Least squares estimation for repeated surveys

Ibrahim Sorie Yansaneh
Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Statistics and Probability Commons

Recommended Citation
https://lib.dr.iastate.edu/rtd/10360
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
Least squares estimation for repeated surveys

Yansaneh, Ibrahim Sorie, Ph.D.

Iowa State University, 1992
Least squares estimation for repeated surveys

by

Ibrahim Sorie Yansaneh

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major: Statistics

Approved:

Signature was redacted for privacy.

\[ \checkmark \]
In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa
1992
DEDICATION

This thesis is affectionately dedicated to my parents:

ALHAJI MOHAMED YANSANEH

and

MAIMOUNATU YANSANEH,

to whom I owe more than I can possibly express, for providing me with the opportunity to have the education that they never had.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1. Repeated Surveys</td>
<td>1</td>
</tr>
<tr>
<td>1.2. Overview of the Thesis</td>
<td>6</td>
</tr>
<tr>
<td>2. LITERATURE REVIEW</td>
<td>9</td>
</tr>
<tr>
<td>2.1. The Classical Approach</td>
<td>9</td>
</tr>
<tr>
<td>2.2. The Time Series Approach</td>
<td>29</td>
</tr>
<tr>
<td>2.3. Best Linear Recursive Estimation</td>
<td>39</td>
</tr>
<tr>
<td>3. PRELIMINARY RESULTS ON CLASSICAL ESTIMATION FOR REPEATED SURVEYS ON</td>
<td>42</td>
</tr>
<tr>
<td>A FIXED POPULATION</td>
<td></td>
</tr>
<tr>
<td>3.1. Notation, Definitions and Assumptions</td>
<td>42</td>
</tr>
<tr>
<td>3.2. Estimation of Current Level and Change for Two—Period Surveys</td>
<td>44</td>
</tr>
<tr>
<td>3.3. Numerical Example</td>
<td>54</td>
</tr>
<tr>
<td>3.4. Extensions and Generalizations</td>
<td>59</td>
</tr>
<tr>
<td>4. LEAST SQUARES ESTIMATION FOR REPEATED SURVEYS</td>
<td>69</td>
</tr>
<tr>
<td>4.1. Introduction</td>
<td>69</td>
</tr>
<tr>
<td>4.2. Some Results in Linear Model Theory</td>
<td>69</td>
</tr>
<tr>
<td>4.3. Best Linear Unbiased Estimation</td>
<td>99</td>
</tr>
<tr>
<td>4.4. The Recursive Regression Estimation Procedure</td>
<td>106</td>
</tr>
<tr>
<td>4.5. State—Space Models and the Kalman Filter</td>
<td>132</td>
</tr>
</tbody>
</table>
5. ALTERNATIVE ESTIMATORS AND ROTATION DESIGNS FOR THE CURRENT POPULATION SURVEY

5.1. Introduction 140
5.2. The Components of Variance Model 142
5.3. Alternative Estimators of Current Level and Change 157
5.4. Time-in—sample Effects 180
5.5. Other Rotation Designs 199
5.6. Estimation of Unemployment Rate 215
5.7. Results and Discussion 217

BIBLIOGRAPHY 284
ACKNOWLEDGEMENTS 292

APPENDIX A GLOSSARY OF TERMS USED IN THE THESIS 294
APPENDIX B MEANS AND VARIANCES FOR THE TRANSFORMED OBSERVATIONS IN THE RECURSIVE REGRESSION PROCEDURE 295
APPENDIX C THE RECURSIVE PROCEDURE FOR COMPUTING LIMITING VARIANCES OF THE ESTIMATORS OF CURRENT LEVEL AND CHANGE 301
APPENDIX D RESULTS FOR THE BREAU AND ERNST (1983) MODEL 312
1. INTRODUCTION

1.1. Repeated Surveys

Like all other statistical designs, sample survey designs are constructed in order to elicit information about the characteristics of a population. Frequently, particularly in sociological and economic research, the population characteristics of interest change with time. Examples of such time-dependent characteristics are labor force characteristics, consumer income and expenditure, retail sales, crop yields, infant mortality, job vacancies, capital investment and stock levels in the stock market. Some of these characteristics have both fixed and time-dependent components. For instance, the frequency and length of duration of unemployment for an individual is a function of fixed characteristics such as age, sex and education, as well as time-dependent characteristics such as previous employment experience and job search activity.

If the principal objective of a survey is the provision of accurate information about the rate and nature of the dynamic social and economic processes underlying populations with time-dependent characteristics, then the usual single-occasion sample surveys or censuses taken over very long time intervals may be inadequate or inappropriate. A more informative procedure is to carry out surveys of one kind or another sequentially over relatively shorter intervals of time. This procedure is referred to as sampling on successive occasions and it falls under the broad category of repeated surveys.

The design and analysis of repeated surveys has received considerable attention in the recent literature. However, as Smith and Holt (1989) noted at the 1989 International Statistical Institute Session in Paris, researchers in such areas as
sociology and health sciences have conducted panel surveys and cohort studies for sometime. They cited Lazarsfield and Fiske (1938) as an example. An example in a health related area is the study by Garcia, Battese and Brewer (1975). Further evidence of the resurgence of interest in repeated surveys is the published proceedings of a conference on panel surveys edited by Kaspryzk, Duncan, Kalton and Singh (1989), sessions at the meetings of the International statistical Institute held in 1987 and 1989, and the Statistics Canada Symposium on the Analysis of Data in Time held in October, 1989.

Duncan and Kalton (1987) list the following objectives of surveys repeated over time:

1. To provide estimates of population parameters at distinct time points (current or longitudinal estimates).
2. To provide estimates of population parameters summed across time (estimates of sum).
3. To measure net change at the aggregate level.
4. To measure components of change, including gross change, change for an individual and variability for an individual.
5. To aggregate individual data over time.
6. To measure the frequency, timing and duration of events.
7. To accumulate information on rare populations.

Several of these objectives implicitly include the estimation of the parameters of subject matter models.

Duncan and Kalton (1987) also define four kinds of surveys:

1. Periodic Surveys, in which no attempt is made to guarantee that particular elements appear in more than one sample (Repeated Independent Sampling).
2. The Pure Panel survey, in which a fixed sample is used on all sampling occasions.

3. The Rotating Panel survey, in which a partial replacement of sampling units is made from occasion to occasion, according to a fixed pattern.

4. The Split Panel survey, which is a combination of a pure panel survey and a repeated survey or a rotating panel survey.

The relative merits of these types of repeated surveys depend on the objectives of the survey and the extent to which any relationship between the values of a particular characteristic observed on the same unit of the population on two different occasions can be used to improve the quality of estimation. For instance, suppose a survey is carried out on the same fixed population over several points in time. Let \( \theta_t \) and \( \theta_h \) be the level of some population characteristic at times \( t \) and \( h \) respectively. We assume without loss of generality that \( t > h \). If \( \hat{\theta}_t \) and \( \hat{\theta}_h \) denote the estimators of \( \theta_t \) and \( \theta_h \) respectively, then

\[
\text{Var}(\hat{\theta}_t + \hat{\theta}_h) = \text{Var}(\hat{\theta}_t) + \text{Var}(\hat{\theta}_h) + 2 \text{Cov}(\hat{\theta}_t, \hat{\theta}_h).
\] (1.1)

It is reasonable to assume that in virtually all repeated surveys, there is positive correlation between the measurements on the same unit on two successive occasions. Under this assumption, it can be seen from (1.1) that if it is desired to estimate the change in level of the characteristic of interest between times \( h \) and \( t \), then the greatest efficiency will be achieved when \( \hat{\theta}_t \) and \( \hat{\theta}_h \) are as highly correlated as possible. The best strategy is then to use the same sample on each occasion (that is, a pure panel survey), whereas the worst strategy is to use independent samples on each
occasion (that is, repeated independent sampling). However, if our primary objective is the estimation of the sum or average of the characteristic of interest over all occasions, then repeated independent sampling is the optimal strategy. (Cochran, 1977, p345). In practice however, we are usually interested in efficient estimation of current level, change in level, and average level over several periods. In this case, neither a panel survey, nor repeated independent sampling would be appropriate. Some form of overlap between the samples on successive occasions is usually desirable, and rotation sampling designs provide a method of accomplishing such an overlap.

Despite the apparent theoretical advantages of repeated surveys over single-occasion or cross-sectional surveys, their design, implementation and analysis involve special problems in addition to the usual problems associated with surveys in general. These include the following:

1. Deriving optimum benefits from repeated surveys requires maintaining field, processing, data management and estimation procedures that are consistent over time.

2. Resistance or lack of cooperation on the part of the sampling units may occur if they are "observed" indefinitely.

3. The composition of the target population may change from one sampling occasion to another, as a result of sampling units dying, being born to the target population, or simply moving from place to place or between different domains of the population over time.

4. In addition to the usual response error present in all surveys, repeated surveys encounter problems of "conditioning" associated with repeated interviews. Respondents may be influenced by information received at earlier interviews.
and this makes them progressively less representative of the population as time
proceeds.

Rotation sampling designs are constructed in order to minimize some of the
drawbacks associated with repeated surveys. Among the advantages of rotation
sampling designs over the other types of repeated surveys are (1) the reduction of
rotation group bias and respondent resistance or non—cooperation relative to panel
surveys, (2) the reduction of sampling costs relative to repeated independent sampling
and (3) greater precision in the estimation of current level and change, relative to
repeated independent sampling. Some examples of rotation sampling designs are the
Current Population Survey conducted by the United States Bureau of Census for the
United States Bureau of Labor Statistics and the Canadian Labor Force Survey,
conducted by Statistics Canada. Other examples not pertaining to the labor force are
the National Crime Survey and the Retail Trade Survey. The two labor force surveys
cited above will be described in detail in Chapter 5.

In this thesis, we address the problem of least squares estimation of selected
parameters in repeated surveys. The theory of least squares is concerned with the
estimation of parameters in a linear model. The parameters of interest are current
level, change in level and average level over a fixed number of periods. Least squares
estimation is a fairly old statistical method, which still attracts considerable interest in
statistics and other scientific disciplines mainly because of its simplicity and practical
significance. An attempt is made here to exploit this simplicity in order to construct
various estimation schemes for repeated surveys. Our strategy throughout is to
formulate the various estimation problems associated with repeated surveys in a
general linear model framework. We then construct our estimation procedures by
extending and generalizing well known techniques from the classical theory of multivariate linear models.

1.2. Overview of The Thesis

In Chapter 2, we briefly review the literature on the analysis of repeated surveys, focusing on those aspects that are pertinent to the present study. The main focus of our review is on the various approaches to the analysis of repeated surveys, namely, the classical approach, the time series approach and best linear recursive estimation. An attempt is made to present the results in chronological order.

Chapter 3 is a chapter of preliminary results. It is a presentation of the complex issues associated with estimation for repeated surveys at a simplified technical level and serves as an introduction to the material presented in subsequent chapters. We start with a brief review of minimum variance estimation of current level and change for two–period surveys. The estimation procedures are compared via a numerical example. The results are then extended to the analysis of repeated surveys over more than two periods.

In Chapter 4, we develop various least squares estimation procedures for repeated surveys. We start by presenting several results from the theory of linear models. These results are then used in the construction of the estimation procedures, which include best linear unbiased estimation based on various periods of data and the recursive regression estimator. The recursive regression estimator is essentially the best linear unbiased estimator based on an infinite number of periods. As its name implies, it is a recursive estimator constructed in order to circumvent the problem of computational complexity associated with best linear unbiased estimation, while at the
same time, producing minimum variance unbiased estimators. The estimator is expressible as a linear combination of the observations obtained at the current level and an appropriate set of previous estimates. Its implementation can be considered as a special case of the Kalman filtering technique. It is shown that the covariance matrix of the recursive least squares estimators converges to a positive definite matrix as the number of periods increases. Several other theoretical results on least squares estimation for repeated surveys are derived. These results are applicable to a wide range of estimation procedures and rotation designs. Some of these applications are illustrated in Chapter 5. Finally, we discuss state space models and Kalman filtering, focusing on the application of the Kalman filter technique to estimation for repeated surveys using time series methods.

The final chapter (Chapter 5) is devoted to a discussion of a wide range of applications of the theoretical results from the preceding chapters to the analysis of data from the Current Population Survey. Our discussion is based on a comparison of the alternative estimation procedures for the labor force characteristics based on the Current Population Survey. Estimators include the present composite estimator, the first order composite estimator, best linear unbiased estimators based on various periods of data and the recursive regression estimator. The comparison is done in terms of estimation expression and the variance of the estimators. The basic data used in the comparison consists of the elementary estimators of the characteristics of interest associated with different rotation groups. An estimated covariance structure of the data based on a components of variance model is used to compare alternative estimators and rotation designs. We also address the issue of revision of previous estimators when additional observations become available. We illustrate the fact that the revision of previous estimates provides optimal estimates of level and change which
are internally consistent. However, the increase in precision of estimation has to be tempered with the fact that the revision of previous estimates is not desirable in many practical situations. In recognition of the drawbacks associated with the revision of previous estimates, we describe a simple procedure for computing unrevised estimates whose variances are very close to those of the optimal estimator of change. Another issue addressed in Chapter 5 is that of rotation-group or time-in-sample effects. We examine the consequences of the time-in-sample effects on the alternative estimators under assumptions of constant time-in-sample effects and under time-varying time-in-sample effects.
2. LITERATURE REVIEW

The interest in the design and analysis of repeated surveys over the past couple of decades has generated a great deal of literature on the subject. In this chapter, we shall briefly review some of the past results that are relevant to the present study. Our review will be divided into three parts. The first two parts correspond to the two methods of analysis of data from repeated surveys, namely the classical approach and the time series approach. The third part is devoted to best linear recursive estimation.

2.1. The Classical Approach

In the classical approach to the analysis of repeated surveys, the population characteristics of interest are considered as fixed parameters. Therefore, for the estimation of the current level, data from previous periods are used only when the surveys are partially overlapping.

Most of the literature on the classical approach to the analysis of repeated surveys makes extensive use of generalized least squares procedures. The basic objective in this procedure is the construction of minimum variance weights for a set of unbiased estimators available for each sampling occasion in the survey.

Jessen (1942), who was influenced by the pioneering work of Cochran (1942), considered the special case of sampling on two occasions with unequal numbers of observations and studied the optimal allocation of units to overlapping and non—overlapping sample groups.

What is considered by many to be the classical paper on this subject is due to Patterson (1950), who considered sampling on T occasions (T > 2) under several
schemes of partial replacement of units. The simplest such sampling plan requires the replacement of a proportion $\mu$ of sampling units on each successive sampling occasion. Also, Patterson (1950) assumed that if, for given $i$, $\theta_{ti}$ is the value of the characteristic of interest on the $i$-th population unit at time $t$, and $\theta_t$ is the corresponding finite population mean, then the differences $\theta_{ti} - \theta_t$, $t = 1, 2, \ldots$, followed a first order autoregressive process. Under the resulting error model, he developed optimal estimators of the fixed $\theta_t$-values (current level) and of the differences $\theta_t - \theta_{t-1}$ (one period change). The main results of this paper can be summarized as follows. Assume the model

$$y_{ti} - \theta_t = \rho(y_{t-1,i} - \theta_{t-1}) + \eta_{ti} \tag{2.1}$$

for all $t$, where and $\{\eta_{ti}\}$ is a sequence of uncorrelated random variables such that:

$$\mathbb{E}\{\eta_{ti}\} = 0$$
$$\text{Var}\{\eta_{ti}\} = (1 - \rho^2)\sigma^2$$
$$\sigma^2 = \mathbb{E}\{y_{ti} - \theta_t\}^2$$

Suppose that on each of $T$ successive occasions which are labelled in order of time: $1, 2, \ldots, T$, a sample of $n$ sampling units is available. Of those included on the $(t-1)$-th occasion, $\lambda n$ are retained for the $t$-th occasion and $\mu n$ are replaced ($\mu = 1 - \lambda; 1 \leq t \leq T$). Furthermore, let

$$\overline{y}_{t-1,m}$$

be the mean of the values on the $(t-1)$-th occasion associated with the units observed on the $(t-1)$-th and $t$-th occasions,
\( \bar{y}_{tm} \) be the mean on the \( t \)-th occasion, of the units observed on the 
\((t-1)\)-th and \( t \)-th occasions.

\( \bar{y}_{t-1,u} \) be the mean on the \((t-1)\)-th occasion of the observations on the \( \mu \) 
sampling units which are not common to both occasions, and

\( \bar{y}_{tu} \) be the means on the \( t \)-th occasion of the observations on the \( \mu \) 
sampling units which are not common to both occasions.

Let \( \hat{\theta}_t \) be a linear unbiased estimator of \( \theta_t \), the population mean at time \( t \). Thus we 
can write \( \hat{\theta}_t \) as:

\[
\hat{\theta}_t = \sum_{i=1}^{n} \sum_{j=1}^{T} w_{ij} y_{ij}
\]

where

\[
\sum_{i=1}^{n} w_{ij} = \begin{cases} 
1 & \text{if } j = t, \\
0 & \text{otherwise}.
\end{cases}
\]

Therefore, the minimum variance linear unbiased estimator of \( \theta_t \) is obtained by 
minimizing the Lagrangian function

\[
\text{Var}\{ \sum_{i=1}^{n} \sum_{j=1}^{T} w_{ij} y_{ij} \} - 2 \sum_{j=1}^{T} \gamma_{tj} \sum_{i=1}^{n} w_{ij}
\]

where \( \gamma_{tj} \) \((1 \leq t, j \leq T)\) are the undetermined Lagrange multipliers. The minimization 
leads to the following set of equations:

\[
\text{Cov}\{ y_{ij}, \hat{\theta}_t \} = \gamma_{tj} \quad \text{for all } i, j.
\]  
(2.2)
Patterson showed that the minimum variance linear unbiased estimator of the population mean at time \( t \) under model (2.1) is of the form:

\[
\hat{\theta}_t = (1 - \varphi_t)\{\bar{y}_{tm} + \rho(\hat{\theta}_{t-1} - \bar{y}_{t-1,m})\} + \varphi_t \bar{y}_{tu}
\]

where \( \varphi_t \) satisfies the recursive relation

\[
1 - \varphi_t = \{1 - (\mu - \lambda) \rho^2 - \lambda \rho^2(1 - \varphi_{t-1})\}^{-1} \lambda
\]

In particular,

\[
1 - \varphi_1 = \lambda
\]

and

\[
\varphi_2 = \{1 - \mu^2 \rho^2\}^{-1} \mu(1 - \mu \rho^2)
\]

Also, the variance of \( \hat{\theta}_t \) is given by

\[
\text{Var}(\hat{\theta}_t) = \text{Cov}(\bar{y}_{tu}, \hat{\theta}_t) = (\mu \nu)^{-1} \varphi_t \sigma^2.
\]

The minimum variance linear unbiased estimator of the change \( \theta_t - \theta_{t-1} \) for the \((t-1)\)-th and \( t \)-th occasions is

\[
\Delta \hat{\theta}_t = (1 + \rho \varphi_{t-1}) \hat{\theta}_t - \rho \varphi_{t-1} \bar{y}_{tu} - \hat{\theta}_{t-1}
\]
and the variance of $\Delta \theta_t$ is

$$\text{Var}(\Delta \theta_t) = (\mu n)^{-1} + (\mu n)^{-2}\left\{1 - \rho^2 \varphi_{t-1}(1 - \varphi_t) - 2\rho(1 - \varphi_t)\right\}$$ \hspace{1cm} (2.6)$$

Patterson further considered the optimal estimation of $\theta_t$ under generalizations of the partial replacement plan to the cases of nonconstant sample sizes, nonconstant replacement proportions and nonconstant sampling error variances. For instance, if

- $n_t$ is the sample size at time $t$,
- $n'_t$ ($\leq n_t$) is the number of a sampling units on the $t$th occasion, which are common with the $(t-1)$th occasion,
- $n''_t$ is the number of new sampling units on the $t$th occasion ($n''_t = n_t - n'_t$),
- $y_{ti}$ ($1 \leq i \leq n_t$) is the $i$th sampled value at time $t$,
- $\bar{y}_{t-1,m}$ is the mean of the $n'_t$ matched units on the $(t-1)$th occasion,
- $\bar{y}_{tm}$ is the mean of the $n'_t$ matched units on the $t$th occasion,
- $\bar{y}_{t-1,u}$ is the mean of the $n_{t-1} - n'_t$ unmatched units on the $(t-1)$th occasion,
- $\bar{y}_{tu}$ is the mean of the $n''_t$ new units,

then the quantities corresponding to (2.3) - (2.4) are

$$\hat{\theta}_t = (1 - \varphi_t)\left\{\bar{y}_{tm} + \rho(\hat{\theta}_{t-1} - \bar{y}_{t-1,m})\right\} + \varphi'_t \bar{y}_{tu}$$ \hspace{1cm} (2.7)$$

where

$$1 - \varphi'_t = \{n_t n''_t - \rho^2 n''_t (n''_{t-1} - \varphi_{t-1} n'_t)\}^{-1} n'_t n''_{t-1}$$
where \( n''_{t-1} \) is not equal to 0, and

\[
\text{Var}(\hat{\theta}_t) = n''_{t-1} \varphi_t' \sigma^2
\]  \hspace{1cm} (2.8)

unless \( n''_t = 0 \). If \( n''_t = 0 \), then,

\[
\text{Var}(\hat{\theta}_t) = \{n^{-1}_t(1 - \rho^2) + n''_{t-1} \rho^2 \varphi_{t-1}'\}
\]  \hspace{1cm} (2.9)

where \( n''_{t-1} \) is not equal to 0. Patterson concluded with brief discussions of optimal sample size selection, optimal number of matching units between two successive occasions, and of non-autoregressive errors.

Eckler (1955) extended Patterson's methods to two and three level rotation sampling designs. For the two level rotation design with fixed sample size on each sampling occasion, he derived recursive formulas for the minimum variance linear unbiased estimator of \( \theta_t \) as a function of the sample averages at times \( t \) and \( t-1 \) as well as the minimum variance linear unbiased estimator of \( \theta_{t-1} \). The estimator of \( \theta_t \) is of the form

\[
\hat{\theta}_t = \bar{y}_{t-1,t} - a_t \bar{y}_{t-1,t-1} + a_t \hat{\theta}_{t-1}
\]  \hspace{1cm} (2.10)

where

\[
\hat{\theta}_{t-1} = \bar{y}_{t-2,t-1} - a_{t-1} \bar{y}_{t-2,t-2} + a_{t-1} \hat{\theta}_{t-2}
\]

and, for any \( h, t, h > t \), \( \bar{y}_{h,t} \) is the sample mean for the \( h \)-th occasion based on the sample drawn on the \( t \)-th occasion. We assume that a sample of \( n \) observations are drawn on each occasion and \( \text{Var}(\bar{y}_{t,t}) = n^{-1} \sigma^2 \) for all \( t \) and that \( \rho \) is the
correlation coefficient between the responses from the same unit on two successive occasions.

The condition for \( \hat{\theta}_t \) to be minimum variance is

\[
\text{Cov}\{\bar{y}_{t-1,t}, \hat{\theta}_t\} = \text{Cov}\{\bar{y}_{t-1,t-1}, \hat{\theta}_t\} 
\]

(2.11)

But

\[
\text{Cov}\{\bar{y}_{t-1,t}, \hat{\theta}_t\} = n^{-1}(\rho - a_t)\sigma^2
\]

and

\[
\text{Cov}\{\bar{y}_{t-1,t-1}, \hat{\theta}_t\} = a_t \text{Cov}\{\bar{y}_{t-1,t-1}, \hat{\theta}_{t-1}\}
\]

\[
= n^{-1}a_t(1-\rho a_{t-1})\sigma^2
\]

Hence from (2.11),

\[
a_t(1-\rho a_{t-1}) = \rho - a_t
\]

which gives

\[
a_t = (2 - \rho a_{t-1})^{-1} \rho \quad \text{for } t = 2,3,\ldots
\]

\[
= 0 \quad \text{for } t = 1
\]

Similarly, for the three-level rotation sampling scheme, Eckler derived expressions for the minimum variance linear unbiased estimator of \( \theta_t \) as a function of the sample averages and the minimum variance linear unbiased estimator of \( \theta_{t-1} \).

Gurney and Daly (1965) generalized the theory of minimum variance unbiased estimation for repeated surveys by using the concept of elementary estimates. An elementary estimate is an estimate which does not make use of the survey data for any
time period except that period to which the estimate refers. Elementary estimates can be incorporated into a linear model framework which uses the correlation structure between the elementary estimates in order to produce minimum variance linear unbiased estimators. Gurney and Daly solved the general problem of minimum variance linear unbiased estimation using Hilbert space theory.

The articles reviewed so far have all assumed an infinite population model in which the portion of the sample introduced at a given time is independent of the sample introduced at any previous time. Rao and Graham (1964) deviated from this assumption and applied concepts of finite populations to the composite estimation of the current population mean \( \theta_t \) and of changes in level between successive occasions \( \theta_t - \theta_{t-1} \) in a general one—level rotation sampling scheme, which allows units, which had been eliminated on previous occasions, to re—enter the sample.

Let \( N \) and \( n \) be the population and sample sizes respectively (both assumed to be the same for all sampling occasions). Also, let \( N \) and \( n \) be multiples of \( r \). A group of \( r \) units remains in the sample for \( h \) occasions \( (n = hr) \), leaves the sample for \( m \) occasions and re—enters it for another \( h \) occasions, and so on. If a unit re—enters the sample after dropping out on \( k - 1 \) previous occasions, it is said to be in the \( k - th \) cycle. Rao and Graham (1964) derived composite estimators of the current population mean \( \theta_t \) and for change \( \theta_t - \theta_{t-1} \) between two successive occasions. The estimator of \( \theta_t \) is

\[
\hat{\theta}_t = Q(\hat{\theta}_{t-1} + \bar{y}_{tm} - \bar{y}_{t-1,m}) + (1-Q)\bar{y}_t
\]

(2.12)

where \( 0 \leq Q < 1 \), \( \hat{\theta}_{t-1} \) is the composite estimator at time \( t-1 \), \( \bar{y}_t \) is the sample mean at time \( t \), and \( \bar{y}_{tm} \) and \( \bar{y}_{t-1,m} \) are defined following model (2.1). The
The estimator of \( \theta_t - \theta_{t-1} \) is

\[ \Delta \hat{\theta}_t = \hat{\theta}_t - \hat{\theta}_{t-1} \]  

(2.13)

They then derived explicit expressions for the variances of these estimators under two different covariance structures:

I. \( \text{Cov}(y_{ti}, y_{t+\ell,i}) = \sigma^2 \) when \( \ell = 0 \)

\[ = \rho |\ell| \sigma^2 \]  
when \( \ell > 0 \)

\( t = 1, 2, \ldots; i = 1, 2, \ldots, N. \)

II \( \text{Cov}(y_{ti}, y_{t+\ell,i}) = \sigma^2 \) when \( \ell = 0 \)

\[ = \{\rho - (|\ell| - 1)d\} \sigma^2 \]  
when \(|\ell| - 1)d \leq \rho, \ell > 0 \)

\[ = 0 \]  
when \(|\ell| - 1)d > \rho, \ell > 0 \)

where \( d \) is a small positive number.

They then carried out an empirical investigation of the gain in efficiency of the composite estimators relative to the simple estimators. The optimum \( Q \) was defined to be the value which gives the maximum percent gain in efficiency. The percent gain in efficiency and the optimum \( Q \) were tabulated for selected values of \( (r, \rho, m) \) and \( ((d, \rho, r), m = \omega) \). Among other thing, this investigation yielded the following results:
1. For model (I), the value of the variance of the composite estimator of $\theta_t$ is virtually the same for moderate $m$ as for $m = \infty$.

2. The optimum value of $Q$ is independent of $m$.

3. The optimum value of $r$ in the estimation of $\theta_t$ is 2, but this may not be the case if we are interested in estimating the change $\theta_t - \theta_{t-1}$ simultaneously.

4. For $m = \infty$, the optimum value of $Q$ for model (I) is either equal to or differs only by 0.1 from the optimum value of $Q$ for model (II), for various values of $d$.

The problem of estimating $\theta_t$ and $\theta_t - \theta_{t-1}$ for two-level rotation schemes under a finite population model was also considered by Wolter (1979). He provided a general estimation theory for two-level rotation schemes with an arbitrary number (>1) of rotation panels. For instance, he considered a survey in which there are three monthly panels that continually rotate in a three-month cycle. That is, one panel reports in January, April, July and October of each year; the second in February, May, August and November; and the third in March, June, September and December. His discussion focussed on the three parameters which he considered to be of great practical interest, viz: monthly total, month-to-month trend and year-to-year trend. Note that "month" and "year" simply mean regular time periods of the survey.

He first considered minimum variance estimation of these parameters by reducing the problem to a special case of the Gauss-Markov model:

$$y = X \beta + \epsilon$$  \hspace{1cm} (2.14)

where $X$ is a $2p \times (p+1)$ "design" matrix,

$$\beta = (\theta_t, \theta_{t-1}, \ldots, \theta_{t-p})'$$
denotes the \((p+1) \times 1\) vector of unknown monthly totals, and the error vector satisfies

\[
E(\epsilon) = 0, \quad \text{and} \quad E(\epsilon \epsilon') = V.
\]

The matrix \(V\) is the covariance matrix of the vector of simple estimators. It is, in general, not diagonal and is assumed to be positive definite. By the Aitken generalization of the Gauss–Markov Theorem, the minimum variance linear unbiased estimator of the vector \(\beta\) of totals is given by:

\[
\hat{\beta} = (X' \ V^{-1} \ X)^{-1} \ X \ V^{-1} \ y
\]

and the covariance matrix of \(\hat{\beta}\) is given by:

\[
\text{Var}\{\hat{\beta}\} = (X' \ V^{-1} \ X)^{-1}.
\]

The minimum variance linear unbiased estimator of any linear combination \(\lambda' \beta\) of monthly totals is \(\lambda' \hat{\beta}\). In particular, if \(\hat{\theta}_t\) and \(\Delta \hat{\theta}_t\) denote respectively the minimum variance linear unbiased estimators of \(\theta_t\) and \(\theta_t - \theta_{t-1}\), then we may write

\[
\hat{\theta}_t = \lambda_1' \ y \quad \text{and} \quad \Delta \hat{\theta}_t = (\lambda_1' - \lambda_2') \ y
\]

where \(\lambda_i'\) is the \(i\)-th row of the weight matrix \((X' \ V^{-1} \ X)^{-1} \ X' \ V^{-1}\). The variances of \(\hat{\theta}_t\) and \(\Delta \hat{\theta}_t\) are \(\lambda' \ V \ \lambda\) and \(\ell' \ V \ \ell\) respectively, where \(\ell = \lambda_1 - \lambda_2\)
In many applied surveys, it is extremely difficult, if not impossible, to compute the minimum variance linear unbiased estimators for all sampling occasions. The reasons for this are outlined by Fuller (1990). They include the following:

1. It is not possible to incorporate all data from the surveys of preceding times into the framework of minimum variance linear unbiased estimation of the population characteristic \( \theta_t \) for the current time because the number of variables exceeds the number of observations.

2. Minimum variance linear unbiased estimation of change inherently involves the revision of previous estimates of level when more data become available. This gives rise to problems of data storage, emanating from the fact that the organizations which keep and publish the longitudinal estimates might be restricted in the number of times they revise previous estimates.

To circumvent the computational complexity of the minimum variance linear unbiased estimators, Wolter (1979) used the procedure of composite estimation as an approximation, and found the resulting estimators to be computationally and statistically efficient. The efficiency of composite estimators was also demonstrated by Gurney and Daly (1965).

Let \( \theta_t \) denote the population total of the characteristic of interest for month \( t \), and let \( y_{t,t} \) and \( y_{t-1,t} \) denote unbiased estimators of \( \theta_t \) and \( \theta_{t-1} \) respectively, obtained from the panel reporting in month \( t \). Wolter considered two composite estimators: The preliminary composite estimator defined by the recursive relation:

\[
\hat{\theta}_{t,p} = (1-\varphi)y_{t,t} + \varphi(\hat{\theta}_{t-1,p} + y_{t,t} - y_{t-1,t})
\]  

(2.16)
and the final composite estimator defined by:

\[
\hat{\theta}_{t-1,F} = (1 - \psi)y_{t-1,t} + \psi\hat{\theta}_{t-1,P}
\]  

(2.17)

where \(\psi \in (0, 1)\) and \(\phi \in (0, 1)\) are obtained by minimizing the variances of \(\hat{\theta}_{t-1,F}\) and \(\hat{\theta}_{t,P}\) respectively. The optimal value of \(\phi\) in (2.16) is

\[
\phi_{opt} = \rho^{-1}[1 - \rho^2]^{1/2}
\]

and the variance of \(\hat{\theta}_{t,P}\) is then equal to \((1 - \rho^2)^{1/2}\sigma^2\). The composite estimators of the month-to-month trend and the year-to-year trend are defined respectively as

\[
\Delta \hat{\theta}_t = \hat{\theta}_{t,P} - \hat{\theta}_{t-1,F}
\]

\[
\delta \hat{\theta}_t = \hat{\theta}_{t,P} - \hat{\theta}_{t-12,F}
\]

He then derived expressions for the approximate variances of these estimators under the following assumptions:

(i) Simple estimates derived from different panels are uncorrelated.

(ii) \(\text{Var}(y_{t,t}) = \text{Var}(y_{t-1,t}) = \sigma^2\), and \(\text{Cov}(y_{t,t}, y_{t-1,t}) = \rho \sigma^2\) for \(t = s-p+1, ..., s\), where \(s\) denotes the most recent month for which data has been collected and \(p\) is the number of months the survey has been in operation.

(iii) The simple estimators are covariance stationary in the sense that:

\[
\text{Cov}(y_{t,t}, y_{t-r,t}) = \text{Cov}(y_{t-1,t}, y_{t-1-r,t})
\]

(2.18)
where $r$ is an integer multiple of the number of rotating panels. For the case of three rotating panels, we have

\[
\text{Var}(\hat{\theta}_{t,p}) = (1-\varphi^2)^{-1}(1-2\varphi \varphi + \varphi^2)\sigma^2 + 2(1+\varphi)^{-1}(1-\varphi)f_3(\varphi)\sigma^2
\]

\[
\text{Var}(\hat{\theta}_{t-1,F}) = (1-\varphi^2)\sigma^2 + \alpha^2\text{Var}(\hat{\theta}_{t-1,p}) + 2\psi^{-1}(1-\psi)(1-\varphi)f_3(\varphi)\sigma^2
\]

where $f_3(\varphi) = \varphi^3 \rho_3 + \varphi^6 \rho_6 + \varphi^9 \rho_9 + \varphi^{12} \rho_{12}$.

Also,

\[
\text{Var}(\Delta \hat{\theta}_t) = \text{Var}(\hat{\theta}_{t,p}) + \text{Var}(\hat{\theta}_{t-1,F}) - 2\text{Cov}(\hat{\theta}_{t,p}, \hat{\theta}_{t-1,F})
\]

where

\[
\text{Cov}(\hat{\theta}_{t,p}, \hat{\theta}_{t-1,F}) = (1-\psi)\{\sigma^2(\rho-\varphi) + \sigma^2(1-\varphi)f_3(\varphi)\} + \psi\text{Cov}(\hat{\theta}_{t,p}, \hat{\theta}_{t-1,p})
\]

and

\[
\text{Cov}(\hat{\theta}_{t,p}, \hat{\theta}_{t-1,p}) = \varphi^{-1}(1-\varphi)^2f_3(\varphi) + \varphi \text{Var}(\hat{\theta}_{t-1,p}).
\]

The expression for the variance of the estimator of the year-to-year change is even more complicated and is omitted.
An empirical study was conducted to determine the optimum values of $\psi$ and $\varphi$ for particular correlation patterns. The study showed that the optimum coefficients of the estimator of the current population total may not be the optimum coefficients of the estimators of the month—month change or the year—to—year change. This suggests that different coefficients should be used for each of these estimators whenever the survey conditions permit this. Wolter concluded with an illustration of the application of this theory in the planning of the Census Bureau's Retail Trade Survey.

Note that if $\bar{y}_{t,t}$ is the sample mean of the elementary estimates for month $t$ and $\delta_{t,t-1}$ is the mean difference of the elementary estimates common to both months $t$ and $t-1$, then, in general, the composite estimator in (2.16) can be written as

$$\hat{\theta}_t = K\hat{\theta}_{t-1} + (\bar{y}_{t,t} - K\bar{y}_{t-1,t})$$

$$= K(\hat{\theta}_{t-1} + \delta_{t,t-1}) + (1-K)\bar{y}_{t,t}$$

where $0 \leq K < 1$. Gurney and Daly (1965) showed the simple composite estimator given by (2.19) can be improved by modifying equation (2.19) so that the linear combination of observations has more weight for the units entering the sample for the first time than those which have been in the sample on previous occasions. Assuming that at time $t$, there are $\ell$ elementary estimates and $m$ of these are obtained from units entering the sample for the first time. Gurney and Daly showed that if more weight is assigned to the $m$ elementary estimates obtained from first time—in—sample units than the remaining $\ell - m$ elementary estimates, then the variance of the resulting estimator becomes very close to that of the minimum variance linear
unbiased estimator. The general form of this estimator, known as the AK composite estimator, is

\[
\hat{\theta}_{t,c} = K(\hat{\theta}_{t-1,c} + \delta_{t-1}) + \ell^{-1}\{(1 - K + A) \sum_{j=1}^{m} y_{i_j,t} \\
+ [1 - K - (1-m)^{-1} \ell A] \sum_{j=m+1}^{\ell} y_{i_j,t}\} \tag{2.20}
\]

where \( K \) and \( A \) are constants, \( 0 \leq K \leq 1 \), \((i_1, t), \ldots, (i_m, t)\) denote the first-time-in-sample panels at time \( t \) and \((i_m+1, t), \ldots, (i_{\ell}, t)\) denote the panels which have been in the sample prior to time \( t \).

Huang and Ernst (1981) studied the variance and bias aspects of the AK composite estimator for the special case of the current Population Survey. They investigated these properties using two intermittent rotation designs, namely the 4–8–4 rotation pattern and the 3–9–3 rotation pattern, for selected labor force characteristics. Assuming constant variance and covariance for all observations at all time periods, Huang and Ernst concluded that for all characteristics studied, the optimum AK composite estimator was more efficient than the simple composite estimator for monthly level, month-to-month change and annual average for both rotation designs. Furthermore, under the 4–8–4 rotation design, if the rotation group bias (see Section 5.4) can be assumed to be constant over time and that the sum of the rotation group effects is zero, the bias of the AK composite estimator is smaller than the bias of the simple composite estimator.

Kumar and Lee (1983) conducted a study similar to Huang and Ernst (1981) for the Canadian Labor Force Survey. The optimal AK composite estimator, that is, the
an estimator having minimum variance among the class of estimators defined by (2.20), was found to be more efficient in terms of mean square error and had smaller bias than the corresponding optimal simple composite estimator. However, for the labor force characteristics with significant rotation group bias, both the optimal AK composite estimator and the optimal simple composite estimator had a relative efficiency (measured as ratios of mean square error) smaller than 110% with respect to the simple poststratified estimator. Kumar and Lee subsequently obtained better composite estimators by minimizing the mean square error instead of the variance. The resulting estimator had gains in efficiency ranging from 0 to 22% over the simple poststratified estimator.

Tam (1987) also considered finite population inference for repeated surveys under a superpopulation model. He presented a model under which sampling errors may be non-trivially correlated with true finite population means. He used a state-space representation of this model to derive maximum likelihood estimators of some superpopulation parameters. He then derived predictors of the finite population totals from the model with estimated superpopulation parameters. In his concluding remarks, he noted the potential effects of model misspecification and short runs of surveys on the proposed estimation and prediction procedures.

Yansaneh (1990) considered the estimation of the parameters of a finite stratified population using a two-stage rotating sampling design. Assuming an exchangeable superpopulation model, he derived ratio and regression estimators of selected population characteristics of interest and showed them to be more efficient than their longitudinal counterparts. He also presented a computer program for the computation of the various estimators and their estimated variances.
Fuller (1990) considered the analysis of repeated surveys in which a portion of the units are observed at more than one time point and some units are not observed at some time points. He presented an extensive discussion of a broad range of topics relevant to the analysis of repeated surveys, including a review of least squares estimation for repeated surveys, a discussion of estimation procedures in which existing estimates are not revised when new data become available and techniques for the estimation of longitudinal parameters, such as gross change tables. Some of the results were applied to estimation for a large scale repeated survey of land use conducted by Iowa State University and the United States Soil Conservation Service. Furthermore, he illustrated the effects of measurement error on gross change and showed that survey designs constructed to enable estimation of the parameters of the measurement error process can be very efficient.

Cantwell (1990) derived explicit expressions for the variances of a general class of composite estimators of current level, change in level and average level over time for one-level as well as multi-level rotation designs. Let $y_{ti}$ be the $i$-th elementary estimator of $\theta_t$, $i=1,2, ..., m$. The generalized composite estimator of current level defined by Breau and Ernst (1983) is given by

$$\hat{\theta}_t,G = \sum_{i=1}^{m} a_i y_{ti} - k \sum_{i=1}^{m} b_i y_{t-1,i} + k \hat{\theta}_{t-1,G}$$  \hspace{1cm} (2.21)$$

where $0 \leq k < 1$, and the $a_i$'s and $b_i$'s may take any values, including negative ones, subject to:

$$\sum_{i=1}^{m} a_i = 1, \text{ and } \sum_{i=1}^{m} b_i = 1.$$
If \( a, b, y_{t-1}, \) and \( y_t \) are \( M \times 1 \) vectors defined by \( a' = (a_1, ..., a_m), b' = (b_1, ..., b_m), \) \( y_t' = (y_{t1}, ..., y_{tm}) \) and \( y_{t-1}' = (y_{t-1,1}, ..., y_{t-1,m}) \), then (2.21) can be written as

\[
\hat{\theta}_{t,G} = a' y_t - k b' y_{t-1} + k \theta_{t-1,G}.
\]  

(2.22)

Cantwell assumed the following covariance structure for the data.

\[
\text{Var}\{y_{ti}\} = \sigma^2 \quad \text{for all } t \text{ and } i,
\]

\[
\text{Cov}\{y_{ti}, y_{hj}\} = \rho_{|t-h|} \sigma^2, \quad \text{if } y_{ti}, y_{hj} \text{ refer to the same rotation group } |t-h| \text{ months apart; or } 0 \text{ otherwise. Take } \rho_0 = 1.
\]

Cantwell then proved that the variance of the generalized composite estimator of the current level is

\[
\text{Var}\{\hat{\theta}_{t,G}\} = \sigma^2 (1-k^2)^{-1} \{a' a + k^2 b' (b - 2a) + 2(a - k^2 b)' Q (a - b)\}.
\]  

(2.23)

where \( Q \) is the \( M \times M \) matrix whose entries are

\[
Q_{ij} = k^{i-j} \rho_{i-j} \quad \text{if } 1 \leq j \leq i \leq M
\]

\[
= 0 \quad \text{otherwise.}
\]

and the variances of the generalized composite estimator of month-to-month change is
\[
\text{Var}\{\hat{\theta}_{t,G} - \hat{\theta}_{t-1,G}\} = 2a'(I - \rho_1L)a\sigma^2 \quad \text{if } k = 0
\]
\[
= k^{-1}(a'a + k^2b'b - 2k\rho_1a'lb)b\sigma^2
\]
\[
- (1-k^2)\text{Var}\{\hat{\theta}_{t,G}\} \quad \text{if } 0 < k < 1.
\]

where \( I \) is the \( M \times M \) identity matrix and \( L \) is the \( M \times M \) matrix with ones on the subdiagonal and zeros elsewhere, that is, the elements of \( L \) are \( L_{ij} = 1 \) if \( i - j = 1 \), and 0 otherwise. Finally, Cantwell showed that if we denote the sum of the generalized composite estimators for the last \( p \) months by \( S_{t,p} \), that is,
\[
S_{t,p} = \sum_{j=0}^{p+1} \hat{\theta}_{t-j,G}
\]

then, \( S_{t,p} - S_{t-p,p} \) and \( \hat{\theta}_t - \hat{\theta}_{t-p} \) can be expressed as \( \sum_{i=0}^{\infty} \nu_i'y_{t-i} \) for some vectors \( \nu_i, i = 0,1,2, ... \). He then proved that
\[
\text{Var}\{\sum_{i=0}^{\infty} \nu_i'y_{t-i}\} = \sigma^2\{\sum_{i=0}^{\infty} \nu_i'\nu_i + 2\sum_{i=0}^{\infty} \nu_i'\sum_{j=1}^{M-1} \rho_jL^j\nu_{i+j}\}. \tag{2.24}
\]

Cantwell then extended his results to a special case of multi level rotation design where the sample consists of a fixed number, \( g \) say, of rotation groups, which are interviewed every \( g \)-th month and the period of reference is the previous \( g \) months.
2.2. The Time Series Approach

Unlike the classical approach which treats \( \{ \theta_t \}, t = 1, 2, \ldots \), as a sequence of fixed values, the time series approach treats \( \{ \theta_t \} \) as a realization of a time series. The objective is then to predict \( \theta_t \), referred to as the "signal" in the time series literature, from the observations available up to and including the current time \( t \).

Several authors have addressed modeling and estimation problems for autoregressive or autoregressive moving average signals corrupted by white noise. Suppose that a survey has been conducted at times \( t = 1, 2, \ldots, T \) to estimate \( \theta_t \), \( t = 1, 2, \ldots, T \) where \( \theta_t \) is the true value of the population characteristic of interest at time \( t \). Then the initial sample survey estimate \( Y_t \) of \( \theta_t \) satisfies

\[
Y_t = \theta_t + u_t
\]

(2.25)

where \( \{ u_t \} \) is a sequence of random survey errors independent of \( \{ \theta_t \} \) and satisfying

\[
E\{u_t\} = 0, \quad \text{and} \quad \text{Var}\{u_t\} = \sigma^2
\]

Walker (1960) noted that if \( \theta_t \) follows a p—th order autoregressive process and \( u_t \) follows an independent white noise process, then \( Y_t \) follows an autoregressive moving average ARMA (p,p) process. He then used this result and properties of the sample autocovariance function to obtain method of moments estimators of the variances of \( \theta_t \) and \( u_t \), and of the autoregressive coefficients. He then assessed the
asymptotic properties of such estimators, paying principal attention to the special case when \( p = 1 \).

Pagano (1974) considered the same problem as did Walker (1960). Under the assumption of normality, Pagano used a non-linear least squares procedure to obtain strongly consistent and asymptotically efficient estimators of the parameters of the \( \theta_t \) and \( u_t \) processes. He sketched an extension of his results to the case in which \( \theta_t \) follows an ARMA \((p, q)\) process, \( p < q \), and \( u_t \) follows an independent white noise process.

Box and Jenkins (1976, Appendix A.4.4) generalized the modeling results of Walker (1960) and Pagano (1974) and a further generalization of these results is provided in Eltinge (1987, Section 5.2).

Blight and Scott (1973) realized that in some survey sampling problems, it may be reasonable to consider \( \theta_t, t = 1, 2, 3, \ldots \), to be a realization of a stochastic process. They retained the Patterson (1950) assumptions regarding sampling errors and rotation patterns, but replaced the assumption of fixed \( \theta_t \) values with the assumption that \( \theta_t \) follows a first order autoregressive autoregressive process independent of the sampling error process. Given the parameters of the resulting model, they derived the the minimum mean square error predictors of \( \theta_t \) and \( \theta_t - \theta_{t-1} \), and developed a formula for the optimal number of units to replace on each sampling occasion.

Scott and Smith (1974) also considered the problem of the analysis of data from repeated surveys using time series methods. The assumed model \((2.25)\) and for both overlapping and non-overlapping surveys, they considered the problem of using the complete set of estimates available at time \( t \), say \( Y_t = (Y_t, Y_{t-1}, \ldots) \) based on the repetition of the survey, in order to obtain improved estimates of the current value \( \theta_t \). They extended the results of Blight and Scott (1973) in two directions:
(i) For non-overlapping surveys (and hence uncorrelated $u_t$'s), they presented a Kalman filter approach to the prediction of the $\theta_t$ values.

(ii) For general covariance stationary $\theta_t$ and $u_t$ processes, they applied the methods of Whittle (1963) to obtain minimum mean square error linear predictors of $\theta_t$ and of linear combinations of $\theta_{t-j}$.

Using normal theory assumptions, Scott and Smith proved that the minimum mean square error estimator of $\theta_t$ is a weighted average of $Y_t$, the usual sample survey estimate of $\theta_t$ and $\hat{Y}_t$, the best linear forecast of $Y_t$ from the previous estimates $Y_t = (Y_{t-1}, Y_{t-2}, ...)$, in other words,

$$\hat{\theta}_t = (1-\pi_t)Y_t + \pi_t \hat{Y}_t$$

(2.26)

where

$$\pi_t = \left\{ \text{Var}(Y_t | Y_{t-1}) \right\}^{-1} \omega_t^2$$

(2.27)

is the ratio of the within survey variance of $Y_t$ to the variance of the linear forecast of $Y_t$ from previous surveys. The mean square error of $\hat{\theta}_t$ is given by:

$$v_t^2 = (1-\pi_t)\omega_t^2$$

(2.28)

Note that the values of $\hat{Y}_t$ and $\text{Var}(Y_t | Y_{t-1})$ can be obtained by time series methods for pure prediction. For instance, suppose that $\omega_t^2 = \omega^2$ for all $t$, and $\theta_t$ follows a first order autoregressive process:

$$\theta_t = \lambda \theta_{t-1} + \epsilon_t$$

(2.29)
where $0 \leq \lambda \leq 1$, $\{\epsilon_t\}$ are uncorrelated random variables with

$$E(\epsilon_t) = 0, \text{ and } \text{Var}(\epsilon_t) = \sigma^2.$$ 

Then, the induced model for $Y_t$ can be written as:

$$Y_t = \lambda Y_{t-1} + \epsilon_t + u_t - \lambda u_{t-1}$$ (2.30)

By Fuller (1976, Theorem 2.6.3), $Y_t$ satisfies the autoregressive moving average [ARMA(1,1)] model

$$Y_t = \lambda Y_{t-1} + a_t - \beta a_{t-1}$$ (2.31)

where $\text{Var}(a_t) = \nu^2$. Equating moments in (2.26) and (2.27), we get

$$\beta \nu^2 = \lambda \omega^2$$ (2.32)

$$(1 + \beta^2) \nu^2 = (1 + \lambda^2) \omega^2 + \sigma^2$$ (2.33)

Thus,

$$\hat{Y}_t = \beta^{-1}(\lambda - \beta) \sum_{j=1}^{\infty} \beta^j Y_{t-j}$$ (2.34)

and the residual variance is:

$$E(\hat{Y}_t - Y_t)^2 = \nu^2$$ (2.35)
Thus,
\[
\pi_t = \pi = \nu^{-2} \omega^2 = \lambda^{-1} \beta
\]  
(2.36)

from (2.31). Thus \( \pi_t = \pi \) is constant, that is, \( \pi_t \) is independent of \( t \). Furthermore, the minimum mean square error estimator of \( \theta_t \) is

\[
\hat{\theta}_t = (1-\pi) \sum_{j=0}^{\infty} \beta^j Y_{t-j}
\]

(2.37)

\[
= (1-\pi) Y_t + \pi \lambda \hat{\theta}_{t+1}
\]

with mean square error given by

\[
v_t^2 = (1-\pi) \omega^2
\]

(2.38)

This shows that the best estimator of the current value \( \theta_t \) is a multiple of an exponentially weighted moving average of the estimates \( Y_t, Y_{t-1}, \ldots \), and is a standard exponentially weighted moving average in the important special case: \( \lambda=1 \).

Scott and Smith concluded with an application of their results to an example of an overlapping survey: The Medical Data Index. The analysis of three series from the survey showed significant gains in efficiency of the estimators obtained by time series methods over the simple estimators.

Scott, Smith and Jones (1977) extended some of the results of Scott and Smith (1974) to multistage surveys in which units may overlap at one or more stages of sampling. Smith (1978) provided a brief review of the classical and time series modeling of \( \theta_t \) and presented theoretical and practical arguments in favor of the latter.
Under the assumption of known process parameters, Jones (1979) compared the relative variances of the Patterson (1950), Blight and Scott (1973) and Scott and Smith (1974) estimators of the current value $\theta_t$, and the change $\theta_t - \theta_{t-1}$, under the Blight and Scott (1973) models for $\theta_t$ and $u_t$. For the estimation of $\theta_t$, he found that the Blight and Scott (1973) and Scott and Smith (1974) methods were generally superior to those of Patterson (1950), especially if the variance of $\theta_t$ was small compared to the variance of $e_t$. Furthermore, the Blight and Scott (1973) estimator of $\theta_t$ was found to be somewhat superior to the Scott and Smith (1974) estimator. However, it was observed that if the deviations $\theta_{ti} - \theta_t$ of the value $\theta_{ti}$ of the $i$-th unit from the finite population mean $\theta_t$ were strongly autocorrelated, then the Scott and Smith (1974) method may be less efficient than the Patterson (1950) and Blight and Scott (1973) methods. Furthermore, if only a relatively short series is available, the time series methods of Blight and Scott (1973) and Scott and Smith (1974) may lose some of their theoretical efficiency through preliminary detrending, deseasonalizing, and estimation of the parameters of the $\theta_t$ and $u_t$ processes.

Jones (1980) used least squares theory to present a unified approach to the problem of obtaining best linear unbiased estimators of parameters in repeated sample surveys. Among other things, he extended the results of Blight and Scott (1973) and clarified the relationship between the results of Blight and Scott (1973), those of Scott and Smith (1974) and the classical approach of Patterson (1950), or its generalization by Gurney and Daly (1965). He proceeded as follows:

Let $Y_{ti}$ be the $i$-th elementary unbiased estimate of $\theta_t$, the population parameter from the sample at time $t$ ($i=1,2,\ldots,I$) and let $Y_T$ be the vector of all such estimates for all time periods, say $t = 1, 2, \ldots, T$. Thus we can write:
where $e_{it}$ is the sampling error. In matrix notation, the model is

$$Y_{it} = \theta_t + e_{it}$$

where $Y_{it}$ is the IT-dimensional vector of elementary estimates, $\theta_T = (\theta_1, \ldots, \theta_T)'$ is the T-dimensional vector of unknown parameters, $X$ is a design matrix of 0's and 1's, which relates the elementary estimates in $Y_T$ to their expected values in $\theta_T$, and $\epsilon_T = (e_{11}, \ldots, e_{1T}, \ldots, e_{1T})'$ is the IT-dimensional vector of sampling errors, such that

$$E(\epsilon_T) = 0, \text{ and } \operatorname{Var}(\epsilon_T) = K_\epsilon,$$

where $K_\epsilon$ is assumed to be known and nonsingular. Under the classical approach, the best linear unbiased estimator of $\theta_T$ is given by

$$\hat{\theta}_T = (X'K_\epsilon^{-1}X)^{-1}X'K_\epsilon Y_T$$

(2.40)

with variance given by:

$$\operatorname{Var}(\hat{\theta}_T) = (X'K_\epsilon^{-1}X)^{-1}$$

(2.41)

Note that this result is more general than the results obtained by Patterson (1950) and Eckler (1955). Under the time series approach, we further assume that $\theta_T$ is a random vector with $E(\theta_T) = \mu$ and $\operatorname{Var}(\theta_T) = V$. The generalized least squares formulation now gives the best linear unbiased estimator of $\theta_T$ as
\[ \hat{\theta}_T = (X'K_{\varepsilon}^{-1}X + V^{-1})^{-1}X'K_{\varepsilon}^{-1}Y_T \]  

(2.42)

with variance given by

\[ \text{Var}(\hat{\theta}_T) = (X'K_{\varepsilon}^{-1}X + V^{-1})^{-1} \]  

(2.43)

Note the following:

1. If the elements of $V$ are of the form: $v_{ij} = (1-\lambda^2)^{-1}\lambda|\lambda|\sigma^2$, for some $\lambda$ and $\sigma^2$, $|\lambda| < 1$, $i, j = 1, 2, \ldots, T$, then (2.42) coincides with the estimates derived by Blight and Scott (1973).

2. If $I = 1$, then the model (2.39) reduces to:

\[ Y_T = \theta_T + \epsilon_T \]

and (2.42) and (2.43) reduce to:

\[ \hat{\theta}_T = (K_{\varepsilon}^{-1} + V^{-1})^{-1}K_{\varepsilon}^{-1}Y_T \]

\[ \text{Var}(\hat{\theta}_T) = (K_{\varepsilon}^{-1} + V^{-1})^{-1} \]

which are exactly the results obtained by Scott and Smith (1974). However, the Scott and Smith (1974) assumption that $Y_t$ is the standard sample survey estimate of $\theta_t$ based on the sample at time $t$ alone is no longer required here. In fact, $Y_t$ can be any sampling or design unbiased estimate of $\theta_t$ provided that the sample errors in $\epsilon_T$.
are independent of the values of the \( \theta_t \) process. Jones also sketched a general mixed linear model approach to the estimation of \( \theta_t \) with non-stationary mean.

The papers reviewed so far devoted primary attention to the estimation of the true \( \theta_t \) values, assuming that the parameter of the \( \theta_t \) and \( u_t \) processes are known. However, in practice, these are not known. The parameters of the \( u_t \) process are generally estimable from replicated survey data, but estimation of the parameters of the \( \theta_t \) process requires additional work. Estimation of \( \theta_t \) when parameters of the \( \theta_t \) process are unknown was considered by Miazaki (1985). She considered estimation in repeated surveys as a time series problem. Assuming that \( \theta_t \) in model (2.25), follows a stochastic model, she interpreted the problem of estimating \( \theta_t \) as that of estimating a time series subject to measurement error. In particular, she observed that since in rotation sampling, the sampling units stay in the survey only for a fixed number of occasions, a moving average model may be an appropriate representation of \( u_t \), the sampling error process. She considered a special case of model (2.25) in which \( \theta_t \) follows a \( p \)-th order autoregressive process and \( u_t \) follows a \( q \)-th order moving average process independent of the \( \theta_t \) process. She extended the non-linear least squares estimation procedure of Pagano (1974) to the estimation of the \( \theta_t \) process. She found conditions under which the resulting final parameter estimators were consistent and asymptotically normal. In addition, she showed that the least squares prediction of \( \theta_t \) with estimated process parameters contains an error term not in the predictor constructed with known parameters.

Eltinge and Fuller (1989) applied a generalized least squares procedure, closely related to maximum likelihood, to the estimation of the parameters of the \( \theta_t \) process. They included a component in the estimated prediction variance arising from the estimation of the parameters of the \( \theta_t \) process.
Bell and Hillmer (1990) also discussed the time series approach estimation for repeated surveys. In many ways this paper can be considered a generalization of earlier work done by authors such as Scott and Smith (1974) and Scott, Smith and Jones (1977). After presenting a brief overview of the basic results and framework for the time series approach, they explored the assumptions on model (2.25) on which the standard time series signal extraction results are based, namely, the stationarity of \( \theta_t \), or a suitable difference thereof, the stationarity of \( u_t \) and the uncorrelatedness of \( \theta_t \) and \( u_t \) at all leads and lags. They note that since \( \theta_t \) and \( u_t \) depend on the same population units the validity of the last assumption is not obvious. In fact, Tam (1987) discussed how this assumption fails under an explicitly model based approach. They then showed that the assumption is valid under fairly general conditions, and derived time series estimators which are strongly consistent. Furthermore, they presented an excellent discussion of the issues involved in modeling the time series for the signal process \( \{ \theta_t \} \) and the survey error process \( \{ u_t \} \). Finally, they illustrated their approach with an example using two time series from the Census Bureau's Retail Trade Survey.

Pfeffermann (1991) considered the estimation and seasonal adjustment of population means based on rotating panel surveys carried out at regular time intervals. He used a dynamic structural model that assumes a decomposition of the mean into a trend–level component and a seasonal component. The parameters of the time series models for the components are unknown. The structural model accounts for the correlations between individual panel estimators and possible rotation group effects. It can be applied both in the case of a primary analysis, for which the individual panel estimates are available, and in the case of secondary analysis, in which only the published aggregate estimates are known. The components in the decomposition of the
mean are assumed to follow a stochastic process which is known up to a set of parameters. These parameters are estimated using the Kalman filter technique. Results from an empirical investigation revealed that the use of primary analysis dominates the use of secondary analysis in almost every aspect studied. Furthermore, the use of the model yields more accurate estimates of the population means than those obtained from the classical approach.

2.3. Best Linear Recursive Estimation

As mentioned earlier, estimation for repeated surveys invariably has to contend with computational difficulty due to the accumulation of large amounts of data as the number of periods increases. Attempts to circumvent this problem, without sacrificing the precision of estimation, have led to recursive schemes which produce minimum variance estimates using only previous estimates and the current data, instead of all the previous data. Recursive schemes are also ideal in situations where data do not all become available at the same time but rather accumulate in time.

Initial interest in recursive estimation was stimulated by navigational problems associated with spacecraft in orbit about the earth (see, for instance, Sorenson, 1966, pp 276–281). Subsequently recursive estimation techniques have been found useful in a wide range of problems, including the monitoring of medical patients and nuclear reactors.

Let \( Y_t \) be the \( n_t \times 1 \) vector of observations at time \( t \), and let

\[
Y_t = X_t \theta_t + \epsilon_t \quad (t = 1, 2, \ldots)
\]
where $X_t$ is the $n_t \times p_t$ known matrix and $\theta_t$ is the $p_t \times 1$ vector of unknown parameters at time $t$. In general, the problem is to express the best linear unbiased estimator of $\theta_t$ as a linear combination of the best linear unbiased estimator for time $t-1$ and the observations at time $t$.

The initial work on recursive estimation was carried out by Plackett (1950). Assuming a fixed effects model, he derived formulas for the adjustments of least squares estimates of the regression coefficients, their covariance matrix and the sum of squared residuals needed to incorporate additional observations.

Kalman (1960) published in the engineering literature a recursive procedure, now known as the Kalman filter, for estimating $\theta_t$ when $\theta_t$ is a stochastic process. Jones (1970) discussed the problem of recursively estimating a subset of the parameter vector $\theta_t$ in the linear model defined in (2.44) and included a recursive formula for the sum of squared residuals. Odell and Lewis (1971) also developed recursive algorithms for the best linear unbiased estimation of fixed effects.

Duncan and Horn (1972) recognized the inherent equivalence of least squares theory and Kalman filter theory. Their work showed Kalman filter theory to be essentially random parameter linear regression theory.

Sallas and Harville (1981) extended the recursive estimation techniques developed by Odell and Lewis (1971) for fixed effects and by Duncan and Horn (1972) for random effects, to the estimation for mixed linear models. They presented a mixed model extension of the state space model and then used the Kalman filter to obtain recursive estimators for a two-part random model where the second factor obeys a generalized autoregressive process. They then derived recursive algorithms for the mixed linear model by passing to the limit in an appropriate way.
An expository article on state–space models and Kalman filtering is provided by Meinhold and Singpurwalla (1983). Using a Bayesian formulation and some well–known results in multivariate statistics, they continued the efforts of Duncan and Horn (1972) by describing the state–space model and the Kalman filter procedure in terms familiar to statisticians. They concluded by giving a simple example illustrating the use of the Kalman filter for quality control work.

Ansley and Kohn (1983) modified the state–space model and the Kalman filter to obtain estimates of the state vectors and predictors and interpolators for missing observations. They then extended their results for a discrete time state–space model to continuous time models, including smoothing splines and continuous time autoregressive processes.

Tiller (1989) presented a Kalman filter approach to labor force estimation using survey data. A "signal plus noise" model was postulated for the monthly Current Population Survey sample data. The signal process is the true labor force series and the noise process is the error generated by the sampling process. Using a model for the unobserved population series and the autocovariances of the sampling error, the Kalman filter was used to estimate the true labor force series. Tiller then presented the unemployment rate models as examples.
3. PRELIMINARY RESULTS ON
CLASSICAL ESTIMATION FOR
REPEATED SURVEYS ON A FIXED POPULATION

The purpose of this chapter is two—fold. First, it is an introduction to the fundamental concepts of estimation for repeated surveys, presented at a simplified technical level. Secondly, the procedures presented are intended to illuminate the basic concepts underlying the more complicated procedures discussed in the next chapter. The presentation permits the exploration of the fundamental issues associated with estimation for repeated surveys without undue mathematical complexity. After some basic notation and definitions, we briefly review some classical estimation procedures for two—period surveys. We then compare the various procedures via a numerical example. We conclude with a discussion of extensions of the results to the case of repeated surveys for more than two periods.

3.1. Notation, Definitions and Assumptions

Suppose a survey is carried out repeatedly and the characteristic of interest is \( \theta \). Assume that \( \theta \) changes over time and that its value at time \( t \) is denoted by \( \theta_t \). If \( t \) is the current period, \( \theta_t \) is referred to as the current level and \( \theta_t - \theta_{t-1} \) is referred to as the change in level from the previous to the current period. Let \( Y \) be the survey variable and let \( y_{ti} \) denote the value of \( Y \) on the \( i \)—th unit of the population at time \( t \). Assume that the population consists of \( N \) units and \( N \) is very large. For a start, we adopt the Patterson (1950) assumptions, namely:

(i) the population variance is constant on all occasions and is denoted by \( \sigma^2 \).
(ii) the correlation between the variate values of the same unit on two different occasions $t$ and $k$ is $\rho^{\lfloor t-k \rfloor}$, where $\rho$ is the correlation between the variate values of the same unit on successive occasions.

Suppose now that the survey has been in operation for $t$ periods and that its principal objectives are the estimation of the current level $\theta_t$ and change in level $\theta_t - \theta_{t-1}$. For these purposes, we consider a sequence of $t$ simple random samples of equal size $n$, say, from the population. The sample obtained at period $t$ can be divided into two parts:

(i) the matched part, which consists of those units that were also observed at time $t-1$.

(ii) the unmatched part, which consists of those units that were not observed on the previous occasion.

If $\lambda$ is the proportion of overlap, that is, the proportion of the sample at time $t$ that is common with time $t-1$, then the size of the matched and the unmatched parts of the sample at time $t$ are respectively $n\lambda$ and $n\mu$, where $\mu = 1-\lambda$. We assume that the proportion of overlap and the sample size are constant on all occasions. The simple means based on the responses from the units in the matched and unmatched subsamples of the sample at time $t$ can be defined as follows:

(a): the mean of the responses on the $n\lambda$ matched units at time $t-1$

$$\bar{y}_{t-1,m} = (n\lambda)^{-1} \sum_{i=1}^{n\lambda} y_{t-1,mi}$$

(b): the mean of the responses on the $n\mu$ unmatched units at time $t-1$

$$\bar{y}_{t-1,u} = (n\mu)^{-1} \sum_{i=1}^{n\mu} y_{t-1,ui}$$
(c): the mean of the responses on the \( n\lambda \) matched units at time \( t \)

\[
\bar{y}_{tm} = (n\lambda)^{-1} \sum_{i=1}^{n\lambda} y_{tmi}
\]

(d): the mean of the responses on the \( n\mu \) unmatched units at time \( t \)

\[
\bar{y}_{tu} = (n\mu)^{-1} \sum_{i=1}^{n\mu} y_{tui}
\]

3.2. Estimation of Current Level and Change for Two — Period Surveys

In this section, we discuss various estimation procedures for current level and change based on two periods only. For all estimation procedures, we assume the class of estimators linear in the observations. However, by the symmetry in the correlation structure of the observations, it is sufficient to restrict attention to linear combinations of the subsample means.

3.2.1. Minimum Variance of Current Level

In this procedure, the estimate of the previous level is not revised at the current period. Consider a linear estimator of \( \theta_2 \) of the form:

\[
\hat{\theta}_2 = a\bar{y}_{1u} + b\bar{y}_{1m} + c\bar{y}_{2m} + d\bar{y}_{2u}.
\]
For \( \hat{\theta}_2 \) to be unbiased for \( \theta_2 \), we must have

\[
a + b = 0 \quad \text{and} \quad c + d = 1
\]

so that

\[
\hat{\theta}_2 = a(\overline{y}_{1u} - \overline{y}_{1m}) + c\overline{y}_{2m} + (1-c)\overline{y}_{2u}.
\]

Now,

\[
\text{Var}(\hat{\theta}_2) = n^{-1}[(\mu \lambda)^{-1}a^2 + \lambda^{-1}c^2 + \mu^{-1}(1-c)^2 - \lambda^{-1}(2ac\rho)]\sigma^2 \tag{3.2.1}
\]

Differentiating (3.2.1) with respect to \( a \) and \( c \) and setting to zero, we see that the values of \( a \) and \( c \) that minimize \( \text{Var}(\hat{\theta}_2) \) are given by

\[
a = (1 - \mu^2 \rho^2)^{-1} \lambda \mu \rho \quad \text{and} \quad c = (1 - \mu^2 \rho^2)^{-1} \lambda.
\]

Thus the optimal estimator of \( \theta_2 \) under this procedure is

\[
\hat{\theta}_2 = (1 - \mu^2 \rho^2)^{-1}[(\mu \rho)(\overline{y}_{1u} - \overline{y}_{1m}) + \lambda \overline{y}_{2m} + \mu(1 - \mu^2 \rho^2)\overline{y}_{2u}] \tag{3.2.2}
\]

with variance

\[
\text{Var}(\hat{\theta}_2) = n^{-1}(1 - \mu^2 \rho^2)^{-1}(1 - \mu^2 \rho^2)\sigma^2. \tag{3.2.3}
\]

Now, the variance of the estimated change is

\[
\text{Var}(\hat{\theta}_2 - \hat{\theta}_1) = \text{Var}(\hat{\theta}_2) + \text{Var}(\hat{\theta}_1) - 2\text{Cov}(\hat{\theta}_1, \hat{\theta}_2).
\]
where \( \hat{\theta}_1 \) is the optimal estimator of \( \theta_1 \) at time 1, defined by

\[
\hat{\theta}_1 = \lambda \overline{y}_{1m} + \mu \overline{y}_{1u},
\]

but

\[
\text{Cov}(\hat{\theta}_1, \hat{\theta}_2) = [n(1-\mu^2 \rho^2)]^{-1} \lambda \rho \sigma^2.
\]

Hence,

\[
\text{Var}(\hat{\theta}_2 - \hat{\theta}_1) = [n(1-\mu^2 \rho^2)]^{-1} [2(1-\lambda \rho) - \mu \rho^2 (1+\mu)] \sigma^2
\]  \hspace{1cm} (3.2.4)

Remarks

(i) If \( \mu = 0 \) (complete overlap) or \( \mu = 1 \) (no overlap), the variance of \( \hat{\theta}_2 \) is \( n^{-1} \sigma^2 \).

(ii) The optimum value of \( \mu \) is found by minimizing (3.2.3) with respect to \( \mu \). This gives

\[
\mu_{\text{opt}} = \frac{1}{1 + (1-\rho^2)^{1/2}} \text{ and } 1 - \mu_{\text{opt}} = \lambda_{\text{opt}} = \frac{1}{1 + (1-\rho^2)^{1/2}} - (1-\rho^2)^{1/2}
\]

and, substituting \( \mu_{\text{opt}} \) in (3.2.3), we get the minimum variance

\[
\text{Var}_{\text{opt}}(\hat{\theta}_2) = (2n)^{-1} [1 + (1-\rho^2)^{1/2}].
\]

3.2.2. Minimum Variance for Change

The optimal estimation of change necessarily involves the revision of previous estimates. Consider the general linear estimator of change
\[ \delta \theta = \hat{\theta}_2 - \hat{\theta}_1 = a\bar{y}_1 + b\bar{y}_{1m} + c\bar{y}_{2m} + d\bar{y}_2. \]

For \( \delta \theta \) to be unbiased for \( \theta_2 - \theta_1 \), we must have

\[ a + b = -1 \text{ and } c + d = 1. \]

Hence

\[ \delta \theta = a\bar{y}_1 - (a + 1)\bar{y}_{1m} + c\bar{y}_{2m} + (1 - c)\bar{y}_2. \]

and

\[ \text{Var}(\delta \theta) = \left( n \mu \lambda \right)^{-1} \left[ \lambda a^2 + \mu(a+1)^2 + \mu c^2 + \lambda(1-c)^2 - 2c(a+1)\mu \right] \sigma^2. \]

The optimal estimator of change under this criterion is obtained by minimizing \( \text{Var}(\delta \theta) \) with respect to \( a \) and \( c \). The resulting estimator is

\[ \hat{\delta \theta} = (1-\mu \rho)^{-1} \left[ \lambda(\bar{y}_{2m} - \bar{y}_{1m}) + \mu(1-\rho)(\bar{y}_2 - \bar{y}_1) \right] \quad (3.2.5) \]

and

\[ \text{Var}(\hat{\delta \theta}) = 2[n(1 - \mu \rho)]^{-1}(1 - \rho) \sigma^2 \quad (3.2.6) \]

< \( 2n^{-1} \sigma^2 \)

Now the estimator of current level is given by

\[ \hat{\theta}_2 = \hat{\theta}_1 + \hat{\delta \theta} \]

and therefore,

\[ \text{Var}(\hat{\theta}_2) = \text{Var}(\hat{\theta}_1) + \text{Var}(\hat{\delta \theta}) - 2\text{Cov}(\hat{\theta}_1, \hat{\delta \theta}) \]

But

\[ \text{Cov}(\hat{\theta}_1, \hat{\delta \theta}) = -[n(1-\mu \rho)]^{-1}(1-\rho) \sigma^2 \]
Therefore,

\[
\text{Var}(\hat{\theta}) = [n(1-\mu\nu)]^{-1}[2(1-\rho) - 2(1-\rho) + (1-\mu\nu)]\sigma^2
\]

\[
= n^{-1}\sigma^2
\]

### 3.2.3. Minimum Sum of the Variances of Current Level and Change

We consider the following general linear estimators of current level and change:

\[
\hat{\theta}_2 = a\bar{y}_{1m} + b\bar{y}_{1u} + c\bar{y}_{2m} + d\bar{y}_{2u}
\]

\[
\Delta \hat{\theta} = e\bar{y}_{1m} + f\bar{y}_{1u} + g\bar{y}_{2m} + h\bar{y}_{2u}
\]

where \(a, b, c, d, e, f, g\) and \(h\) are constants. Under the restrictions that the estimator of current level is unbiased for \(\theta_2\) and the estimator of the previous level \(\theta_1\) is not revised at the second period, we may write the estimators \(\hat{\theta}_2\) and \(\Delta \hat{\theta}\) as

\[
\hat{\theta}_2 = (e+\lambda)\bar{y}_{1m} - (e+\lambda)\bar{y}_{1u} + c\bar{y}_{2m} + (1-c)\bar{y}_{2u}
\]

\[
\Delta \hat{\theta} = e\bar{y}_{1m} - (e+1)\bar{y}_{1u} + c\bar{y}_{2m} + (1-c)\bar{y}_{2u}
\]

Therefore,

\[
\text{Var}(\hat{\theta}_2) = (n\mu\lambda)^{-1}[(\lambda+\mu)(e+\lambda)^2 + \mu c^2 + \lambda(1-c)^2 + 2\mu c(e+\lambda)\rho]\sigma^2
\]

(3.2.9)
and

\[ \text{Var}(\Delta \hat{\theta}) = (n\mu\lambda)^{-1}\mu e^2 + \lambda(e+1)^2 + \mu c^2 + \lambda(1-c)^2 + 2ecp\sigma^2 \quad (3.2.10) \]

If \( L(\hat{\theta}_2, \Delta \hat{\theta}) \) is the sum of \( \text{Var}(\hat{\theta}_2) \) and \( \text{Var}(\Delta \hat{\theta}) \), then we have

\[ L(\hat{\theta}_2, \Delta \hat{\theta}) = (n\mu\lambda)^{-1}\{e^2 + (e+\rho)^2\}(\lambda+\mu) + (2e+1)\lambda \]

\[ + 2c^2\mu + 2\lambda(1-c)^2 + 2c\rho(2e+\lambda)\sigma^2 . \]

Now, differentiating \( L(\hat{\theta}_2, \Delta \hat{\theta}) \) with respect to \( e \) and \( c \) and equating to 0, we get the solutions

\[ e = [2(1-\mu^2\rho^2)]^{-1} \lambda(\mu^2\rho^2 - 2\mu\rho - 2) \]

and

\[ c = [2(1-\mu^2\rho^2)]^{-1} \lambda(\mu\rho + 2) \]

Thus the optimal estimators of current level and change are

\[ \Delta \hat{\theta} = [2(1-\mu^2\rho^2)]^{-1}\{\lambda(2+\mu\rho)(\overline{y}_1u - \overline{y}_{1m}) + (\overline{y}_{2m} - \overline{y}_{2u})\} + \overline{y}_{2u} \quad (3.2.11) \]

and

\[ \Delta \hat{\theta} = [2(1-\mu^2\rho^2)]^{-1}\{\lambda(2+2\mu\rho-\mu^2\rho^2)(\overline{y}_1u - \overline{y}_{1m}) + \lambda(2+\mu\rho)(\overline{y}_{2m} - \overline{y}_{2u})\} + \overline{y}_{2u} - \overline{y}_{1u} \quad (3.3.12) \]
and the variances of these estimators are obtained by substituting the optimal values of $e$ and $c$ into (3.2.9) and (3.2.10) respectively.

### 3.2.4. Full Generalized Least Squares Estimation

Let $y_1$ and $y_2$ denote the data vectors obtained at times 1 and 2 respectively. If we assume equal sample sizes at each period and the common sample size is $n$, then the linear model at time 2 may be written as

$$y = X\beta + \epsilon$$  \hspace{1cm} (3.2.13)

where $y = (y_1, y_2)'$, $\beta = (\theta_1, \theta_2)'$, $X = I_{2 \times 2} \otimes J_{n \times 1}$, the Kronecker product of the 2 x 2 identity matrix $I_{2 \times 2}$ and the $n \times 1$ vector of ones $J_{n \times 1}$. Assume that $\epsilon$ is the vector of error terms such that $E(\epsilon) = 0$, and $E(\epsilon \epsilon') = V$. Then the minimum variance linear unbiased estimator of $\beta$ is

$$\hat{\beta} = (X' V^{-1} X)^{-1} X' V^{-1} y$$ \hspace{1cm} (3.2.14)

with covariance matrix of

$$\Sigma = (X' V^{-1} X)^{-1}$$ \hspace{1cm} (3.2.15).

Note that this procedure is equivalent to the procedure discussed in Section 3.2.1. However, in this procedure, it is possible to obtain both revised and unrevised estimators of change as well as their variances from (3.2.14) and (3.2.15).
3.2.5. Restricted Generalized Least Squares Estimation

In the unrestricted generalized least squares estimation discussed in Section 3.2.4, the estimate of the parameter at time $t = 1$ will depend on the data at time $t = 2$. In general, this information is not available at time $t = 1$. We therefore modify the procedure by considering restricted generalized least squares estimators where the restriction is that the estimator for the first period must be the estimator obtained from the initial sample. A detailed discussion of this procedure and its consequences on the efficiency of estimation is provided in Chapter 4.

We assume the linear model (3.2.13), where $y_1$ and $y_2$ denote the data vectors at times 1 and 2 respectively. Let the linear models corresponding to the two time periods be

model at time $t = 1$: \[ y_1 = X_1 \beta_1 + \epsilon_1, \] (3.2.16)

model at time $t = 2$: \[ y_2 = X_1 \beta_1 + X_2 \beta_2 + \epsilon_2, \] (3.2.17)

where $X_1$ is $n \times p_1$, $X_2$ is $n \times p_2$, $\beta_1$ is $p_1 \times 1$, $\beta_2$ is $p_2 \times 1$ and $p_1 + p_2 = p$. We impose the restriction that certain parametric functions must be estimated by specific linear functions of the data:

\[ \lambda_i' \beta = g_i' y, \quad 1 \leq i \leq k, \]

where $\lambda_i$ and $g_i$ are fixed column vectors and are specified. Note that models (3.2.16) and (3.2.17) constitute a partition of a generalization of the model (3.2.13) with $X$ and $\beta = (\beta_1, \beta_2)'$ are of dimensions $2n \times p$ and $p$, respectively. Let the
resulting partition of $X$, $V$ and $\epsilon$ be as follows:

$$\epsilon = (\epsilon_1, \epsilon_2)^\prime,$$
$$X = \begin{bmatrix} X_1 & 0 \\ X_1 & X_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Note that if we do full generalized least squares estimation, $\hat{\beta}_1$ will depend on $y_2$, the data at time $t = 2$, which is not available at time $t = 1$. We therefore impose the restriction

$$\hat{\beta}_1 = (X_1'V_{11}^{-1}X_1)^{-1}X_1'V_{11}^{-1}y_1 \tag{3.2.18}$$

$$= (X_1'V_{11}^{-1}X_1)^{-1}X_1'V_{11}^{-1}(I, 0)y$$

The full generalized least squares estimator of $\beta$ is the value that minimizes the following quadratic form in $\mathbb{R}^p$:

$$Q(\beta_1, \beta_2) = (y - X\beta)' V^{-1}(y - X\beta) \tag{3.2.19}$$

In general, for restricted generalized least squares estimation, we minimize the following Lagrangian function

$$Q(\beta, \mu) = (y - X\beta)' V^{-1}(y - X\beta) - 2 \Sigma_{i=1}^{k} \mu_i(\lambda_i'\beta - g_i' y) \tag{3.2.20}$$
where \( k \) is the number of restrictions, and \( \mu = (\mu_1, \mu_2, \ldots, \mu_k)' \) is the vector of Lagrange multipliers. Differentiating (3.2.20) with respect to \( \mu_i, i=1,2,\ldots,k \), we get the system of equations:

\[
2X'V^{-1}(y-X\beta) + \sum_{i=1}^{k} \mu_i \lambda_i = 0
\]

\[
\lambda_i \beta - g_i y = 0
\]

\[
\vdots
\]

\[
\lambda_k \beta - g_k y = 0.
\]  \hspace{1cm} (3.2.21)

Thus the normal equations are given in matrix form by:

\[
\begin{bmatrix}
X'V^{-1}X & \Lambda \\
\Lambda' & 0
\end{bmatrix}
\begin{bmatrix}
\beta \\
\mu
\end{bmatrix}
= 
\begin{bmatrix}
X'V^{-1} \\
0
\end{bmatrix}y
\]  \hspace{1cm} (3.2.22)

where \( \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) and \( g = (g_1, g_2, \ldots, g_k) \). From (3.2.22), we have:

\[
\begin{bmatrix}
\hat{\beta} \\
\hat{\mu}
\end{bmatrix}
= 
\begin{bmatrix}
X'V^{-1}X & \Lambda \\
\Lambda' & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
X'V^{-1} \\
0
\end{bmatrix}y
\]  \hspace{1cm} (3.2.23)

and the first component of the solution vector above gives the restricted least squares estimator of \( \beta \) as a linear function of the observation vector \( y \). Furthermore, the variance of the restricted least squares estimator of \( \beta \) is the upper \( p \times p \) submatrix of the matrix
Wolter (1979) presents an alternative method (via composite estimators) of computing the restricted generalized least squares estimator of level and change that leaves the previous estimator unchanged.

In the next section, we use an example of a two—period survey, constructed by Fuller (1990), to compare the various estimation procedures discussed in Section 3.2.

3.3. Numerical Example

This example illustrates the problems associated with the estimation of selected parameters in a two—period survey. The characteristic of interest is the proportion of employed in the population. To simplify matters, we assume that:

(1): the population is constant over time.
(2): an equal number of elements are observed at each of the two times such that one half of the elements observed at the first time are also observed at the second time. That is, of the elements observed at the second time, one half were observed at the first time and one half are new to the sample.

The hypothetical observations can be presented in a two—way classification table such as Table 3.1 below.

Let \( S_1 \) be the half sample observed at time \( t = 1 \) only, \( S_2 \) is the half sample observed at time \( t = 2 \) only, and \( S_{12}(i) \) is the sample overlap for both time periods observed at.
Table 3.1. Hypothetical proportions of employed and unemployed for a two period survey

<table>
<thead>
<tr>
<th>TIME 1</th>
<th>TIME 2</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Employed</td>
<td>Unemployed</td>
<td>Total</td>
</tr>
<tr>
<td>Employed</td>
<td>0.85</td>
<td>0.05</td>
<td>0.90</td>
</tr>
<tr>
<td>Unemployed</td>
<td>0.03</td>
<td>0.07</td>
<td>0.10</td>
</tr>
<tr>
<td>Total</td>
<td>0.88</td>
<td>0.12</td>
<td>1.00</td>
</tr>
</tbody>
</table>

period $i$, $i = 1, 2$. Thus the complete sample observed at time $t = i$ is $S_i \cup S_{12}(i)$, $i = 1, 2$. Let $p_{1u}$, $p_{1m}$, $p_{12m}$, $p_{2u}$, $p_{2u}$ denote the observed proportions of employed in samples $S_1$, $S_{12}(1)$, $S_{12}(1) \cap S_{12}(2)$, $S_{12}(2)$ and $S_2$ respectively. We assume simple random sampling at each period. Let

$$p = (p_{1u}, p_{1m}, p_{12m}, p_{2u}, p_{2u})'$$

and

$$\pi = (\pi_1, \pi_1, \pi_{12}, \pi_2, \pi_2)'$$

where $\pi_1$, $\pi_{12}$, $\pi_2$ denote respectively the proportions of employed at time $t = 1$ only, both times $t = 1$ and $t = 2$, and time $t = 2$ only. Then, since the elements in $p$ are sample proportions, we have

$$\frac{(n/2)^{1/2}}{N(0, \Sigma)}.$$

(3.3.1)
where $0$ is the $5 \times 1$ vector of zeros, and

$$
\Sigma = 
\begin{bmatrix}
\pi_1(1-\pi_1) & 0 & 0 & 0 & 0 \\
0 & \pi_1(1-\pi_1) & \pi_{12}(1-\pi_1) & \pi_{12}-\pi_1\pi_2 & 0 \\
0 & \pi_{12}(1-\pi_1) & \pi_{12}(1-\pi_{12}) & \pi_{12}(1-\pi_2) & 0 \\
0 & \pi_{12}-\pi_1\pi_2 & \pi_{12}(1-\pi_2) & \pi_2(1-\pi_2) & 0 \\
0 & 0 & 0 & 0 & \pi_2(1-\pi_2)
\end{bmatrix}
$$

Thus in large samples, $p$ is approximately distributed as $N(\pi, V)$ where $V = 2n^{-1} \Sigma$.

We can now incorporate the sample proportions in a linear model in order to obtain the best linear unbiased estimator or the full generalized least squares estimator of $\pi$. The linear model may be written as (3.2.13), where the vector of observations $y$ consists of the sample proportions in $p$,

$$
X' = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}, \quad \beta = (\pi_1, \pi_{12}, \pi_2)'
$$

and $\epsilon$ is a vector of random errors such that $E(\epsilon) = 0$ and $\text{Var}(\epsilon) = V$. The full generalized least squares estimator of $\beta$ is

$$
\hat{\beta}_{\text{GLS}} = (X' \hat{V}^{-1} X)^{-1} X' \hat{V}^{-1} y
$$

where $\hat{V}$ is an estimate of $V$ obtained by replacing the population proportions in $V$ by their sample counterparts.
Table 3.2. Variance of alternative estimation procedures for sample of size 200 at each period of a two—period survey

Multiply all entries by $10^{-2}$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimation Procedure</th>
<th>Simple</th>
<th>Restricted GLS</th>
<th>Full GLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_1$</td>
<td></td>
<td>0.0450</td>
<td>0.0450</td>
<td>0.0406</td>
</tr>
<tr>
<td>$\pi_{12}$</td>
<td></td>
<td>0.1275</td>
<td>0.0610</td>
<td>0.0584</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td></td>
<td>0.0528</td>
<td>0.0483</td>
<td>0.0477</td>
</tr>
<tr>
<td>$\pi_{12}/\pi_2$</td>
<td></td>
<td>0.1010</td>
<td>0.0317</td>
<td>0.0308</td>
</tr>
<tr>
<td>$\pi_{2}-\pi_1$</td>
<td></td>
<td>0.0688</td>
<td>0.0581</td>
<td>0.0565</td>
</tr>
</tbody>
</table>

The estimated variance of the full generalized least squares estimator is

$$\hat{\text{Var}}(\hat{\beta}_{\text{GLS}}) = (X'\hat{V}^{-1}X)^{-1}$$

The variances of the estimators under the other procedures are computed in accordance with the formulas presented in Section 3.2. The results for this specific example are presented in the Table 3.2. The first column of Table 3.2 gives the variance of the simple estimator. We define the simple estimator of a parameter of interest at time $t$ to be the simple mean of the observations obtained at time $t$ only. The last two columns give the variances of the restricted and generalized least squares estimators respectively. This small example clearly illustrates the following points:

1. For estimation of current level of employment ($\pi_2$), there is a modest gain in efficiency as a result of using the generalized least squares procedures instead of
the simple estimator. The gain in efficiency is about 9% for the restricted generalized least squares procedure and about 11% for the full generalized least squares procedure. The low gain in precision can be attributed to the weak correlation of about 0.595 between unemployment at the two time periods.

2. There is a very large improvement in the precision of the estimation of $\pi_{12}$ and $\pi_{12}/\pi_2$ from using the generalized least square procedures. For instance, the variance of the full generalized least squares estimator of $\pi_{12}$ and $\pi_{12}/\pi_2$ are respectively 46% and 31% of the variances of the corresponding simple estimator.

3. The use of the restricted generalized least squares procedure produces estimates of $\pi_{12}$, $\pi_2$, $\pi_{12}/\pi_2$, and $\pi_2 - \pi_1$, whose variances are very close to those of the full generalized least squares estimates. The variance of the restricted generalized least squares estimate of $\pi_2$ is 99% of the variance of the full generalized least squares estimates. The corresponding figures for $\pi_{12}$, $\pi_{12}/\pi_2$ and $\pi_2 - \pi_1$ are, respectively, 96%, 97%, and 97%.

Next, we compare the procedures outlined in Sections 3.2.1, 3.2.2, and 3.2.3 in terms of the variances of the estimators. Recall that the variances of the estimators for these procedures were computed under the assumption that the population variances on the two time periods were the same, and that the sample sizes are constant over time. To satisfy this condition, we adjust the data in Table 3.1 so as to have the same variance at both time periods. This can be accomplished, for instance, by choosing the (1, 1) element of Table 3.1 to be 0.91, with equal marginal unemployment proportions of 0.93. The resulting variances are presented in Table 3.3.
Table 3.3. Variances of alternative estimators of current level and change

Multiply entries by 10^{-2}

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Procedure</th>
<th>Minimum variance of current level</th>
<th>Minimum variance of change</th>
<th>Minimum sum of variances</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_2$</td>
<td>Simple</td>
<td>0.0326</td>
<td>0.0300</td>
<td>0.0306</td>
</tr>
<tr>
<td>$\pi_2 - \pi_1$</td>
<td>Simple</td>
<td>0.0476</td>
<td>0.0436</td>
<td>0.0417</td>
</tr>
</tbody>
</table>

The last three columns of Table 3.3 give the variances of the estimators of current level and change based on the procedures outlined in Sections 3.2.1, 3.2.2 and 3.2.3, respectively.

It can be seen from Table 3.3 that although the variances of the estimators under the "minimum sum of the variances" procedure are higher than the optimal estimators of current level and change, the procedure exhibits considerable advantage over the simple estimation procedure. The gain in precision is about 6% for current level and about 15% for change in level.

3.4. Extensions and Generalizations

3.4.1. Repeated Surveys for More Than Two Periods

The results obtained in the previous sections can be extended to repeated surveys over more than two periods. For simplicity, we shall retain all the assumptions in Section 3.1. We start by stating the following useful characterizations of minimum variance linear unbiased estimators.
Lemma 3.4.1. Assume that $X_1, X_2, ..., X_n$ are random variables with

\[
\begin{align*}
E\{X_i\} &= \mu &\text{for all } i, \\
Var\{X_i\} &= \sigma_i^2 &\text{for all } i, \\
Cov\{X_i, X_j\} &= 0 &\text{for all } i \neq j.
\end{align*}
\]

Let $a_1, a_2, ..., a_n$ denote some unknown constants. Then the minimum variance linear unbiased estimator of $\mu$ is given by

\[
T = \sum_{i=1}^{n} a_i X_i,
\]

where

\[
a_i = \left(\sum_{i=1}^{n} \sigma_i^{-2}\right)^{-1} \sigma_i^{-2}
\]

and the variance of $T$ is

\[
Var\{T\} = \left(\sum_{i=1}^{n} \sigma_i^{-2}\right)^{-1}
\]

Proof. This lemma is a generalization of the result in Lehmann (1983, page 57). and can be easily proved by minimizing $Var\{T\}$ subject to $E\{T\} = \mu$ by the Lagrange multiplier method.

Lemma 3.4.2. A necessary and sufficient condition for an estimator $T_0$ of a parameter to be a minimum variance linear unbiased estimator is that $T_0$ be uncorrelated with all unbiased estimators of zero, that is,

\[
Cov\{T_0, z\} = 0, \text{ for all } z, \text{ such that } E\{z\} = 0.
\]

Now, suppose we are interested in estimating the current level of a certain parameter \( \theta \). Our presentation closely follows Patterson (1950). The following statistics are available at time \( t \): the minimum variance linear unbiased estimator of \( \theta \) at time \( t-1 \), denoted by \( \hat{\theta}_{t-1} \), the sample means based on the matched and unmatched subsamples of the sample at time \( t-1 \), denoted respectively by \( \bar{y}_{t-1,m} \) and \( \bar{y}_{t-1,u} \), and the sample means based on the matched and unmatched subsamples of the sample at time \( t \), denoted respectively by \( \bar{y}_{t,m} \) and \( \bar{y}_{t,u} \). We consider a general estimator which is a linear combination of these observations

\[
\hat{\theta}_t = a\hat{\theta}_{t-1} + b\bar{y}_{t-1,u} + c\bar{y}_{t-1,m} + d\bar{y}_{tu} + e\bar{y}_{tm}
\]

For \( \hat{\theta}_t \) to be unbiased for \( \theta_t \), we must have

\[
c = -(a + b) \quad \text{and} \quad e = 1 - d.
\]

Thus \( \hat{\theta}_t \) may be written as

\[
\hat{\theta}_t = a\hat{\theta}_{t-1} + b\bar{y}_{t-1,u} - (a+b)\bar{y}_{t-1,m} + d\bar{y}_{tu} + (1-d)\bar{y}_{tm} \tag{3.4.1}
\]

If \( \hat{\theta}_t \) is the minimum variance linear unbiased estimator of \( \theta_t \), then applying Lemma 3.4.2, we have

\[
\text{Cov}\{\hat{\theta}_t, \bar{y}_{t-1,u} - \bar{y}_{t-1,m}\} = 0 \quad \text{and} \quad \text{Cov}\{\hat{\theta}_t, \bar{y}_{t-2,u} - \bar{y}_{t-2,m}\} = 0.
\]
In other words,

\[ \text{Cov}\{\hat{\theta}_t, \bar{y}_{t-1,u}\} = \text{Cov}\{\hat{\theta}_t, \bar{y}_{t-1,m}\} \]  

(3.4.2)

and

\[ \text{Cov}\{\hat{\theta}_t, \bar{y}_{t-2,u}\} = \text{Cov}\{\hat{\theta}_t, \bar{y}_{t-2,m}\} \]  

(3.4.3).

But

\[ \text{Cov}\{\hat{\theta}_t, \bar{y}_{t-1,u}\} = a\text{Cov}\{\hat{\theta}_{t-1}, \bar{y}_{t-1,u}\} + (n\mu)^{-1}b\sigma^2 \]  

(3.4.4)

and

\[ \text{Cov}\{\hat{\theta}_t, \bar{y}_{t-1,m}\} = a\text{Cov}\{\hat{\theta}_{t-1}, \bar{y}_{t-1,m}\} + (n\lambda)^{-1}\{(1-d)\rho - (a+b)\sigma^2\} \]  

(3.4.5).

Equating (3.4.4) and (3.4.5) as suggested by (3.4.2), and simplifying, we obtain

\[ b = \mu[\rho(1-d) - a] \]  

(3.4.6)

Similarly, from (3.4.3), we obtain \( b = 0 \) and hence from (3.4.6), we get

\[ a = \rho(1-d) \]  

(3.4.7)

Substituting for \( a \) and \( b \) in (3.4.1), we get

\[ \hat{\theta}_t = d\bar{y}_{tu} + (1-d)[\bar{y}_{tm} + \rho(\hat{\theta}_{t-1} - \bar{y}_{t-1,m})] \]
or, in general,

\[ \hat{\theta}_t = \varphi_t \overline{y}_{tu} + (1 - \varphi_t) [\overline{y}_{tm} + \rho(\hat{\theta}_{t-1} - \overline{y}_{t-1,m})] \quad (3.4.8) \]

Furthermore, from Lemma 3.4.2, we have

\[ \text{Cov}(\hat{\theta}_t, \overline{y}_{tu}) = \text{Cov}(\hat{\theta}_t, \overline{y}_{tm}) . \quad (3.4.9) \]

In other words,

\[ \varphi_t (n\mu)^{-1} \sigma^2 = (1 - \varphi_t) [(n\lambda)^{-1} (1 - \rho^2) \sigma^2 + \rho \text{Cov}(\hat{\theta}_{t-1}, \overline{y}_{tm})] , \quad (3.4.10) \]

where

\[ \text{Cov}(\hat{\theta}_{t-1}, \overline{y}_{tm}) = \rho \text{Cov}(\hat{\theta}_{t-1}, \overline{y}_{t-1,m}) = \rho \text{Var}(\hat{\theta}_{t-1}) . \]

Substituting in (3.4.10), and simplifying, we get

\[ \varphi_t = \left[ \rho^2 \text{Var}(\hat{\theta}_{t-1}) + (n\lambda)^{-1} (1 - \rho^2) \sigma^2 + (n\mu)^{-1} \sigma^2 \right]^{-1} \]

\[ [\rho^2 \text{Var}(\hat{\theta}_{t-1}) + (n\lambda)^{-1} (1 - \rho^2) \sigma^2] , \]
and the variance of the minimum variance linear unbiased estimator of $\theta_t$ is

$$\text{Var}\{\hat{\theta}_t\} = \varphi_t(n\mu)^{-1}\sigma^2. \quad (3.4.11)$$

It is straightforward to show that when the proportion of overlap is fixed,

$$\lim_{t \to \infty} \text{Var}\{\hat{\theta}_t\} = \varphi_\infty(n\mu)^{-1}\sigma^2, \quad (3.4.12)$$

where

$$\varphi_\infty = 2\rho^{-1}\left[(1-\rho^2)^{1/2} - (1-\rho^2)\right],$$

and that if the proportion of overlap varies with time, where $\lambda_t$ denotes the proportion of overlap at time $t$, then

$$\lim_{t \to \infty} \lambda_t = 1/2.$$

Remarks.

1. The reader will recognize the expression $\bar{y}_t + \rho(\hat{\theta}_{t-1} - \bar{y}_{t-1,m})$ in (3.4.8) as the double sampling regression estimator based on the matched subsample of the sample at time $t$. We have thus expressed the minimum variance linear unbiased estimator of $\theta_t$ as a weighted average of the unmatched subsample of the sample at time $t$, and the double sampling regression estimator based on the matched subsample of the sample at time $t$. 
2. Expanding (3.4.8) by substituting for \( \hat{\theta}_{t-1} \) recursively, we may express \( \hat{\theta}_t \) as a linear combination of the subsample means from time 1 through time \( t \) as follows:

\[
\hat{\theta}_t = \varphi_t \bar{y}_{tu} + (1-\varphi_t) \bar{y}_{tm} + \rho(1-\varphi_t) \hat{\theta}_{t-1}
\]

\[
= \varphi_t \bar{y}_{tu} + (1-\varphi_t) \bar{y}_{tm} + \rho(1-\varphi_t)
\]

\[
[\varphi_{t-1} \bar{y}_{t-1,u} + (1-\varphi_{t-1}) \bar{y}_{t-1,m} + \rho(1-\varphi_{t-1}) \hat{\theta}_{t-2}].
\]

Continuing in this way, we get

\[
\hat{\theta}_t = \varphi_t \bar{y}_{tu} + (1-\varphi_t) \bar{y}_{tm}
\]

\[
+ \sum_{k=1}^{t-2} \rho^k \prod_{r=k+1}^{t} (1-\varphi_r) \{\varphi_{t-k} \bar{y}_{t-k-1,u} + (1-\varphi_{t-k}) \bar{y}_{t-k-1,m} - \bar{y}_{t-k-1,m}\} + [\rho^{t-1} \prod_{r=2}^{t} (1-\varphi_r)](\bar{y}_1 - \bar{y}_{1m})
\]

where \( \bar{y}_1 \) is the mean of the total sample on the first occasion.

Under the more general correlation pattern

\[
\text{Cov}\{y_{ti}, y_{t+h;i}\} = \sigma^2
\]

for \( h = 0 \) and all \( t, i \)

\[
= \rho_h \sigma^2
\]

if \( h \neq 0 \) and all \( t, i \), and \( \rho_0 = 1 \).
we may express $\hat{\theta}_t$ as

$$
\hat{\theta}_t = \varphi_t \bar{y}_{tu} + (1-\varphi_l)\bar{y}_{tm} + \sum_{k=2}^{t-2} \prod_{r=k+1}^{t} \rho_r (1-\varphi_r) \nonumber
$$

$$
[\varphi_{t-k} \bar{y}_{t-k,u} + (1-\varphi_{t-k})\bar{y}_{t-k,m} - \bar{y}_{t-k-1,m}]
$$

$$
+ \prod_{r=2}^{t} \rho_r (1-\varphi_r) (\bar{y}_1 - \bar{y}_{1m}) 
$$

(3.4.14)

3.4.2. More General Correlation Patterns

We now discuss an easier procedure of obtaining the minimum variance linear unbiased estimators of current level. The procedure involves the construction of a linear model from which the minimum variance linear unbiased estimators are obtained via the usual least squares procedures. The procedure is equivalent to the Patterson (1950) procedure, but it is more amenable to extensions and generalizations and less computationally intensive. We shall illustrate the procedure with the case when the observations have a second order autoregressive error structure, that is,

$$
\text{Cov}\{y_{ti}, y_{t+h}\} = \sigma^2 
$$

if $h = 0$, for all $i$,

$$
= \rho_h \sigma^2 
$$

if $|h| \leq 2$ for all $i$,

$$
= 0 
$$

if $|h| > 2$ for all $i$. 

The following statistics are available at time $t$:

1. the best estimates for the previous two occasions based on data through time $t-1$, denoted by $\hat{\theta}_{t-2}(t-1)$ and $\hat{\theta}_{t-1}(t-1)$ respectively,
2. the covariance matrix $C_t$ of $\hat{\theta}_t = (\hat{\theta}_{t-2}(t-1), \hat{\theta}_{t-1}(t-1))'$, and
3. the observations obtained at time $t$.

The units in the sample at time $t$ can be divided into three groups, namely, the units entering the sample for the first time at time $t$, the units which are in the sample for the second time at time $t$, and the units which are in the sample for the third time at time $t$. Let the elementary estimates (that is, simple means) based on these observations be denoted by $y_{t1}$, $y_{t2}$, and $y_{t3}$ respectively. These observations can be transformed so that they are uncorrelated with previous observations as follows:

$$
\begin{align*}
z_{t1} &= y_{t1} \\
z_{t2} &= y_{t2} - \rho_1 y_{t-1,1} \\
z_{t3} &= y_{t3} - \alpha_1 y_{t-1,2} - \alpha_2 y_{t-2,1},
\end{align*}
$$

where $\alpha_1$ and $\alpha_2$ are functions of $\rho_1$ and $\rho_2$, constructed so that $z_{ti}$ is uncorrelated with $y_{t-j,i}$ for all $j > 0$. Let the variances of $z_{ti}$ be $\sigma_i^2$, $i = 1, 2, 3$ and let $V_0 = \text{Diag}\{\sigma_1^2, \sigma_2^2, \sigma_3^2\}$. Assuming constant proportion of overlap, we can write the linear model at time $t$ as

$$
Y = X\beta + \epsilon,
$$

(3.6.1)
where \( \mathbf{Y} = (\hat{\theta}_{t-2(t-1)}, \hat{\theta}_{t-1(t-2)}, z_{t1}, z_{t2}, z_{t3})', \beta = (\hat{\theta}_{t-2}, \hat{\theta}_{t-1}, \theta_t)', \)

and

\[
\mathbf{X}' = \begin{bmatrix} 1 & 0 & 0 & 0 & -\alpha_2 \\ 0 & 1 & 0 & -\rho_1 & -\alpha_1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.
\]

The covariance matrix of \( \mathbf{Y} \) is \( \mathbf{V} = \text{Blockdiag} \{\mathbf{C}_t, \mathbf{V}_0\}. \) The minimum variance linear unbiased estimator of \( \beta_t \) is then given by

\[
\hat{\beta}_t = (\mathbf{X}' \mathbf{V}^{-1} \mathbf{X})^{-1} \mathbf{X}' \mathbf{V}^{-1} \mathbf{Y}
\]

with variance

\[
\text{Var} \{\beta\} = (\mathbf{X}' \mathbf{V}^{-1} \mathbf{X}).
\]

The minimum variance linear unbiased estimator of any linear combination in particular of change \( \hat{\theta}_t - \theta_{t-1} \), and the variance of the estimate can then be computed. Note that varying proportions of overlap can be incorporated in this framework by varying the number of observations in each of the three categories described above.

Finally, we note that the simple procedure described in this section illustrates a very efficient method of obtaining minimum variance linear unbiased estimators. The procedure serves as a basis for the recursive regression estimator described in Chapter 4.
4. LEAST SQUARES ESTIMATION
FOR REPEATED SURVEYS

4.1. Introduction

In this chapter, we develop some least squares estimation procedures for repeated surveys. We begin by presenting some results on general linear model theory. These results are presented in Section 4.2. and will be useful in the sequel. In Section 4.3, we discuss the best linear unbiased estimation procedure. Best linear unbiased estimation becomes progressively more complicated computationally as the number of periods of the survey increases. To circumvent this problem and, at the same time, produce minimum variance estimators, we introduce the recursive regression estimation procedure. This procedure is described in Section 4.4. Some theoretical results associated with the procedure are also presented. In Section 4.5, we discuss State–Space models and Kalman filtering, with particular emphasis on those aspects that are applicable to the problem of estimation for repeated surveys using time series methods. Some convergence results associated with the Kalman filter procedure are also presented. The theoretical results are applications of standard least squares theory to our particular rotating panel survey estimation problem. Applications of the theoretical results are presented and discussed in Chapter 5.

4.2. Some Results in Linear Model Theory

Lemma 4.2.1. (Partitioned Matrix Inversion). Let $T$ be a $p \times p$ nonsingular matrix, $U$ a $p \times q$ matrix, $V$ a $q \times p$ matrix, and $W$ a $q \times q$ matrix. Let $Q = W - VT^{-1}U$. If $Q$ is nonsingular, then
\[
\begin{bmatrix}
T & U \\
V & W
\end{bmatrix}^{-1} = \begin{bmatrix}
T^{-1} + T^{-1}UQ^{-1}VT^{-1} & -T^{-1}UQ^{-1} \\
- Q^{-1}VT^{-1} & Q^{-1}
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
W & V \\
U & T
\end{bmatrix}^{-1} = \begin{bmatrix}
Q^{-1} & - Q^{-1}VT^{-1} \\
- T^{-1}UQ^{-1} & T^{-1} + T^{-1}UQ^{-1}VT^{-1}
\end{bmatrix}.
\]

Proof. As the inverse of a matrix is unique, this result can be proved directly by checking that the given matrix, pre- or post-multiplied by its inverse, gives the identity matrix.

Before stating our next results, we define the following special types of linear models:

**Definition 4.2.1.**

(1) **The Gauss—Markov Model**

The \( n \times 1 \) vector of observations \( Y \) is said to follow a Gauss—Markov model if it can be expressed as

\[
Y = X\beta + \epsilon
\]

(4.2.1)

where \( X \) is an \( n \times p \) full rank matrix of known constants, \( \beta \) is a \( p \times 1 \) vector of unknown parameters, and \( \epsilon \) is an \( n \times 1 \) vector of random errors such that \( E\{\epsilon\} = 0 \) and \( E\{\epsilon\epsilon'\} = \sigma^2I \) (\( \sigma^2 > 0 \) is an unknown constant).
The Aitken Model

The $n \times 1$ vector of observations $Y$ is said to follow the Aitken model if it can be expressed as in equation (4.2.1), where $X$ and $\beta$ retain their definitions, but

$$E\{e\} = 0$$

and

$$E\{ee'\} = \sigma^2 V,$$

where $V$ is a positive definite matrix whose elements are known.

Theorem 4.2.1. (Gauss–Markov). Under the Gauss–Markov model, the ordinary least squares estimator

$$\hat{\beta} = (X'X)^{-1}X'Y$$

(4.2.2)

of an estimable parametric function $\lambda' \hat{\beta}$ is the best linear unbiased estimator in the sense that it has uniformly minimum variance among all linear unbiased estimators of $\lambda' \beta$.


The following theorem is a generalization of the Gauss–Markov Theorem to the case when the covariance matrix of the vector of errors is an arbitrary positive definite matrix.
**Theorem 4.2.2.** (Aitken). Under the Aitken Model, the generalized least squares estimator

\[ \lambda' \hat{\beta} = \lambda'(X'V^{-1}X)^{-1}X'V^{-1}Y \]  

(4.2.3)

of the estimable parametric function \( \lambda' \beta \) is the best linear unbiased estimator of \( \lambda' \beta \).


In general, the ordinary least squares estimator of an estimable linear function \( \lambda' \beta \) will not be the best linear unbiased estimator under the Aitken model. The following theorem specifies one of the circumstances under which the ordinary least squares estimator of every estimable function of the vector of parameters is the best linear unbiased estimator. A detailed discussion of essentially equivalent conditions is given in Puntanen and Styan (1989).

**Theorem 4.2.3.** (Zyskind). Under the Aitken model, the ordinary least squares estimator of \( \lambda' \beta \) is the best linear unbiased estimator of \( \lambda' \beta \) for every \( \lambda \) for which \( \lambda' \beta \) is estimable if and only if there exists a matrix \( Q \) such that

\[ VX = XQ \]

**Proof.** See Graybill (1976, page 209).
The following theorem is essentially a matrix theoretic result. It shows that if we augment a least squares problem by adding new observations which are functions of additional parameters, then the least squares estimators of the original parameters are identical to those of the original problem. Furthermore, the least squares estimators of the new parameters can be expressed as a function of new observations, the least squares estimators of the original parameters and the deviations from the original model.

**Theorem 4.2.4.** Assume the linear model

$$Y_1 = X\alpha + \epsilon_1 \quad (4.2.4)$$

where

- $Y_1$ is an $n \times 1$ vector of observations,
- $X$ is an $n \times p$ full rank matrix of known constants ($n > p$),
- $\alpha$ is a $p \times 1$ vector of regression parameters, and
- $\epsilon_1$ is an $n \times 1$ vector of random variables.

Let the augmented model be

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X & 0 \\ A & B \end{bmatrix} \begin{bmatrix} \alpha \\ \theta \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \quad (4.2.5)$$
where

\[ Y_2 \text{ is a } k \times 1 \text{ vector of additional observations,} \]
\[ A \text{ is a } k \times p \text{ matrix of known constants,} \]
\[ B \text{ is a } k \times k \text{ nonsingular matrix of known constants,} \]
\[ \theta \text{ is a } k \times 1 \text{ vector of additional parameters, and} \]
\[ \epsilon_2 \text{ is a } k \times 1 \text{ vector of random variables.} \]

Let

\[ \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \]

be a positive definite \((n + k) \times (n + k)\) symmetric matrix.

Let

\[ \hat{\alpha} = (X' \Sigma_{11}^{-1} X)^{-1} X' \Sigma_{11}^{-1} Y_1 \] \hspace{1cm} (4.2.6)

and

\[ \hat{\beta} = \begin{bmatrix} \hat{\alpha} \\ \bar{\theta} \end{bmatrix} = (W' \Sigma^{-1} W)^{-1} W' \Sigma^{-1} Y \] \hspace{1cm} (4.2.7)

where

\[ W = \begin{bmatrix} X & 0 \\ A & B \end{bmatrix}, \]

and

\[ Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \]

and all inverses exist. Then \( \hat{\alpha} = \bar{\alpha} \).
Proof. Let $T$ be a lower triangular matrix such that

$$\Sigma^{-1} = T^T T \quad (4.2.8)$$

and let

$$T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}, \quad (4.2.9)$$

where $T_{22}$ is nonsingular. Then,

$$W^T \Sigma^{-1} W = \begin{bmatrix} X^T T_{11}^T T_{11} X + P P & P^T T_{22} B \\ B^T T_{22} P & B^T T_{22}^T T_{22} B \end{bmatrix}$$

where $P = T_{21} X + T_{22} A$. Therefore, from Lemma (4.2.1), we have

$$(W^T \Sigma^{-1} W)^{-1}$$

$$= \begin{bmatrix} (X^T \Sigma_{11}^{-1} X)^{-1} & -(X^T \Sigma_{11}^{-1} X)^{-1} P^T (T_{22} B)^{-1} \\ -(T_{22} B)^{-1} P (X^T \Sigma_{11}^{-1} X)^{-1} & (B^T T_{22}^T T_{22} B)^{-1} + (T_{22} B)^{-1} P Q^{-1} P^T (T_{22} B)^{-1} \end{bmatrix} \quad (4.2.10)$$

Furthermore,

$$W^T \Sigma^{-1} Y = \begin{bmatrix} X^T T_{11} T_{11}^T Y_1 + P^T T_{21} Y_1 + P^T T_{22} Y_2 \\ B^T T_{22}^T T_{21} Y_1 + B^T T_{22}^T T_{22} Y_2 \end{bmatrix}. \quad (4.2.11)$$
Therefore

\[
\begin{bmatrix}
\hat{\theta}
\end{bmatrix} = \begin{bmatrix}
(X'\Sigma_1^{-1}X)^{-1}X'\Sigma_1^{-1}Y_1 \\
B^{-1}(Y_2 - A\hat{\alpha}) + (T_{22}B)^{-1}T_{21}(Y_1 - X\hat{\alpha})
\end{bmatrix}
\] (4.2.12)

and \( \hat{\alpha} = \tilde{\alpha} \).

**Remark 4.2.1.** Note that Theorem 4.2.4 is a result of matrix algebra. The assumptions on \( Y_1, \alpha, \) and \( \epsilon_1 \) are superfluous, in the sense that no properties of these vectors are used in the proof of Theorem 4.2.4. We can show directly that if \( \hat{\alpha} \) and \( \tilde{\alpha} \) are defined as in (4.2.6) and (4.2.7), then \( \hat{\alpha} = \tilde{\alpha} \). However, we shall show in the following corollary that with additional statistical assumptions on the vector of errors, we can obtain more general and interesting results.

**Corollary 4.2.4.** Let the model (4.2.4) – (4.2.5) hold and assume that

\[
E\left[\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix}\right] = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\] (4.2.13)

and

\[
E\left[\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix}\begin{bmatrix}
\epsilon_1' \\
\epsilon_2'
\end{bmatrix}\right] = \sigma^2 \Sigma,
\] (4.2.14)

where

\[
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}
\]
is a known positive definite matrix, and $\sigma^2$ is an unknown positive constant. Then the following results hold:

(i) The best linear unbiased estimators of $\alpha$ based on models (4.2.4) and (4.2.5) are the same.

(ii) The estimated variance of $\hat{\alpha}$ based on model (4.2.4) is the same as the estimated variance of $\tilde{\alpha}$ based on model (4.2.5). In other words,

$$\hat{V}\{\hat{\alpha}\} = \hat{V}\{\tilde{\alpha}\},$$

where

$$\hat{V}\{\hat{\alpha}\} = \hat{\sigma}^2 X' \Sigma^{-1}_1 X^{-1},$$

$$\hat{V}\{\tilde{\alpha}\} = \sigma^2 K,$$

$K$ is the $p \times p$ upper left submatrix of the $(p + k) \times (p + k)$ matrix $(W' \Sigma^{-1} W)^{-1}$, and $\hat{\sigma}^2$ is the residual mean square from fitting model (4.2.4) and is equal to the residual mean square from fitting model (4.2.5).

Proof.

(i) Under the assumptions of (4.2.13) and (4.2.14), we conclude from Theorem 4.2.2 that $\hat{\alpha}$ is the best linear unbiased estimator of $\alpha$ based on model (4.2.4) and the result follows from Theorem 4.2.4.

(ii) Note that

$$Q(\tilde{\alpha}) = (Y - W\tilde{\alpha})' \Sigma^{-1}(Y - W\tilde{\alpha})$$
\[ Y \Sigma^{-1} (Y - W\tilde{\theta}) \]  \hspace{1cm} (4.2.16)

\[ = (Y_1', Y_2') \begin{bmatrix} T_{11}' T_{11} + T_{21}' T_{21} & T_{21}' T_{22} \\ T_{22} T_{21} & T_{22}' T_{22} \end{bmatrix} \begin{bmatrix} Y_1 - X\tilde{\alpha} \\ Y_2 - A\tilde{\alpha} - B\tilde{\theta} \end{bmatrix}, \]

\[ = Y_1' T_{11} T_{11} (Y_1 - X\tilde{\alpha}) + Y_1' T_{21} T_{21} (Y_1 - X\tilde{\alpha}) \]

\[ + Y_2' T_{22} T_{21} (Y_1 - X\tilde{\alpha}) + Y_1' T_{21} T_{22} Y_2 \]

\[ + Y_2' T_{22} T_{22} Y_2 - Y_1' T_{21} T_{22} (A\tilde{\alpha} + B\tilde{\theta}) \]

\[ - Y_2' T_{22} T_{22} (A\tilde{\alpha} + B\tilde{\theta}). \]  \hspace{1cm} (4.2.17)

Now, from (4.2.12), we see that

\[ A\tilde{\alpha} + B\tilde{\theta} = Y_2 + T_{22}^{-1} T_{21} (Y_1 - X\tilde{\alpha}). \]

Substituting in (4.2.17), we get

\[ Q(\tilde{\theta}) = Y_1' T_{11} (Y_1 - X\tilde{\alpha}). \]

Since \( \tilde{\alpha} = \hat{\alpha} \) [from ((i))] and \( \Sigma_{11}^{-1} = T_{11}' T_{11} \), we have

\[ Q(\tilde{\theta}) = Y_1' \Sigma_{11}^{-1} (Y_1 - X\tilde{\alpha}) \]
\[ = Q(\hat{\alpha}), \]

as required.

Now, if \( df_1 \) and \( df_2 \) denote the degrees of freedom from fitting models (4.2.4) and (4.2.5), respectively, then

\[
df_2 = (n + k) - (p + k) = n - p = df_1. \]

Therefore, the residual mean square from fitting either model is

\[
\hat{\sigma}^2 = \frac{Q(\hat{\alpha})}{n-p} = \frac{Q(\hat{\beta})}{n-p}. \tag{4.2.18} \]

The result now follows from (4.2.10) and (4.2.18).

Remark 4.2.2. The results of Corollary 4.2.4 can be shown directly as follows.

Given the vector \((\epsilon_1, \epsilon_2)'\) the transformed vector

\[
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} =
\begin{bmatrix}
T_{11} & 0 \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix} =
\begin{bmatrix}
T_{11}\epsilon_1 \\
T_{21}\epsilon_1 + T_{22}\epsilon_2
\end{bmatrix}
\]

is distributed with mean 0 and variance \(\sigma^2 I_{n+k}\). Applying the transformation \(T\) to the model (4.2.5), we get
\[
\begin{bmatrix}
T_{11}Y_1 \\
T_{21}Y_1 + T_{22}Y_2
\end{bmatrix} = \begin{bmatrix}
T_{11}X & 0 \\
T_{21}X + T_{22}A & T_{22}B
\end{bmatrix} \begin{bmatrix}
\alpha \\
\theta
\end{bmatrix} + \begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} \quad (4.2.19)
\]

\[
= \begin{bmatrix}
T_{11}X & 0 \\
P & T_{22}B
\end{bmatrix} \begin{bmatrix}
\alpha \\
\theta
\end{bmatrix} + \begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} \quad (4.2.20)
\]

where \( P = T_{21}X + T_{22}A \). Model (4.2.19) can be written as

\[ Z = M\beta + \eta \quad (4.2.21) \]

where

\[
Z = \begin{bmatrix}
T_{11}Y_1 \\
T_{21}Y_1 + T_{22}Y_2
\end{bmatrix},
\]

\[
M = \begin{bmatrix}
T_{11}X & 0 \\
P & T_{22}B
\end{bmatrix},
\]

\[
\eta = \begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix}
\alpha \\
\theta
\end{bmatrix}
\]

Therefore, from Theorem 4.2.1, the best linear unbiased estimator of \( \beta \), based on model (4.2.21), is

\[
\hat{\beta} = \begin{bmatrix}
\hat{\alpha} \\
\hat{\theta}
\end{bmatrix} = (M'M)^{-1}M'Z \quad (4.2.22)
\]
Now, note that

\[ M'M = W^\prime \Sigma^{-1} W \]

\[ M'Z = W^\prime \Sigma^{-1} Y. \]

Hence, from (4.2.10) and (4.2.11), we have

\[
\bar{\beta} = \begin{bmatrix} \bar{\alpha} \\ \bar{\theta} \end{bmatrix} = \begin{bmatrix} (X^\prime \Sigma_{11}^{-1} X)^{-1} X^\prime \Sigma_{11}^{-1} Y_1 \\ B^{-1} (Y_2 - A\bar{\alpha}) + (T_{22} B)^{-1} T_{21} (Y_1 - X\bar{\alpha}) \end{bmatrix}.
\]

Thus, as before, \( \hat{\alpha} = \bar{\alpha} \) and then \( \hat{V}(\bar{\alpha}) = \bar{V}(\bar{\alpha}). \)

Special Case 4.2.4. (Fuller's Method of Obtaining Predictions). Fuller (1980) discusses a method of obtaining predictions in regression models. The result can be obtained as a special case of Theorem 4.2.4. Given the linear model (4.2.4), consider the problem of predicting the unknown vector

\[ \tau = X_0 \alpha \]

where \( X_0 \) is \( k \times p \) vector of known constants. Fuller (1980) suggests forming the augmented model

\[
\begin{bmatrix} Y \\ 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ X_0 & -I \end{bmatrix} \begin{bmatrix} \alpha \\ \tau \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_0 \end{bmatrix}, \quad (4.2.23)
\]
where $I$ is the $k \times k$ identity matrix, and

$$
E\left[ \begin{pmatrix} \epsilon_1 \\ \epsilon_0 \end{pmatrix} \right] = \sigma^2 \Omega = \sigma^2 \begin{bmatrix} V_{11} & V_{10} \\ V_{01} & V_{00} \end{bmatrix}.
$$

Letting $\Omega^{-1} = T'T$, where

$$
T = \begin{bmatrix} T_{11} & 0 \\ T_{01} & T_{00} \end{bmatrix}
$$

and $T_{00}$ is nonsingular, he proved that the generalized least squares estimator of $(\alpha', \tau')$ can be expressed as

$$
\begin{bmatrix} \hat{\alpha} \\ \hat{\tau} \end{bmatrix} = \left[ (X'V^{-1}X)^{-1}X'V^{-1}Y_1 \right] X_0 \hat{\alpha} - T_{00}^{-1}T_{01}(Y_1 - X\hat{\alpha}).
$$

Thus, the generalized least squares estimator of $\alpha$ is the same as the estimator of the original problem and the best linear unbiased predictor of $\tau$ is given by

$$
\hat{\tau} = X_0 \hat{\alpha} - T_{00}^{-1}T_{01}(Y_1 - X\hat{\alpha}).
$$

where $\hat{\tau}$ is the estimator defined by the least squares solution to equation (4.2.24).

This result can be obtained by letting $Y_2 = 0$, $A = X_0$, $B = -I$, and $\theta = \tau$ in Theorem 4.2.4.

□
Remark 4.2.3. From Theorem 4.2.4 and Corollary 4.2.4, we conclude that, given a least squares problem, if we add $k$ new observations ($k \geq 1$) that are functions of $k$ parameters, and perhaps of other parameters, then the least squares estimators of the original parameters and their estimated variances are identical to those of the original problem.

Theorem 4.2.5. Let $A$ be an $m \times n$ matrix, $B$, an $n \times m$ matrix, $C$, an $n \times n$ matrix and $D$, an $m \times m$ matrix. If $C$, $D$, and $E = C^{-1} + BD^{-1}A$ are nonsingular, then

$$(D + ACB)^{-1} = D^{-1} - D^{-1}AE^{-1}BD^{-1}$$


Theorem 4.2.6. Let $C$, $D$, and $C + D$ be $k \times k$ nonsingular matrices. Then,

$$(C^{-1} + D^{-1})D^{-1} = C(C + D)^{-1}.$$  

Proof. We have

$$C(C + D)^{-1} = [(C + D)C^{-1}]^{-1}$$

$$= (I + DC^{-1})^{-1}$$

$$= (DD^{-1} + DC^{-1})^{-1}$$
Next, we use Theorem 4.2.3 to prove that under the covariance structure of all rotation designs considered in this thesis, the best linear unbiased estimator of current level is obtained by minimizing the sum of squared deviations from the model based on the assumption that observations in different streams are uncorrelated. The assumption is the assumption that the covariance matrix of the vech of the data matrix is block diagonal. The efficacy of this result lies in the reduced complexity of the estimation problem. To obtain the best linear unbiased estimator of current level, we need only invert an $n \times n$ block diagonal matrix instead of an $n \times n$ matrix.

**Theorem 4.2.6.** Assume the model

$$Y = X\theta + \epsilon,$$

(4.2.26)

where $Y$ is an $n \times 1$ vector of observations, $\theta$ is a $p \times 1$ vector of regression parameters,

$$X = J_{s \times 1} \otimes I_{p \times p}$$
is an \( n \times p \) full rank matrix of known constants, where \( J_{s \times 1} \) is the \( s \times 1 \) vector of ones, \( I_{p \times p} \) is the \( p \times p \) identity matrix, \( s \) is the number of streams, and \( p \) is the number of periods of the survey, so that \( n = sp \). Assume \( \epsilon \) is an \( n \times 1 \) vector of random variables, such that

\[
E\{\epsilon\} = 0,
\]

and

\[
E\{\epsilon\epsilon'\} = \Sigma = V_{bb} + cJ_{n \times 1}J_{n \times 1}', \tag{4.2.27}
\]

where

\[
V_{bb} = \text{Blockdiag}\{V_1, V_2, \ldots, V_s\},
\]

\( J_{n \times 1} \) is the \( n \times 1 \) vector of ones and \( c \) is an unknown positive constant. Then,

(i) the best linear unbiased estimator of \( \theta \) can be obtained by minimizing

\[
Q(\theta) = (Y - X\theta)'V_{bb}^{-1}(Y - X\theta). \tag{4.2.28}
\]

(ii) the variance of this estimator is

\[
(X'V_{bb}^{-1}X)^{-1}X'V_{bb}^{-1}(V_{bb} + cJ_{n \times 1}J_{n \times 1}')V_{bb}^{-1}X(X'V_{bb}^{-1}X)^{-1} \tag{4.2.29}
\]

Proof. For part (i), it is sufficient to show that the best linear unbiased estimator of \( \theta \), based on model (4.2.26) — (4.2.27), is the same as that based on the model

\[
Y = X\theta + \epsilon, \tag{4.2.30}
\]
where $Y$, $X$, and $\theta$ are as defined in model (4.2.26), and $e$ is a $n \times 1$ vector of random variables such that

$$E\{e\} = 0,$$

and

$$E\{ee'\} = V_{bb}. \quad (4.2.31)$$

Now, since $V_{bb}$ is symmetric and positive definite, we may write

$$V_{bb}^{-1} = R' R \quad (4.2.32)$$

for some nonsingular matrix $R$.

Applying the transformation $R$ to the models (4.2.26) — (4.2.27), we get

$$Z = W\theta + \eta, \quad (4.2.33)$$

where

$$Z = RY, \quad W = RX, \quad \eta = R\epsilon,$$

and

$$E\{\eta\} = 0,$$

$$E\{\eta\eta'\} = V = R\Sigma R' = I + cRJ_{n \times 1}J_{n \times 1}'R'. \quad (4.2.34)$$
Applying the same transformation $R$ to the model (4.2.30) – (4.2.31), we get

$$Z = W\theta + \delta, \quad (4.2.35)$$

where $\delta = Re$, and

$$E\{\delta\} = 0,$$

$$E\{\delta^T\} = I, \quad (4.2.36)$$

where $I$ is the $n \times n$ identity matrix.

By Theorem 4.2.3, the ordinary least squares estimator of $\theta$, based on model (4.2.35) – (4.2.36), is the best linear unbiased estimator of $\theta$ under models (4.2.33) – (4.2.34) if and only if there exists a matrix $Q$ such that

$$VW = WQ$$

or

$$RSR^T RX = RXQ.$$ 

Since $R$ is nonsingular and $V_{bb}^{-1} = R^T R$, this condition is equivalent to

$$\Sigma V_{bb}^{-1} X = XQ. \quad (4.2.37)$$

Now

$$\Sigma V_{bb}^{-1} X = X + c_{n \times 1} J_{n \times 1} V_{bb}^{-1} X. \quad (4.2.38)$$

Furthermore,

$$J_{n \times 1} J_{n \times 1} V_{bb}^{-1} = J_{n \times 1} \otimes C_{1 \times n}.$$
where $C_{1 \times n}$ is the $1 \times n$ vector of column sums of $V_{bb}^{-1}$. Therefore,

$$J_{n \times 1} J'_{n \times 1} V_{bb}^{-1} X = (J_{n \times 1} \otimes C_{1 \times n})(J_{s \times 1} \otimes I_{p \times p}),$$

$$= J_{n \times 1} \otimes B_{1 \times p},$$

(4.2.39)

where $B_{1 \times p}$ is the $1 \times p$ row vector whose $j$-th component is given by

$$\lambda_j = \sum_{\ell} c_{1\ell}, \quad 1 \leq j \leq p,$$

$c_{1i}$ is the $i$-th entry of $C_{1 \times n}$ and $\ell \equiv j \mod p$, $1 < \ell \leq n$. Now,

$$J_{n \times 1} \otimes B_{1 \times p} = (J_{s \times 1} \otimes I_{p \times p})(J_{p \times 1} \otimes B_{1 \times p})$$

$$= XD,$$

where $D = J_{p \times 1} \otimes B_{1 \times p}$. Therefore,

$$J_{n \times 1} J'_{n \times 1} V_{bb}^{-1} X = XD.$$

(4.2.40)

Substituting into (4.2.38), we have

$$\Sigma V_{bb}^{-1} X = X + cXD$$
for \( Q = I_{p \times p} + cD \). Therefore, the ordinary least squares estimator of \( \beta \), based on models (4.2.35) – (4.2.36), is the best linear unbiased estimator of \( \beta \) based on models (4.2.33) – (4.2.34). However, the ordinary least squares estimator of \( \beta \), based on models (4.2.35) – (4.2.36), is the best linear unbiased estimator of \( \beta \) based on models (4.2.30) – (4.2.31). Also, the best linear unbiased estimator of \( \beta \), based on models (4.2.33) – (4.2.34), is the same as the best linear unbiased estimator of \( \beta \) based on models (4.2.26) – (4.2.27). Therefore, the best linear unbiased estimator of \( \beta \), based on models (4.2.26) – (4.2.27), is the same as that based on models (4.2.30) – (4.2.31). Hence, the result (i) follows.

For part (ii), note that the best linear unbiased estimator of \( \beta \) can be written as

\[
\hat{\beta} = PY
\]  

(4.2.42)

where

\[
P = (X'V_{bb}^{-1}X)^{-1}X'V_{bb}^{-1}
\]  

(4.2.43)

Therefore, the covariance matrix of \( \hat{\beta} \) is

\[
\text{Var}\{\hat{\beta}\} = PVP'
\]
Next, we prove a matrix result that will be useful in the proof of the next theorem.

**Lemma 4.2.2.** Let $V$ be an $n \times n$ positive definite symmetric matrix, and $X$ be an $n \times p$ non-null matrix. Then, the matrix

$$V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}$$

is positive semidefinite.

**Proof.** Suppose $Y$ is an $n \times 1$ random vector such that $E\{Y\} = 0$ and $E\{YY'\} = V^{-1}$. Define a linear transformation of $Y$ as $Z = X'Y$. Then,

$$E\{Z\} = 0, \quad E\{ZZ'\} = X'V^{-1}X \quad \text{and} \quad E\{ZY'\} = X'V^{-1}.$$

Therefore, the regression coefficient in the regression of $Y$ on $Z$ is $(X'V^{-1}X)^{-1}X'V^{-1}$ and the mean squared deviations from this regression is

$$Q = V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}.$$
Note that $Q$ is the residual covariance matrix of the regression of $Y$ on $Z$ and, hence, is positive semidefinite.

In the following theorem, we show that, given a least squares problem, if we add new observations which are functions of additional parameters and possibly of other parameters in the original model, the variances of the estimators of the original parameters are less than or equal to their variances in the original problem.

**Theorem 4.2.7.** Assume the following models:

**Original Model:**

\[ Y_2 = X_2 \beta_2 + \epsilon_2, \]

\[ E\{\epsilon_2\} = 0, \quad \text{and} \quad E\{\epsilon_2 \epsilon_2'\} = \Sigma_{22}, \]

where $\Sigma_{22}$ is nonsingular.

**Augmented Model:**

\[
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} =
\begin{bmatrix}
A & B \\
0 & X_2
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix} +
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix},
\]

\[ E\left(\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix}\right) = \begin{bmatrix} 0 \\
0 \end{bmatrix}, \]

and

\[
E\left(\begin{bmatrix}
\epsilon_1 \\
\epsilon_1', \epsilon_2'
\end{bmatrix}\right) = \Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix},
\]
where $\Sigma$ is nonsingular. Assume the row rank of $A$ is equal to the number of
elements in $\beta_1$ and that $X_2^T X_2$ is nonsingular. Let $\hat{\beta}_2$ and $\bar{\beta} = (\hat{\beta}_1, \hat{\beta}_2)^T$ be the
least squares estimators of $\beta_2$ and of $\beta = (\beta_1, \beta_2)^T$ based on models (4.2.45) and
(4.2.46), respectively. Then

$$\text{Var}\{\hat{\beta}_2\} \geq \text{Var}\{\bar{\beta}_2\},$$

in the sense that $\text{Var}\{\hat{\beta}_2\} - \text{Var}\{\bar{\beta}_2\}$ is nonnegative definite.

Proof. First, note that $\text{Var}\{\hat{\beta}_2\} = (X_2^T \Sigma_2^{-1} X_2)^{-1}$. Let us transform the
observations in the augmented model so that the original observations and the new
observations are uncorrelated. Define the transformed observations as

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} I & -X_2 \Sigma_{12}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}. $$

Thus, the transformed model is given by

$$Z = W\beta + \xi,$$  \hspace{1cm} (4.2.47)

where

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

$$W = \begin{bmatrix} A & B - \Sigma_{12} \Sigma_{22}^{-1} X_2 \\ 0 & X_2 \end{bmatrix}. $$
and

\[
\xi = \begin{bmatrix}
I & -\Sigma_{12}\Sigma_{22}^{-1}X_2 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix}.
\]

Then, \( \mathbb{E}\{\xi\} = 0 \), and

\[
\mathbb{E}\{\xi'\} = Q = \begin{bmatrix}
Q_{11} & 0 \\
0 & \Sigma_{22}
\end{bmatrix},
\]

where \( Q_{11} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \). Therefore,

\[
\text{Var}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = (W'Q^{-1}W)^{-1}
\]

\[
= \begin{bmatrix}
A'Q_{11}^{-1}A & A'Q_{11}^{-1}P \\
P'Q_{11}^{-1}A & P'Q_{11}^{-1}P + X_2'\Sigma_{22}^{-1}X_2
\end{bmatrix}^{-1},
\]

where \( P = B - \Sigma_{12}\Sigma_{22}^{-1}X_2 \). Therefore,

\[
\text{Var}\{\beta_2\} = \left[X_2'\Sigma_{22}^{-1}X_2 + P'Q_{11}^{-1}P - P'Q_{11}^{-1}A(A'Q_{11}^{-1}A)^{-1}A'Q_{11}^{-1}P\right]^{-1}
\]

\[
= \left[X_2'\Sigma_{22}^{-1}X_2 + P'[Q_{11}^{-1} - Q_{11}^{-1}A(A'Q_{11}^{-1}A)^{-1}A'Q_{11}^{-1}]P\right]^{-1}
\]

\[
= \left(X_2'\Sigma_{22}^{-1}X_2 + P'\Omega P\right)^{-1},
\]
where
\[ \Omega = Q_{11}^{-1} - Q_{11}^{-1} A(A'Q_{11}^{-1}A)^{-1}A'Q_{11}^{-1}. \]

Now, \( Q_{11} \) is the covariance matrix of \( Y_1 - \Sigma_{12} \Sigma_{22}^{-1} Y_2 \) and, hence, is positive definite and symmetric. Therefore, from Lemma 4.2.2, we conclude that \( \Omega \) is positive semidefinite. This implies that \( P'\Omega P \) is positive semidefinite, and therefore,

\[ \text{Var}\{ \hat{\beta}_2 \} = (X'_2 \Sigma_{22}^{-1}X_2 + P'\Omega P)^{-1} \leq (X'_2 \Sigma_{22}^{-1}X_2)^{-1} = \text{Var}\{ \hat{\beta}_2 \} \]

**Alternative Proof.** This proof does not require that the row rank of \( A \) be equal to the number of elements in \( \beta_1 \). Consider any linear function, say \( \lambda' \beta_2 \), of the parameters in the vector \( \beta_2 \). Under the augmented model,

\[ \text{E}\{ \lambda' \hat{\beta}_2 \} = \text{E}\{ \lambda' (X'_2 \Sigma_{22}^{-1}X_2)^{-1}X'_2 \Sigma_{22}^{-1}Y_2 \} \]

\[ = \lambda' (X'_2 \Sigma_{22}^{-1}X_2)^{-1}X'_2 \Sigma_{22}^{-1}X_2 \beta_2 \]

\[ = \lambda' \beta_2. \]

In other words, \( \lambda' \hat{\beta}_2 \) is an unbiased estimator of \( \lambda' \beta_2 \) under the augmented model. Furthermore, we may write \( \lambda' \hat{\beta}_2 \) as a linear function of \( (Y'_1, Y'_2) \) as follows:

\[ \lambda' \hat{\beta}_2 = \begin{bmatrix} 0, \lambda' (X'_2 \Sigma_{22}^{-1}X)^{-1}X'_2 \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}. \]
That is, $\lambda' \hat{\beta}_2$ belongs to the class $\mathcal{C}$ of all unbiased estimators of $\lambda' \beta_2$, which are linear functions of $(Y_1', Y_2')'$. However, from Theorem 4.2.2, $\lambda' \hat{\beta}_2$ is the best linear unbiased estimator of $\lambda' \beta_2$ (in the sense of minimum variance) under the augmented model. Therefore, necessarily,

$$\text{Var}(\lambda' \hat{\beta}_2) \geq \text{Var}(\lambda' \tilde{\beta}_2)$$

for any $\lambda$. \(\square\)

In the next lemma, we consider a least squares estimation problem to which is added new observations which are (i) uncorrelated with the original observations, and (ii) functions of new parameters and possibly of the original parameters. We show that if the original parameters are known, then the variances of the least squares estimators constructed treating the original parameters as unknown are greater than or equal to those treating the original parameters as known.

**Lemma 4.2.3.** Assume the model

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}, \tag{4.2.48}$$

where $Y_1$ and $Y_2$ are vectors of observations of dimensions $n_1 \times 1$ and $n_2 \times 1$, respectively, $X_{11}$, $X_{21}$, and $X_{22}$ are matrices whose entries are known and whose dimensions are $n_1 \times p_1$, $n_2 \times p_1$, and $n_2 \times p_2$, respectively, $\theta_1$ and $\theta_2$ are vectors of parameters of dimensions $p_1 \times 1$ and $p_2 \times 1$, respectively, and $(\epsilon_1', \epsilon_2')'$ is an $(n_1 + n_2) \times 1$ vector of errors such that
and

$$E\{(\epsilon_1, \epsilon_2)\} = (0, 0)$$

where \( \Sigma_{11} \) and \( \Sigma_{22} \) are positive definite symmetric matrices of dimensions \( p_1 \times p_1 \) and \( p_2 \times p_2 \), respectively. Consider the following reduced form of the second equation in (4.2.48) in which \( \theta_1^0 \) is the known true value of \( \theta_1 \):

$$Y_2 - X_{21} \theta_1^0 = X_{22} \theta_2 + \epsilon_2. \quad (4.2.49)$$

If \( \hat{\theta}_2 \) and \( \bar{\theta}_2 \) are the least squares estimates of \( \theta_2 \) based on models (4.2.48) and (4.2.49), respectively, then

$$\text{Var}\{\hat{\theta}_2\} \geq \text{Var}\{\bar{\theta}_2\}.$$
Now, suppose \( \theta_1 \) is unknown and has to be estimated. If we rewrite model (4.2.48) as

\[
Y = W\theta + \epsilon,
\]

where

\[
Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad W = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix},
\]

\[
\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix},
\]

then, the covariance matrix of the least squares estimator of \( \theta \) is

\[
(W\Sigma^{-1}W)^{-1} = \begin{bmatrix}
    X_{11}' \Sigma_{11}^{-1}X_{11} + X_{21}' \Sigma_{22}^{-1}X_{21} & X_{21}' \Sigma_{22}^{-1}X_{22} \\
    X_{22}' \Sigma_{22}^{-1}X_{21} & X_{22}' \Sigma_{22}^{-1}X_{22}
\end{bmatrix}^{-1}.
\]

Now, the covariance matrix of \( \hat{\theta}_2 \) is the lower right \( p_2 \times p_2 \) submatrix of the \((p_1 + p_2) \times (p_1 + p_2)\) matrix \((W\Sigma^{-1}W)^{-1}\). Therefore,

\[
[\text{Var} \{ \hat{\theta}_2 \}]^{-1} = X_{22}' \Sigma_{22}^{-1}X_{22}
\]

\[
- X_{22}' \Sigma_{22}^{-1}X_{21} (X_{11}' \Sigma_{11}^{-1}X_{11} + X_{21}' \Sigma_{22}^{-1}X_{21})^{-1} X_{21}' \Sigma_{22}^{-1}X_{22}
\]  \hspace{1cm} (4.2.50)

\[
\leq X_{22}' \Sigma_{22}^{-1}X_{22}
\]
since the second term on the right hand side of (4.2.50) is nonnegative definite. Therefore,

\[ \text{Var} \{ \hat{\theta}_2 \} \geq (X_{22}'X_{22}^{-1}X_{22})^{-1} = \text{Var} \{ \hat{\theta}_2 \}. \]

Hence, the result. \( \square \)

Before describing the least squares estimation procedures for repeated surveys, we give a general description of the class of repeated surveys under consideration in this thesis, and state some assumptions that will be useful in the sequel. The assumptions stem from the special structure of repeated survey data under consideration in this thesis.

4.2.1. Basic Assumptions

Assume that in each period of the survey, a fixed number of rotation groups are included in the sample. The basic data consist of the elementary estimators of the parameters of interest associated with different rotation groups. If the number of rotation groups included in the sample in each period is \( s \), where \( s \geq 1 \), then the data obtained over a time interval of \( p \) periods can be arranged in a \( p \times s \) data matrix denoted by \( M \). The total number of observations is \( n = ps \), where \( n \) is the number of entries in \( M \). We shall refer to the columns of \( M \) as "streams." Note that the number of rotation groups in the sample each period corresponds to the number of streams. Assume that:

1. A given rotation group in a stream is observed over a period of total length \( m + 1 \) and the observation pattern is fixed.
2. The columns of $M$ are independent.

3. The covariance structure of the observations in a stream is constant over time, and it is the same for all streams.

4. Of the $s$ rotation groups included in the sample at each time, one group is being observed for the first time, one is being observed for the second time, ..., one is being observed for the last time, where the last time is $m$ periods from the first observation.

4.3. Best Linear Unbiased Estimation

4.3.1. Introduction

The best linear unbiased estimator of the current level is defined to be the best unbiased linear combination of all of the elementary estimators from the rotation groups available for estimation, where the best linear estimator is the linear estimator with the smallest variance. It is possible in the process of computing the best linear unbiased estimator for the current level, to also compute the best linear unbiased estimators for all periods using data available at the current time.

Our discussion of best linear unbiased estimation will proceed according to the generalized least squares approach described by Wolter (1979) and Fuller (1990). A linear model will be constructed to define the estimator.

Suppose that a repeated survey has been in operation for $p$ periods. Assume that $s$ streams of data collected over $p$ periods are available for estimation. For any $t$, let $y_{i,0,t}$, $i = 1, 2, ..., s$, be the elementary estimate of the parameter of interest, obtained from the rotation group which is in stream $i$ at time $t$. Let
be the vector of \( p \) observations in the \( i \)-th stream. Let \( Y_p \) be the data vector formed by the streams or columns of the \( p \times s \) data matrix \( M \) arranged chronologically. In other words, let

\[
Y_p = (y_1, \ldots, y_s)'
\]

be the \( n \times 1 \) vector of observations where \( n = s \times p \), let

\[
\theta_p = (\theta_1, \ldots, \theta_p)
\]

be the \( p \times 1 \) vector of true but unknown parameters of interest. Let

\[
X = J_{s \times 1} \otimes I_{p \times p}
\]

be the \( n \times p \) design matrix which relates the estimates in \( Y_p \) to their expected values in \( \theta_p \). Thus, \( X \) is the Kronecker product \( J_{s \times 1} \otimes I_{p \times p} \) where \( J_{s \times 1} \) is the \( s \times 1 \) vector of ones, and \( I_{p \times p} \) is the \( p \times p \) identity matrix. The linear model is then

\[
Y_p = X\theta_p + \epsilon_p
\]
where $\epsilon_p$ is the vector of error terms satisfying

$$E\{\epsilon_p\} = 0$$

and

$$E\{\epsilon_p \epsilon'_p\} = V_p,$$

where $V_p$ is assumed to be symmetric and positive definite. From Theorem 4.2.2, the best linear unbiased estimator of $\theta_p$ is

$$\hat{\theta}_p = GY_p \quad (4.3.6)$$

where

$$G = (X'V_p^{-1}X)^{-1}X'V_p^{-1}$$

and the covariance matrix of $\hat{\theta}_p$ is

$$\text{Var}\{\hat{\theta}_p\} = \Sigma_p = (X'V_p^{-1}X)^{-1}. \quad (4.3.7)$$

The matrix $G$ is the weight matrix, and is composed of the coefficients that are applied to the elementary estimators in forming the best linear unbiased estimator of $\theta_p$.

The best linear unbiased estimator of any linear combination $\lambda'\theta_p$ of the components of the parameter vector $\theta_p$ is $\lambda' \hat{\theta}_p$, with variance $\lambda' \Sigma_p \lambda$. In particular, if $G_i, i = 1, 2, \ldots, p$ denotes the $i$-th row of $G$, then the best linear unbiased
estimator of the current level based on data through time \( p \) is

\[
\hat{\theta}_p(p) = G_p Y_p. \tag{4.3.8}
\]

The best linear unbiased estimator of change in level \( \theta_p - \theta_{p-q} \), \( 1 \leq q \leq p - 1 \), is

\[
\hat{\theta}_p(p) - \hat{\theta}_{p-q}(p) = D_{p,p-q} Y_p \tag{4.3.9}
\]

where \( D_{p,p-q} = G_p - G_{p-q} \). Therefore, the variance of \( q \)-period change in level is

\[
V\{\hat{\theta}_p(t) - \hat{\theta}_{p-q}(t)\} = D_{p,p-q} V_p D_{p,p-q}'. \tag{4.3.10}
\]

4.3.2. The Revision of Previous Estimates

The computation of the optimal estimators of the levels preceding \( p \) in a \( p \)-period survey [that is, the components \( \theta_1, \ldots, \theta_{p-1} \) in the parameter vector \( \theta_p \) defined in (4.3.3)] uses all the data through the \( p \)-th period. This procedure yields updated or revised estimates of previous levels to reflect the use of more observations which become available in succeeding periods.

There are some advantages associated with the revision of previous estimates when more data become available. First, it provides estimators for current level and change that are internally consistent, that is, the optimal estimator of change, say \( \theta_t - \theta_{t-q} \) at any time \( t \), is the difference between the optimal estimates of \( \theta_t \) and \( \theta_{t-q} \) at time \( t \). A second advantage, which is demonstrated in Chapter 5, is that revision of
previous estimates leads to increased precision in the estimation of change. Despite these advantages, there are situations in which revision of previous estimates is neither practical, nor desirable. Repeated revision of published time series of survey estimates may present difficulties in terms of cost of record-keeping and bookkeeping for users of the entire time series. There is also the potential for a reduction in the public credibility of the published estimates. These drawbacks have led some survey organizations to not revise previous estimates. For instance, the Bureau of Labor Statistics of the United States does not revise unemployment statistics. Once released, they are official estimates.

The unrevised estimate of change in level \( \theta_t - \theta_{t-q} \), \( 1 \leq q \leq t - 1 \), is the difference between the optimal estimate of \( \theta_t \) based on data through time \( t \) and the optimal estimate of \( \theta_{t-q} \) based on data through time \( t - q \). The ideal procedure for constructing weights for the unrevised estimator of \( \theta_t - \theta_{t-q} \) is to take the difference between the weights for the estimate of \( \theta_t \) at time \( t \) and the weights for the estimate of \( \theta_{t-q} \) at time \( t - q \). Note that the weights for all rotation groups introduced in the sample after time \( t - q \) up until time \( t \) are 0. Patterson (1950) and Gurney and Daly (1965) considered the use of unrevised estimators of change in level and concluded that their use may result in considerable loss of efficiency relative to the optimal estimators of change (that is, change under revision of previous estimates) if the inter-period correlation is high or the rotation rate (that is, the proportion of the sample that is replaced from one period to the next) is high.

We now describe a simple procedure for computing estimators of change with no revision of the previous estimates. The procedure yields unrevised estimators of change whose variances are very close to those of the optimal estimator of change.
The idea of the computation is to express both the previous estimate and the current estimate as linear functions of an extended data vector. For instance, suppose for fixed \( t \), an estimate of the change \( \theta_t - \theta_{t-q}, 1 \leq q \leq t - 1 \), is required, without revising previous estimates. Let \( Y_t \) denote the data vector of the length \( st \), where \( s \) is the number of streams and \( t \) is the number of periods, and let the \( t \)-th period be the current period, that is, the last period of observation. Then the best linear unbiased estimator of \( \theta_t \) can be expressed as a linear function of \( Y_t \) as

\[
\hat{\theta}_t(t) = L_t^* Y_t. \tag{4.3.11}
\]

where \( L_t^* \) is a \( 1 \times st \) row vector of appropriate coefficients. Now, the unrevised estimate of \( \theta_{t-q} \) is constructed using the coefficients in \( L_t \). The estimate can be expressed as

\[
\hat{\theta}_{t-q}(t-q) = L_{t-q}^* Y_t, \tag{4.3.12}
\]

where \( L_{t-q}^* \) is the \( 1 \times st \) row vector whose entries are such that the coefficients for the elementary estimates from the rotation group at time \( t - q \) are exactly the coefficients of the elementary estimates from the rotation groups at time \( t \) in \( L_t \), the coefficients for the elementary estimates from the rotation groups at time \( t - q - 1 \) in \( L_{t-q} \) are exactly those at time \( t - 1 \) in \( L_t \), and so on. In general, the coefficients for the elementary estimates from the rotation groups at time \( t - q - j (j > 0) \) in \( L_{t-q} \) are exactly equal to the coefficients for the elementary estimates from the rotation groups at time \( t - j \) in \( L_t \). The coefficients in \( L_{t-q} \) for the elementary estimates from the rotation groups at time \( t \), where \( t - q < \ell \leq t \), are zero. The unrevised
estimator of $\theta_t - \theta_{t-q}$ is then given by

$$\hat{\theta}_t(t) - \hat{\theta}_{t-q}(t-q) = \lambda_t Y_t,$$  \hspace{1cm} (4.3.13)

where

$$\lambda_t = L_t - L_{t-q}.$$  

If $V_t$ is the covariance matrix of $Y_t$, then the variance of the estimate (4.3.13) is $\lambda_t V_t \lambda_t$.

The reader will note that the procedures described here are general and applicable to any rotation design. No assumptions are made regarding the functional form of the elementary estimators associated with the rotation groups, or their covariance structure. The only requirements are that the elementary estimators for each period be unbiased for the parameter of interest at that period and that their error process be stationary. To construct the best linear unbiased estimator under any rotation scheme, one simply forms the data vector $Y$ of elementary estimators, specifies the corresponding covariance matrix $V$, and constructs the design matrix $X$. The best linear unbiased estimator is then constructed from the usual least squares formulas.

In the following lemma, we show that under the data and covariance structures described in this section, the variances of the least squares estimators of the parameters of interest at the current level decrease as the number of periods increases.

**Lemma 4.3.1.** Let $V\{\hat{\theta}_h | Y_i, ..., Y_j\}$ denote the variance of the least squares estimator of the parameter $\theta$ at time $h$ based on the data from period $i$ through
period \( j \) \((j > i)\). Then, under the assumptions of Section 4.2.1, we have

\[
V\{\hat{\theta}_t | Y_1, ..., Y_t\} \geq V\{\hat{\theta}_{t+1} | Y_1, ..., Y_{t+1}\}.
\]

**Proof.** Note that by the symmetry in the rotation design described in Section 4.2.1, the covariance structure of \((Y_1, ..., Y_t)\) is exactly the same as the covariance structure of \((Y_2, ..., Y_{t+1})\). Furthermore, \(\hat{\theta}_t\) is the same linear function of \((Y_1, ..., Y_t)\) as \(\hat{\theta}_{t+1}\) is of \((Y_2, ..., Y_{t+1})\). Therefore,

\[
V\{\hat{\theta}_t | Y_1, ..., Y_t\} = V\{\hat{\theta}_{t+1} | Y_2, ..., Y_{t+1}\}.
\]

However, from Theorem 4.2.7, we know that

\[
V\{\hat{\theta}_{t+1} | Y_2, ..., Y_{t+1}\} \geq V\{\hat{\theta}_{t+1} | Y_1, ..., Y_{t+1}\}.
\]

Therefore,

\[
V\{\hat{\theta}_t | Y_1, ..., Y_t\} \geq V\{\hat{\theta}_{t+1} | Y_1, ..., Y_{t+1}\}.
\]

\(\Box\)

4.4. The Recursive Regression Estimation Procedure

4.4.1. Introduction

The recursive regression estimation procedure is a computationally efficient method of producing minimum variance estimators in repeated surveys. Instead of
using all the available information in a large least squares computation, the recursive regression estimation procedure uses a linear combination of an appropriate set of initial estimates and the new observations at the current level to produce the best linear unbiased estimators of current level and change.

Suppose the survey has been in operation for \( m \) periods. At the current time, denoted by \( c \), where \( c \geq m \), a set of \( s \) elementary estimators of the parameter \( \theta_c \), are observed. These observations can be transformed so that they are uncorrelated with previous observations. After transformation, the expected values of the transformed observations are functions of parameters in addition to \( \theta_c \). Assume the best linear unbiased estimators of the parameters for the previous \( m \) periods, as well as the \( m \times m \) covariance matrix of these estimators, are available. Thus, at the current time \( c \), we have:

1. \( m \) initial estimates, \( \hat{\theta}_{c-1}(m) = (\hat{\theta}_{c-m}, \ldots, \hat{\theta}_{c-1})' \),
2. the covariance matrix \( \Sigma_{11,c-1}(m) \) of \( \hat{\theta}_{c-1}(m) \),
3. \( s \) independent observations, which are obtained by transforming the observations at the current time so that they are uncorrelated with previous observations. Let us denote these transformed observations by \( z_{ic} \), \( i = 1, 2, \ldots, s \), where

\[
z_{ic} = y_{i,0,c} - \sum_{j=1}^{m} b_{k(i,c),j} y_{i,0,c-j}^j \]

where \( k(i, c) = k \) defines time-in-sample as a function of \( i \) and \( c \). For \( k = 1, 2, \ldots, s \), we have

\[
z_{1c} = y_{1,0,c} \]
where \( y_{i,0,t} \) denotes the elementary estimator from the rotation group which is in stream \( i \) at time \( t \). The \( b_{ij} \)'s are constructed so that \( z_{ic} \) is uncorrelated with \( y_{i,0,c-j} \) for all \( j > 0 \). The expected values of the \( z_{1c}, z_{2c}, \ldots, z_{sc} \) are \( \theta_c, \theta_c - b_{21} \theta_{c-1}, \ldots, \theta_c - \sum_{j=1}^{m} b_{sj} \theta_{c-j} \), respectively. Our objective is to construct the minimum variance estimator of \( \theta_c \), the current level of the parameter of interest. The linear model at the current time may be written as

\[
Z_c = W \theta_{c(m+1)} + \epsilon_c, \tag{4.4.1}
\]

where

\[
Z_c' = (\theta_{c-1(m)}, z_{sc})',
\]

\[
W = \begin{bmatrix} I_m & 0 \\ X_{21} & J_{s \times 1} \end{bmatrix},
\]

\[
\theta_{c(m+1)} = (\theta_{c-m}, \ldots, \theta_{c-1}, \theta_c),
\]

\[
z_c' = (z_{1c}, z_{2c}, \ldots, z_{sc}),
\]

\( I_m \) is the \( m \times m \) identity matrix, \( J_{s \times 1} \) is the \( s \times 1 \) column vector of ones, and \( X_{21} \) is an \( s \times m \) matrix which is constant over time.
The covariance matrix of $Z_c$ is:

$$V_c = \begin{bmatrix} \Sigma_{11,c-1(m)} & 0 \\ 0 & Q_{00} \end{bmatrix},$$

where

$$Q_{00} = \text{Var}(z_c) = \text{Diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_s^2\},$$

$$\sigma_i^2 = \text{var}(z_{ic}), \ i = 1, 2, \ldots, s.$$ 

It is assumed that $\sigma_i^2, \ i = 1, 2, \ldots, s$, are positive.

**Definition 4.4.1.** (The Recursive Regression Estimator). The recursive regression estimator of the vector $\theta_{c(m+1)}$ of parameters at time $c$ is the least squares estimator of based on model (4.4.1), that is, constructed with $Z_c$.

If $\hat{\theta}_{c(m+1)}$ denotes the recursive regression estimator of $\theta_{c(m+1)}$, we may write

$$\hat{\theta}_{c(m+1)} = Pz_c,$$  \hspace{1cm} (4.4.2)

where

$$P = (W' V_c^{-1} W)^{-1} W' V_c^{-1},$$  \hspace{1cm} (4.4.3) 

and $W, V_c$, and $Z_c$ are defined in model (4.4.1). The covariance matrix of $\hat{\theta}_{c(m+1)}$ is

$$Q_R = (W' V_c^{-1} W)^{-1}$$  \hspace{1cm} (4.4.4)
The matrix $P$ is the weight matrix and is composed of the coefficients that are applied to the elementary estimators in forming the recursive regression estimator. The recursive regression estimator of any linear combination $\lambda' \theta_{c(m+1)}$ of components of the parameter vector $\theta_{c(m+1)}$ is $\lambda' \theta_{c(m+1)}$, with variance $\lambda' Q_{R} \lambda$. In particular, if $P_i$ denotes the $i$-th row of $P$, then the recursive regression estimator of the current level is

$$\hat{\theta}_{c,R} = P_c Z_c$$

and the recursive regression estimator of $q$-period change, $1 \leq q \leq c - 1$, is

$$\hat{\theta}_c(c) - \hat{\theta}_{c-q}(c) = HZ_c$$

where

$$H = P_c - P_{c-q}.$$ 

The variances of the estimators (4.4.5) and (4.4.6) are $P_c V_c P'_c$ and $HV_c H'$, respectively. The unrevised estimate of $q$-period change is:

$$\hat{\theta}_{c,R} - \hat{\theta}_{c-q,R} = DZ_c$$

where $D = P_t - P_0$ and $P_0$ is the $1 \times (m+s)$ row vector with its $(m-1)$-th component equal to one and all other components equal to zero. The variance of this estimator is $DV_c D'$. 
We shall refer to the \( m \) initial estimates used in the recursive regression estimation procedure as the recursive least squares estimates. We now state and prove a theorem which shows that estimators of current level and change obtained from the recursive regression procedure are optimal in the sense of minimum variance. The theorem shows that at any fixed time \( t \) in a repeated survey, the recursive least squares estimates

\[ \hat{\theta}_{t-m}(t-1), \hat{\theta}_{t-m+1}(t-1), \ldots, \hat{\theta}_{t-1}(t-1), \]

and the new independent observations \( z_{it}, i = 1, \ldots, s \) derived from the observations obtained from the \( s \) streams in the sample at time \( t \), contain all the relevant information for the problem of estimating \( \theta_t \) and \( \theta_t - \theta_{t-q}, q > 0 \). This means that for estimating \( \theta_t \) and \( \theta_t - \theta_{t-q}, q > 0 \), we need only use the initial estimates and the new independent observations instead of using all the observations available at time \( t \).

**Theorem 4.4.1. (Optimality of the recursive regression estimator)** Suppose at time \( t \), the survey has been in operation for \( t \) periods and we have the estimators \( \hat{\theta}_1(t-1), \hat{\theta}_2(t-1), \ldots, \hat{\theta}_{t-m}(t-1), \hat{\theta}_{t-m+1}(t-1), \ldots, \hat{\theta}_{t-1}(t-1) \), which are the least squares estimators of \( \theta_1, \theta_2, \ldots, \theta_{t-m}, \theta_{t-m+1}, \ldots, \theta_{t-1} \), respectively at time \( t \), and \( m < t \). Then, the recursive regression estimator of current level \( \theta_t \) is the best linear unbiased estimator of \( \theta_t \) based on data for periods \( 1, 2, \ldots, t \).
Proof. The model at time $t$ in all of the unknown parameters may be written as

$$Y_t = X\theta_t + \epsilon_t, \quad (4.4.8)$$

where

$$Y_t' = [\hat{\theta}_1(t-1), ..., \hat{\theta}_{t-m+1}(t-1),$$

$$\hat{\theta}_{t-m}(t-1), ..., \hat{\theta}_{t-1}(t-1), z_t'],$$

$$z_t' = (z_{1t}, ..., z_{st}),$$

$$\theta_t' = (\theta_1, ..., \theta_{t-m+1}, \theta_{t-m}, ..., \theta_{t-1}, \theta_t),$$

$$X = \begin{bmatrix} I_{t-m-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & X_{32} & J_{s \times 1} \end{bmatrix}.$$ 

$I_v$ is the $v \times v$ identity matrix, $J_{s \times 1}$ is the $s \times 1$ column vector of ones and $X_{32}$ is an $s \times m$ matrix which is constant over time, and $\epsilon_t$ is an $(t + s) \times 1$ vector of random errors such that

$$E\{\epsilon\} = 0 \text{ and } E\{\epsilon \epsilon'\} = \Sigma = \begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 \\ \Omega_{12}' & \Omega_{22} & 0 \\ 0 & 0 & \Omega_{33} \end{bmatrix},$$

where

$$\Omega_{11} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}, \quad \Omega_{12} = \begin{bmatrix} \rho_{12}' & \rho_{13}' \end{bmatrix}, \quad \Omega_{22} = \begin{bmatrix} \rho_{22}' & \rho_{23}' \end{bmatrix}, \quad \Omega_{33} = \begin{bmatrix} \rho_{33}' \end{bmatrix}.$$
where
\[
\begin{bmatrix}
\Omega_{11} & \Omega_{12} \\
\Omega_{12} & \Omega_{22}
\end{bmatrix}
\]
is such that \(\Omega_{11}\) is the covariance matrix of \(\mathbf{\theta}_1 = [\hat{\theta}_1(t - 1), \ldots, \hat{\theta}_{t-m+1}(t - 1)]',\)
\(\Omega_{22}\) is the covariance matrix of \(\mathbf{\theta}_{\Pi} = [\hat{\theta}_{t-m}(t - 1), \ldots, \hat{\theta}_{t-1}(t - 1)]',\)
\(\Omega_{12}\) is the cross covariance matrix between \(\mathbf{\theta}_1\) and \(\mathbf{\theta}_{\Pi}\), and \(\Omega_{33} = \text{Var}\{z_t\} = Q_{00}\) defined in model 4.4.1. We conclude from Theorem 4.2.2 that the best linear unbiased estimator of \(\mathbf{\theta}_t\) based on model (4.4.8) is
\[
\hat{\theta}_t(t) = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y_t \\
\]
(4.4.9)
with variance
\[
\text{Var}\{\hat{\theta}_t\} = (X'\Sigma^{-1}X)^{-1}. \\
\]
(4.4.10)

The best linear unbiased estimator of \(\mathbf{\theta}_t\) and its variance are respectively, the \(t\)-th element of \(\hat{\mathbf{\theta}}_t\) and the \(t\)-th diagonal element of \(\text{Var}\{\hat{\mathbf{\theta}}_t\}\). In our recursive regression procedure, we retain only \(m\) initial estimates. The reduced model thus becomes
\[
Z_t = W\theta_{t(m+1)} + \eta_t, \\
\]
(4.4.11)
where
\[ W = \begin{bmatrix} I_m & 0 \\ X_{32} & J_{s \times 1} \end{bmatrix}, \]
\[ Z_t = (\hat{\theta}_{t-1}(m), z_t') \]
\[ = (\hat{\theta}_{t-m}(t-1), \ldots, \hat{\theta}_{t-1}(t-1), z_{t1}, z_{t2}, \ldots, z_{ts}), \]
\[ \hat{\theta}_t^{(m+1)} = (\hat{\theta}_{t-m}, \hat{\theta}_{t-m+1}, \ldots, \hat{\theta}_t), \]
\[ \eta_t \text{ is a } (m + s + 1) \times 1 \text{ vector of random errors, such that } E\{\eta_t\} = 0, \text{ and} \]
\[ E\{\eta_t \eta_t'\} = V_t = \begin{bmatrix} \Sigma_{11,t-1(m)} & 0 \\ 0' & \Sigma_{00} \end{bmatrix}, \]
where \( \Sigma_{11,t-1(m)} = \Omega_{22} \), the covariance matrix of \( \hat{\eta}_t \). Also, from Theorem 4.2.4, we conclude that the best linear unbiased estimator of \( \theta_t = (\theta_{t-m}, \theta_{t-m+1}, \ldots, \theta_{t-1})' \) at time \( t \) based on model (4.4.8) is the same as that based on model (4.4.11). Therefore, the best linear unbiased estimator of \( \theta_t \) based on model (4.4.11) is the same as that based on model (4.4.8). But the best linear unbiased estimator of \( \theta_t \), based on model (4.4.11), is precisely the recursive regression estimator of \( \theta_t \), denoted
by \( \hat{\theta}_{t,R} \). Therefore,

\[
\hat{\theta}_{t,R} = \hat{\theta}_t(t).
\]

Hence, the recursive regression estimator is the best linear unbiased estimator of \( \theta_t \).

**Remark 4.4.1.** It follows from Theorem 4.2.2 that the best linear unbiased estimator of any parametric function \( \lambda' \theta_{t(m+1)} \) is \( \lambda' \hat{\theta}_{t(m+1)} \), where

\[
\hat{\theta}_{t(m+1)} = (W'V_t^{-1}W)^{-1}W'V_t^{-1}Z_t. \tag{4.4.12}
\]

In particular, the best linear unbiased estimator of \( q \)-period change \( \theta_t - \theta_{t-q} \), where \( 1 \leq q \leq t - 1 \), is \( \hat{\theta}_{t,R} - \hat{\theta}_{t-q,R} \), which is \( \hat{\theta}_t(t) - \hat{\theta}_{t-q}(t) \) under revision of previous estimates. The variances of the recursive regression estimators of change over several periods constructed with no revision of previous estimates can be computed by the procedure described in the appendix.

We now illustrate the recursive regression estimation procedure. The best linear unbiased estimator of

\[
\theta_{t(m+1)} = (\theta_{t-m}, ..., \theta_{t-1}, \theta_t)',
\]
on the basis of model (4.4.11) is

\[
\hat{\theta}_{t(m+1)} = (W^tV_t^{-1}W)^{-1}W^tV_t^{-1}Z_t,
\]

and the \((m + 1) \times (m + 1)\) covariance matrix of \(\hat{\theta}_{t(m+1)}\) is

\[
\Sigma_{t(m+1)} = (W^tV_t^{-1}W)^{-1}.
\] (4.4.13)

Let us partition \(\Sigma_{t(m+1)}\) as

\[
\Sigma_{t(m+1)} = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix},
\]

(4.4.14)

where

\[
\Sigma_{11} = \text{Var}\{\hat{\theta}_{t-m}(t)\}, \quad \Sigma_{11,1}(m) = \text{Var}\{\hat{\theta}_{t-m+1}(t), ..., \hat{\theta}_t(t)\}',
\]

\[
\Sigma_{12} = \text{Cov}\{\hat{\theta}_{t-m}(t), (\hat{\theta}_{t-m+1}(t), ..., \hat{\theta}_t(t))'\}.
\]

In the recursive regression procedure, the model for time \(t + 1\) is constructed as follows: We drop the initial estimate \(\hat{\theta}_{t-m}(t)\) from the data vector, and drop \(\theta_{t-m}\) from the parameter vector. The parameter \(\theta_{t+1}\) is added to the parameter vector. This way, the dimension of the estimation problem is kept constant over time. Thus, the model at time \(t + 1\) may be written as:
\( Z_{t+1} = W \theta_{t+1(m+1)} + \epsilon_{t+1} \) \hspace{1cm} (4.4.15)

where

\[ Z_{t+1}' = (\theta_t'(m), z_{t+1}) \]

\[ \theta_{t+1(m+1)} = (\theta_{t-m+1}, \ldots, \theta_t, \theta_{t+1}) \]

and the covariance matrix of \( Z_{t+1} \) is

\[ V_{t+1} = \begin{bmatrix} \Sigma_{11,t(m)} & 0 \\ 0 & Q_{00} \end{bmatrix} \]

From model (4.4.15), we obtain the best linear unbiased estimator of \( \theta_{t+1(m+1)} \) and its covariance matrix from the usual least squares formulas. The least squares estimates of the last \( m \) elements of \( \theta_{t+1(m+1)} \) are then used as the initial estimates in the model for the next iteration.

We now state and prove a theorem which shows that the covariance matrix of the vector of recursive least squares estimators obtained in the recursive regression procedure converges to a positive definite matrix as the number of periods increases.

**Theorem 4.4.2.** At any time \( t \), let the vector of recursive least squares estimators, denoted by

\[ \hat{\theta}_t(m) = (\hat{\theta}_{t-m+1}(t), \ldots, \hat{\theta}_{t-1}(t), \hat{\theta}_t(t))' \]
be the least squares estimator of the vector of parameters

$$\theta_t(m) = (\theta_{t-m+1}, \ldots, \theta_{t-1}, \theta_t)'$$

based on data through time $t$, with covariance matrix denoted by $\Sigma_{11,t(m)}$. Assume that:

(a) the survey sampling error is such that all covariances for observations in a stream greater than $m$ periods apart are zero,

(b) if $V_n$ is the covariance matrix of any $n$ observations, then the elements of $V_n^{-1}$ are bounded for all $n$,

(c) the streams are independent.

Then,

(i) The limit of $\Sigma_{11,t(m)}$ exists.

(ii) If

$$\lim_{t \to \infty} \Sigma_{11,t(m)} = \Sigma_{11(m)}'$$

then $\Sigma_{11(m)}$ is an $m \times m$ positive definite matrix.

Proof. The proof of part (i) of this theorem will be based on the following lemmas.

Lemma 4.4.1. Let the assumptions of Theorem 4.4.2 hold. Then the variance of the estimator of current level $\theta_c$ converges to a positive number as the number of periods increases.
Proof. If the means \( \theta_{c-1}, \theta_{c-2}, \ldots, \theta_{c-m} \) were known, then the following are all unbiased estimators of \( \theta_c \):

\[
g_{1c} = y_{1,0,c}
\]

\[
g_{2c} = y_{2,0,c} - b_1(y_{2,0,c-1} - \theta_{c-1})
\]

\[
\vdots
\]

\[
g_{sc} = y_{s,0,c} - \sum_{j=1}^{m} b_s j(y_{s,0,c-j} - \theta_{c-j})
\]

Furthermore, \( g_{1c}, g_{2c}, \ldots, g_{sc} \) are independent with variances \( \sigma_1^2, \sigma_2^2, \ldots, \sigma_s^2 \), respectively. These variances are bounded below by assumption (b) of Theorem 4.4.2. Thus, we may write

\[
g = J_{s \times 1} \theta_c + \epsilon, \tag{4.4.17}
\]

where

\[
g = (g_{1c}, g_{2c}, \ldots, g_{sc})',
\]

\( J_{s \times 1} \) is the \( s \times 1 \) column vector of ones, and \( \epsilon \) is the \( s \times 1 \) vector of errors with

\[
E\{\epsilon\} = 0
\]

and

\[
E\{\epsilon \epsilon'\} = V = \text{Diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_s^2\}.
\]
Therefore, from Theorem 4.2.2, the best linear unbiased estimator of $\theta_c$ is

$$\hat{\theta}_c = (J_{s \times 1} V^{-1} J_{s \times 1}^{-1})^{-1} J_{s \times 1} V^{-1} g.$$  \hspace{1cm} (4.4.18)

The variance of $\hat{\theta}_c$ is

$$\hat{\theta}_c = (J_{s \times 1} V^{-1} J_{s \times 1}^{-1})^{-1}$$

$$= \left[ \sum_{i=1}^{s} \sigma_i^2 \right]^{-1}.$$ \hspace{1cm} (4.4.19)

From Lemma 4.2.3, since the transformed observations are uncorrelated with previous observations, this variance is a positive lower bound for the variance of the estimator of $\theta_c$.

Consider now the variance of the estimator of the current level as the total number of periods increases. Increasing the number of periods corresponds to increasing the total number of observations. By Theorem 4.2.7 and Lemma 4.3.1, the variance of the estimator of current level is nonincreasing as the number of observations increases. Since this variance is bounded below by (4.4.19), we conclude that the variance of the estimator of current level converges to a positive number as the number of periods increases.

\[\square\]

**Lemma 4.4.2.** Let the assumptions of Theorem 4.4.2 hold. Then, the variance of each of the least squares estimators of $\theta_{t-m}$, $\theta_{t-m+1}$, ..., $\theta_{t-1}$ based on data from period 1 to $t$ converges to a positive number as $t$ increases.
Proof. First, suppose at a fixed time $\tau$, at least $m$ periods of observations are available both prior to $\tau$ and after $\tau$. Let a transformation of the following form be defined for the observations in each of the $s$ streams at time $\tau$:

$$u_{i\tau} = y_{i,0,\tau} - \sum_{j=-m}^{m} b_{k(i,\tau),j} y_{i,0,\tau-j'}$$

(4.4.20)

where for any $t$, $y_{i,0,t}$ is the elementary estimate of $\theta_t$ which is based on the rotation group that is in stream $i$ at time $t$, $b_{k(i,\tau),0} = 0$, and $u_{i\tau}$ is uncorrelated with all observations preceding and succeeding $y_{i,0,\tau}$ in the stream $i$. Let the variances of $u_{i\tau}$, $i = 1, 2, ..., s$ be $\lambda_1^2, \lambda_2^2, ..., \lambda_s^2$, respectively. These variances are bounded below by assumption (b) of Theorem 4.4.2.

Then, assuming that $\theta_{\tau-m}, \theta_{\tau-1}, \theta_{\tau+1}, ..., \theta_{\tau+m}$ are all known, we conclude from Lemma 4.2.3 that the lower bound for the variance of the estimator of $\theta_\tau$ at time $\tau$ is

$$\left[ \sum_{i=1}^{s} \lambda_i^2 \right]^{-1}.$$

(4.4.20a)

This is a lower bound for the diagonal elements of the covariance matrix of the vector of initial estimates $\hat{\theta}_{\tau-m}, ..., \hat{\theta}_{\tau-1}$ that are used in the estimation problem for estimating $\theta_\tau$.

Now, for any time $t$, let us assume that we begin our sequence of estimators with the least squares estimators.
based on data from the preceding \( m \) periods, and the vector of transformed observations \( z_t \), where

\[
z_t = (az_{1t}, az_{2t}, \ldots, az_{st})
\]

and

\[
z_{it} = y_{i,0,t} - \sum_{j=1}^{m} b_{k(i,t),j} y_{i,0,t-j} \quad i = 1, 2, \ldots, s.
\]

Then, the linear model at time \( t \) may be written as

\[
Z_t = W\dot{\theta}_{t(m+1)} + \epsilon_t,
\]

where

\[
Z_t = (\dot{\theta}_{t-1(m)}, z_t).
\]

with fixed dimension,

\[
\dot{\theta}_{t(m+1)} = (\theta_{t-m}, \ldots, \theta_{t-1}, \theta_t),
\]

\[
W = (I_{m+1}, X_{22}).
\]

\( I_{m+1} \) is the \((m + 1) \times (m + 1)\) identity matrix, \( X_{22} \) is an \((s - 1) \times (m + 1)\) matrix which is constant over time, and the covariance matrix of \( Z_t \) is given by
\[
V_t = \begin{bmatrix}
Q_{11,t(m+1)} & 0 \\
0 & \Delta_0
\end{bmatrix},
\]

where
\[
Q_{11,t(m+1)} = \text{Block diag}\{\Sigma_{11,t(m)}, \sigma_i^2\},
\]
and
\[
\Delta_0 = \text{Diag}\{\sigma_2^2, ..., \sigma_s^2\},
\]
\[
\sigma_i^2 = \text{Var}\{z_{it}\}, \quad i = 1, 2, ..., s.
\]

Then, by Theorem 4.4.1, the covariance matrix of the best linear unbiased estimators of the components of the parameter vector \( \theta_t(m+1) \) is
\[
\Sigma_{t+1(m+1)} = (W' V_t^{-1} W)^{-1}
\]
\[
= (Q_{11,t(m+1)} + X_{22} \Delta_0^{-1} x_{22})^{-1}
\]
\[
= Q_{11,t(m+1)} - Q_{11,t(m+1)} x_{22} D_t^{-1} x_{22} Q_{11,t(m+1)}
\]
\[
(4.4.24)
\]

where \( D_t(s-1) = \Delta_0 + X_{22} Q_{11,t(m+1)} x_{22} \) (by Theorem 4.2.5). Since the second term on the right hand side of (4.4.24) is positive definite, we conclude that the first \( m \) diagonal elements of \( \Sigma_{t+1(m+1)} \) are less than or equal to the original diagonal entries of \( \Sigma_{11,t(m)} \). This means that as the number of periods (and hence, the number
of observations) increases, the variances of the estimators of \( \theta_{t-1}, \theta_{t-2}, \ldots, \theta_{t-m} \) decrease. Since these variances are bounded below by the positive quantity (4.4.20a), we conclude that for any time \( t \), the variance of each of the estimators of \( \theta_{t-1}, \theta_{t-2}, \ldots, \theta_{t-m} \) converges to a positive number as \( t \) increases.

\[ \square \]

**Lemma 4.4.3.** Let the assumptions of Theorem 4.4.2 hold. For any time point \( t \), the variances of each of the least squares estimators of \( \theta_t - \theta_{t-1}, \ldots, \theta_{t-m+1} - \theta_{t-m}, \theta_{t-m} \) based on data from periods 1 through \( t \) converge to a positive number as the number of periods increases.

**Proof.** First, we show that the variance of \( \theta_c - \theta_{c-1} \) (where \( c \) is the current period), converges as the number of periods increases. Define

\[ r_{ic} = y_{i,0,c} - y_{i,0,c-1}, \quad i = 1, 2, \ldots, s. \]

Next, we transform the observations \( r_{ic}, i = 1, \ldots, s \) so that they are uncorrelated with previous observations. The transformed observations are defined as

\[ q_{ic} = r_{ic} - \sum_{j=1}^{m} b_{k(i,c),j} r_{i,c-j}, \quad k = 1, \ldots, s, \]

For \( k = 1, \ldots, s \), we have

\[ q_{1c} = r_{1c} = y_{1,0,c}. \]
\[ q_{1c} = r_{2c} - b_{21}r_{2,c-1} = y_{2,0,c} - y_{2,0,c-1} - b_{21}y_{2,0,c-1}, \]
\[ q_{3c} = r_{3c} - b_{31}r_{3,c-1} - b_{32}r_{3,c-2} = y_{3,0,c} - y_{3,0,c-1} - b_{31}y_{3,0,c-1} - (b_{31} + b_{32})y_{3,0,c-2}, \]
\[ q_{4c} = r_{4c} - b_{41}r_{4,c-1} - b_{42}r_{4,c-2} - b_{43}r_{4,c-3} = y_{4,0,c} - y_{4,0,c-1} - b_{41}y_{4,0,c-1} - (b_{41} + b_{42})y_{4,0,c-2} - (b_{42} + b_{43})y_{4,0,c-3}, \]
\[ \vdots \]
\[ q_{sc} = y_{s,0,c} - y_{s,0,c-1} - b_{s1}y_{s,0,c-1} - \frac{m}{j=1} (b_{s,j} + b_{s,j+1})y_{s,0,c-j-1}. \]

Let the variances of \( q_{1c}, \ldots, q_{sc} \) be \( \gamma_{1}^{2}, \ldots, \gamma_{s}^{2} \), respectively. If the parameters \( \theta_{c-1}^{1}, \theta_{c-2}^{2}, \ldots, \theta_{c-m}^{s} \) were known, then the following are all unbiased estimators of \( \theta_{c} - \theta_{c-1}^{1} \):

\[ v_{1c} = y_{1,0,c} - \theta_{c-1}^{1} \]
\[ v_{2c} = y_{2,0,c} - y_{2,0,c-1} - b_{21}(y_{2,0,c-1} - \theta_{c-1}^{1}) \]
\[ \vdots \]
\[ v_{sc} = y_{s,0,c} - y_{s,0,c-1} - b_{s1}(y_{s,0,c-1} - \theta_{c-1}^{1}) \]
\[ - \frac{m}{j=1} (b_{s,j} + b_{s,j+1})(y_{s,0,c-j-1} - \theta_{c-j-1}^{1}). \]
Furthermore, the transformed observations $v_{ic}$, $i = 1, ..., s$, are independent with variances $\gamma_1^2, \gamma_2^2, ..., \gamma_s^2$, respectively. By assumption (b) of Theorem 4.4.2, these variances are bounded below. Therefore, as in the proof of Lemma 4.4.3, the variance of the best linear unbiased estimator of $\theta_c - \theta_{c-1}$ is $[\Sigma \gamma_i^{-2}]^{-1}$. Since the transformed observations are uncorrelated with previous observations, we conclude from Lemma 4.2.3 that this variance is a positive lower bound for the variance of the estimator of $\theta_c - \theta_{c-1}$. Next, we note that if

$$\Delta \theta_c(m+1) = (\theta_{c-m}, \theta_{c-m+1} - \theta_{c-m}, ..., \theta_{c} - \theta_{c-1})'$$

and

$$\theta_c(m+1) = (\theta_{c-m}, \theta_{c-m+1}, ..., \theta_{c})',$$

then

$$\Delta \theta_c(m+1) = F \theta_c(m+1)'$$

where

$$F = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & 0 & \cdots & 0 & 0 \\
& & & & & & \\
& & & & & & \\
0 & 0 & 0 & 0 & \cdots & -1 & 1
\end{bmatrix}$$

Thus, if we define

$$\Delta g = (q_{1c}, q_{2c}, ..., q_{sc})' \quad \text{and} \quad \delta \theta_c = \theta_c - \theta_{c-1},$$
then the linear model,

\[ \Delta g = J_{s \times 1} \delta \theta_c + \Delta \epsilon_c \]  

(4.4.25)

is a reparameterization of the model in Lemma 4.4.1. By mimicking the procedure in the proof of Lemma 4.4.1, we can show that the variance of the least squares estimate of \( \theta_c - \theta_{c-1} \) converges as the number of periods increases indefinitely. Similarly, we can show, by mimicking the proof of Lemma 4.4.2 that for any time \( t \), the variances of the least squares estimators of \( \theta_{t-m}, \theta_{t-m+1} - \theta_{t-m}, \ldots, \theta_{t-1} - \theta_{t-2} \) all converge as the number of periods increases.

We may now prove Theorem 4.4.2.

Proof of Theorem 4.4.2. The matrix \( \Sigma_{11,t(m)} \) is the covariance matrix of the least squares estimators of \( \theta_{t-m}, \ldots, \theta_{t-1} \) at time \( t-1 \). Thus, it is enough to show that the covariance matrix \( \Sigma_{t(m+1)} \) of the least squares estimators of \( \theta_{t-m}, \ldots, \theta_{t-1}, \theta_t \) at time \( t \) converges as \( t \to \infty \). From Lemma 4.4.1 and Lemma 4.4.2, the variance of each of the least squares estimators of \( \theta_{t-m}, \theta_{t-m+1}, \ldots, \theta_{t-1}, \theta_t \) converges to a positive number as \( t \to \infty \). In other words, the diagonal elements of \( \Sigma_{t(m+1)} \) converge as \( t \to \infty \). From Lemma 4.4.3, the variance of each of the least squares estimators of \( \theta_{t-m}, \theta_{t-m+1} - \theta_{t-m}, \ldots, \theta_t - \theta_{t-1} \) converges to a positive number as \( t \to \infty \). Now, note that for each \( j, 1 \leq j \leq m \),

\[ \text{Var}(\tilde{\theta}_t - \tilde{\theta}_{t-j}) = \text{Var}(\tilde{\theta}_t) + \text{V}(\tilde{\theta}_{t-j}) - 2\text{Cov}(\tilde{\theta}_t, \tilde{\theta}_{t-j}). \]  

(4.4.26)
Since $\text{Var}\{\hat{\theta}_t - \hat{\theta}_{t-j}\}$, $\text{Var}\{\hat{\theta}_t\}$, and $\text{V}\{\hat{\theta}_{t-j}\}$ all converge as $t \to \infty$, we conclude that $\text{Cov}\{\hat{\theta}_t, \hat{\theta}_{t-j}\}$ converges as $t \to \infty$, that is, the covariance between any of the least squares estimators of any pair of the parameters $\theta_{t-m}$, $\theta_{t-m+1}$, ..., $\theta_{t-1}$, $\theta_t$ converges as $t \to \infty$. Thus, the off-diagonal elements of $\Sigma_{t(m+1)}$ converge as $t \to \infty$ and therefore, $\lim_{t \to \infty} \Sigma_{t(m+1)}$ exists. Hence, $\lim_{t \to \infty} \Sigma_{11(t(m))}$ exists.

We now prove that the limiting covariance matrix $\Sigma_{11(m)}$ is positive definite. It is enough to show that the variance of any nontrivial linear combination of the recursive least squares estimators $\hat{\theta}_{t-j}(t-1)$, $j = 1, 2, ..., m$ is bounded below by a positive quantity.

Let $n$ be the number of observations at time $t-1$, where $n = s \times (t-1)$, and let the vector of observations be denoted by

$$Y_n = (Y_1, Y_2, ..., Y_n)^\prime$$

so that

$$\text{V}\{Y_n\} = V_n,$$

where $V_n$ is an $n \times n$ positive definite symmetric matrix. Let $\beta_{ij}$ denote the regression coefficient of $Y_j$ in the population regression of $Y_i$ on $Y_1, Y_2, ..., Y_{i-1}$, where $1 \leq j < i - 1$. 


Let \( T \) be the nonsingular matrix defined by

\[
T = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\beta_{21} & 1 & 0 & 0 & \cdots & 0 & 0 \\
-\beta_{31} & -\beta_{32} & 1 & 0 & \cdots & 0 & 0 \\
-\beta_{41} & -\beta_{42} & -\beta_{43} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\beta_{n1} & -\beta_{n2} & -\beta_{n3} & -\beta_{n4} & \cdots & -\beta_{n,n-1} & 1
\end{bmatrix}
\]

Then,

\[
TV_n T' = D = \text{Diag}\{d_{11}, d_{22}, \ldots, d_{nn}\}.
\]

Therefore,

\[
V_n^{-1} = T' D^{-1} T.
\]

By assumption (b), all elements of \( V_n^{-1} \) are bounded. In particular, the \((n, n)\) element of \( V_n^{-1} \), which is \( d_{nn}^{-1} \), is bounded. Therefore, \( d_{nn} \) is bounded below. Since the ordering of the observations in the data vector \( Y_n \) is arbitrary, we conclude that every linear combination of the observations with one of its coefficients equal to 1 has variance bounded below. Note that the linear combinations, with one coefficient equal to 1, which yield minimum variance are precisely those whose coefficients are provided by the last row of the matrix \( T \), because these coefficients are obtained from the population regression of one of the observations on the others. Let \( v_{mm} \) be the lower bound of every linear combination of the observations with one of the coefficients equal to 1.
Now, every estimator of a parameter $\theta_{t-j}$, $j = 1, ..., m$ is a linear combination of all observations such that the sum of the coefficients for the observations in the $s$ streams at time $t-j$ is 1 and the sum of the coefficients for the observations in the $s$ streams at any other time is 0. For the sum of the coefficients of the $s$ observations at time $t-j$ to be equal to 1, at least one of the coefficients must be greater than or equal to $s^{-1}$. The minimum variance of any linear combination with first coefficient $s^{-1}$ is $s^{-2}v_{mm}$. Therefore, for each $j$, $j = 1, 2, ..., m$,

$$V\{\hat{\theta}_{t-j}(t-1)\} \geq s^{-2}v_{mm}.$$ 

Now, consider an arbitrary, nontrivial linear combination of the recursive least square estimators $\hat{\theta}_{t-j}(t-1)$, $j = 1, ..., m$, given by

$$\sum_{j=1}^{m} \gamma_j \hat{\theta}_{t-j}(t-1),$$

where, without loss of generality, we choose $\gamma_1 = 1$. Therefore,

$$\sum_{j=1}^{m} \gamma_j \hat{\theta}_{t-j}(t-1) = \hat{\theta}_{t-1}(t-1) + \sum_{j=2}^{m} \gamma_j \hat{\theta}_{t-j}(t-1)$$

$$= \sum_{i=1}^{s} \sum_{h=1}^{t-1} c_{ih} y_{i,0,h} + \sum_{j=2}^{m} \sum_{i=1}^{s} \gamma_j \sum_{h=1}^{t-1} f_{i h(t-j)} y_{i,0,h}$$

$$= \sum_{i=1}^{s} \left[ c_{i,t-1} + \sum_{j=2}^{m} \gamma_j f_{i,t-1(t-j)} \right] y_{i,0,t-1}$$
where \( c_{i,t-1}, i = 1, \ldots, s \) are the coefficients of \( y_{i,0,t-1} \) in \( \hat{\theta}_{t-1}(t-1) \) and \( f_{i,t-1}(t-j), j = 2, \ldots, m \) are the coefficients of \( y_{i,0,t-1} \) in \( \hat{\theta}_{t-2}, \ldots, \hat{\theta}_{t-m} \), respectively. Therefore,

\[
\sum_{i=1}^{s} c_{i,t-1} = 1
\]

and

\[
\sum_{i=1}^{s} f_{i,t-1}(t-j) = 0 \quad \text{for } j = 2, \ldots, m.
\]

Thus,

\[
\sum_{i=1}^{s} \left[ c_{i,t-1} + \sum_{j=2}^{m} \gamma_{j} f_{i,t-1}(t-j) \right] = 1.
\]

In other words, in the linear combination (4.4.27), the sum of the coefficients for the observations \( y_{i,0,t-1}, i = 1, \ldots, s \), in the s streams at time \( t-1 \) is 1. Therefore, at least one of the coefficients of the observations in the linear combination (4.4.27) is greater than or equal to \( s^{-1} \). Hence,

\[
\sum_{j=1}^{m} \gamma_{j} \hat{\theta}_{t-j}(t-1) \geq s^{-1} v_{mm},
\]

and we conclude that \( \Sigma_{11}(m) \) is positive definite.
4.5. State Space Modeling and the Kalman Filter

In this section, we briefly review the basic structure of state-space models and their accompanying Kalman filter equations (Kalman, 1960), emphasizing only those aspects that are relevant to the analysis presented in Section 5.4.3.

State-space models consist, in general, of two sets of linear equations which define how the observable and unobservable processes in the models evolve stochastically in time. One set of equations, called the measurement or observation equation, describes the functional relationship between the observations and the current state of the process parameters. The second set of equations, known as the state or transition equation, shows how the process parameters evolve over time. Our presentation follows Fuller (1992).

Suppose the r-dimensional observation vector $Y_t$ is the sum of a linear function of a p-dimensional vector time series $X_t$ and measurement error. Let

$$Y_t = H_t X_t + u_t, \quad t = 1, 2, ... \quad (4.5.1)$$

$$X_t = A_t X_{t-1} + \epsilon_t, \quad t = 1, 2, ... \quad (4.5.2)$$

where $\{H_t\}$ is a sequence of known $r \times p$ matrices, $A_t$ is a sequence of known $p \times p$ matrices, and $\{u_t, \epsilon_t\}$ is a sequence of uncorrelated random vectors with zero mean and known covariances matrix

$$E\{(u_t', \epsilon_t')(u_t', \epsilon_t')\} = \text{Blockdiag}\{\Sigma_{uut}, \Sigma_{eett}\}.$$
Equations (4.5.1) and (4.5.2) are referred to, respectively, as the *measurement* or *observation equation* and the *state* or *transition equation*.

Suppose \( \hat{X}_0 \) is a known initial vector. Then, we can write

\[
\hat{X}_0 = X_0 + v_0
\]

where \( v_0 \) is a \((0, \Sigma_{vv00})\) random vector, and we assume that \( v_0 \) is uncorrelated with \( \epsilon_t \), \( t = 1, 2, \ldots \). We also assume that the initial estimator \( \hat{X}_0 \) and the parameters \( \Sigma_{uut}, \Sigma_{cct}, A_t, H_t, t = 1, 2, \ldots \) are known. Now, we can write

\[
A_t\hat{X}_{t-1} = A_tX_{t-1} + A_t(\hat{X}_{t-1} - X_{t-1})
\]

\[
= A_tX_{t-1} + A_tv_{t-1}
\]

\[
= X_t - \epsilon_t + A_tv_{t-1},
\]

where we have used (4.5.2) and the fact that \( \hat{X}_t = X_t + v_t \). Therefore,

\[
A_t\hat{X}_{t-1} = X_t + w_t,
\]

where \( w_t = A_tv_{t-1} - \epsilon_t \), and so \( A_t\hat{X}_{t-1} \) is an unbiased estimator of \( X_t \) with error \( w_t \). Thus, we get the system of equations

\[
Y_t = H_tX_t + u_t,
\]  \hspace{1cm} (4.5.3)
\[ A_t \hat{X}_{t-1} = X_t + w_t. \quad (4.5.4) \]

If we assume that \( \Sigma_{\text{uutt}} \) is nonsingular, the best linear unbiased estimator of \( X_t \) is

\[ \hat{X}_t = [H_t^{\prime} \Sigma_{\text{uutt}}^{-1} H_t + \Sigma_{\text{wwtt}}^{-1}]^{-1} [H_t^{\prime} \Sigma_{\text{uutt}}^{-1} Y_t + \Sigma_{\text{wwtt}}^{-1} A_t \hat{X}_{t-1}], \quad (4.5.5) \]

where

\[ \Sigma_{\text{wwtt}} = E\{w_t w_t^{\prime}\} = \Sigma_{\text{cett}} + A_t \Sigma_{\text{vv},t-1} A_t^{\prime} \quad (4.5.6) \]

\[ \Sigma_{\text{vvtt}} = E\{(\hat{X}_t - X_t)(\hat{X}_t - X_t)^{\prime}\} \]

\[ = [H_t^{\prime} \Sigma_{\text{uutt}}^{-1} H_t + \Sigma_{\text{wwtt}}^{-1}]^{-1} \quad (4.5.7) \]

\[ = \Sigma_{\text{wwtt}} - \Sigma_{\text{wwtt}} H_t D_t^{-1} H_t^{\prime} \Sigma_{\text{wwtt}} \quad (4.5.8) \]

and

\[ D_t = \Sigma_{\text{uutt}} + H_t \Sigma_{\text{wwtt}} H_t^{\prime}. \quad (4.5.9) \]

The estimator (4.5.5) can also be written as

\[ \hat{X}_t = A_t \hat{X}_{t-1} + \Sigma_{\text{wwtt}} H_t D_t^{-1} (Y_t - H_t A_t \hat{X}_{t-1}) \quad (4.5.10) \]

where \( D_t \) is the covariance matrix of \( Y_t - H_t A_t \hat{X}_{t-1} \) and \( \Sigma_{\text{wwtt}} H_t^{\prime} \) is the covariance between \( w_t \) and \( Y_t - H_t A_t \hat{X}_{t-1} \). Thus, the estimator (4.5.10) is the
difference between the unbiased estimator $A_t \hat{X}_{t-1}$ of $X_t$ and an unbiased estimator of the error $w_t$ incurred in estimating $X_t$ by $A_t \hat{X}_{t-1}$. Thus, if we are given $Y_t$, $\hat{X}_{t-1}$, and $\Sigma_{v,v,t-1}$, then we can use the system of equations (4.5.10), (4.5.7), and (4.5.6) to construct $\hat{X}_t$, the best unbiased estimator of $X_t$, as well as the error covariance matrix $\Sigma_{v,v,t}$.

This system of equations is usually referred to as the system of updating equations. Note that the formula (4.5.9) is preferable to (4.5.8) for computing $\Sigma_{v,v,t}$ because in (4.5.9), only the matrix $D_t$ is required to be nonsingular and, hence, this formula would be more efficient from a computational standpoint.

Next, we derive an alternative expression for the error covariance matrix $\Sigma_{v,v,t}$. Rewriting (4.5.10) as

$$\hat{X}_t = A_t \hat{X}_{t-1} + K_t Z_t,$$

where $K_t = \Sigma_{w,w,t} H_t D_t^{-1}$ and $Z_t = Y_t - H_t A_t \hat{X}_{t-1}$, we have

$$X_t - \hat{X}_t = A_t X_{t-1} + \epsilon_t - A_t \hat{X}_{t-1} - K_t Z_t$$

$$= \epsilon_t + A_t v_{t-1} - K_t H_t A_t (X_{t-1} - \hat{X}_{t-1}) - K_t Z_t.$$

Thus,

$$v_t = \epsilon_t + (A_t - K_t H_t A_t) v_{t-1} - K_t Z_t.$$  \hspace{1cm} (4.5.11)
Note that the summands in (4.5.11) are uncorrelated. Therefore,

\[ \Sigma_{vV,tt} = \Sigma_{eett} + B_t \Sigma_{vV,t-1,t-1} B_t' + K_t D_t K_t', \]

where

\[ B_t = A_t - K_t H_t A_t. \]

We now state a theorem which shows that under certain conditions, the sequence of error covariance matrices in the Kalman filter procedure converges as \( t \) increases.

**Theorem 4.5.1.** Assume model \((4.5.1) - (4.5.2)\), and suppose \( H_t \equiv I, A_t \equiv A, \Sigma_{uu,t} \equiv \Sigma_{uu}, \Sigma_{e\epsilon,t} \equiv \Sigma_{e\epsilon}, \Sigma_{uu} > 0, \Sigma_{e\epsilon} > 0 \), and the roots of \( A \) are less than one in absolute value. Then, \( \lim_{t \to \infty} \Sigma_{vV,tt} \) exists and is positive definite.

**Proof.** The process \( X_t \) is converging to a stationary autoregressive process under the assumptions that \( \Sigma_{e\epsilon} \) is constant and that the roots of \( A \) are less than one in absolute value. The covariance matrix of the \( X_t \)-process is positive definite, and the elements of the inverse of the covariance matrix are bounded. The variance of \( v_t \) is the variance of the error in a least squares estimator of \( X_t \) based on \( Y_t, Y_{t-1}, ..., Y_1 \) and \( \hat{X}_0 \). The variance is nonincreasing as \( t \) increases because more observations are included in the estimation. The variance is bounded below because the elements of the inverse of the covariance matrix of \((Y_1', ..., Y_t')\) are uniformly bounded. Hence, the result. ☐
Remark. For the special case of univariate time series, the model (4.5.1) - (4.5.2) with $H_t = I$ reduces to

\[ Y_t = X_t + u_t, \quad t = 1, 2, \ldots \]

\[ X_t = \alpha X_{t-1} + e_t, \quad t = 1, 2, \ldots \]  
\[ (4.5.12) \]

where we assume that $|\alpha| < 1$ and that \( \{u_t, e_t\} \) is a sequence of uncorrelated random variables satisfying

\[ E\{(u_t, e_t)\} = (0, 0) \]

and

\[ E\left[\begin{bmatrix} u_t \\ e_t \end{bmatrix} (u_t, e_t)\right] = \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix}. \]

The corresponding updating equations for the best linear unbiased estimator $\hat{X}_t$ of $X_t$ and the variance of the error $\nu_t$ are

\[ \hat{X}_t = \alpha \hat{X}_{t-1} + c_t(Y_t - \alpha \hat{X}_{t-1}) \]  
\[ (4.5.13) \]

\[ \sigma_{\nu t}^2 = (1 - c_t)\sigma_{\omega t}^2, \]  
\[ (4.5.14) \]

where $\sigma_{\omega t}^2 = \sigma_u^2 + \alpha^2 \sigma_{\nu t-1}^2$ and $c_t = (\sigma_u^2 + \sigma_{\omega t}^2)^{-1} \sigma_{\omega t}^2$. Equation (4.5.14) may be rewritten as
We now prove Theorem 4.5.1 for the special case of a univariate time series, and specify the limiting error variance explicitly. The following lemma is useful for the proof.

Lemma 4.5.1. For $a > 0$, $b > 0$, $c > 0$, the function

$$g(x) = \frac{a(b + cx)}{a + b + cx}$$

is a nondecreasing function of $x$.

Proof. The proof is immediate from the fact that $g'(x) > 0$ for all $x \in \mathbb{R}$.

\[\square\]

Theorem 4.5.2. Let the conditions of model (4.5.12) hold and let $\sigma^2_{v,t}$ be as defined in (4.5.14). Then

$$\lim_{t \to \infty} \sigma^2_{v,t} = \sigma^2_{v,w'}$$

where

$$\sigma^2_{v,w'} = (2\alpha^2)^{-1} \{ (1 - \alpha^2) \sigma^2_u - \sigma^2_e + \left[ (\sigma^2_u(1 - \alpha^2))^2 + (\sigma^2_e)^2 + 2\sigma^2_u\sigma^2_e(1 + \alpha^2) \right]^{1/2} \}.$$
Proof. In the light of Lemma 4.5.1, we see from equation (4.5.15) that the sequence \( \{\sigma_{v,t}^2\} \) is monotone. It is nonincreasing or nondecreasing depending on the start value. Furthermore, the expression (4.5.15) for \( \sigma_{v,t}^2 \) may be rewritten as

\[
\sigma_{v,t}^2 = \sigma_u^2 - \frac{(\sigma_v^2)^2}{\sigma_u^2 + \sigma_e^2 + \alpha^2 \sigma_{v,t-1}^2}.
\]

Therefore,

\[
\sigma_{v,t}^2 \leq \sigma_u^2 \quad \text{for all } t.
\]

Thus, \( \{\sigma_{v,t}^2\} \) is a monotone sequence which is bounded. Therefore, its limit exists.

Now, suppose \( \lim_{t \to \infty} \sigma_{v,t}^2 = \sigma_{v,\infty}^2 \). Then, from (4.5.15),

\[
\sigma_{v,\infty}^2 = \frac{\sigma_u^2 (\sigma_e^2 + \alpha^2 \sigma_{v,\infty}^2)}{\sigma_u^2 + \sigma_e^2 + \alpha^2 \sigma_{v,\infty}^2}.
\]

Solving for \( \sigma_{v,\infty}^2 \) and simplifying, we get the expression for \( \sigma_{v,\infty}^2 \), given in the statement of the theorem. \( \square \)
5. COMPARISON OF ALTERNATIVE ESTIMATORS AND
ROTATION DESIGNS FOR THE CURRENT
POPULATION SURVEY

5.1. Introduction

The Current Population Survey is a household survey conducted by the United States Census Bureau in cooperation with the Bureau of Labor Statistics. It is designed to generate estimates of labor force characteristics (including unemployment), demographic characteristics, and other characteristics of the noninstitutionalized civilian population. The sample design of the Current Population Survey contains a rotation scheme that permits the replacement of a fraction of the households in the sample each month.

For any given month, the sample consists of eight time—in—sample panels or rotation groups, of which one is being interviewed for the first time, one is being interviewed for the second time, one is being interviewed for the third time, one is being interviewed for the fourth time, one is being interviewed for the fifth time, one is being interviewed for the sixth time, one is being interviewed for the seventh time, and one is being interviewed for the eighth time. Households in a rotation group are interviewed for four consecutive months, dropped for the next eight succeeding months, and then interviewed for another 4 consecutive months. They are then dropped from the sample entirely. This system of interviewing is called the 4–8–4 rotation scheme, and is a special case of the scheme described by Rao and Graham (1964).

In this chapter, we consider the estimation of the following characteristics of the labor force: employed, unemployed, Civilian Labor Force, and unemployment rate.
Estimation procedures are applied not only to the intermittent 4—8—4 rotation scheme used in the Current Population Survey, but also to the two continuous schemes: the 8—in—then—out rotation scheme and the 6—in—then—out rotation scheme. The 6—in—then—out scheme is used in Canadian Labor Force Survey. See Section 5.5.

We introduce a procedure for the estimation of current level and change for all the rotation schemes under consideration. This estimation procedure is recursive in nature and is designed to minimize the problem of computational complexity associated with best linear unbiased estimation. The recursive regression estimator is compared to the following estimators of current level and change:

1. The Present Composite Estimator,
2. The First Order Composite Estimator (an estimator that is optimal under the first order autoregressive model),
3. The Best Linear Unbiased Estimator based on 12, 16 and 24 periods.

Both the estimation expression and the variance of the estimators are compared.

In the next section, we describe the correlation structure of the 4—8—4 rotation scheme of the Current Population Survey. In Section 5.3, we give a detailed description of the alternative estimators of current level and change considered in this chapter. For each estimation procedure, we consider the consequences of revising previous estimates. This is a process of updating estimates of previous levels using data collected up to and including the current level. In Section 5.4, we consider the effect of time—in—sample effects on the alternative estimators under assumptions of constant and time—varying time—in—sample group effects. In Section 5.5, we describe the two continuous rotation schemes, and discuss the previously described estimation procedures for these special cases. Estimation of the current level and change in
the unemployment rate is considered in Section 5.6. In Section 5.7, we present results on the comparison of the alternative estimators. Our discussion will include a comparison of both the optimal coefficients used in the construction of the estimators and the variances of the estimators. We shall also present an analysis of the relative precision of both revised and unrevised estimates under each estimation procedure.

5.2. The Components Of Variance Model

The rotation scheme in the Current Population Survey is schematically described in Table 5.1 for 24 periods. The survey is assumed to start from time $t-23$ to $t$ when 24 periods are studied.

Suppose the survey has been in operation for $p$ periods and that the sample for each period consists of $s$ rotation groups. For computational convenience, the data can be arranged in a $p \times s$ data matrix in such a way that all of the observations on a rotation group appear in a single column. We use the term "stream" to describe the sequence of observations created by a sequence of rotation groups, where all observations on a rotation group are in a single stream. For instance, the columns of the $p \times s$ data matrix shown in Table 5.1 correspond to streams. Because each rotation group is observed for a fixed number of periods at a time, the number of rotation groups in a stream of $p$ periods varies. For instance, for the 4–8–4 rotation scheme, the first stream in Table 5.1 contains four rotation groups while the second stream contains five rotation groups.

Breau and Ernst (1983) estimated the covariance structure of the Current Population Survey based on data for 1976–1977. Their estimates are described in detail in Appendix D. In this section, we estimate the variance of alternative
Table 5.1. Data from 24 periods of a survey collected by the 4—8—4 rotation sampling scheme of the current population survey.

<table>
<thead>
<tr>
<th>Month</th>
<th>Streams</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A₁₁,1</td>
</tr>
<tr>
<td>2</td>
<td>A₂₁,2</td>
</tr>
<tr>
<td>3</td>
<td>A₃₁,3</td>
</tr>
<tr>
<td>4</td>
<td>A₄₁,4</td>
</tr>
<tr>
<td>5</td>
<td>B₅₅,5</td>
</tr>
<tr>
<td>6</td>
<td>B₆₆₆</td>
</tr>
<tr>
<td>7</td>
<td>B₇₇₇</td>
</tr>
<tr>
<td>8</td>
<td>B₈₈₈</td>
</tr>
<tr>
<td>9</td>
<td>C₉₉₉</td>
</tr>
<tr>
<td>10</td>
<td>C₁₀₁₀</td>
</tr>
<tr>
<td>11</td>
<td>C₁₁₁₁</td>
</tr>
<tr>
<td>12</td>
<td>C₁₂₁₂</td>
</tr>
<tr>
<td>13</td>
<td>A₁₃₁₃</td>
</tr>
<tr>
<td>14</td>
<td>A₁₄₁₄</td>
</tr>
<tr>
<td>15</td>
<td>A₁₅₁₅</td>
</tr>
<tr>
<td>16</td>
<td>A₁₆₁₆</td>
</tr>
<tr>
<td>17</td>
<td>Π₁₇₁₇</td>
</tr>
<tr>
<td>18</td>
<td>Π₁₈₁₈</td>
</tr>
<tr>
<td>19</td>
<td>Π₁₉₁₉</td>
</tr>
<tr>
<td>20</td>
<td>Π₂₀₁₀</td>
</tr>
<tr>
<td>21</td>
<td>C₂₁₁₅</td>
</tr>
<tr>
<td>22</td>
<td>C₂₂₁₆</td>
</tr>
<tr>
<td>23</td>
<td>C₂₃₁₇</td>
</tr>
<tr>
<td>24</td>
<td>C₂₄₈</td>
</tr>
</tbody>
</table>

*Notation: Aᵢᵢ denotes a household which is in its j—th time—in—sample in month t of the survey.
estimators using a correlation structure based on the model postulated by Adam and Fuller (1991). The data analyzed by Adam and Fuller were composed of 48 replicates constructed from the twelve months of the Current Population Survey during 1987. The replicates were based upon the primary sampling units of the survey. Their model can be written as

\[ y_{tjk} = \mu + u_j + \alpha_t + r_k + \gamma_g + \zeta_{tk} + \epsilon_{tjk}, \tag{5.2.1} \]

\[ \sum_{u_j} = \sum_{\alpha_t} = \sum_{r_k} = \sum_{\gamma_g} = 0, \]

\[ \sum_{\zeta_{tk}} = 0 \text{ for all } k, \]

\[ \sum_{\epsilon_{tjk}} = 0 \text{ for all } t, \]

where \( y_{tjk} \) is the elementary estimate of a characteristic such as Civilian Labor Force, obtained from the \( j \)-th replicate for the \( k \)-th time-in-sample at month \( t \), \( \mu \) represents the overall mean, \( u_j \) is the replicate effect, \( \alpha_t \) represents the time effect, sometimes called the month effect, \( r_k \) is the time-in-sample effect, \( \gamma_g \) is the effect of rotation groups identified by Latin letters in Table 5.1, \( \zeta_{tk} \) represents interactions among time-in-sample, time and group effects, and \( \epsilon_{tjk} \) is the error arising from different rotation groups at different time points. Now, let

\[ r_{tjk} = u_j + \epsilon_{tjk}, \tag{5.2.2} \]
where \( u_j \) and \( \epsilon_{tjk} \) are defined in model (5.2.1). Thus, \( r_{tjk} \) is the original observation with time, time-in-sample, groups and their interactions effects removed. With a slight abuse of notation, let \( r_{gjk} \) be the value of \( r_{tjk} \) obtained when we use the rotation group index in place of the time index. Thus, \( r_{gjk} \) is the original observation for the \( k \)-th time-in-sample of the \( g \)-th rotation group in the \( j \)-th replicate when the effects of factors of model (5.2.1) except replicate are removed. We assume

\[
\begin{align*}
    r_{gjk} &= u_j + e_{gj} + a_{gjk}, \\
    a_{gjk} &= \sum_{\ell=1}^{3} \xi_{g,j,k-\ell} + b_{gjk},
\end{align*}
\]

where \( u_j \) is the replicate effect \( e_{gj} \) is the permanent effect of rotation group \( g \), and \( a_{gjk} \) is a transient effect associated with rotation group \( g \). It is assumed that the transient rotation group effect is a stationary third order autoregressive process. It is assumed that \( \{ u_j \} \), \( \{ e_{gj} \} \), and \( \{ a_{gjk} \} \) are independent sequences. It is also assumed that

\[
\begin{align*}
    E\{ u_j \} &= 0 \quad \text{for all } j, \\
    E\{ u_j u_s \} &= \sigma_u^2 \quad \text{for all } j \text{ and } s, \\
    e_{gj} &\sim \text{Ind.}(0, \sigma_e^2),
\end{align*}
\]
\[ b_{gjk} \sim \text{Ind.}(0, \sigma_b^2). \]

Assuming that the method of construction of the replicates is such that the \( u_j \) are nearly independent, it follows that

\[
\gamma_r(h) = E[r_{gjk}r_{g,ij,k+h}] = \sigma_u^2 + \sigma_e^2 + \rho_a(h)\sigma_a^2,
\]

where \( \rho_r(h) \) is the autocorrelation function of \( r_{gjk} \), \( \gamma_r(h) \) is the autocovariance function of \( r_{gjk} \), and \( \rho_a(h) \) is the autocorrelation function of \( a_{gjk} \). Thus, \( \rho_r(h) \) is the correlation between \( r_{gjk} \) and \( r_{g,ij,k+h} \), where \( r_{gjk} \) is the observation on a single rotation group at month \( k \). The estimated autocorrelations and autocovariances of the three variables for lags up to 16 for observations on a single rotation group of a replicate are given in Table 5.2, for employed, unemployed, Civilian Labor Force, and unemployment rate, respectively. These correlations and covariances are defined by (5.2.5) and (5.2.6).

The estimated autocorrelations for the first stream of Table 5.1 are given in Tables 5.3, 5.4, 5.7, and 5.8 for employed, unemployed, Civilian Labor Force, and unemployment rate, respectively. The autocorrelations for the stream are
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Emp.</td>
<td>Unemp.</td>
<td>CLF</td>
</tr>
<tr>
<td>0</td>
<td>0.2513</td>
<td>0.0561</td>
<td>0.2162</td>
</tr>
<tr>
<td>1</td>
<td>0.2028</td>
<td>0.0279</td>
<td>0.1702</td>
</tr>
<tr>
<td>2</td>
<td>0.1843</td>
<td>0.0212</td>
<td>0.1555</td>
</tr>
<tr>
<td>3</td>
<td>0.1723</td>
<td>0.0181</td>
<td>0.1441</td>
</tr>
<tr>
<td>4</td>
<td>0.1691</td>
<td>0.0150</td>
<td>0.1373</td>
</tr>
<tr>
<td>5</td>
<td>0.1682</td>
<td>0.0134</td>
<td>0.1331</td>
</tr>
<tr>
<td>6</td>
<td>0.1682</td>
<td>0.0124</td>
<td>0.1305</td>
</tr>
<tr>
<td>7</td>
<td>0.1684</td>
<td>0.0118</td>
<td>0.1288</td>
</tr>
<tr>
<td>8</td>
<td>0.1685</td>
<td>0.0114</td>
<td>0.1277</td>
</tr>
<tr>
<td>9</td>
<td>0.1685</td>
<td>0.0111</td>
<td>0.1271</td>
</tr>
<tr>
<td>10</td>
<td>0.1686</td>
<td>0.0110</td>
<td>0.1267</td>
</tr>
<tr>
<td>11</td>
<td>0.1686</td>
<td>0.0109</td>
<td>0.1265</td>
</tr>
<tr>
<td>12</td>
<td>0.1686</td>
<td>0.0109</td>
<td>0.1263</td>
</tr>
<tr>
<td>13</td>
<td>0.1686</td>
<td>0.0108</td>
<td>0.1262</td>
</tr>
<tr>
<td>14</td>
<td>0.1686</td>
<td>0.0108</td>
<td>0.1261</td>
</tr>
<tr>
<td>15</td>
<td>0.1686</td>
<td>0.0108</td>
<td>0.1261</td>
</tr>
<tr>
<td>16</td>
<td>0.1686</td>
<td>0.0108</td>
<td>0.1261</td>
</tr>
</tbody>
</table>

<sup>1</sup>Source: Breau and Ernst (1983)

<sup>2</sup>Multiply entries for autocovariances by 10<sup>-6</sup>. NA: Not available.

<sup>3</sup>Source: Adam and Fuller (1991)
Table 5.3. Estimates of parameters of transient processes $a_{gjk}$

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>$\hat{\xi}_1$</th>
<th>$\hat{\xi}_2$</th>
<th>$\hat{\xi}_3$</th>
<th>$\sigma_b^2$</th>
<th>$\sigma_a^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Employed</td>
<td>0.40481</td>
<td>0.04270</td>
<td>-0.04945</td>
<td>0.06841</td>
<td>0.08263</td>
</tr>
<tr>
<td>Unemployed</td>
<td>0.33422</td>
<td>0.08452</td>
<td>0.05267</td>
<td>0.03831</td>
<td>0.04508</td>
</tr>
<tr>
<td>Civilian labor force</td>
<td>0.43415</td>
<td>0.11345</td>
<td>0.00318</td>
<td>0.06756</td>
<td>0.08962</td>
</tr>
<tr>
<td>Unemployment rate (%)</td>
<td>0.32855</td>
<td>0.07524</td>
<td>0.04382</td>
<td>0.01579</td>
<td>0.01934</td>
</tr>
</tbody>
</table>


Table 5.4. Estimates of $\sigma_u^2$, $\sigma_e^2$, and $\sigma_a^2$

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>$\sigma_u^2$</th>
<th>$\sigma_e^2$</th>
<th>$\sigma_a^2$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Employed</td>
<td>0.00552</td>
<td>0.16319</td>
<td>0.08263</td>
<td>0.25134</td>
</tr>
<tr>
<td>Unemployed</td>
<td>0.00065</td>
<td>0.01037</td>
<td>0.04508</td>
<td>0.05610</td>
</tr>
<tr>
<td>Civilian labor force</td>
<td>0.00499</td>
<td>0.12159</td>
<td>0.08962</td>
<td>0.21620</td>
</tr>
<tr>
<td>Unemployment rate (%)</td>
<td>0.00028</td>
<td>0.00466</td>
<td>0.01934</td>
<td>0.02428</td>
</tr>
</tbody>
</table>

Table 5.5. Estimated autocorrelations for one replicate of the first stream of Table 5.2; characteristic = employed

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0.8068</th>
<th>0.7332</th>
<th>0.6856</th>
<th>0.0219</th>
<th>0.0219</th>
<th>0.0219</th>
<th>0.0219</th>
<th>0.0219</th>
<th>0.0219</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.8068</td>
<td>0.7332</td>
<td>0.6856</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.8068</td>
<td>0.7332</td>
<td>0.6856</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.8068</td>
<td>0.7332</td>
<td>0.6856</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.8068</td>
<td>0.7332</td>
<td>0.6856</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.8068</td>
<td>0.7332</td>
<td>0.6856</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.8068</td>
<td>0.7332</td>
<td>0.6856</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.8068</td>
<td>0.7332</td>
<td>0.6856</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
<td>0.0219</td>
</tr>
</tbody>
</table>
Table 5.5.

(Continued)

0.6708

0.6708

0.6708

0.6708

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.6708

0.6708

0.6708

0.6708

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.6708

0.6708

0.6708

0.6708

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.6708

0.6708

0.6708

0.6708

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.6708

0.6708

0.6708

0.6708

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.6708

0.6708

0.6708

0.6708

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.6708

0.6708

0.6708

0.6708

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.6708

0.6708

0.6708

0.6708

1

0.8068

0.7332

0.6856

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

1

0.8068

0.7332

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

1

0.8068

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

1

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

0.0219

1

0.8068

0.7332

0.6856

0.0219

0.0219

0.0219

0.0219

1

0.8068

0.7332

0.0219

0.0219

0.0219

0.0219

1

0.8068

0.0219

0.0219

0.0219

0.0219

1

0.0219

0.0219

0.0219

0.0219

1

0.8068

0.7332

0.6856

1

0.8068

0.7332

1

0.8068


Table 5.6. Estimated autocorrelations for one replicate of the first stream of Table 5.1; characteristic = unemployed

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0.4979</th>
<th>0.3788</th>
<th>0.3230</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
</tbody>
</table>
Table 5.6. (Continued)

<table>
<thead>
<tr>
<th>0.1938</th>
<th>0.1933</th>
<th>0.1929</th>
<th>0.1927</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.0116</th>
<th>0.1938</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1948</td>
<td>0.1938</td>
<td>0.1933</td>
<td>0.1929</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.1948</td>
</tr>
<tr>
<td>0.1964</td>
<td>0.1948</td>
<td>0.1933</td>
<td>0.1929</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.1964</td>
</tr>
<tr>
<td>0.1989</td>
<td>0.1964</td>
<td>0.1948</td>
<td>0.1933</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.1989</td>
</tr>
<tr>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.1938</td>
<td>0.1933</td>
<td>0.1929</td>
<td>0.1927</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.1948</td>
<td>0.1933</td>
<td>0.1929</td>
<td>0.1927</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>0.1989</td>
<td>0.1964</td>
<td>0.1948</td>
<td>0.1933</td>
<td>0.1929</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
<tr>
<td>1</td>
<td>0.4979</td>
<td>0.3788</td>
<td>0.3230</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0116</td>
</tr>
</tbody>
</table>
Table 5.7. Estimated autocorrelations for one replicate of the first stream of Table 5.1; characteristic = Civilian Labor Force

<table>
<thead>
<tr>
<th></th>
<th>0.7879</th>
<th>0.7197</th>
<th>0.6668</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
</tbody>
</table>
Table 5.7. (Continued)

<table>
<thead>
<tr>
<th></th>
<th>0.5843</th>
<th>0.5838</th>
<th>0.5836</th>
<th>0.5834</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
<th>0.0231</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5850</td>
<td>0.5843</td>
<td>0.5838</td>
<td>0.5836</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>0.5862</td>
<td>0.5850</td>
<td>0.5843</td>
<td>0.5838</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>0.5881</td>
<td>0.5862</td>
<td>0.5850</td>
<td>0.5843</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
<tr>
<td>1</td>
<td>0.7879</td>
<td>0.7197</td>
<td>0.6668</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
<td>0.0231</td>
</tr>
</tbody>
</table>
Table 5.8. Estimated autocorrelations for one replicate of the first stream of Table 5.1; characteristic = unemployment rate

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0.5187</th>
<th>0.4019</th>
<th>0.3484</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
<th>0.0115</th>
</tr>
</thead>
</table>
Table 5.8.  (Continued)

<p>| | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2426</td>
<td>0.2423</td>
<td>0.2421</td>
<td>0.5834</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.2432</td>
<td>0.2426</td>
<td>0.2423</td>
<td>0.2421</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.2442</td>
<td>0.2432</td>
<td>0.2426</td>
<td>0.2423</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.2459</td>
<td>0.2442</td>
<td>0.2432</td>
<td>0.2426</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5187</td>
<td>0.4019</td>
<td>0.3484</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td>1</td>
<td>0.5187</td>
<td>0.4019</td>
<td>0.3484</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
<td>0.0115</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
if observations are \( h \) distance apart and made on the same rotation group, and

\[
\frac{\sigma_u^2 + \sigma_e^2 + \rho_a(h)\sigma_a^2}{\sigma_u^2 + \sigma_e^2 + \sigma_a^2},
\]

if observations \( h \) distance apart in the same stream are made on different rotation groups.

5.3. Alternative Estimators of Current Level and Change

In this section, we describe various alternative estimation procedures for current level and change for the Current Population Survey, which uses the 4–8–4 rotation design. Estimators considered are the present composite estimator, the recursive regression estimator, and the best linear unbiased estimator for various periods of data.

5.3.1. The Present Composite Estimator

Composite estimators combine the estimators of previous periods with data from the current period to form an estimate of the current period. Up until 1985, the composite estimator used in the Current Population Survey was a weighted average of the direct estimator for the current period and a second estimator which is the sum of
the composite estimator for the previous period and the estimated change from the previous to the current period [Breau and Ernst (1983)]. With the Current Population Survey, six of the eight rotation groups observed at time \( t \) were observed at time \( t - 1 \).

The composite estimator used until 1985 is a special case \((\varphi = 0.5)\) of the class of simple composite estimators which have the general form

\[
\hat{\theta}_{tc} = (1 - \varphi)\hat{y}_{t,0,0} + \varphi(\hat{\theta}_{t-1,c} + \hat{\delta}_{t,t-1}),
\]

where

\[
\hat{\delta}_{t-1,c} \quad \text{is the composite estimator for time} \quad t - 1, \quad \text{and} \quad \hat{\delta}_{t,t-1} \quad \text{is an estimator of change in level from time} \quad t - 1 \quad \text{to} \quad t, \quad \text{based on the six rotation groups common to both periods. In other words, the composite estimator used up until 1985 is}
\]

\[
\hat{\theta}_{tc} = 0.5\hat{y}_{t,0,0} + 0.5(\hat{\theta}_{t-1,c} + \hat{\delta}_{t,t-1}).
\]
In 1985, a new composite estimator, which we shall call the "Present Composite Estimator", was introduced. It is of the general form

\[ \hat{\theta}_{tc} = \phi_1 \bar{y}_{t,0,1} + \phi_2 (\hat{\theta}_{t-1,c} + \hat{\delta}_{t,t-1}) + \phi_3 \hat{\delta}_t, \quad (5.3.4) \]

where \( \bar{y}_{t,0,1} \) and \( \hat{\theta}_{t-1,c} \) are defined following (5.3.1), \( \hat{\delta}_{t,t-1} \) is defined in (5.3.3) and

\[ \hat{\delta}_t = 6^{-1} \left\{ \frac{4}{\sum_{k=2}^{3} (y_{t,0,k} + y_{t,0,k+4})} - \frac{4}{\sum_{k=2}^{3} (y_{t-1,0,k-1} + y_{t-1,0,k+3})} \right\}. \]

The estimator currently uses the constants \( \phi_1 = 0.6 \), \( \phi_2 = 1 - \phi_1 = 0.4 \), and \( \phi_3 = 0.2 \).

This is an extension of the old composite estimator in the sense that a new term has been added. The first two terms which correspond to the old composite estimator now have weights 0.6 and 0.4, respectively, and the new term has weight 0.2. The new term is an estimator of the net difference between the incoming part (first and fifth times in sample) and the continuing parts of the sample for the current period. The effect of this modification is to make the expected value of this estimator closer to
the expected value of the basic estimator, which is the mean of the elementary estimates obtained from the eight rotation groups at the current period.

We shall describe the procedure for computing variances of current level and change for the present composite estimator. The same procedure is applicable to the old composite estimator. The composite estimator is a function of current and past observations, where the weight declines as the distance from the current period increases. We compute the weights to be used in the construction of the estimator, associated with 24 periods of observations. Our discussion will be based on the 24 × 8 data matrix M given in Table 5.1. Notice that it is sufficient to compute weights only for rotation groups in the first four streams because the weights for the rotation groups in stream i are exactly the same as the weights for the rotation groups in stream i + 4, i = 1, 2, 3, 4.

If \( \lambda_1 = \frac{1}{8}, \lambda_2 = -\frac{1}{6}, \) and \( \lambda_3 = \frac{1}{3} \), then the general form (5.3.4) of the present composite estimator may be expressed as

\[
\hat{\theta}_{tc} = \omega_1 (y_{t,0,2} + y_{t,0,3} + y_{t,0,4} + y_{t,0,6} + y_{t,0,7} + y_{t,0,8}) \\
+ \omega_2 (y_{t-1,0,1} + y_{t-1,0,2} + y_{t-1,0,3} + y_{t-1,0,5} + y_{t-1,0,6} + y_{t-1,0,7}) \\
+ \omega_3 (y_{t,0,1} + y_{t,0,5}) + \phi_2 \hat{\theta}_{t-1},
\]

(5.3.5)

where \( \omega_1 = \phi_1 \lambda_1 \), \( \omega_2 = \phi_2 \lambda_2 \), \( \omega_3 = \phi_2 \lambda_3 \), \( \omega_2 = \phi_2 \lambda_2 \), and \( \omega_3 = \lambda_1 (\phi_1 + \phi_3) \). We can see from the expression (5.3.5) that the present composite estimator assigns weight \( \omega_1 \)
to elementary estimates of level obtained from rotation groups which were in the survey in the previous period. The estimate in the previous period is then assigned weight $\omega_2$. The estimates from rotation groups which were not in the sample in the previous period are assigned weight $\omega_3$ and the present composite estimator for the previous period is assigned weight $\phi_2$. The weights have the properties

$$6\omega_1 + 2\omega_3 = 1,$$

$$6\omega_2 + \phi_2 = 0.$$

Let $\hat{\theta}_{tci}$ be the component of $\hat{\theta}_t$ corresponding to the $i$-th stream. Then, we may write $\hat{\theta}_{tci}$ as a linear combination of the elementary estimators from all rotation groups in the $i$-th stream, as well as previous composite estimators. For instance, for the first stream, we have

$$\hat{\theta}_{t1} = \omega_1 y_{t,0,8} + \omega_2 y_{t-1,0,7} + \phi_2 \hat{\theta}_{t-1,c,1}$$

Substituting into (5.3.6) recursively going from bottom to top in the first stream of Table 5.1, we get

$$\hat{\theta}_{t1} = \omega_1 y_{t,0,8} + \beta_1 y_{t-1,0,7} + \phi_2 \beta_1 y_{t-2,0,6} + \phi_2^2 \beta y_{t-3,0,5}$$

$$+ \phi_2 \{\omega_1 y_{t-4,0,4} + \beta_1 y_{t-5,0,3} + \phi_2 \beta_1 y_{t-6,0,2} + \phi_2^2 \beta y_{t-7,0,1}\}.$$
$$+ \phi_2^8 \{ \omega_1 y_t - 8,0,8 + \beta_1 y_t - 9,0,7 + \phi_2 \beta_1 y_t - 10,0,6 + \phi_2^2 \beta_2 y_t - 11,0,5 \}$$

$$+ \phi_2^{12} \{ \omega_1 y_t - 12,0,4 + \beta_1 y_t - 13,0,3 + \phi_2 \beta_1 y_t - 14,0,2 + \phi_2^2 \beta_2 y_t - 15,0,1 \}$$

$$+ \phi_2^{16} \{ \omega_1 y_t - 16,0,8 + \beta_1 y_t - 17,0,7 + \phi_2 \beta_1 y_t - 18,0,6 + \phi_2^2 \beta_2 y_t - 19,0,5 \}$$

$$+ \phi_2^{20} \{ \omega_1 y_t - 20,0,4 + \beta_1 y_t - 21,0,3 + \phi_2 \beta_1 y_t - 22,0,2 + \phi_2^2 \beta_2 y_t - 23,0,1 \} ,$$

where $\beta_1 = \omega_2 + \phi_2 \omega_1$, $\beta_2 = \omega_2 + \phi_2 \omega_3$ and, since $|\phi_2| < 1$ implies $\phi_2^l \approx 0$ for large $l$, we have taken $\phi_2^{24} \beta_{t-24} \approx 0 \cdot (\phi_2^{24} = (0.4)^{24} = 2.82 \times 10^{-10})$. Similarly, we can find the weights for all rotation groups in all of the other streams. Note that the weights for the eight rotation groups for the current period are, respectively,

$$\omega_1, \omega_3, \omega_1, \omega_1, \omega_3, \omega_1, \omega_1 ,$$

and, for all other periods, the weight for $y_{t-j,0,k}$ in stream $i$ is exactly $\phi_2$ times the weight of $y_{t-j+1,0,k}$ in stream $i-1$.

Thus, we could write the pre-1985 composite estimator of current level as

$$\hat{\theta}_c = \omega^c Y_t , \quad (5.3.7)$$

where $Y_t$ is the vech of $M$, $\omega^c = (\ell_1, \ell_2, ..., \ell_{24})$, and $\ell_i$ is the row vector of weights corresponding to the eight rotation groups in period $i$, $i = 1, ..., 24$. 

Therefore, the variance of the current composite estimator is, approximately,

$$\text{Var}\{\hat{\theta}_{tc}\} = \omega_c' V \omega_c,$$

where $V$ is the covariance matrix of $Y_t$.

Estimators for previous levels and of period-to-period change may then be computed by expressing them as linear functions of $Y_t$ in an appropriate way. The weights for the present composite estimators of the previous levels $t - h$, $h = 1, 2, \ldots$, $t - 1$, are derived from those of the present composite estimator of the current level. For instance, the weights for the present composite estimator of the previous level are obtained as follows: Each of the elementary estimators obtained from the rotation groups in the sample for the current period is assigned weight 0. For all other rotation groups, the weight assigned to the estimate $y_{t-h,0,k}$ is exactly the weight assigned to the estimate $y_{t-h+1,0,k'}$ for $k = 1, 2, \ldots, 8$, by the present composite estimator of current level. If we denote the resulting weight vector by $\omega_{c-1}$, then the present composite estimator of one-period change is

$$\hat{\theta}_{tc} - \hat{\theta}_{t-1,c} = D_{c,c-1} Y_t,$$

where $D_{c,c-1} = \omega_c - \omega_{c-1}$, and its variance is

$$V\{\hat{\theta}_{tc} - \hat{\theta}_{t-1,c}\} = D_{c,c-1} V D_{c,c-1}'.$$

This procedure is described in detail for the best linear unbiased estimators in Section 4.3.
5.3.2. The Best Linear Unbiased Estimator of Current Level

Under the 4—8—4 rotation design, best linear unbiased estimation can be treated as a special case of the general least squares estimation procedure described in Section 4.3. We start by assuming that there are no time—in—sample effects or rotation group bias in the survey. This means that all elementary estimators for a given period have the same expected value. The effect of time—in—sample effects rotation group bias on the various estimators is investigated in Section 5.4.

A linear model will be constructed to define the estimator. Let

\[ y_i = (y_{i,0,1}, y_{i,0,2}, \ldots, y_{i,0,p})' \]  

where, for any \( t \), \( y_{i,0,t} \) is an estimate for month \( t \) based on a rotation group in the \( i \)—th stream. Eight streams of data collected over \( p \) periods are available. Let \( Y_p \) be the data vector formed by the streams of the data matrix arranged chronologically. In other words, let

\[ Y_p = (y_1', y_2', \ldots, y_8')' \]  

be the \( n \times 1 \) vector of observations, \( (n = s \times p) \) let

\[ \theta_p = (\theta_1, \theta_2, \ldots, \theta_{p-1}, \theta_p)' \]
be the $p \times 1$ vector of true but unknown parameters of interest, let

$$X_1 = J_{s \times 1} \otimes I_{p \times p}$$

be the $n \times p$ design matrix which relates the estimates in $Y_p$ to the parameters in $\theta_p$. Thus, $X_1$ is the Kronecker product $J_{s \times 1} \otimes I_{p \times p}$, where $J_{s \times 1}$ is the $s \times 1$ vector of ones, and $I_{p \times p}$ is the $p \times p$ identity matrix.

The linear model with no time—in—sample effects is

$$Y_p = X_1 \theta_p + \epsilon_p,$$

where $\epsilon_p$ is the vector of error terms and we assume that:

$$E\{\epsilon_p\} = 0.$$

Let $V_p$ be the covariance matrix of $Y_p$. The elements of $V_p$ can be computed using the fact that $Y_p$ consists of $s$ streams. If we assume that there are no replicate effects, then the columns, where columns correspond to streams, of the data matrix in Table 5.1 are independent, and the covariance matrix of $Y_p$ is block diagonal. In particular, if $Q_i$ is the covariance matrix of stream $i$, $i = 1, 2, ..., s$, then the covariance matrix of $Y_p$ is

$$\Omega_{bb} = \text{Blockdiag}\{Q_1, Q_2, ..., Q_s\}.$$
On the basis of model (5.2.3) and under the assumption of no replicate effects, the
autocovariances in each stream are

\[ \sigma_e^2 + \rho_a(h)\sigma_a^2 \]

if observations are \( h \) distance apart in the same rotation group, and are zero if the
observations are in different rotation groups. The corresponding expression for the
autocorrelation in each stream is

\[ \frac{\sigma_e^2 + \rho_a(h)\sigma_a^2}{\sigma_e^2 + \sigma_a^2} . \]

In the presence of replicate effects, the covariance matrix of \( Y_p \) is

\[ V_p = \Omega_{bb} + J_n \cdot J_n^\prime \cdot \sigma_u^2 . \] (5.3.15)

Alternatively, in terms of correlations, we have

\[ \rho_p = R_{bb} + J_n \cdot J_n^\prime \cdot \delta_u^2 , \]

where \( \rho_p \) and \( R_{bb} \) are respectively the correlation matrices of \( Y_p \) under the
assumptions of replicate effects and no replicate effects, and

\[ \delta_u^2 = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_e^2 + \sigma_a^2} . \]
With the covariance matrix in (5.3.15), we conclude from Theorem 4.2.6 that the best linear unbiased estimator \( \hat{\theta}_p \) of \( \theta_p \) is obtained by minimizing the sum of squared deviations from the model that assumes no replicate effects. In other words,

\[
\hat{\theta}_p = CY_p ,
\]

(5.3.16)

where

\[
C = (X_1^{'\Omega_{bb}^{-1}X_1})^{-1}X_1^{'\Omega_{bb}^{-1}}
\]

and the covariance matrix of \( \hat{\theta}_p \) is

\[
\text{Var}\{\hat{\theta}_p\} = \Sigma_p = (X_1^{'\Omega_{bb}^{-1}X_1})^{-1}X_1^{'}\Omega_{bb}^{-1}(\Omega_{bb} + J_{n\times 1}J_{n\times 1}^{'}\sigma_u^2)\Omega_{bb}^{-1}X_1(X_1^{'\Omega_{bb}^{-1}X_1})^{-1} .
\]

(5.3.17)

The matrix \( C \) is the weight matrix, and is composed of the coefficients that are applied to the elementary estimators in forming the best linear unbiased estimator of \( \theta_p \).

The best linear unbiased estimator of any linear combination \( \lambda^{'\hat{\theta}_p} \) of the components of the parameter vector \( \theta_p \) is \( \lambda^{'\hat{\theta}_p} \), with variance \( \lambda^{'\Sigma_p}\lambda \). In particular, if \( C_i \) denotes the i-th row of \( C \), then the best linear unbiased estimator of the current level is

\[
\hat{\theta}_p(p) = C_p Y_p ,
\]

(5.3.18)
where \( \hat{\theta}_p(p) \) denotes the best estimator of \( \theta_p \) using data through time \( p \). The best linear unbiased estimator of the period-to-period change is

\[
\hat{\theta}_p(p) - \hat{\theta}_{p-1}(p) = D_{p,p-1} Y_p,
\]

(5.3.19)

where \( D_{p,p-1} = C_p - C_{p-1} \). The variances of the best linear unbiased estimators (5.3.18) and (5.3.19) are

\[
C_p V_p C_p' \quad \text{and} \quad D_{p,p-1} V_p D_{p,p-1}',
\]

respectively.

Finally, note that

\[
C_p V_p C_p' = C_p (\Omega_{bb} + \sigma_u^2 J_{n \times 1} J_{n \times 1}') C_p'
\]

\[
= C_p \Omega_{bb} C_p' + \sigma_u^2 C_p J_{n \times 1} J_{n \times 1}' C_p'
\]

\[
= C_p \Omega_{bb} C_p' + \sigma_u^2,
\]

because \( C_p J_{n \times 1} \) is the sum of the coefficients of the elementary estimates from the rotation groups at the current level which is equal to one. This shows that for any characteristic, the variance of the estimator of current level is obtained by adding \( \sigma_u^2 \) to the variance of the estimate of current level under the assumption of no replicate effects.
5.3.3. The Recursive Regression Estimator for the 4–8–4 Rotation Scheme

The best linear unbiased estimation procedure described in the preceding section may not be computationally efficient when it is necessary to repeat the procedure for each month over a long period of time. The computation of the estimators becomes progressively more difficult as the number of periods increases. Wolter (1979) circumvented the computational problem by considering the simpler composite estimators. Composite estimators are, in general, less efficient than the best linear unbiased estimators.

The recursive regression procedure of estimation is an attempt to address the problem of computational complexity associated with best linear unbiased estimation, while, at the same time, producing minimum variance estimators.

Let us assume that the 4–8–4 rotation scheme of the Current Population Survey has been in operation for \( m \) months. Then the best linear unbiased estimation procedure described in the preceding section provides us with \( m \) estimates of \( \theta_h \), \( h = p - m + 1, \ldots, p \) which are the best linear unbiased estimators at time \( p \), as well as the \( m \times m \) covariance matrix of these estimates. Thus, at time \( p + 1 \), we have:

(a) \( p \) initial estimates \( \hat{\theta}_p(p) = (\hat{\theta}_{p-m+1}(p), \ldots, \hat{\theta}_p(p))' \)

(b) the covariance matrix \( \Sigma_{11,p} \) of \( \hat{\theta}_p \)

(c) the eight independent elementary estimates of \( \theta_{p+1} \), obtained from the rotation group totals for time \( p + 1 \).

To obtain the best linear unbiased estimator of \( \theta_{p+1} \), our procedure adopts a linear model approach in which the vector of observations consists of the \( p \) initial
estimates given in (a) and the eight independent estimates of $\theta_{p+1}$ obtained from the rotation groups introduced at time $p + 1$, given in (c).

**Notation and definitions 5.3.1.**

Define the following eight independent linear combinations of observations for the eight time–in–sample groups observed at time $p + 1$:

$$z_{i,p+1} = y_{i,0,p+1} - \sum_{j=1}^{m} \alpha_{k(i,p+1),j} y_{i,0,p+1-j},$$

where $k(i,p+1) = k$ gives time–in–sample as a function of $(i,p+1)$, and $\alpha_{k(i,p+1)} = 0$ for all $i, j, k$ if $j < i$ and for all $k$, if $j > i$. For any $t$, $z_{i,t}$, $i = 1, 2, \ldots, 8$, denotes the "residual" from the regression of $y_{i,0,t}$ on its predecessors in the same stream: $y_{i,0,h}$ for $h < t$. Thus, $z_{i,t}$ is a linear combination of $y_{i,0,h}$, where,

- $h = t, t-1, \ldots, t-j+1$ for $1 \leq k \leq 4$
- $h = t, t-9, t-10, t-11, t-12$ for $k=5$
- $h = t, t-1, t-10, t-11, t-12, t-13$ for $k=6$
- $h = t, t-1, t-2, t-10, t-11, t-12, t-13, t-14$ for $k=7$
- $h = t, t-1, t-2, t-3, t-12, t-13, t-14, t-15$ for $k=8$.

Note that $z_{i,p+1}$ is that linear combination of all observations obtained from the rotation group which is in stream $i$ at time $p + 1$ such that the $z$'s are uncorrelated with the previous $y$'s. Hence, they are uncorrelated with $(\hat{\theta}_{p-m+1}(p), \ldots, \hat{\theta}_p(p))$. Expressions for the $z_{i,p+1}$, $k=1,2,\ldots,8$ as well as their means and variances, are given in Appendix B. From (i) – (viii), of Appendix B, we may write the model at time $p + 1$ as
\[ Z_{p+1} = X_2 \theta_{p+1(m+1)} + \epsilon_{p+1}, \]  

where

\[ Z_{p+1} = (\hat{\theta}_{p-m+1(p)}, \ldots, \hat{\theta}_p(p), z_{1,p+1}, z_{2,p+1}, \ldots, z_{8,p+1}) \]

\[ = \text{the } (m + 8) \times 1 \text{ vector consisting of } m \text{ initial estimates of } m \]
\[ \text{parameters and the } 8 \text{ independent observations constructed from the} \]
\[ \text{8 elementary estimators obtained at time } p + 1, \]

\[ \hat{\theta}_{p+1(m+1)} = (\hat{\theta}_{p-m+1}, \ldots, \hat{\theta}_p, \hat{\theta}_{p+1}) \]

\[ = \text{the } (m + 1) \times 1 \text{ vector of population parameters.} \]

\[ X_2 = \begin{bmatrix} I_{m \times m} & 0 \\ X_{21} & J_{8 \times 1} \end{bmatrix}, \]  

\[ X_{21} = \begin{bmatrix} O_{4 \times 7} & O_{4 \times 5} & C_{4 \times 3} \\ B_{4 \times 7} & O_{4 \times 5} & D_{4 \times 3} \end{bmatrix}, \]

\[ B = \begin{bmatrix} 0 & 0 & 0 & -\alpha_{44} & -\alpha_{43} & -\alpha_{42} & -\alpha_{41} \\ 0 & 0 & -\alpha_{55} & -\alpha_{54} & -\alpha_{53} & -\alpha_{52} & 0 \\ 0 & -\alpha_{66} & -\alpha_{65} & -\alpha_{64} & -\alpha_{63} & 0 & 0 \\ -\alpha_{77} & -\alpha_{76} & -\alpha_{75} & -\alpha_{74} & 0 & 0 & 0 \end{bmatrix}, \]
Now, let

\[ \psi_1 = 1 - \alpha_{11}, \]
\[ \psi_2 = 1 - \alpha_{21} - \alpha_{22}, \]
\[ \psi_3 = 1 - \alpha_{31} - \alpha_{32} - \alpha_{33}, \]
\[ \psi_4 = 1 - \alpha_{41} - \alpha_{42} - \alpha_{43} - \alpha_{44}, \]
\[ \psi_5 = 1 - \alpha_{51} - \alpha_{52} - \alpha_{53} - \alpha_{54} - \alpha_{55}, \]
\[ \psi_6 = 1 - \alpha_{61} - \alpha_{62} - \alpha_{63} - \alpha_{64} - \alpha_{65} - \alpha_{66}, \]
\[ \psi_7 = 1 - \alpha_{71} - \alpha_{72} - \alpha_{73} - \alpha_{74} - \alpha_{75} - \alpha_{76} - \alpha_{77}, \]

and let

\[ \psi' = (1, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7). \]  \hfill (5.3.24)

We may rewrite \( Z_{p+1} \) as

\[ Z'_{p+1} = (\hat{\theta}_{p(m)}', \ z'_{p+1}). \]
where

\[ \hat{\theta}_p(m) = (\hat{\theta}_{p-m+1}(p), \ldots, \hat{\theta}_p(p)) \text{ and } z_{p+1} = (z_{1,p+1}, \ldots, z_{8,p+1}) . \]

Let \( \Sigma_{11,p(m)} \) be the \( m \times m \) covariance matrix of \( \hat{\theta}_p(m) \). We have

\[ \text{Cov}\{\hat{\theta}_p(m), z_{p+1}\} = J_{m \times 1} \psi \sigma_u^2 \text{ and } \text{Var}\{z_{p+1}\} = \psi \psi' \sigma_u^2 , \]

where \( J_{m \times 1} \) is the \( m \times 1 \) vector of ones. Therefore,

\[ \text{Var}\{Z_{p+1}\} = V_{p+1} = \begin{bmatrix} \hat{\Sigma}_{11,p(m)} & 0 \\ 0 & \sigma_u^2 \end{bmatrix} + \sigma_u^2 \begin{bmatrix} J_{m \times 1}J_{m \times 1}' & J_{m \times 1} \psi' \\ \psi J_{m \times 1}' & \psi \psi' \end{bmatrix} \]

(5.3.25)

where \( \hat{\Sigma}_{11,p(m)} = \Sigma_{11,p(m)} - \sigma_u^2 J_{m \times 1}J_{m \times 1}' , \)

\[ Q_{00} = \text{diag}\{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2, \sigma_7^2, \sigma_8^2\} \]

(5.3.26)

and \( \sigma_k^2 = \text{Var}\{z_{i,p+1}\}, i = 1, 2, \ldots, 8 . \)

The recursive regression estimator of the vector \( \theta_{p+1(m+1)} \) of the population characteristics at time \( p + 1 \) is

\[ \hat{\theta}_{p+1(m+1)} = PZ_{p+1} \]

(5.3.27)

where

\[ P = (X_2'V_{p+1}^{-1}X_2)^{-1}X_2'V_{p+1}^{-1} \]

(5.3.28)
and the covariance matrix of $\hat{\theta}_{p+1(m+1)}$ is

$$
\Sigma_{p+1(m+1)} = (X'_{2} V^{-1} X_{2})^{-1} .
$$

We now describe our recursive procedure for computing the estimate of the current level, which is the last element of $\theta_{p+1(m+1)}$. Recall that at time $p+1$, our data vector, given by (5.3.21), consists of $m$ initial estimates, all of them optimal for their respective periods using data collected up to and including the current period, as well as the eight elementary estimates obtained from the eight rotation groups introduced at time $p+1$.

The linear model for the data at time $p+1$ is given in (5.3.20) – (5.3.25). The recursive procedure is implemented as follows: From model (5.3.20) – (5.3.25), the best linear unbiased estimator of $\theta_{p+1(m+1)}$ using data through time $t+1$ is given by

$$
\hat{\theta}_{p+1(m+1)} = (\hat{\theta}_{p-m+1(p+1)}, \ldots, \hat{\theta}_{p(p+1)}, \hat{\theta}_{p+1(p+1)}) ,
$$

with covariance matrix $\Sigma_{p+1(m+1)}$ given in (5.3.29).

At time $p+2$, the linear model is

$$
Z_{p+2} = X_{2} \theta_{p+2(m+1)} + \varepsilon_{p+2} ,
$$

(5.3.30)
where

\[
\begin{align*}
Z_{p+2}^\prime &= (\hat{\theta}_{p-m+2(p+1)}, \ldots, \hat{\theta}_{p(p+1)}, \hat{\theta}_{p+1(p+1)}, z_{1,p+2}^\prime, \ldots, z_{8,p+2}^\prime) \\
(5.3.31) \\
\theta_{p+2(m+1)}^\prime &= (\theta_{p-m+2}^\prime, \ldots, \theta_{p}^\prime, \theta_{p+1}^\prime, \theta_{p+2}^\prime) \\
(5.3.32) \\
\text{Var}\{Z_{p+2}\} &= V_{p+2} = \begin{bmatrix}
\hat{\Sigma}_{11,p+1(m)} & 0 \\
0 & Q_{00}
\end{bmatrix} + \sigma_u^2 \begin{bmatrix}
J_{m \times 1}J^\prime_{m \times 1} & J_{m \times 1}\psi^\prime \\
\psi J^\prime_{m \times 1} & \psi\psi^\prime
\end{bmatrix},
(5.3.33)
\end{align*}
\]

where \(\hat{\Sigma}_{11,p+1(m)} = \Sigma_{11,p+1(m)} - \sigma_u^2 J_{m \times 1}J^\prime_{m \times 1}\) and \(\Sigma_{11,p+1(m)}\) is the \(m \times m\) lower right submatrix of the \((m+1) \times (m+1)\) matrix \(\Sigma_{p+1(m+1)}\) given in (5.3.29).

From this model, we obtain the best linear unbiased estimator of \(\theta_{p+2(m+1)}\) based on data through time \(p+2\). Then the last \(m\) elements of \(\hat{\theta}_{p+2(m+1)}\) are used as initial estimates for the next iteration. The general procedure is as follows:

1. the initial estimate of the earliest level is dropped from the data vector and the parameter corresponding to this estimate is also dropped from the parameter vector. The least squares estimate of the immediately preceding level is then added to the data vector, thus ensuring that the dimension of the data vector remains the same from iteration to iteration. The new observations obtained at this stage are functions of an additional parameter which corresponds to the current level and this parameter is added to the parameter vector. Thus, from iteration to iteration, the number of parameters does not change, the vector of
random errors changes in accordance with changes in the data vector, and the model matrix remains the same. This means that the dimension of the estimation problem remains the same from iteration to iteration.

(2) The covariance matrix of the data vector at the \( \ell \)-th iteration is obtained as follows: Let the model at the \((\ell-1)\)-th iteration be

\[
Z_{p+\ell-1} = X_2 \theta_{p+\ell-1(m+1)} + \epsilon_{p+\ell-1} \tag{5.3.34}
\]

and let the covariance matrix of \( Z_{p+\ell-1} \) be

\[
\Sigma_{p+\ell-1(m+1)} = (X_2' V_{p+\ell-1}^{-1} X_2)^{-1}. \tag{5.3.36}
\]

Then, the covariance matrix of \( Z_{p+\ell} \), the data vector at the \( \ell \)-th iteration is

\[
\Sigma_{11,p+\ell-1(m)} = \begin{bmatrix} \Sigma_{11,p+\ell-1(m+1)} & 0 \\ 0 & \Sigma_{00} \end{bmatrix} + \sigma_u^2 \begin{bmatrix} J_{m \times 1} & J_{m \times 1} \\ \psi J_{m \times 1} & \psi \psi' \end{bmatrix} \tag{5.3.37}
\]

where \( \Sigma_{11,p+\ell-1(m)} \) is the covariance matrix of the \( m \) recursive least squares estimates in \( Z_{p+\ell-1} \) under the assumption of no replicate effects, that is, the \( m \times m \) lower right submatrix of

\[
\Sigma_{p+\ell-1(m+1)} = (X_2' V_{p+\ell-1}^{-1} X_2)^{-1}.
\]
where \( \Sigma_{11,p+\ell-1}(m) = \Sigma_{11,p+\ell-1}(m) - \sigma^2_1 J_{m \times 1} J_{m \times 1}' \) and \( \Sigma_{11,p+\ell-1}(m) \) is the \( m \times m \) lower right submatrix of \( \Sigma_{p+\ell-1}(m+1) \) given in (5.3.36).

From Theorem 4.4.2., \( \Sigma_{11,p}(m) \) stabilizes at \( \Sigma_0 \), say, as the number of periods increases. Let the limiting covariance matrix of \( Z_{p+1} \) be \( \Sigma \), where

\[
\Sigma = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & Q_{00} \end{bmatrix} + \sigma^2_0 \begin{bmatrix} J_{m \times 1} J_{m \times 1}' & J_{m \times 1}' \psi' \\ \psi J_{m \times 1} & \psi' \end{bmatrix}
\]  
(5.3.38)

and \( Q_{00} \) is defined in (5.3.26). Therefore, the recursive regression estimator of the vector \( \theta_{p+1(m+1)} \) of the parameters converges to

\[
\hat{\theta}_{m+1,R} = P Z_{p+1},
\]  
(5.3.39)

where

\[
P = (X_2' \Sigma^{-1} X_2)' X_2 \Sigma^{-1}
\]  
(5.3.40)

\( X_2 \) is defined in (5.3.23), \( Z_{p+1} \) is defined in (5.3.21) and \( \Sigma \) is defined in (5.3.38).

The covariance matrix of \( \hat{\theta}_{m+1,R} \) is

\[
Q = (X_2' \Sigma^{-1} X_2)^{-1}
\]  
(5.3.41)

The matrix \( P \) is the weight matrix and is composed of the coefficients that are applied to the elementary estimators in forming the recursive regression estimator. The recursive regression estimator of any linear combination \( \lambda' \theta_{p+1(m+1)} \) of components of the parameter vector \( \theta_{p+1(m+1)} \) converges to \( \lambda' \hat{\theta}_{p+1,R} \), with
variance $\lambda' Q \lambda$. In particular, if $P_i$ denotes the $i$-th row of $P$, then the recursive regression estimator of the current level converges to

$$\hat{\theta}_{p+1,R} = \hat{\theta}_{p+1}(p+1) = P_{m+1}Z_{p+1}$$  \hspace{1cm} (5.3.42)$$

and the recursive regression estimator of one-period change converges to

$$\hat{\theta}_{p+1}(p+1) - \hat{\theta}_p(p+1) = RZ_{p+1}$$  \hspace{1cm} (5.3.43)$$

where

$$R = P_{m+1} - P_m.$$ 

The variances of the estimators (5.3.42) and (5.3.43) are $P_{m+1} \Sigma P_{m+1}'$ and $R \Sigma R'$ respectively. Now, let

$$P_{m+1} = (\pi_0', \pi_1')$$

where $\pi_1$ is the $s \times 1$ vector of coefficients of the elementary estimates obtained from the $s$ rotation groups for the current month, and $\pi_0$ is the $m \times 1$ vector of coefficients of the $m$ initial estimates. Then, we know that $\pi_1' J_{s \times 1} = 1$ and $\pi_0' J_{m \times 1} = 1 - \pi_1' \psi$ or $\pi_0' J_{m \times 1} + \pi_1' \psi = 1$. Therefore,

$$P_{m+1} \Sigma P_{m+1}' = (\pi_0', \pi_1')[\Sigma_0 \ 0 \ 0 \ Q_{00}][\pi_0 \ \pi_1]$$
Now,

\[
\begin{pmatrix} \pi_0, \pi_1' \end{pmatrix} \begin{bmatrix} J_{m \times 1}' J'_{m \times 1} & J_{m \times 1}' \psi' \\ J_{m \times 1}' & \psi' \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \pi_0' J_{m \times 1}' J'_{m \times 1} \pi_0 + \pi_1' \psi' \pi_1' + \pi_0' J_{m \times 1}' \psi \pi_1 + \pi_1' \psi' \pi_0' \pi_1' \\
+ \pi_0' J_{m \times 1}' \psi' \pi_1 + \pi_1' \psi' \pi_0' \pi_1' \\
= (\pi_0' J_{m \times 1} + \pi_1' \psi)^2 \\
= 1.
\]

Therefore,

\[
P_{m+1} \Sigma P_{m+1} = P_{m+1} \begin{bmatrix} \Sigma_0 & 0 \\ 0 & Q_{00} \end{bmatrix} P_{m+1}' + \sigma_u^2.
\]

Thus, we arrive at the same conclusion as in Section 5.3.2, that the variance of the estimate of the current level is obtained by adding \( \sigma_u^2 \) to the variance of the estimator of current level constructed under the assumption of no replicate effects.

The unrevised estimate of one - period change is:

\[
\hat{\theta}_{p+1}(p+1) - \hat{\theta}_p(p) = DZ_{p+1}, \quad (5.3.44)
\]
where $D = P_{m+1} - P_0$ and $P_0$ is the $1 \times (m+s)$ row vector with its $m$-th component equal to one and all other components equal to zero. The variance of this estimator is $D \Sigma D'$, where $\Sigma$ is the covariance matrix of $Z_{p+1}$ defined in (5.3.38).

5.4. Time-in-Sample Effects

A major problem with most periodic surveys is that of time-in-sample or rotation group effects. This refers to the phenomenon by which estimates of current level for a given period obtained from different rotation groups have different expected values, depending on the length of time they have been included in the sample. The effects of this bias on the estimates of current level and change have been studied by Bailar (1975 and 1978) using labor force data from the Current Population Survey, by Ghangurde (1982) and Tessier and Tremblay (1976) for the Canadian Labor Force Survey, and by Pfeffermann (1991).

We shall now examine the effect of rotation group bias or time-in-sample effects on the least squares estimators of current level and change. The least squares procedures described in Section 5.3 can be modified to incorporate time-in-sample effects. Our discussion will focus on the 4–8–4 rotation scheme, but our procedure can be easily modified and applied to any rotation design. We assume that data for 24 periods are available and have been arranged in the $p \times s$ data matrix $M$ shown in Table 5.1, where $p = 24$ and $s = 8$. Thus, the total number of observations is $n = p \times s$. 
5.4.1. Best Linear Unbiased Estimation

We proceed exactly as in Section 5.3.1, where we used elementary estimators in a linear model to produce best linear unbiased estimators. In the presence of time-in-sample effects, the components of the linear model (5.3.7) in Section 5.3.1 that change are the design matrix $X_l$ and the parameter vector $\theta_p$. The design matrix changes because the elementary estimator obtained from rotation groups at time $t$ are no longer unbiased estimates of the parameter of interest $\theta_t$ at time $t$.

Suppose $\tau_{tk}$ is the rotation group effect for time $t$ associated with the rotation group which is in its $k$-th time-in-sample. Then, for each time $t$, we may write the model

$$ y_{tjk} = \theta_t + \tau_{tk} + \epsilon_{tjk}, $$

(5.4.1)

where $y_{tjk}$ and $\epsilon_{tjk}$ are defined in Section 5.2 and $\epsilon_{tjk}$ has the same covariance structure as in Section 5.2. In some cases, we assume that the time-in-sample effects are constant. That is,

(i) $\tau_{tk} = \tau_k$ for all $t$.

We generally assume that the sum of the time-in-sample effects is zero. That is,

(ii) $\sum_{k=1}^{s} \tau_{tk} = 0$
for all \( t \). The second assumption is one condition that gives estimability of the current level.

If time—in—sample effects are present, then the linear model (5.3.7) is modified as follows. If the time—in—sample effects are constant, the parameter vector becomes

\[
\beta_p^* = (\tau_1, \ldots, \tau_s, \theta_1, \theta_2, \ldots, \theta_{p-1}, \theta_p)',
\]

(5.4.2)

where \( \theta_1, \ldots, \theta_{p-1}, \theta_p \) are the true but unknown parameters of interest and \( \tau_1, \ldots, \tau_s \) are the time—in—sample effects. The design matrix is

\[
X_{1,\tau}^* = (J_{s \times 1} \otimes I_{s \times s}, X_1).
\]

(5.4.3)

Thus, \( X_{1,\tau}^* \) is an \( n \times (p + s) \) design matrix where the last \( p \) columns are identical to those of the design matrix \( X_1 = J_{s \times 1} \otimes I_{p \times p} \) of model (5.3.7). The covariance matrix \( V \) of the data vector (the vech of \( M \)) is exactly the same as that defined in (5.3.15). However, since the matrix \( (X_{1,\tau}^*, V^{-1}X_{1,\tau}^*) \) is now singular, the components of the parameter vector \( \beta_p^* \) are nonestimable (Searle, 1971). In other words, the current level of a characteristic is not estimable in the model without the restriction (ii). However, change is estimable. Imposing a constraint such as assumption (ii) will enable us to obtain an estimation for \( \beta_p^* \). With this restriction, the design matrix, denoted by \( X_{1,\tau} \) may now be explicitly written as:

\[
X_{1,\tau} = (T, X_1),
\]

(5.4.4)
where $T$ is an $n \times (s-1)$ matrix,

$$T = (M_3', A_7', M_2', A_6', M_2', A_5', M_2', A_4', M_2', A_3', M_2', A_2', M_2', A_1', M_2', I_7),$$

$$M_3 = J_{3 \times 1} \otimes K_{(s-1)}, \quad K_{(s-1)} = (I_{(s-1) \times (s-1)}, -J_{(s-1) \times 1}),$$

$$M_2 = J_{2 \times 1} \otimes K_{(s-1)}, \quad A_1 = \begin{bmatrix} I_{6 \times 6} & 0 \\ -J_{1 \times 6} & -1 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & I_{(s-2) \times (s-2)} \\ -1 & -J_{1 \times (s-2)} \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_{(s-3) \times (s-3)} \\ -1 & -1 & -J_{1 \times (s-3)} \end{bmatrix},$$

$$A_5 = \begin{bmatrix} I_{2 \times 2} & 0 & 0 \\ 0' & 0 & I_4 \\ -J_{1 \times 2} & -1 & -J_{1 \times 4} \end{bmatrix}, \quad A_4 = \begin{bmatrix} I_{3 \times 3} & 0 & 0 \\ 0' & 0 & I_{3 \times 3} \\ -J_{1 \times 3} & -1 & -J_{1 \times 3} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} I_{4 \times 4} & 0 & 0 \\ 0' & 0 & I_{2 \times 2} \\ -J_{1 \times 4} & -1 & -J_{1 \times 2} \end{bmatrix}, \text{ and } \quad A_2 = \begin{bmatrix} I_{5 \times 5} & 0 & 0 \\ 0' & 0 & 1 \\ -J_{1 \times 5} & -1 & -1 \end{bmatrix}.$$
Therefore, the best linear unbiased estimator of

$$\beta_p = (\tau, \ldots, \tau_{s-1}, \theta_1, \ldots, \theta_{p-1}, \theta_p)'$$  \hspace{1cm} (5.4.5)$$
is

$$\hat{\beta}_p = (X_{1,\tau}' X_{1,\tau}^{-1})^{-1} X_{1,\tau}' V^{-1} y_p$$  \hspace{1cm} (5.4.6)$$

and the covariance matrix of $\hat{\beta}_p$ is

$$\Sigma_{p,\tau} = (X_{1,\tau}' V^{-1} X_{1,\tau})^{-1}.$$  \hspace{1cm} (5.4.7)$$

Note that if $\hat{\tau}_k$ is the least squares estimator of $\tau_k$, $k = 1, \ldots, s-1$ obtained from the $(p+k)$-th component of $\hat{\beta}_p$, then, from the restriction (ii), the least squares estimator of $\tau_s$ is

$$\hat{\tau}_s = - \sum_{k=1}^{s-1} \hat{\tau}_k.$$  \hspace{1cm} (5.4.8)$$

The variances of the estimators of current level and change can then be computed in the manner described in Sections 4.3.2 and 5.7. The variances of the best linear unbiased estimator based on 24 periods and in the presence of rotation group bias are presented in Table 5.51 for employed, unemployed, and Civilian Labor Force.
5.4.2. The Recursive Regression Procedure

To construct a recursive estimator in the presence of time-in-sample effects, we proceed as in Section 5.3.3 with appropriate modifications in the design matrix and parameter vector of the corresponding linear model. We assume that at time $t$, the following quantities are available:

(a) The initial estimates

\[ \hat{\beta}_{t-1}(\ell) = (\hat{\tau}_{t-1(s-1)}, \hat{\theta}_{t-1(m)})' \]

where $\hat{\tau}_{t-1(s-1)} = (\hat{\tau}_{1(t-1)}, ..., \hat{\tau}_{s-1(t-1)})$ and $\hat{\theta}_{t-1(m)} = (\hat{\theta}_{t-m(t-1)}, ..., \hat{\theta}_{t-1(t-1)})$.

(b) The covariance matrix of $\hat{\beta}_{t-1}(\ell)$ given by

\[ \Gamma_{11,t-1(\ell)} = \begin{bmatrix} \Omega_{11,t-1(s-1)} & \Omega_{12,t-1} \\ \Omega_{12,t-1}' & \Omega_{22,t-1(m)} \end{bmatrix}, \quad (5.4.9) \]

where $\Omega_{11,t-1(s-1)} = \text{Var}\{\hat{\tau}_{t-1(s-1)}\}$, $\Omega_{12,t-1} = \text{Cov}\{\hat{\tau}_{t-1(s-1)}, \hat{\theta}_{t-1(m)}\}$, and $\Omega_{22,t-1} = \text{Var}\{\hat{\theta}_{t-1(m)}\}$.

(c) $s$ independent observations obtained by a suitable transformation of the elementary estimates obtained from the rotation groups at time $t$. Let these observations be denoted by $z_t = (z_{1t}, ..., z_{st})'$. 
Then, the linear model at time $t$ is

$$Z_{t, \tau} = W_2 \beta_t(\ell+1) + \epsilon_{t, \tau},$$

(5.4.9a)

where

$$Z_{t, \tau} = (\tilde{\beta}_{t-1}(\ell), z_t)$$

and

$$\beta_t(\ell+1) = (\tau_1, \ldots, \tau_{s-1}, \theta_{t-m}, \ldots, \theta_{t-1}, \theta_t).$$

The design matrix $W_2$ is given by

$$W_2 = \begin{bmatrix}
I_{(s-1) \times (s-1)} & 0 & 0 \\
0 & I_{p \times p} & 0 \\
W_{31} & W_{32} & J_{s \times 1}
\end{bmatrix},$$

(5.4.10)

where

$$W_{31} = \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}.$$
\[ \beta_{\tau_j} = 1 + \alpha_{\tau_j}, \quad j = 1, ..., 7 \] and \[ W_{32} = X_{21}, \] which is defined following (5.3.23).

First, we write down the covariance matrix of \( Z_{t,T} \) explicitly. Note that every initial estimate \( \hat{\theta}_{t-i}(t-1) \), \( i = 1, ..., t-1 \) can be written as a linear combination of all the elementary estimates that constitute the data vector at time \( t-1 \). That is,

\[
\hat{\theta}_{t-i}(t-1) = \sum_{h=1}^{t-1} \sum_{k=1}^{s} c_{hk} y_{hjk}, \quad i = 1, 2, ..., t-1
\]

(5.4.11)

\[
= \sum_{h=1}^{t-1} \sum_{k=1}^{s} c_{hk} (u_j + e_{hj} + a_{hjk})
\]

where, for convenience, we have used the expression of \( y_{hjk} \) given by (5.2.3) and \( u_j \), \( e_{hj} \), and \( a_{hjk} \) are defined following (5.2.3). We have also used the fact that in (5.4.11),

\[
\sum_{h=1}^{t-1} \sum_{k=1}^{s} c_{hk} = 1.
\]

Therefore, from (5.4.12), we see that

\[
\text{Cov}\{\hat{\theta}_{t-i}(t-1), y_{tjk}\} = \sigma_u^2, \quad i = 1, ..., t-1.
\]

(5.4.13)
Now, if we define $Y_{tj}$ as $Y_{tj} = (y_{tj1}, \ldots, y_{tjs})$, then

$$z_t = \Delta Y_{tj},$$

(5.4.14)

where $\Delta$ is the lower triangular matrix defined by

$$
\Delta = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-\alpha_{11} & 1 & 1 & 1 & 1 & 1 & 1 \\
-\alpha_{22} & -\alpha_{21} & 1 & 1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha_{66} & -\alpha_{65} & -\alpha_{64} & -\alpha_{63} & -\alpha_{62} & -\alpha_{61} & 1 \\
-\alpha_{77} & -\alpha_{76} & -\alpha_{75} & -\alpha_{74} & -\alpha_{73} & -\alpha_{72} & -\alpha_{71} & 1
\end{bmatrix}
$$

Note that $\Delta J_{s \times 1} = \psi$, where $\psi$ is defined in (5.3.24) and $J_{s \times 1}$ is the $s \times 1$ vector of ones. From (5.4.13), we see that

$$\text{Cov}\{\hat{\theta}_{t-1}(t-1), Y_{tj}\} = \sigma_u^2 J_{s \times 1}.$$ 

Therefore, from (5.4.14),

$$\text{Cov}\{\hat{\theta}_{t-1}(t-1), z_t\} = \sigma_u^2 J_{s \times 1} \Delta^\prime = \sigma_u^2 \psi,$$
and hence,

\[ \text{Cov}\{ \theta_t, z_t \} = \sigma_u^2 J_{t-1} \psi , \]

where \( J_{t-1} \) is the \((t-1) \times 1\) vector of ones. Next, we show that the covariance between the initial estimate of each time-in-sample effect and the new observations is zero. Note that each initial estimate of the time-in-sample effects can be written as a linear combination of the elementary estimates that constitute the data vector at time \( t - 1 \), that is,

\[
\hat{\tau}_k(t-1) = \sum_{h=1}^{t-1} \sum_{k=1}^{s} d_{hk} y_{hjk}, \quad k = 1, \ldots, s , \quad (5.4.15)
\]

where

\[
\sum_{h=1}^{t-1} \sum_{k=1}^{s} d_{hk} = 0 .
\]

Therefore,

\[
\hat{\tau}_k(t-1) = s \sum_{h=1}^{t-1} d_{hk} e_{hj} + \sum_{h=1}^{t-1} \sum_{k=1}^{s} d_{hk} a_{tjk} ,
\]

and

\[
\text{Cov}\{ \hat{\tau}_k(t-1), y_{tjk} \} = 0 , \quad k = 1, 2, \ldots, s .
\]

Hence,

\[
\text{Cov}\{ \hat{\tau}_t, z_t \} = 0 ,
\]
where $0$ is the $s \times s$ matrix of zeros. Finally, we note that

$$\text{Var}\{z_t\} = Q_{00} + \sigma_u^2 \psi \psi' ,$$

where $Q_{00}$ is defined in (5.3.26). From the preceding calculations, we see that the covariance matrix of $Z_{t,\tau}$, the data vector at time $t$, is given by:

$$V_{t,\tau} = \begin{bmatrix}
\Omega_{11,t-1(s-1)} & \Omega_{12,t-1} & 0 \\
\Omega_{12,t-1} & \Omega_{22,t-1(m)} & 0 \\
0' & 0' & Q_{00}
\end{bmatrix} + \sigma_u^2 \begin{bmatrix}
0' & 0 & 0 \\
0' & 0 & J_{t-1} \psi' \\
0' & \psi J_{t-1} \psi' & \psi \psi'
\end{bmatrix}.$$

In implementing the recursive procedure, the current estimates of the time—in—sample effects are in the data vector throughout the iteration process. It therefore follows that the variance of each of the time—in—sample effects will converge to zero as the number of periods increases.

One may be unwilling to assume that the time—in—sample effects are constant over a long period. One way of permitting the time—in—sample effects to change slowly over time is to do a kind of "exponential smoothing" by adjusting the covariance matrix of the estimated effects at time $t$, $\Gamma_{11,t-1(\ell)}$ of (5.4.9) used to construct the estimator (5.4.7). One procedure is to multiply the covariance matrix of the initial estimates of the time—in—sample effects, $\Omega_{11,t-1(s-1)}$, by a constant bigger than one, so that the diagonal elements increase slightly as the number of periods increases. Since, if no modification is introduced, the variance of each of the estimators of the time—in—sample effects decreases at the rate $n^{-1}$, the factor that is
used to increase $\Omega_{11,t-1(s-1)}$ after, say $T$ periods (including the number of periods on which the initial estimates for the recursive procedure are based) is $1 + T^{-1}$.

For the particular case under consideration, the initial estimates are based on 15 periods. The number of iterations before adjusting $\Omega_{11,t-1(s-1)}$ was chosen to be 21. This gives estimates based on 36 periods and the results correspond to those of the best linear unbiased estimator based on 36 periods.

In this procedure, it is important to distinguish between the matrix used to define the estimator and the actual covariance matrix of the estimators.

Suppose $\Sigma_{11,t,\tau}$ converges to $\Sigma_{11,\tau}$ where

$$
\Sigma_{11,\tau} = \begin{bmatrix}
\Omega_{11(s-1)} & \Omega_{12} \\
\Omega'_{12} & \Omega_{22(m)}
\end{bmatrix},
$$

as the number of periods increases. Then, the recursive regression estimator of $\beta_t(t+1)$ converges to

$$
\hat{\beta}_t(t+1) = P_{\tau} Z_t, \tau',
$$

(5.4.16)

where

$$
P_{\tau} = \left[ W_2' V^{-1} W_2 \right]^{-1} W_2' V^{-1},
$$
and

\[
V_\tau = \begin{bmatrix}
\Omega_{11}^{(s-1)} & \Omega_{12} & 0 \\
\Omega_{12} & \Omega_{22}^{(m)} & 0 \\
0 & 0 & Q_{00}
\end{bmatrix} + \sigma_u^2 \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & J_{t-1} \psi' \\
0 & \psi \psi_{t-1} & \psi \psi'
\end{bmatrix}.
\]

However, the limiting variance of \( \hat{t}_{t+1} \) is not equal to

\[
\Sigma_{t+1,R} = \left[ W_2 V_{\tau}^{-1} W_2 \right]^{-1}.
\] (5.4.17)

Since \( \hat{t}_{t+1} \) is a function of preceding estimates, one can use the procedure outlined in Section 5.3.1 to calculate the coefficients of the observations that define the estimator. The recursive regression estimator of current level converges to the \( \ell \)-th element of \( \hat{t}_{t+1} \) with variance which converges to the last diagonal element of

\[
P_{(t+1),\tau} V_{(t+1),\tau} P'_{(t+1),\tau}.
\]

The recursive regression estimators of several period change both with and without revision of previous estimates are computed exactly as described in Section 5.3.3. The results based on an application of this procedure are presented and discussed in Section 5.7.

5.4.3. Time Varying Rotation Group Effects

We now extend model (5.4.1) to the case of time-varying time-in-sample effects by permitting the time-in-sample effects to vary stochastically in time. We assume that the time-in-sample effects follow a first order autoregressive process with
a nonzero mean, and that the sum of the time—in—sample effects is zero. The model for the time—in—sample effects is

$$\tau_{tk} = \tau_k + \alpha(\tau_{t-1,k} - \tau_k) + e_{tk}, \quad (5.4.18)$$

where $e_t = (e_{t1}, \ldots, e_{ts})$ is a vector of errors which satisfies $E(e_t) = 0$ and $E(e_t e'_t) = \Sigma_{ee}$. If $R_{ee}$ is the correlation matrix of $e_t$, and $D_e$ is the diagonal matrix whose diagonal entries are the square roots of the variances of $e_{tk}$, $k = 1, \ldots, s$, then we may write

$$\Sigma_{ee} = D_e R_{ee} D'_e. \quad (5.4.19)$$

where $R_{ee} = (r_{ij})$ and

$$r_{ij} = \begin{cases} 1 & \text{if } i = j, \\ (s-1)^{-1} & \text{if } i \neq j. \end{cases}$$

We assume that $\alpha$ and $\Sigma_{ee}$ are known. Our objective is to construct an updating procedure for the best linear unbiased estimation of $\theta_t$ as the number of periods increases. To do this, we first formulate the estimation problem in state—space form and then apply the Kalman filter techniques described in Section 4.5 to obtain the updating procedure for the estimator of $\theta_t$, as well as the variance of the estimation error. Models (5.4.1) and (5.4.18) can be combined to form a state—space model as

$$y_{tjk} = x_{tk} + u_{tjk} \quad (5.4.20)$$
where \( \lambda_{tk} = \tau_{tk} - \tau_k \), for all \( t \) and \( k \), and \( \{e_{tk}\} \) is independent of \( \{u_{tjk}\} \). The first two equations constitute the measurement equation and the last equation is the state equation. We assume that \( \tau_k \) and \( \theta_k \) are fixed for all \( t \) and \( k \), and that \( u_{tjk} \) satisfies a first order autoregressive model with parameter \( \rho \). Thus, the observations at time \( t \) can be transformed so as to be uncorrelated with previous observations. The transformed observations are \( z_{t1} = y_{tj1} \), \( z_{t5} = y_{tj5} \), and \( z_{tk} = y_{tjk} - \rho y_{t-1,j,k-1} \), \( k = 2, 3, 4, 6, 7, 8 \). Let \( Y_t(s) = z_t = (z_{t1}, ..., z_{ts})' \). Then, the covariance matrix of \( Y_t(s) \) is

\[
\Sigma_{uu} = \text{Diag}\{\sigma_1^2, \sigma_2^2, \sigma_2^2, \sigma_1^2, \sigma_2^2, \sigma_2^2, \sigma_2^2\},
\]

where

\[
\sigma_1^2 = \text{Var}\{z_{t1}\} = \text{Var}\{z_{t5}\}
\]

and

\[
\sigma_2^2 = (1 - \rho^2)\sigma_1^2 = \text{Var}\{z_{tk}\}, \quad k = 2, 3, 4, 6, 7, 8.
\]
We may now express (5.4.20) – (5.4.22) as

\[ Y_t(s) = H_t \beta_t + u_t, \] (5.4.24)

\[ \beta_t = A_t \beta_{t-1} + \epsilon_t, \] (5.4.25)

where

\[ H_t = H = (H_{11} \ H_{12} \ H_{13} \ H_{14}), \]

\[
H_{11} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\rho & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\rho & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\rho & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\rho & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -\rho & 1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -(1-\rho)
\end{bmatrix},
\]

\[
H_{12} = \begin{bmatrix}
0 & 1 \\
-\rho & 1 \\
-\rho & 1 \\
-\rho & 1 \\
0 & 1 \\
-\rho & 1 \\
-\rho & 1 \\
-\rho & 1
\end{bmatrix}, \quad H_{13} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\rho & 0 & 0 & 0 & 0 \\
0 & 0 & -\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\rho & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\rho & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\rho & 0
\end{bmatrix},
\]
\[
H_{14} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\end{bmatrix},
\]

\[
\beta_t = (\tau_1, \ldots, \tau_7, \theta_{t-1}, \theta_t, \lambda_{t-1,1}, \ldots, \lambda_{t-1,7}, \lambda_{t1}, \ldots, \lambda_{t7})',
\]

\[
\beta_{t-1} = (\tau_1, \ldots, \tau_7, \theta_{t-1}, \theta_t, \lambda_{t-2,1}, \ldots, \lambda_{t-2,7}, \lambda_{t-1,1}, \ldots, \lambda_{t-1,7})',
\]

\[
A_t = A = \begin{bmatrix}
I_{7 \times 7} & 0_{7 \times 2} & 0_{7 \times 7} & 0_{7 \times 7} \\
0_{2 \times 7} & I_{2 \times 2} & 0_{2 \times 7} & 0_{2 \times 7} \\
0_{7 \times 7} & 0_{7 \times 2} & I_{7 \times 7} & 0_{7 \times 7} \\
0_{7 \times 7} & 0_{7 \times 2} & 0_{7 \times 7} & \alpha I_{7 \times 7} \\
\end{bmatrix},
\]

and \( \epsilon_t = (0, \ldots, 0, 0, 0, \ldots, 0, \epsilon_{t1}, \ldots, \epsilon_{t7})' \), which satisfies
\[ E\{\epsilon_t\} = 0 \text{, and } E\{\epsilon_t \epsilon_t^\prime\} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0' & 0 & 0 & 0 \\
0' & 0' & 0 & 0 \\
0' & 0' & 0 & \Sigma_{ee}
\end{bmatrix}, \]

where \( \Sigma_{ee} \) is defined following 5.4.19 and \( 0 \) is a matrix of zeros of appropriate dimension. Further, note that \( u_t \) and \( \epsilon_t \) are uncorrelated random vectors with zero mean and known covariance matrix given by

\[ E\{(u_t', \epsilon_t')' (u_t', \epsilon_t')\} = \text{Block diag}\{\Sigma_{uu'}, \Sigma_{\epsilon\epsilon'}\}. \]

Let \( \hat{\beta}_{t-1} \) be the best linear unbiased estimator of \( \beta_{t-1} \) based on data through time \( t-1 \) with error covariance matrix \( \Sigma_{vv,t-1,t-1} \). Then the best linear unbiased estimator of \( \beta_t \) is

\[ \hat{\beta}_t = A \hat{\beta}_{t-1} + \Sigma_{\omega\omega t t} H' D_t^{-1} (Y_t - HA \hat{\beta}_{t-1}), \quad (5.4.27) \]

where

\[ \Sigma_{\omega\omega tt} = \Sigma_{\epsilon\epsilon} + A \Sigma_{vv,t-1,t-1} A', \]

\[ D_t = \Sigma_{uu} + HE_{\omega\omega tt} H'. \]
The error covariance matrix is

$$\Sigma_{vtt} = E\{ (\hat{\theta}_t - \beta_t)(\hat{\theta}_t - \beta_t)' \}$$

$$= \Sigma_{\omega tt} - \Sigma_{\omega tt} H' D_t^{-1} H \Sigma_{\omega tt} \ .$$  \hspace{1cm} (5.4.26)

The best linear unbiased estimator of $\theta_t$ is the ninth element of the vector of estimates $\hat{\theta}_t$ in (5.4.26), and the error variance is the ninth diagonal element of $\Sigma_{vtt}$ in (5.4.27).

To initiate the filter, we implement the best linear unbiased procedure based on model (5.4.1) for 24 periods. This gives us the initial estimates of $\theta_{t-1}$ and the time–in–sample effects $\tau = (\tau_1, ..., \tau_7)'$ as well as the covariance matrix of these estimates. Let $\hat{\theta}_{t-1,0}$ be the initial estimator of $\theta_{t-1}$ with variance $v_0$. Since we don’t have an initial estimator of $\theta_t$ at time $t-1$, we assume that the initial estimate of $\theta_t$ is 0 with infinite variance. Let $\hat{\tau} = (\hat{\tau}_1, ..., \hat{\tau}_7)'$ be the initial estimate of $\tau$ such that $\text{Var} \{ \hat{\tau}_i \} = s_i^2$, $i = 1, ..., 7$ and let $D\tau = \text{Diag} \{ s_1, s_2, ..., s_7 \}$ . Then, if $\hat{\beta}_0$ is the initial estimator of $\beta_t$, the initial error covariance matrix is

$$\Sigma_{v00} = \begin{bmatrix}
\Sigma_{ee,0} & 0 & 0 & 0 \\
0 & \mathbf{s}_0 & 0 & 0 \\
0 & 0 & \Sigma_{ee,0} & \alpha\Sigma_{ee,0} \\
0 & 0 & \alpha\Sigma'_{ee,0} & \Sigma_{ee,0}
\end{bmatrix}.$$
where $\Sigma_{ee,0} = D_r R_{ee} D_r$ and

$$
\hat{\xi}_0 = \begin{bmatrix} v_0 & 0 \\ 0 & \infty \end{bmatrix}.
$$

Operationally, we take

$$
\hat{\xi}_0 = \begin{bmatrix} v_0 & 0 \\ 0 & N \end{bmatrix},
$$

where $N$ is an arbitrarily large number, say $10^6$. At the $j$-th step, the matrix $\hat{\xi}_j$ is

$$
\hat{\xi}_j = \begin{bmatrix} v_j & 0 \\ 0 & N \end{bmatrix}.
$$

5.5. Other Rotation Designs

In this section, we consider two continuous rotation schemes as alternatives to the intermittent 4–8–4 rotation scheme used in the Current Population Survey and described in Section 5.1. The continuous rotation schemes under consideration are the 8-in-then-out rotation scheme and the 6-in-then-out rotation scheme. These rotation schemes are investigated on the presumption that they are easier to implement in the field and require less record keeping in the office. Also, the optimal least squares estimator is easier to construct for the continuous schemes than for an intermittent rotation scheme such as the 4–8–4 rotation scheme. We shall discuss
various estimation procedures and compare both estimation expression and the
variances of estimators under the three rotation schemes.

5.5.1. The 6-in-then-out Scheme

The 6-in-then-out scheme is the rotation scheme used in the Canadian Labor
Force Survey. See Kumar, S., and Lee, H. (1983). For each period of the survey, the
sample consists of six rotation groups. A rotation group remains in the sample for six
consecutive periods and then drops out of the sample for good. The rotation pattern is
illustrated in Table 5.9 for 24 periods.

The recursive regression estimation procedure for this scheme is implemented as
follows. Suppose at time $t + 1$, we have the following:

(i) five initial estimates $\hat{\theta}_t^{(5)} = (\hat{\theta}_{t-4}^{(t)}, ..., \hat{\theta}_t^{(t)})$, which are the best
    estimates of the characteristic of interest at times $t - 4, ..., t$ based on
data through time $t$,

(ii) the covariance matrix $\Sigma_{11,t}^{(5)}$ of $\hat{\theta}_t^{(5)}$,

(iii) six independent observations $z_{t+1,1}, ..., z_{t+1,6}$ which are transformations
    of the observations at $t + 1$. If $y_{t,0,k}$ denotes the elementary estimate
    based on a rotation group which is in its $k-$th time-in-sample at time $t$,
    these transformations are defined by

\[
\begin{align*}
    z_{t+1,1} &= y_{t+1,0,1} \\
    z_{t+1,k} &= y_{t+1,0,k} - \sum_{j=1}^{k-1} f_{k-1,j} y_{t+1-j,0,k-j}, \quad k = 2, 3, 4, 5, 6.
\end{align*}
\]  

(5.5.1)
Table 5.9. Data from 24 periods of a survey collected by the 6-in-then-out rotation scheme

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A_1,1</td>
<td>E_1,2</td>
<td>I_1,3</td>
<td>M_1,4</td>
<td>Q_1,5</td>
<td>V_1,6</td>
</tr>
<tr>
<td>2</td>
<td>A_2,2</td>
<td>E_2,3</td>
<td>I_2,4</td>
<td>M_2,5</td>
<td>Q_2,6</td>
<td>W_2,1</td>
</tr>
<tr>
<td>3</td>
<td>A_3,3</td>
<td>E_3,4</td>
<td>I_3,5</td>
<td>M_3,6</td>
<td>R_3,1</td>
<td>W_3,2</td>
</tr>
<tr>
<td>4</td>
<td>A_4,4</td>
<td>E_4,5</td>
<td>I_4,6</td>
<td>N_4,1</td>
<td>R_4,2</td>
<td>W_4,3</td>
</tr>
<tr>
<td>5</td>
<td>A_5,5</td>
<td>E_5,6</td>
<td>J_5,1</td>
<td>N_5,2</td>
<td>R_5,3</td>
<td>W_5,4</td>
</tr>
<tr>
<td>6</td>
<td>A_6,6</td>
<td>F_6,1</td>
<td>J_6,2</td>
<td>N_6,3</td>
<td>R_6,4</td>
<td>W_6,5</td>
</tr>
<tr>
<td>7</td>
<td>B_7,1</td>
<td>F_7,2</td>
<td>J_7,3</td>
<td>N_7,4</td>
<td>R_7,5</td>
<td>W_7,6</td>
</tr>
<tr>
<td>8</td>
<td>B_8,2</td>
<td>F_8,3</td>
<td>J_8,4</td>
<td>N_8,5</td>
<td>R_8,6</td>
<td>X_8,1</td>
</tr>
<tr>
<td>9</td>
<td>B_9,3</td>
<td>F_9,4</td>
<td>J_9,5</td>
<td>N_9,6</td>
<td>S_9,1</td>
<td>X_9,2</td>
</tr>
<tr>
<td>10</td>
<td>B_10,4</td>
<td>F_10,5</td>
<td>J_10,6</td>
<td>O_10,1</td>
<td>S_10,2</td>
<td>X_10,3</td>
</tr>
<tr>
<td>11</td>
<td>B_11,5</td>
<td>F_11,6</td>
<td>K_11,1</td>
<td>O_11,2</td>
<td>S_11,3</td>
<td>X_11,4</td>
</tr>
<tr>
<td>12</td>
<td>B_12,6</td>
<td>G_12,1</td>
<td>K_12,2</td>
<td>O_12,3</td>
<td>S_12,4</td>
<td>X_12,5</td>
</tr>
<tr>
<td>13</td>
<td>C_13,1</td>
<td>G_13,2</td>
<td>K_13,3</td>
<td>O_13,4</td>
<td>S_13,5</td>
<td>X_13,6</td>
</tr>
<tr>
<td>14</td>
<td>C_14,2</td>
<td>G_14,3</td>
<td>K_14,4</td>
<td>O_14,5</td>
<td>S_14,6</td>
<td>Y_14,1</td>
</tr>
<tr>
<td>15</td>
<td>C_15,3</td>
<td>G_15,4</td>
<td>K_15,5</td>
<td>O_15,6</td>
<td>T_15,1</td>
<td>Y_15,2</td>
</tr>
<tr>
<td>16</td>
<td>C_16,4</td>
<td>G_16,5</td>
<td>K_16,6</td>
<td>P_16,1</td>
<td>T_16,2</td>
<td>Y_16,3</td>
</tr>
<tr>
<td>17</td>
<td>C_17,5</td>
<td>G_17,6</td>
<td>L_17,1</td>
<td>P_17,2</td>
<td>T_17,3</td>
<td>Y_17,4</td>
</tr>
<tr>
<td>18</td>
<td>C_18,6</td>
<td>H_18,1</td>
<td>L_18,2</td>
<td>P_18,3</td>
<td>T_18,4</td>
<td>Y_18,5</td>
</tr>
<tr>
<td>19</td>
<td>D_19,1</td>
<td>H_19,2</td>
<td>L_19,3</td>
<td>P_19,4</td>
<td>T_19,5</td>
<td>Y_19,6</td>
</tr>
<tr>
<td>20</td>
<td>D_20,2</td>
<td>H_20,3</td>
<td>L_20,4</td>
<td>P_20,5</td>
<td>T_20,6</td>
<td>Z_20,1</td>
</tr>
<tr>
<td>21</td>
<td>D_21,3</td>
<td>H_21,4</td>
<td>L_21,5</td>
<td>P_21,6</td>
<td>U_21,1</td>
<td>Z_21,2</td>
</tr>
<tr>
<td>22</td>
<td>D_22,4</td>
<td>H_22,5</td>
<td>L_22,6</td>
<td>R_22,1</td>
<td>U_22,2</td>
<td>Z_22,3</td>
</tr>
<tr>
<td>23</td>
<td>D_23,5</td>
<td>H_23,6</td>
<td>E_23,1</td>
<td>R_23,2</td>
<td>U_23,3</td>
<td>Z_23,4</td>
</tr>
<tr>
<td>24</td>
<td>D_24,6</td>
<td>A_24,1</td>
<td>E_24,2</td>
<td>R_24,3</td>
<td>U_24,4</td>
<td>Z_24,5</td>
</tr>
</tbody>
</table>

*Notation: $A_{t,j}$ denotes a household which is in its $j$-th time-in-sample in period $t$ of the survey.*
Then, we may write the model at time $t+1$ as

$$Z_{t+1,6} = X_6 \theta_{t+1}(6) + \epsilon_{t+1,6}, \quad (5.5.2)$$

where

$$Z_{t+1,6}' = (\hat{\theta}_{t-4}(t), ..., \hat{\theta}_t(t), z_{t+1,1}, ..., z_{t+1,6}),$$

and

$$\hat{\theta}_{t+1}(6) = (\theta_{t-4}, ..., \theta_t, \theta_{t+1}),$$

and

$$X_6 = \begin{bmatrix} I_5 & 0 \\ K_{21} & J_{6 \times 1} \end{bmatrix}$$

$I_5$ is the $5 \times 5$ identity matrix, $J_{6 \times 1}$ is the $6 \times 1$ column of ones, and

$$K_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f_{11} \\ 0 & 0 & 0 & -f_{22} & -f_{21} \\ 0 & 0 & -f_{33} & -f_{32} & -f_{31} \\ 0 & -f_{44} & -f_{43} & -f_{42} & -f_{41} \\ -f_{55} & -f_{54} & -f_{53} & -f_{52} & -f_{51} \end{bmatrix}$$

a $6 \times 5$ matrix which is constant over time. Furthermore, if

$$\psi_{k,6} = 1 - \sum_{j=1}^{k} f_{kj}, \quad k = 1, ..., 5,$$
and

$$\Psi_6 = (1, \psi_{1,6}, \psi_{2,6}, \psi_{3,6}, \psi_{4,6}, \psi_{5,6})^T,$$

then the covariance matrix of $Z_{t+1,6}$ is

$$V_{t+1,6} = \begin{bmatrix} \Sigma_{11,t}(5) & 0 \\ 0 & Q_{00,6} \end{bmatrix} + \sigma_u^2 \begin{bmatrix} 0 & J_{5 \times 1} \Psi_6 \\ \Psi_6^T J_{5 \times 1} & \Psi_6^T \Psi_6 \end{bmatrix},$$

where $J_{5 \times 1}$ is the $5 \times 1$ vector of ones, $Q_{00,6} = \text{Diag}\{\sigma_1^2, \ldots, \sigma_6^2\}$, and $\sigma_k^2 = \text{Var}\{z_{t+1,k}\}$, $k = 1, 2, 3, 4, 5, 6$.

By Theorem 4.4.1, the best linear unbiased estimator of $\theta_{t+1}(6)$ is

$$\hat{\theta}_{t+1}(6) = (X_6 V_{t+1,6} X_6)^{-1} X_6 V_{t+1,6} Z_{t+1,6},$$

and the covariance matrix of $\hat{\theta}_{t+1}(6)$ is

$$\Sigma_{t+1}(6) = (X_6 V_{t+1,6} X_6)^{-1}.$$

To keep the dimension of our estimation problem constant from time $t + 1$ to time $t + 2$, we drop the best estimator of $\theta_{t-4}$ at time $t + 1$ from the vector of observations and add the best estimator of $\theta_{t+1}$ at time $t + 1$. Thus, the data vector $Z_{t+2,6}$ at time $t + 2$ consists of the five estimates $\hat{\theta}_{t-3}(t+1), \ldots, \hat{\theta}_{t+1}(t+1)$, (which are the best estimates of $\theta_{t-3}, \ldots, \theta_{t+1}$ based on data through time $t + 1$) and the six independent observations $z_{t+2,1}, \ldots, z_{t+2,6}$ at time $t + 2$. 
Thus, the model at time \( t + 2 \) is

\[
Z_{t+2,6} = X_6 \theta_{t+2(6)} + \epsilon_{t+2,6},
\]

where

\[
\theta_{t+2(6)} = (\theta_{t-3}, \ldots, \theta_{t+1}, \theta_{t+2}),
\]

\[
Z'_{t+2,6} = (\hat{\theta}_{t-3(t+1)}, \ldots, \hat{\theta}_{t+1(t+1)}, z_{t+2,1}, \ldots, z_{t+2,6}),
\]

and

\[
\text{Var}(Z_{t+2,6}) = V_{t+2,6} = \begin{bmatrix}
\Sigma_{11,t+1(5)} & 0 \\
0 & Q_{00,6}
\end{bmatrix} + \sigma_u^2 \begin{bmatrix}
0 & J_{5 \times 1} \Psi' \\
\Psi' J_{5 \times 1} & \Psi_6 \Psi_6'
\end{bmatrix},
\]

where \( \Sigma_{11,t+1(5)} \) is the \( 5 \times 5 \) lower right submatrix of the \( 6 \times 6 \) matrix \( \Sigma_{t+1(6)} \) given in (5.5.4). From model (5.5.5), we obtain the best linear unbiased estimator of \( \theta_{t+2(6)} \) from the usual least squares formulas.

From Theorem 4.4.2, the covariance matrix \( \Sigma_{11,t(5)} \) of the recursive least squares estimates in this recursive procedure converges to a positive definite matrix as the number of periods increases indefinitely. Suppose \( \Sigma_{11,t(5)} \) stabilizes at \( \Sigma_{11(5)} \), and let

\[
\Sigma_{0,6} = \begin{bmatrix}
\Sigma_{11(5)} & 0 \\
0 & Q_{00,6}
\end{bmatrix} + \sigma_u^2 \begin{bmatrix}
0 & J_{5 \times 1} \Psi' \\
\Psi' J_{5 \times 1} & \Psi_6 \Psi_6'
\end{bmatrix}.
\]
Then, the recursive regression estimator of $\theta_{t+1}(6)$ converges to

$$\hat{\theta}_{6,R} = (X'_6 \Sigma_{0,6}^{-1} X_6)^{-1} X'_6 \Sigma_{0,6}^{-1} Z_{t+1,6}, \tag{5.5.6}$$

and the covariance matrix of $\hat{\theta}_{6,R}$ converges to

$$\Sigma_{6, R} = (X'_6 \Sigma_{0,6}^{-1} X_6)^{-1}. \tag{5.5.7}$$

Estimators of current level and change over several periods can be computed as described in Section 5.3 for the 4—8—4 rotation scheme.

### 5.5.2. The 8—in—then—out Scheme

In this rotation scheme, there are eight rotation groups in the sample for each period, one rotation group in its first time in sample, ..., and one rotation group in its eighth time in sample. A rotation group stays in the sample for eight consecutive periods and then drops out of the sample for good. The rotation pattern for this scheme is illustrated in Table 5.10 for 24 periods. We now describe the recursive regression procedure for the 8—in—then—out rotation scheme.

Suppose that at time $t + 1$, we have the following:

(i) seven initial estimates $\hat{\theta}'_t(7) = (\hat{\theta}_{t-6}(t), \ldots, \hat{\theta}_t(t))$ which are the best estimates of $\hat{\theta}'_t(7) = (\theta_{t-6}, \ldots, \theta_t)$ based on data through time $t$,

(ii) the covariance matrix $\Sigma_{11,t}(7)$ of $\hat{\theta}_t(7)$,

(iii) eight independent observations obtained from the rotation groups in the sample for the last eight periods.
Table 5.10. Data from 24 periods of a survey collected by the 8-in—then—out rotation scheme

<table>
<thead>
<tr>
<th>Month</th>
<th>Streams</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>A_{1,1}</td>
</tr>
<tr>
<td>2</td>
<td>A_{2,2}</td>
</tr>
<tr>
<td>3</td>
<td>A_{3,3}</td>
</tr>
<tr>
<td>4</td>
<td>A_{4,4}</td>
</tr>
<tr>
<td>5</td>
<td>A_{5,5}</td>
</tr>
<tr>
<td>6</td>
<td>A_{6,6}</td>
</tr>
<tr>
<td>7</td>
<td>A_{7,7}</td>
</tr>
<tr>
<td>8</td>
<td>A_{8,8}</td>
</tr>
<tr>
<td>9</td>
<td>B_{9,1}</td>
</tr>
<tr>
<td>10</td>
<td>B_{10,2}</td>
</tr>
<tr>
<td>11</td>
<td>B_{11,3}</td>
</tr>
<tr>
<td>12</td>
<td>B_{12,4}</td>
</tr>
<tr>
<td>13</td>
<td>B_{13,5}</td>
</tr>
<tr>
<td>14</td>
<td>B_{14,6}</td>
</tr>
<tr>
<td>15</td>
<td>B_{15,7}</td>
</tr>
<tr>
<td>16</td>
<td>B_{16,8}</td>
</tr>
<tr>
<td>17</td>
<td>C_{17,1}</td>
</tr>
<tr>
<td>18</td>
<td>C_{18,2}</td>
</tr>
<tr>
<td>19</td>
<td>C_{19,3}</td>
</tr>
<tr>
<td>20</td>
<td>C_{20,4}</td>
</tr>
<tr>
<td>21</td>
<td>C_{21,5}</td>
</tr>
<tr>
<td>22</td>
<td>C_{22,6}</td>
</tr>
<tr>
<td>23</td>
<td>C_{23,7}</td>
</tr>
<tr>
<td>24</td>
<td>C_{24,8}</td>
</tr>
</tbody>
</table>

\(^1\text{Notation: } A_{t,j} \text { denotes a household which is in its } j-\text{th time— in— sample in period } t \text{ of the survey.}\)
These are defined as

\[ z_{t+1,1} = y_{t+1,0,1} \]

\[ z_{t+1,k} = y_{t+1,0,k} - \sum_{j=1}^{k-1} c_{k-1,j} y_{t+1-j,0,k-j}, \quad k = 2, 3, ..., 8. \]

(5.5.8)

Thus, we may write the model at time \( t + 1 \) as

\[ Z_{t+1,8} = X_8 \theta_{t+1(8)} + \epsilon_{t+1,8}, \]

where

\[ Z_{t+1(8)} = (\dot{\theta}_{t-6}(t), ..., \dot{\theta}_t(t), z_{t+1,1}, ..., z_{t+1,8}), \]

\[ \theta_{t+1(8)} = (\theta_{t-6}, ..., \theta_t, \theta_{t+1}), \]

\[ X_8 = \begin{bmatrix} I_7 & 0 \\ W_{21} & J_{8 \times 1} \end{bmatrix}, \]

\( I_7 \) is the \( 7 \times 7 \) identity matrix, \( J_{8 \times 1} \) is the \( 8 \times 1 \) vector of ones, and
Furthermore, if

\[ \Gamma = (1, v')', \]

where \( v = J_{8 \times 1} - W_{21} J_{7 \times 1} \), then the covariance matrix of \( Y_{t+1} \) is

\[
V_{t+1,8} = \begin{bmatrix}
\Sigma_{11,t} \cdot (7) & 0 \\
0 & Q_{00,8}
\end{bmatrix} + \sigma^2 \begin{bmatrix}
0 & J_{7 \times 1} \Gamma' \\
\Gamma J_{7 \times 1} & \Gamma \Gamma'
\end{bmatrix},
\]

\( Q_{00,8} = \text{Diag}\{\sigma^2_1, \ldots, \sigma^2_8\} \), and \( \sigma^2_k = \text{Var}\{z_{t+1,k}\}, k = 1, 2, \ldots, 8 \). Therefore, by Theorem 4.4.2, the best linear unbiased estimator of \( \theta_{t+1}(8) \) and its variance are given by the usual generalized least squares formulas. The recursive regression estimation procedure for this scheme is then implemented in a manner analogous to that for the scheme described in Section 5.5.1.
5.5.3. The First Order Composite Estimator

The first order composite estimator is a composite estimator, of the present composite type, constructed to give approximately optimum estimates of current level under a first order autoregressive model. The variances of the estimators will be based on the covariances estimated from the 1987 data, not on the autoregressive model.

We start with the 4–8–4 rotation scheme. The weights to be used in the construction of the first order composite estimator of current level are obtained as follows. Assume that at time \( t \), we have the following:

(i) The present composite estimator of the previous level, denoted by \( \hat{\theta}_{t-1} \)

(ii) the eight independent observations \( y_{t,0,1}, y_{t,0,5} \) and \( y_{t,0,k} - \alpha y_{t-1,0,k-1}, k = 2, 3, 4, 6, 7, 8 \).

Therefore, at time \( t \), we may write the model

\[
g_t = X\beta_t(2) + \epsilon_t, \tag{5.5.9}
\]

where

\[
\begin{align*}
\mathbf{e}_t' &= (\hat{\theta}_{t-1}, r_{t1}, r_{t2}, r_{t3}, r_{t4}, r_{t5}, r_{t6}, r_{t7}, r_{t8})', \\
r_{t1} &= y_{t,0,1}, \quad r_{t5} = y_{t,0,5}, \\
\text{and} \\
r_{ti} &= y_{t,0,i} - \alpha y_{t-1,0,i-1}, \quad i = 2, 3, 4, 6, 7, 8,
\end{align*}
\]
\[
X' = \begin{bmatrix}
1 & 0 & -\alpha & -\alpha & -\alpha & 0 & -\alpha & -\alpha & -\alpha \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix},
\]
and
\[
\beta'_t(2) = (\theta_{t-1}, \theta_t).
\]

If we assume that all entries in \( g_t \) are independent, then the covariance matrix of \( g_t \) is

\[
\Omega_t = \text{Diag}\{V(\hat{\theta}_{t-1}), \sigma^2_1, \sigma^2_2, \sigma^2_2, \sigma^2_2, \sigma^2_2, \sigma^2_2, \sigma^2_2, \sigma^2_2\},
\]
where
\[
\sigma^2_1 = V(r_{t1}) = V(r_{t5}),
\]
and
\[
\sigma^2_2 = (1 - \alpha^2)\sigma^2_1 = V(r_{ti}), \quad i = 2, 3, 4, 6, 7, 8.
\]

The variance of the optimal estimator of \( \beta_t(2) \) in the sense of minimum variance is

\[
R_t = (X' \Omega_t^{-1} X)^{-1}.
\]  \hspace{1cm} (5.5.10)

Starting with an appropriate initial value for \( V(\hat{\theta}_{t-1}) \), we iterate this procedure in exactly the same way as for the recursive regression estimation procedure. By Theorem 4.4.2, \( R_t \) converges to \( R_\infty \), say, as \( t \to \infty \). The coefficients for the optimal estimator of \( \beta_t(2) \) converge to

\[
W_\infty = (X' R_\infty^{-1} X)^{-1} X' R_\infty^{-1}.
\]  \hspace{1cm} (5.5.11)
If we denote the weight vector for the best composite estimator of current level $\theta_t$ by $\ell_c$, then $\ell_c$ is the second row of the $2 \times 9$ limiting weight matrix $W_\infty$. Thus, in the notation of Section 5.3.1, we may write $\ell_c$ as

$$\ell_c = (\phi_2, \omega_1, \omega_3, \omega_1, \omega_1, \omega_3, \omega_1, \omega_1),$$

where $\phi_2$, $\omega_1$, and $\omega_3$ are, respectively, the weights assigned to the present composite estimator for the previous level, the elementary estimator from the rotation groups which were in the sample in the previous period and the elementary estimator from the rotation groups which were not in the sample in the previous period. Note that the entries in $\ell_c$ are not absolutely the best weights since $r_{ti}$, $i = 2, 3, 4, 6, 7, 8$ are not completely independent of $\hat{\theta}_{t-1}$. The $r_{ti}$ are independent of $\hat{\theta}_{t-1}$ under the assumption of a first order autoregression with parameter $\alpha$.

The computation of the variances of the best first order composite estimator of current level and change over several periods can now be carried out in exactly the same manner as that described in Section 5.3.1 for the present composite estimator.

We now consider the continuous rotation schemes. For the 8—in—then-out rotation scheme, the linear model that generates the weights for the best first order composite estimator is given by (5.5.9) with

$$r_{t1} = y_{t,0,1},$$

$$r_{ti} = y_{t,0,i} - \alpha y_{t-1,0,i-1}, \quad i = 2, 3, ..., 8$$
For the 6-in-then-out rotation scheme, again the linear model is given by (5.5.9), where

\[ g_t' = \left( \hat{\theta}_{t-1}, r_{t1}, r_{t2}, \ldots, r_{t6} \right), \]

where

\[ r_{t1} = y_{t,0,1}, \]

\[ r_{ti} = y_{t,0,i} - \alpha y_{t-1,0,i-1}, \quad i = 2, 3, \ldots, 6, \]

and

\[ X' = \begin{bmatrix} 1 & 0 & -\alpha & -\alpha & -\alpha & -\alpha & -\alpha & -\alpha \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \]

In both cases, we assume that the components of the data vector \( g_t \) are independent. Let the weights assigned to the elementary estimators from the rotation groups in the current period be denoted by

\[ p = (\alpha_2, p_1, p_3, p_1, p_1, p_1, p_1, p_1, p_1) \]
and
\[ q = (\alpha_3, q_1, q_3, q_1, q_1, q_1) \]

for the 8-in-then-out and 6-in-then-out rotation schemes, respectively. Thus, the weights in \( p \) and \( q \) are such that \( \alpha_2, \alpha_3 \) correspond to \( \hat{\theta}_{t-1} \), the weights \( p_1, q_1 \) correspond to \( y_{t,0i} - \alpha y_{t-1,0,i-1}, i = 2, 3, ..., s \), (where \( s \) is the number of streams; \( s \) is 6 for the 6-in-then-out rotation scheme and 8 for the 8-in-then-out rotation scheme) and \( p_3, q_3 \) correspond to \( y_{t,0,1} \). Then, for the 8-in-then-out rotation scheme, the component of the best first order composite estimator corresponding to first stream of Table 5.10 is derived as follows. We start from the last observation in the first stream and work our way to the top of the stream. Recall that the weight assigned to an observation in the stream is determined by the time-in-sample status of the observation. Now,

\[ \hat{\theta}_{t_{c1,8}} = p_1 y_{t,0,8} + p_2 y_{t-1,0,7} + \alpha_2 \hat{\theta}_{t-1,c,1,8} \]

\[ = p_1 y_{t,0,8} + (p_2 + \alpha_2 p_1) y_{t-1,0,7} + \alpha_2 p_2 y_{t-2,0,6} + \alpha_2^2 \hat{\theta}_{t-2,c,1} \]

\[ = p_1 y_{t,0,8} + (p_2 + \alpha_2 p_1) y_{t-1,0,7} + \alpha_2 (p_2 + \alpha_2 p_1) y_{t-2,0,6} \]

\[ + \alpha_2 p_2 y_{t-3,0,5} + \alpha_2^2 \hat{\theta}_{t-3,c,1} \]

\[ = p_1 y_{t,0,8} + \beta_1 p y_{t-1,0,7} + \alpha_2 \beta_1 p y_{t-2,0,6} \]

\[ + \alpha_2^2 p_2 y_{t-3,0,5} + \alpha_2^3 \hat{\theta}_{t-3,c,1} \]  

(5.5.10)
where $p_2 = -\alpha p_1$ and $\beta_{1p} = p_2 + \alpha_2 p_1$. Substituting into (5.5.10) recursively up to the observation at the top of the first stream of Table 5.10, we get

\begin{equation}
\hat{\vartheta}_{tc1,8} = p_1 y_{t,0,8} + \beta_{1p} \sum_{j=0}^{5} \alpha_2^j y_{t-j-1,0,8-j-1} + \alpha_2^6 \beta_{2p} y_{t-7,0,1} \\
+ \alpha_2^{8} p_1 y_{t-8,0,8} + \beta_{1p} \sum_{j=0}^{5} \alpha_2^j y_{t-j-9,0,8-j-1} + \alpha_2^{14} \beta_{2p} y_{t-15,0,1} \\
+ \alpha_2^{16} p_1 y_{t-16,0,8} + \beta_{1p} \sum_{j=0}^{5} \alpha_2^j y_{t-j-17,0,8-j-1} + \alpha_2^{22} \beta_{2p} y_{t-23,0,1},
\end{equation}

(5.5.11)

where $\beta_{2p} = p_2 + \alpha_2 p_3$.

For the 6-in-then-out rotation scheme, the corresponding expression (that is, the component of the best composite estimator corresponding to the first stream of Table 5.9) is

\begin{equation}
\hat{\vartheta}_{tc1,6} = q_1 y_{t,0,6} + \beta_{1q} \sum_{j=0}^{3} \alpha_3^j y_{t-j-1,6-j-1} + \alpha_3^{4} \beta_{2q} y_{t-5,0,1} \\
+ \alpha_3^{6} q_1 y_{t-6,0,6} + \beta_{1q} \sum_{j=0}^{3} \alpha_3^j y_{t-j-7,0,6-j-1} + \alpha_3^{10} \beta_{2q} y_{t-11,0,1} \\
+ \alpha_3^{12} q_1 y_{t-12,0,6} + \beta_{1q} \sum_{j=0}^{3} \alpha_3^j y_{t-j-13,0,6-j-1} + \alpha_3^{16} \beta_{2q} y_{t-17,0,1}.
\end{equation}
where \( \beta_{1q} = q_2 + \alpha_3 q_1 \), \( \beta_{2q} = q_2 + \alpha_3 q_3 \), and \( q_2 = -\alpha q_1 \). The weights corresponding to the rotation groups in the remaining streams are computed in a similar way. Once all the weights for the one-period best composite estimator of the current level have been obtained, the variances of the best composite estimators of current level and several period change under both continuous rotation schemes can be computed by the procedure described in Section 5.3.1.

5.6. Estimation of Unemployment Rate

The unemployment rate is defined to be the ratio of the total unemployed to the Civilian Labor Force. Let \( \hat{\theta}_{1t} \) and \( \hat{\theta}_{2t} \) be, respectively, the estimates of unemployed and Civilian Labor Force at time \( t \). Under the assumption that the estimator of unemployed and Civilian Labor Force are independent, the variance of the estimator of unemployment rate is

\[
V\{\hat{\theta}_{1t}^{-1}\hat{\theta}_{1t}\} = V\{\hat{\theta}_{1t}^{-1}(\hat{\theta}_{1t} - R_t\hat{\theta}_{2t})\} = \sigma_{2t}^2(V\{\hat{\theta}_{1t}\} + R_t^2V\{\hat{\theta}_{2t}\}), \tag{5.6.1}
\]

where \( R_t = \sigma_{2t}^{-1}\hat{\theta}_{1t} \), and \( \hat{\theta}_{1t} = E\{\hat{\theta}_{1t}\} \), \( i = 1, 2 \).

Furthermore, if \( \hat{\theta}_{1h} \) and \( \hat{\theta}_{2h} \) are, respectively, the estimates of unemployment and Civilian Labor Force at time \( h \), \( t \neq h \), then the covariance between the unemployment rates at time \( t \) and \( h \) is
Therefore, the variance of the estimator of $s$-period change in unemployment rate may be written as

$$V\{\hat{\theta}_{2t, t-s}^{-1} \hat{\theta}_{1t} - \hat{\theta}_{2, t-s}^{-1} \hat{\theta}_{1, t-s} \} = V\{\hat{\theta}_{2t, 1t}^{-1} \hat{\theta}_{1t}\} + V\{\hat{\theta}_{2, t-s, 1, t-s}^{-1} \hat{\theta}_{1, t-s}\} - 2\text{Cov}\{\hat{\theta}_{2, t-s, 1, t-s}^{-1} \hat{\theta}_{1, t-s}, \hat{\theta}_{2t, t-s}^{-1} \hat{\theta}_{1t}\},$$

(5.6.3)

where $V\{\hat{\theta}_{2t, 1t}^{-1} \hat{\theta}_{1t}\}$ is given by (5.6.1), $V\{\hat{\theta}_{2, t-s, 1, t-s}^{-1} \hat{\theta}_{1, t-s}\}$ is defined similarly, and $\text{Cov}\{\hat{\theta}_{2t, 1t}^{-1} \hat{\theta}_{1t}, \hat{\theta}_{2, t-s, 1, t-s}^{-1} \hat{\theta}_{1, t-s}\}$ is given by (5.6.2) with $h$ replaced by $t-s$. If we assume that for all $0 < s < t$,

$$\theta_{2t} = \theta_{2s} = \theta_2$$

and

$$R_t = R_s = R,$$

then the variance of the estimator of $s$-period change in unemployment rate is given by (5.6.3), where

$$\text{Cov}\{\hat{\theta}_{2t, 1t}^{-1} \hat{\theta}_{1t}, \hat{\theta}_{2, t-s, 1, t-s}^{-1} \hat{\theta}_{1, t-s}\} = \theta_2^{-2} \text{Cov}\{\hat{\theta}_{1, t}, \hat{\theta}_{1, t-s}\} + R^2 \text{Cov}\{\hat{\theta}_{2t}, \hat{\theta}_{2, t-s}\}.$$

(5.6.4)
The estimated variances of the estimators of current level of, and several-period change in unemployment rate using approximation (5.6.4) are presented in Section 5.7 for the 4—8—4 rotation scheme.

5.7. Results and Discussion

In this section, we present empirical results comparing alternative estimators of current level, change in level and yearly average level for components of the labor force based on the Current Population Survey. The coefficients defining the elementary estimators from the rotation groups and the variances of the estimators are compared.

The comparison is based on three rotation designs, one of which is intermittent and the other two are continuous. The intermittent rotation design is the 4—8—4 scheme, which is used in the Current Population Survey. The continuous designs are the 8-in—then—out scheme and the 6—in—then—out scheme, which is used in the Canadian Labor Force Survey (Kumar and Lee, 1983).

First, we compare alternative estimators under the 4—8—4 rotation scheme. Estimators include the present composite estimator, the recursive regression estimator and best linear unbiased estimators based on various periods of data. Characteristics of interest are employed, unemployed, Civilian Labor Force and unemployment rate. The construction of the regression estimator for all rotation schemes is based on the model described in Section 5.2. Both revision and nonrevision of previous estimators are considered.

As previously mentioned, the present composite estimation procedure considered in this chapter does not involve the revision of previous estimates. In
general, the present composite estimator for the $p$–th period only uses data from periods 1 through $p$ in estimation, even if $h$ ($h > p$) periods of data are available. The weights used in the construction of the present composite estimator are exhibited in Table 5.11 for Civilian Labor Force in the special case of data from a 24–period survey under the 4–8–4 rotation design. These weights are obtained by the procedure described in Section 5.3.1.

We consider the cases of constructing the regression estimator for the 4–8–4 rotation scheme using the model described in Section 5.2. The estimators considered for comparison are the present composite estimator, the best linear unbiased estimators for $p = 2, 3, 12, 16, 24$, and the recursive regression estimator. For each characteristic, the parameters of interest are the current level and several periods change up to twelve periods.

The best linear unbiased procedures correspond to special cases of model (5.3.13) with $p = 2, 3, 12, 16, 24$. Therefore, the best linear unbiased estimator of

$$\theta_p = (\theta_1, \ldots, \theta_{p-1}, \theta_p)$$

is given by (5.3.16) and the covariance matrix is given by (5.3.17) with $p = 2, 3, 12, 16, 24$, respectively. The variances of the estimators of current level and one–period change under revision of previous estimates are computed, respectively, in accordance with the formulas following (5.3.19). The variances of the estimators of change in level over an interval of several periods by the procedure described in Section 4.3, both for the case of revision and on revision of previous estimates.

In the best linear unbiased procedure for estimating current level, the sum of the coefficients of the rotation group totals for the current month equals one, and the
Table 5.11. Coefficients for the present composite estimator of current level of Civilian Labor Force based on 24 periods

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-5 \times 10^{-11}$</td>
<td>$-2 \times 10^{-11}$</td>
<td>$-2 \times 10^{-11}$</td>
<td>$-9 \times 10^{-11}$</td>
</tr>
<tr>
<td>2</td>
<td>$-6 \times 10^{-11}$</td>
<td>$-6 \times 10^{-11}$</td>
<td>$2 \times 10^{-10}$</td>
<td>$-1 \times 10^{-10}$</td>
</tr>
<tr>
<td>3</td>
<td>$-2 \times 10^{-10}$</td>
<td>$6 \times 10^{-10}$</td>
<td>$-3 \times 10^{-10}$</td>
<td>$-2 \times 10^{-10}$</td>
</tr>
<tr>
<td>4</td>
<td>$1.5 \times 10^{-10}$</td>
<td>$-7 \times 10^{-10}$</td>
<td>$-4 \times 10^{-10}$</td>
<td>$-1.5 \times 10^{-10}$</td>
</tr>
<tr>
<td>5</td>
<td>$-1.8 \times 10^{-9}$</td>
<td>$-9 \times 10^{-10}$</td>
<td>$9 \times 10^{-10}$</td>
<td>$3.8 \times 10^{-9}$</td>
</tr>
<tr>
<td>6</td>
<td>$-2.3 \times 10^{-9}$</td>
<td>$-2.3 \times 10^{-9}$</td>
<td>$9.2 \times 10^{-9}$</td>
<td>$-4.6 \times 10^{-9}$</td>
</tr>
<tr>
<td>7</td>
<td>$-5.7 \times 10^{-9}$</td>
<td>$2.3 \times 10^{-8}$</td>
<td>$-1.2 \times 10^{-8}$</td>
<td>$-5.7 \times 10^{-9}$</td>
</tr>
<tr>
<td>8</td>
<td>$5.7 \times 10^{-8}$</td>
<td>$-2.9 \times 10^{-8}$</td>
<td>$-1.4 \times 10^{-8}$</td>
<td>$-1.4 \times 10^{-8}$</td>
</tr>
<tr>
<td>9</td>
<td>$-7.2 \times 10^{-8}$</td>
<td>$-3.6 \times 10^{-8}$</td>
<td>$-3.6 \times 10^{-8}$</td>
<td>$1.4 \times 10^{-7}$</td>
</tr>
<tr>
<td>10</td>
<td>$-9.0 \times 10^{-8}$</td>
<td>$-9.0 \times 10^{-8}$</td>
<td>$3.8 \times 10^{-7}$</td>
<td>$-1.8 \times 10^{-7}$</td>
</tr>
<tr>
<td>11</td>
<td>$2.2 \times 10^{-7}$</td>
<td>$9.0 \times 10^{-7}$</td>
<td>$-4.5 \times 10^{-7}$</td>
<td>$-2.2 \times 10^{-7}$</td>
</tr>
<tr>
<td>12</td>
<td>$2.2 \times 10^{-6}$</td>
<td>$-1.1 \times 10^{-6}$</td>
<td>$-5.6 \times 10^{-7}$</td>
<td>$-5.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>13</td>
<td>$-2.8 \times 10^{-6}$</td>
<td>$-1.4 \times 10^{-6}$</td>
<td>$-1.4 \times 10^{-6}$</td>
<td>$5.6 \times 10^{-6}$</td>
</tr>
<tr>
<td>14</td>
<td>$-3.5 \times 10^{-6}$</td>
<td>$-3.5 \times 10^{-6}$</td>
<td>$0.000014$</td>
<td>$-7.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>15</td>
<td>$-8.7 \times 10^{-6}$</td>
<td>$0.000035$</td>
<td>$-0.000017$</td>
<td>$-8.7 \times 10^{-6}$</td>
</tr>
<tr>
<td>16</td>
<td>$0.000087$</td>
<td>$-0.000044$</td>
<td>$-0.000022$</td>
<td>$-0.000052$</td>
</tr>
<tr>
<td>17</td>
<td>$-0.000109$</td>
<td>$-0.000055$</td>
<td>$-0.000055$</td>
<td>$0.000219$</td>
</tr>
<tr>
<td>18</td>
<td>$-0.000137$</td>
<td>$-0.000137$</td>
<td>$0.000546$</td>
<td>$-0.000273$</td>
</tr>
<tr>
<td>19</td>
<td>$-0.000341$</td>
<td>$0.001365$</td>
<td>$-0.000683$</td>
<td>$-0.000341$</td>
</tr>
<tr>
<td>20</td>
<td>$0.003413$</td>
<td>$-0.001707$</td>
<td>$-0.000853$</td>
<td>$-0.000853$</td>
</tr>
<tr>
<td>21</td>
<td>$-0.004267$</td>
<td>$-0.002133$</td>
<td>$-0.002133$</td>
<td>$0.008533$</td>
</tr>
<tr>
<td>22</td>
<td>$-0.005333$</td>
<td>$-0.005333$</td>
<td>$0.021333$</td>
<td>$-0.010667$</td>
</tr>
<tr>
<td>23</td>
<td>$-0.013333$</td>
<td>$0.053333$</td>
<td>$-0.026667$</td>
<td>$-0.013333$</td>
</tr>
<tr>
<td>24</td>
<td>$0.133333$</td>
<td>$0.100000$</td>
<td>$0.133333$</td>
<td>$0.133333$</td>
</tr>
<tr>
<td>Period</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>--------</td>
<td>----------</td>
<td>----------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>1</td>
<td>$-4 \times 10^{-11}$</td>
<td>$-2 \times 10^{-11}$</td>
<td>$-2 \times 10^{-11}$</td>
<td>$-9 \times 10^{-11}$</td>
</tr>
<tr>
<td>2</td>
<td>$-6 \times 10^{-11}$</td>
<td>$-6 \times 10^{-11}$</td>
<td>$2 \times 10^{-11}$</td>
<td>$-1 \times 10^{-11}$</td>
</tr>
<tr>
<td>3</td>
<td>$-2 \times 10^{-10}$</td>
<td>$6 \times 10^{-10}$</td>
<td>$3 \times 10^{-10}$</td>
<td>$-2 \times 10^{-10}$</td>
</tr>
<tr>
<td>4</td>
<td>$1.5 \times 10^{-9}$</td>
<td>$-7 \times 10^{-10}$</td>
<td>$-4 \times 10^{-10}$</td>
<td>$-4 \times 10^{-10}$</td>
</tr>
<tr>
<td>5</td>
<td>$-1.8 \times 10^{-9}$</td>
<td>$-9 \times 10^{-10}$</td>
<td>$-9 \times 10^{-10}$</td>
<td>$3.8 \times 10^{-9}$</td>
</tr>
<tr>
<td>6</td>
<td>$-2.3 \times 10^{-9}$</td>
<td>$-2.3 \times 10^{-9}$</td>
<td>$9.2 \times 10^{-9}$</td>
<td>$-4.6 \times 10^{-9}$</td>
</tr>
<tr>
<td>7</td>
<td>$-5.7 \times 10^{-9}$</td>
<td>$2.3 \times 10^{-8}$</td>
<td>$-1.2 \times 10^{-8}$</td>
<td>$-5.7 \times 10^{-9}$</td>
</tr>
<tr>
<td>8</td>
<td>$5.7 \times 10^{-8}$</td>
<td>$-2.9 \times 10^{-8}$</td>
<td>$-1.4 \times 10^{-8}$</td>
<td>$-1.4 \times 10^{-8}$</td>
</tr>
<tr>
<td>9</td>
<td>$-7.2 \times 10^{-8}$</td>
<td>$-3.6 \times 10^{-8}$</td>
<td>$-3.6 \times 10^{-8}$</td>
<td>$1.4 \times 10^{-7}$</td>
</tr>
<tr>
<td>10</td>
<td>$-9.0 \times 10^{-8}$</td>
<td>$-9.0 \times 10^{-8}$</td>
<td>$3.6 \times 10^{-7}$</td>
<td>$-1.8 \times 10^{-7}$</td>
</tr>
<tr>
<td>11</td>
<td>$-2.2 \times 10^{-7}$</td>
<td>$8.9 \times 10^{-7}$</td>
<td>$-4.5 \times 10^{-7}$</td>
<td>$-2.2 \times 10^{-7}$</td>
</tr>
<tr>
<td>12</td>
<td>$2.2 \times 10^{-6}$</td>
<td>$-1.1 \times 10^{-6}$</td>
<td>$-5.6 \times 10^{-7}$</td>
<td>$-5.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>13</td>
<td>$-2.8 \times 10^{-6}$</td>
<td>$-1.4 \times 10^{-6}$</td>
<td>$-1.4 \times 10^{-6}$</td>
<td>$5.6 \times 10^{-6}$</td>
</tr>
<tr>
<td>14</td>
<td>$-3.5 \times 10^{-6}$</td>
<td>$-3.5 \times 10^{-6}$</td>
<td>$0.000014$</td>
<td>$-7.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>15</td>
<td>$-8.7 \times 10^{-6}$</td>
<td>$0.000035$</td>
<td>$-0.000017$</td>
<td>$-8.7 \times 10^{-6}$</td>
</tr>
<tr>
<td>16</td>
<td>$0.000087$</td>
<td>$-0.000044$</td>
<td>$-0.00022$</td>
<td>$-0.000022$</td>
</tr>
<tr>
<td>17</td>
<td>$0.000109$</td>
<td>$-0.000055$</td>
<td>$-0.000055$</td>
<td>$0.000219$</td>
</tr>
<tr>
<td>18</td>
<td>$-0.000137$</td>
<td>$-0.000137$</td>
<td>$0.000546$</td>
<td>$-0.000273$</td>
</tr>
<tr>
<td>19</td>
<td>$-0.000341$</td>
<td>$0.001365$</td>
<td>$-0.000683$</td>
<td>$-0.000341$</td>
</tr>
<tr>
<td>20</td>
<td>$0.003413$</td>
<td>$-0.001707$</td>
<td>$-0.000853$</td>
<td>$-0.000853$</td>
</tr>
<tr>
<td>21</td>
<td>$-0.004267$</td>
<td>$-0.002133$</td>
<td>$-0.002133$</td>
<td>$0.008533$</td>
</tr>
<tr>
<td>22</td>
<td>$-0.005333$</td>
<td>$-0.005333$</td>
<td>$0.021333$</td>
<td>$-0.010667$</td>
</tr>
<tr>
<td>23</td>
<td>$-0.013333$</td>
<td>$0.053333$</td>
<td>$-0.026667$</td>
<td>$-0.013333$</td>
</tr>
<tr>
<td>24</td>
<td>$0.133333$</td>
<td>$0.100000$</td>
<td>$0.133333$</td>
<td>$-0.133333$</td>
</tr>
</tbody>
</table>
Table 5.12. Best linear unbiased coefficients of the estimator of current level of Civilian Labor Force based on 24 periods

<table>
<thead>
<tr>
<th>Period</th>
<th>Rotation group by time in sample</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>24</td>
<td>0.142843</td>
<td>0.081964</td>
<td>0.116937</td>
<td>0.127440</td>
</tr>
<tr>
<td>23</td>
<td>-0.008086</td>
<td>0.060502</td>
<td>-0.044672</td>
<td>-0.026119</td>
</tr>
<tr>
<td>22</td>
<td>-0.015000</td>
<td>0.010133</td>
<td>0.044221</td>
<td>-0.040587</td>
</tr>
<tr>
<td>21</td>
<td>-0.023221</td>
<td>-0.005554</td>
<td>0.006797</td>
<td>0.030225</td>
</tr>
<tr>
<td>20</td>
<td>0.019489</td>
<td>-0.015081</td>
<td>-0.004812</td>
<td>0.003638</td>
</tr>
<tr>
<td>19</td>
<td>0.002139</td>
<td>0.012554</td>
<td>-0.010089</td>
<td>-0.003307</td>
</tr>
<tr>
<td>18</td>
<td>-0.002130</td>
<td>0.001095</td>
<td>0.006995</td>
<td>-0.005754</td>
</tr>
<tr>
<td>17</td>
<td>-0.001833</td>
<td>-0.000940</td>
<td>-0.000170</td>
<td>0.002393</td>
</tr>
<tr>
<td>16</td>
<td>-0.000135</td>
<td>0.002551</td>
<td>-0.000651</td>
<td>-0.001392</td>
</tr>
<tr>
<td>15</td>
<td>0.002315</td>
<td>0.001200</td>
<td>0.008514</td>
<td>0.003352</td>
</tr>
<tr>
<td>14</td>
<td>0.005439</td>
<td>0.003183</td>
<td>0.002893</td>
<td>0.010886</td>
</tr>
<tr>
<td>13</td>
<td>0.013167</td>
<td>0.006151</td>
<td>0.004716</td>
<td>0.005399</td>
</tr>
<tr>
<td>12</td>
<td>-0.012612</td>
<td>0.013802</td>
<td>0.009006</td>
<td>0.008376</td>
</tr>
<tr>
<td>11</td>
<td>-0.006384</td>
<td>-0.000972</td>
<td>0.009202</td>
<td>0.006423</td>
</tr>
<tr>
<td>10</td>
<td>-0.008976</td>
<td>-0.001387</td>
<td>0.003976</td>
<td>0.005055</td>
</tr>
<tr>
<td>9</td>
<td>-0.021773</td>
<td>-0.004616</td>
<td>0.00355</td>
<td>0.006256</td>
</tr>
<tr>
<td>8</td>
<td>0.009968</td>
<td>-0.014248</td>
<td>-0.003947</td>
<td>0.000138</td>
</tr>
<tr>
<td>7</td>
<td>0.002575</td>
<td>0.007990</td>
<td>-0.010341</td>
<td>-0.003007</td>
</tr>
<tr>
<td>6</td>
<td>0.000183</td>
<td>0.001975</td>
<td>0.006188</td>
<td>-0.007187</td>
</tr>
<tr>
<td>5</td>
<td>-0.000512</td>
<td>0.000356</td>
<td>0.001588</td>
<td>0.004729</td>
</tr>
<tr>
<td>4</td>
<td>-0.001185</td>
<td>0.000556</td>
<td>0.000661</td>
<td>0.001416</td>
</tr>
<tr>
<td>3</td>
<td>-0.001281</td>
<td>-0.002904</td>
<td>0.001520</td>
<td>0.001025</td>
</tr>
<tr>
<td>2</td>
<td>-0.001337</td>
<td>-0.002177</td>
<td>-0.004375</td>
<td>0.002226</td>
</tr>
<tr>
<td>1</td>
<td>-0.002057</td>
<td>-0.003075</td>
<td>-0.004291</td>
<td>-0.007407</td>
</tr>
</tbody>
</table>
Table 5.12. Continued

<table>
<thead>
<tr>
<th>Period</th>
<th>Rotation group by time in sample</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td>24</td>
<td>0.131537</td>
</tr>
<tr>
<td>23</td>
<td>-0.014495</td>
</tr>
<tr>
<td>22</td>
<td>-0.021459</td>
</tr>
<tr>
<td>21</td>
<td>-0.034790</td>
</tr>
<tr>
<td>20</td>
<td>0.019714</td>
</tr>
<tr>
<td>19</td>
<td>0.002264</td>
</tr>
<tr>
<td>18</td>
<td>-0.002013</td>
</tr>
<tr>
<td>17</td>
<td>-0.001654</td>
</tr>
<tr>
<td>16</td>
<td>-0.001467</td>
</tr>
<tr>
<td>15</td>
<td>0.001560</td>
</tr>
<tr>
<td>14</td>
<td>0.004678</td>
</tr>
<tr>
<td>13</td>
<td>0.011804</td>
</tr>
<tr>
<td>12</td>
<td>0.009434</td>
</tr>
<tr>
<td>11</td>
<td>0.006098</td>
</tr>
<tr>
<td>10</td>
<td>0.003538</td>
</tr>
<tr>
<td>9</td>
<td>0.000372</td>
</tr>
<tr>
<td>8</td>
<td>0.005762</td>
</tr>
<tr>
<td>7</td>
<td>0.000203</td>
</tr>
<tr>
<td>6</td>
<td>-0.002186</td>
</tr>
<tr>
<td>5</td>
<td>-0.004700</td>
</tr>
<tr>
<td>4</td>
<td>0.003705</td>
</tr>
<tr>
<td>3</td>
<td>0.001423</td>
</tr>
<tr>
<td>2</td>
<td>0.001331</td>
</tr>
<tr>
<td>1</td>
<td>0.002641</td>
</tr>
</tbody>
</table>
sum of the coefficients of the rotation group totals for each of the previous months equals zero. In general, when there are \( h \) months of data available for estimation, then the estimate for month \( s \) (\( s \leq h \)) is a linear combination of all rotation group totals, where the sum of the coefficients of the rotation group totals for month \( s \) is equal to one and the sum of the coefficients of the rotation group totals for every other month is equal to zero. This point is illustrated in Table 8.b for the special case of \( s = h = 24 \), for Civilian Labor Force. The construction of minimum variance weights under both revision and nonrevision of previous estimates is also illustrated for this special case in Tables 5.13 and 5.14.

For the recursive regression procedure, we first implement the best linear unbiased procedure for 15 periods in order to obtain 15 initial estimates. Our model at time \( p + 1 \) is

\[
Z_{p+1} = X_2 \theta_{p+1}(16) + \epsilon_{p+1},
\]

where \( X_2 \) is defined in (5.3.23),

\[
Z_{p+1} = (\hat{\theta}_{p-m+1}(p), \ldots, \hat{\theta}_p(p), z_{p+1,1}, \ldots, z_{p+1,8}) = (\hat{\theta}_{p}(15), z_{p+1}^\prime)
\]

= 23 \times 1 \text{ vector of 15 initial estimates and the 8 elementary estimates obtained from the rotation groups introduced during the 16th period,}

and \( \theta_{p+1}(16) \) and \( V_{p+1} \), the covariance matrix of \( Z_{p+1} \), are defined in (5.3.25), respectively, with \( m = 15 \). Therefore, the recursive regression estimator of \( \theta_{p+1}(16) \) and its covariance matrix are given, respectively, by (5.3.27) and (5.3.29) with \( m = 15 \).
Table 5.13. Best linear unbiased coefficients of the estimator of the previous level of Civilian Labor Force based on 24 periods under revision of previous estimates

<table>
<thead>
<tr>
<th>Period</th>
<th>Rotation group by time in sample</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td>-0.009024</td>
</tr>
<tr>
<td>23</td>
<td>0.135794</td>
</tr>
<tr>
<td>22</td>
<td>-0.011508</td>
</tr>
<tr>
<td>21</td>
<td>-0.024558</td>
</tr>
<tr>
<td>20</td>
<td>0.026603</td>
</tr>
<tr>
<td>19</td>
<td>0.002762</td>
</tr>
<tr>
<td>18</td>
<td>-0.003321</td>
</tr>
<tr>
<td>17</td>
<td>-0.005030</td>
</tr>
<tr>
<td>16</td>
<td>0.002088</td>
</tr>
<tr>
<td>15</td>
<td>0.000874</td>
</tr>
<tr>
<td>14</td>
<td>0.004314</td>
</tr>
<tr>
<td>13</td>
<td>0.011842</td>
</tr>
<tr>
<td>12</td>
<td>-0.014843</td>
</tr>
<tr>
<td>11</td>
<td>-0.005216</td>
</tr>
<tr>
<td>10</td>
<td>-0.007111</td>
</tr>
<tr>
<td>9</td>
<td>-0.018164</td>
</tr>
<tr>
<td>8</td>
<td>0.011320</td>
</tr>
<tr>
<td>7</td>
<td>0.003107</td>
</tr>
<tr>
<td>6</td>
<td>0.000235</td>
</tr>
<tr>
<td>5</td>
<td>-0.001147</td>
</tr>
<tr>
<td>4</td>
<td>0.000013</td>
</tr>
<tr>
<td>3</td>
<td>-0.001020</td>
</tr>
<tr>
<td>2</td>
<td>-0.001336</td>
</tr>
<tr>
<td>1</td>
<td>-0.002060</td>
</tr>
</tbody>
</table>
Table 5.13. Continued

<table>
<thead>
<tr>
<th>Period</th>
<th>Rotation group by time in sample</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td>24</td>
<td>-0.019321</td>
</tr>
<tr>
<td>23</td>
<td>0.129957</td>
</tr>
<tr>
<td>22</td>
<td>-0.017395</td>
</tr>
<tr>
<td>21</td>
<td>-0.035119</td>
</tr>
<tr>
<td>20</td>
<td>0.027342</td>
</tr>
<tr>
<td>19</td>
<td>0.003178</td>
</tr>
<tr>
<td>18</td>
<td>-0.002913</td>
</tr>
<tr>
<td>17</td>
<td>-0.004333</td>
</tr>
<tr>
<td>16</td>
<td>0.001085</td>
</tr>
<tr>
<td>15</td>
<td>0.000306</td>
</tr>
<tr>
<td>14</td>
<td>0.003743</td>
</tr>
<tr>
<td>13</td>
<td>0.010823</td>
</tr>
<tr>
<td>12</td>
<td>0.006021</td>
</tr>
<tr>
<td>11</td>
<td>0.006579</td>
</tr>
<tr>
<td>10</td>
<td>0.004604</td>
</tr>
<tr>
<td>9</td>
<td>0.002528</td>
</tr>
<tr>
<td>8</td>
<td>0.005889</td>
</tr>
<tr>
<td>7</td>
<td>0.000094</td>
</tr>
<tr>
<td>6</td>
<td>-0.002778</td>
</tr>
<tr>
<td>5</td>
<td>-0.006477</td>
</tr>
<tr>
<td>4</td>
<td>0.004505</td>
</tr>
<tr>
<td>3</td>
<td>0.001466</td>
</tr>
<tr>
<td>2</td>
<td>0.001119</td>
</tr>
<tr>
<td>1</td>
<td>0.002265</td>
</tr>
</tbody>
</table>
Table 5.14. Best linear unbiased coefficients of the estimator of the previous level of Civilian Labor Force based on 24 periods under no revision of previous estimates

<table>
<thead>
<tr>
<th>Period</th>
<th>Rotation group by time in sample</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>23</td>
<td></td>
<td>0.141045</td>
<td>0.142843</td>
<td>0.081964</td>
<td>0.116937</td>
</tr>
<tr>
<td>22</td>
<td></td>
<td>-0.011830</td>
<td>-0.008086</td>
<td>0.065020</td>
<td>-0.044672</td>
</tr>
<tr>
<td>21</td>
<td></td>
<td>-0.026638</td>
<td>-0.015000</td>
<td>0.010133</td>
<td>0.044221</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.029342</td>
<td>-0.023221</td>
<td>-0.005554</td>
<td>0.006797</td>
</tr>
<tr>
<td>19</td>
<td></td>
<td>0.003140</td>
<td>0.019489</td>
<td>-0.015081</td>
<td>-0.004812</td>
</tr>
<tr>
<td>18</td>
<td></td>
<td>-0.003797</td>
<td>0.002139</td>
<td>0.012554</td>
<td>-0.010089</td>
</tr>
<tr>
<td>17</td>
<td></td>
<td>-0.006594</td>
<td>-0.002130</td>
<td>0.001095</td>
<td>0.006995</td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>0.003134</td>
<td>-0.001833</td>
<td>-0.000940</td>
<td>-0.000170</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>-0.000972</td>
<td>-0.000135</td>
<td>0.002551</td>
<td>-0.000651</td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>0.003773</td>
<td>0.002315</td>
<td>0.001200</td>
<td>0.008514</td>
</tr>
<tr>
<td>13</td>
<td></td>
<td>0.011635</td>
<td>0.005439</td>
<td>0.003183</td>
<td>0.002893</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>-0.019865</td>
<td>0.013167</td>
<td>0.006151</td>
<td>0.004716</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>-0.005906</td>
<td>-0.012612</td>
<td>0.013802</td>
<td>0.009006</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>-0.007793</td>
<td>-0.006384</td>
<td>-0.000972</td>
<td>0.009202</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>-0.020102</td>
<td>-0.008976</td>
<td>-0.001387</td>
<td>0.003976</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>0.011930</td>
<td>-0.021773</td>
<td>-0.004516</td>
<td>0.000355</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>0.003348</td>
<td>0.009968</td>
<td>-0.014248</td>
<td>-0.003947</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0.000206</td>
<td>0.002575</td>
<td>0.007990</td>
<td>-0.010341</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>-0.001504</td>
<td>0.000183</td>
<td>0.001975</td>
<td>0.006188</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.000598</td>
<td>-0.000512</td>
<td>0.000356</td>
<td>0.001588</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>-0.000868</td>
<td>-0.001185</td>
<td>0.000556</td>
<td>0.000661</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>-0.001231</td>
<td>-0.001281</td>
<td>-0.002904</td>
<td>0.001520</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>-0.001748</td>
<td>-0.001337</td>
<td>-0.002177</td>
<td>-0.004375</td>
</tr>
</tbody>
</table>
Table 5.14. Continued

<table>
<thead>
<tr>
<th>Period</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>23</td>
<td>0.127440</td>
<td>0.131537</td>
<td>0.121972</td>
<td>0.136262</td>
</tr>
<tr>
<td>22</td>
<td>-0.021119</td>
<td>-0.014495</td>
<td>0.060182</td>
<td>-0.024999</td>
</tr>
<tr>
<td>21</td>
<td>-0.040587</td>
<td>-0.021460</td>
<td>0.007392</td>
<td>0.041938</td>
</tr>
<tr>
<td>20</td>
<td>0.030225</td>
<td>-0.034790</td>
<td>-0.008307</td>
<td>0.005506</td>
</tr>
<tr>
<td>19</td>
<td>0.003638</td>
<td>0.019714</td>
<td>-0.019987</td>
<td>-0.006101</td>
</tr>
<tr>
<td>18</td>
<td>-0.003307</td>
<td>0.002264</td>
<td>0.012599</td>
<td>-0.012363</td>
</tr>
<tr>
<td>17</td>
<td>-0.005754</td>
<td>-0.002013</td>
<td>0.001119</td>
<td>0.007282</td>
</tr>
<tr>
<td>16</td>
<td>0.002393</td>
<td>-0.001654</td>
<td>-0.000922</td>
<td>-0.000008</td>
</tr>
<tr>
<td>15</td>
<td>-0.001392</td>
<td>-0.001467</td>
<td>0.002560</td>
<td>-0.000492</td>
</tr>
<tr>
<td>14</td>
<td>0.003352</td>
<td>0.001560</td>
<td>-0.029499</td>
<td>0.008786</td>
</tr>
<tr>
<td>13</td>
<td>0.010886</td>
<td>0.004678</td>
<td>-0.013741</td>
<td>-0.024974</td>
</tr>
<tr>
<td>12</td>
<td>0.005399</td>
<td>0.011804</td>
<td>-0.010525</td>
<td>-0.010847</td>
</tr>
<tr>
<td>11</td>
<td>0.008376</td>
<td>0.009434</td>
<td>-0.015619</td>
<td>-0.006481</td>
</tr>
<tr>
<td>10</td>
<td>0.006423</td>
<td>0.006098</td>
<td>0.011491</td>
<td>-0.018065</td>
</tr>
<tr>
<td>9</td>
<td>0.005055</td>
<td>0.003538</td>
<td>0.005668</td>
<td>0.012229</td>
</tr>
<tr>
<td>8</td>
<td>0.006256</td>
<td>0.000372</td>
<td>0.002448</td>
<td>0.005029</td>
</tr>
<tr>
<td>7</td>
<td>0.000138</td>
<td>0.005762</td>
<td>-0.001754</td>
<td>0.000733</td>
</tr>
<tr>
<td>6</td>
<td>-0.003007</td>
<td>0.000203</td>
<td>0.004437</td>
<td>-0.002062</td>
</tr>
<tr>
<td>5</td>
<td>-0.007187</td>
<td>-0.002186</td>
<td>-0.000016</td>
<td>0.0002549</td>
</tr>
<tr>
<td>4</td>
<td>0.004729</td>
<td>-0.004700</td>
<td>-0.001625</td>
<td>-0.000434</td>
</tr>
<tr>
<td>3</td>
<td>0.001416</td>
<td>0.003705</td>
<td>-0.002944</td>
<td>-0.001340</td>
</tr>
<tr>
<td>2</td>
<td>0.001025</td>
<td>0.001423</td>
<td>0.003459</td>
<td>-0.002012</td>
</tr>
<tr>
<td>1</td>
<td>0.002226</td>
<td>0.001331</td>
<td>0.002036</td>
<td>0.004044</td>
</tr>
</tbody>
</table>
Table 5.15. Optimal coefficients of the rotation group totals for the current level used in forming the recursive regression estimator of the current level (see equation 5.7.5)

<table>
<thead>
<tr>
<th>Month of survey</th>
<th>Rotation group by month in sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 (Current month)</td>
<td>0.081295 0.1166201 0.1272257 0.1313674 0.122211 0.1365682 0.1414242 0.1432884</td>
</tr>
</tbody>
</table>

Table 5.16. Optimal coefficients of the estimates of the preceding levels used in forming the recursive regression estimator of the current level (see equation 5.7.5)

<table>
<thead>
<tr>
<th>Month of survey</th>
<th>Coefficient of estimate</th>
<th>Month of survey</th>
<th>Coefficient of estimate</th>
<th>Month of survey</th>
<th>Coefficient of estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0092565</td>
<td>6</td>
<td>0.0274388</td>
<td>11</td>
<td>4.808 × 10⁻¹⁶</td>
</tr>
<tr>
<td>2</td>
<td>0.0149575</td>
<td>7</td>
<td>0.0289080</td>
<td>12</td>
<td>3.859 × 10⁻¹⁶</td>
</tr>
<tr>
<td>3</td>
<td>0.0221344</td>
<td>8</td>
<td>6.229 × 10⁻¹⁷</td>
<td>13</td>
<td>0.0249200</td>
</tr>
<tr>
<td>4</td>
<td>0.0486536</td>
<td>9</td>
<td>3.409 × 10⁻¹⁶</td>
<td>14</td>
<td>0.1070189</td>
</tr>
<tr>
<td>5</td>
<td>0.0316992</td>
<td>10</td>
<td>3.524 × 10⁻¹⁶</td>
<td>15</td>
<td>0.4673483</td>
</tr>
</tbody>
</table>

Table 5.17. Entries of the vector \( \psi \) defined in (5.3.26)

<table>
<thead>
<tr>
<th>( \psi_1 )</th>
<th>( \psi_2 )</th>
<th>( \psi_3 )</th>
<th>( \psi_4 )</th>
<th>( \psi_5 )</th>
<th>( \psi_6 )</th>
<th>( \psi_7 )</th>
<th>( \psi_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000000</td>
<td>0.2171110</td>
<td>0.1609311</td>
<td>0.1419749</td>
<td>0.2797028</td>
<td>0.1048410</td>
<td>0.0851749</td>
<td>0.0794024</td>
</tr>
</tbody>
</table>
for computing the limiting variances of both revised and unrevised estimates of change over several periods is described in detail in the appendix.

The variances of the alternative estimation procedures under no revision of previous estimates, relative to the variance of the basic estimator of current level, for each of the characteristics of interest, are presented in Tables 5.18 — 5.20. We define the basic estimator of the current level as the simple mean of the elementary estimates obtained from the 8 rotation groups at the current period. That is, if \( \bar{y}_t \) denotes the basic estimator of \( \theta_t \), the parameter of interest at time \( t \), then \( \bar{y}_t = (1/8)\sum_{k=1}^{8}y_{t,0,k} \) and \( \text{Var}(\bar{y}_t) = \sigma^2/8 \), where \( \sigma^2 = \text{Var}(y_{t,0,k}) \) for all \( t \) and \( k \). The basic estimator of change between two periods is the difference between the simple means of the 8 elementary estimators obtained at the two periods. The variances of the alternative estimators of current level and change over several periods under no revision of previous estimates are given in Table 5.18 for employed, Table 5.19 for unemployed, and Table 5.20 for Civilian Labor Force. The variances under revision of previous estimates are given in Table 5.27 for employed, Table 5.28 for unemployed, and Table 5.29 for Civilian Labor Force. Tables 5.21, 5.22, and 5.23 show, respectively, the variances of the alternative estimators relative to the present composite estimator for employed, unemployed, and Civilian Labor Force, while Tables 5.33, 5.34, and 5.35 show, respectively, the relative efficiencies of the alternative estimators under revision of previous estimates relative to that under no revision of previous estimates, for each of the three characteristics. In Tables 5.24, 5.25, and 5.26, we present relative efficiencies of the alternative estimators under no revision of previous estimates relative to the best linear unbiased estimator for 12 periods for the three characteristics. The corresponding values for the case of revision of previous estimates are presented in Tables 5.30, 5.31, and 5.32.
The recursive regression estimator of current level at time \( p + 1 \) may be written as

\[
\hat{\theta}_{p+1,R} + \sum_{j=1}^{m} c_j \hat{\theta}_{p-j+1(p)} + \sum_{k=1}^{s} c_{k+m} z_{p+1,k} \]  
\tag{5.7.5}
\]

where \( s \) is the number of streams. The coefficients \( c_j, j = 1, 2, ..., p + s \), used in the expression (5.7.5) are given in Tables 5.15 and 5.16 for the special case of civilian labor force, \( m = 15 \) and \( s = 8 \).

The estimator of current level obtained from the recursive regression procedure is a more complicated function of the elementary estimators from the rotation groups than that obtained from the best linear unbiased estimation. It is a linear function of the elementary estimators from the rotation groups at the current period and the initial estimates, which are in turn linear combinations of elementary estimates from rotation groups for previous periods. The sum of the coefficients of the elementary estimators from the rotation groups in the current period is equal to one, but unlike the case of best linear unbiased estimation, the sum of the coefficients of the estimates of the previous periods is not equal to zero. This sum is equal to 1 minus the sum of the products of the coefficients of the rotation group totals for the current month and the elements of the vector \( \psi \) defined in (5.3.24). This point is illustrated by Tables 5.15, 5.16 and 5.17.

The limiting variances of the recursive regression estimators of current level and one-period change under both revision and nonrevision of previous estimates, can be computed in accordance with the formulas following (5.3.43). A recursive procedure
Table 5.18. Variances of alternative estimators of employed relative to the variance of the basic estimator of current level; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Basic estimator</th>
<th>Present composite</th>
<th>BLUE 2 periods</th>
<th>BLUE 3 periods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>1.000</td>
<td>0.862</td>
<td>0.893</td>
<td>0.836</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.677</td>
<td>0.511</td>
<td>0.570</td>
<td>0.520</td>
</tr>
<tr>
<td>2—period change</td>
<td>1.083</td>
<td>0.813</td>
<td>0.873</td>
<td>0.776</td>
</tr>
<tr>
<td>3—period change</td>
<td>1.413</td>
<td>1.065</td>
<td>1.117</td>
<td>0.969</td>
</tr>
<tr>
<td>4—period change</td>
<td>1.701</td>
<td>1.279</td>
<td>1.329</td>
<td>1.137</td>
</tr>
<tr>
<td>5—period change</td>
<td>1.701</td>
<td>1.363</td>
<td>1.487</td>
<td>1.269</td>
</tr>
<tr>
<td>6—period change</td>
<td>1.701</td>
<td>1.390</td>
<td>1.487</td>
<td>1.374</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.701</td>
<td>1.388</td>
<td>1.487</td>
<td>1.323</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.701</td>
<td>1.353</td>
<td>1.410</td>
<td>1.263</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.560</td>
<td>1.255</td>
<td>1.311</td>
<td>1.188</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.419</td>
<td>1.154</td>
<td>1.213</td>
<td>1.114</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.278</td>
<td>1.061</td>
<td>1.115</td>
<td>1.048</td>
</tr>
<tr>
<td>12—period change</td>
<td>1.137</td>
<td>0.992</td>
<td>1.030</td>
<td>0.985</td>
</tr>
</tbody>
</table>
Table 5.18. Continued

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.704</td>
<td>0.672</td>
<td>0.661</td>
<td>0.650</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.457</td>
<td>0.437</td>
<td>0.432</td>
<td>0.432</td>
</tr>
<tr>
<td>2—period change</td>
<td>0.646</td>
<td>0.613</td>
<td>0.603</td>
<td>0.604</td>
</tr>
<tr>
<td>3—period change</td>
<td>0.763</td>
<td>0.724</td>
<td>0.709</td>
<td>0.711</td>
</tr>
<tr>
<td>4—period change</td>
<td>0.830</td>
<td>0.800</td>
<td>0.782</td>
<td>0.784</td>
</tr>
<tr>
<td>5—period change</td>
<td>0.880</td>
<td>0.847</td>
<td>0.827</td>
<td>0.829</td>
</tr>
<tr>
<td>6—period change</td>
<td>0.913</td>
<td>0.873</td>
<td>0.853</td>
<td>0.855</td>
</tr>
<tr>
<td>7—period change</td>
<td>0.930</td>
<td>0.884</td>
<td>0.863</td>
<td>0.865</td>
</tr>
<tr>
<td>8—period change</td>
<td>0.932</td>
<td>0.884</td>
<td>0.857</td>
<td>0.860</td>
</tr>
<tr>
<td>9—period change</td>
<td>0.912</td>
<td>0.854</td>
<td>0.828</td>
<td>0.832</td>
</tr>
<tr>
<td>10—period change</td>
<td>0.895</td>
<td>0.824</td>
<td>0.799</td>
<td>0.806</td>
</tr>
<tr>
<td>11—period change</td>
<td>0.883</td>
<td>0.795</td>
<td>0.772</td>
<td>0.782</td>
</tr>
<tr>
<td>12—period change</td>
<td>0.883</td>
<td>0.767</td>
<td>0.747</td>
<td>0.761</td>
</tr>
</tbody>
</table>
Table 5.19. Variances of alternative estimators of unemployed relative to the variance of the basic estimator of current level; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Basic estimator</th>
<th>Present composite</th>
<th>BLUE 2 periods</th>
<th>BLUE 3 periods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>1.000</td>
<td>0.947</td>
<td>0.958</td>
<td>0.941</td>
</tr>
<tr>
<td>1—period change</td>
<td>1.155</td>
<td>1.070</td>
<td>1.107</td>
<td>1.090</td>
</tr>
<tr>
<td>2—period change</td>
<td>1.490</td>
<td>1.361</td>
<td>1.399</td>
<td>1.368</td>
</tr>
<tr>
<td>3—period change</td>
<td>1.686</td>
<td>1.528</td>
<td>1.569</td>
<td>1.516</td>
</tr>
<tr>
<td>4—period change</td>
<td>1.830</td>
<td>1.645</td>
<td>1.695</td>
<td>1.630</td>
</tr>
<tr>
<td>5—period change</td>
<td>1.830</td>
<td>1.691</td>
<td>1.746</td>
<td>1.685</td>
</tr>
<tr>
<td>6—period change</td>
<td>1.830</td>
<td>1.708</td>
<td>1.746</td>
<td>1.713</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.830</td>
<td>1.710</td>
<td>1.746</td>
<td>1.705</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.830</td>
<td>1.701</td>
<td>1.731</td>
<td>1.690</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.787</td>
<td>1.671</td>
<td>1.695</td>
<td>1.659</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.745</td>
<td>1.641</td>
<td>1.662</td>
<td>1.631</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.703</td>
<td>1.614</td>
<td>1.629</td>
<td>1.605</td>
</tr>
<tr>
<td>12—period change</td>
<td>1.660</td>
<td>1.593</td>
<td>1.600</td>
<td>1.581</td>
</tr>
<tr>
<td>Parameter</td>
<td>BLUE 12 periods</td>
<td>BLUE 16 periods</td>
<td>BLUE 24 periods</td>
<td>Recursive regression estimator</td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>Current level</td>
<td>0.924</td>
<td>0.918</td>
<td>0.918</td>
<td>0.918</td>
</tr>
<tr>
<td>1-period change</td>
<td>1.077</td>
<td>1.073</td>
<td>1.073</td>
<td>1.073</td>
</tr>
<tr>
<td>2-period change</td>
<td>1.345</td>
<td>1.338</td>
<td>1.338</td>
<td>1.338</td>
</tr>
<tr>
<td>3-period change</td>
<td>1.481</td>
<td>1.473</td>
<td>1.473</td>
<td>1.473</td>
</tr>
<tr>
<td>4-period change</td>
<td>1.569</td>
<td>1.563</td>
<td>1.562</td>
<td>1.562</td>
</tr>
<tr>
<td>5-period change</td>
<td>1.614</td>
<td>1.607</td>
<td>1.606</td>
<td>1.606</td>
</tr>
<tr>
<td>6-period change</td>
<td>1.637</td>
<td>1.628</td>
<td>1.628</td>
<td>1.628</td>
</tr>
<tr>
<td>7-period change</td>
<td>1.646</td>
<td>1.637</td>
<td>1.636</td>
<td>1.636</td>
</tr>
<tr>
<td>8-period change</td>
<td>1.645</td>
<td>1.635</td>
<td>1.634</td>
<td>1.634</td>
</tr>
<tr>
<td>9-period change</td>
<td>1.624</td>
<td>1.614</td>
<td>1.613</td>
<td>1.614</td>
</tr>
<tr>
<td>10-period change</td>
<td>1.606</td>
<td>1.595</td>
<td>1.594</td>
<td>1.595</td>
</tr>
<tr>
<td>11-period change</td>
<td>1.590</td>
<td>1.578</td>
<td>1.577</td>
<td>1.578</td>
</tr>
<tr>
<td>12-period change</td>
<td>1.577</td>
<td>1.563</td>
<td>1.563</td>
<td>1.564</td>
</tr>
</tbody>
</table>
Table 5.20. Variances of alternative estimators of Civilian Labor Force relative to the variance of the basic estimator of current level; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Basic estimator</th>
<th>Present composite</th>
<th>BLUE 2 periods</th>
<th>BLUE 3 periods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>1.000</td>
<td>0.868</td>
<td>0.899</td>
<td>0.845</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.697</td>
<td>0.538</td>
<td>0.597</td>
<td>0.548</td>
</tr>
<tr>
<td>2—period change</td>
<td>1.086</td>
<td>0.828</td>
<td>0.889</td>
<td>0.794</td>
</tr>
<tr>
<td>3—period change</td>
<td>1.410</td>
<td>1.076</td>
<td>1.129</td>
<td>0.986</td>
</tr>
<tr>
<td>4—period change</td>
<td>1.688</td>
<td>1.283</td>
<td>1.337</td>
<td>1.150</td>
</tr>
<tr>
<td>5—period change</td>
<td>1.688</td>
<td>1.364</td>
<td>1.487</td>
<td>1.278</td>
</tr>
<tr>
<td>6—period change</td>
<td>1.688</td>
<td>1.392</td>
<td>1.487</td>
<td>1.377</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.688</td>
<td>1.391</td>
<td>1.487</td>
<td>1.334</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.688</td>
<td>1.362</td>
<td>1.421</td>
<td>1.283</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.565</td>
<td>1.277</td>
<td>1.335</td>
<td>1.217</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.445</td>
<td>1.191</td>
<td>1.250</td>
<td>1.153</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.324</td>
<td>1.111</td>
<td>1.166</td>
<td>1.096</td>
</tr>
<tr>
<td>12—period change</td>
<td>1.203</td>
<td>1.052</td>
<td>1.093</td>
<td>1.042</td>
</tr>
</tbody>
</table>
Table 5.20. Continued

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.739</td>
<td>0.717</td>
<td>0.709</td>
<td>0.704</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.492</td>
<td>0.478</td>
<td>0.474</td>
<td>0.474</td>
</tr>
<tr>
<td>2—period change</td>
<td>0.682</td>
<td>0.658</td>
<td>0.651</td>
<td>0.652</td>
</tr>
<tr>
<td>3—period change</td>
<td>0.811</td>
<td>0.782</td>
<td>0.773</td>
<td>0.774</td>
</tr>
<tr>
<td>4—period change</td>
<td>0.890</td>
<td>0.870</td>
<td>0.858</td>
<td>0.859</td>
</tr>
<tr>
<td>5—period change</td>
<td>0.946</td>
<td>0.925</td>
<td>0.912</td>
<td>0.914</td>
</tr>
<tr>
<td>6—period change</td>
<td>0.983</td>
<td>0.957</td>
<td>0.945</td>
<td>0.946</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.004</td>
<td>0.972</td>
<td>0.960</td>
<td>0.962</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.009</td>
<td>0.976</td>
<td>0.960</td>
<td>0.963</td>
</tr>
<tr>
<td>9—period change</td>
<td>0.994</td>
<td>0.953</td>
<td>0.938</td>
<td>0.942</td>
</tr>
<tr>
<td>10—period change</td>
<td>0.981</td>
<td>0.931</td>
<td>0.917</td>
<td>0.921</td>
</tr>
<tr>
<td>11—period change</td>
<td>0.971</td>
<td>0.910</td>
<td>0.896</td>
<td>0.903</td>
</tr>
<tr>
<td>12—period change</td>
<td>0.968</td>
<td>0.891</td>
<td>0.878</td>
<td>0.887</td>
</tr>
</tbody>
</table>
Table 5.21. Relative efficiencies of alternative estimators of employed relative to the variance of the present composite estimator; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>122</td>
<td>128</td>
<td>130</td>
<td>133</td>
</tr>
<tr>
<td>1—period change</td>
<td>112</td>
<td>117</td>
<td>118</td>
<td>118</td>
</tr>
<tr>
<td>2—period change</td>
<td>126</td>
<td>125</td>
<td>135</td>
<td>135</td>
</tr>
<tr>
<td>3—period change</td>
<td>140</td>
<td>147</td>
<td>150</td>
<td>150</td>
</tr>
<tr>
<td>4—period change</td>
<td>154</td>
<td>160</td>
<td>164</td>
<td>163</td>
</tr>
<tr>
<td>5—period change</td>
<td>155</td>
<td>161</td>
<td>165</td>
<td>164</td>
</tr>
<tr>
<td>6—period change</td>
<td>152</td>
<td>159</td>
<td>163</td>
<td>163</td>
</tr>
<tr>
<td>7—period change</td>
<td>149</td>
<td>157</td>
<td>161</td>
<td>161</td>
</tr>
<tr>
<td>8—period change</td>
<td>145</td>
<td>153</td>
<td>158</td>
<td>157</td>
</tr>
<tr>
<td>9—period change</td>
<td>138</td>
<td>147</td>
<td>152</td>
<td>151</td>
</tr>
<tr>
<td>10—period change</td>
<td>129</td>
<td>140</td>
<td>144</td>
<td>143</td>
</tr>
<tr>
<td>11—period change</td>
<td>120</td>
<td>134</td>
<td>137</td>
<td>136</td>
</tr>
<tr>
<td>12—period change</td>
<td>112</td>
<td>129</td>
<td>133</td>
<td>130</td>
</tr>
</tbody>
</table>
Table 5.22. Relative efficiencies of alternative estimators of unemployed relative to the variance of the present composite estimator; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>1—period change</td>
<td>100</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>2—period change</td>
<td>105</td>
<td>105</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>3—period change</td>
<td>108</td>
<td>109</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>4—period change</td>
<td>110</td>
<td>110</td>
<td>110</td>
<td>110</td>
</tr>
<tr>
<td>5—period change</td>
<td>111</td>
<td>111</td>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>6—period change</td>
<td>111</td>
<td>111</td>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>7—period change</td>
<td>111</td>
<td>111</td>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>8—period change</td>
<td>111</td>
<td>111</td>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>9—period change</td>
<td>111</td>
<td>111</td>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>10—period change</td>
<td>110</td>
<td>111</td>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>11—period change</td>
<td>110</td>
<td>111</td>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>12—period change</td>
<td>110</td>
<td>110</td>
<td>110</td>
<td>111</td>
</tr>
</tbody>
</table>
Table 5.23. Relative efficiencies of alternative estimators of Civilian Labor Force relative to the present composite estimator; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>118</td>
<td>121</td>
<td>122</td>
<td>123</td>
</tr>
<tr>
<td>1-period change</td>
<td>109</td>
<td>113</td>
<td>114</td>
<td>114</td>
</tr>
<tr>
<td>2-period change</td>
<td>121</td>
<td>126</td>
<td>127</td>
<td>127</td>
</tr>
<tr>
<td>3-period change</td>
<td>133</td>
<td>138</td>
<td>139</td>
<td>139</td>
</tr>
<tr>
<td>4-period change</td>
<td>144</td>
<td>148</td>
<td>150</td>
<td>149</td>
</tr>
<tr>
<td>5-period change</td>
<td>144</td>
<td>148</td>
<td>150</td>
<td>149</td>
</tr>
<tr>
<td>6-period change</td>
<td>142</td>
<td>146</td>
<td>147</td>
<td>147</td>
</tr>
<tr>
<td>7-period change</td>
<td>139</td>
<td>143</td>
<td>145</td>
<td>145</td>
</tr>
<tr>
<td>8-period change</td>
<td>135</td>
<td>140</td>
<td>142</td>
<td>141</td>
</tr>
<tr>
<td>9-period change</td>
<td>129</td>
<td>134</td>
<td>136</td>
<td>136</td>
</tr>
<tr>
<td>10-period change</td>
<td>121</td>
<td>128</td>
<td>130</td>
<td>129</td>
</tr>
<tr>
<td>11-period change</td>
<td>114</td>
<td>122</td>
<td>124</td>
<td>123</td>
</tr>
<tr>
<td>12-period change</td>
<td>109</td>
<td>118</td>
<td>120</td>
<td>119</td>
</tr>
</tbody>
</table>
Table 5.24. Relative efficiencies of alternative estimators of employed relative to the 12-period best linear unbiased estimator; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>105</td>
<td>107</td>
<td>108</td>
</tr>
<tr>
<td>1-period change</td>
<td>105</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>2-period change</td>
<td>105</td>
<td>107</td>
<td>107</td>
</tr>
<tr>
<td>3-period change</td>
<td>105</td>
<td>108</td>
<td>107</td>
</tr>
<tr>
<td>4-period change</td>
<td>104</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>5-period change</td>
<td>104</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>6-period change</td>
<td>105</td>
<td>107</td>
<td>107</td>
</tr>
<tr>
<td>7-period change</td>
<td>105</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>8-period change</td>
<td>105</td>
<td>109</td>
<td>108</td>
</tr>
<tr>
<td>9-period change</td>
<td>107</td>
<td>110</td>
<td>110</td>
</tr>
<tr>
<td>10-period change</td>
<td>109</td>
<td>112</td>
<td>111</td>
</tr>
<tr>
<td>11-period change</td>
<td>111</td>
<td>114</td>
<td>113</td>
</tr>
<tr>
<td>12-period change</td>
<td>115</td>
<td>118</td>
<td>116</td>
</tr>
</tbody>
</table>
Table 5.25. Relative efficiencies of alternative estimators of unemployed relative to the 12-period best linear unbiased estimator 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>1-period change</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>2-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>3-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>4-period change</td>
<td>100</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>5-period change</td>
<td>100</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>6-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>7-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>8-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>9-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>10-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>11-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>12-period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
</tbody>
</table>
Table 5.26. Relative efficiencies of alternative estimators of Civilian Labor Force relative to the 12-period best linear unbiased estimator; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>103</td>
<td>104</td>
<td>105</td>
</tr>
<tr>
<td>1—period change</td>
<td>103</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>2—period change</td>
<td>104</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>3—period change</td>
<td>102</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>4—period change</td>
<td>102</td>
<td>103</td>
<td>104</td>
</tr>
<tr>
<td>5—period change</td>
<td>102</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>6—period change</td>
<td>103</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>7—period change</td>
<td>103</td>
<td>105</td>
<td>104</td>
</tr>
<tr>
<td>8—period change</td>
<td>103</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>9—period change</td>
<td>104</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>10—period change</td>
<td>105</td>
<td>107</td>
<td>107</td>
</tr>
<tr>
<td>11—period change</td>
<td>107</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>12—period change</td>
<td>109</td>
<td>110</td>
<td>109</td>
</tr>
</tbody>
</table>
Table 5.27. Variances of alternative estimators of employed relative to the variance of the basic estimator of current level; 1987 correlation pattern and revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.704</td>
<td>0.672</td>
<td>0.661</td>
<td>0.650</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.402</td>
<td>0.398</td>
<td>0.397</td>
<td>0.397</td>
</tr>
<tr>
<td>2—period change</td>
<td>0.567</td>
<td>0.555</td>
<td>0.553</td>
<td>0.553</td>
</tr>
<tr>
<td>3—period change</td>
<td>0.679</td>
<td>0.658</td>
<td>0.653</td>
<td>0.652</td>
</tr>
<tr>
<td>4—period change</td>
<td>0.765</td>
<td>0.737</td>
<td>0.723</td>
<td>0.722</td>
</tr>
<tr>
<td>5—period change</td>
<td>0.821</td>
<td>0.790</td>
<td>0.767</td>
<td>0.766</td>
</tr>
<tr>
<td>6—period change</td>
<td>0.854</td>
<td>0.824</td>
<td>0.793</td>
<td>0.790</td>
</tr>
<tr>
<td>7—period change</td>
<td>0.866</td>
<td>0.842</td>
<td>0.803</td>
<td>0.799</td>
</tr>
<tr>
<td>8—period change</td>
<td>0.861</td>
<td>0.833</td>
<td>0.797</td>
<td>0.792</td>
</tr>
<tr>
<td>9—period change</td>
<td>0.829</td>
<td>0.793</td>
<td>0.764</td>
<td>0.758</td>
</tr>
<tr>
<td>10—period change</td>
<td>0.805</td>
<td>0.753</td>
<td>0.732</td>
<td>0.726</td>
</tr>
<tr>
<td>11—period change</td>
<td>0.802</td>
<td>0.717</td>
<td>0.702</td>
<td>0.696</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA¹</td>
<td>0.688</td>
<td>0.678</td>
<td>0.670</td>
</tr>
</tbody>
</table>

NA¹: Not available
Table 5.28. Variances of alternative estimators of unemployed relative to the variance of the basic estimator of current level; 1987 correlation pattern and revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.924</td>
<td>0.918</td>
<td>0.918</td>
<td>0.918</td>
</tr>
<tr>
<td>1—period change</td>
<td>1.043</td>
<td>1.041</td>
<td>1.041</td>
<td>1.041</td>
</tr>
<tr>
<td>2—period change</td>
<td>1.298</td>
<td>1.295</td>
<td>1.294</td>
<td>1.294</td>
</tr>
<tr>
<td>3—period change</td>
<td>1.432</td>
<td>1.426</td>
<td>1.425</td>
<td>1.425</td>
</tr>
<tr>
<td>4—period change</td>
<td>1.524</td>
<td>1.516</td>
<td>1.513</td>
<td>1.513</td>
</tr>
<tr>
<td>5—period change</td>
<td>1.569</td>
<td>1.562</td>
<td>1.556</td>
<td>1.556</td>
</tr>
<tr>
<td>6—period change</td>
<td>1.592</td>
<td>1.585</td>
<td>1.578</td>
<td>1.578</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.601</td>
<td>1.595</td>
<td>1.586</td>
<td>1.586</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.599</td>
<td>1.593</td>
<td>1.584</td>
<td>1.584</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.581</td>
<td>1.570</td>
<td>1.563</td>
<td>1.562</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.571</td>
<td>1.549</td>
<td>1.543</td>
<td>1.541</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.581</td>
<td>1.531</td>
<td>1.525</td>
<td>1.523</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA¹</td>
<td>1.515</td>
<td>1.510</td>
<td>1.507</td>
</tr>
</tbody>
</table>

NA¹: Not available
Table 5.29. Variances of alternative estimators of Civilian Labor Force relative to the basic estimator of current level; 1987 correlation pattern and revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.739</td>
<td>0.717</td>
<td>0.709</td>
<td>0.704</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.437</td>
<td>0.435</td>
<td>0.435</td>
<td>0.435</td>
</tr>
<tr>
<td>2—period change</td>
<td>0.600</td>
<td>0.593</td>
<td>0.592</td>
<td>0.592</td>
</tr>
<tr>
<td>3—period change</td>
<td>0.721</td>
<td>0.708</td>
<td>0.704</td>
<td>0.704</td>
</tr>
<tr>
<td>4—period change</td>
<td>0.812</td>
<td>0.795</td>
<td>0.786</td>
<td>0.785</td>
</tr>
<tr>
<td>5—period change</td>
<td>0.873</td>
<td>0.855</td>
<td>0.839</td>
<td>0.838</td>
</tr>
<tr>
<td>6—period change</td>
<td>0.911</td>
<td>0.892</td>
<td>0.871</td>
<td>0.870</td>
</tr>
<tr>
<td>7—period change</td>
<td>0.930</td>
<td>0.913</td>
<td>0.886</td>
<td>0.884</td>
</tr>
<tr>
<td>8—period change</td>
<td>0.933</td>
<td>0.911</td>
<td>0.886</td>
<td>0.883</td>
</tr>
<tr>
<td>9—period change</td>
<td>0.914</td>
<td>0.882</td>
<td>0.861</td>
<td>0.857</td>
</tr>
<tr>
<td>10—period change</td>
<td>0.905</td>
<td>0.853</td>
<td>0.836</td>
<td>0.832</td>
</tr>
<tr>
<td>11—period change</td>
<td>0.915</td>
<td>0.827</td>
<td>0.814</td>
<td>0.809</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA†</td>
<td>0.826</td>
<td>0.795</td>
<td>0.788</td>
</tr>
</tbody>
</table>

NA†: Not available
Table 5.30. Relative efficiencies of alternative estimators of employed relative to the 12-period best linear unbiased estimator; 1987 correlation pattern and revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>105</td>
<td>107</td>
<td>108</td>
</tr>
<tr>
<td>1—period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>2—period change</td>
<td>102</td>
<td>103</td>
<td>102</td>
</tr>
<tr>
<td>3—period change</td>
<td>103</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>4—period change</td>
<td>104</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>5—period change</td>
<td>104</td>
<td>107</td>
<td>107</td>
</tr>
<tr>
<td>6—period change</td>
<td>104</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>7—period change</td>
<td>103</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>8—period change</td>
<td>103</td>
<td>108</td>
<td>109</td>
</tr>
<tr>
<td>9—period change</td>
<td>105</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>10—period change</td>
<td>107</td>
<td>110</td>
<td>111</td>
</tr>
<tr>
<td>11—period change</td>
<td>112</td>
<td>114</td>
<td>115</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA (^1)</td>
<td>NA (^1)</td>
<td>NA (^1)</td>
</tr>
</tbody>
</table>

NA \(^1\): Not available
Table 5.31. Relative efficiencies of alternative estimators of unemployed relative to the 12-period best linear unbiased estimator; 1987 correlation pattern and revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>1—period change</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>2—period change</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>3—period change</td>
<td>100</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>4—period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>5—period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>6—period change</td>
<td>100</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>7—period change</td>
<td>100</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>8—period change</td>
<td>100</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>9—period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>10—period change</td>
<td>101</td>
<td>102</td>
<td>102</td>
</tr>
<tr>
<td>11—period change</td>
<td>103</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA¹</td>
<td>NA¹</td>
<td>NA¹</td>
</tr>
</tbody>
</table>

NA¹: Not available
Table 5.32. Relative efficiencies of alternative estimators of Civilian Labor Force relative to the 12—period best linear unbiased estimator; 1987 correlation pattern and revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>103</td>
<td>104</td>
<td>105</td>
</tr>
<tr>
<td>1—period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>2—period change</td>
<td>101</td>
<td>101</td>
<td>101</td>
</tr>
<tr>
<td>3—period change</td>
<td>102</td>
<td>102</td>
<td>102</td>
</tr>
<tr>
<td>4—period change</td>
<td>102</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>5—period change</td>
<td>102</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>6—period change</td>
<td>102</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>7—period change</td>
<td>102</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>8—period change</td>
<td>102</td>
<td>105</td>
<td>106</td>
</tr>
<tr>
<td>9—period change</td>
<td>104</td>
<td>106</td>
<td>107</td>
</tr>
<tr>
<td>10—period change</td>
<td>106</td>
<td>108</td>
<td>109</td>
</tr>
<tr>
<td>11—period change</td>
<td>111</td>
<td>112</td>
<td>113</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA¹</td>
<td>NA¹</td>
<td>NA¹</td>
</tr>
</tbody>
</table>

NA¹: Not available
Breau and Ernst (1983) determined the number of months necessary for the best linear unbiased estimator to achieve a reduction in variance greater than the generalized composite estimator (See Section 2.1.). They discovered that no more than seven months (in the case of estimation of current level), and only three months (in the case of estimation of one-period change) are necessary for the best linear unbiased estimator to do better than the generalized composite estimator. In accordance with this result, the best linear unbiased estimation procedure for twelve, sixteen, and twenty-four periods produce more efficient estimators of both current level and one-period change than the present composite estimator, which is a special case of the generalized composite estimator.

5.7.1. Estimation of Current Level

In general, the best linear unbiased procedure becomes more statistically efficient as the number of periods increases. For all the characteristics of interest, the results reveal that more than three periods are required for the best linear unbiased procedure to be uniformly more efficient than the present composite estimator, in estimating current level. Therefore, for our comparison of the alternative estimators, we shall concentrate on the best linear unbiased estimators for periods twelve and above.

For twelve periods or more, there is a substantial improvement in the precision of the estimate of current level from using the best linear unbiased procedure. The gain in precision is highest for Employed: 22% for the best linear unbiased estimator for 12 periods, 28% for the best linear unbiased estimator for 16 periods, 30% for the best linear unbiased estimator for 24 periods, and 33% for the recursive regression
estimator. The corresponding figures for Civilian Labor Force are 18%, 21%, 22%, and 23%. The gain in precision is least for unemployed, where there is a gain in precision of 3% for all the estimators. See Tables 5.21, 5.22, and 5.23. These results are a reflection of the correlation patterns of the characteristics. The average time-in-sample correlations are strongest for employed, moderate for Civilian Labor Force, and weakest for unemployed. See Table 5.2.

It can be seen from Tables 5.24, 5.25, and 5.26 that relative to the best linear unbiased estimator for 12 periods, for employed, the gain in precision is 5% for the best linear unbiased estimator for 16 periods, 7% for the best linear unbiased estimator for 24 periods, and 8% for the recursive regression estimator. For Civilian Labor Force, the corresponding figures are 3%, 4%, and 5%, and for unemployed, there is hardly any improvement in the precision of the estimates (only 1% for best linear unbiased estimators for 16 and 24 periods, as well as for the recursive regression estimator).

5.7.2. Estimation of Change Under No Revision of Previous Estimates

For all the characteristics under consideration, there is a substantial improvement in the efficiency of the estimation of change from using the alternative estimators instead of the present composite estimator. The gain in precision increases as the number of periods increases and the length of the interval of change increases. The gain is modest for one-period change, then increases substantially, reaching a maximum value at the five-period change for all characteristics. The gain then decreases slightly. In the case of the recursive regression estimator, the maximum gain in estimated change is 64% for employed, 11% for unemployed, and 49% for Civilian Labor Force.
Relative to the twelve-period best linear unbiased estimator, there is only a modest gain inefficiency of the alternative estimators for employed and Civilian Labor Force. There is hardly any gain in precision for unemployed.

5.7.3. Estimation of Change Under Revision of Previous Estimates

The relative precisions of the alternative estimators relative to the best linear unbiased estimator for 12 periods, under revision of previous estimates are given in Table 5.30 for employed, Table 5.31 for unemployed, and Table 5.32 for Civilian Labor Force. In general, the gains in precision of the alternative estimators are slightly less than those obtained relative to the present composite estimator when previous estimates are not revised. However, for all the characteristics under consideration, this gain in precision is monotonically increasing as the interval of change increases. As in the case of estimation of change under no revision (Section 5.7.2), the gains in precision of estimators of unemployed under revision of previous estimates are smaller than those for Civilian Labor Force and employed.

An examination of the last columns of Tables 5.33, 5.34, and 5.35 reveals that a modest gain in efficiency is possible if one is willing to revise previous estimates when more data become available. The maximum gain is for twelve-period change and is about 14% for employed, about 4% for unemployed, and about 13% for Civilian Labor Force.
Table 5.33. Relative efficiencies of alternative estimators of employed under revision of previous estimates relative to the variance of the estimators under no revision of previous estimates; 1987 correlation pattern

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>1—period change</td>
<td>114</td>
<td>110</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>2—period change</td>
<td>114</td>
<td>111</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>3—period change</td>
<td>112</td>
<td>110</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>4—period change</td>
<td>109</td>
<td>109</td>
<td>108</td>
<td>109</td>
</tr>
<tr>
<td>5—period change</td>
<td>107</td>
<td>107</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>6—period change</td>
<td>107</td>
<td>106</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>7—period change</td>
<td>107</td>
<td>105</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>8—period change</td>
<td>108</td>
<td>106</td>
<td>108</td>
<td>109</td>
</tr>
<tr>
<td>9—period change</td>
<td>110</td>
<td>108</td>
<td>108</td>
<td>110</td>
</tr>
<tr>
<td>10—period change</td>
<td>111</td>
<td>109</td>
<td>109</td>
<td>111</td>
</tr>
<tr>
<td>11—period change</td>
<td>110</td>
<td>111</td>
<td>110</td>
<td>112</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA¹</td>
<td>112</td>
<td>110</td>
<td>114</td>
</tr>
</tbody>
</table>

NA¹: Not available
Table 5.34. Relative efficiencies of alternative estimators of unemployed under revision of previous estimates relative to the variance of the estimators under no revision of previous estimates; 1987 correlation pattern

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>1—period change</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>2—period change</td>
<td>104</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>3—period change</td>
<td>104</td>
<td>103</td>
<td>104</td>
<td>103</td>
</tr>
<tr>
<td>4—period change</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>5—period change</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>6—period change</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>7—period change</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>8—period change</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>9—period change</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>10—period change</td>
<td>102</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>11—period change</td>
<td>101</td>
<td>103</td>
<td>103</td>
<td>104</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA¹</td>
<td>103</td>
<td>104</td>
<td>104</td>
</tr>
</tbody>
</table>

NA¹: Not available
Table 5.35. Relative efficiencies of alternative estimators of Civilian Labor Force under revision of previous estimates relative to the variance of the estimators under no revision of previous estimates; 1987 correlation pattern

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>1—period change</td>
<td>113</td>
<td>110</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>2—period change</td>
<td>114</td>
<td>111</td>
<td>110</td>
<td>110</td>
</tr>
<tr>
<td>3—period change</td>
<td>113</td>
<td>111</td>
<td>110</td>
<td>110</td>
</tr>
<tr>
<td>4—period change</td>
<td>110</td>
<td>109</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>5—period change</td>
<td>108</td>
<td>108</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>6—period change</td>
<td>108</td>
<td>107</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>7—period change</td>
<td>108</td>
<td>107</td>
<td>108</td>
<td>109</td>
</tr>
<tr>
<td>8—period change</td>
<td>108</td>
<td>107</td>
<td>108</td>
<td>109</td>
</tr>
<tr>
<td>9—period change</td>
<td>109</td>
<td>108</td>
<td>109</td>
<td>110</td>
</tr>
<tr>
<td>10—period change</td>
<td>108</td>
<td>109</td>
<td>110</td>
<td>111</td>
</tr>
<tr>
<td>11—period change</td>
<td>106</td>
<td>110</td>
<td>110</td>
<td>112</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA^1</td>
<td>108</td>
<td>110</td>
<td>113</td>
</tr>
</tbody>
</table>

NA^1: Not available
5.7.4. Estimation of Unemployment Rate

The variances of the alternative estimators of the rate of unemployment under no revision of previous estimates are given in Table 5.36. The relative efficiencies of the alternative estimators relative to the present composite estimator and to the best linear unbiased estimator for 2 periods are given respectively in Table 5.37 and Table 5.38. The computations are based on the assumption that unemployed and Civilian Labor Force are independent. Since the variance of the estimator of unemployed is the dominant component in the formulas for the variance of unemployment rate [See (5.6.1) and (5.6.3)], the variances of alternative estimators of unemployment rate are approximately the same as those for unemployed. There are only modest gains in the precision of the alternative estimators relative to both the present composite estimator and the best linear unbiased estimator based on 2 periods.

5.7.5. Comparison of Rotation Designs

The results of the comparison of alternative estimators and rotation designs are given in Tables 5.39 — 5.50. Alternative estimators of current level, change in level over an interval of several periods, and average level over time are compared. The performance of the estimators under the intermittent 4—8—4 rotation design and two continuous rotation designs are also compared. The continuous rotation designs were investigated on the presumption that they are easier to implement in the field and require less record keeping in the office. Also, the optimal least squares estimator is
Table 5.36. Variances of alternative estimators ($\times 10^{-7}$) of the unemployment rate under no revision of previous estimates; 1987 correlation pattern

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Basic estimator</th>
<th>Present composite</th>
<th>BLUE 2 periods</th>
<th>BLUE 3 periods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>1.844</td>
<td>1.743</td>
<td>1.764</td>
<td>1.732</td>
</tr>
<tr>
<td>1—period change</td>
<td>2.115</td>
<td>1.956</td>
<td>2.024</td>
<td>1.992</td>
</tr>
<tr>
<td>2—period change</td>
<td>2.735</td>
<td>2.493</td>
<td>2.564</td>
<td>2.503</td>
</tr>
<tr>
<td>3—period change</td>
<td>3.100</td>
<td>2.802</td>
<td>2.878</td>
<td>2.777</td>
</tr>
<tr>
<td>4—period change</td>
<td>3.370</td>
<td>3.021</td>
<td>3.113</td>
<td>2.989</td>
</tr>
<tr>
<td>5—period change</td>
<td>3.370</td>
<td>3.107</td>
<td>3.211</td>
<td>3.093</td>
</tr>
<tr>
<td>6—period change</td>
<td>3.370</td>
<td>3.138</td>
<td>3.211</td>
<td>3.147</td>
</tr>
<tr>
<td>7—period change</td>
<td>3.370</td>
<td>3.143</td>
<td>3.211</td>
<td>3.131</td>
</tr>
<tr>
<td>8—period change</td>
<td>3.370</td>
<td>3.125</td>
<td>3.181</td>
<td>3.103</td>
</tr>
<tr>
<td>9—period change</td>
<td>3.286</td>
<td>3.068</td>
<td>3.113</td>
<td>3.044</td>
</tr>
<tr>
<td>10—period change</td>
<td>3.207</td>
<td>3.012</td>
<td>3.050</td>
<td>2.991</td>
</tr>
<tr>
<td>11—period change</td>
<td>3.128</td>
<td>2.959</td>
<td>2.988</td>
<td>2.943</td>
</tr>
<tr>
<td>12—period change</td>
<td>3.047</td>
<td>2.920</td>
<td>2.934</td>
<td>2.897</td>
</tr>
<tr>
<td>Parameter</td>
<td>BLUE 12 periods</td>
<td>BLUE 16 periods</td>
<td>BLUE 24 periods</td>
<td>Recursive regression estimator</td>
</tr>
<tr>
<td>-------------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>Current level</td>
<td>1.697</td>
<td>1.687</td>
<td>1.686</td>
<td>1.685</td>
</tr>
<tr>
<td>1—period change</td>
<td>1.966</td>
<td>1.959</td>
<td>1.958</td>
<td>1.958</td>
</tr>
<tr>
<td>2—period change</td>
<td>2.458</td>
<td>2.445</td>
<td>2.444</td>
<td>2.444</td>
</tr>
<tr>
<td>3—period change</td>
<td>2.710</td>
<td>2.695</td>
<td>2.693</td>
<td>2.693</td>
</tr>
<tr>
<td>4—period change</td>
<td>2.871</td>
<td>2.860</td>
<td>2.858</td>
<td>2.858</td>
</tr>
<tr>
<td>5—period change</td>
<td>2.955</td>
<td>2.940</td>
<td>2.939</td>
<td>2.939</td>
</tr>
<tr>
<td>6—period change</td>
<td>2.997</td>
<td>2.981</td>
<td>2.979</td>
<td>2.979</td>
</tr>
<tr>
<td>7—period change</td>
<td>3.015</td>
<td>2.996</td>
<td>2.995</td>
<td>2.995</td>
</tr>
<tr>
<td>8—period change</td>
<td>3.011</td>
<td>2.993</td>
<td>2.991</td>
<td>2.991</td>
</tr>
<tr>
<td>9—period change</td>
<td>2.974</td>
<td>2.955</td>
<td>2.953</td>
<td>2.953</td>
</tr>
<tr>
<td>10—period change</td>
<td>2.940</td>
<td>2.919</td>
<td>2.918</td>
<td>2.918</td>
</tr>
<tr>
<td>11—period change</td>
<td>2.911</td>
<td>2.888</td>
<td>2.886</td>
<td>2.887</td>
</tr>
<tr>
<td>12—period change</td>
<td>2.888</td>
<td>2.860</td>
<td>2.859</td>
<td>2.861</td>
</tr>
</tbody>
</table>
Table 5.37. Relative efficiencies of alternative estimators of unemployment rate relative to the present composite estimator; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 3 periods</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>101</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>1—period change</td>
<td>98</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>2—period change</td>
<td>100</td>
<td>101</td>
<td>102</td>
<td>102</td>
<td>102</td>
</tr>
<tr>
<td>3—period change</td>
<td>101</td>
<td>103</td>
<td>104</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>4—period change</td>
<td>101</td>
<td>105</td>
<td>106</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>5—period change</td>
<td>101</td>
<td>105</td>
<td>106</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>6—period change</td>
<td>100</td>
<td>105</td>
<td>105</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>7—period change</td>
<td>100</td>
<td>104</td>
<td>105</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>8—period change</td>
<td>101</td>
<td>104</td>
<td>104</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>9—period change</td>
<td>101</td>
<td>103</td>
<td>104</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>10—period change</td>
<td>101</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>11—period change</td>
<td>101</td>
<td>102</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>12—period change</td>
<td>101</td>
<td>101</td>
<td>102</td>
<td>102</td>
<td>102</td>
</tr>
</tbody>
</table>
Table 5.38. Relative efficiencies of alternative estimators of unemployment rate relative to the 2-period best linear unbiased estimator; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 3 periods</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>102</td>
<td>104</td>
<td>105</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>1-period change</td>
<td>102</td>
<td>103</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
<tr>
<td>2-period change</td>
<td>102</td>
<td>104</td>
<td>105</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>3-period change</td>
<td>104</td>
<td>106</td>
<td>107</td>
<td>107</td>
<td>107</td>
</tr>
<tr>
<td>4-period change</td>
<td>104</td>
<td>108</td>
<td>109</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>5-period change</td>
<td>104</td>
<td>109</td>
<td>109</td>
<td>109</td>
<td>109</td>
</tr>
<tr>
<td>6-period change</td>
<td>102</td>
<td>107</td>
<td>108</td>
<td>108</td>
<td>108</td>
</tr>
<tr>
<td>7-period change</td>
<td>103</td>
<td>107</td>
<td>107</td>
<td>107</td>
<td>107</td>
</tr>
<tr>
<td>8-period change</td>
<td>103</td>
<td>106</td>
<td>106</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>9-period change</td>
<td>102</td>
<td>105</td>
<td>105</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>10-period change</td>
<td>102</td>
<td>104</td>
<td>105</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>11-period change</td>
<td>102</td>
<td>103</td>
<td>104</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>12-period change</td>
<td>101</td>
<td>102</td>
<td>103</td>
<td>103</td>
<td>103</td>
</tr>
</tbody>
</table>
Table 5.39. Variances of alternative estimators of employed; variance of basic estimator of current level equals one

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Present composite estimator</th>
<th>Best estimator 4-8-4</th>
<th>Best estimator 8-in-then-out</th>
<th>Best estimator 6-in-then-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.862</td>
<td>0.653</td>
<td>0.761</td>
<td>0.759</td>
</tr>
<tr>
<td>1-period change</td>
<td>0.511</td>
<td>0.432</td>
<td>0.395</td>
<td>0.434</td>
</tr>
<tr>
<td>2-period change</td>
<td>0.813</td>
<td>0.604</td>
<td>0.559</td>
<td>0.619</td>
</tr>
<tr>
<td>3-period change</td>
<td>1.065</td>
<td>0.710</td>
<td>0.669</td>
<td>0.747</td>
</tr>
<tr>
<td>4-period change</td>
<td>1.279</td>
<td>0.783</td>
<td>0.731</td>
<td>0.829</td>
</tr>
<tr>
<td>5-period change</td>
<td>1.363</td>
<td>0.828</td>
<td>0.782</td>
<td>0.901</td>
</tr>
<tr>
<td>6-period change</td>
<td>1.390</td>
<td>0.854</td>
<td>0.828</td>
<td>0.970</td>
</tr>
<tr>
<td>7-period change</td>
<td>1.388</td>
<td>0.863</td>
<td>0.874</td>
<td>1.026</td>
</tr>
<tr>
<td>8-period change</td>
<td>1.353</td>
<td>0.858</td>
<td>0.960</td>
<td>1.071</td>
</tr>
<tr>
<td>9-period change</td>
<td>1.255</td>
<td>0.830</td>
<td>0.960</td>
<td>1.108</td>
</tr>
<tr>
<td>10-period change</td>
<td>1.154</td>
<td>0.803</td>
<td>0.993</td>
<td>1.139</td>
</tr>
<tr>
<td>11-period change</td>
<td>1.061</td>
<td>0.779</td>
<td>1.021</td>
<td>1.165</td>
</tr>
<tr>
<td>12-period change</td>
<td>0.992</td>
<td>0.758</td>
<td>1.046</td>
<td>1.186</td>
</tr>
<tr>
<td>12-period average</td>
<td>0.369</td>
<td>0.326</td>
<td>0.440</td>
<td>0.394</td>
</tr>
<tr>
<td>Change in 12-period averages</td>
<td>0.248</td>
<td>0.162</td>
<td>0.365</td>
<td>0.403</td>
</tr>
</tbody>
</table>
Table 5.40. Variances of alternative estimators of unemployed; variance of basic estimator of current level equals one

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Present composite estimator</th>
<th>Best estimator 4-in-8-out</th>
<th>Best estimator 8-in-then-out</th>
<th>Best estimator 6-in-then-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.947</td>
<td>0.918</td>
<td>0.944</td>
<td>0.938</td>
</tr>
<tr>
<td>1-period change</td>
<td>1.070</td>
<td>1.073</td>
<td>1.003</td>
<td>1.051</td>
</tr>
<tr>
<td>2-period change</td>
<td>1.361</td>
<td>1.338</td>
<td>1.250</td>
<td>1.312</td>
</tr>
<tr>
<td>3-period change</td>
<td>1.528</td>
<td>1.473</td>
<td>1.372</td>
<td>1.443</td>
</tr>
<tr>
<td>4-period change</td>
<td>1.645</td>
<td>1.562</td>
<td>1.473</td>
<td>1.543</td>
</tr>
<tr>
<td>5-period change</td>
<td>1.691</td>
<td>1.606</td>
<td>1.533</td>
<td>1.607</td>
</tr>
<tr>
<td>6-period change</td>
<td>1.708</td>
<td>1.628</td>
<td>1.577</td>
<td>1.655</td>
</tr>
<tr>
<td>7-period change</td>
<td>1.710</td>
<td>1.636</td>
<td>1.612</td>
<td>1.686</td>
</tr>
<tr>
<td>8-period change</td>
<td>1.701</td>
<td>1.634</td>
<td>1.642</td>
<td>1.705</td>
</tr>
<tr>
<td>9-period change</td>
<td>1.671</td>
<td>1.614</td>
<td>1.663</td>
<td>1.719</td>
</tr>
<tr>
<td>10-period change</td>
<td>1.641</td>
<td>1.595</td>
<td>1.678</td>
<td>1.727</td>
</tr>
<tr>
<td>11-period change</td>
<td>1.614</td>
<td>1.578</td>
<td>1.688</td>
<td>1.733</td>
</tr>
<tr>
<td>12-period change</td>
<td>1.593</td>
<td>1.564</td>
<td>1.696</td>
<td>1.737</td>
</tr>
<tr>
<td>12-period average</td>
<td>0.255</td>
<td>0.249</td>
<td>0.301</td>
<td>0.266</td>
</tr>
<tr>
<td>Change in 12-period averages</td>
<td>0.273</td>
<td>0.262</td>
<td>0.372</td>
<td>0.359</td>
</tr>
</tbody>
</table>
Table 5.41. Variances of alternative estimators of Civilian Labor Force; variance of basic estimator of current level equals one

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Present composite estimator</th>
<th>Best estimator 4–8–4 then–out</th>
<th>Best estimator 8–in–then–out</th>
<th>Best estimator 6–in–then–out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.868</td>
<td>0.706</td>
<td>0.796</td>
<td>0.783</td>
</tr>
<tr>
<td>1–period change</td>
<td>0.538</td>
<td>0.474</td>
<td>0.430</td>
<td>0.470</td>
</tr>
<tr>
<td>2–period change</td>
<td>0.828</td>
<td>0.652</td>
<td>0.589</td>
<td>0.651</td>
</tr>
<tr>
<td>3–period change</td>
<td>1.076</td>
<td>0.774</td>
<td>0.709</td>
<td>0.786</td>
</tr>
<tr>
<td>4–period change</td>
<td>1.283</td>
<td>0.859</td>
<td>0.793</td>
<td>0.883</td>
</tr>
<tr>
<td>5–period change</td>
<td>1.364</td>
<td>0.913</td>
<td>0.858</td>
<td>0.962</td>
</tr>
<tr>
<td>6–period change</td>
<td>1.392</td>
<td>0.946</td>
<td>0.913</td>
<td>1.032</td>
</tr>
<tr>
<td>7–period change</td>
<td>1.391</td>
<td>0.961</td>
<td>0.963</td>
<td>1.088</td>
</tr>
<tr>
<td>8–period change</td>
<td>1.362</td>
<td>0.962</td>
<td>1.010</td>
<td>1.133</td>
</tr>
<tr>
<td>9–period change</td>
<td>1.277</td>
<td>0.941</td>
<td>1.049</td>
<td>1.170</td>
</tr>
<tr>
<td>10–period change</td>
<td>1.191</td>
<td>0.920</td>
<td>1.083</td>
<td>1.199</td>
</tr>
<tr>
<td>11–period change</td>
<td>1.111</td>
<td>0.902</td>
<td>1.111</td>
<td>1.223</td>
</tr>
<tr>
<td>12–period change</td>
<td>1.052</td>
<td>0.886</td>
<td>1.135</td>
<td>1.242</td>
</tr>
<tr>
<td>12–period average</td>
<td>0.369</td>
<td>0.346</td>
<td>0.448</td>
<td>0.396</td>
</tr>
<tr>
<td>Change in 12–period averages</td>
<td>0.259</td>
<td>0.206</td>
<td>0.393</td>
<td>0.413</td>
</tr>
</tbody>
</table>
Table 5.42. Relative efficiencies of alternative estimators of employed relative to the present composite estimator

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Best estimator 4–8–4</th>
<th>Best estimator 8–in–then–out</th>
<th>Best estimator 6–in–then–out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>132</td>
<td>113</td>
<td>114</td>
</tr>
<tr>
<td>1–period change</td>
<td>118</td>
<td>129</td>
<td>118</td>
</tr>
<tr>
<td>2–period change</td>
<td>135</td>
<td>145</td>
<td>131</td>
</tr>
<tr>
<td>3–period change</td>
<td>150</td>
<td>159</td>
<td>143</td>
</tr>
<tr>
<td>4–period change</td>
<td>163</td>
<td>175</td>
<td>154</td>
</tr>
<tr>
<td>5–period change</td>
<td>165</td>
<td>174</td>
<td>151</td>
</tr>
<tr>
<td>6–period change</td>
<td>163</td>
<td>168</td>
<td>143</td>
</tr>
<tr>
<td>7–period change</td>
<td>161</td>
<td>159</td>
<td>135</td>
</tr>
<tr>
<td>8–period change</td>
<td>158</td>
<td>147</td>
<td>126</td>
</tr>
<tr>
<td>9–period change</td>
<td>151</td>
<td>131</td>
<td>113</td>
</tr>
<tr>
<td>10–period change</td>
<td>144</td>
<td>116</td>
<td>101</td>
</tr>
<tr>
<td>11–period change</td>
<td>136</td>
<td>104</td>
<td>91</td>
</tr>
<tr>
<td>12–period change</td>
<td>131</td>
<td>95</td>
<td>84</td>
</tr>
<tr>
<td>12–period average</td>
<td>113</td>
<td>84</td>
<td>94</td>
</tr>
<tr>
<td>Change in</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12–period averages</td>
<td>153</td>
<td>68</td>
<td>62</td>
</tr>
</tbody>
</table>
Table 5.43. Relative efficiencies of alternative estimators of unemployed relative to the present composite estimator

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Best estimator 4-8-4</th>
<th>Best estimator 8-in-then-out</th>
<th>Best estimator 6-in-then-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>103</td>
<td>100</td>
<td>101</td>
</tr>
<tr>
<td>1-period change</td>
<td>100</td>
<td>107</td>
<td>102</td>
</tr>
<tr>
<td>2-period change</td>
<td>102</td>
<td>109</td>
<td>104</td>
</tr>
<tr>
<td>3-period change</td>
<td>104</td>
<td>111</td>
<td>106</td>
</tr>
<tr>
<td>4-period change</td>
<td>105</td>
<td>112</td>
<td>107</td>
</tr>
<tr>
<td>5-period change</td>
<td>105</td>
<td>110</td>
<td>105</td>
</tr>
<tr>
<td>6-period change</td>
<td>105</td>
<td>108</td>
<td>103</td>
</tr>
<tr>
<td>7-period change</td>
<td>105</td>
<td>106</td>
<td>101</td>
</tr>
<tr>
<td>8-period change</td>
<td>104</td>
<td>104</td>
<td>100</td>
</tr>
<tr>
<td>9-period change</td>
<td>104</td>
<td>101</td>
<td>97</td>
</tr>
<tr>
<td>10-period change</td>
<td>103</td>
<td>98</td>
<td>95</td>
</tr>
<tr>
<td>11-period change</td>
<td>102</td>
<td>96</td>
<td>93</td>
</tr>
<tr>
<td>12-period change</td>
<td>102</td>
<td>94</td>
<td>92</td>
</tr>
<tr>
<td>12-period average</td>
<td>102</td>
<td>85</td>
<td>96</td>
</tr>
<tr>
<td>Change in</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12-period averages</td>
<td>104</td>
<td>73</td>
<td>76</td>
</tr>
</tbody>
</table>
Table 5.44. Relative efficiencies of alternative estimators of Civilian Labor Force relative to the present composite estimator

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Best estimator 4–8–4</th>
<th>Best estimator 8–in–then–out</th>
<th>Best estimator 6–in–then–out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>123</td>
<td>109</td>
<td>111</td>
</tr>
<tr>
<td>1–period change</td>
<td>114</td>
<td>125</td>
<td>115</td>
</tr>
<tr>
<td>2–period change</td>
<td>127</td>
<td>141</td>
<td>127</td>
</tr>
<tr>
<td>3–period change</td>
<td>139</td>
<td>141</td>
<td>137</td>
</tr>
<tr>
<td>4–period change</td>
<td>149</td>
<td>162</td>
<td>145</td>
</tr>
<tr>
<td>5–period change</td>
<td>149</td>
<td>159</td>
<td>142</td>
</tr>
<tr>
<td>6–period change</td>
<td>147</td>
<td>153</td>
<td>135</td>
</tr>
<tr>
<td>7–period change</td>
<td>145</td>
<td>144</td>
<td>128</td>
</tr>
<tr>
<td>8–period change</td>
<td>142</td>
<td>135</td>
<td>120</td>
</tr>
<tr>
<td>9–period change</td>
<td>136</td>
<td>122</td>
<td>109</td>
</tr>
<tr>
<td>10–period change</td>
<td>130</td>
<td>110</td>
<td>99</td>
</tr>
<tr>
<td>11–period change</td>
<td>123</td>
<td>100</td>
<td>91</td>
</tr>
<tr>
<td>12–period change</td>
<td>119</td>
<td>93</td>
<td>85</td>
</tr>
<tr>
<td>12–period average</td>
<td>107</td>
<td>82</td>
<td>93</td>
</tr>
<tr>
<td>Change in 12–period averages</td>
<td>126</td>
<td>66</td>
<td>63</td>
</tr>
</tbody>
</table>
easier to construct for the continuous designs than for an intermittent design such as the 4–8–4 rotation design.

For the rest of our discussion, we consider only the estimation of employed, unemployed, and Civilian Labor Force. As already mentioned, results for unemployment rate are virtually identical to those of unemployed and are therefore omitted from the comparison. Estimators include the present composite estimator, the first order composite estimator and the best estimator. The best estimator used in these comparisons is the best linear unbiased estimator of current level based on 36 periods. The efficiency of the 36—period estimator is virtually the same as that of the recursive regression estimator. The variances of the various estimators relative to the ratio estimator of the current level are given in Table 5.39 for employed, Table 5.40 for unemployed, and Table 5.41 for Civilian Labor Force. None of the estimators are adjusted for time—in—sample effects. Furthermore, previous estimates are not revised when more observations are obtained.

For all rotation schemes under consideration, there is some improvement in the precision of the estimators of current level from using the best estimator relative to the present composite estimator. As seen in Table 5.42, the gain is highest for employed where, under the 4–8–4 rotation scheme, the variance of the best estimator of current level is only 92% of that of the present composite estimator.

The relative precision of the best estimators of change relative to the present composite estimator depends on the rotation design. From Table 5.42, we see that under the 4–8–4 rotation scheme, there is some gain in precision, which increases as the interval of change increases. For employed, the variance of the best estimator is 85% of the variance of the present composite estimator in estimating one—period
change, 61% of the variance of the present composite estimator in estimating six-period change, and 96% of the variance of the present composite estimator in estimating twelve-period change.

For estimating twelve-period averages, the present composite estimator is about 13% less efficient than the best estimator and for estimating change in twelve-period averages, it is about 53% less efficient, as can be seen from the first column of Table 5.42. Similar results are obtained for Civilian Labor Force and these are shown in Tables 5.41 and 5.44. For unemployed, there are only modest gains in precision from using the best estimator relative to the present composite estimator, as shown in the first column of Tables 5.40 and 5.43.

It can be seen from Tables 5.42 and 5.44 that for both employed and Civilian Labor Force, there is some gain in precision for estimation of current level from using the best estimator instead of the present composite estimator. Again, the highest gains in precision are for employed (13% and 14%, respectively under the 8-in-then-out and 6-in-then-out scheme). There is no such gain in precision for unemployed (see first row of Table 5.43). The relative precision of the best estimators of change relative to the present composite estimator depends on the rotation scheme. For the 8-in-then-out scheme, for employed, we see from the second columns of Tables 5.39 and 5.42 that the gain in precision steadily increases as the interval of change increases, reaching a maximum value of 75% for four-period change and then declining to about 4% for eleven-period change.

The results for unemployed and Civilian Labor Force are similar but with the magnitudes of the gains reduced for Civilian Labor Force and further reduced for unemployed (see second column of Tables 5.43 and 5.44).
For the 6—in—then—out scheme and for all characteristics, the gain in precision for the best estimator of change relative to the present composite estimator increases as the interval of change increases, reaching a maximum of 54% for employed, 7% for unemployed, and 45% for Civilian Labor Force, for four—period change in each case. It then declines steadily. The parameters for which the best estimator of change becomes less efficient than the present composite estimator under the 6—in—then—out scheme are change beyond an interval of ten periods for employed and an interval of nine periods for unemployed and Civilian Labor Force (see the third columns of Tables 5.42, 5.43 and 5.44).

For estimation of twelve—period change, twelve—period average and change in twelve—period averages, the best estimators (where best is for current level) for both continuous rotation designs and for all characteristics are less efficient than the present composite estimator, as can be seen from the last three rows, second and third columns of Tables 5.42, 5.43, and 5.44. The greatest loss in precision as a result of using the best estimator instead of the present composite estimator is for employed where, under the 6—in—then—out scheme, the best composite estimator of change in twelve—period averages is 38% less efficient than the present composite estimator (see the last column of Table 5.42).

In Tables 5.45, 5.46, and 5.47, we compare the variances of the estimators of the form of the present composite estimator for employed, unemployed, and Civilian Labor Force, respectively. In each case, the first column contains estimated variances for the present composite estimator. The other columns contain estimated variances of the first order composite estimator under the various rotation designs. The first order composite estimator is constructed to give approximately optimum estimators of current level under the assumption of autoregressive errors.
The relative efficiencies of the alternative estimators relative to the present composite estimator are given in Table 5.42—5.44 and Tables 5.48—5.50 for all the characteristics and rotation designs under consideration.

Under the 4—8—4 rotation scheme, there are modest gains in the precision of estimation of current level from using the first order composite estimator instead of the present composite estimator. The gain is 9% for employed, 1% for unemployed, and 8% for Civilian Labor Force (see Tables 5.48, 5.49, and 5.50). However, for employed and Civilian Labor Force, the first order composite estimator of change under the 4—8—4 rotation scheme is clearly superior to the present composite estimator. For both characteristics, the maximum gain in precision of the first order composite estimator of change relative to the present composite estimator occurs in the estimation of four—period change (28% for employed and 23% for Civilian Labor Force). In estimating current level and change up to twelve periods under the 4—8—4 rotation scheme, the first order composite estimator has roughly the same efficiency as the present composite estimator (see the first column of Table 5.49).

The results for the continuous rotation schemes are similar. For employed and Civilian Labor Force, there are modest gains in precision from using the first order composite estimator instead of the present composite estimator (see second and third rows of Tables 5.48 and 5.50). For estimating change, the gain in precision of the first order composite estimator relative to the present composite estimator increases as the interval of estimation increases, reaching a maximum for four—period change for all characteristics and both rotation schemes. It then declines steadily. For employed, the maximum gain in precision of the first order composite estimator of change relative to the present composite estimator is 54% for the 8—in—then—out scheme and 38% for the 6—in—then—out scheme (see Table 5.48). For unemployed, the corresponding
Table 5.45. Variances of alternative composite estimators of employed; variance of the basic estimator of current level equals one

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Present composite estimator</th>
<th>First order composite estimator 4-in-8-out</th>
<th>First order composite estimator 8-in-then-out</th>
<th>First order composite estimator 6-in-then-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.862</td>
<td>0.790</td>
<td>0.826</td>
<td>0.800</td>
</tr>
<tr>
<td>1-period change</td>
<td>0.511</td>
<td>0.482</td>
<td>0.417</td>
<td>0.457</td>
</tr>
<tr>
<td>2-period change</td>
<td>0.813</td>
<td>0.706</td>
<td>0.605</td>
<td>0.666</td>
</tr>
<tr>
<td>3-period change</td>
<td>1.065</td>
<td>0.868</td>
<td>0.742</td>
<td>0.819</td>
</tr>
<tr>
<td>4-period change</td>
<td>1.279</td>
<td>1.003</td>
<td>0.832</td>
<td>0.929</td>
</tr>
<tr>
<td>5-period change</td>
<td>1.363</td>
<td>1.084</td>
<td>0.912</td>
<td>1.031</td>
</tr>
<tr>
<td>6-period change</td>
<td>1.390</td>
<td>1.127</td>
<td>0.989</td>
<td>1.133</td>
</tr>
<tr>
<td>7-period change</td>
<td>1.388</td>
<td>1.141</td>
<td>1.067</td>
<td>1.205</td>
</tr>
<tr>
<td>8-period change</td>
<td>1.353</td>
<td>1.128</td>
<td>1.147</td>
<td>1.254</td>
</tr>
<tr>
<td>9-period change</td>
<td>1.255</td>
<td>1.086</td>
<td>1.204</td>
<td>1.289</td>
</tr>
<tr>
<td>10-period change</td>
<td>1.154</td>
<td>1.042</td>
<td>1.246</td>
<td>1.313</td>
</tr>
<tr>
<td>11-period change</td>
<td>1.061</td>
<td>0.996</td>
<td>1.276</td>
<td>1.329</td>
</tr>
<tr>
<td>12-period change</td>
<td>0.992</td>
<td>0.951</td>
<td>1.298</td>
<td>1.341</td>
</tr>
<tr>
<td>12-period average</td>
<td>0.369</td>
<td>0.380</td>
<td>0.454</td>
<td>0.388</td>
</tr>
<tr>
<td>Change in 12-period averages</td>
<td>0.248</td>
<td>0.263</td>
<td>0.473</td>
<td>0.450</td>
</tr>
</tbody>
</table>
Table 5.46. Variances of alternative composite estimators of unemployed; variance of the basic estimator of current level equals one

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Present composite estimator</th>
<th>First order composite estimator 4–8–4</th>
<th>First order composite estimator 8–in–then–out</th>
<th>First order composite estimator 6–in–then–out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.947</td>
<td>0.942</td>
<td>0.961</td>
<td>0.952</td>
</tr>
<tr>
<td>1–period change</td>
<td>1.070</td>
<td>1.090</td>
<td>1.011</td>
<td>1.060</td>
</tr>
<tr>
<td>2–period change</td>
<td>1.361</td>
<td>1.370</td>
<td>1.267</td>
<td>1.331</td>
</tr>
<tr>
<td>3–period change</td>
<td>1.528</td>
<td>1.526</td>
<td>1.399</td>
<td>1.474</td>
</tr>
<tr>
<td>4–period change</td>
<td>1.645</td>
<td>1.643</td>
<td>1.510</td>
<td>1.588</td>
</tr>
<tr>
<td>5–period change</td>
<td>1.691</td>
<td>1.687</td>
<td>1.581</td>
<td>1.666</td>
</tr>
<tr>
<td>6–period change</td>
<td>1.708</td>
<td>1.702</td>
<td>1.635</td>
<td>1.729</td>
</tr>
<tr>
<td>7–period change</td>
<td>1.710</td>
<td>1.703</td>
<td>1.679</td>
<td>1.755</td>
</tr>
<tr>
<td>8–period change</td>
<td>1.701</td>
<td>1.694</td>
<td>1.720</td>
<td>1.766</td>
</tr>
<tr>
<td>9–period change</td>
<td>1.671</td>
<td>1.662</td>
<td>1.738</td>
<td>1.770</td>
</tr>
<tr>
<td>10–period change</td>
<td>1.641</td>
<td>1.633</td>
<td>1.746</td>
<td>1.772</td>
</tr>
<tr>
<td>11–period change</td>
<td>1.614</td>
<td>1.606</td>
<td>1.749</td>
<td>1.773</td>
</tr>
<tr>
<td>12–period change</td>
<td>1.593</td>
<td>1.580</td>
<td>1.751</td>
<td>1.774</td>
</tr>
<tr>
<td>12–period average</td>
<td>0.255</td>
<td>0.250</td>
<td>0.301</td>
<td>0.261</td>
</tr>
<tr>
<td>Change in 12–period averages</td>
<td>0.273</td>
<td>0.264</td>
<td>0.388</td>
<td>0.363</td>
</tr>
</tbody>
</table>
Table 5.47. Variances of alternative composite estimators of Civilian Labor Force; variance of the basic estimator of current level equals one

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>Present composite estimator</th>
<th>First order composite estimator 4-in-4</th>
<th>First order composite estimator 8-in-then-out</th>
<th>First order composite estimator 6-in-then-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.868</td>
<td>0.806</td>
<td>0.844</td>
<td>0.818</td>
</tr>
<tr>
<td>1-period change</td>
<td>0.538</td>
<td>0.516</td>
<td>0.449</td>
<td>0.491</td>
</tr>
<tr>
<td>2-period change</td>
<td>0.828</td>
<td>0.737</td>
<td>0.628</td>
<td>0.693</td>
</tr>
<tr>
<td>3-period change</td>
<td>1.076</td>
<td>0.904</td>
<td>0.769</td>
<td>0.851</td>
</tr>
<tr>
<td>4-period change</td>
<td>1.283</td>
<td>1.042</td>
<td>0.875</td>
<td>0.973</td>
</tr>
<tr>
<td>5-period change</td>
<td>1.364</td>
<td>1.124</td>
<td>0.964</td>
<td>1.080</td>
</tr>
<tr>
<td>6-period change</td>
<td>1.392</td>
<td>1.167</td>
<td>1.045</td>
<td>1.181</td>
</tr>
<tr>
<td>7-period change</td>
<td>1.391</td>
<td>1.181</td>
<td>1.121</td>
<td>1.249</td>
</tr>
<tr>
<td>8-period change</td>
<td>1.362</td>
<td>1.170</td>
<td>1.197</td>
<td>1.296</td>
</tr>
<tr>
<td>9-period change</td>
<td>1.277</td>
<td>1.130</td>
<td>1.250</td>
<td>1.327</td>
</tr>
<tr>
<td>10-period change</td>
<td>1.191</td>
<td>1.090</td>
<td>1.288</td>
<td>1.348</td>
</tr>
<tr>
<td>11-period change</td>
<td>1.111</td>
<td>1.048</td>
<td>1.314</td>
<td>1.363</td>
</tr>
<tr>
<td>12-period change</td>
<td>1.052</td>
<td>1.007</td>
<td>1.333</td>
<td>1.372</td>
</tr>
<tr>
<td>12-period average</td>
<td>0.369</td>
<td>0.378</td>
<td>0.454</td>
<td>0.388</td>
</tr>
<tr>
<td>Change in 12-period averages</td>
<td>0.259</td>
<td>0.270</td>
<td>0.473</td>
<td>0.449</td>
</tr>
</tbody>
</table>
Table 5.48. Relative efficiencies of composite estimators of employed relative to the present composite estimator

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>First-order composite estimator</th>
<th>First-order composite estimator</th>
<th>First-order composite estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4–8–4</td>
<td>8—in–then–out</td>
<td>6—in–then–out</td>
</tr>
<tr>
<td>Current level</td>
<td>109</td>
<td>104</td>
<td>108</td>
</tr>
<tr>
<td>1–period change</td>
<td>106</td>
<td>123</td>
<td>112</td>
</tr>
<tr>
<td>2–period change</td>
<td>115</td>
<td>134</td>
<td>122</td>
</tr>
<tr>
<td>3–period change</td>
<td>123</td>
<td>144</td>
<td>130</td>
</tr>
<tr>
<td>4–period change</td>
<td>128</td>
<td>154</td>
<td>138</td>
</tr>
<tr>
<td>5–period change</td>
<td>126</td>
<td>150</td>
<td>132</td>
</tr>
<tr>
<td>6–period change</td>
<td>123</td>
<td>141</td>
<td>123</td>
</tr>
<tr>
<td>7–period change</td>
<td>122</td>
<td>130</td>
<td>115</td>
</tr>
<tr>
<td>8–period change</td>
<td>120</td>
<td>118</td>
<td>108</td>
</tr>
<tr>
<td>9–period change</td>
<td>116</td>
<td>104</td>
<td>97</td>
</tr>
<tr>
<td>10–period change</td>
<td>111</td>
<td>93</td>
<td>88</td>
</tr>
<tr>
<td>11–period change</td>
<td>107</td>
<td>83</td>
<td>80</td>
</tr>
<tr>
<td>12–period change</td>
<td>104</td>
<td>76</td>
<td>74</td>
</tr>
<tr>
<td>12–period average</td>
<td>103</td>
<td>81</td>
<td>95</td>
</tr>
<tr>
<td>Change in 12–period averages</td>
<td>94</td>
<td>52</td>
<td>55</td>
</tr>
<tr>
<td>Quantity estimated</td>
<td>First-order composite estimator 4–8–4</td>
<td>First-order composite estimator 8-in-then-out</td>
<td>First-order composite estimator 6-in-then-out</td>
</tr>
<tr>
<td>--------------------</td>
<td>--------------------------------------</td>
<td>-----------------------------------------------</td>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>Current level</td>
<td>101</td>
<td>99</td>
<td>100</td>
</tr>
<tr>
<td>1–period change</td>
<td>98</td>
<td>106</td>
<td>101</td>
</tr>
<tr>
<td>2–period change</td>
<td>99</td>
<td>107</td>
<td>102</td>
</tr>
<tr>
<td>3–period change</td>
<td>100</td>
<td>109</td>
<td>104</td>
</tr>
<tr>
<td>4–period change</td>
<td>100</td>
<td>109</td>
<td>104</td>
</tr>
<tr>
<td>5–period change</td>
<td>100</td>
<td>107</td>
<td>102</td>
</tr>
<tr>
<td>6–period change</td>
<td>100</td>
<td>105</td>
<td>99</td>
</tr>
<tr>
<td>7–period change</td>
<td>100</td>
<td>102</td>
<td>97</td>
</tr>
<tr>
<td>8–period change</td>
<td>100</td>
<td>99</td>
<td>96</td>
</tr>
<tr>
<td>9–period change</td>
<td>101</td>
<td>96</td>
<td>94</td>
</tr>
<tr>
<td>10–period change</td>
<td>101</td>
<td>94</td>
<td>93</td>
</tr>
<tr>
<td>11–period change</td>
<td>101</td>
<td>92</td>
<td>91</td>
</tr>
<tr>
<td>12–period change</td>
<td>101</td>
<td>91</td>
<td>90</td>
</tr>
<tr>
<td>12–period average</td>
<td>102</td>
<td>85</td>
<td>102</td>
</tr>
<tr>
<td>Change in 12–period averages</td>
<td>103</td>
<td>70</td>
<td>75</td>
</tr>
</tbody>
</table>
Table 5.50. Relative efficiencies of alternative composite estimators of Civilian Labor Force relative to the present composite estimator

<table>
<thead>
<tr>
<th>Quantity estimated</th>
<th>First-order composite estimator 4-8-4</th>
<th>First-order composite estimator 8-in-then-out</th>
<th>First-order composite estimator 6-in-then-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>108</td>
<td>103</td>
<td>106</td>
</tr>
<tr>
<td>1-period change</td>
<td>104</td>
<td>120</td>
<td>110</td>
</tr>
<tr>
<td>2-period change</td>
<td>112</td>
<td>132</td>
<td>120</td>
</tr>
<tr>
<td>3-period change</td>
<td>119</td>
<td>140</td>
<td>126</td>
</tr>
<tr>
<td>4-period change</td>
<td>123</td>
<td>147</td>
<td>132</td>
</tr>
<tr>
<td>5-period change</td>
<td>121</td>
<td>142</td>
<td>126</td>
</tr>
<tr>
<td>6-period change</td>
<td>119</td>
<td>133</td>
<td>118</td>
</tr>
<tr>
<td>7-period change</td>
<td>118</td>
<td>124</td>
<td>111</td>
</tr>
<tr>
<td>8-period change</td>
<td>116</td>
<td>114</td>
<td>105</td>
</tr>
<tr>
<td>9-period change</td>
<td>113</td>
<td>102</td>
<td>96</td>
</tr>
<tr>
<td>10-period change</td>
<td>109</td>
<td>93</td>
<td>88</td>
</tr>
<tr>
<td>11-period change</td>
<td>106</td>
<td>85</td>
<td>82</td>
</tr>
<tr>
<td>12-period change</td>
<td>105</td>
<td>79</td>
<td>77</td>
</tr>
<tr>
<td>12-period average</td>
<td>98</td>
<td>81</td>
<td>95</td>
</tr>
<tr>
<td>Change in 12-period averages</td>
<td>96</td>
<td>55</td>
<td>58</td>
</tr>
</tbody>
</table>
figures are 9% and 4% (see Table 5.49), and for Civilian Labor Force, they are 47% and 32% (see Table 5.50).

For estimating twelve-period averages and change in twelve-period averages, the first order composite estimator is less efficient than the present composite estimator for all characteristics and all rotation designs except for twelve-period average for employed under the 4–8–4 rotation scheme and both twelve-period averages and change in twelve-period averages for unemployed under the 4–8–4 rotation scheme. In these instances, there are modest gains in efficiency from using the first order composite estimator rather than the present composite estimator.

Finally, we discuss the effects of time-in-sample effects on the variances of the alternative estimators of current level and change, using the best linear unbiased estimator based on 24 periods and the recursive regression estimator for illustration. As mentioned in Section 5.4, in the presence of time-in-sample effects, the alternative estimators of current level and change are biased relative to the mean of the basic estimator. Under the added restriction that the sum of the time-in-sample effects is zero, the variances of all estimators of current level and change are expected to be greater than those obtained under the assumption of no time-in-sample effects. The loss in efficiency due to estimating time-in-sample effects is a function of the estimability restriction and of the length of period used to estimate the time-in-sample effects.

The variances of the 24-period best linear unbiased estimator of current level and change over several periods in the presence of time-in-sample effects are presented in Table 5.51 for employed, unemployed, and Civilian Labor Force.

It can be seen from Tables 5.51 and 5.54 that in estimating current level, there is an increase in variance of about 10% for employed and about 8% for Civilian Labor
Force as a result of incorporating time-in-sample effects. For estimation of change, the increase in variance rises monotonically as the interval of change increases for both characteristics. For employed, the increase in variance is about 4% for one-period change, about 10% for six-period change and about 20% for twelve-period change. The corresponding values for Civilian Labor Force are 4%, 8%, and 14%. For unemployed, there is virtually no increase in the variances of the alternative estimators of current level and change due to time-in-sample effects (see second column of Table 5.4.9).

The variances of the recursive regression estimator of current level and change for all characteristics, in the presence of time-in-sample effects are presented in Table 5.52 (for the case of revision of previous estimates) and Table 5.53 (for the case of nonrevision of previous estimates. When previous estimates are revised, there is virtually no increase in the variances of the recursive regression estimator in estimating change and only a modest increase in the variance in estimating current level (compare Table 5.52 with the last columns of Tables 5.21, 5.22, and 5.23). When previous estimates are not revised, there are modest increases in the variance of the recursive regression estimator of change due to time-in-sample effects for employed and Civilian Labor Force, but no increase in variance due to time-in-sample effects for unemployed. When previous estimates are revised, there is virtually no increase in the variances of the recursive regression estimator in estimating change and only a modest increase in the variance in estimating current level (compare Table 5.52 with the last columns of Tables 5.21, 5.22, and 5.23). When previous estimates are not revised, there are modest increases in the variance of the recursive regression estimator of change due to time-in-sample effects for employed and Civilian Labor Force, but no increase in variance due to time-in-sample effects for unemployed.
Table 5.51. Variances of the best linear unbiased estimator based on 24 periods in the presence of time-in-sample effects; variance of the basic estimator of current level equals one; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Characteristic</th>
<th>Employed</th>
<th>Unemployed</th>
<th>Civilian Labor Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td></td>
<td>0.729</td>
<td>0.928</td>
<td>0.763</td>
</tr>
<tr>
<td>1—period change</td>
<td></td>
<td>0.449</td>
<td>1.089</td>
<td>0.490</td>
</tr>
<tr>
<td>2—period change</td>
<td></td>
<td>0.635</td>
<td>1.348</td>
<td>0.682</td>
</tr>
<tr>
<td>3—period change</td>
<td></td>
<td>0.754</td>
<td>1.487</td>
<td>0.816</td>
</tr>
<tr>
<td>4—period change</td>
<td></td>
<td>0.839</td>
<td>1.578</td>
<td>0.911</td>
</tr>
<tr>
<td>5—period change</td>
<td></td>
<td>0.895</td>
<td>1.623</td>
<td>0.974</td>
</tr>
<tr>
<td>6—period change</td>
<td></td>
<td>0.931</td>
<td>1.646</td>
<td>1.015</td>
</tr>
<tr>
<td>7—period change</td>
<td></td>
<td>0.952</td>
<td>1.656</td>
<td>1.039</td>
</tr>
<tr>
<td>8—period change</td>
<td></td>
<td>0.958</td>
<td>1.655</td>
<td>1.047</td>
</tr>
<tr>
<td>9—period change</td>
<td></td>
<td>0.943</td>
<td>1.635</td>
<td>1.035</td>
</tr>
<tr>
<td>10—period change</td>
<td></td>
<td>0.930</td>
<td>1.617</td>
<td>1.023</td>
</tr>
<tr>
<td>11—period change</td>
<td></td>
<td>0.920</td>
<td>1.602</td>
<td>1.014</td>
</tr>
<tr>
<td>12—period change</td>
<td></td>
<td>0.914</td>
<td>1.589</td>
<td>1.009</td>
</tr>
</tbody>
</table>

1The estimator is constructed so that the expected value of the estimator equals the expected value of the average of the eight elementary estimators.
Table 5.52. Variances of the recursive regression estimator in the presence of time—in—sample effects; variance of the basic estimator of current level equals one; 1987 correlation pattern and revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Characteristic</th>
<th>Employed</th>
<th>Unemployed</th>
<th>Civilian Labor Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td></td>
<td>0.688</td>
<td>0.923</td>
<td>0.733</td>
</tr>
<tr>
<td>1—period change</td>
<td></td>
<td>0.398</td>
<td>1.042</td>
<td>0.436</td>
</tr>
<tr>
<td>2—period change</td>
<td></td>
<td>0.555</td>
<td>1.296</td>
<td>0.595</td>
</tr>
<tr>
<td>3—period change</td>
<td></td>
<td>0.656</td>
<td>1.427</td>
<td>0.708</td>
</tr>
<tr>
<td>4—period change</td>
<td></td>
<td>0.727</td>
<td>1.516</td>
<td>0.791</td>
</tr>
<tr>
<td>5—period change</td>
<td></td>
<td>0.771</td>
<td>1.559</td>
<td>0.845</td>
</tr>
<tr>
<td>6—period change</td>
<td></td>
<td>0.797</td>
<td>1.581</td>
<td>0.877</td>
</tr>
<tr>
<td>7—period change</td>
<td></td>
<td>0.806</td>
<td>1.590</td>
<td>0.892</td>
</tr>
<tr>
<td>8—period change</td>
<td></td>
<td>0.800</td>
<td>1.587</td>
<td>0.892</td>
</tr>
<tr>
<td>9—period change</td>
<td></td>
<td>0.766</td>
<td>1.565</td>
<td>0.866</td>
</tr>
<tr>
<td>10—period change</td>
<td></td>
<td>0.735</td>
<td>1.544</td>
<td>0.841</td>
</tr>
<tr>
<td>11—period change</td>
<td></td>
<td>0.706</td>
<td>1.526</td>
<td>0.819</td>
</tr>
<tr>
<td>12—period change</td>
<td></td>
<td>0.683</td>
<td>1.511</td>
<td>0.801</td>
</tr>
</tbody>
</table>

The estimator is constructed so that the expected value of the estimator equals the expected value of the average of the eight elementary estimators.
Table 5.53. Variances of the recursive regression estimator in the presence of time-in-sample effects; variance of the basic estimator of current level equals one; 1987 correlation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Employed</th>
<th>Unemployed</th>
<th>Civilian Labor Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current level</td>
<td>0.688</td>
<td>0.923</td>
<td>0.733</td>
</tr>
<tr>
<td>1-period change</td>
<td>0.438</td>
<td>1.075</td>
<td>0.480</td>
</tr>
<tr>
<td>2-period change</td>
<td>0.614</td>
<td>1.342</td>
<td>0.663</td>
</tr>
<tr>
<td>3-period change</td>
<td>0.725</td>
<td>1.479</td>
<td>0.789</td>
</tr>
<tr>
<td>4-period change</td>
<td>0.801</td>
<td>1.569</td>
<td>0.877</td>
</tr>
<tr>
<td>5-period change</td>
<td>0.849</td>
<td>1.613</td>
<td>0.934</td>
</tr>
<tr>
<td>6-period change</td>
<td>0.878</td>
<td>1.635</td>
<td>0.970</td>
</tr>
<tr>
<td>7-period change</td>
<td>0.891</td>
<td>1.644</td>
<td>0.987</td>
</tr>
<tr>
<td>8-period change</td>
<td>0.889</td>
<td>1.642</td>
<td>0.990</td>
</tr>
<tr>
<td>9-period change</td>
<td>0.864</td>
<td>1.622</td>
<td>0.972</td>
</tr>
<tr>
<td>10-period change</td>
<td>0.842</td>
<td>1.603</td>
<td>0.954</td>
</tr>
<tr>
<td>11-period change</td>
<td>0.822</td>
<td>1.587</td>
<td>0.939</td>
</tr>
<tr>
<td>12-period change</td>
<td>0.807</td>
<td>1.573</td>
<td>0.926</td>
</tr>
</tbody>
</table>

*The estimator is constructed so that the expected value of the estimator equals the expected value of the average of the eight elementary estimators.*
Table 5.54. Percentage increase in variances of alternative estimators due to estimation of time-in-sample effects; 1987 rotation pattern and no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE for 24 periods</th>
<th>Recursive regression estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Employed</td>
<td>Unemployed</td>
</tr>
<tr>
<td>Current level</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>1—period change</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>2—period change</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>3—period change</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>4—period change</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>5—period change</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>6—period change</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>7—period change</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>8—period change</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>9—period change</td>
<td>14</td>
<td>1</td>
</tr>
<tr>
<td>10—period change</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>11—period change</td>
<td>19</td>
<td>2</td>
</tr>
<tr>
<td>12—period change</td>
<td>22</td>
<td>2</td>
</tr>
</tbody>
</table>
5.7.6. Conclusions

The main conclusions emerging from the variance computations in Section 5.7 can be summarized as follows:

1. For all rotation designs and all characteristics under consideration, there are alternative estimation procedures with a variance of the current level smaller than that of the present composite estimator.

2. For estimation of change under the 4—8—4 rotation design, the gain in precision of the alternative estimators relative to the present composite estimator increases as the interval of change increases. For the continuous rotation schemes, there is a gain in precision up to about 7 periods (for the 8—in—then—out scheme) and to about 5 periods (for the 6—in—then—out scheme) for all characteristics.

3. For estimation of change under all rotation designs, a modest gain in efficiency is possible if one is willing to revise previous estimates. This is due to the fact that revised estimates are based on more observations than unrevised estimates.

4. There is no appreciable gain in the precision of the alternative estimators as the number of periods of data used in the estimation is increased beyond 24 periods. The lack of appreciable gain in precision of the recursive regression estimator relative to the 24—period best linear unbiased estimator can be attributed to two factors: the low correlations after the first few times—in—sample and the fact that the recursive regression estimation procedure converges very quickly.
5. The best estimator for the 8-in-then-out rotation design is superior to the present composite estimator for current level and for changes up to about one year for both employed and Civilian Labor Force. For unemployed, the superiority of the best estimator relative to the present composite estimator is for current level and change up to nine periods.

6. For the continuous rotation designs, the variances of the first order composite estimators of current level and change are very similar to the variances of the best estimator for all characteristics.

7. For all characteristics, the variances of estimators of twelve-period averages and change in twelve-period averages are smaller for the present rotation-estimation procedure than for the best estimators constructed under the continuous rotation designs.

8. The first order composite estimator combined with the 8-in-then-out rotation design gives efficiencies very close to those of the best estimator for the same rotation design.

9. Generally, the 8-in-then-out rotation design provides superior estimates of changes up to six periods, but inferior estimates of current level, relative to the 4-8-4 rotation design.

10. For both employed and Civilian Labor Force, there is some increase in the variances of the alternative estimators of current level and change when time-in-sample effects with zero sum are included in the model. The increase in the variance of the estimator of change increases as the interval of change increases. There is virtually no increase in the variances of the alternative estimators of change due to time-in-sample effects, for unemployed.
BIBLIOGRAPHY


Cochran, W. G. (1942), Sampling theory when the sampling units are of unequal sizes. *Journal of the American Statistical Association* 37, 199–212.


ACKNOWLEDGEMENTS

The research for this thesis was directed by my major professor, Dr. Wayne A. Fuller. He contributed immensely to every aspect of the research, from technical content to exposition. It is difficult to imagine the completion of this thesis without his encouragement, meticulous guidance, experience and insight into the subject matter of the research. For this, and the many other ways in which he has been so helpful to me, I wish to hereby record my profound gratitude to him.

I wish to thank the other members of my committee: Dr. D. A. Harville, Dr. K. J. Koehler, Dr. F. Jay Breidt, and Dr. G. Lieberman, for reading the original draft of the thesis and making many useful suggestions. I am grateful to Dr. D. L. Isaacson, Dr. Roy Hickman, Dr. S. M. Nusser, Ms. Helen Nelson, and the entire faculty and staff of the department, particularly those in the Survey Section of the Statistical Laboratory, for creating an academic environment conducive to study and research.

Many of my fellow graduate students not only contributed to some aspects of my research, but helped convert the potentially difficult and frustrating years of graduate study into a very enjoyable and memorable period of my life. My numerous discussions with Abdoulaye Adam on areas of mutual research interest enhanced my understanding of the subject matter. Rohit Deo carefully read portions of Chapter 4 and made some valuable suggestions. Philip Iversen and Kevin Dodd helped me with some of the SAS programming that produced the results in Chapter 5. Joe Croos generously shared his knowledge of computers with me.

I now wish to say a special word of thanks and appreciation to my wife Kadiatu, for providing an incredible amount of love, understanding and moral support throughout my graduate program, as she has done in many other areas of my life. Her
unwavering devotion to our family made it easy for me to concentrate on my studies. I am also grateful to our children Halima, Maimounatu, Bunturabi, Ahmed and Aisha for sharing both the joys and frustrations of graduate study with me. They perhaps made more sacrifices than I, towards the successful completion of my graduate program.

Judy Shafer did most of the typing for both the original and final drafts of the thesis. I appreciate her expertise and patience under the pressure arising from Graduate College deadlines.

I would like to thank the USAID/AAI/AFGRAD program for sponsoring my graduate studies at Iowa State University. The research for this thesis was partly supported by joint statistical agreement with the United States Bureau of the Census.
GLOSSARY OF TERMS USED IN THE THESIS

**Rotation Group**: A set of households which comprises \( \frac{1}{8} \) of the sample in any given time or period of the survey. For the Current Population Survey, the unit of time is month.

**Time—in—sample**: The number of times a particular rotation group has been interviewed.

**Panel**: Synonymous with rotation group.

**Stream**: A sequence of observations created by a sequence of rotation groups, where all observations on a rotation group are in a single stream.
In this appendix, we define the transformations of the 8 observations introduced into the sample at the current period and present their means and variances. These transformed observations are used in the recursive regression estimation procedure. Recall that $y_{t,0,k}$ denotes the elementary estimate of the parameter of interest based on a rotation group which is in its $k$-th time-in-sample at time $t$. Let $\rho_j$ denote the estimated autocorrelations of lag $j$ for a rotation group. The estimated autocorrelations are given in Table 5.2(a) for $j = 1, 2, \ldots, 16$. Let $\text{Var}(y_{t,0,k}) = \sigma^2$ for all $t$ and $k$.

\[
\begin{align*}
z_{t+1,1} &= y_{t+1,0,1}, \\
z_{t+1,2} &= y_{t+1,0,2} - \alpha_{11}y_{t,0,1}, \\
z_{t+1,3} &= y_{t+1,0,3} - \alpha_{21}y_{t,0,2} - \alpha_{22}y_{t-1,0,1}, \\
z_{t+1,4} &= y_{t+1,0,4} - \alpha_{31}y_{t,0,3} - \alpha_{32}y_{t-1,0,2} - \alpha_{33}y_{t-2,0,1}, \\
z_{t+1,5} &= y_{t+1,0,5} - \alpha_{41}y_{t-8,0,4} - \alpha_{42}y_{t-9,0,3} - \alpha_{43}y_{t-10,0,2} - \alpha_{44}y_{t-11,0,1},
\end{align*}
\]
\( z_{t+1,6} = y_{t+1,0,6} - \alpha_{51} y_{t,0,5} - \alpha_{52} y_{t-1,0,4} - \alpha_{53} y_{t-10,0,3} - \alpha_{54} y_{t-11,0,2} - \alpha_{55} y_{t-12,0,1} \),

\( z_{t+1,7} = y_{t+1,0,7} - \alpha_{61} y_{t,0,6} - \alpha_{62} y_{t-1,0,5} - \alpha_{63} y_{t-10,0,4} - \alpha_{64} y_{t-11,0,3} - \alpha_{65} y_{t-12,0,2} - \alpha_{66} y_{t-13,0,1} \),

\( z_{t+1,8} = y_{t+1,0,8} - \alpha_{71} y_{t,0,7} - \alpha_{72} y_{t-1,0,6} - \alpha_{73} y_{t-10,0,5} - \alpha_{74} y_{t-11,0,4} - \alpha_{75} y_{t-12,0,3} - \alpha_{76} y_{t-13,0,2} - \alpha_{77} y_{t-14,0,1} \).

For computational convenience, let us assume that there are no replicate effects. The \( \alpha \)'s are constructed so that \( z_{t+1,k} \) is uncorrelated with \( y_{t+1-j,0,k} \) for all \( j > 0 \).

Thus,

\( \alpha_{11} = \rho_1 \)

\[
\alpha_2 = \begin{bmatrix}
\alpha_{21} \\
\alpha_{22}
\end{bmatrix} = \begin{bmatrix}
1 & \rho_1 \\
\rho_1 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
\rho_1 \\
\rho_2
\end{bmatrix} = \Sigma_{22}^{-1} \begin{bmatrix}
\rho_1 \\
\rho_2
\end{bmatrix}
\]

\[
\alpha_3 = \begin{bmatrix}
\alpha_{31} \\
\alpha_{32} \\
\alpha_{33}
\end{bmatrix} = \begin{bmatrix}
1 & \rho_1 & \rho_2 \\
\rho_1 & 1 & \rho_1 \\
\rho_2 & \rho_1 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{bmatrix} = \Sigma_{33}^{-1} \begin{bmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{bmatrix}
\]
\[ \alpha_4 = \begin{bmatrix} \alpha_{41} \\ \alpha_{42} \\ \alpha_{43} \\ \alpha_{44} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 \\ \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho_9 \\ \rho_{10} \\ \rho_{11} \\ \rho_{12} \end{bmatrix} = \Sigma_{44}^{-1} \Sigma_{14} \]

\[ \alpha_5 = \begin{bmatrix} \alpha_{51} \\ \alpha_{52} \\ \alpha_{53} \\ \alpha_{54} \\ \alpha_{55} \end{bmatrix} = \begin{bmatrix} 1 & \rho_9 & \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_9 & 1 & \rho_1 & \rho_2 & \rho_3 \\ \rho_{10} & \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_{11} & \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_{12} & \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho_1 \\ \rho_{10} \\ \rho_{11} \\ \rho_{12} \\ \rho_{13} \end{bmatrix} = \Sigma_{55}^{-1} \Sigma_{15} \]

\[ \alpha_6 = \begin{bmatrix} \alpha_{61} \\ \alpha_{62} \\ \alpha_{63} \\ \alpha_{64} \\ \alpha_{65} \\ \alpha_{66} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_1 & 1 & \rho_9 & \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{10} & \rho_9 & 1 & \rho_1 & \rho_2 & \rho_3 \\ \rho_{11} & \rho_{10} & \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_{12} & \rho_{11} & \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_{13} & \rho_{12} & \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho_1 \\ \rho_{10} \\ \rho_{11} \\ \rho_{12} \\ \rho_{13} \\ \rho_{14} \end{bmatrix} = \Sigma_{66}^{-1} \Sigma_{16} \]

and

\[ \alpha_7 = \begin{bmatrix} \alpha_{71} \\ \alpha_{72} \\ \alpha_{73} \\ \alpha_{74} \\ \alpha_{75} \\ \alpha_{76} \\ \alpha_{77} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_1 & 1 & \rho_1 & \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_2 & \rho_1 & 1 & \rho_9 & \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{11} & \rho_{10} & \rho_9 & 1 & \rho_1 & \rho_2 & \rho_3 \\ \rho_{12} & \rho_{11} & \rho_{10} & \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_{13} & \rho_{12} & \rho_{11} & \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_{14} & \rho_{13} & \rho_{12} & \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_{12} \\ \rho_{13} \\ \rho_{14} \\ \rho_{15} \end{bmatrix} = \Sigma_{77}^{-1} \Sigma_{17} \]
Furthermore,

(i) \[ E\{z_{t+1,1}\} = \theta_{t+1} \quad \text{and} \quad \text{Var}\{z_{t+1,1}\} = \sigma_1^2 = \sigma^2 \]

(ii) \[ E\{z_{t+1,2}\} = \theta_{t+1} - \alpha_{11} \theta_t \quad \text{and} \quad \text{Var}\{z_{t+1,2}\} = \sigma_2^2 = (1 - \alpha_{11}^2)\sigma^2 \]

(iii) \[ E\{z_{t+1,3}\} = \theta_{t+1} - \alpha_{21} \theta_t - \alpha_{22} \theta_{t-1} \]

\[ \text{Var}\{z_{t+1,3}\} = \sigma_3^2 = (1 - \alpha_{22}^2) \Sigma_{33}(1, -\alpha_2^2)^r = 1 - \alpha_2^2 \Sigma_{12} \]

(iv) \[ E\{z_{t+1,4}\} = \theta_{t+1} - \alpha_{31} \theta_t - \alpha_{32} \theta_{t-1} - \alpha_{33} \theta_{t-2} \]

\[ \text{Var}\{z_{t+1,4}\} = \sigma_4^2 = (1, -\alpha_3^2) \Sigma_{44}(1, -\alpha_3^2)^r = 1 - \alpha_3^2 \Sigma_{13} \]

(v) \[ E\{z_{t+1,5}\} = \theta_{t+1} - \alpha_{41} \theta_t - \alpha_{42} \theta_{t-9} - \alpha_{43} \theta_{t-10} - \alpha_{44} \theta_{t-11} \]

\[ \text{Var}\{z_{t+1,5}\} = \sigma_5^2 = (1, -\alpha_4^2) \Sigma_{55}(1, -\alpha_4^2)^r = 1 - \alpha_4^2 \Sigma_{14} \]

(vi) \[ E\{z_{t+1,6}\} = \theta_{t+1} - \alpha_{51} \theta_t - \alpha_{52} \theta_{t-9} - \alpha_{53} \theta_{t-10} - \alpha_{54} \theta_{t-11} - \alpha_{55} \theta_{t-12} \]

\[ \text{Var}\{z_{t+1,6}\} = \sigma_6^2 = (1, -\alpha_5^2) \Sigma_{66}(1, -\alpha_5^2)^r = 1 - \alpha_5^2 \Sigma_{15} \]
(vii) \[ E\{z_{t+1,7}\} = \theta_{t+1} - \alpha_6 \theta_t - \alpha_7 \theta_{t-1} - \alpha_8 \theta_{t-10} - \alpha_9 \theta_{t-11} \]

\[ - \alpha_{10} \theta_{t-12} - \alpha_{11} \theta_{t-13} \]

\[ \text{Var}\{z_{t+1,7}\} = \sigma^2 T = (1, -\alpha_6) \Sigma_{T_7}(1, -\alpha_6)' = 1 - \alpha_6 \Sigma_{16} \]

(viii) \[ E\{z_{t+1,8}\} = \theta_{t+1} - \alpha_{11} \theta_t - \alpha_{12} \theta_{t-1} - \alpha_{13} \theta_{t-2} - \alpha_{14} \theta_{t-11} \]

\[ - \alpha_{15} \theta_{t-12} - \alpha_{16} \theta_{t-13} - \alpha_{17} \theta_{t-14} \]

\[ \text{Var}\{z_{t+1,8}\} = \sigma^2 8 = (1, -\alpha_7) \Sigma_{88}(1, -\alpha_7)' = 1 - \alpha_7 \Sigma_{17} \]

where

\[
\Sigma_{88} = \begin{bmatrix}
1 & \rho_1 & \rho_2 & \rho_3 & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\
\rho_1 & 1 & \rho_1 & \rho_2 & \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_2 & \rho_1 & 1 & \rho_1 & \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\
\rho_3 & \rho_{12} & \rho_{11} & 1 & \rho_9 & \rho_{10} & \rho_{11} & \rho_{12} \\
\rho_{12} & \rho_{11} & \rho_{10} & 1 & \rho_9 & 1 & \rho_2 & \rho_3 \\
\rho_{13} & \rho_{12} & \rho_{11} & 1 & \rho_1 & 1 & \rho_1 & \rho_2 \\
\rho_{14} & \rho_{13} & \rho_{12} & \rho_{11} & \rho_2 & \rho_1 & 1 & \rho_1 \\
\rho_{15} & \rho_{14} & \rho_{13} & \rho_{12} & \rho_3 & \rho_2 & \rho_1 & 1
\end{bmatrix}
\]

Therefore, the covariance matrix of the vector

\[ z_{t+1} = (z_{t+1,1}, z_{t+1,2}, z_{t+1,3}, z_{t+1,4}, z_{t+1,5}, z_{t+1,6}, z_{t+1,7}, z_{t+1,8})' \]
is

$$\text{Var}\{z_{t+1}\} = Q_{00} = \text{Blockdiag}\{\sigma^2_1, \sigma^2_2, \sigma^2_3, \sigma^2_4, \sigma^2_5, \sigma^2_6, \sigma^2_7, \sigma^2_8\}.$$ 

In the presence of replicate effects, it is easy to see that the covariance matrix of $z_{t+1}$ is

$$\text{Var}\{z_{t+1}\} = Q_{00} + \sigma^2_u \psi \psi',$$

where $\psi$ is defined in (5.3.24).
APPENDIX C

THE RECURSIVE REGRESSION PROCEDURE FOR COMPUTING LIMITING VARIANCES OF THE ESTIMATORS OF CURRENT LEVEL AND CHANGE

In this section, we describe a recursive scheme for computing the limiting variances of the estimates of current level and change from the recursive regression procedure. We shall restrict attention to the 4—8—4 rotation design but the procedure is applicable to any rotation design. We assume that the survey has been in operation for m months at time t. For computational convenience, we assume that there are no replicate effects.

Notation:

Let the model at time \( t + j \), \( j = 1, 2, \ldots \) be of the form

\[
Y_{t+j} = X_{t+j} \theta_{t+j} + \epsilon_{t+j}
\]

where \( Y_{t+j} \) is the data vector, \( X_{t+j} \) is the model matrix, \( \theta_{t+j} \) is the vector of the parameters of interest, \( \epsilon_{t+j} \) is the vector of errors with mean 0 and covariance matrix \( V_{t+j} \). Then, the best linear unbiased estimator of \( \theta_{t+j} \) at time \( t+j \) is

\[
\hat{\theta}_{t+j}(t+j) = P_{t+j} Y_{t+j}
\]

where

\[
P_{t+j} = (X_{t+j} V_{t+j}^{-1} X_{t+j})^{-1} X_{t+j} V_{t+j}^{-1}
\]
Let \( m \) be the number of periods for which initial estimates are available, and let \( s \) be the number of streams in the rotation scheme. At time \( t + j \), let \( C_{ij} \) denotes the \( i \)-th row of \( P_{t+j} \), \( \lambda_{ij} \) denote the \( 1\times(m+s+j-1) \) row vector with its \( i \)-th component equal to one, and all other components equal to 0.

Let \( \hat{\theta}_{t-m+1}(t), \ldots, \hat{\theta}_t(t) \) be the best linear unbiased estimators of \( \theta_{t-m+1}, \ldots, \theta_t \), respectively, at time \( t \) with covariance matrix \( \Sigma_{11,t(m)} \). Then, the data vector at time \( t + 1 \) is given by

\[
Y_{t+1} = (\hat{\theta}_{t-m+1}(t), \ldots, \hat{\theta}_t(t), z_{t+1,1}, \ldots, z_{t+1,8})
\]  
(C.1)

where \( z_{t+1,k} \mid k = 1, \ldots, 8 \) are defined in Appendix B. The corresponding linear model is

\[
Y_{t+1} = X_{t+1} \theta_{t+1} + \epsilon_{t+1},
\]

where

\[
\theta_{t+1} = (\theta_{t-p+1}, \ldots, \theta_t, \theta_{t+1})
\]  
(C.3)

and

\[
\text{Var}\{Y_{t+1}\} = V_{t+1} = \begin{bmatrix} \Sigma_{11,t(m)} & 0 \\ 0 & Q_{00} \end{bmatrix},
\]

and \( X_{t+1} = X_2 \) defined in (5.3.23), \( \Sigma_{11,m} \) is the covariance matrix of the recursive least squares estimators \( \hat{\theta}_{t-m+1}(t), \ldots, \hat{\theta}_t(t) \), and \( Q_{00} \) is defined in Appendix B.

Therefore, the best linear unbiased estimator of \( \theta_{t+1} \) at \( t + 1 \) is
\[
\hat{\beta}_{t+1}(t+1) = (\hat{\beta}_{t-m+1}(t+1), \ldots, \hat{\beta}_t(t+1), \hat{\beta}_{t+1}(t+1))'
\]

\[= P_{t+1} Y_{t+1},\]

where

\[P_{t+1} = (X_{t+1} \cdot V_{t+1}^{-1} X_{t+1})^{-1} X_{t+1} \cdot V_{t+1}^{-1} .\]

Note that the elements of \( \hat{\beta}_{t+1}(t+1) \) are linear combinations of the data vector at time \( t+1 \). We now define the following vector of estimates at time \( t + 1 \).

\[
\hat{\beta}_{t+1} = A_{t+1} Y_{t+1},
\]

where

\[A_{t+1} = (C_{21}, \ldots, C_{m-1,1}, \lambda_{m,1}, C_{m,1}, C_{m+1,1})'.\]

Then,

\[
\text{Var}\{\hat{\beta}_{t+1}\} = \Sigma_{t+1} = A_{t+1} V_{t+1} A_{t+1}'.
\]

Note that the elements of \( \hat{\beta}_{t+1} \) are precisely the estimates we need for the estimation of current level as well as one-period change under revision and no revision of previous estimates. Since we are interested in estimating change over an interval of several periods, it is important from this stage onwards to retain in our data vector at any given time, those previous estimates which are necessary for the computation of the desired estimates of change. For instance, our data vector for time \( t + 2 \) should contain \( \hat{\beta}_{t+1} \), as well as the transformed observations from the rotation groups introduced at time \( t + 2 \). The linear model for time \( t + 2 \) analogous to (C.2) is
then obtained by appropriately augmenting both the design matrix and the covariance matrix in the linear model for time $t + 1$. Thus, the linear model for time $t + 2$ is

$$Y_{t+2} = X_{t+2} \theta_{t+2} + \epsilon_{t+2}$$  \hspace{1cm} (C.8)

where

$$Y'_{t+2} = (\hat{\beta}'_{t+1}, x_{t+2,1}, \ldots, x_{t+2,8}),$$

$$\theta_{t+2} = (\theta_{t-m+2}, \ldots, \theta_t, \theta_{t+1}, \theta_{t+2}),$$

$$X_{t+2} = \begin{bmatrix} W_{t+2} \\ X_{22} \end{bmatrix},$$

where $W_{t+2} = \text{Diag}\{I_{(m-2)\times(m-2)}, J_{2\times1}, I_{2\times2}\}$, $X_{22}$ is the 7x16 submatrix consisting of the last 7 rows of the $23 \times 16$ matrix $X_2$ defined in (5.3.23) and

$$\text{Var}\{Y_{t+2}\} = V_{t+2} = \begin{bmatrix} \Sigma_{t+1} & 0 \\ 0 & Q_{00} \end{bmatrix}.$$ \hspace{1cm} (C.9)

Therefore, the best linear unbiased estimator of $\theta_{t+2}$ at time $t + 2$ is

$$\hat{\theta}_{t+2(t+2)} = (\hat{\theta}_{t-m+2(t+2)}, \ldots, \hat{\theta}_t(t+2), \hat{\theta}_{t+1(t+2), \hat{\theta}_{t+2(t+2)}}),$$

$$= P_{t+2} Y_{t+2}.$$ \hspace{1cm} (C.10)
where
\[
P_{t+2} = (X_{t+2}' V_{t+2}^{-1} X_{t+2})^{-1} X_{t+2}' V_{t+2}^{-1}.
\]

We now define
\[
\hat{\beta}_{t+2} = (\hat{\varphi}_{t-p-3(t+2)}, ..., \hat{\varphi}_{t}(t), \hat{\varphi}_{t}(t+2), \hat{\varphi}_{t+1}(t+1), \hat{\varphi}_{t+1}(t+2), \hat{\varphi}_{t+2}(t+2)) = A_{t+2}^\prime Y_{t+2},
\]
where
\[
A_{t+2} = (C_{2,2}, ..., C_{m-2,2}, \lambda_{m-1,2}, C_{m-1,2}, \lambda_{m,2}, C_{m,2}, C_{m+1,2})^\prime,
\]
The covariance matrix of $\hat{\beta}_{t+2}$ is
\[
\text{Var}\{\hat{\beta}_{t+2}\} = \Sigma_{t+2} = A_{t+2} V_{t+2} A_{t+2}^\prime.
\]
Both $\hat{\beta}_{t+2}$ and $\Sigma_{t+2}$ are then used in the model for time $t + 3$, and so on.

In general, suppose we are interested in estimating current level and change in level over an interval of up to $t + p$ periods, $p \geq 1$, both with and without revision of previous estimates. Then, we need a vector of $2p + 1$ estimates consisting of an estimate of current level, $p$ revised estimates and $p$ unrevised estimates.

In the recursive regression estimation procedure, we start with $m$ initial estimates and $s$ independent elementary estimates obtained from the rotation groups introduced into the sample at time $t + 1$. The data vector at time $t + 1$ is given by (C.1).
The model at time $t + j$, $j \geq 1$, is

$$Y_{t+j} = X_{t+j} \theta_{t+j} + \epsilon_{t+j},$$  \hspace{1cm} (C.13)

where

$$Y'_{t+j} = (\hat{\beta}_{t+j-1}, z_{t+j,1}, \ldots, z_{t+j,s}),$$

$$\theta'_{t+j} = (\theta_{t-p+j}, \ldots, \theta_{t+j-1}, \theta_{t+j}),$$

$$X_{t+j} = \begin{bmatrix} W_{t+j} \\ X_{22} \end{bmatrix},$$

$$W_{t+j} = \text{Blockdiag}\{I_{(m-j)\times(m-j)}, I_{(j-1)\times(j-1)} \otimes J_{2\times1}, I_{2\times2}\},$$

and

$$\text{Var}\{Y_{t+j}\} = V_{t+j} = \begin{bmatrix} \Sigma_{t+j-1} & 0 \\ 0 & Q_{00} \end{bmatrix},$$ \hspace{1cm} (C.14)

where $\Sigma_{t+j-1}$ is the covariance matrix of $\hat{\beta}_{t+j-1}$. Therefore, the best linear unbiased estimator of $\theta_{t+j}$ is

$$\hat{\theta}_{t+j}(t+j) = (\hat{\theta}_{t-m+j}(t+j), \ldots, \hat{\theta}_{t+j-1}(t+j), \hat{\theta}_{t+j}(t+j))$$ \hspace{1cm} (C.15)

$$= P_{t+j} Y_{t+j},$$

where

$$P_{t+j} = (X'_{t+j} V_{t+j}^{-1} X_{t+j})^{-1} X'_{t+j} V_{t+j}^{-1}.$$
We now define

\[ \hat{\beta}_{t+j} = \begin{bmatrix} \hat{\beta}_{t-m+j+1}(t+j) , \ldots , \hat{\beta}_t(t) , \hat{\beta}_t(t+j) , \hat{\beta}_{t+1}(t+1) , \hat{\beta}_{t+1}(t+j) , \ldots \end{bmatrix} = A_{t+j} Y_{t+j}, \]

where

\[ A_{t+j} = \begin{bmatrix} C_{2,j} , \ldots , C_{m-j,j} , \lambda_{m-j+1,j} C_{m-j+1,j} , \ldots \end{bmatrix}, \]

and

\[ \text{Var}\{\hat{\beta}_{t+j}\} = \Sigma_{t+j} = A_{t+j} V_{t+j} A_{t+j}^\dagger. \]  

In particular, for \( j = 12 \), that is, at time \( t + 12 \), we have the model

\[ Y_{t+12} = X_{t+12} \theta_{t+12} + \epsilon_{t+12}, \]

where

\[ Y_{t+12} = (\hat{\beta}_{t+11} , z_{t+12,1} , \ldots , z_{t+12,8} ), \]

\[ \theta_{t+12} = (\theta_{t-m+12} , \ldots , \theta_{t+11} , \theta_{t+12} ), \]
\[ X_{t+12} = \begin{bmatrix} W_{t+12} \\ X_{22} \end{bmatrix}, \]

\[ W_{t+12} = \text{Blockdiag}\{I_{(m-12)\times(m-12)}, I_{11\times11} \otimes J_{2\times1}, I_{2\times2}\}, \]

\[ \text{Var}\{Y_{t+12}\} = V_{t+12} = \begin{bmatrix} \Sigma_{t+11} & 0 \\ 0 & Q_{00} \end{bmatrix}. \] (C.19)

Therefore, the best linear unbiased estimator of \( \theta_{t+12} \) at time \( t + 12 \) is

\[ \hat{\theta}_{t+12}(t+12) = (\hat{\theta}_{t-m+12}(t+12), ..., \hat{\theta}_{t+10}(t+12), \hat{\theta}_{t+11}(t+12), \hat{\theta}_{t+12}(t+12)) \]

\[ = P_{t+12} Y_{t+12}, \] (C.20)

where

\[ P_{t+12} = (X_{t+12}^\prime V_{t+12}^{-1} X_{t+12})^{-1} X_{t+12}^\prime V_{t+12}^{-1}. \]

Define

\[ \hat{\theta}_{t+12}^\prime = [\hat{\theta}_{t-m+13}(t+12), ..., \hat{\theta}_{t}(t+12), \hat{\theta}_{t+1}(t+1), \hat{\theta}_{t+1}(t+12), \hat{\theta}_{t+2}(t+2), \hat{\theta}_{t+2}(t+12), ..., \hat{\theta}_{t+11}(t+11), \]

\[ \hat{\theta}_{t+1}(t+12), \hat{\theta}_{t+2}(t+2), \hat{\theta}_{t+2}(t+12), ..., \hat{\theta}_{t+11}(t+11), \]
\[ \hat{t}_{t+11(t+12)}, \hat{t}_{t+12(t+12)} \]
\[ = A_{t+12} Y_{t+12}, \]

where

\[ A_{t+12} = (C_{2,12}, \ldots, C_{m-12,12}, \lambda_{m-11,12}, C_{m-11,12}, \lambda_{m-9,12}, C_{m-10,12}, \lambda_{m-7,12}, C_{m-9,12}, \ldots, \lambda_{m+11,12}, C_{m+12}, C_{m+1,12}) \]

and

\[ \text{Var}\{\hat{\beta}_{t+12}\} = \Sigma_{t+12} = A_{t+12} Y_{t+12} A^{'t+12}. \]

Therefore, at time \( t + 12 \), we have all the estimates required for the computation of estimates of current level and several period change up to twelve periods. The vector consisting of such estimates is

\[ \hat{\theta} = [\hat{t}_{t+12(t+12)}, \hat{t}_{t+11(t+12)}, \hat{t}_{t+11(t+11)}, \hat{t}_{t+10(t+12)}, \hat{t}_{t+10(t+10)}, \hat{t}_{t+9(t+12)}, \hat{t}_{t+9(t+9)}, \hat{t}_{t+8(t+12)}, \hat{t}_{t+8(t+12)}, \hat{t}_{t+7(t+12)}, \hat{t}_{t+7(t+7)}, \hat{t}_{t+6(t+12)}, \hat{t}_{t+6(t+12)}]. \]
\[ \hat{\theta}_{t+6}(t+6), \hat{\theta}_{t+5}(t+12), \hat{\theta}_{t+5}(t+5), \hat{\theta}_{t+4}(t+12), \]
\[ \hat{\theta}_{t+4}(t+4), \hat{\theta}_{t+3}(t+12), \hat{\theta}_{t+3}(t+3), \hat{\theta}_{t+2}(t+12), \]
\[ \hat{\theta}_{t+2}(t+2), \hat{\theta}_{t+1}(t+12), \hat{\theta}_{t+1}(t+1), \hat{\theta}_{t}(t+12), \hat{\theta}_{t}(t) \]

(C.23)

Each component in \( \hat{\theta} \) can be expressed as a linear combination of the data vector \( Y_{t+12} \) at time \( t + 12 \). That is, we may write

\[ \hat{\theta} = B_{t+12} Y_{t+12}, \]  

(C.24)

where \( B_{t+12} \) is obtained from \( A_{t+12} \) by deleting its first two rows. The covariance matrix of \( \hat{\theta} \) is

\[ \text{Var}\{\hat{\theta}\} = Q_{t+12} = B_{t+12} V_{t+12} B'_{t+12}. \]  

(C.25)

Note that the elements of \( \hat{\theta} \) are of two types, viz: \( \hat{\theta}_{t+j}(t+j) \), and \( \hat{\theta}_{t+j}(t+12) \), \( j = 0, 1, 2, ..., 12 \). Thus, \( \hat{\theta} \) provides us with all the estimates we need for the computation of the best linear unbiased estimators for current level (time \( t + 12 \)) and change in level over an interval to 12 periods, both with and without revision of previous estimates. For instance, suppose we define the matrix \( D \) to be
where \( p \) is the largest interval between levels for which an estimate of change is required. For the special case considered in this section, \( p = 12 \). Then,

\[
\hat{\beta} = D\hat{\theta},
\]

is the \((2p + 1) \times (2p + 1)\) column vector such that the first element is the best linear unbiased estimator for the current level and for \( 2 \leq r \leq (2p + 1) \), the \( r \)-th element is the best linear unbiased estimator for the \((r - 1)\)-period change in level under revision of previous estimates, and the \((r + 1)\)-st element is the same estimate of change in level under no revision of previous estimates. The covariance matrix of \( \hat{\beta} \) is

\[
\text{Var}\{\hat{\beta}\} = D Q_{t+12} D'.
\]
APPENDIX D

RESULTS FOR THE BREAU AND ERNST (1983) MODEL.


Let $\rho_j$ be as defined in Appendix C. The Breau and Ernst (1983) model is defined as follows:

1. The correlations between different rotation groups in the same month are zero, that is,

$$\text{Corr}\{y_{t,0,k}, y_{t,0,s}\} = 0 \quad \text{for} \quad k \neq s = 1, \ldots, 8$$

2. All correlations between different rotation groups in different months are zero except the following:

$$\text{Corr}\{y_{t,0,k}, y_{r,0,s}\} = \begin{cases} \rho_1 & \text{if } r = t - 1, \\ \rho_2 & \text{if } r = k - 2, \\ \rho_3 & \text{if } r = t - 3, \\ & \text{if } k = 1, 2, 3, 5, 6, 7 \\ & \text{if } s = k - 1 \\ & \text{if } k = 1, 2, 5, 6 \\ & \text{if } s = k - 2 \\ & \text{if } k = 1, 5 \\ & \text{if } s = k - 3 \end{cases}$$
\[ \rho_9 \quad \text{if } r = t - 9, \quad k = 5, s = 4 \]
\[ \rho_{10} \quad \text{if } r = t - 10, \quad k = 3, 4 \]
\[ s = k - 2 \]
\[ \rho_{11} \quad \text{if } r = t - 11, \quad k = 2, 3, 4 \]
\[ s = k - 3 \]
\[ \rho_{12} \quad \text{if } r = t - 12, \quad k = 1, 2, 3, 4 \]
\[ s = k - 4 \]
\[ \rho_{13} \quad \text{if } r = t - 13, \quad k = 1, 2, 3 \]
\[ s = k - 5 \]
\[ \rho_{14} \quad \text{if } r = t - 14, \quad k = 1, 2 \]
\[ s = k - 6 \]
\[ \rho_{15} \quad \text{if } r = t - 15, \quad k = 8, s = 1 \]

D.2. Results

In the following tables we present the results of the comparison of alternative estimators of current level and change, corresponding to the Breau and Ernst (1983) model.
Table D.1. Variance of alternative estimators of unemployed relative to the variance of the basic estimator of current level; no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Basic Estimator</th>
<th>Current Composite</th>
<th>BLUE 2 periods</th>
<th>BLUE 3 periods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Level</td>
<td>1.000</td>
<td>0.997</td>
<td>0.949</td>
<td>0.929</td>
</tr>
<tr>
<td>1—period change</td>
<td>1.228</td>
<td>1.134</td>
<td>1.170</td>
<td>1.150</td>
</tr>
<tr>
<td>2—period change</td>
<td>1.605</td>
<td>1.494</td>
<td>1.497</td>
<td>1.459</td>
</tr>
<tr>
<td>3—period change</td>
<td>1.828</td>
<td>1.708</td>
<td>1.687</td>
<td>1.623</td>
</tr>
<tr>
<td>4—period change</td>
<td>2.000</td>
<td>1.849</td>
<td>1.835</td>
<td>1.756</td>
</tr>
<tr>
<td>5—period change</td>
<td>2.000</td>
<td>1.918</td>
<td>1.899</td>
<td>1.823</td>
</tr>
<tr>
<td>6—period change</td>
<td>2.000</td>
<td>1.950</td>
<td>1.899</td>
<td>1.858</td>
</tr>
<tr>
<td>7—period change</td>
<td>2.000</td>
<td>1.961</td>
<td>1.899</td>
<td>1.848</td>
</tr>
<tr>
<td>8—period change</td>
<td>2.000</td>
<td>1.955</td>
<td>1.881</td>
<td>1.832</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.951</td>
<td>1.931</td>
<td>1.841</td>
<td>1.797</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.904</td>
<td>1.901</td>
<td>1.804</td>
<td>1.766</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.855</td>
<td>1.871</td>
<td>1.766</td>
<td>1.736</td>
</tr>
<tr>
<td>12—period change</td>
<td>1.805</td>
<td>1.855</td>
<td>1.731</td>
<td>1.706</td>
</tr>
<tr>
<td>Parameter</td>
<td>BLUE 12 periods</td>
<td>BLUE 16 periods</td>
<td>BLUE 24 periods</td>
<td>Recursive Regression Estimator</td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>Current Level</td>
<td>0.907</td>
<td>0.902</td>
<td>0.901</td>
<td>0.901</td>
</tr>
<tr>
<td>1—period change</td>
<td>1.133</td>
<td>1.129</td>
<td>1.129</td>
<td>1.129</td>
</tr>
<tr>
<td>2—period change</td>
<td>1.429</td>
<td>1.423</td>
<td>1.422</td>
<td>1.422</td>
</tr>
<tr>
<td>3—period change</td>
<td>1.579</td>
<td>1.571</td>
<td>1.571</td>
<td>1.571</td>
</tr>
<tr>
<td>4—period change</td>
<td>1.681</td>
<td>1.675</td>
<td>1.674</td>
<td>1.674</td>
</tr>
<tr>
<td>5—period change</td>
<td>1.735</td>
<td>1.727</td>
<td>1.727</td>
<td>1.727</td>
</tr>
<tr>
<td>6—period change</td>
<td>1.763</td>
<td>1.754</td>
<td>1.753</td>
<td>1.753</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.774</td>
<td>1.765</td>
<td>1.765</td>
<td>1.765</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.774</td>
<td>1.764</td>
<td>1.764</td>
<td>1.763</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.753</td>
<td>1.743</td>
<td>1.743</td>
<td>1.742</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.734</td>
<td>1.726</td>
<td>1.725</td>
<td>1.723</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.716</td>
<td>1.707</td>
<td>1.707</td>
<td>1.704</td>
</tr>
<tr>
<td>12—period change</td>
<td>1.700</td>
<td>1.692</td>
<td>1.691</td>
<td>1.686</td>
</tr>
</tbody>
</table>
Table D.2. Variances of alternative estimators of unemployed relative to the variance of the basic estimator of current level; revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive Regression Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Level</td>
<td>0.907</td>
<td>0.902</td>
<td>0.901</td>
<td>0.901</td>
</tr>
<tr>
<td>1—period change</td>
<td>1.093</td>
<td>1.092</td>
<td>1.092</td>
<td>1.092</td>
</tr>
<tr>
<td>2—period change</td>
<td>1.374</td>
<td>1.371</td>
<td>1.371</td>
<td>1.371</td>
</tr>
<tr>
<td>3—period change</td>
<td>1.520</td>
<td>1.514</td>
<td>1.513</td>
<td>1.513</td>
</tr>
<tr>
<td>4—period change</td>
<td>1.625</td>
<td>1.618</td>
<td>1.615</td>
<td>1.615</td>
</tr>
<tr>
<td>5—period change</td>
<td>1.679</td>
<td>1.672</td>
<td>1.667</td>
<td>1.667</td>
</tr>
<tr>
<td>6—period change</td>
<td>1.707</td>
<td>1.701</td>
<td>1.693</td>
<td>1.693</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.719</td>
<td>1.714</td>
<td>1.704</td>
<td>1.704</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.718</td>
<td>1.712</td>
<td>1.703</td>
<td>1.703</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.701</td>
<td>1.689</td>
<td>1.680</td>
<td>1.680</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.693</td>
<td>1.668</td>
<td>1.661</td>
<td>1.659</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.706</td>
<td>1.646</td>
<td>1.640</td>
<td>1.638</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA</td>
<td>1.627</td>
<td>1.621</td>
<td>1.618</td>
</tr>
</tbody>
</table>

NA¹: Not available
Table D.3. Variance of alternative estimators of Civilian Labor Force relative to the variance of the basic estimator of current level; no revision of previous estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Basic Estimator</th>
<th>Current Composite</th>
<th>BLUE 2 periods</th>
<th>BLUE 3 periods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Level</td>
<td>1.000</td>
<td>0.812</td>
<td>0.870</td>
<td>0.780</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.776</td>
<td>0.523</td>
<td>0.646</td>
<td>0.586</td>
</tr>
<tr>
<td>2—period change</td>
<td>1.255</td>
<td>0.870</td>
<td>1.002</td>
<td>0.884</td>
</tr>
<tr>
<td>3—period change</td>
<td>1.647</td>
<td>1.164</td>
<td>1.288</td>
<td>1.107</td>
</tr>
<tr>
<td>4—period change</td>
<td>2.000</td>
<td>1.390</td>
<td>1.543</td>
<td>1.309</td>
</tr>
<tr>
<td>5—period change</td>
<td>2.000</td>
<td>1.500</td>
<td>1.740</td>
<td>1.469</td>
</tr>
<tr>
<td>6—period change</td>
<td>2.000</td>
<td>1.547</td>
<td>1.740</td>
<td>1.599</td>
</tr>
<tr>
<td>7—period change</td>
<td>2.000</td>
<td>1.554</td>
<td>1.740</td>
<td>1.547</td>
</tr>
<tr>
<td>8—period change</td>
<td>2.000</td>
<td>1.527</td>
<td>1.660</td>
<td>1.486</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.857</td>
<td>1.450</td>
<td>1.561</td>
<td>1.411</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.717</td>
<td>1.359</td>
<td>1.464</td>
<td>1.339</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.570</td>
<td>1.271</td>
<td>1.362</td>
<td>1.268</td>
</tr>
<tr>
<td>12—period change</td>
<td>1.431</td>
<td>1.229</td>
<td>1.280</td>
<td>1.206</td>
</tr>
</tbody>
</table>
### Table D.3. Continued

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive Regression Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Level</td>
<td>0.667</td>
<td>0.647</td>
<td>0.641</td>
<td>0.637</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.509</td>
<td>0.495</td>
<td>0.494</td>
<td>0.493</td>
</tr>
<tr>
<td>2—period change</td>
<td>0.733</td>
<td>0.709</td>
<td>0.706</td>
<td>0.705</td>
</tr>
<tr>
<td>3—period change</td>
<td>0.876</td>
<td>0.847</td>
<td>0.843</td>
<td>0.841</td>
</tr>
<tr>
<td>4—period change</td>
<td>0.973</td>
<td>0.954</td>
<td>0.948</td>
<td>0.946</td>
</tr>
<tr>
<td>5—period change</td>
<td>1.044</td>
<td>1.026</td>
<td>1.019</td>
<td>1.016</td>
</tr>
<tr>
<td>6—period change</td>
<td>1.092</td>
<td>1.071</td>
<td>1.064</td>
<td>1.061</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.120</td>
<td>1.096</td>
<td>1.090</td>
<td>1.086</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.132</td>
<td>1.105</td>
<td>1.098</td>
<td>1.093</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.122</td>
<td>1.091</td>
<td>1.084</td>
<td>1.077</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.113</td>
<td>1.079</td>
<td>1.071</td>
<td>1.061</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.099</td>
<td>1.063</td>
<td>1.057</td>
<td>1.039</td>
</tr>
<tr>
<td>12—period change</td>
<td>1.098</td>
<td>1.062</td>
<td>1.057</td>
<td>1.027</td>
</tr>
</tbody>
</table>
### Table D.4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BLUE 12 periods</th>
<th>BLUE 16 periods</th>
<th>BLUE 24 periods</th>
<th>Recursive Regression Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Level</td>
<td>0.667</td>
<td>0.647</td>
<td>0.641</td>
<td>0.637</td>
</tr>
<tr>
<td>1—period change</td>
<td>0.444</td>
<td>0.442</td>
<td>0.442</td>
<td>0.442</td>
</tr>
<tr>
<td>2—period change</td>
<td>0.634</td>
<td>0.629</td>
<td>0.628</td>
<td>0.628</td>
</tr>
<tr>
<td>3—period change</td>
<td>0.761</td>
<td>0.752</td>
<td>0.749</td>
<td>0.749</td>
</tr>
<tr>
<td>4—period change</td>
<td>0.867</td>
<td>0.855</td>
<td>0.848</td>
<td>0.847</td>
</tr>
<tr>
<td>5—period change</td>
<td>0.940</td>
<td>0.927</td>
<td>0.915</td>
<td>0.914</td>
</tr>
<tr>
<td>6—period change</td>
<td>0.989</td>
<td>0.974</td>
<td>0.958</td>
<td>0.957</td>
</tr>
<tr>
<td>7—period change</td>
<td>1.018</td>
<td>1.003</td>
<td>0.982</td>
<td>0.981</td>
</tr>
<tr>
<td>8—period change</td>
<td>1.030</td>
<td>1.008</td>
<td>0.987</td>
<td>0.986</td>
</tr>
<tr>
<td>9—period change</td>
<td>1.022</td>
<td>0.985</td>
<td>0.967</td>
<td>0.965</td>
</tr>
<tr>
<td>10—period change</td>
<td>1.025</td>
<td>0.963</td>
<td>0.947</td>
<td>0.945</td>
</tr>
<tr>
<td>11—period change</td>
<td>1.043</td>
<td>0.934</td>
<td>0.921</td>
<td>0.917</td>
</tr>
<tr>
<td>12—period change</td>
<td>NA(^1)</td>
<td>0.920</td>
<td>0.907</td>
<td>0.900</td>
</tr>
</tbody>
</table>

NA\(^1\): Not available