} E_{jqik} v_q (\bar{\xi}) \tag{3.72}
\]

The third kind identity is obtained by substituting the first kind identity (3.55) into equation (3.72) as

\[
\int_{S} T_{pi}^* r_k \, dS = \int_{S} E_{jqik} [U_{pj}^* n_q (\bar{x}) \, dS + T_{pj} v_q (\bar{\xi})] \, dS \tag{3.73}
\]
In Chapter 4, the basic and the modified forms of the identities developed above will be used to regularize the integral representations and to get the computational forms of the traction and the displacement boundary integral equations.

**Characteristics of the Kernels**

As the source point approaches the field point \((r \to 0)\), all fundamental solution kernels contained in the displacement representation (3.20) and the traction representation (3.52) become singular. The singularity orders of the kernels in both representations are summarized in Table 3.1.

<table>
<thead>
<tr>
<th>Kernel</th>
<th>Singularity order 2-D</th>
<th>Singularity order 3-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U_{ij})</td>
<td>(O(\ln r))</td>
<td>(O(1/r))</td>
</tr>
<tr>
<td>(U_{ij}^*, T_{ij})</td>
<td>(O(1/r))</td>
<td>(O(1/r^2))</td>
</tr>
<tr>
<td>(T_{ij}^*)</td>
<td>(O(1/r^2))</td>
<td>(O(1/r^3))</td>
</tr>
</tbody>
</table>

The weakly singular kernels, \(O(\ln r)\) in 2-D and \(O(1/r)\) in 3-D, can be integrated by a coordinate transformation method. The appropriate transformations in both 2-D and 3-D are presented in Chapter 5. The integrals for the strongly singular kernels, \(O(1/r)\) in 2-D and \(O(1/r^2)\) in 3-D, can be obtained through a surface exclusion and a Cauchy principal value interpretation. The hypersingular kernels, \(O(1/r^2)\) in 2-D and \(O(1/r^3)\) in 3-D,
contained in the traction representation have been poorly understood and special techniques are required to integrate them numerically.

Regularization

All fundamental solutions \((U_{ij} \text{ and } T_{ij})\) and their derivatives \((U_{ij,k} \text{ and } T_{ij,k})\) contain singular kernels and they are multiplied by the density functions \(u(\vec{x})\) and \(t(\vec{x})\). The singular integrals containing these singular kernels can be regularized by certain key terms of a Taylor expansion of the density functions.

Mathematically, the Taylor expansion for a function of one variable may be written as

\[
f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n
\]

(3.74)

As \(x \rightarrow x_0\), the second term on the right side of equation (3.74) goes to 0 and is of order \(O(r)\).

\[
f(x) - f(x_0) = 0 \quad (3.75)
\]

By multiplying equation (3.75) onto the strongly singular kernels, the \(O(1/r)\) singularity of the product becomes \(O(1)\) and is thus integrable in the ordinary sense. The inner integral for the singular kernel of \(O(1/r)\) can be expressed as

\[
\int_{S_0} \frac{f(\vec{x})}{r(\vec{x}, \vec{\xi})} dS(\vec{x}) = \int_{S_0} \frac{f(\vec{x}) - f(\vec{\xi})}{r(\vec{x}, \vec{\xi})} dS(\vec{x}) + \int_{S_0} \frac{1}{r(\vec{x}, \vec{\xi})} dS(\vec{x}) f(\vec{\xi})
\]

(3.76)

where \(S_0\) denotes an isolated part of the surface \(S\) which contains the singular point \(\vec{\xi}\).

The first integral in the right side of equation (3.76) has a regular integrand. If the second
term of equation (3.76) has the closed-form expression or is handled properly, the singular integral could be evaluated numerically.

To isolate the singular kernels with the order of $O(1/r)$, Cruse and Vanburen [30] and Jeng and Wexler [47] subtracted the first term of a Taylor expansion and then added back the corresponding term with no density function. Aliabadi et al. [48, 49] have checked the accuracy of the regularized integration with plane parallelograms and curved elements and also considered higher-order terms of a Taylor expansion for the same singularity order kernels. Brandão [50] and Kaya and Erdogan [51] used up to the first derivative terms of a Taylor expansion to evaluate one-dimensional hypersingular integrals in the Hadamard sense. Krishnasamy et al. [17] have extended this idea going up to the first derivatives of the surface integrals and regularized the hypersingular integrals to at most weakly singular ones. Then, they converted the added back $O(1/r^2)$ terms from the surface integrals to line integrals by using Stokes' theorem.

**Regularization of the Displacement Representation**

By ignoring the body force term, one can rewrite the interior representation (3.20) in the form

$$u_i(\vec{x}) + \int_S T_{ij}(\vec{x}, \vec{\xi}) u_j(\vec{x}) dS(\vec{x}) = \int_S U_{ij}(\vec{x}, \vec{\xi}) t_j(\vec{x}) dS(\vec{x}) \quad \forall \vec{x} \in S, \quad \forall \vec{\xi} \in S \cup V \tag{3.77}$$

To remove the strong singularity in $T_{ij}$ kernel, subtract and add back the first term of a Taylor expansion of the density function $\vec{u}$ as given in equation (3.76).

$$u_i(\vec{\xi}) + \int_S T_{ij} [u_j(\vec{x}) - u_j(\vec{\xi})] dS + \int_S T_{ij} dS u_j(\vec{\xi}) = \int_S U_{ij} t_j(\vec{x}) dS \tag{3.78}$$
By applying the first kind interior identity (3.55), one proceeds to obtain regularized displacement representation for the interior region as equation (3.80).

\[
u_i(\xi) + \int_S T_{ij}[u_j(\bar{x}) - u_j(\xi)]dS + (-\delta_{ij})u_j(\xi) = \int_S U_{ij} t_j(\bar{x})dS\]  (3.79)

\[
\int_S T_{ij}[u_j(\bar{x}) - u_j(\xi)]dS = \int_S U_{ij} t_j(\bar{x})dS \quad (\forall \bar{x}, \xi \in S, S \subseteq \Omega)  (3.80)
\]

Now, to form the displacement boundary integral equation for the interior region, let \( \xi \) go to \( \bar{z} \) from the interior to the surface by the limiting process (see Figure 3.6(a)).

\[
\int_S T_{ij}[u_j(\bar{x}) - u_j(\bar{z})]dS = \int_S U_{ij} t_j(\bar{x})dS \quad (\forall \bar{x}, \bar{z} \in S)  (3.81)
\]

To form the displacement boundary integral equation for the exterior region, one may rewrite equation (3.21), by ignoring the body force term, and apply the exterior identity (3.57) to get,

\[
\int_S T_{ij}[u_j(\bar{x}) - u_j(\bar{\xi})]dS + \int_S T_{ij} dS u_j(\bar{\xi}) = \int_S U_{ij} t_j(\bar{x})dS  (3.82)
\]

\[
\int_S T_{ij}[u_j(\bar{x}) - u_j(\bar{\xi})]dS = \int_S U_{ij} t_j(\bar{x})dS \quad (\because \int_S T_{ij} dS = 0)  (3.83)
\]

By taking the limit \( \bar{\xi} \to \bar{z} \) in equation (3.83), one gets the displacement boundary integral equation for the exterior region (see Figure 3.6(b)) as

\[
\int_S T_{ij}[u_j(\bar{x}) - u_j(\bar{z})]dS = \int_S U_{ij} t_j(\bar{x})dS \quad (\forall \bar{x}, \bar{z} \in S)  (3.84)
\]
(a) Source point from the interior region

(b) Source point from the exterior region

Figure 3.6: The interior and exterior regions
Notice that the regularized boundary integral equations (3.81) and (3.84) for the interior and the exterior regions are identical. Now, all variables in the displacement boundary integral equation, equations (3.81) or (3.84), are on the surface and the left-side integral has $O(1)$ singularity and is integrable in the ordinary sense. Here, the right-side integral is in the sense of a Cauchy principal value.

**Regularization of the Traction Representation**

Similar to the displacement representation, the traction representation can be regularized to form the traction boundary integral equation. The only difference from the displacement representation is that the traction representation requires the subtraction of the first two terms of the Taylor expansion to regularize the kernel. By subtracting two terms of a Taylor expansion, using the identities on the added back terms and taking the limit $\xi \to \bar{z}$, the traction boundary integral equation can be formulated. Finally, one will have at most weakly singular integrals in the traction boundary integral equation.

By virtue of the identity (3.55), the traction representation can be written as

$$t_p(\bar{\xi}) = \delta_{pj} t_j(\bar{\xi}) = - \int_S T_{pj}(\bar{x}, \bar{\xi}) dS t_j(\bar{\xi}) = - \int_S T_{pj}(\bar{x}, \bar{\xi}) t_j(\bar{\xi}) dS$$

(3.85)

$$\int_S T_{pj}^* u_j(\bar{x}) dS(\bar{x}) = \int_S \left[ u_{pj}^* t_j(\bar{x}) + T_{pj} t_j(\bar{\xi}) \right] dS(\bar{x})$$

(3.86)

Now, consider a Taylor expansion of $u_I(\bar{x})$ at $\bar{\xi}$, assuming that the displacement field in the neighborhood of $\bar{\xi}$ is sufficiently continuous. The Taylor expansion of $u_I$ up to the first derivative terms is
\[ u_i(\bar{x}) = u_i(\bar{\zeta}) + \frac{\partial u_i(\bar{\zeta})}{\partial \zeta_k} \bar{\zeta}_k \cdot \bar{F} + \cdots \]

\[ = u_i(\bar{\zeta}) + \frac{\partial u_i(\bar{\zeta})}{\partial \zeta_m} \bar{\zeta}_m \cdot \bar{F} + \frac{\partial u_i(\bar{\zeta})}{\partial \nu} \bar{\nu} \cdot \bar{F} + \cdots \]

\[ = u_i(\bar{\zeta}) + \frac{\partial u_i(\bar{\zeta})}{\partial \zeta_m} b_{mn} r_n + \frac{\partial u_i(\bar{\zeta})}{\partial \nu} b_{3n} r_n + \cdots \] (3.87)

where \((\zeta_1, \zeta_2, \nu)\) are the orthogonal coordinates, \(\bar{\zeta}_m\) and \(\bar{\nu}\) are the unit tangential and normal vector components at the point \(\bar{\zeta}\) respectively. Subscript \(i, k, n\) range over 1, 2, 3 and \(m\) over 1, 2. The unit vector \(\vec{e}_i\) is for the global system and \(b_{mn}\) denotes the direction cosines from \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\) to \((\bar{\zeta}_1, \bar{\zeta}_2, \bar{\nu})\). Here, notice that at the limiting point \(\bar{x}\) on the surface, \(\bar{\zeta}_m \cdot \bar{F}\) is of \(O(r)\) and \(\bar{\nu} \cdot \bar{F}\) is of \(O(r)\) for smooth surface curvatures.

By subtracting the first two terms of a Taylor expansion from \(u(\bar{x})\) on the left side of equation (3.86), one gets

\[ \int_{S} T_{p j}^* u_j(\bar{x}) \, dS(\bar{x}) = \int_{S} T_{p j}^* \left[ u_j(\bar{x}) - u_j(\bar{\zeta}) - \frac{\partial u_j(\bar{\zeta})}{\partial \zeta_m} b_{mn} r_n \right] \, dS(\bar{x}) \]

\[ + \int_{S} T_{p j}^* dS(\bar{x}) u_j(\bar{\zeta}) + \int_{S} T_{p j}^* r_n dS(\bar{x}) b_{mn} \frac{\partial u_j(\bar{\zeta})}{\partial \zeta_m} \] (3.88)

The second term of the right side in equation (3.88) is 0 by the identity (3.68). By applying the third kind identity (3.73) to equation (3.88), one obtains

\[ \int_{S} T_{p j}^* u_j(\bar{x}) \, dS(\bar{x}) = \int_{S} T_{p j}^* \left[ u_j(\bar{x}) - u_j(\bar{\zeta}) - \frac{\partial u_j(\bar{\zeta})}{\partial \zeta_m} b_{mn} r_n \right] \, dS(\bar{x}) \]
By substituting equation (3.89) into equation (3.86), following regularized traction representation is formed.

\[
\sum_{i} E_{ijm} \left[ n_{pi} u_{q} (\vec{x}) + T_{pi} v_{q} (\vec{\zeta}) \right] dS(\vec{x}) b_{mn} \frac{\partial u_{j}(\vec{\zeta})}{\partial \zeta_{m}}
\]  

(3.89)

By taking the limit \( \vec{\zeta} \to \vec{z} \), as before in the displacement equation, the source point \( \vec{z} \) is moved to the boundary to form the regularized traction boundary integral equation.

\[
\int_{S} T_{pj}^{*} \left[ u_{j}(\vec{x}) - u_{j}(\vec{\zeta}) - \frac{\partial u_{j}(\vec{\zeta})}{\partial \zeta_{m}} b_{mn} r_{n} \right] dS(\vec{x})
\] 

(3.90)

\[
\int_{S} E_{ijm} \left[ n_{pi} u_{q} (\vec{x}) + T_{pi} v_{q} (\vec{\zeta}) \right] dS(\vec{x}) b_{mn} \frac{\partial u_{j}(\vec{\zeta})}{\partial \zeta_{m}} \ \ \ \ (\forall \ \vec{\zeta} \in S, \ \forall \ \vec{\xi} \in S \cup V)
\] 

(3.91)

All variables in equation (3.91) are now on the surface. Here, if \( \alpha \) denotes the number of dimensions in Euclidean space, subscripts \( i, j, p, q \) and \( n \) take up to \( \alpha \) value and \( m \) takes up to \( \alpha - 1 \) value. Now, by defining new variable \( W_{pjim} \) as
\[ W_{pj} = E_{ijm} \left[ U^*_{pj} n_q(\bar{x}) + T_{p} v_q(\bar{x}) \right] b_{mn} \]  

(3.92)

one has following regularized traction boundary integral equation on the boundary.

\[
\int_S T^*_{pj} \left[ u_j(\bar{x}) - u_j(\bar{z}) - \frac{\partial u_j(\bar{z})}{\partial \zeta_m} b_{mn} r_n \right] dS(\bar{x}) + \int_S W_{pj} dS(\bar{x}) \frac{\partial u_j(\bar{z})}{\partial \zeta_m} \\
= \int_S \left[ U^*_{pj} t_j(\bar{x}) + T_{pj} t_j(\bar{z}) \right] dS(\bar{x}) \quad (\forall \bar{x}, \bar{z} \in S)  
\]  

(3.93)
CHAPTER 4. COMPUTATIONAL FORMS OF THE BOUNDARY INTEGRAL EQUATIONS

Equations (3.81) and (3.93) represent the formal forms of the displacement and the traction boundary integral equations. For the computational and programming purposes, the rearrangement of these equations is necessary. It is convenient to partition the whole integration range \( S \) into several parts, that is, \( S = S' + S_o \) or \( S = S' + S_o^+ + S_o^- \) where \( S_o \) is that part of the boundary containing the collocation point and \( S' \) is the remainder of \( S \). In the following, \( S_o^+ \) denotes one crack surface and \( S_o^- \) denotes the opposing crack surface. The displacement boundary integral equations are collocated at points on the "+" side of the crack surface and the traction boundary integral equations are collocated at points on the "-" side of the crack. On the outer boundary, that is, not on the crack surface, the displacement boundary integral equations are always to be used.

The Displacement Boundary Integral Equation

On the Outer Surface

When the collocation point is on the outer surface, the total boundary will be partitioned into two parts, that is, \( S = S' + S_o \). With partitioned integral ranges, one has the computational form of the displacement boundary integral equation (3.81) as follows.
\[
\int_{S'} T_{ij} u_j(x) dS + \int_{S_0} T_{ij} [u_j(x) - u_j(z)] dS - \int_{\Gamma_0} T_{ij} dS u_j(z)
= \int_{S'} U_{ij} t_j(x) dS + \int_{S_0} U_{ij} [t_j(x) - t_j(z)] dS + \int_{S_0} U_{ij} dS t_j(z)
\] (4.1)

In applications, this type of equation is collocated only at points on continuous elements that are not on crack surfaces.

If the auxiliary surface is introduced, one can determine another computational form of the displacement boundary integral equation by applying the modified identity (3.60) to equation (4.1). It can be written as

\[
\int_{S'} T_{ij} u_j(x) dS + \int_{S_0} T_{ij} [u_j(x) - u_j(z)] dS - \int_{\Gamma_0} T_{ij} dS u_j(z)
= \int_{S'} U_{ij} t_j(x) dS + \int_{S_0} U_{ij} [t_j(x) - t_j(z)] dS + \int_{S_0} U_{ij} dS t_j(z)
\] (4.2)

This equation is to be used or collocated at points which are on the outer and/or on one of the crack surfaces.

For unbounded regions, the integration range is divided into three parts \( S = S' + S_0 + S_\infty \) (Figure 4.1(b)). With the partitioned boundaries, the interior representation (3.20) can be written, by ignoring the body force term, as

\[
u_i(\xi) + \int_{S' + S_0} T_{ij}(\xi, \bar{\xi}) u_j(\bar{x}) dS + \int_{S_\infty} T_{ij}(\bar{x}, \bar{\xi}) u_j(\bar{x}) dS \]

\[
= \int_{S' + S_0} U_{ij}(\bar{x}, \bar{\xi}) t_j(\bar{x}) dS + \int_{S_\infty} U_{ij}(\bar{x}, \bar{\xi}) t_j(\bar{x}) dS
\] (4.3)
From the rigid body translation condition, $u(\bar{x}) = \text{constant}$, one gets the following identity.

$$\int_{S' + S_0} T(\bar{x}, \bar{z}) dS = 0$$  \hspace{1cm} (4.4)

Also, the third of the left side and the second of the right side integrals of equation (4.3) vanish on the infinite boundary. Thus, one can write equation (4.3) as
\[ u_t(\bar{x}) + \int_{S'} T_{ij}(\bar{x}, \tilde{x}) u_j(\tilde{x}) dS = \int_{S'} U_{ij}(\bar{x}, \tilde{x}) t_j(\tilde{x}) dS \quad (4.5) \]

With the identity (4.4) and the first term of a Taylor expansion, one has the regularized form of equation (4.5) as
\[ u_t(\bar{x}) + \int_{S'} T_{ij}(\bar{x}, \tilde{x}) [u_j(\bar{x}) - u_j(\tilde{x})] dS = \int_{S'} U_{ij}(\bar{x}, \tilde{x}) t_j(\tilde{x}) dS \quad (4.6) \]
\[
\int_{S'} T_{ij} u_j(\bar{x}) dS + \int_{S'} T_{ij} [u_j(\bar{x}) - u_j(\tilde{x})] dS - \int_{S'} T_{ij} dS u_j(\tilde{x}) + u_j(\tilde{x}) \delta_{ij} \\
= \int_{S'} U_{ij} t_j(\bar{x}) dS + \int_{S'} U_{ij} [t_j(\bar{x}) - t_j(\tilde{x})] dS + \int_{S'} U_{ij} dS t_j(\tilde{x}) \quad (4.7) \]

By rearranging the third and the fourth terms in the left side and taking the limit \( \tilde{x} \to \bar{z} \), one gets the displacement boundary integral equation for an unbounded region problem as
\[
\int_{S'} T_{ij} u_j(\bar{x}) dS + \int_{S'} T_{ij} [u_j(\bar{x}) - u_j(\tilde{x})] dS - \left[ \int_{S'} T_{ij} dS - \delta_{ij} \right] u_j(\bar{z}) \\
= \int_{S'} U_{ij} t_j(\bar{x}) dS + \int_{S'} U_{ij} [t_j(\bar{x}) - t_j(\tilde{x})] dS + \int_{S'} U_{ij} dS t_j(\tilde{x}) \quad (4.8) \]

If one considers the two conditions of Figure 4.1(a) and (b) for the interior and the exterior, respectively, and introduces the auxiliary boundary, one gets following two identities.
\[
\int_{S' + S_o} T(\bar{x}, \tilde{x}) dS = 0 \quad (4.9) \\
\int_{S_o + \Gamma_o} T(\bar{x}, \tilde{x}) dS = -\delta_{ij} \quad (4.10) \]
By subtracting equation (4.10) from equation (4.9), one obtains

\[ \int_{S'} \mathbf{T}(\tilde{x}, \tilde{\xi}) dS - \delta_{ij} = \int_{\Gamma_0} \mathbf{T}(\tilde{x}, \tilde{\xi}) dS \]  

(4.11)

Finally, by substituting equation (4.11) into (4.8), the computational form of the displacement boundary integral equation for an unbounded region by the auxiliary boundary becomes

\[
\int_{S'} T_{ij} \mathbf{u}_j(\tilde{x}) dS + \int_{S_o} T_{ij} [u_j(\tilde{x}) - u_j(\tilde{z})] dS - \int_{\Gamma_0} T_{ij} dS u_j(\tilde{z}) \\
= \int_{S'} U_{ij} t_j(\tilde{x}) dS + \int_{S_o} U_{ij} [t_j(\tilde{x}) - t_j(\tilde{z})] dS + \int_{S_o} U_{ij} dS t_j(\tilde{z})
\]  

(4.12)

One can see that equation (4.2) and equation (4.12) for an interior region and for an unbounded region, respectively, with the auxiliary surfaces, are identical.

**On the Crack Surface**

Figure 4.2 shows schematic view of a two-dimensional crack. The crack could be in a bounded or in an unbounded region. When the collocation point is on the crack surface, the integration over the opposite crack surface becomes singular as \( \tilde{\xi} \to \tilde{\xi}^+ \), even though the collocation point is not on the integration element. In this situation, one needs to partition the integration range into three parts \( S = S' + S_o^+ + S_o^- \). One may write the left side of equation (3.81) for the collocation point \( \tilde{\xi}^+ \) with partitioned integration ranges (when the collocation point is on the opposite surface, change the sign from "+" to "-") as

\[
\int_{S} T_{ij}(\tilde{x},\tilde{\xi})[u_j(\tilde{x}) - u_j(\tilde{z})] dS
\]
\[ J = \int_{S'} T_{ij} [u_j(x) - u_j(z^+)] dS + \int_{S_0^+} T_{ij} [u_j(x) - u_j(z^+)] dS + \int_{S_0^-} T_{ij} [u_j(x) - u_j(z^+)] dS \]

\[ = \int_{S} T_{ij} u_j(x) dS + \int_{S_0^+} T_{ij} [u_j(x) - u_j(z^+)] dS - \int_{S} T_{ij} dS u_j(z^+) \]

\[ + \int_{S_0^-} T_{ij} [u_j(x) - u_j(z^-)] dS + \int_{S_0^-} T_{ij} dS u_j(z^-) - \int_{S_0^-} T_{ij} dS u_j(z^+) \]  

(4.13)

With the partitioned integration ranges and applying the identity (3.58) for equation (4.13) at the point \( z^+ (z^+ \in E) \), one has the modified form of the second kind identity as

\[ \int_{S} T_{ij} dS = 0 \Rightarrow \int_{S_0^+ + S_0^-} T_{ij} dS = 0 \]  

\[ \Rightarrow \int_{S_0^-} T_{ij} dS = - \int_{S_0^+} T_{ij} dS \]  

(4.15)

By the use of the identity (4.15), equation (4.13) can be written as

Figure 4.2: Embedded crack in a 2-D domain
Here, one may introduce an auxiliary surface for the added back term integration through the modified identity (3.62) for each crack surface.

\[
\int_{\Gamma_o^+} T_{ij} \, dS = - \int_{\Gamma_o^-} T_{ij} \, dS \quad (4.17)
\]

\[
\int_{\Gamma_o^-} T_{ij} \, dS = - \int_{\Gamma_o^+} T_{ij} \, dS \quad (4.18)
\]

Then, one arrives at the computational form of the displacement boundary integral equation for the crack surface as

\[
\int_{S^+} T_{ij} \, u_j(\bar{x}) \, dS + \int_{S^+} T_{ij} [u_j(\bar{x}) - u_j(\bar{z}^+)] \, dS - \int_{\Gamma_o^+} T_{ij} \, dS \, u_j(\bar{z}^+) - \int_{S^-} T_{ij} [u_j(\bar{x}) - u_j(\bar{z}^-)] \, dS - \int_{\Gamma_o^-} T_{ij} \, dS \, u_j(\bar{z}^-) = \int_{S^+} U_{ij} \, t_j(\bar{x}) \, dS + \int_{S^-} U_{ij} [t_j(\bar{x}) - t_j(\bar{z}^+)] \, dS + \int_{\Gamma_o^+} U_{ij} \, dS \, t_j(\bar{z}^+) + \int_{S^-} U_{ij} [t_j(\bar{x}) - t_j(\bar{z}^-)] \, dS + \int_{\Gamma_o^-} U_{ij} \, dS \, t_j(\bar{z}^-) \quad (4.19)
\]

Notice that equation (4.19) is a form similar to equation (4.2). If one use equation (4.19) for the collocation points on one crack surface, the corresponding traction boundary integral equation is to be used for points on the opposite crack surface.
The Traction Boundary Integral Equation

On the Outer Surface

Similar to the displacement equation, the integration range is divided into two parts $S = S' + S_o$. From the regularized traction equation given in equation (3.93), one has

$$
\int_{S'} T^*_{pj} u_j(\bar{x}) \, ds - \int_{S'} T^*_{pj} dS u_j(\bar{z}) - \int_{S'} T^*_{pj} r_n \, ds \frac{\partial u_j(\bar{z})}{\partial \xi_m} b_{mn} + \int_{S'} W_{pjm} \, ds \frac{\partial u_j(\bar{z})}{\partial \xi_m} 
+ \int_{S_o} T^*_{pj} \left[ u_j(\bar{x}) - u_j(\bar{z}) - \frac{\partial u_j(\bar{z})}{\partial \xi_m} b_{mn} r_n \right] \, ds + \int_{S_o} W_{pjm} \, ds \frac{\partial u_j(\bar{z})}{\partial \xi_m}
(4.20)
$$

$$
= \int_{S'} U^*_{pj} t_j(\bar{x}) \, ds + \int_{S'} T_{pj} \, ds t_j(\bar{z}) + \int_{S_o} \left[ U^*_{pj} t_j(\bar{x}) + T_{pj} \, t_j(\bar{z}) \right] \, ds
$$

By applying modified second kind identities, equations (3.60) and (3.69), to equation (4.20), one gets the computational form of the traction boundary integral equation on the outer surface that is given by

$$
\int_{S'} T^*_{pj} u_j(\bar{x}) \, ds - \int_{S'} T^*_{pj} dS u_j(\bar{z}) - \int_{S'} T^*_{pj} r_n \, ds \frac{\partial u_j(\bar{z})}{\partial \xi_m} b_{mn} + \int_{S'} W_{pjm} \, ds \frac{\partial u_j(\bar{z})}{\partial \xi_m} 
+ \int_{S_o} T^*_{pj} \left[ u_j(\bar{x}) - u_j(\bar{z}) - \frac{\partial u_j(\bar{z})}{\partial \xi_m} b_{mn} r_n \right] \, ds + \int_{S_o} W_{pjm} \, ds \frac{\partial u_j(\bar{z})}{\partial \xi_m}
(4.21)
$$

$$
= \int_{S'} U^*_{pj} t_j(\bar{x}) \, ds + \int_{\Gamma_o} T_{pj} \, dS t_j(\bar{z}) + \int_{S_o} \left[ U^*_{pj} t_j(\bar{x}) + T_{pj} \, t_j(\bar{z}) \right] \, ds
$$

Equation (4.21) is to be used for collocation points on the outer and/or on the crack surfaces. For the regularization by subtraction to be effective, one cannot use the traction equation at points where the displacement derivatives are discontinuous, or, at the collo-
cation point, the displacement must have continuous first derivatives. If a collocation point is located on the geometric end nodes in a 2-D or on the corner nodes in a 3-D element, one cannot construct the auxiliary surface for that collocation point and the normal and the tangential vector components may not have unique values at common points for connected elements. For these reasons, equation (4.21) will be used only at collocation points which are located inside of boundary elements.

On the Crack Surface

As before for the displacement equation on the crack surface, the integration range is divided into three parts \( S = S' + S_0^+ + S_0^- \). Then, equation (3.93) at the point \( \tilde{z}^+ \) can be written as

\[
\int_{S'} T_{pj}^* u_j(\tilde{x}) dS - \int_{S'} T_{pj}^* dS \int_{S'} W_{pjm} dS \frac{\partial u_j(\tilde{x}^+)}{\partial \zeta_m} b_{mn} + \int_{S'} W_{pjm} dS \frac{\partial u_j(\tilde{x}^+)}{\partial \zeta_m} b_{mn} r_n dS + \int_{S_0^+} T_{pj}^* \left[ u_j(\tilde{x}) - u_j(\tilde{x}^+) - \frac{\partial u_j(\tilde{x}^+)}{\partial \zeta_m} b_{mn} r_n \right] dS + \int_{S_0^+} W_{pjm} dS \frac{\partial u_j(\tilde{x}^+)}{\partial \zeta_m} r_n dS
\]

\[
+ \int_{S_0^-} T_{pj}^* \left[ u_j(\tilde{x}) - u_j(\tilde{x}^+) - \frac{\partial u_j(\tilde{x}^+)}{\partial \zeta_m} b_{mn} r_n \right] dS + \int_{S_0^-} W_{pjm} dS \frac{\partial u_j(\tilde{x}^+)}{\partial \zeta_m} r_n dS \tag{4.22}
\]

\[
= \int_{S'} U_{pj}^* t_j(\tilde{x}) dS + \int_{S'} T_{pj} dS \int_{S'} t_j(\tilde{x}^+) dS + \int_{S_0^+} \left[ U_{pj}^* t_j(\tilde{x}) + T_{pj} t_j(\tilde{x}^+) \right] dS + \int_{S_0^-} \left[ U_{pj}^* t_j(\tilde{x}) + T_{pj} t_j(\tilde{x}^+) \right] dS
\]
One may subtract and add back some terms for an alternate arrangement as

\[
\int_{S'} T_{pj}^* u_j(\bar{x}) \, dS \\
- \int_{S'} T_{pj}^* dS u_j(\bar{z}^+) - \int_{S'} T_{pj}^* r_n \, dS \frac{\partial u_j(\bar{z}^+)}{\partial \zeta_m} b_{mn} + \int_{S'} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^+)}{\partial \zeta_m} \\
+ \int_{S_o^+} T_{pj}^* \left[ u_j(\bar{x}) - u_j(\bar{z}^+) - \frac{\partial u_j(\bar{z}^+)}{\partial \zeta_m} b_{mn} r_n \right] \, dS + \int_{S_o^+} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^+)}{\partial \zeta_m} \\
+ \int_{S_o^-} T_{pj}^* \left[ u_j(\bar{x}) - u_j(\bar{z}^-) - \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} b_{mn} r_n \right] \, dS + \int_{S_o^-} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} \\
- \int_{S_o^-} T_{pj}^* dS u_j(\bar{z}^-) - \int_{S_o^-} T_{pj}^* r_n \, dS \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} b_{mn} + \int_{S_o^-} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} \tag{4.23} \\
+ \int_{S_o^-} T_{pj}^* dS u_j(\bar{z}^-) + \int_{S_o^-} T_{pj}^* r_n \, dS \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} b_{mn} - \int_{S_o^-} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m}
\]

\[
= \int_{S'} U_{pj}^* t_j(\bar{x}) \, dS + \int_{S'} T_{pj}^* \, dS t_j(\bar{z}^+) \\
+ \int_{S_o^+} \left[ U_{pj}^* t_j(\bar{x}) + T_{pj} t_j(\bar{z}^+) \right] \, dS \\
+ \int_{S_o^-} \left[ U_{pj}^* t_j(\bar{x}) - T_{pj} t_j(\bar{z}^-) \right] \, dS + \int_{S_o^-} T_{pj} \, dS t_j(\bar{z}^-) + \int_{S_o^-} T_{pj} \, dS t_j(\bar{z}^+)
\]

For some terms in equation (4.23), one can sum the integration ranges \( S' + S_o^- \) by applying the identities on the auxiliary surface.
\[
\int_{S'} T_{pj}^* u_j(\bar{x}) \, dS \quad - \quad \int_{S'} T_{pj}^* dS u_j(\bar{z}^+) - \int_{S'} T_{pj}^* r_n \, dS \frac{\partial u_j(\bar{z}^+)}{\partial \zeta_m} b_{mn} + \int_{S'} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^+)}{\partial \zeta_m} \\
+ \int_{S_0^+} T_{pj}^* \left[ u_j(\bar{x}) - u_j(\bar{z}^+) - \frac{\partial u_j(\bar{z}^+)}{\partial \zeta_m} b_{mn} r_n \right] \, dS + \int_{S_0^+} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^+)}{\partial \zeta_m} \\
+ \int_{S_0^-} T_{pj}^* dS u_j(\bar{z}^-) + \int_{S_0^-} T_{pj}^* r_n \, dS \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} b_{mn} - \int_{S_0^-} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} \\
+ \int_{S_0^-} T_{pj}^* \left[ u_j(\bar{x}) - u_j(\bar{z}^-) - \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} b_{mn} r_n \right] \, dS + \int_{S_0^-} W_{pjm} \, dS \frac{\partial u_j(\bar{z}^-)}{\partial \zeta_m} \tag{4.24}
\]

By the identities (4.15) and (4.17), the integration range for the summed up terms can be replaced by the auxiliary surface \( \Gamma_o^+ \) as \( S' + S_0^- \rightarrow \Gamma_o^+ \). Also, by the identity (4.18), the opposite crack surfaces can be replaced by the auxiliary surface \( \Gamma_o^- \) as \( S_0^- \rightarrow -\Gamma_o^- \). Thus, one gets the computational form of the traction boundary integral equation on the crack surface expressed as
\[ \int_{S} T_{pj}^{*} u_{j}(\bar{x}) \, dS \]

\[ - \int_{\Gamma_{o}^{+}} T_{pj}^{*} dS u_{j}(\bar{z}^{+}) - \int_{\Gamma_{o}^{+}} T_{pj}^{*} r_{n} \, dS \frac{\partial u_{j}(\bar{z}^{+})}{\partial \zeta_{m}} b_{mn} \]

\[ + \int_{S_{o}^{+}} T_{pj}^{*} \left[ u_{j}(\bar{x}) - u_{j}(\bar{z}) - \frac{\partial u_{j}(\bar{z})}{\partial \zeta_{m}} b_{mn} r_{n} \right] \, dS + \int_{S_{p}} W_{pj} dS \frac{\partial u_{j}(\bar{z})}{\partial \zeta_{m}} \]

\[ - \int_{\Gamma_{o}^{-}} T_{pj}^{*} dS u_{j}(\bar{z}^{-}) - \int_{\Gamma_{o}^{-}} T_{pj}^{*} r_{n} \, dS \frac{\partial u_{j}(\bar{z}^{-})}{\partial \zeta_{m}} b_{mn} \]

\[ + \int_{S_{o}^{-}} T_{pj}^{*} \left[ u_{j}(\bar{x}) - u_{j}(\bar{z}) - \frac{\partial u_{j}(\bar{z})}{\partial \zeta_{m}} b_{mn} r_{n} \right] \, dS + \int_{S_{p}} W_{pj} dS \frac{\partial u_{j}(\bar{z})}{\partial \zeta_{m}} \]  \hspace{0.5cm} (4.25)

\[ = \int_{S} U_{pj}^{*} t_{j}(\bar{x}) \, dS \]

\[ + \int_{\Gamma_{o}^{+}} T_{pj} dS t_{j}(\bar{z}^{+}) + \int_{S_{o}^{+}} \left[ U_{pj}^{*} t_{j}(\bar{x}) + T_{pj} t_{j}(\bar{z}^{+}) \right] \, dS \]

\[ - \int_{\Gamma_{o}^{-}} T_{pj} dS t_{j}(\bar{z}^{-}) + \int_{S_{o}^{-}} \left[ U_{pj}^{*} t_{j}(\bar{x}) - T_{pj} t_{j}(\bar{z}^{-}) \right] \, dS \]

One can see that equation (4.25) has a form similar to equation (4.21). As mentioned in the previous section for the outer surface, this traction boundary integral equation (4.25) is to be used only at points where the displacements have continuous first derivatives. From the right side of equation (4.25), one finds that the directions of the traction components at the collocation points on the opposite crack surface have opposite signs. If
one uses the traction equation (4.25) for a collocation point on one crack surface, one should use the displacement equation (4.19) for the collocation point on the opposite crack surface to ensure a unique solution.

In this chapter, all computational forms of the boundary integral equations in elastostatics have been developed. They include the displacement boundary integral equations (4.1), (4.2) and (4.19) for a bounded region, the displacement boundary integral equations (4.8) and (4.12) for an unbounded region and the traction boundary integral equations (4.21) and (4.25) for both of bounded and unbounded regions. Actually, the displacement boundary integral equations (4.2) and (4.12) for a bounded and an unbounded region, respectively, are identical when auxiliary surfaces are used. One can see that all equations have similar forms. The developed equations can be used for many areas including general elastostatic problems or the crack problems in fracture mechanics, a bounded or an unbounded region problem and a single region or multiple region problems.
CHAPTER 5. SINGULAR INTEGRATION

The singularity order of the fundamental solution kernels varies from \( O(\ln r) \) to \( O(1/r^2) \) in two-dimensional and from \( O(1/r) \) to \( O(1/r^3) \) in three-dimensional problems as summarized in Table 3.1. In Chapter 3, the integrals with the higher singularities were regularized so that at most weakly singular integrals occur.

When the body has no crack and the source point is not on the integration element, one may simply compute all integrals using regular numerical integration method such as Gaussian quadrature. However, when the collocation point is on the integration element, the integrals become singular due to the fundamental solution kernels and one needs special integration techniques. Figure 5.1 shows typical singular elements on two- and three-dimensional surfaces. However, such a singularity on the outer surface can be removed through coordinate transformations.

If a body has a crack and when the collocation point is on one of the crack surfaces, two opposing points occupy same spatial coordinates (Figure 5.2). As the result, the numerical integration for the opposite surface is also singular. In this case, one can not simply compute all integrals over the opposite crack surface. If the collocation point is positioned at inside of the element, as for discontinuous elements, the singular integrals are replaced by integrals over the artificial auxiliary surfaces.
Figure 5.1: Typical singular element
Figure 5.2: A singular point \( \bar{z} \), a field point \( \bar{x} \) and an auxiliary surface \( \Gamma_0 \) on the crack surface.

The auxiliary surface can be any shape and may be interpreted as a contracted form of the actual surfaces except for the singular element where the collocating point is located.

**Weak Singularity Removal by Coordinate Transformation**

In two-dimensional problem, the weakly singular integrals that are \( O(\ln r) \) in singularity can be integrated as an improper integral through the following two steps. The first step is to divide the integration element into two sub-elements at the singular point \( \bar{z} \), see Figure 5.3. The second step is to remove the logarithmic singularity by the coordinate transformation.

\[
\begin{align*}
    r &= \xi^2, \quad dr = 2\xi d\xi \\
    \int_{r} f(r) dr &= \int_{\xi} f[r(\xi)] 2\xi d\xi \\
    O(\ln r) &= O(r \ln r)
\end{align*}
\]
In three-dimensional problems, the weakly singular integrals that are $O(1/r)$ in singularity can be similarly integrated through a polar coordinates transformation. To do this, one can divide a singular element into three or four, depending on the element type, sub-triangles by joining the collocation point to three or four geometric corner nodes as shown in Figure 5.4. The sub-triangles allow the use of the polar coordinates system that removes the weak singularity in the integrands [70]. Figure 5.5 shows the actual domain and the transformed domain in the polar coordinates system.

$$I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) \, dx \, dy : O(1/r)$$

$$= \int_{\phi=0}^{\phi=\hat{\phi}} \int_{r=0}^{r=\hat{r}} f[x(r, \phi), y(r, \phi)] r \, dr \, d\phi$$

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\hat{\rho}} f[x(r(\rho, \theta), \phi(\rho, \theta)), y(r(\rho, \theta), \phi(\rho, \theta))] \rho \, d\rho \, d\theta : O(1)$$

where $\hat{\rho} = \frac{1}{\cos\theta + \sin\theta}$ from $\xi + \eta = 1$ and $|J| = 2 \times \text{triangle area in an actual domain.}$
Figure 5.4: Sub-elements for a singular element in a 3-D problem

(a) Triangular element

(b) Quadrilateral element

Figure 5.5: Polar coordinates transformation in a 3-D problem

(a) Actual domain

(b) Transformed domain
Strong Singularity Integration by Use of the Auxiliary Surface

The added back integrals appearing after the regularization process are not singular even though they contain strongly singular fundamental solution kernels because the integration range does not include the element where the collocation points are positioned. When a crack exists and the collocation point is on one of the crack surfaces, the integration point is on the opposing crack surface element, then the integral is singular.

Figure 5.6 and Table 5.1 show a schematic view and all possible integration cases for a 2-D and 3-D cracked body, respectively. In case 2, in Table 5.1, the integrals are always not singular because the collocation point is never on same element as the integration point. In cases 1 and 3, the integrals are weakly singular when the collocation points and the field points are located on same element. This weak singularity can be removed by the coordinates transformation as explained in the previous section. In case 4, the integrations for the added back terms over the elements on the opposite crack surface are always strongly singular.

To remove stronger singularities in case 4, the actual surface containing the opposite crack surface is replaced by a smoothly curved auxiliary surface which isolates the singular integration path. The auxiliary surface can be any shape in the mathematical sense. The numerical integration over the auxiliary surface will have an ordinary value when the auxiliary surface is smooth, especially for the traction boundary integral equations that the surface gradients are contained. For an optimal integration path for every singular collocation point, the auxiliary surface may have different shapes depending on the collocation point position on the singular element.
Figure 5.6: Schematic view of a cracked body

Table 5.1: Possible integration cases for a cracked body

<table>
<thead>
<tr>
<th>case</th>
<th>collocation point on</th>
<th>field point on</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$S_1$</td>
<td>$S_1$</td>
</tr>
<tr>
<td>2</td>
<td>$S_1$</td>
<td>$S_2$ or $S_3$</td>
</tr>
<tr>
<td></td>
<td>or $S_2$ or $S_3$</td>
<td>$S_1$</td>
</tr>
<tr>
<td>3</td>
<td>$S_2$</td>
<td>$S_2$</td>
</tr>
<tr>
<td></td>
<td>or $S_3$</td>
<td>$S_3$</td>
</tr>
<tr>
<td>4</td>
<td>$S_2$</td>
<td>$S_3$</td>
</tr>
<tr>
<td></td>
<td>or $S_3$</td>
<td>$S_2$</td>
</tr>
</tbody>
</table>
One way of constructing the auxiliary surface is to displace the physical location of the collocation point in the opposite direction of the normal of the singular element at \( \vec{x} \) [71]. Here, one should notice that if the displaced location is close to the collocation point, the integrals become nearly singular.

In two-dimensional problems, the highest point for the auxiliary boundary over the collocation point is determined by a fractional length of the singular element and the opposite normal to the element at \( \vec{x} \). If the geometric length of the singular element is \( L \), the moved distance is \( \alpha L \), where \( \alpha \) is an arbitrary constant. The value used for \( \alpha \) in the example problems in Chapter 7 is 0.25. The curved boundary is constructed by connecting two geometric end nodes and the relocated center point using the shape functions for 3-nodes quadratic continuous element (see Appendix B for used shape functions). Two examples of the auxiliary boundaries on the intrinsic elements are shown in Figure 5.7.

In three-dimensional problems, the highest point for the auxiliary boundary over the collocation point is determined by a fraction of two diagonal lengths of the singular element and the opposite normal to the element at \( \vec{x} \). If two diagonals of the singular element are \( D_1 \) and \( D_2 \), the moved distance is \( \beta (D_1 + D_2) \), where \( \beta \) is an arbitrary constant. The value used for \( \beta \) in the example problems in Chapter 7 is 0.1. The curved auxiliary surface is constructed with the relocated center point and eight geometric nodes of the singular element using the shape functions for 9-nodes quadrilateral continuous element (see Appendix C for used shape functions). Three different auxiliary surfaces on the intrinsic elements are shown in Figure 5.8.

By virtue of the constructed auxiliary surfaces, the added back integrals always have non-singular integration paths. Now, all integrals in the displacement and traction boundary integral equations are numerically integrable for both the outer and crack surfaces.
Figure 5.7: An auxiliary surface over a singular element in a 2-D problem
Figure 5.8: An auxiliary surface over a singular element in a 3-D problem
Figure 5.9 shows how one divides the primitive auxiliary surface into several auxiliary sub-surfaces or sub-elements in two dimensions. Consider an arbitrary auxiliary surface \( \Gamma_o \) in the actual domain \( \mathbf{x} \) (Figure 5.9(a)) and in the transformed domain \( \xi \) (Figure 5.9(b)). Here, for the convenience, consider only the "equally spaced" \( N \) number of auxiliary surfaces in the intrinsic element. From the transformed domain or the intrinsic element in \( \xi \) coordinate, one may transform the divided auxiliary surfaces into the intrinsic sub-elements in the coordinate \( \eta \) once again.

Figure 5.9: Coordinates transformation for an auxiliary surface in a 2-D problem
On the transformed domain $\xi$, one has

$$\int_{-1}^{+1} f(\xi) \, d\xi = \sum_{i=1}^{N} \int_{-1}^{+1} f[\xi(\eta)] \frac{1}{N} \, d\eta$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{M} f[\xi(\eta_j)] \frac{w_j}{N}$$

(5.4)

where $M$, $\eta_j$ and $w_j$ are the number of quadrature points, the quadrature point coordinates and the Gaussian weights on the $i-$th intrinsic sub-element, respectively.

The variable in the transformed domain is given by

$$\xi(\eta_j) = -1 + \Delta \xi \left( i + \frac{\eta_j - 1}{2} \right)$$

(5.5)

where

$$\Delta \xi = \frac{2}{N} \quad (N = 1, 2, 3, \ldots, N)$$

(5.6)

Thus, one has

$$\int_{\Gamma_o} f(x) \, dS = \int_{-1}^{+1} f(\xi) |J|_{x, \xi} \, d\xi = \sum_{i=1}^{N} \sum_{j=1}^{M} f[\xi(\eta_j)] |J|_{x, \xi} \frac{w_j}{N}$$

(5.7)

where $|J|_{x, \xi}$ denotes the Jacobian in the $x - \xi$ coordinates system.

The above coordinate transformation method for one-dimensional functions can easily be extended to two-dimensional functions for the auxiliary surfaces in three-dimensional space by iterated integrals. The convergence tests of the following section show that only one element on the auxiliary surface is sufficient for accurate and stable results.
Convergence Tests for the Auxiliary Surfaces

Two kinds of convergence tests were performed. The first test was to determine the necessary integration order for the outer surface to obtain converged results (Figure 5.10). In this test, only one element was used on the auxiliary surface. The second test was for varying numbers of sub-elements on the auxiliary surface. There are three data sets for each fixed integration order for the outer surface (Figures 5.11-5.13).

For a convergence test, a penny-shaped crack embedded in a cylindrical bar under tension model was chosen. The outer and the crack surfaces were modelled with quadratic elements. For each of the top and bottom outer surface, 8-triangular continuous elements were used. For the circumferential outer surface, 80-quadrilateral continuous elements were used. Each crack surface was modelled with 12-quadrilateral discontinuous elements. One end of the cylinder was fixed and a uniform tensile force was applied on the other end.

The displacement boundary integral equation (equation (4.1)) was collocated at all points on the outer surface. On the crack surfaces, the traction boundary integral equation (equation (4.15)) was collocated at the points on one crack surface while the displacement boundary integral equation (equation (4.9)) was collocated at the opposite points. Numerical integrations are carried out by Gaussian quadrature.

As part of the calculated results, stress intensity factors were computed from the nodal displacements on the crack surfaces and were compared with the analytic solutions. One can see from Figures 5.10-5.13 that with more than 20 x 20 integration points on the auxiliary surface, the stress intensity factors are converged and are not sensitive to the number of elements but do depend on the integration order for the outer elements.
Figure 5.10: Stress intensity factors versus the integration order for the outer surface:
One element on the auxiliary surface

Figure 5.11: Stress intensity factors versus the number of sub-elements on the auxiliary surface: Integration order for the outer surface = 7 x 7
Figure 5.12: Stress intensity factors versus the number of sub-elements on the auxiliary surface: Integration order for the outer surface = 9 x 9

Figure 5.13: Stress intensity factors versus the number of sub-elements on the auxiliary surface: Integration order for the outer surface = 11 x 11
CHAPTER 6. NUMERICAL IMPLEMENTATION

To evaluate the developed boundary integral equations, equations (4.1), (4.2), (4.8), (4.19), (4.21) and (4.25), a program named BEM was developed. The BEM program has the capabilities of solving embedded or edge crack problems, in single or multiple regions, on bounded or unbounded domains, in two and three dimensions.

This chapter describes the geometric discretization process, numerical integration, integration on the crack surfaces, normalization of linear equations, stress intensity factor calculation and data interfacing with I-DEAS and AutoCAD.

Discretization

In two-dimensional problems, the surface geometry is discretized by the line elements represented by linear, quadratic and cubic shape functions. In three-dimensional problems, the surface is discretized by the triangular and quadrilateral elements parameterized by the linear or quadratic shape functions. In Appendix B and Appendix C, one will find various shape functions for different types of elements in two and three dimensional problems, respectively, as used in the BEM program.

The continuous elements (Figure 6.1(a) and Figure 6.2(a)), which belong to the $C^0$ class, are widely used in both the finite element and boundary element methods. The traction boundary integral equations require $C^1$ continuity. To satisfy the continuity require-
ments, the traction boundary integral equation cannot be collocated at corners or at ends of the continuous element. When continuous elements are used, the derivatives of the unknown functions do not have a unique value at the points where two elements are connected. One possible way to treat this non-uniqueness is to use the averaged values of two derivatives for the connected elements. Another is to use discontinuous elements (Figure 6.1(b) and Figure 6.2(b)) on the crack surfaces. Partially discontinuous elements (Figure 6.1(c) and Figure 6.2(c)) are needed to join two different types of elements. Figures 6.3 and 6.4 show two typical edge crack models which are discretized using all three different kinds of elements. In the cracked body, overlaying crack surfaces occupy the same spatial coordinates are to be discretized by the separated geometric entities.

If one tries to input the collocation node coordinates for the discontinuous element directly, it is cumbersome. The \textit{BEM} program reads the geometry data for the region which is discretized by continuous elements, together with the essential and natural boundary conditions prescribed on geometric nodes. Then the \textit{BEM} program generates the collocation node numbers, its coordinates and the element connectivities internally depending on assigned element types, that is, continuous or discontinuous. The \textit{BEM} program also reassigns all boundary conditions on the generated collocation nodes from the values on the geometric nodes. On the basis of the generated data, all calculations are carried out on the collocation nodes. Output of the displacements, stresses and reaction forces is at the collocation nodes. In Chapter 7, the deformed shapes seem disjointed on the crack surfaces since the graphics connect the collocation points which are inside the elements.
Figure 6.1: Different types of 2-D elements

(a) Continuous  
(b) Discontinuous  
(c) Partially discontinuous

Figure 6.2: Different types of 3-D elements

(a) Continuous  
(b) Discontinuous  
(c) Partially discontinuous
Figure 6.3: Sample mesh for a 2-D edge crack model

Figure 6.4: Sample mesh for a 3-D edge crack model
Numerical Integration

The displacement $u_j(x)$ and the traction $t_j(x)$ components on the element, whether known or unknown, can be approximated by the linear combinations of the shape functions $H_k(\xi_1, \xi_2)$ and the nodal values $u_j^k$ and $t_j^k$ at the collocation points. Gaussian quadrature provides us the choice of weights and abscissas to make the integral exact for a class of integrands polynomial times some known function $W(x)$. By doing this, Gaussian integration has an additional freedom to choose the abscissas that are not in the usual of polynomial integrands which the functional values are evaluated at a set of equally spaced points. Here, a brief explanation of the Gaussian integration scheme for a function with one variable is given. The approximation for a function $f(x) = g(x)W(x)$ is exact if $g(x)$ is a one-dimensional polynomial as

$$\int_{x_1}^{x_2} f(x) \, dx = \int_{x_1}^{x_2} g(x)W(x) \, dx \approx \sum_{i=1}^{N} g(x_i)v_i$$

(6.1)

By defining $w_i = v_i/W(x_i)$, equation (6.1) becomes

$$\int_{x_1}^{x_2} f(x) \, dx = \sum_{i=1}^{N} w_i f(x_i)$$

(6.2)

where $x_i$ are the abscissas and $w_i$ are the weights. On the intrinsic element, two dimensional integrands can be written as

$$\int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) \, d\xi \, d\eta = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j f(\xi_i, \eta_j)$$

(6.3)

where $w_i, w_j$ are one-dimensional Gaussian weights and $\xi_i, \eta_j$ are the Gaussian points for the integration of order $N$. 
Consider next the local coordinate system \((\xi_1, \xi_2, n)\) for a 3-D element where \(\xi_1\) and \(\xi_2\) denote two tangential directions and \(n\) denotes the normal direction to the surface. If \(\phi_j(\xi_1, \xi_2)\) represents one of \(u_j(\xi_1, \xi_2)\) or \(t_j(\xi_1, \xi_2)\) on the surface, it can be approximated by the linear combinations as

\[
\phi_j(\xi_1, \xi_2) = \sum_{k=1}^{K} H_k(\xi_1, \xi_2) \phi_j^k \quad (j = 1, 2, 3) \tag{6.4}
\]

where \(K\) is the total number of nodes on one element and the subscript \(j\) is associated with the degree of freedom. The derivatives of \(\phi_j(\xi_1, \xi_2)\) with respect to each of the tangential coordinates can be expressed by the following equations.

\[
\frac{\partial \phi_j(\xi_1, \xi_2)}{\partial \xi_1} = \sum_{k=1}^{K} \frac{\partial H_k(\xi_1, \xi_2)}{\partial \xi_1} \phi_j^k \tag{6.5}
\]

\[
\frac{\partial \phi_j(\xi_1, \xi_2)}{\partial \xi_2} = \sum_{k=1}^{K} \frac{\partial H_k(\xi_1, \xi_2)}{\partial \xi_2} \phi_j^k \tag{6.6}
\]

From the derivatives on the local coordinates system, one obtains the derivatives on the global coordinates system as

\[
\begin{bmatrix}
\frac{\partial \phi_j(x)}{\partial x_1} \\
\frac{\partial \phi_j(x)}{\partial x_2} \\
\frac{\partial \phi_j(x)}{\partial x_3}
\end{bmatrix}
= [J]^{-1}
\begin{bmatrix}
\frac{\partial H_1}{\partial \xi_1} & \frac{\partial H_2}{\partial \xi_1} & \cdots & \frac{\partial H_k}{\partial \xi_1} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\phi_1^j \\
\phi_2^j \\
\cdots \\
\phi_k^j \\
\frac{\partial \phi_j}{\partial n}
\end{bmatrix} \tag{6.7}
\]

where the inverse Jacobian matrix is given by

\[
[J]^{-1} = \begin{bmatrix}
\frac{\partial \xi_i}{\partial x_j}
\end{bmatrix} = [J]^{ji} \quad (i, j = 1, 2, 3; \xi_3 = n) \tag{6.8}
\]
Compared to the traction boundary integral equations, to form the algebraic equations for the displacement boundary integral equations is somewhat easier. The algebraic equations for the regularized traction boundary integral equation on the outer surface (4.21) will be formed through the following procedures. The surface $S$ is divided into a couple of sub-surfaces, that is, $S = S^1 + S^0$ or $S = \Gamma^1 + S^0$. Let total number of elements corresponding to each sub-surface be $P (\in S^1)$, $Q (\in S^0)$ and $R (\in \Gamma^1)$. Also, let the number of nodes on each element be $L (\in S^1)$, $M (\in S^0)$ and $N (\in \Gamma^1)$, respectively. Then, equation (4.21) can be approximated by

$$\sum_{p=1}^{P} \int_{S_p} \left[T^*_{pj}\right] u_j(\bar{x}) \, dS(\bar{x}) - \sum_{r=1}^{R} \int_{\Gamma_r} \left[T^*_{pj}\right] dS(\bar{x}) \left[u_j(\bar{\xi})\right]_I$$

$$- \sum_{r=1}^{R} \int_{\Gamma_r} \left[T^*_{pj} r_n\right] dS(\bar{x}) \left[\frac{\partial u_j(\bar{\xi})}{\partial \xi_m} b_{mn}\right]_I + \sum_{r=1}^{R} \int_{\Gamma_r} \left[W_{pj}\right] dS(\bar{x}) \left[\frac{\partial u_j(\bar{\xi})}{\partial \xi_m}\right]_I$$

$$+ \sum_{q=1}^{Q} \int_{S_q} \left[T^*_{pj}\right] \left[u_j(\bar{x}) - u_j(\bar{\xi})\right]_I - \left\{ \frac{\partial u_j(\bar{\xi})}{\partial \xi_m} b_{mn} r_n \right\}_I dS(\bar{x})$$

$$+ \sum_{q=1}^{Q} \int_{S_q} \left[W_{pj}\right] dS(\bar{x}) \left[\frac{\partial u_j(\bar{x})}{\partial \xi_m}\right]_I$$

$$= \sum_{p=1}^{P} \int_{S_p} \left[U^*_{pj}\right] t_j(\bar{x}) \, dS(\bar{x}) + \sum_{r=1}^{R} \int_{\Gamma_r} \left[T_{pj}\right] dS(\bar{x}) \left[t_j(\bar{\xi})\right]_I$$

$$+ \sum_{q=1}^{Q} \int_{S_q} \left[U_{pj}\right] t_j(\bar{x}) + \left[T_{pj}\right] \left[t_j(\bar{\xi})\right]_I \, dS(\bar{x})$$
where the subscript \( I \) denotes the summation on the global node number. By using the associated shape functions and the Jacobian, one transforms the integration over the global coordinates into the local coordinates to get

\[
\begin{align*}
&= \sum_{p=1}^{P} \sum_{\alpha=1}^{L} \int \left[ T_{P}^{\ast} \right]_{1} H_{\alpha} |J| dS(\tilde{\xi}) \phi_{j}^{\alpha} \\
&- \sum_{r=1}^{R} \sum_{\gamma=1}^{N} \int \left[ T_{P}^{\ast} \right]_{1} \left[ H_{\gamma} + \frac{\partial H_{\gamma}}{\partial \xi_{1}} J_{1} r_{n}^{\gamma} b_{n} + \frac{\partial H_{\gamma}}{\partial \xi_{2}} J_{2} r_{n}^{\gamma} b_{2n} \right] |J| dS(\tilde{\xi}) \phi_{j}^{\gamma} \\
&+ \sum_{r=1}^{R} \sum_{\gamma=1}^{N} \int \left[ \left[ W_{P}^{1} \right]_{1} \left( \frac{\partial H_{\gamma}}{\partial \xi_{1}} J_{1} \right) + \left[ W_{P}^{2} \right]_{1} \left( \frac{\partial H_{\gamma}}{\partial \xi_{2}} J_{2} \right) \right] |J| dS(\tilde{\xi}) \phi_{j}^{\gamma} \\
&+ \sum_{q=1}^{Q} \sum_{\beta=1}^{M} \int \left[ \left[ W_{P}^{1} \right]_{1} \left( \frac{\partial H_{\beta}}{\partial \xi_{1}} J_{1} \right) + \left[ W_{P}^{2} \right]_{1} \left( \frac{\partial H_{\beta}}{\partial \xi_{2}} J_{2} \right) \right] |J| dS(\tilde{\xi}) \phi_{j}^{\beta} \\
&= \sum_{p=1}^{P} \sum_{\alpha=1}^{L} \int \left[ U_{P}^{\ast} \right]_{1} H_{\alpha} |J| dS(\tilde{\xi}) \phi_{j}^{\alpha} \\
&+ \sum_{r=1}^{R} \sum_{\gamma=1}^{N} \int \left[ T_{P} \right]_{1} H_{\gamma} |J| dS(\tilde{\xi}) \phi_{j}^{\gamma} \\
&+ \sum_{q=1}^{Q} \sum_{\beta=1}^{M} \int \left[ U_{P}^{\ast} \right]_{1} H_{\beta} + \left[ T_{P} \right]_{1} H_{\beta} |J| dS(\tilde{\xi}) \phi_{j}^{\beta}
\end{align*}
\]

which is ready for programming. Summation over the subscript \( n (n = 1, 2, 3) \) is implied and the summation over \( p, q \) and \( r \) is carried out on the elements surrounding the node \( I \).
Integration on the Crack Surfaces

Numerical integration on the crack surfaces requires the special attention. On the crack surfaces, when the collocation point is opposite to the integration element, one has to know the correct local node number on the opposite integration element to use the correct shape functions. It can be determined by coordinate comparison of two opposite nodes. Figure 6.5 shows all possible integration cases on the 2-D crack surfaces. In Figure 6.5, two normals at the points on opposite crack surfaces may be in same or in opposite direction, depending on the collocation and the field point positions. For any case, the auxiliary surface will be constructed in the direction opposite the normal at nodes on the integration element. The integration ranges over the remaining elements and all elements on whole surfaces, except singular element, are replaced by the auxiliary surface.

In 2-D problems, the node numbering direction on the auxiliary surface may be same as the node numbering direction for the remainder since it starts from the singular element geometric ending point and ends at the starting point of the singular element. From this, one can easily get the local node numbers on the auxiliary surface by the reverse counting of the local node numbers on the collocation element. In 3-D problems, the situation is more complex. One can not know the correct local node number on the opposite crack surface so the local node number on the opposite surface is determined by coordinate comparison. If two points are on the overlaying crack surfaces, the distance between two opposite points should be zero. In the case of nearly contacted crack surfaces, the opposite node among all other nodes on the opposite crack surface will be the one positioned at the shortest position.
Figure 6.5: Singular or nearly singular integration for the nodes on the opposite crack surfaces
Normalization of Linear Equations

From boundary integral equations, there results a set of linear equations that can be written in matrix form as

\[ A \cdot x = b \]  \hspace{1cm} (6.11)

where \( A = aA_D + \beta A_T \), \( b = \alpha b_D + \beta b_T \) and \( \alpha \) and \( \beta \) are constants. The subscript \( D \) denotes the displacement boundary integral equation and \( T \) for the traction boundary integral equation. The vector \( x \) is the unknown nodal displacements or tractions and \( b \) is the prescribed boundary conditions at the nodes.

The equation at any collocation point can be expressed as

\[ [\alpha A_D + \beta A_T] [x_D + T] = [\alpha b_D + \beta b_T] \]  \hspace{1cm} (6.13)

Now, depending on the problem type and whether there is a crack or not, the coefficients \( \alpha \) and \( \beta \) at a collocation point will be one of following three cases: (1) \( \alpha \neq 0, \beta = 0 \), (2) \( \alpha = 0, \beta \neq 0 \) and (3) \( \alpha \neq 0, \beta \neq 0 \). When one collocates the traction boundary integral equations at a point for cases 2 or 3, the coefficients in the traction boundary integral equations will be a factor of Young's modulus larger than those of the displacement boundary integral equation. A system matrix which contains coefficients with these two different scales may be difficult to solve.

Thus, there are two reasons for normalizing the system matrix. The first reason is that the system matrix is fully populated and all coefficients may not be uniform. The sec-
ond reason is that there are two different types of equations. From the independent use of the displacement and traction boundary integral equations, the system matrix coefficients in the traction equations will be a factor of Young's modulus larger than with those by the displacement equations.

Assuming LU decomposition method is used to invert the system matrix, the first normalization is concerned with Crout's method with partial pivoting [72]. One finds the largest element in each column as the diagonal element and scales all other elements in that column by the largest element to yield a low condition number.

\[ \frac{A}{\gamma} \cdot x' = b, \quad (x' = \gamma x) \]  
\[ x = \frac{x'}{\gamma} \]  

\[ \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \gamma_1 & \gamma_2 & \ldots & \gamma_N \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{bmatrix} \begin{bmatrix} \gamma_1 x_1 \\ \gamma_2 x_2 \\ \vdots \\ \gamma_N x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} \]  
\[ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \frac{1}{\gamma_1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} \]  

The second normalization is the scaling of the coefficients of the traction boundary integral equations by Young's modulus to give the same order values as those of the dis-
placement boundary integral equations. By dividing both sides of the traction boundary integral equation by Young's modulus before to form a system matrix, every equation in the system matrix will have same order coefficients.

**Computation of the Stress Intensity Factors**

The stress intensity factors $K_1$, $K_{II}$, and $K_{III}$ represent the strength of the stress field singularity at the crack tip. Each subscript of K denotes a mode of deformation. Term $K_1$ is related to the mode I deformation where the crack surface displacement is normal to the crack face and which tends to open the crack. $K_{II}$ is related to in-plane shear deformation, that is, to slide one crack surface with respect to the other. $K_{III}$ corresponds to out-of-plane shear. The stress intensity factors are determined by the crack geometry and the loads imposed but are not functions of the coordinates. Mathematically, the stress intensity factors may be interpreted as the strength of the $1/\sqrt{r}$ stress singularities at the crack tip. Physically, they may be interpreted as the intensity of the load transmitted through the crack tip region due to the introducing of the crack.

Analytic expressions for the stress fields ahead of the crack tip in a mixed-mode problem can be written in terms of the stress components

$$
\lim_{r \to 0} \sigma_{ij} = \frac{1}{\sqrt{r}} \left[ K_1 f_{ij}^{(I)}(\theta) + K_{II} f_{ij}^{(II)}(\theta) + K_{III} f_{ij}^{(III)}(\theta) \right]
$$

or by the displacement components

$$
\lim_{r \to 0} u_i = \sqrt{r} \left[ K_1 g_i^{(I)}(\theta) + K_{II} g_i^{(II)}(\theta) + K_{III} g_i^{(III)}(\theta) \right]
$$

where $f_{ij}$ and $g_i$ are dimensionless function of $\theta$. 
In the numerical analysis, the stress intensity factors are computed from the displacement components by following approximate equations

\[
\begin{bmatrix}
K_I \\
K_{II} \\
K_{III}
\end{bmatrix} = \begin{bmatrix}
\Delta u_n \\
\Delta u_t \\
\Delta u_s(1+v)
\end{bmatrix} \sqrt{\frac{\pi}{2}} \frac{E}{4(1-v^2)} \frac{1}{\sqrt{r}}
\]

where \( r \) is the distance from the crack tip to the point where the displacement is calculated on the crack surface. The variable \( \Delta u \) indicates the crack opening displacement at each point and the subscript \( n \) is for the component normal to the crack surface and \( t \) and \( s \) are the tangent components, respectively.

In the \textit{BEM} program developed herein, the displacement components used for the stress intensity factor calculation points are extracted from boundary elements at the crack tip and used for to the stress intensity factor calculation routines internally. In addition to the normal output file, the \textit{BEM} program writes an external file which contains the stress intensity factors only.

**Interface with I-DEAS and AutoCAD**

To prepare an input data deck and to display the output results, a translator named \textit{BEMUNV} was developed. \textit{BEMUNV} can convert the geometry data contained in a universal file written by SDRC I-DEAS\(^1\) into an input data file for the program \textit{BEM}. With a \textit{BEM} output result file, \textit{BEMUNV} can write the universal file for I-DEAS from which the nodal displacements and the stresses may be graphically displayed. \textit{BEMUNV} can also

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\(^1\) Integrated Design Engineering Analysis Software developed by Structural Dynamics Research Corporation.
write a DXF\textsuperscript{2} file for AutoCAD\textsuperscript{3} from a universal file, so that the \textit{BEM} input and output data files can be displayed on the initial and/or deformed geometry.

I-DEAS is a comprehensive mechanical computer aided engineering software system consisting of many families includes drafting, solid modeling, computer aided design, system dynamics and engineering analysis by the finite element method. An engineering analysis module called Pre-/Post-processor has especially powerful functions in generating and writing an input deck for the finite element analysis. It can write an input file with several different formats, that is, for its own solver SDRC Superb and the third party finite element solvers, such as MSC/NASTRAN, Swanson ANSYS and others such as an external file named a universal file\textsuperscript{4} in ASCII format. A universal file contains all kinds of data sets necessary for the finite element analysis including geometry, material properties and boundary condition data. A universal file can be used to prepare an input deck for different solvers, such as the \textit{BEM} program, through a translator by extracting the necessary data sets. Calculations resulting from third party programs can also be read into I-DEAS through a universal file for a visual inspection referred to post-processing. While developing the \textit{BEM} program, an input data deck for a complicated model was generated by I-DEAS using a program file\textsuperscript{5} The geometry data, that is, node numbers and nodal coordinates, element numbers, element types and connectivities data, contained in a universal file written by I-DEAS can then be transferred into the \textit{BEM} program via the translator \textit{BE-}

\textsuperscript{2} Drawing interchange file developed by Autodesk Incorporation.
\textsuperscript{3} Trademarks of Autodesk Incorporation.
\textsuperscript{4} A physical file containing symbolic data in pre-described physical records. Universal files contain the definition of nearly all I-DEAS entities and information.
\textsuperscript{5} A permanent file which contains I-DEAS language commands that can be used to automatically operate program.
After solving problems by the *BEM* program, a universal file can be written to put the nodal displacements and the stresses at nodes into I-DEAS by *BEMUNV*.

AutoCAD is a personal computer based program for drafting and a solid modeling. Recently, a main frame version has been released. AutoCAD has a standard DXF file which uses ASCII data to communicate the graphics data with other graphic programs. AutoCAD does not have the engineering analysis data, such as element connectivities and boundary conditions, thus one can not use AutoCAD as a pre-processor for an engineering analysis purpose, but it is useful for checking input geometry prepared manually and displaying the deformed shape on personal computers. Once a DXF file is read into AutoCAD, one can change the view point and can make a hard copy by using most printers and put the graphics data into most popular word processors to make documents containing the graphics. The translator *BEMUNV* can write a DXF file by extracting several kinds of data sets from the *BEM* input and/or output files. The data set which *BEMUNV* can handle includes geometric node coordinates, collocation node coordinates, node numbers, element shapes (initial and deformed) and element numbers as an independent layer.
CHAPTER 7. APPLICATIONS TO THE CRACK PROBLEMS

To evaluate and check the present method, a number of well-known cracks in two and three dimensions, such as a crack in an infinite medium, an embedded crack, an edge crack and a mixed-mode crack, were simulated. The developed program *BEM* is general enough to handle single or multiple crack problems. One can analyze a generally cracked body, with general shapes of cracks, with any number of embedded or edge cracks, by the *BEM* program. In this thesis, cracked bodies with two equal sized embedded cracks, each for two- and three-dimensional problems, were analyzed. For all crack problems, both sides of the crack surfaces are modelled with discontinuous elements and the traction boundary integral equations are collocated at points on one crack surface while the displacement boundary integral equations are collocated at points on the opposite surface.

For the problem of a crack in an infinite medium, a model of only the crack surfaces with discontinuous elements is required. On each crack surface, equal and opposite traction forces are applied which tend to open the crack. For embedded crack problems, the outer boundaries are modelled with continuous elements upon which the displacement boundary integral equations are collocated.

In the edge crack problems, the outer boundaries are modelled with continuous elements when the elements are not connected to the crack surfaces. When the crack surface intersects the outer surfaces, partially discontinuous elements are used to satisfy the continuity requirements at the crack edges. The displacement boundary integral equations are
collocated at all points on the outer boundaries which are modelled with continuous and partially discontinuous elements.

For all models, the stress intensity factors are determined from the nodal displacement components near the crack tip points which are on the crack front elements. The computed stress intensity factors are normalized and compared to analytic or empirical solutions. Also, all deformed shapes as well as the initial shapes are presented.

Two-dimensional Crack Problems

In the two-dimensional crack problems, six typical crack models were chosen. The stress intensity factors were calculated at the middle node of the crack front element as shown in Figure 7.1. The first five examples model a through thickness crack in an infinite plate under internal pressure (Figure 7.2), a center cracked plate subjected to tension (Figure 7.3(a)), a center slant cracked plate subjected to tension (Figure 7.3(b)), a single edge cracked plate subjected to tension (Figure 7.4(a)) and a double edge cracked plate subjected to tension (Figure 7.4(b)). As a last example of a two-dimensional crack problem, two central cracks in a rectangular plate were analyzed to demonstrate the capabilities of the program and algorithm for the multiple crack problems.

Through Thickness Crack in an Infinite Plate under Internal Pressure

A through thickness crack in an infinite plate subjected to internal pressure is a classical and typical crack problem for the mode I deformation in fracture mechanics analysis. This problem was solved for different number of elements with same geometry.
Figure 7.1: Stress intensity factor calculation point in 2-D crack problems

Figure 7.2: A through thickness crack in an infinite plate under internal pressure
(a) A centered crack \((h/b = 3.5, a/b = 0.5)\)

(b) A centered slant crack \((h/b = 3.5, a/b = 0.5, \theta = 45^\circ)\)

Figure 7.3: Typical 2-D crack problems: A rectangular plate with an internal crack
(a) Single edge crack (h/b = 3.5, a/b = 0.5)

(b) Double edge crack (h/b = 3.5, a/b = 0.5)

Figure 7.4: Typical 2-D crack problems: A rectangular plate with edge crack
Near the crack tips, a fine mesh is used compared to the central part (Figure 7.5). Boundary conditions at the infinite boundary are not required. Both essential and natural boundary conditions at the infinite boundary are satisfied by the integral equation of both the displacement and the traction type.

The stress intensity factor for this loading condition is the same as that of single crack in an infinite plate subjected to tension where the analytic solution is given by

$$K_1 = \sigma \sqrt{\pi a}$$

(7.1)

where $\sigma$ is the applied internal pressure (remotely applied tension stress for the single crack in an infinite plate) and $a$ is the half crack length. The calculated stress intensity factors for mode I deformations are given in Table 7.1. One can see from Table 7.1 that the results, with only a few elements by the program BEM, are very accurate in comparison to the analytic solution.

**Center Cracked Plate Subjected to Tension**

Figure 7.6 shows the initial and the deformed shapes of a center cracked plate subjected to tension. This is a typical two-dimensional embedded crack model. The bottom end of the model has been fixed in the vertical direction and uniform tensile loads are applied on the top end in the upward direction. Analyzed crack length to plate width ratios are $a/b = 0.2, 0.3, 0.4, \ldots, 0.8$, where $a$ denotes one-half of the crack length and $b$ denotes one-half of the plate width. The calculated stress intensity factors for mode I deformations are given in Table 7.2.
Figure 7.5: Initial and deformed shape of a through thickness crack in an infinite plate under internal pressure

Table 7.1: \( \frac{K_I}{\sigma\sqrt{\pi a}} \) for a through thickness crack in an infinite plate under internal pressure

<table>
<thead>
<tr>
<th>No. of elem.</th>
<th>Analytic</th>
<th>BEM</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>0.778</td>
<td>-22.20</td>
</tr>
<tr>
<td>3</td>
<td>&quot;</td>
<td>0.971</td>
<td>-2.90</td>
</tr>
<tr>
<td>5</td>
<td>&quot;</td>
<td>1.010</td>
<td>-1.00</td>
</tr>
<tr>
<td>7</td>
<td>&quot;</td>
<td>1.021</td>
<td>+2.10</td>
</tr>
<tr>
<td>9</td>
<td>&quot;</td>
<td>1.026</td>
<td>+2.60</td>
</tr>
</tbody>
</table>

\( a \) Number of elements on each crack surface.

\( b \) \( K_I = \sigma\sqrt{\pi a} \).

\( c \) \( K_I \) is calculated at the middle node of a crack front element.
Figure 7.6: Initial and deformed shape of a central crack in a rectangular plate subjected to tension \((\frac{h}{b} = 3.5, \frac{a}{b} = 0.5)\)
Table 7.2: \( \frac{K_1}{\sigma \sqrt{\pi a}} \) for a central crack in a rectangular plate subjected to tension

<table>
<thead>
<tr>
<th>( a/b )^a</th>
<th>Analytic^b</th>
<th>BEM^c</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.025</td>
<td>1.051</td>
<td>+2.54</td>
</tr>
<tr>
<td>0.3</td>
<td>1.058</td>
<td>1.085</td>
<td>+2.55</td>
</tr>
<tr>
<td>0.4</td>
<td>1.109</td>
<td>1.139</td>
<td>+2.70</td>
</tr>
<tr>
<td>0.5</td>
<td>1.187</td>
<td>1.219</td>
<td>+2.70</td>
</tr>
<tr>
<td>0.6</td>
<td>1.303</td>
<td>1.340</td>
<td>+2.84</td>
</tr>
<tr>
<td>0.7</td>
<td>1.488</td>
<td>1.533</td>
<td>+3.02</td>
</tr>
<tr>
<td>0.8</td>
<td>1.816</td>
<td>1.878</td>
<td>+3.41</td>
</tr>
</tbody>
</table>

^a See Figure 7.3(a) for the ratio of a crack length to the plate width.

^b [73, Tada, p. 2.1]: \( K_1 = \sigma \sqrt{\pi a} \cdot F(a/b) \).

^c \( K_1 \) is calculated at the middle node of a crack front element.

Number of elements:
(38) on the outer surfaces, (9) on each crack surface.

Center Slant Cracked Plate Subjected to Tension

This is a typical two-dimensional mixed mode crack problem. Figure 7.7 shows the initial and the deformed shapes of a 45-degrees center slant cracked plate subjected to tension. The bottom end has been fixed in the longitudinal direction and a uniform tensile load is applied at the other end in the upward direction. Analyzed crack length to plate width ratios are \( a/b = 0.1, 0.2, 0.3, \ldots, 0.8 \), where \( a \) denotes one-half of the crack length and \( b \) denotes one-half of the plate width. The stress intensity factors are given in Table 7.3 for the opening mode \( K_1 \) and in Table 7.4 for the in-plane shear mode \( K_{II} \).
Figure 7.7: Initial and deformed shape of a 45° slant crack in a rectangular plate subjected to tension \( \frac{h}{b} = 3.5, \frac{a}{b} = 0.5 \)
Table 7.3: \( \frac{K_I}{\sigma \sqrt{\pi a}} \) for a center slant crack in a rectangular plate subjected to tension 
\((\theta = 45^\circ)\)

<table>
<thead>
<tr>
<th>(a/b^a)</th>
<th>Analytic(^b)</th>
<th>BEM(^c)</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.505</td>
<td>0.517</td>
<td>+2.38</td>
</tr>
<tr>
<td>0.2</td>
<td>0.518</td>
<td>0.532</td>
<td>+2.70</td>
</tr>
<tr>
<td>0.3</td>
<td>0.541</td>
<td>0.555</td>
<td>+2.59</td>
</tr>
<tr>
<td>0.4</td>
<td>0.572</td>
<td>0.589</td>
<td>+2.97</td>
</tr>
<tr>
<td>0.5</td>
<td>0.612</td>
<td>0.632</td>
<td>+3.27</td>
</tr>
<tr>
<td>0.6</td>
<td>0.661</td>
<td>0.687</td>
<td>+3.93</td>
</tr>
<tr>
<td>0.7</td>
<td>0.721</td>
<td>0.755</td>
<td>+4.72</td>
</tr>
<tr>
<td>0.8</td>
<td>0.795</td>
<td>0.841</td>
<td>+5.79</td>
</tr>
</tbody>
</table>

\(^a\) See Figure 7.3(b) for the ratio of a crack length to the plate width.

\(^b\) [74, Murakami, p. 911].

\(^c\) \(K_I\) is calculated at the middle node of a crack front element.
Number of elements:
(38) on the outer surfaces, (9) on each crack surface.
Table 7.4: \( \frac{K_{II}}{\sigma \sqrt{\pi a}} \) for a center slant crack in a rectangular plate subjected to tension

\((\theta = 45^\circ)\)

<table>
<thead>
<tr>
<th>(a/b^a)</th>
<th>Analytic(^b)</th>
<th>BEM(^c)</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.502</td>
<td>0.515</td>
<td>+2.59</td>
</tr>
<tr>
<td>0.2</td>
<td>0.507</td>
<td>0.520</td>
<td>+2.56</td>
</tr>
<tr>
<td>0.3</td>
<td>0.516</td>
<td>0.529</td>
<td>+2.52</td>
</tr>
<tr>
<td>0.4</td>
<td>0.529</td>
<td>0.541</td>
<td>+2.27</td>
</tr>
<tr>
<td>0.5</td>
<td>0.546</td>
<td>0.558</td>
<td>+2.20</td>
</tr>
<tr>
<td>0.6</td>
<td>0.567</td>
<td>0.578</td>
<td>+1.94</td>
</tr>
<tr>
<td>0.7</td>
<td>0.595</td>
<td>0.604</td>
<td>+1.51</td>
</tr>
<tr>
<td>0.8</td>
<td>0.630</td>
<td>0.635</td>
<td>+0.79</td>
</tr>
</tbody>
</table>

\(^a\) See Figure 7.3(b) for the ratio of a crack length to the plate width.

\(^b\) [74, Murakami, p. 911].

\(^c\) \(K_{II}\) is calculated at the middle node of a crack front element.

Number of elements:
   (38) on the outer surfaces, (9) on each crack surface.
Single Edge Cracked Plate Subjected to Tension

Figure 7.8 shows the initial and the deformed shapes of a single edge cracked plate subjected to tension. At the crack edge on the outer boundary, partially discontinuous elements are used to join two different element types, continuous and discontinuous elements. The model has been fixed at the side opposite the edge crack for both degrees of freedom. Equal and opposite uniform tensile loads are applied on both ends of the plate. Analyzed crack length to plate width ratios are $a/b = 0.2, 0.3, 0.4, \cdots, 0.8$, where $a$ denotes one-half of the crack length and $b$ denotes one-half of the plate width. The calculated stress intensity factors for the mode I deformations are given in Table 7.5.

Double Edge Cracked Plate Subjected to Tension

Figure 7.9 shows the initial and the deformed shapes of a double edge cracked plate subjected to tension. At the crack edge on the outer boundary, partially discontinuous elements are used to join two different types of elements, continuous and discontinuous elements. The model has been fixed at one end in the longitudinal direction and uniform tensile loads are applied at the other end in the upward direction. Analyzed crack length to plate width ratios are $a/b = 0.2, 0.3, 0.4, \cdots, 0.8$, where $a$ denotes one-half of the crack length and $b$ denotes one-half of the plate width. The calculated stress intensity factors for the mode I deformations are given in Table 7.6.
Figure 7.8: Initial and deformed shape of single edge crack in a rectangular plate subjected to tension \( \frac{h}{b} = 4.0, \frac{a}{b} = 0.5 \)
Table 7.5: $\frac{K_I}{\sigma \sqrt{\pi a}}$ for single edge crack in a rectangular plate subjected to tension

<table>
<thead>
<tr>
<th>a/b$^a$</th>
<th>Analytic$^b$</th>
<th>BEM$^c$</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.366</td>
<td>1.433</td>
<td>+4.90</td>
</tr>
<tr>
<td>0.3</td>
<td>1.655</td>
<td>1.741</td>
<td>+5.20</td>
</tr>
<tr>
<td>0.4</td>
<td>2.108</td>
<td>2.219</td>
<td>+5.26</td>
</tr>
<tr>
<td>0.5</td>
<td>2.827</td>
<td>2.972</td>
<td>+5.13</td>
</tr>
<tr>
<td>0.6</td>
<td>4.043</td>
<td>4.261</td>
<td>+5.39</td>
</tr>
<tr>
<td>0.7</td>
<td>6.376</td>
<td>6.684</td>
<td>+4.83</td>
</tr>
<tr>
<td>0.8</td>
<td>11.997</td>
<td>12.220</td>
<td>+1.86</td>
</tr>
</tbody>
</table>

$^a$ See Figure 7.4(a) for the ratio of a crack length to the plate width.

$^b$ [73, Tada, p. 2.11]: $K_I = \sigma \sqrt{\pi a} \cdot F(a/b)$.

where,

$$F(a/b) = \sqrt{\frac{2b}{\pi a}} \tan \frac{\pi a}{2b} \left[ 0.752 + 2.02\left(\frac{a}{b}\right) + 0.37(1 - \sin \frac{\pi a}{2b})^3 \right].$$

$^c$ $K_I$ is calculated at the middle node of a crack front element.

Number of elements:

(29) on the outer surfaces, $\left(\frac{a}{b} \times 10 + 3\right)$ on each crack surface.
Figure 7.9: Initial and deformed shape of double edge crack in a rectangular plate subjected to tension $\frac{h}{b} = 3.5, \frac{a}{b} = 0.5$
Table 7.6: $\frac{K_I}{\sigma \sqrt{\pi a}}$ for double edge crack in a rectangular plate subjected to tension

<table>
<thead>
<tr>
<th>a/b</th>
<th>Analytic</th>
<th>BEM</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.118</td>
<td>1.149</td>
<td>+2.77</td>
</tr>
<tr>
<td>0.3</td>
<td>1.119</td>
<td>1.158</td>
<td>+3.48</td>
</tr>
<tr>
<td>0.4</td>
<td>1.132</td>
<td>1.174</td>
<td>+3.71</td>
</tr>
<tr>
<td>0.5</td>
<td>1.163</td>
<td>1.212</td>
<td>+4.21</td>
</tr>
<tr>
<td>0.6</td>
<td>1.226</td>
<td>1.281</td>
<td>+4.49</td>
</tr>
<tr>
<td>0.7</td>
<td>1.343</td>
<td>1.399</td>
<td>+4.17</td>
</tr>
<tr>
<td>0.8</td>
<td>1.567</td>
<td>1.622</td>
<td>+3.51</td>
</tr>
</tbody>
</table>

- **a** See Figure 7.4(b) for the ratio of a crack length to the plate width.
- **b** [73, Tada, p. 2.7]: $K_I = \sigma \sqrt{\pi a} \cdot F(a/b)$.

where,

\[ F(a/b) = \sqrt{\frac{2b}{\pi a}} \tan \frac{\pi a}{2b} \left[ 1 + 0.122 \cos^4 \frac{\pi a}{2b} \right]. \]

- **c** $K_I$ is calculated at the middle node of a crack front element.

Number of elements:

- (36) on the outer surfaces,
- on each crack surface, (4) case for $\frac{a}{b} = 0.2$ and (6) for the others.
Two Central Cracks in a Plate Subjected to Tension

Figure 7.10 shows the initial and the deformed shapes of a multiple centered crack in a plate subjected to tension. The bottom end of the model has been fixed in the vertical direction and uniform tensile loads are applied on the top end in the upward direction. In Figure 7.10, the parametric dimensions for the analyzed model are $h/b = 1.0$ and $a/b = 0.5$. A total of 76-elements are used for the model, that is, 40-elements for the outer surface and 9-elements for each crack surface. The normalized analytic solution for the stress intensity factor at the crack tip is 1.18 [73, p18.2]. The calculated stress intensity factor by the BEM is 1.209 and the error to the analytic solution is +2.46 percent. In the reference [73], the equation used for the normalization is given by

$$\frac{K_I}{\sigma \sqrt{\pi a}} = F\left(\frac{a}{b}, \frac{h}{b}\right)$$

Three-dimensional Crack Problems

In three-dimensional crack problems, four well-known single crack problems, one of a multiple crack problem and one of a practical crack problem were analyzed. Stress intensity factors were calculated at the points on the crack front element as shown in Figure 7.11 and are compared to the analytic solutions.

The first example problem is a penny-shaped crack embedded in an infinite solid medium (Figure 7.12). Figure 7.13 illustrates that the stress intensity factor $K_I$ at the crack tip under the remotely applied uniform stress is the same as that of similar cracks under the uniform internal pressure.
Figure 7.10: Initial and deformed shape of two central cracks in a rectangular plate subjected to tension ($\frac{h}{b} = 1.0$, $\frac{a}{b} = 0.5$)
The penny-shaped crack is modelled with several different meshes. All meshes are shown in Figure 7.14. The most coarse mesh case, shown in Figure 7.14(a), is modelled with only 1 element on each crack surface and the fine mesh case of Figure 7.14(f) is modelled with 33 elements on each crack surface. Among all mesh cases, Figure 7.14(d) is selected for the embedded penny-shaped crack in a cylinder problem. Then, a penny-shaped crack embedded in a cylindrical bar under tension, a penny-shaped crack embedded in a cylindrical bar under torsion and a circumferential crack in a solid cylindrical under tension follow. As a multiple crack model, two penny-shaped cracks in a solid cylinder under tension was chosen. When the crack is in the cylindrical body, a total 80-elements were used on the outer surfaces of the cylinder.

After evaluating the performances of the proposed scheme, an hourglass shaped bar with an edge crack was analyzed as a practical application. An hourglass shaped bar model has three deformation modes. On the outer surfaces, 224 quadrilateral and 64 triangular quadratic elements were used. On each crack surface, 6 quadrilateral quadratic elements were used.

**Penny-shaped Crack in an Infinite Medium Subjected to Tension**

Figure 7.15 shows the top and the front views of the initial and the deformed shape of a penny-shaped crack embedded in an infinite solid medium under internal pressure. The mesh in Figure 7.14(e) was used for the deformation shown. As explained in the through thickness crack in an infinite plate problem, no boundary conditions are required for the infinite boundary.
Figure 7.11: Stress intensity factor calculation point in 3-D crack problems
Figure 7.12: Penny-shaped crack embedded in an infinite solid medium

\[ K_I^A = K_I^B \]

Figure 7.13: Penny-shaped crack subjected to load
Figure 7.14: Mesh for a penny-shaped crack in an infinite medium
Figure 7.15: Initial and deformed shape of a penny-shaped crack in an infinite medium under internal pressure
Table 7.7 shows the stress intensity factors as determined from the displacement components at the points on the crack front elements. The analytic solution of the stress intensity factor is given by

\[ K_I = \frac{2}{\pi} \sigma \sqrt{\pi a} \quad \text{(7.3)} \]

**Penny-shaped Crack Embedded in a Cylindrical Bar under Tension**

Before solving the problem of a cylindrical bar with an embedded penny-shaped crack, a solid cylinder model without a crack is first prepared and solved. By adding the mesh for a penny-shaped crack shown in Figure 14(d) to the solid cylinder model, the mesh for the crack problem is complete. The bottom surface of the cylinder is fixed and uniform traction forces were applied to the top surface in the upward direction. Analyzed ratios of crack radius to the cylinder outer radii are \( a/b = 0.05, 0.1, 0.2, \ldots, 0.9 \), where \( a \) denotes the crack radius and \( b \) denotes the cylinder outer radius. Figure 7.16 shows the mesh and the zoomed out deformed shape of the crack portion for the case \( a/b = 0.5 \). The calculated stress intensity factors \( K_I \) are given in Table 7.8.

**Penny-shaped Crack Embedded in a Cylindrical Bar under Torsion**

The model used for the previous analysis of an embedded penny-shaped crack in a solid cylinder was reused for the in-plane shear mode \( K_{III} \) example. The geometry data and the essential boundary conditions are the same as the previous model. Only the loading conditions are changed from tension to torsion. A twisted shape for \( a/b = 0.5 \) is shown in Figure 7.17. The calculated stress intensity factors \( K_{III} \) are given in Table 7.9.
Table 7.7: \( \frac{K_I}{\sigma \sqrt{\pi a}} \) for a penny-shaped crack in an infinite medium under internal pressure

<table>
<thead>
<tr>
<th>Case</th>
<th>No. of elem.</th>
<th>Analytic</th>
<th>BEM</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>1</td>
<td>0.637</td>
<td>0.644</td>
<td>+1.10</td>
</tr>
<tr>
<td>(b)</td>
<td>4</td>
<td>&quot;</td>
<td>0.602</td>
<td>-5.39</td>
</tr>
<tr>
<td>(c)</td>
<td>5</td>
<td>&quot;</td>
<td>0.624</td>
<td>-2.04</td>
</tr>
<tr>
<td>(d)</td>
<td>12</td>
<td>&quot;</td>
<td>0.627</td>
<td>-1.57</td>
</tr>
<tr>
<td>(e)</td>
<td>20</td>
<td>&quot;</td>
<td>0.638</td>
<td>+0.16</td>
</tr>
<tr>
<td>(f)</td>
<td>33</td>
<td>&quot;</td>
<td>0.640</td>
<td>+0.47</td>
</tr>
</tbody>
</table>

\( a \) Case number indicates mesh case in Figure 7.14.

\( b \) Number of elements on each crack surface.

\( c \) \( K_I = \frac{2}{\pi} \sigma \sqrt{\pi a} \).

\( d \) \( K_I \) is calculated at the side node in cases (a) and (b) and at the center of the crack front element for all other cases, see Figure 7.11.
Figure 7.16: Initial and deformed shape of an embedded penny-shaped crack in a cylinder under tension ($\frac{a}{b} = 0.5$)
Table 7.8: $\frac{K_I}{\sigma_{net} \sqrt{\pi a}}$ for a penny-shaped crack in a solid cylinder under tension

<table>
<thead>
<tr>
<th>$a/b^a$</th>
<th>Analytic$^b$</th>
<th>BEM$^c*$</th>
<th>% error</th>
<th>BEM$^d*$</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.635</td>
<td>0.620</td>
<td>-2.36</td>
<td>0.621</td>
<td>-2.20</td>
</tr>
<tr>
<td>0.1</td>
<td>0.631</td>
<td>0.619</td>
<td>-1.90</td>
<td>0.630</td>
<td>-0.16</td>
</tr>
<tr>
<td>0.2</td>
<td>0.614</td>
<td>0.604</td>
<td>-1.63</td>
<td>0.616</td>
<td>+0.33</td>
</tr>
<tr>
<td>0.3</td>
<td>0.589</td>
<td>0.579</td>
<td>-1.70</td>
<td>0.591</td>
<td>+0.34</td>
</tr>
<tr>
<td>0.4</td>
<td>0.556</td>
<td>0.546</td>
<td>-1.80</td>
<td>0.557</td>
<td>+0.18</td>
</tr>
<tr>
<td>0.5</td>
<td>0.516</td>
<td>0.507</td>
<td>-1.74</td>
<td>0.517</td>
<td>+0.19</td>
</tr>
<tr>
<td>0.6</td>
<td>0.469</td>
<td>0.461</td>
<td>-1.71</td>
<td>0.469</td>
<td>0.0</td>
</tr>
<tr>
<td>0.7</td>
<td>0.414</td>
<td>0.406</td>
<td>-1.93</td>
<td>0.413</td>
<td>-0.24</td>
</tr>
<tr>
<td>0.8</td>
<td>0.346</td>
<td>0.342</td>
<td>-1.16</td>
<td>0.346</td>
<td>0.0</td>
</tr>
<tr>
<td>0.9</td>
<td>0.252</td>
<td>0.256</td>
<td>+1.59</td>
<td>0.256</td>
<td>+1.59</td>
</tr>
</tbody>
</table>

$^a$ Ratio of the centered penny-shaped crack radius to the cylinder outer radius.

$^b$ $K_I = \sigma_{net} \sqrt{\pi a} * F_1(\frac{a}{b})$ [73, Tada, p. 28.1].

where,

$$\sigma_{net} = \frac{p \pi b^2}{\pi (b^2 - a^2)}$$

$$F_1(\frac{a}{b}) = \sqrt{1 - \frac{a}{b} + \frac{2}{\pi} \left\{1 + \frac{1}{2} \cdot \frac{a}{b} - \frac{5}{8} \left(\frac{a}{b}\right)^2 + 0.421 \left(\frac{a}{b}\right)^3\right\}}.$$  

$^c$ See Figure 7.14(d) for the mesh on each crack surface.

$^d$ See Figure 7.14(f) for the mesh on each crack surface.

$^* K_I$ is calculated at the center of the crack front element, see Figure 7.11(a).
Figure 7.17: Initial and twisted shape of a cylinder with an embedded penny-shaped crack under torsion ($\frac{a}{b} = 0.5$)
Table 7.9: $\frac{K_{III}}{\tau_{net} \sqrt{\pi a}}$ for a penny-shaped crack in a solid cylinder under torsion

<table>
<thead>
<tr>
<th>Case$^a$</th>
<th>a/b$^b$</th>
<th>Analytic$^c$</th>
<th>BEM$^d$</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>0.424</td>
<td>0.437</td>
<td>+3.07</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.424</td>
<td>0.437</td>
<td>+3.07</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.424</td>
<td>0.437</td>
<td>+3.07</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
<td>0.421</td>
<td>0.434</td>
<td>+3.09</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>0.415</td>
<td>0.428</td>
<td>+3.13</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
<td>0.402</td>
<td>0.414</td>
<td>+2.99</td>
</tr>
<tr>
<td>7</td>
<td>0.6</td>
<td>0.379</td>
<td>0.391</td>
<td>+3.17</td>
</tr>
<tr>
<td>8</td>
<td>0.7</td>
<td>0.342</td>
<td>0.355</td>
<td>+3.80</td>
</tr>
<tr>
<td>9</td>
<td>0.8</td>
<td>0.288</td>
<td>0.298</td>
<td>+3.47</td>
</tr>
<tr>
<td>10</td>
<td>0.9</td>
<td>0.205</td>
<td>0.205</td>
<td>0.0</td>
</tr>
</tbody>
</table>

$^a$ All cases use same mesh on each crack surface in Figure 7.14(d).

$^b$ Ratio of the centered penny-shaped crack radius to the cylinder outer radius.

$^c$ $K_{III} = \tau_{net} \sqrt{\pi a} \cdot F_1(\frac{a}{b})$ [73, Tada, p. 28.3].

where,

$$\tau_{net} = \frac{2Ta}{\pi (b^4 - a^4)},$$

$$F_1(\frac{a}{b}) = \sqrt{\frac{1 - \frac{a}{b}}{\frac{4}{3\pi}}} \left\{ 1 + \frac{1}{2(b^3)} + \frac{3}{8(b^5)} + \frac{5}{16(b^7)} - \frac{93}{128(b^9)} + 0.038(\frac{a}{b})^5 \right\}.$$
A Circumferential Crack in a Cylindrical Bar under Tension

This edge crack problem requires a complex programming algorithm and mesh compared to an embedded crack problem. When a crack edge meets regular surface, partially discontinuous elements are used to join two different types of elements. For the outer surface, the mesh and the boundary conditions are the same as those of the penny-shaped crack embedded in a cylindrical bar under tension. The difference is a circumferential edge crack instead of an embedded penny-shaped crack. Analyzed ratios of a circumferential edge crack inner radius to the cylinder outer radius are \( a/b = 0.3, 0.4, 0.5, \ldots, 0.7 \).

The crack surfaces were modelled with three different meshes depending on the ratio of a crack front radius to the cylinder outer radius (Figure 7.18). The mesh and the opened crack shapes for \( a/b = 0.3 \) are shown in Figures 7.19 and 7.20, respectively. The calculated stress intensity factors \( K_1 \) are given in Table 7.10.

Two Penny-shaped Cracks Embedded in a Cylindrical Bar under Tension

By adding the penny-shaped crack mesh shown in Figure 14(d) to the solid cylinder model, the mesh for this problem was prepared without re-meshing the whole cylinder. Figure 7.21 shows the mesh and the deformed shapes of two penny-shaped cracks embedded in a cylindrical bar under tension. The bottom surface of the cylinder is fixed and uniform traction forces are applied on the top surface in the upward direction. The ratio of the crack radius to the cylinder outer radius is \( a/b = 0.5 \), where \( a \) denotes the crack radius and \( b \) the cylinder outer radius. The two penny-shaped cracks are far enough apart to cause almost no mutual interaction effects.
Figure 7.18: Mesh on a circumferential crack surface in a solid cylinder
Figure 7.19: Mesh for a solid cylinder with a circumferential crack \( \frac{a}{b} = 0.3 \)
Figure 7.20: Deformed shape of a solid cylinder with a circumferential crack under tension ($\frac{a}{b} = 0.3$)
Table 7.10: \( \frac{K_1}{\sigma_{net}\sqrt{\pi a}} \) for a circumferential crack in a cylinder under tension

<table>
<thead>
<tr>
<th>Case</th>
<th>( a/b )(^a)</th>
<th>Analytic(^b)</th>
<th>BEM(^c)</th>
<th>% error</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.3*</td>
<td>0.494</td>
<td>0.456</td>
<td>-7.69</td>
</tr>
<tr>
<td>2</td>
<td>0.4**</td>
<td>0.486</td>
<td>0.452</td>
<td>-7.00</td>
</tr>
<tr>
<td>3</td>
<td>0.5**</td>
<td>0.475</td>
<td>0.442</td>
<td>-6.95</td>
</tr>
<tr>
<td>4</td>
<td>0.6***</td>
<td>0.459</td>
<td>0.420</td>
<td>-8.49</td>
</tr>
<tr>
<td>5</td>
<td>0.7***</td>
<td>0.434</td>
<td>0.402</td>
<td>-7.37</td>
</tr>
</tbody>
</table>

\(^a\) Ratio of the cylinder net section radius to the cylinder outer radius.

\(^b\) \( K_1 = \sigma_{net}\sqrt{\pi a} \cdot F_1\left(\frac{a}{b}\right) [73, Tada, p. 27.1]. \)

where,

\[ \sigma_{net} = \frac{p\pi b^2}{\pi a^2}, \]

\[ F_1\left(\frac{a}{b}\right) = \sqrt{1 - \frac{a}{b}} \cdot \frac{1}{2} \left[ 1 + \frac{1}{2} \frac{a}{b} + \frac{3}{8} \left(\frac{a}{b}\right)^2 - 0.363 \left(\frac{a}{b}\right)^3 + 0.731 \left(\frac{a}{b}\right)^4 \right]. \]

\(^c\) \( K_1 \) is calculated at the center of the crack front element, see Figure 7.11(a).

* see Figure 7.18(a) for the mesh on each crack surface.

** see Figure 7.18(b) for the mesh on each crack surface.

*** see Figure 7.18(c) for the mesh on each crack surface.
Figure 7.21: Initial and deformed shape of two embedded penny-shaped cracks in a cylinder under tension ($\frac{a}{b} = 0.5$)
The crack tip stress intensity factors $K_1$ for both penny-shaped cracks were calculated at the center of the crack front elements (Figure 11(a)) and the normalized stress intensity factor $K_1$ for $a/b = 0.5$ is 0.507 for both cracks. This calculated stress intensity factor is the same as that of the penny-shaped crack case and is -1.74 percent in error compared to the analytic solution given in Table 7.8.

**Hourglass-shaped Bar with an Edge Crack under One-end Deflection**

Figure 7.22 shows a detailed geometry of an hourglass-shaped bar with an edge crack. The bar was modelled with three artificial regions to partition the coefficient matrix into three smaller ones. The lower part of the bar, up to $H$ from the bottom surface, is fixed as a cantilever. The prescribed boundary condition is a uniform deflection of the upper part, up to $H$ from the top surface. Figure 7.23(a) shows the mesh and the collocation points on the crack surfaces. The opened crack shape as caused by the deflection of the upper end is shown in Figure 7.23(b).

The three stress intensity factors are determined at the side nodes on the crack front element (see Figure 7.11(b) for the stress intensity factor calculation point). The calculated stress intensity factors for three different modes are given in Figure 7.24. The opening mode $K_1$ and the in-plane shear mode $K_{II}$ are symmetric with respect to the center of the crack front. Two crack front end points are positioned far from the neutral axis of the bar and $K_1$ at these points is expected to be larger than at other points on the crack front. The stress intensity factors for the out-of-plane shear mode $K_{III}$ are anti-symmetric with respect to the crack front center.
Figure 7.22: An hourglass-shaped bar with an edge crack modelled with 3-regions

(dimension unit = inch)

\[ L = 2.06000 \quad H = 0.37500 \]
\[ D = 0.33840 \quad h = 0.76800 \]
\[ d = 0.12634 \quad r = 0.12467 \]
\[ E = 1.0 \quad \nu = 0.3 \]
Figure 7.23: Deformed shape of an hourglass-shaped bar with an edge crack: One-end fixed, one-end deflected ($\delta = 0.05''$)
Figure 7.24: Stress intensity factors for an hourglass-shaped bar with an edge crack

sections along an edge crack front: see Figure 7.22(a)
CHAPTER 8. CONCLUSIONS

A dual boundary integral equation method for a single-region solution of a general mixed-mode crack in two- and three-dimensional elastostatics has been developed. Also, modified integral identities were developed from the formal integral identities by using auxiliary surfaces.

The displacement and the traction boundary integral equations were developed by taking the limit of the relevant representation equations. The strongly singular and the hypersingular integrals in the displacement and the traction boundary integral equations, respectively, are regularized using the key terms of the Taylor expansion of the density (displacements) functions and the identities pertaining to the fundamental solutions.

To isolate the added back, strongly singular integrals, surface elements containing the collocation point and the opposing crack surface elements are replaced by auxiliary surfaces which provide non-singular integration paths. Convergence tests for the introduced auxiliary surfaces show that the stress intensity factors are not sensitive to the number of elements on the auxiliary surface but do depend on the integration order for elements on curved outer surfaces. With a smooth enough auxiliary surface, fairly accurate results were obtained without subdividing the auxiliary surface.

On the crack surfaces, discontinuous quadrilateral elements were used to satisfy the continuity requirement. The traction boundary integral equations are collocated at points
on one crack surface while the displacement boundary integral equations are collocated at points on the opposite crack surface.

When the crack shape changes, the developed method requires mesh modifications only on the crack surfaces. The developed BEM program has the capabilities of data checking only or solving the problem. Also, one can run the BEM program in an interactive mode or a batch mode as a background job to solve many problems without rerunning the BEM program. Through the developed translator BEMUNV, one can interface the boundary element data with two popular pre-/post-processor and graphics programs -- I-DEAS and AutoCAD -- to prepare input data and to analyze output data.

A number of two- and three-dimensional crack problems were analyzed by the present technique and accurate stress intensity factors were obtained. The example problems include a crack in an unbounded region, an embedded crack, an edge crack, a single crack and a multiple crack.

For the future work, following points should be addressed.

First, during the evaluation of the present scheme, it was found that when the crack surface is modelled with large aspect ratio elements (greater than 2), convergence on the auxiliary surface elements was not easy. This may be due to the abrupt slope change on the curved auxiliary surfaces which are constructed on narrow quadrilaterals. One alternative solution is to convert the stronger integrals on the surface to the contour integrals through Stoke's theorem prior to numerical computations.

Second, the boundary element method constructs the matrices with fully populated coefficients. This requires large core memory to solve and to reserve the coefficient matrices. Three-dimensional large models with more than 1000 nodes (3000 unknowns) re-
quire more than 60 MB core memory. Partitioning the coefficient matrices into small ones to reduce required core memory and computing time might be considered.

Third, by use of a parallel processing algorithm, one could reduce the computation time, especially for three dimensional problems.
BIBLIOGRAPHY


11. Choi, J. H. and Kwak, B. M., A Boundary Integral Equation Formulation in Deri­


APPENDIX A. SURFACE GRADIENTS ON 3-D SURFACE

Consider the local coordinates system on three-dimensional surface which has the orthonormal base vectors $(\vec{\zeta}_1, \vec{\zeta}_2)$ at $\vec{z}$ (Figure A.1). Also, consider the non-orthogonal curvilinear system $(\eta^1, \eta^2)$ which has the basis vectors $(\vec{\eta}_1, \vec{\eta}_2)$, such that

\[ \vec{\zeta}_1 \times \vec{\zeta}_2 = \vec{n} \quad (A.1) \]

\[ \vec{\zeta}_1 \cdot \vec{\zeta}_2 = 0 \quad (\vec{\zeta}_1 \perp \vec{\zeta}_2) \quad (A.2) \]

Figure A.1: Surface coordinates system for an element on 3-D surface
\[ |\vec{\xi}_1| = |\vec{\xi}_2| = 1 \quad (A.3) \]
\[ \vec{e}_1 = \frac{\partial \vec{r}}{\partial \eta^1} \quad (A.4) \]
\[ \vec{e}_2 = \frac{\partial \vec{r}}{\partial \eta^2} \quad (A.5) \]

where \( \vec{r}(x_1, x_2, x_3) \) is the position vector of \( \vec{z} \) on the surface.

The normal vectors for two coordinates systems are not, in general, parallel.

\[ \vec{e}_3 \parallel \vec{n} \quad (A.6) \]

The unit vectors \( \vec{e}^i \ (i = 1, 2, 3) \) are such that

\[ \vec{e}^i \cdot \vec{e}^j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad (A.7) \]

\[ \vec{e}^3 = \frac{\vec{e}_1 \times \vec{e}_2}{\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)} \quad (A.8) \]

\[ \vec{e}^3 \parallel \vec{n} \quad (A.9) \]

Also, consider the gradients of an arbitrary function \( \phi \) as follow.

\[ \nabla \phi = \frac{\partial \phi}{\partial \eta^1} \vec{e}^1 + \frac{\partial \phi}{\partial \eta^2} \vec{e}^2 + \frac{\partial \phi}{\partial \eta^3} \vec{e}^3 \quad (A.10) \]

\[ \frac{d\phi}{d\ell} = \nabla \phi \cdot \vec{\ell} \]

\[ = \frac{\partial \phi}{\partial \eta^1} \vec{e}^1 \cdot \vec{\ell} + \frac{\partial \phi}{\partial \eta^2} \vec{e}^2 \cdot \vec{\ell} + \frac{\partial \phi}{\partial \eta^3} \vec{e}^3 \cdot \vec{\ell} \quad (A.11) \]

\[ = \frac{\partial \phi}{\partial \eta^1} \vec{e}^1 \cdot \vec{\ell} + \frac{\partial \phi}{\partial \eta^2} \vec{e}^2 \cdot \vec{\ell} \quad (\because \vec{e}^3 \parallel \vec{n} \text{ and } \vec{\ell} \perp \vec{n}) \]
The base vectors satisfy the following relation.

\[ \mathbf{e}_j = (\mathbf{e}_i \cdot \mathbf{e}_j) \mathbf{e}_i \]  \hspace{2cm} (A.12)

\[ \mathbf{e}_i \cdot \mathbf{e}_j = g_{ij} \]  \hspace{2cm} (A.13)

\[ \mathbf{e}_1 = g_{11} \mathbf{e}_1^1 + g_{12} \mathbf{e}_2^1 + g_{13} \mathbf{e}_3^1 \]  \hspace{2cm} (A.14)

\[ \mathbf{e}_2 = g_{21} \mathbf{e}_1^1 + g_{22} \mathbf{e}_2^1 + g_{23} \mathbf{e}_3^1 \]  \hspace{2cm} (A.15)

From the above, one has

\[ \mathbf{e}_j \cdot \mathbf{\ell} = g_{i1} (\mathbf{e}_1 \cdot \mathbf{\ell}) + g_{i2} (\mathbf{e}_2 \cdot \mathbf{\ell}) + g_{i3} (\mathbf{e}_3 \cdot \mathbf{\ell}) \]  \hspace{2cm} (i = 1, 2) \hspace{2cm} (A.16)

\[ = g_{i1} (\mathbf{e}_1 \cdot \mathbf{\ell}) + g_{i2} (\mathbf{e}_2 \cdot \mathbf{\ell}) \hspace{2cm} (\because \mathbf{e}_3 \parallel \mathbf{n} \text{ and } \mathbf{\ell} \perp \mathbf{n}) \]

In matrix form, one can write

\[ \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{\ell} \\ \mathbf{e}_2 \cdot \mathbf{\ell} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{\ell} \\ \mathbf{e}_2 \cdot \mathbf{\ell} \end{pmatrix} \]  \hspace{2cm} (A.17)

\[ g_{ij} = \frac{\partial \mathbf{\ell}_i}{\partial \eta^i} \cdot \frac{\partial \mathbf{\ell}_j}{\partial \eta^j} \]

\[ = \frac{\partial x_\alpha}{\partial \eta^i} \mathbf{\ell}_i \cdot \frac{\partial x_\beta}{\partial \eta^j} \mathbf{\ell}_j \]

\[ = \frac{\partial x_\alpha}{\partial \eta^i} \frac{\partial x_\beta}{\partial \eta^j} \]  \hspace{2cm} (\because \mathbf{\ell}_i \cdot \mathbf{\ell}_j = \delta_{ij}) \hspace{2cm} (A.18)

The determinant \( G \) is

\[ G = \left| g_{ij} \right| = g_{11} g_{22} - g_{12} g_{21} \]  \hspace{2cm} (A.19)

and from equations (A.17) and (A.19), one has

\[ \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{\ell} \\ \mathbf{e}_2 \cdot \mathbf{\ell} \end{pmatrix} = \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{\ell} \\ \mathbf{e}_2 \cdot \mathbf{\ell} \end{pmatrix} \]  \hspace{2cm} (A.20)
With equations (A.21) and (A.22), equation (A.11) becomes

\[ \frac{d\phi}{d\ell} = \frac{1}{G} \left[ \frac{\partial \phi}{\partial \eta^1} \{ g_{22} \varepsilon_1 - g_{12} \varepsilon_2 \} + \frac{\partial \phi}{\partial \eta^2} \{ g_{11} \varepsilon_2 - g_{21} \varepsilon_1 \} \right] \cdot \vec{\ell}, \]  

(A.23)

Now, let \( \phi \to u_j(\vec{x}) \) (\( j = 1, 2, 3 \)) and \( \ell \to \zeta_m \) (\( m = 1, 2 \)), so the surface gradients are

\[ \frac{\partial u_j}{\partial \zeta_m} = \nabla_{\eta^i} u_j \cdot \vec{\zeta}_m = \frac{1}{G} \left[ \frac{\partial u_j}{\partial \eta^1} \{ g_{22} \varepsilon_1 - g_{12} \varepsilon_2 \} + \frac{\partial u_j}{\partial \eta^2} \{ g_{11} \varepsilon_2 - g_{21} \varepsilon_1 \} \right] \cdot \vec{\zeta}_m, \]  

(A.24)

where

\[ g_{ij} = \frac{\partial x_\alpha}{\partial \eta^i} \frac{\partial x_\beta}{\partial \eta^j}, \]

\[ G = g_{11} g_{22} - g_{12} g_{21}, \quad (g_{12} = g_{21}), \]  

(A.25)

\[ \varepsilon_i = \frac{\partial \vec{r}}{\partial \eta^i} = \frac{\partial x_\alpha}{\partial \eta^i} \vec{r}_\alpha. \]
## 2-nodes Linear Elements

<table>
<thead>
<tr>
<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Node 1" /></td>
<td>( H_1(r) = \frac{1}{2}(1-r) )</td>
</tr>
<tr>
<td><img src="image2" alt="Node 2" /></td>
<td>( H_1(r) = \frac{2}{3}(1-r) )</td>
</tr>
<tr>
<td><img src="image3" alt="Node 3" /></td>
<td>( H_1(r) = \frac{2}{3}(1-r) )</td>
</tr>
<tr>
<td><img src="image4" alt="Node 4" /></td>
<td>( H_1(r) = \frac{1}{2} - r )</td>
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### 3-nodes Quadratic Elements

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<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
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<tbody>
<tr>
<td>1 - 2 - 3</td>
<td>$H_1(r) = -\frac{1}{2}r(1-r)$</td>
</tr>
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<td>$H_2(r) = (1-r)(1+r)$</td>
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<tr>
<td></td>
<td>$H_3(r) = \frac{1}{2}r(1+r)$</td>
</tr>
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<td>$H_3(r) = \frac{9}{10}r(1+r)$</td>
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<tr>
<td>1 - 2/3 - 3</td>
<td>$H_1(r) = -\frac{9}{10}r(1-r)$</td>
</tr>
<tr>
<td></td>
<td>$H_2(r) = \frac{3}{2}(1-r)(\frac{2}{3} + r)$</td>
</tr>
<tr>
<td></td>
<td>$H_3(r) = \frac{3}{5}r\left(\frac{2}{3} + r\right)$</td>
</tr>
<tr>
<td>1 - 2/3 - 3</td>
<td>$H_1(r) = -\frac{9}{8}r\left(\frac{2}{3} - r\right)$</td>
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<td></td>
<td>$H_2(r) = \frac{9}{4}(\frac{2}{3} - r)(\frac{2}{3} + r)$</td>
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<tr>
<td></td>
<td>$H_3(r) = \frac{9}{8}r\left(\frac{2}{3} + r\right)$</td>
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### 4-nodes Cubic Elements

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<th>Shape Functions</th>
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</thead>
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<tr>
<td>1 2 3 4</td>
<td>( H_1(r) = -\frac{9}{16}(1-r)(\frac{1}{3}+r)(\frac{1}{3}-r) )</td>
</tr>
<tr>
<td>-1 -1/3 0 1/3 1</td>
<td>( H_2(r) = \frac{27}{16}(1+r)(1-r)(\frac{1}{3}-r) )</td>
</tr>
<tr>
<td></td>
<td>( H_3(r) = \frac{27}{16}(1+r)(1-r)(\frac{1}{3}+r) )</td>
</tr>
<tr>
<td></td>
<td>( H_4(r) = -\frac{9}{16}(1+r)(\frac{1}{3}+r)(\frac{1}{3}-r) )</td>
</tr>
</tbody>
</table>

-1 -1/3 0 1/3 1

| 1 2 3 4                 | \( H_1(r) = -\frac{9}{14}(-r)(\frac{1}{3}-r)(\frac{1}{3}+r) \) |
| -1 -1/3 0 1/3 3/4 1     | \( H_2(r) = \frac{27}{13}(-r)(\frac{1}{3}-r)(1+r) \) |
|                         | \( H_3(r) = \frac{27}{10}(-r)(\frac{1}{3}+r)(1+r) \) |
|                         | \( H_4(r) = -\frac{576}{455}(-r)(\frac{1}{3}+r)(1+r) \) |

-1 -1/3 0 1/3 3/4 1

| 1 2 3 4                 | \( H_1(r) = -\frac{576}{455}(\frac{1}{3}+r)(\frac{1}{3}+r)(1-r) \) |
| -1 -3/4 -1/3 0 1/3 1    | \( H_2(r) = \frac{27}{10}((-r)(\frac{1}{3}+r)(1-r) \) |
|                         | \( H_3(r) = \frac{27}{13}((-r)(\frac{1}{3}+r)(1-r) \) |
|                         | \( H_4(r) = -\frac{9}{14}(\frac{1}{3}+r)(\frac{1}{3}+r)(\frac{1}{3}-r) \) |

-1 -3/4 -1/3 0 1/3 1

| 1 2 3 4                 | \( H_1(r) = -\frac{4}{3}(-r)(\frac{1}{4}-r)(\frac{1}{4}+r) \) |
| -1 -3/4 -1/4 0 1/4 3/4 1| \( H_2(r) = 4(\frac{3}{4}-r)(\frac{3}{4}+r)(\frac{1}{4}-r) \) |
|                         | \( H_3(r) = 4(\frac{3}{4}-r)(\frac{3}{4}+r)(\frac{1}{4}+r) \) |
|                         | \( H_4(r) = -\frac{4}{3}(\frac{3}{4}+r)(\frac{1}{4}+r)(\frac{1}{4}-r) \) |
### APPENDIX C. SHAPE FUNCTIONS AND NODE NUMBERING SEQUENCE FOR 3-D ELEMENTS

#### 3-nodes Triangular Linear Elements

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<th>Shape Functions</th>
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</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>$H_1(r,s) = 1 - r - s$</td>
</tr>
<tr>
<td></td>
<td>$H_2(r,s) = r$</td>
</tr>
<tr>
<td></td>
<td>$H_3(r,s) = s$</td>
</tr>
</tbody>
</table>

![Diagram of 3-nodes Triangular Linear Elements]

<table>
<thead>
<tr>
<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>$H_1(r,s) = \frac{5}{3} - 2(r + s)$</td>
</tr>
<tr>
<td>(1/6,2/3)</td>
<td>$H_2(r,s) = 2r - \frac{1}{3}$</td>
</tr>
<tr>
<td>(1/6,1/6)</td>
<td>$H_3(r,s) = 2s - \frac{1}{3}$</td>
</tr>
</tbody>
</table>

![Diagram of 3-nodes Triangular Linear Elements]
### 6-nodes Triangular Quadratic Elements

<table>
<thead>
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<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>$H_1(r,s) = (1-r-s)(1-2r-2s)$</td>
</tr>
<tr>
<td>(0,1/2)</td>
<td>$H_2(r,s) = r(2r-1)$</td>
</tr>
<tr>
<td>(0,0)</td>
<td>$H_3(r,s) = s(2s-1)$</td>
</tr>
<tr>
<td>(1/2,0)</td>
<td>$H_4(r,s) = 4r(1-r-s)$</td>
</tr>
<tr>
<td>(1,0)</td>
<td>$H_5(r,s) = 4rs$</td>
</tr>
<tr>
<td>(1/2,1/2)</td>
<td>$H_6(r,s) = 4s(1-r-s)$</td>
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</tbody>
</table>

![Node Numbering Sequence Diagram](image)

<table>
<thead>
<tr>
<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>$H_1(r,s) = \left(\frac{5}{3} - 2(r+s)\right)\left[\frac{7}{3} - 4(r+s)\right]$</td>
</tr>
<tr>
<td>(1/6,5/12)</td>
<td>$H_2(r,s) = \left(\frac{1}{3} - 2r\right)\left(\frac{5}{3} - 4r\right)$</td>
</tr>
<tr>
<td>(1/6,2/3)</td>
<td>$H_3(r,s) = \left(\frac{1}{3} - 2s\right)\left(\frac{5}{3} - 4s\right)$</td>
</tr>
<tr>
<td>(2/3,1/6)</td>
<td>$H_4(r,s) = -4\left(\frac{1}{3} - 2r\right)\left(\frac{5}{3} - 2(r+s)\right)$</td>
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<tr>
<td>(5/12,5/12)</td>
<td>$H_5(r,s) = 4\left(\frac{1}{3} - 2r\right)\left(\frac{1}{3} - 2s\right)$</td>
</tr>
<tr>
<td>(5/12,1/6)</td>
<td>$H_6(r,s) = -4\left(\frac{5}{3} - 2(r+s)\right)\left(\frac{1}{3} - 2s\right)$</td>
</tr>
</tbody>
</table>
## 4-nodes Quadrilateral Linear Elements

**Node Numbering Sequence**

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<thead>
<tr>
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<th>(1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(-1,-1)</td>
<td>(1,-1)</td>
</tr>
</tbody>
</table>

**Shape Functions**

<table>
<thead>
<tr>
<th>Node</th>
<th>Shape Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>H₁(r,s)</td>
<td>( \frac{1}{4} (1-r)(1-s) )</td>
</tr>
<tr>
<td>H₂(r,s)</td>
<td>( \frac{1}{4} (1+r)(1-s) )</td>
</tr>
<tr>
<td>H₃(r,s)</td>
<td>( \frac{1}{4} (1+r)(1+s) )</td>
</tr>
<tr>
<td>H₄(r,s)</td>
<td>( \frac{1}{4} (1-r)(1+s) )</td>
</tr>
</tbody>
</table>

**Diagram:**

- Nodes labeled 1 through 4
- Nodes are labeled with coordinates
- Shape functions illustrate the mapping of the elements

**Diagram:**

- Nodes labeled with coordinates
- Shape functions illustrate the mapping of the elements

### 4-nodes Quadrilateral Linear Elements (continued)

<table>
<thead>
<tr>
<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
</thead>
</table>
| ![Node Numbering Diagram](image) | \[ \begin{align*}
H_1(r,s) &= \frac{1}{3}(1-r)(1-s) \\
H_2(r,s) &= \frac{1}{3}(1+r)(1-s) \\
H_3(r,s) &= \frac{1}{3}(1+r)(1+s) \\
H_4(r,s) &= \frac{1}{3}(1-r)(1+s)
\end{align*} \] |

| ![Node Numbering Diagram](image) | \[ \begin{align*}
H_1(r,s) &= \frac{4}{9}\left(\frac{1}{2} - r\right)\left(\frac{1}{2} - s\right) \\
H_2(r,s) &= \frac{4}{9}\left(\frac{1}{2} + r\right)\left(\frac{1}{2} - s\right) \\
H_3(r,s) &= \frac{4}{9}\left(\frac{1}{2} + r\right)\left(1 + s\right) \\
H_4(r,s) &= \frac{4}{9}\left(\frac{1}{2} - r\right)\left(1 + s\right)
\end{align*} \] |
8-nodes Quadrilateral Quadratic Elements

<table>
<thead>
<tr>
<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1,1))</td>
<td>((-2/3,0))</td>
</tr>
<tr>
<td>((-1,0))</td>
<td>((-2/3,2/3))</td>
</tr>
<tr>
<td>((-1,-1))</td>
<td>((-2/3,-2/3))</td>
</tr>
</tbody>
</table>

\begin{align*}
H_1(r,s) &= -\frac{1}{4}(1-r)(1-s)(1+r+s) \\
H_2(r,s) &= -\frac{1}{4}(1+r)(1-s)(1-r+s) \\
H_3(r,s) &= -\frac{1}{4}(1+r)(1+s)(1-r-s) \\
H_4(r,s) &= -\frac{1}{4}(1-r)(1+s)(1+r-s) \\
H_5(r,s) &= \frac{1}{2}(1-r^2)(1-s) \\
H_6(r,s) &= \frac{1}{2}(1+r)(1-s^2) \\
H_7(r,s) &= \frac{1}{2}(1-r^2)(1+s) \\
H_8(r,s) &= \frac{1}{2}(1-r)(1-s^2) \\
\end{align*}
### 8-nodes Quadrilateral Quadratic Elements (continued)

<table>
<thead>
<tr>
<th>Node Numbering Sequence</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(-1/3)</td>
<td></td>
</tr>
<tr>
<td>(-1,0)</td>
<td></td>
</tr>
<tr>
<td>(-1,-1)</td>
<td></td>
</tr>
<tr>
<td>(1/3)</td>
<td></td>
</tr>
<tr>
<td>(1,0)</td>
<td></td>
</tr>
<tr>
<td>(1,-1)</td>
<td></td>
</tr>
</tbody>
</table>

| $H_1(r,s)$ | $= \frac{3}{10} (1-r)(\frac{2}{3} - s) - \frac{1}{2} \ H_5 - \frac{2}{5} \ H_8$ |
| $H_2(r,s)$ | $= \frac{9}{25} (1+r)(\frac{2}{3} - s) - \frac{3}{5} \ H_5 - \frac{2}{5} \ H_6$ |
| $H_3(r,s)$ | $= \frac{9}{25} (1+r)(1+s) - \frac{3}{5} \ H_6 - \frac{3}{5} \ H_7$ |
| $H_4(r,s)$ | $= \frac{9}{25} (\frac{2}{3} - r)(1+s) - \frac{2}{5} \ H_7 - \frac{3}{5} \ H_8$ |
| $H_5(r,s)$ | $= \frac{9}{25} (\frac{2}{3} - r)(1+r)(\frac{2}{3} - s)$ |
| $H_6(r,s)$ | $= \frac{9}{10} (\frac{2}{3} - r)(1+r)(1+s)$ |
| $H_7(r,s)$ | $= \frac{9}{10} (\frac{2}{3} - r)(1+r)(1+s)$ |
| $H_8(r,s)$ | $= \frac{9}{10} (\frac{2}{3} - r)(\frac{2}{3} - s)(1+s)$ |
9-nodes Quadrilateral Quadratic Elements

<table>
<thead>
<tr>
<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1,1)</td>
<td>( H_1(r,s) = \frac{1}{4} rs(1-r)(1-s) )</td>
</tr>
<tr>
<td>(0,1)</td>
<td>( H_2(r,s) = -\frac{1}{4} rs(1+r)(1-s) )</td>
</tr>
<tr>
<td>(1,1)</td>
<td>( H_3(r,s) = \frac{1}{4} rs(1+r)(1+s) )</td>
</tr>
<tr>
<td>(-1,0)</td>
<td>( H_4(r,s) = -\frac{1}{4} rs(1-r)(1+s) )</td>
</tr>
<tr>
<td>(0,0)</td>
<td>( H_5(r,s) = -\frac{1}{2} s(1-r^2)(1-s) )</td>
</tr>
<tr>
<td>(1,0)</td>
<td>( H_6(r,s) = \frac{1}{2} r(1+r)(1-s^2) )</td>
</tr>
<tr>
<td>(-1,-1)</td>
<td>( H_7(r,s) = \frac{1}{2} s(1-r^2)(1+s) )</td>
</tr>
<tr>
<td>(0,-1)</td>
<td>( H_8(r,s) = -\frac{1}{2} r(1-r)(1-s^2) )</td>
</tr>
<tr>
<td>(1,-1)</td>
<td>( H_9(r,s) = (1-r^2)(1-s^2) )</td>
</tr>
</tbody>
</table>

For the second quadrilateral:

| (-2/3,2/3)             | \( H_1(r,s) = \frac{9}{16} \left( \frac{2}{3} - r \right) \left( \frac{2}{3} - s \right) - \frac{1}{2} H_5 - \frac{1}{2} H_8 - \frac{1}{4} H_9 \) |
| (-2/3,0)               | \( H_2(r,s) = \frac{9}{16} \left( \frac{2}{3} + r \right) \left( \frac{2}{3} - s \right) - \frac{1}{2} H_5 - \frac{1}{2} H_6 - \frac{1}{4} H_9 \) |
| (-2/3,-2/3)            | \( H_3(r,s) = \frac{9}{16} \left( \frac{2}{3} + r \right) \left( \frac{2}{3} + s \right) - \frac{1}{2} H_6 - \frac{1}{2} H_7 - \frac{1}{4} H_9 \) |
| (-1,1)                 | \( H_4(r,s) = \frac{9}{16} \left( \frac{2}{3} - r \right) \left( \frac{2}{3} + s \right) - \frac{1}{2} H_7 - \frac{1}{2} H_8 - \frac{1}{4} H_9 \) |
| (0,2/3)                | \( H_5(r,s) = \frac{27}{16} \left( 4 - r^2 \right) \left( \frac{2}{3} - s \right) - \frac{1}{2} H_9 \) |
| (1,1)                  | \( H_6(r,s) = \frac{27}{16} \left( 4 - r^2 \right) \left( \frac{4}{3} + s \right) - \frac{1}{2} H_9 \) |
| (2/3,2/3)              | \( H_7(r,s) = \frac{27}{16} \left( 4 - r^2 \right) \left( \frac{2}{3} + s \right) - \frac{1}{2} H_9 \) |
| (2/3,0)                | \( H_8(r,s) = \frac{27}{16} \left( 4 - r^2 \right) \left( \frac{4}{3} - s \right) - \frac{1}{2} H_9 \) |
| (2/3,-2/3)             | \( H_9(r,s) = \frac{27}{16} \left( 4 - r^2 \right) \left( 4 - s^2 \right) \) |
### 9-nodes Quadrilateral Quadratic Elements (continued)

<table>
<thead>
<tr>
<th>Node Numbering Sequence</th>
<th>Shape Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Node Diagram" /></td>
<td>$H_1(r,s) = \frac{3}{10}(1-r)(\frac{2}{3}-s) - \frac{1}{2}H_5 - \frac{2}{5}H_8 - \frac{1}{5}H_9$</td>
</tr>
<tr>
<td></td>
<td>$H_2(r,s) = \frac{3}{10}(1+r)(\frac{2}{3}-s) - \frac{1}{2}H_5 - \frac{2}{5}H_6 - \frac{1}{5}H_9$</td>
</tr>
<tr>
<td></td>
<td>$H_3(r,s) = \frac{3}{10}(1+r)(1+s) - \frac{1}{2}H_6 - \frac{3}{10}H_7 - \frac{3}{10}H_9$</td>
</tr>
<tr>
<td></td>
<td>$H_4(r,s) = \frac{3}{10}(1-r)(1+s) - \frac{1}{2}H_7 - \frac{3}{10}H_8 - \frac{3}{10}H_9$</td>
</tr>
<tr>
<td></td>
<td>$H_5(r,s) = \frac{3}{5}(1-r^2)(\frac{2}{3}-s) - \frac{2}{5}H_9$</td>
</tr>
<tr>
<td></td>
<td>$H_6(r,s) = \frac{3}{4}(1+r)(\frac{2}{3}-s)(1+s) - \frac{1}{2}H_9$</td>
</tr>
<tr>
<td></td>
<td>$H_7(r,s) = \frac{3}{5}(1-r^2)(1+s) - \frac{3}{5}H_9$</td>
</tr>
<tr>
<td></td>
<td>$H_8(r,s) = \frac{3}{4}(1-r)(\frac{2}{3}-s)(1+s) - \frac{1}{2}H_9$</td>
</tr>
<tr>
<td></td>
<td>$H_9(r,s) = \frac{3}{2}(1-r^2)(\frac{2}{3}-s)(1+s)$</td>
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</table>

![Node Diagram](image)