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Aspects of statistical multiple tolerancing

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Aspects of statistical multiple tolerancing

Yu, Yunn-Hwu, Ph.D.
Iowa State University, 1992
Aspects of statistical multiple tolerancing

by

Yunn-Hwu Yu

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CHAPTER 1. INTRODUCTION AND PRELIMINARIES

Tolerance Technology

Manufacturing of exactly equal parts is known from experience to be impossible. Usually, a part may include holes, pegs, or screw threads. "Exactly equal" means both the dimension and position of every holes (or pegs) in each part are manufactured as exactly designed. In order to assure that mating parts "fit well", the designer must work with safety margins.

The importance of the statistician's role in tolerancing can be illustrated by the following example. Suppose that we are manufacturing three gears of 2 inch face width, and that gears are to be assembled next to each other on a transmission case shaft. If the overall length of the assembled gears must be held within a tolerance of ±0.015 inch, then to how much tolerance should the width of each of the three gears be held?

A designer in manufacturing practice might say, "To be absolutely sure that the assembly will not exceed the ±0.015 inch tolerance, ±0.005 inch of tolerance is the largest tolerance we can afford to give the individual parts." On the other hand, a statistician might say, "It seldom happens that three different parts are all at their extreme (maximum or minimum) dimension at the same time, and therefore a tolerance greater than ±0.005 inch can be applied to each part without exceeding
the assembly tolerance of ±0.015 inch." However, the extreme cases do happen, and
the probability of acceptable assemblies (confidence level) depends on the choice of
the tolerance of the individual parts. Usually, the machining costs can be reduced
by increasing the individual tolerances, but increasing tolerances will decrease our
confidence in the assemblies's likelihood of meeting design specifications. Hence we
not only need knowledge about statistics but we also need to know the relationship
between tolerances and costs, in order to choose a proper (economic and mechanical)
tolerance of the individual parts.

In a engineering process, tolerance calculations are involved in product specifi­
cations. The product specifications made by the designer pertain mainly to form,
dimension, position, material requirements, each of which has a basic target and tol­
erance. The aspect of tolerances that most concerns us in this dissertation is position
tolerance.

A position tolerance is the total permissible variation in the location of a feature
(e.g., hole or slot) about its exact true position. Position tolerancing techniques are
most effective and appropriate in mating part situations. For circular features (holes,
pegs, etc.) the position tolerance is the diameter of the tolerance zone within which
the center of the feature must lie, and the center of the tolerance zone is at the design
position. For some kinds of features (slots, tabs, etc.) the position tolerance is the
total width of the tolerance zone within which the center plane of the feature must
lie and the center of the tolerance zone is at the design position (see Figure 1.1).

It is useful to define th concept MMC (maximum material condition)as the
condition in which a feature composed of material contains the maximum amount of
material, within the stated limits of design. The MMC size of a feature usually is its
Figure 1.1: Illustration of position tolerance and tolerance zone
high or low limit size. Suppose the dimensions of a hole and a peg must be held to within $0.250 \pm 0.005$ inch and $0.235 \pm 0.005$ inch respectively. The MMC size of the hole is 0.245 inch (low limit), but the MMC size of the peg is 0.240 inch (high limit). Hence MMC size of a feature depends not only on the limit of size but also on the type of feature. In this instance, when both hole and peg are at MMC the position tolerance is 0.005 inch ($0.245 - 0.240$).

Consider two simple configurations featuring a plane containing circular holes and a plate containing corresponding circular pegs. The holes and pegs in the two planes will have location misplacements within their own location tolerances. In order to simplify this situation, we can treat the true positions of the pegs in the peg-plane as the true-design positions; that is, we assume no location tolerance in the peg-plane, and correspondingly enlarge the location tolerance of holes in the hole-plane. If the distances between the centers of each hole and its corresponding peg are within certain limits, then we may say the assembly is acceptable. When the location of a pattern of features (e.g., holes) from a coordinate axes (datum lines) is less important than the accuracy required within the pattern of features, composite position tolerancing may be used. Composite position tolerancing incorporates two types of position tolerances. One specifies feature-relating position tolerance, defining permissible deviations of hole center positions, and the other one specifies pattern-locating position tolerance, the former is of primary concern in this dissertation.

Figure 1.2 shows an example of a composite position tolerance for a 4-hole pattern. The feature-relating position tolerance, for each hole is 0.3 and the pattern locating tolerance is 0.9 at MMC relative to datums A, B, and C. Holes are indicated by large circles, with each hole axis marked by a large cross. Intermediate-sized cir-
ircles represent pattern-locating tolerance zones with their centers individually fixed relative to datum surfaces. The smaller circles represent feature-relating tolerance zones. Centers of the latter circles can be located anywhere, provided their relative positions form a pattern (peg-pattern) identical to that of design positions, since the location of the pattern is disregarded. So, with regard to feature-relating positional tolerance, we can move the peg-pattern over the hole-pattern to find a new location such that the distances $d_i$ of hole center $i$ and its corresponding peg center will satisfy a certain condition. The interference is the maximum overlap between the holes and their Suppose $R_i$ and $r_i$ are the radii of the $i$th hole and its corresponding peg respectively. If $d_i$ denotes the distance between the center of the $i$th peg and the center of its corresponding hole, then we define interference $I_i$ as:

$$I_i = d_i - (R_i - r_i).$$

The intended fit is possible if $I_i$ is negative, for all $i$.

Suppose the centers of the pegs are located by coordinates $(u_i, v_i)$, $i = 1, \ldots, 4$, and the corresponding holes are labeled $1, \ldots, 4$. The centers of the holes are located by coordinates $(u_i + \epsilon_i, v_i + \eta_i)$, $i = 1, \ldots, 4$, where

$$2\sqrt{\epsilon_i^2 + \eta_i^2}$$

should be less than the locating tolerance. Foster (1986) and Liggett (1970) describe a graphical method to verify composite position tolerance. In this graphical method, we can choose any hole center and move the center of its corresponding peg on top of it, and then pick mutually orthogonal $X$- and $Y$-coordinate measurements. The new coordinates of the centers of the pegs after translation are denoted as $(u_i', v_i')$, $i = 1, \ldots, 4$. The deviations of centers $((u_i + \epsilon_i - u_i', v_i + \eta_i - v_i'))$ are plotted on graph
0.3 feature-relating zones  
(4 zones, basically related to each other)

0.9 pattern-locating tolerance zone  
(4 zones, basically oriented to datum lines)

Figure 1.2: Interpretation of composite position tolerance
Figure 1.3: Foster's graphical method for composite tolerance

paper (see Figure 1.3). If there exists a circle with radius not greater than half of the feature-relating position tolerance limit such that every deviation is inside that circle, then we can accept this part. The disadvantage of this method is that some rejected parts may become acceptable by choosing different X- and Y- coordinates, since this method does not allow any freedom of rotation.

In order to solve this problem, Gunasena (1991) derived an algorithm for the verification of composite position tolerances by considering rotational freedom. Once rotation was considered, the problem became complicated. Since Gunasena believed the optimal rotation to be small, he used approximations

\[
\sin \theta \simeq \theta
\]

and

\[
\cos \theta \simeq 1
\]

in his model. However, a possibility remains that the algorithm may reject a few "good" parts by using the rotation approximations. McCann (1988) developed an
exact algorithm for this sort of fitting problem. The objective of McCann's method is to minimize a three-variable non-convex interference function \( f(\theta, x, y) \) in which the first variable is rotation, and the remaining variables are mutually orthogonal translations X and Y. With fixed \( \theta_0 \), the interference function is convex in \((X, Y)\) and the optimal \( f(\theta_0, X^*(\theta_0), Y^*(\theta_0)) \) can be derived in closed form by means of Kuhn-Tucker conditions. Then the optimal value of rotation can be found by searching for the minimum value of \( f(\theta, X^*(\theta), Y^*(\theta)) \) over \( \theta \).

David and McCann (1988) addressed "upstream" and "downstream" problems for assembling process. The "upstream" (tolerancing) problem concerns in part the relative importance of tight tolerancing of component position vs. tight tolerancing of component dimension within an assembly part on the role that the tolerancing of that component plays in determining the probability that the part to which it belongs will fit acceptably at the next stage of assembly (p. 1),

and the "downstream" (inspection) problem concerns in part the notion (as in David, 1985) of accepting part according to "the probability of it fitting acceptably at the next stage of assembly" (p. 1).

"Fit acceptably" depends on what kind of fit is required in the assembling process. The notations \( I_i, d_i, R_i, \), and \( r_i \) are defined as before. Figure 1.4 illustrates that \( R_i + d_i \) is the largest distance between the center of a peg and any point inside the hole. Hence the maximum clearance, \( C_i \), between the peg and hole is:

\[
C_i = d_i + (R_i - r_i)
\]

and the minimum clearance is \(-I_i\). Suppose \( U_i \) is the design upper bound of the
and the minimum clearance is \(-I_i\). Suppose \(U_i\) is the design upper bound of the maximum clearance and \(L_i\) is the design lower bound of the minimum clearance; then the following inequalities express three possible requirements for the fitting process:

\[
I_i = d_i - (R_i - r_i) \leq 0 \tag{1.1}
\]

\[
C_i - U_i = d_i + (R_i - r_i) - U_i \leq 0 \tag{1.2}
\]

\[
I_i + L_i = d_i - (R_i - r_i) + L_i \leq 0 \tag{1.3}
\]

Note that Equation 1.1 is true if Equation 1.2 or 1.3 is true. In David and McCann’s paper, an assembling process is concerned with (1) “fit” if Equation 1.1 is satisfied; (2) “tight fit” if Equation 1.2 is satisfied; and (3) “good fit” if Equations 1.2 and 1.3 are both satisfied. Both “upstream” and “downstream” tolerancing problems were discussed in the sense of “fit.”

A fourth type of fit, which concerns the subject of this dissertation is “point-to-point” fit. When the “points” involved are the respective centers of the hole and
fact corresponds, much in the spirit of the “good fit” above, to the aim of creating as uniform as possible a clearance space of the peg within the hole.

Prospectus

Recall the “fit” requirement of assembly in Equation 1.1. Suppose the dimension of the radii of holes and pegs are fixed and $R_i - r_i = c$, $i = 1, \ldots, n$, where $c$ is constant. Then Equation 1.1 becomes

$$I_i = d_i \leq c,$$ (1.4)

where $d_i$ is the distance between the $i$th peg center and its corresponding hole center. Also note that $I_i$ in the previous inequality indeed is so call the point-to-point interference.

The point-to-point interference of a peg and its corresponding hole is the interference without considering the difference between the radius of a hole and the radius of its corresponding peg. In other words, the point-to-point interference is equal to the distance between the center of a peg and the center of its corresponding hole.

The centers of pegs (design centers) locate at $(u_i, v_i)$, $i = 1, \ldots, n$, and the centers of their corresponding holes locate at $(u_i + \epsilon_i, v_i + \eta_i)$, $i = 1, \ldots, n$, where $\epsilon_i$'s are independently and identically distributed random variables with a common distribution function $F_1$, and $\eta_i$'s are independently and identically distributed random variables with a common distribution function $F_2$. In this dissertation, we emphasize the case where the distributions of $\epsilon_i$ and $\eta_i$ are normal, with mean 0 and variance $\sigma^2$, and the $\epsilon_i$'s and $\eta_i$'s are independent. Hence we have $2n$ independently and identically distributed random variables.
Recall that when the location of a pattern from its coordinate datum lines is less important than the accuracy required within the pattern of features, we may adjust the locations of pegs in the peg-plane without changing the relative positions of pegs to search a new position for the peg-pattern, for which the peg-plane and the hole-plane will fit well. Suppose \((u'_i, v'_i)\)'s denote the new locations of peg centers, then the point-to-point interference for \(i\)th peg and its corresponding hole becomes:

\[
\sqrt{[(u'_i + \epsilon_i) - u_i']^2 + [(v'_i + \eta_i) - v_i']^2}.
\]

Since the new positioning may involve two mutually orthogonal parameters \(x\) and \(y\) of translation, and one parameter \(\theta\) of rotation, \(u'_i\) and \(v'_i\) may be better rewritten as \(u'_i(x, y, \theta)\) and \(v'_i(x, y, \theta)\) respectively. Consequently, \(I_i\) should be rewritten as \(I_i(x, y, \theta)\).

This dissertation focuses on two kinds of point-to-point interference fitting problems for large \(n\). One is called minmax point-to-point interference and the other is called least squares point-to-point interference. The minmax problem is discussed in Chapter 2 and the least squares problem is discussed in Chapter 3. The minmax point-to-point interference problem studies the ways in which manufacturing tolerances affect the likelihood of (1) meeting composite tolerances, and (2) acceptable fits for making parts; the least squares point-to-point interference problem studies the ways in which manufacturing affect Tauchi-type squared-error penalties for failing to meet exact design specifications.

Suppose the requirement of two planes fitting is that every point-to-point interference should be within an upper limit, say \(U\), then Equation 1.4 can be expressed by

\[
I_i(x, y, \theta) \leq U, \quad \forall i
\]
Note that every $I_i$ is less than or equal to the upper limit is equivalent to that the maximum of $I_i$ is less or equal to the upper limit, so Equation 1.6 can be written as

$$I(x, y, \theta) = \max_{1 \leq i \leq n} I_i(x, y, \theta) \leq U$$

where $I(x, y, \theta)$ denotes the maximum interference if the positioning adjustment parameters are $x$, $y$, and $\theta$.

In Chapter 2, we discuss in more detail the case of the positioning adjustment without rotation. The motivation for Chapter 2 can be viewed in several ways; any them is the investigation of how positioning tolerance, as expressed by linear standard deviation, is propagated to assembly interference, after an assembly procedure is used, in a certain sense, at making the best of any manufacture discrepancies. However, the aim of Chapter 2 is to find the minmax point-to-point interference $I$ such that

$$I = \min_{(x, y) \in \mathbb{R}^2} I(x, y) = \min_{(x, y) \in \mathbb{R}^2} \max_{1 \leq i \leq n} I_i(x, y)$$

From Equation 1.5, we have

$$I_i(x, y) = \sqrt{[(u_i + \epsilon_i) - (u_i + x)]^2 + [(v_i + \eta_i) - (v_i + y)]^2}$$

$$= \sqrt{(\epsilon_i - x)^2 + (\eta_i - y)^2},$$

hence the problem is reduced to search a point $(x^*, y^*)$ such that the maximum distance between this point and any $(\epsilon_i, \eta_i)$ is minimum; this minimum value is called minmax point-to-point interference. In the reduced problem, the design locations of the centers of the holes and pegs no longer appear, and hence, do not affect the fitting process. We now only consider a pattern with $n$ points located at $(\epsilon_i, \eta_i)$,
$i = 1, \ldots, n$, and the problem of studying the large-sample distribution of

$$I = \min_{(x,y) \in \mathcal{R}^2} \max_{1 \leq i \leq n} \sqrt{(\epsilon_i - x)^2 + (\eta_i - y)^2}.$$  

In Chapter 2, that task is approached by a upper- and lower-bounding processes. With respect to the lower-bounding process, it is shown that

$$I \geq \max_{1 \leq i < j \leq n} \min_{(x,y) \in \mathcal{R}^2} \max_{l=i,j} \sqrt{(\epsilon_l - x)^2 + (\eta_l - y)^2}.$$  

It should be noted here, that the important role of pair of point-to-point interference was already discussed by McCann (1988), in connection with a certain Kuhn-Tucker analysis. With respect to the upper-bounding process, we use a certain elemental fact from the theory of extreme for a Rayleigh($\sqrt{2}$) distribution. However, even this pairwise lower bound of the point-to-point interference turns out to be difficult to deal with, and we use two partial approaches: a further lower-bounding, and a heuristic approach to the asymptotic distribution of that lower bound.

The least squares point-to-point interference approach of Chapter 3 deals with the case where the penalty for mis-positioning is a function of the sum of the squares distances between peg centers and corresponding hole centers. Such a penalty could well be one which Tauchchi would approve. The optimal adjustment $(x^*, y^*, \theta^*)$ now is such that

$$\sum_{i=1}^{n} I_i^2(x^*, y^*, \theta^*) = \min_{x,y,\theta} \sum_{i=1}^{n} I_i^2(x, y, \theta).$$  

The easy least-squares theory allows including a rotational adjustment $\theta$, in addition to the translation. The analysis involves partitioning the 2n-df sum of squares

$$\sum_{i=1}^{n} (\epsilon_i^2 + \eta_i^2)$$.
into 4 asymptotically independent parts, respective among with horizontal adjustment (1 df), vertical adjustment (1 df), rotational adjustment (1 df), and a (2n-3)-df residual sum of squares.

**Elements of Extreme Value Theory**

**Elementary results**

The two types of point-to-point interference fitting problems which are discussed in this dissertation are for the case of large \( n \); that is, there are many pegs and holes on the peg-plane and the hole-plane, respectively. It is therefore useful for the minmax part of the work (Chapter 2) to base the analysis on the asymptotic theory of extreme values, and some basic relevant concepts and theorems are now given.

Assume that \( X_1, X_2, \ldots, X_n \) are identically distributed random variables. We put

\[
F(x) = P(X < x)
\]

and

\[
Z_n = \max(X_1, X_2, \ldots, X_n).
\]

By assumption,

\[
H_n(x) = P(Z_n < x) = F^n(x).
\]

Suppose there exist sequences \( a_n \), and \( b_n \) (> 0) of constants, such that, as \( n \to \infty \)

\[
\lim H_n(a_n + b_n x) = H(x)
\]

exists for all continuity points of \( H(x) \) where \( H(x) \) is nondegenerate distribution functions. In other words, \( F(x) \) is in the domain of attraction of \( H(x) \).
It is well known (Fisher and Tipper (1928), Gumbel (1958), Gnedenko (1943),
de Hann (1970), and Galambos (1987) ) that $H(x)$ must be one of following limit
distribution functions, for some $r > 0$:

\begin{align*}
(1) \ H_1(x) &= \begin{cases}
\exp(-x^r) & \text{if } x > 0; \\
0 & \text{otherwise};
\end{cases} \\
(2) \ H_2(x) &= \begin{cases}
1 & \text{if } x \geq 0; \\
\exp(-(-x)^r) & \text{otherwise};
\end{cases} \\
(3) \ H_3(x) &= \exp(-e^{-x}), -\infty < x < \infty.
\end{align*}

The first result addresses the non-uniqueness of the sequences $a_n$, and $b_n$ in
Equations 1.7. The following are some lemmas about asymptotic extreme value
theory, and asymptotic distribution theory in general, which we will use through this
dissertation. They are all to be found in either Cramer (1946), von Mises (1939),
Gnedenko (1943), Galambos (1987), or de Hann (1970). Through all lemmas of this
section, we assume that there exist a pair of sequences, $a_n$ and $b_n$ ($> 0$), such that

\[
\lim_{n \to \infty} P(Z_n \leq a_n + b_n x) = H(x) \tag{1.8}
\]

where $H(x)$ is one of $H_1(x)$, $H_2(x)$, and $H_3(x)$.

**Lemma 1.1** Suppose $F_n(x)$ and $\Phi(x)$ are distribution functions, and $\Phi(x)$ is non-
degenerate. Suppose constant sequences $a_n$, $b_n$ ($> 0$), $a_n^*$, and $b_n^*$ ($> 0$) are such that

as $n \to \infty$

\[
F_n(a_n + b_n x) \longrightarrow \Phi(x),
\]

and

\[
F_n(a_n^* + b_n^* x) \longrightarrow \Phi(x).
\]
Then
\[ \lim_{n \to \infty} \frac{a_n - a_n^*}{b_n} = 0 \]
and
\[ \lim_{n \to \infty} \frac{b_n}{b_n^*} = 1. \]

Lemma 1.1 is a general lemma of non-uniqueness, where \( \{F_n(x)\} \) could be any sequence of distribution functions. For example,
\[ F_n(x) = F^m(x) \]
is just a special case for this lemma.

**Lemma 1.2** Let \( X_1 \)'s be independently and identically distributed random variable with a common distribution function. Then the \( k \)th upper extreme and the \( k \)th lower extreme are asymptotically independently distributed.

The next two lemmas provide equivalent methods to verify that \( F(x) \) is in the domain of attraction of \( H(x) \). The location and scale sequences in the second method are called normalizing constants.

**Lemma 1.3** Assume that \( \alpha \) and \( \beta \) are the lower and upper endpoints of \( F(x) \) respectively, and for some \( a \),
\[ \int_a^\beta (1 - F(x))dx < \infty. \]

Suppose there exists a function \( g(t) \), \( \alpha < t < \beta \), such that as \( t \to \beta \)
\[ \frac{1 - F(t + xg(t))}{1 - F(t)} \to e^{-x}, \quad \forall x. \]
Then,
\[ \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} X_i \leq a_n + b_n x \right) = H_3(x), \]
where
\[ \begin{cases} a_n = \inf \{ x : 1 - F(x) \leq \frac{1}{n} \} \\ b_n = g(a_n). \end{cases} \]

**Lemma 1.4** Assume that \( \alpha \) and \( \beta \) are the lower and upper endpoints of \( F(x) \) respectively, and for some \( a, \)
\[ \int_a^\beta (1 - F(x))dx < \infty. \]

For \( \alpha < t < \beta \), define
\[ R(t) = (1 - F(t))^{-1} \int_t^\beta (1 - F(x))dx. \]

Assume that, for all \( x \), as \( n \to \beta \)
\[ \frac{1 - F(t + xR(t))}{1 - F(t)} \to e^{-x}, \]
then
\[ \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} X_i \leq a_n + b_n x \right) = H_3(x), \]
where
\[ \begin{cases} a_n = \inf \{ x : 1 - F(x) \leq \frac{1}{n} \} \\ b_n = R(a_n). \end{cases} \]

**Lemma 1.5** Suppose there exists a function \( g(t) \), \( \alpha < t < \beta \), where \( \alpha \) and \( \beta \) are the lower and upper endpoints of \( F(x) \), such that as \( t \to \beta \)
\[ \frac{1 - F(t + xg(t))}{1 - F(t)} \to e^{-x}, \quad \forall x. \]
If there exists a constant sequence \( a_n \) such that as \( n \to \infty \)

\[
n(1 - F(a_n)) \to 1,
\]

then

\[
\lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} X_i \leq a_n + g(a_n)x \right) = H_3(x).
\]

**Further results on the limiting distribution functions**

Although the location and scale constants can not be found easily for some distributions, there is no difficulty to find those for random variables \( |X_i| \), \( X_i^2 \), and \( \sqrt{X_i} \), if \( a_n \) and \( b_n \) are known, under some certain conditions.

**Lemma 1.6** Let \( Y_i = |X_i|, i = 1, \ldots, n \). If \( F(-x) = 1 - F(x) \), then

\[
\lim_{n \to \infty} P( \max_{1 \leq i \leq n} Y_i \leq a_n + b_n x) = H^2(x).
\]

Moreover,

\[
\lim_{n \to \infty} (Y_i \leq a'_{n} + b'_{n} x) = H(x),
\]

where

\[
\begin{aligned}
  a'_{n} &= a_{n}, & b'_{n} &= 2^\frac{1}{\tau} b_{n}, & \text{if } H(x) = H_1(x); \\
  a'_{n} &= a_{n}, & b'_{n} &= 2^\frac{-1}{\tau} b_{n}, & \text{if } H(x) = H_2(x); \\
  a'_{n} &= a_{n} + (\ln 2)b_{n}, & b'_{n} &= b_{n}, & \text{if } H(x) = H_3(x).
\end{aligned}
\]

**Proof** We observe that

\[
\max_{1 \leq i \leq n} |X_i| \leq t \iff |X_i| \leq t \forall i
\]

\[
\iff -t \leq X_i \leq t \forall i
\]

\[
\iff \max_{1 \leq i \leq n} X_i \leq t \text{ and } \min_{1 \leq i \leq n} \geq -t
\]
By the assumption of symmetry, 
\[ P \left( \max_{1 \leq i \leq n} X_i \leq a_n + b_n x \right) = P \left( \min_{1 \leq i \leq n} X_i \geq -a_n + b_n (-x) \right), \]
and also by Lemma 1.2, max \( X_i \) and min \( X_i \) are asymptotically independent, hence 
\[ P \left( \max_{1 \leq i \leq n} Y_i \leq a_n + b_n x \right) = P \left( \max_{1 \leq i \leq n} X_i \leq a_n + b_n x \right) P \left( \min_{1 \leq i \leq n} X_i \geq -a_n + b_n (-x) \right) = \left[ P \left( \max_{1 \leq i \leq n} X_i \leq a_n + b_n x \right) \right]^2. \]

Therefore, 
\[ \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} Y_i \leq a_n + b_n x \right) = H^2(x). \]

Note that \( H^2(x) \) still belongs to one of three types of limiting distribution function after translating variable.

1. If \( H(x) = H_1(x) \) and \( x > 0 \),
\[ H^2(x) = \exp(-2x^{-r}) = \exp(-\left(2^{-\frac{1}{r}} x\right)^{-r}) \]
Let \( y = 2^{-\frac{1}{r}} x \), then Equation 1.9 becomes
\[ \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} Y_i \leq a_n + (b_n 2^{-\frac{1}{r}}) y \right) = H(y); \]

2. If \( H(x) = H_2(x) \) and \( x < 0 \),
\[ H^2(x) = \exp(-2(-x)^{-r}) = \exp(-\left(-2^{-\frac{1}{r}} x\right)^{-r}) \]
Let \( y = 2^{-\frac{1}{r}} x \), then Equation 1.9 becomes
\[ \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} Y_i \leq a_n + (b_n 2^{-\frac{1}{r}}) y \right) = H(y); \]
3. If $H(x) = H_3(x)$,

$$H^2(x) = \exp(-2e^{-x}) = \exp(-e^{-(x-ln 2)})$$

Let $y = \frac{1}{2^r} x$, then Equation 1.9 becomes

$$\lim_{n \to \infty} P(\max_{1 \leq i \leq n} Y_i \leq a_n + b_n(y + \ln 2)) = \lim_{n \to \infty} P(\max_{1 \leq i \leq n} Y_i \leq (a_n + (\ln 2)b_n) + b_n y) = H(y).$$

Lemma 1.7 Assume that the distribution of $X$ is symmetric and $\lim_{n \to \infty} \frac{b_n}{a_n} = 0$. If $Y_i = X_i^2$, $i = 1, \ldots, n$, then

$$\lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} Y_i \leq c_n + d_n x \right) = H^2(x)$$

where $c_n = a_n^2$ and $d_n = 2a_nb_n$, or

$$\lim_{n \to \infty} (Y_i \leq c'_n + d'_n x) = H(x),$$

where

$$\begin{cases} c'_n = c_n, & d'_n = 2^r d_n, \quad \text{if } H(x) = H_1(x); \\ c'_n = c_n, & d'_n = 2^{-\frac{1}{r}} d_n, \quad \text{if } H(x) = H_2(x); \\ c'_n = c_n + (\ln 2)d_n, & d'_n = d_n, \quad \text{if } H(x) = H_3(x). \end{cases}$$

[Proof] Since

$$\max_{1 \leq i \leq n} X_i^2 = \left( \max_{1 \leq i \leq n} |X_i| \right)^2.$$
\[ P\left( \max_{1 \leq i \leq n} |X_i| \leq a_n + b_n x \right) = P\left( \left( \max_{1 \leq i \leq n} X_i \right)^2 \leq (a_n + b_n x)^2 \right) = P\left( \max_{1 \leq i \leq n} X_i^2 \leq a_n^2 + (2a_n b_n + b_n^2 x)x \right). \]

Let \( d_n = 2a_nb_n \), then \( \forall x, \)
\[
\lim_{n \to \infty} \frac{2a_nb_n + b_n^2 x}{d_n} = 1 + \lim_{n \to \infty} \frac{b_n x}{a_n} \\
\text{since } b_n \neq 0 \\
= 1, \\
\text{since } \lim_{n \to \infty} \frac{b_n}{a_n} = 0.
\]

Therefore, by Lemma 1.1,
\[
\lim_{n \to \infty} P\left( \max_{1 \leq i \leq n} Y_i \leq a_n^2 + (2a_n b_n) x \right) = \lim_{n \to \infty} P\left( \max_{1 \leq i \leq n} |X_i| \leq a_n + b_n x \right) = H^2(x), \text{ by Lemma 1.6.}
\]

Similarly, as we have done in the proof of Lemma 1.6, \( H^2(x) \) can be rewritten as a form of \( H(x) \).

**Lemma 1.8** Suppose that \( X_1, X_2, \ldots, X_n \) are positive variables and \( Y_i = \sqrt{X_i}, i = 1, \ldots, n. \) If \( \lim_{n \to \infty} \frac{b_n}{a_n} = 0, \) then
\[
\lim_{n \to \infty} P\left( \max_{1 \leq i \leq n} Y_i \leq c_n + d_n x \right) = H(x),
\]
where \( c_n = \sqrt{a_n} \) and \( d_n = \frac{b_n}{2\sqrt{a_n}}. \)

**Proof** We know that, for all \( t > 0, \)
\[
X_i \leq t \quad \forall i \iff \sqrt{X_i} \leq \sqrt{t}, \quad \forall i;
\]
that is,
\[
\max_{1 \leq i \leq n} X_i \leq a_n + b_n x \iff \max_{1 \leq i \leq n} \sqrt{X_i} \leq \sqrt{a_n} \sqrt{1 + \frac{b_n}{a_n} x}.
\]

Let \( X_n = \frac{2an}{bn} \left( \sqrt{1 + \frac{b_n}{an} x} - 1 \right) \), then
\[
\sqrt{a_n} \sqrt{1 + \frac{b_n}{an} x} = \sqrt{a_n} \left( \frac{b_n x}{2 \sqrt{a_n} x} \right).
\]

We need to show that
\[
\lim_{n \to \infty} \frac{x_n}{x} \to 1.
\]

Let \( \epsilon_n = \sqrt{1 + \frac{b_n}{an} x} \to 1 \) if \( \frac{b_n}{an} \to 0 \).

Also note that, for any \( x \), as \( n \to \infty \)
\[
\epsilon_n = \sqrt{1 + \frac{b_n}{an} x} \to 1.
\]

Hence,
\[
\frac{x_n}{x} = \frac{2}{\epsilon_n + 1} \to 1.
\]

since
\[
\lim_{n \to \infty} \frac{b_n x_n}{2 \sqrt{a_n} x} = \lim_{n \to \infty} \frac{x_n}{x} = 1
\]

by Lemma 1.1, we have
\[
\lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} Y_i \leq \sqrt{a_n} + \left( \frac{b_n x_n}{2 \sqrt{a_n} x} \right) \right)
\]
\[
= \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} Y_i \leq \sqrt{a_n} + \frac{b_n}{2 \sqrt{a_n} x} \right).
\]
Illustrations

Example 1. ($X^2_2$ distribution) Let $X_1, X_2, \ldots, X_n$ be iid random variables with common distribution function $F(t)$.

$$F(t) = 1 - e^{-t/2} \quad t \geq 0.$$ 

Define a function $g(t) = 2, \forall t \geq 0$. Since 

$$\frac{1 - F(t + 2x)}{1 - F(t)} = \frac{e^{-(t+2x)/2}}{e^{-t/2}} = e^{-x}$$

is true for any $x$, then by Lemma 1.3 we have 

$$\lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} X_i \leq a_n + b_n x \right) = H_3(x),$$

that is, $F(t)$ is the domain of attraction of $H_3(x)$, and one choice of the sequences $a_n$ and $b_n > 0$ is: $a_n = \inf \{ x : 1 - F(x) \leq \frac{1}{n} \}$ and $b_n = g(a_n)$. That is 

$$\begin{cases} a_n = 2 \log n; \\ b_n = 2. \end{cases}$$

Example 2 and 3 are two different approaches to search location constants $a_n$ and scale constants $b_n$ for the Rayleigh($\sqrt{2}$) distribution. Example 2 does so by using Lemma 1.3, while Example 3 does so by using Example 1, plus Lemma 1.8.

Example 2. (Rayleigh($\sqrt{2}$) distribution) Let $X_1, X_2, \ldots, X_n$ be iid random variables with common distribution $F(t)$.

$$F(t) = 1 - e^{-t^2/2}, \quad \forall t > 0.$$
Let 
\[ g(t) = \frac{1}{t}, \quad \forall t > 0, \]
then
\[ \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x - \frac{x^2}{2t}} \]
\[ \rightarrow e^{-x} \]
as \( t \rightarrow \infty \)

Hence \( F(t) \) is in the domain of attraction of \( H_3(x) \), and one choice of the sequences \( a_n \) and \( b_n > 0 \) such that

\[ \lim_{n \rightarrow \infty} P \left( \max_{1 \leq i \leq n} X_i \leq a_n + b_n x \right) = H_3(x) \]

are

\[ a_n = \inf \left\{ x : e^{-x^2/2} \leq \frac{1}{n} \right\} \]
\[ = \inf \{ x : x^2 \geq 2 \log n \} \]
\[ = \sqrt{2 \log n}; \]

\[ b_n = g(a_n) = 1/\sqrt{2 \log n}. \]

Example 3. Let \( X_i \sim \chi_2^2 \) and \( Y_i = \sqrt{X_i} \), then \( Y_i \sim \text{Rayleigh}(\sqrt{2}) \). For this example, we would like to use Lemma 1.8 to find the sequences \( a_n \) and \( b_n \).

From example 1, we know that

\[ P \left( \max_{1 \leq i \leq n} X_i \leq 2 \log n + 2x \right) = H_3(x). \]
The distribution function of $X_i$ is symmetric and

\[
\frac{b_n}{a_n} = \frac{1}{\log n} \rightarrow 0, \text{ as } n \rightarrow \infty,
\]

so

\[
P \left( \max_{1 \leq i \leq n} Y_i \leq a_n^* + b_n^* x \right) = H_3(x),
\]

where

\[
\begin{align*}
a_n^* &= \sqrt{2 \log n}; \\
b_n^* &= \frac{1}{\sqrt{2 \log n}}.
\end{align*}
\]
CHAPTER 2. MINMAX POINT-TO-POINT INTERFERENCE

Introductory

Considering two simple configurations featuring a plane containing circular holes and a plate containing corresponding circular pegs. The centers of pegs in the peg-plane are located at

\[(u_i, v_i) \quad \forall i = 1, \ldots, n,\]

and the centers of their corresponding holes in the hole-plane are located at

\[(u_i + \epsilon_i, v_i + \eta_i) \quad \forall i = 1, \ldots, n.\]

Here \(\epsilon_i\) and \(\eta_i\) are horizontal and vertical random variables, respectively. Recall that the point-to-point interferences are the distances between the centers of holes and the centers of their corresponding pegs (see Chapter 1). Suppose we put the hole-plane on the top of the peg-plane and move the origin of the hole-plane to \((x, y)\) in the peg-plane; then the coordinates of the centers of the pegs on the hole-plane become

\[(u_i + x, v_i + y) \quad \forall i = 1, \ldots, n,\]

and the point-to-point interference for the \(i\)th peg and its corresponding hole can be written as

\[\sqrt{(\epsilon_i - x)^2 + (\eta_i - y)^2} \quad \forall i = 1, \ldots, n. \quad (2.1)\]
Equation 2.1 shows that the design locations for the centers of pegs and holes need not be considered. The minmax interference problem is reduced to minimizing the maximal distance between \((x, y)\) and any point \((\epsilon_i, \eta_i)\).

The following inequalities express the main concepts of this chapter.

\[
\begin{align*}
\max_{1 \leq i \leq n} \sqrt{(\epsilon_i - x_o)^2 + (\eta_i - y_o)^2} & \geq \min_{(x, y) \in \mathbb{R}^2} \max_{1 \leq i \leq n} \sqrt{(\epsilon_i - x)^2 + (\eta_i - y)^2} \equiv I_n \quad (2.2) \\
\min_{1 \leq i < j \leq n} \max_{l = i, j} \sqrt{(\epsilon_l - x)^2 + (\eta_l - y)^2} & \geq \max_{1 \leq i < j \leq n} \min_{(x, y) \in \mathbb{R}^2} \max_{l = i, j} \sqrt{(\epsilon_l - x)^2 + (\eta_l - y)^2} \equiv I_{2,n} \quad (2.5)
\end{align*}
\]

In the next section, the relation between the first two inequalities is used to find an upper bound for the minmax interference, \(I_n\). The population means \((0, 0)\) of \(\epsilon_i\) and \(\eta_i\) are used for \(x_o\) and \(y_o\) respectively in Equation 2.2. This leads to the maximum of \(n\) Rayleigh\((\sqrt{2})\) random variables as an upper bound for \(I_n\). In the following section, an equivalent form for \(I_{2,n}\) is given, and \(I_{2,n}\) is further lower bounded from below. The aim of the rest sections of this chapter is to discuss an approximating process for an even further lower bound, namely a lower bound for (2.5).

**An Upper Bound Argument for Minmax Interference**

Let \((\epsilon_i, \eta_i), i = 1, \ldots, n\), be \(n\) points in the plane and denote the minmax interference by \(I_n\). Assume that \(\epsilon_i\) and \(\eta_i\) are independently and identically distributed random variables with a common normal distribution function. Without loss of generality we suppose this normal distribution function is a standard normal distribution.
function. Recall the definition of minmax interference:

\[ I_n = \min_{(x,y) \in \mathcal{R}^2} \max_{1 \leq i \leq n} \sqrt{(\epsilon_i - x)^2 + (\eta_i - y)^2}. \quad (2.6) \]

It is clear that any \((x, y)\) in \(\mathcal{R}^2\) determines an upper bound for \(I_n\). Using \((x_0, y_0) = (0, 0)\) in Equation 2.2 means no location adjustment leading to a corresponding upper bound \(r_1\) for \(I_n\), where \(r_1\) is the largest distance between any \((\epsilon_i, \eta_i)\) and the origin. This upper bound provides us with the information how bad the maximum interference will be if we don't do any adjustment at all. Another reason for choosing \((0, 0)\) is that \(\epsilon_i\) and \(\eta_i\) are both symmetrically distributed with mean 0, so that \((0, 0)\) should not give us a very bad upper bound. Of course, the limiting distribution of \(r_1\) then also provides a ready-made limiting distribution function for an upper bound to minmax interference.

**Theorem 2.1** Suppose that \(\epsilon_i \sim \text{i.i.d.} \ N(0, 1)\), \(\eta_i \sim \text{i.i.d.} \ N(0, 1)\), for all \(i = 1, \ldots, n\), and that \(\epsilon_i\)'s and \(\eta_i\)'s are also independent. For \(i = 1, \ldots, n\) let

\[ r_i = \sqrt{\epsilon_i^2 + \eta_i^2} \]

and reorder the \(r_i\) as \(r_1 \geq \ldots \geq r_n\). Then

\[ \lim_{n \to \infty} P \left( r_1 \leq \sqrt{2 \log n} + \frac{1}{\sqrt{2 \log n}} \right) = \exp(-e^{-t}), \quad (2.7) \]

\[ \forall -\infty < t < \infty. \]

**Proof** By assumption, \(r_i\)'s are independently and identically distributed, according to the Rayleigh(\(\sqrt{2}\)) distribution. Then Equation 2.7 follows by Example 2 in Chapter 1.
A Lower Bound Argument for Minmax Interference

We note first, as also noted by McCann (1988), that

\[
\begin{align*}
\min_{(x, y) \in \mathbb{R}^2} \max_{l=i,j} \sqrt{(\epsilon_l - x)^2 + (\eta_l - y)^2} &= \sqrt{(\epsilon_i - x_{i,j}^0)^2 + (\eta_i - y_{i,j}^0)^2} \\
&= \sqrt{(\epsilon_j - x_{i,j}^0)^2 + (\eta_j - y_{i,j}^0)^2} \\
&\equiv I(i, j),
\end{align*}
\]

where \((x_{i,j}^0, y_{i,j}^0)\) is the middle point of points \(i\) and \(j\), and

\[
(x_{i,j}^0, y_{i,j}^0) = \left( \frac{1}{2}(\epsilon_i + \epsilon_j), \frac{1}{2}(\eta_i + \eta_j) \right),
\]

so that \(I_{2,n}\) may be written

\[
I_{2,n} = \max_{1 \leq i < j \leq n} I(i, j),
\]

and the right hand side of Equation 2.8 provides a further form for a lower bound to \(I_n\).

The following theorem gives a further useful expression equivalent to Equation 2.8, namely (2.11), where \(R_{n, \theta}\) is given by (2.9) and (2.10).

**Theorem 2.2** Suppose \(I_{2,n}\) is defined as before, and define

\[
\begin{aligned}
M_{n, \theta} &= \max_{1 \leq i \leq n} (\epsilon_i \cos \theta + \eta_i \sin \theta), \\
m_{n, \theta} &= \min_{1 \leq i \leq n} (\epsilon_i \cos \theta + \eta_i \sin \theta),
\end{aligned}
\]

and define the "directional range" \(R_{n, \theta}\) by

\[
R_{n, \theta} = M_{n, \theta} - m_{n, \theta}.
\]
Then
\[ 2I_{2,n} = \max_{0 \leq \theta < \pi} R_{n,\theta}. \] (2.11)

[Proof] This proof includes two parts, the first part shows that \(2I_{2,n}\) is larger than or equal to \(\max R_{n,\theta}\) and the second part shows that \(\max R_{n,\theta}\) is at least larger than or equal to \(2I_{2,n}\).

(I) For any \(\theta\), where \(0 \leq \theta < \pi\), there exists \(1 \leq l(\theta) < k(\theta) \leq n\) and \(l(\theta) \neq k(\theta)\), such that
\[ M_{n,\theta} = \epsilon l(\theta) \cos \theta + \eta l(\theta) \sin \theta, \]
and
\[ M_{n,\theta} = \epsilon k(\theta) \cos \theta + \eta k(\theta) \sin \theta. \]

Then
\[ R_{n,\theta} = M_{n,\theta} - M_{n,\theta} \leq d(l(\theta), k(\theta)), \]
where \(d(l(\theta), k(\theta))\) denotes the distance between points \(l(\theta)\) and \(k(\theta)\). Define a subset of \(\{1, 2, \ldots, n\}\), \(S_n\), as:
\[ S_n = \{i : \epsilon_i \cos \theta + \eta_i \sin \theta = M_{n,\theta} \text{ or } \epsilon_i \cos \theta + \eta_i \sin \theta = M_{n,\theta} \text{ for some } 0 \leq \theta < \pi\} \]

Then,
\[ 2I_{2,n} = \max_{1 \leq i < j \leq n} d_{i,j} \geq \max_{i,j \in S_n} \max_{0 \leq \theta < \pi} d_{i,j} \]
\[ \geq \max_{0 \leq \theta < \pi} R_{n,\theta}. \]

(II) Denote \(P_i\) as \((\epsilon_i, \eta_i)\) and \(\theta(i, j)\) as the angle of \(P_i \overrightarrow{P_j}\). Then for any \(1 \leq i < j \leq n\), we have
\[ d_{i,j} \leq R_{n,\theta(i,j)}. \]
Consequently,

\[ 2I_{2,n} = \max_{1 \leq i < j \leq n} d_{i,j} \leq \max_{0 \leq \theta < \pi} R_{n,\theta}. \]

Combine the results of (I) and (II), we have

\[ 2I_{2,n} = \max_{0 \leq \theta < \pi} R_{n,\theta}. \]

The lower-bounding process now goes on: Three different quantities no greater than \(2I_{2,n}\), together with their asymptotic distributions, are considered below. These quantities are produced, by restricting the number of \(\theta\) in Equation 2.11, respectively, to a single angle, to a finite number of angle, and to a number of angle growing with \(n\) at a suitable rate.

The Case of a Single Angle — The Limiting Distribution Function for a Range

In this section, we consider a range of the set of \((\epsilon_i, \eta_i)\)'s with respect to direction \(\theta\). Without loss of generality, we set \(\theta = 0\), and write \(R_{n,\theta}\) as \(R_n\) in this moment. For any other \(\theta\)'s, the "axes rotation" process can be used to translate them to "\(\theta = 0\) case." If \(\theta = 0\), then \(M_{n,\theta} = \max_{1 \leq i \leq n} \epsilon_i\) and \(m_{n,\theta} = \min_{1 \leq i \leq n} \epsilon_i\), and, again, rewrite them as \(M_n\) and \(m_n\) respectively.

Lemma 2.1 gives the limiting distribution function of a range. It follows directly from Lemma 1.2. The proof of Lemma 2.1 can be found in several articles (de Hann, Galambos, etc.).

**Lemma 2.1** Suppose \(\epsilon_i\)'s are independently and identically distributed random variables, and, additionally, suppose there exist sequences \(a_n, c_n,\) and \(b_n (> 0)\) such
that
\[ \lim_{n \to \infty} P(M_n \leq a_n + b_n t) = \lim_{n \to \infty} H_n(t) = H(t) \]
and
\[ \lim_{n \to \infty} P(m_n \leq c_n + b_n t) = \lim_{n \to \infty} L_n(t) = L(t) \]
where \( H(t) \) is one of upper limiting distribution functions, and \( L(t) \) is one of lower limiting distribution functions. Then
\[ \lim_{n \to \infty} P(R_n \leq (a_n + c_n) + b_n t) = \int_{-\infty}^{\infty} [1 - L(s - t)] dH(s). \]

The following example gives the limiting distribution function for \( R_n \), using Lemma 2.1.

**Example 1. (Normal distribution)**

If \( \epsilon_i \sim i.i.d. \mathcal{N}(0,1) \), then \( \forall -\infty < t < \infty \),
\[ \lim_{n \to \infty} P(M_n \leq a_n + b_n t) = \exp(-e^{-t}) \]
and
\[ \lim_{n \to \infty} P(m_n \leq -a_n + b_n t) = 1 - \exp(-e^t), \]
where one choice of \( a_n \) and \( b_n \) can be expressed by
\[ \begin{cases} a_n = \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} \\ b_n = \frac{1}{\sqrt{2 \log n}} \end{cases} \] (2.12)

Then the limiting distribution function for \( R_n \) can be expressed as:
\[ \lim_{n \to \infty} P(R_n \leq 2a_n + b_n t) = \int_{-\infty}^{\infty} \exp\left(-e^{(s-t)}\right) d\left[\exp\left(-e^{-t}\right)\right]. \]
From Theorem 2.2, half of \( R_n \) is a lower bound for \( I_{2,n} \). Consequently, it is a lower bound for \( I_n \), by Equation 2.5. Although \( \frac{1}{2} R_n \) may not be a good lower bound, we can still compare the location and scale constant sequences for \( r[1] \) with those for \( \frac{1}{2} R_n \). The location constant of \( r[1] \), say \( a_{r,n} \), is \( \sqrt{2 \log n} \) (see Example 2 in Chapter 1), and that of \( R_n \), say \( a_{R,n} \), is
\[
\frac{\sqrt{2 \log n} - \log \log n + \log 4\pi}{2 \sqrt{2 \log n}}.
\]
In both limiting distribution functions, the scale constants are the same, namely \( b_n \) equal to \( 1/\sqrt{2 \log n} \). If \( \lim_{n \to \infty} (a_{r,n} - a_{R,n})/b_n = 0 \), according to Lemma 1.1, we can say that the two limiting distribution functions are equivalent. Unfortunately, this is not the case, although \( a_{r,n} - a_{R,n} \to 0 \) as \( n \to \infty \).

The Case of a Finite Number \( k \) of Angles — The Limiting Distribution Function for the Maximum of \( k \) Directional Ranges

In this section, we are going to consider \( k \) directional ranges instead of one range to find a better lower bound for \( I_{2,n} \). As we know, a directional range is the difference between maximum and minimum extreme values with respect to a particular direction, and, for the case of a symmetric distribution, the limiting distribution function of the maximum extreme value is equal to the limiting complement distribution function of minimum extreme value. Also, because of the asymptotic independence of the maximum extreme value and minimum extreme value, it is relevant to consider the joint distribution function of two maximum extreme values, \( M_n, \theta_1 \) and \( M_n, \theta_2 \), \( \theta_1 \neq \theta_2 \). Then it is possible to extend the result for the case of 2 ranges to the
case of \( k \) ranges. Theorem 2.3 shows that for any two different directions, say \( \theta_1 \) and \( \theta_2 \), \( M_{n, \theta_1} \) and \( M_{n, \theta_2} \) are asymptotically independent in the normal case. Theorem 2.3 explicates the technique to be used in Theorem 2.4, which treats the case of \( q = 2k \). In order to simplify notation, we denote \( M_{n, \theta_1} \) and \( M_{n, \theta_2} \) by \( M_{n, 1} \) and \( M_{n, 2} \), respectively.

**Theorem 2.3** Suppose \( M_{n, 1} \) and \( M_{n, 2} \) are the maximum extreme values with respect to \( \theta_1 \) and \( \theta_2 \) respectively and the assumption on \( \epsilon_i \) and \( \eta_i \) in Theorem 2.1 hold. Then \( M_{n, 1} \) and \( M_{n, 2} \) are independent as \( n \to \infty \), with marginal asymptotic distribution equal to \( H_2(x) \).

**[Proof]** Without loss of generality, the axes are rotated in order to translate \( M_{n, 1} \) and \( M_{n, 2} \) to be the extreme values with respect to \( \theta \) and \(-\theta\) (according to the new axes), where

\[
\theta = \frac{|\theta_1 - \theta_2|}{2}.
\]

Note that the event of \( M_{n, 1} \leq x \) is equivalent to the event

\[
\epsilon_i \cos \theta + \eta_i \sin \theta \leq x, \quad \forall i = 1, \ldots, n,
\]

and, similarly, the event of \( M_{n, 2} \leq y \) is equivalent to the event

\[
\epsilon_i \cos \theta - \eta_i \sin \theta \leq y, \quad \forall i = 1, \ldots, n.
\]

In addition, the intersection of \( \epsilon \cos \theta + \eta \sin \theta = x \) and \( \epsilon \cos \theta - \eta \sin \theta = y \) is at

\[
\left( \frac{x + y}{2 \cos \theta}, \frac{x - y}{2 \cos \theta} \right). \tag{2.13}
\]
Then the joint distribution function of $M_{n,1}$ and $M_{n,2}$ becomes

$$ P \left( \max_{1 \leq i \leq n} (\epsilon_i \cos \theta + \eta_i \sin \theta) \leq x; \max_{1 \leq i \leq n} (\epsilon_i \cos \theta - \eta_i \sin \theta) \leq y \right) $$

$$ = P(\epsilon_i \cos \theta + \eta_i \sin \theta \leq x \text{ and } \epsilon_i \cos \theta - \eta_i \sin \theta \leq y; \forall i = 1, \ldots, n) $$

$$ = [P(\epsilon_i \cos \theta + \eta_i \sin \theta \leq x \text{ and } \epsilon_i \cos \theta - \eta_i \sin \theta \leq y)]^n $$

(by the i.i.d. assumption on $(\epsilon_i, \eta_i)$)

$$ = [1 - P(A_\theta(x,y))]^n, $$

where $A_\theta(x,y)$ (see Figure 2.1) is a region such that, for $\forall (\epsilon, \eta) \in A_\theta(x,y)$,

$$ \epsilon \cos \theta + \eta \sin \theta > x $$

or

$$ \epsilon \cos \theta - \eta \sin \theta > y. $$

Hence, the problem is reduced to finding the probability of $A_\theta(x,y)$.

The first step is to translate coordinates $(\epsilon_i, \eta_i)$ to $(u_{1,i}, v_{1,i})$ by setting

$$ u_{1,i} = \epsilon_i \cos \theta + \eta_i \sin \theta $$

and

$$ v_{1,i} = \eta_i \cos \theta - \epsilon_i \sin \theta, $$

then the intersection point (see Equation 2.13) becomes $(x, \lambda_\theta(x,y))$, where

$$ \lambda_\theta(x,y) = \frac{x + y}{2} \cot \theta - \frac{x - y}{2} \tan \theta. $$

Similarly, translate $(\epsilon_i, \eta_i)$ to $(u_{2,i}, v_{2,i})$ by setting

$$ u_{2,i} = \epsilon_i \cos \theta - \eta_i \sin \theta $$
Figure 2.1: Illustration of a region of $A_g(x, y)$

$A_g(x, y) = A_{1,g}(x, y) + A_{2,g}(x, y) + R_g(x, y)$
and

\[ v_{2,i} = \eta_i \cos \theta + \epsilon_i \sin \theta. \]

Then, again, the intersection point becomes \((y, \rho\theta(x, y))\), where

\[ \rho\theta(x, y) = \frac{x - y}{2} \cot \theta + \frac{x + y}{2} \tan \theta. \]

We call \(\lambda\theta(x, y)\) the "endpoint" with respect to direction \(\theta\). Similarly, \(\rho\theta(x, y)\) is the "endpoint" with respect to direction \(-\theta\).

Since the random variables \(u_{k,i}\) and \(v_{k,i}\) are independent, for \(k = 1\), as well as for \(k = 2\), \(A\theta(x, y)\) can be partitioned into three regions, say \(A_1,\theta(x, y), A_2,\theta(x, y),\) and \(R_{n,\theta}(x, y)\) (see Figure 2.1), such that

\[ A_1,\theta(x, y) = \{(u_{1,i}, v_{k,i}) : u_{1,i} \geq x, v_{1,i} \geq \lambda\theta(x, y)\} \]

\[ A_2,\theta(x, y) = \{(u_{2,i}, v_{2,i}) : u_{2,i} \geq y, v_{2,i} \leq \rho\theta(x, y)\}, \]

and

\[ R_{n,\theta}(x, y) = A\theta(x, y) \setminus \left( A_1,\theta(x, y) + A_2,\theta(x, y) \right), \]

where \(A_1,\theta(x, y)\) and \(A_2,\theta(x, y)\) represent the half-infinite rectangular areas, and \(R_{n,\theta}(x, y)\) represents the residual area of \(A\theta(x, y)\).

Let \(\xi = n(1 - \Phi(x))\) and \(\zeta = n(1 - \Phi(y))\) where \(\Phi(\cdot)\) is the standard normal distribution function. Then by Cramer (1946), \(x\) and \(y\) can be expressed as functions of \(\xi\) and \(\zeta\) respectively:

\[ x = a_n - b_n \log \xi + O\left( \frac{1}{\log n} \right) \tag{2.14} \]

\[ y = a_n - b_n \log \zeta + O\left( \frac{1}{\log n} \right), \tag{2.15} \]
where
\[
\begin{align*}
    a_n &= \sqrt{2} \log n - \log \log n + \log 4 \pi \frac{1}{2 \sqrt{2} \log n} \\
    b_n &= \frac{1}{\sqrt{2} \log n}.
\end{align*}
\]

Now, we need go back to the endpoints \( \lambda_\theta(x, y) \) and \( \rho_\theta(x, y) \). By using Equations 2.14 and 2.15, \( 2\lambda_\theta(x, y) \) can be rewritten as:
\[
\left[ b_n \left( \log \frac{\zeta}{n} \right) + O \left( \frac{1}{\log n} \right) \right] \cot \theta - \left[ 2a_n - b_n (\log \xi \zeta) + O \left( \frac{1}{\log n} \right) \right] \tan \theta.
\]
Hence for any positive number \( K \), there exists \( N_{\theta, \xi, \zeta}(K) \) such that
\[
\lambda_\theta(x, y) < -K, \quad \forall n \geq N_{\theta, \xi, \zeta}(K),
\]
and then
\[
P(v_{1,i} < \lambda_\theta(x, y)) = o_n(1).
\]
Note that \( x \) and \( y \) will increase as \( n \) increases. In addition, by the independence of \( u_{1,i} \) and \( v_{1,i} \), we have
\[
P(A_{1,\theta}(x, y)) = P(u_{1,i} \geq x)P(v_{1,i} \geq \lambda_\theta(x, y))
\]
\[
= (1 - \Phi(x))(1 + o_n(1))
\]
\[
= \frac{\xi}{n}(1 + o_n(1)).
\]
Similar analysis for \( \rho_\theta(x, y) \) yields the fact that, for any positive number \( K' \), there exists \( N_{\theta, \xi, \zeta}(K') \) such that
\[
\rho_\theta(x, y) > K', \quad \forall n \geq N_{\theta, \xi, \zeta}(K'),
\]
and then
\[
P(v_{2,i} > \rho_\theta(x, y)) = o_n(1).
\]
Therefore, we have
\[
P(A_{2, \theta}(x, y)) = \frac{\zeta}{n}(1 + o_n(1)).
\] (2.20)

Further, because
\[
P(R_{n, \theta}(x, y)) \leq P(u_{1,i} \geq x, v_{1,i} \leq \lambda_{\theta}(x, y)) = \frac{\xi}{n}(o_n(1)),
\]
we can write that
\[
P(R_{n, \theta}(x, y)) = \frac{1}{n}o_n(1).
\] (2.21)

Therefore, by Equations 2.19, 2.20, and 2.21,
\[
P(A_\theta(x, y)) = \frac{1}{n}[(\xi(1 + o_n(1)) + \zeta(1 + o_n(1)) + o_n(1)]
\]
and as \(n \to \infty\),
\[
\xi(1 + o_n(1)) + \zeta(1 + o_n(1)) + o_n(1) \to \xi + \zeta.
\] (2.22)

Finally, the limiting joint distribution function of \(M_{n,1}\) and \(M_{n,2}\) can be obtained, using Equation 2.22, as
\[
\lim_{n \to \infty} P(M_{n,1} \leq x, M_{n,2} \leq y) = \exp(-e^{-\xi})\exp(-e^{-\zeta}).
\]

Now, while still adhering to Equations 2.14 and 2.15, let \(\xi\) and \(\zeta\) be fixed, not dependent on \(n\), and let \(t = -\log \xi\) and \(s = -\log \zeta\). Then also by Lemma 1.1, we have
\[
\lim_{n \to \infty} P(M_{n,1} \leq a_n + b_n t, M_{n,2} \leq a_n + b_n s) = \exp(-e^{-t})\exp(-e^{-s})
\]
Theorem 2.4 Suppose we have $q$ different directions $0 < \theta_1 < \theta_2 < \ldots < \theta_q < 2\pi$ and the assumption on $\epsilon_i$ and $\eta_i$ in Theorem 2.1 hold. Let $M_{n,i}$ represent the maximum value with respect to direction $\theta_i$, that is, for all $i$

$$M_{n,i} = \max_{1 \leq j \leq n} (\epsilon_j \cos \theta_i + \eta_j \sin \theta_i).$$

Then $M_{n,1}, \ldots, M_{n,q}$ are asymptotically independent, with marginal asymptotic distributions equal to $H_3(x)$, and

$$\lim_{n \to \infty} P\left( \max_{1 \leq i \leq q} M_{n,i} \leq c_n(q) + b_n t \right) = \exp \left( -e^{-t} \right),$$

where

$$\begin{cases}
    c_n(q) = \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2 \sqrt{2 \log n}} + \frac{\log q}{\sqrt{2 \log n}} \\
    b_n = \frac{1}{\sqrt{2 \log n}}.
\end{cases}$$

[Proof] The joint distribution function of $M_{n,1}, \ldots, M_{n,q}$ can be expressed by

$$P(M_{n,1} \leq x_1, \ldots, M_{n,q} \leq x_q) = [1 - P(A(x_1, \ldots, x_q))]^n,$$

where $A(x_1, \ldots, x_q)$ is the “outside region” bounded by $\epsilon \cos \theta_i - \eta \sin \theta_i = x_i$, $i = 1, \ldots, q$. Techniques similar to those used in the proof of Theorem 2.3 are used here. By using “axis-rotation” and “coordinate-translation” processes, we can get two endpoints, $E_{1,i}$ and $E_{2,i}$, with respect to each direction and new axes. Note that $E_{1,i}$ is a function of $x_i$ and $x_{i+1}$ and $E_{2,i}$ is a function of $x_{i-1}$ and $x_i$. Then $A(x_1, \ldots, x_q)$ can naturally be partitioned into $q + 1$ sub-regions. With $x_0$ equal to $x_q$ and $x_{q+1}$ equal to $x_1$,

$$A(x_1, \ldots, x_q) = \sum_{i=1}^{q} A_i(x_{i-1}, x_i, x_{i+1}) + R(x_1, \ldots, x_q),$$
Figure 2.2: Illustration of a region of $A_i(x_{i-1}, x_i, x_{i+1})$
where $A_i(x_{i-1}, x_i, x_{i+1})$ is the rectangular region between two endpoints $E_{1,i}$ and $E_{2,i}$, and $R(x_1, \ldots, x_q)$ is the residual region of $A(x_1, \ldots, x_q)$ (see Figure 2.2).

Let $\xi_i = n(1 - \Phi(x_i))$, then, again,

$$x_i = a_n - b_n \log \xi_i + O \left( \frac{1}{\log n} \right),$$

where

$$\begin{cases} a_n = \sqrt{2\log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2\log n}} \\ b_n = \frac{1}{\sqrt{2\log n}} \end{cases}$$

We have, for $i = 1, \ldots, q$

$$P(A_i(x_{i-1}, x_i, x_{i+1})) = \frac{\xi_i}{n}(1 + o_n(1))$$

and

$$P(R(x_1, \ldots, x_q)) = \frac{1}{n} o_n(1),$$

since the distances $l_i$ between two endpoints satisfy

$$l_i \to \infty, \quad \text{as } n \to \infty$$

(see the proof in Theorem 2.3). Then,

$$\lim_{n \to \infty} P(M_{n,1} \leq x_1, \ldots, M_{n,q} \leq x_q) = \lim_{n \to \infty} \left[ 1 - \frac{1}{n} \left( \sum_{i=1}^{q} \xi_i (1 + o_n(1)) + o_n(1) \right) \right]^n$$

$$= \exp \left( - \sum_{i=1}^{q} \xi_i \right).$$

In addition, from Equation 2.23 and by setting $t_i = -\log \xi_i$, the limiting joint distribution function becomes:

$$\lim_{n \to \infty} P(M_{n,i} \leq an + bnt_i, i = 1, \ldots, q)$$

$$= \prod_{i=1}^{q} \left[ \lim_{n \to \infty} P(M_{n,i} \leq an + bnt_i) \right]$$

$$= \prod_{i=1}^{q} \exp(-e^{-t_i}),$$
which shows that \( M_{n,1}, \ldots, M_{n,q} \) are asymptotically independent. With regard to the \( \max_{1 \leq i \leq q} M_{n,i} \),

\[
\lim_{n \to \infty} P \left( \max_{1 \leq i \leq q} M_{n,i} \leq a_n + b_n s \right) = \left[ \lim_{n \to \infty} P(M_{n,1} \leq a_n + b_n s) \right]^q = [\exp(-e^{-s})]^q = \exp \left( -e^{-(s - \log q)} \right).
\]

Now let \( t = s - \log q \); then

\[
\lim_{n \to \infty} P \left( \max_{1 \leq i \leq q} M_{n,i} \leq c_n + b_n t \right) = \exp \left( -e^{-t} \right),
\]

\[ \forall -\infty < t < \infty, \]

where

\[
\begin{align*}
c_n(q) &= a_n + b_n \log q \\
&= \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} + \frac{\log q}{\sqrt{2 \log n}} \\
b_n &= \frac{1}{\sqrt{2 \log n}}.
\end{align*}
\]

Theorem 2.4 shows that the location constant for \( \max_{1 \leq i \leq k} M_{n,i} \) enlarges slightly, by amount \( k/\sqrt{2 \log n} \). The following theory shows the asymptotical independence of \( k \) ranges. A similar phenomenon occurs in the treating of \( k = q/2 \) directional ranges, as shown by Theorem 2.15.

**Theorem 2.5** Suppose \( R_{n,1}, \ldots, R_{n,k} \) are the directional ranges with respect to \( \theta_1, \ldots, \theta_k \) where \( 0 \leq \theta_1 < \ldots < \theta_k < \pi \), and the assumption of \( \epsilon_i \) and \( \eta_i \) in
Theorem 2.1 holds. Then $R_{n,1}, \ldots, R_{n,k}$ are asymptotically independent and

$$
\lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} R_{n,i} \leq 2a_n + b_n t \right) = \left\{ \int_{-\infty}^{\infty} \exp \left( -e^{(s-t)} \right) d[\exp(-e^{-s})] \right\}^k,
$$

(2.24)

where $a_n$ and $b_n$ are the same as in Equation 2.12.

[Proof] For $i = 1, \ldots, k$, let $\theta_{k+i} = \pi + \theta_i$ in Theorem 2.4. Then

$$
P(M_{n,k+i} \leq x_{k+i}) = P(M_{n,i} \geq -x_{k+i})
$$

and $R_{n,i}$ can be represented as:

$$
R_{n,i} = M_{n,i} + M_{n,k+i}.
$$

(2.25)

Since $M_{n,1}, \ldots, M_{n,2k}$ are asymptotically independent (Theorem 2.4), we have

$$
\lim_{n \to \infty} P(R_{n,i} \leq 2a_n + b_n t; i = 1, \ldots, k) = \prod_{i=1}^{k} \left[ \lim_{n \to \infty} P(R_{n,i} \leq a_n + b_n t) \right].
$$

Moreover,

$$
\lim_{n \to \infty} P \left( \max_{1 \leq i \leq k} R_{n,i} \leq 2a_n + b_n t \right) = \lim_{n \to \infty} P(R_{n,i} \leq 2a_n + b_n t; i = 1, \ldots, k)
$$

$$
= \left\{ \int_{-\infty}^{\infty} \exp \left( -e^{(s-t)} \right) d[\exp(-e^{-s})] \right\}^k.
$$

\hfill \Box

A Heuristically Derived Limiting Distribution Function for a Lower Bound of $I_{2,n}$

At the end of the third section, we dealt with quantities

$$
2I_{2,n} = \max_{0 \leq \theta < \pi} R_{n,\theta}.
$$
The relation makes clear that a grid of equally spaced $\theta_i$ values between 0 and $\pi$, of decreasing mesh size, will provide increasingly better approximations for $2I_{2,n}$.

In the previous section, we provided the limiting distribution of the maximum of $k$ directional ranges, for $k$ equal to any finite fixed number. Consider the right hand side of Equation 2.24. Since

$$\left\{ \int_{-\infty}^{\infty} \exp(-e^{(s-t)}) d \left[ \exp \left( -e^{-s} \right) \right] \right\}^k,$$

(2.26)

decreases with $k$, it is reasonable to suppose that the location constant for $k = k_n$ increasing with $n$ will be larger than

$$\sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}},$$

Such behavior is also suggested by Theorem 2.4.

Define

$$L_{n,i,k} = \frac{R_{n,i,k} - 2an}{b_n} \quad i = 1, \ldots, k,$$

then from the results of last section, for $k$ fixed, we have

$$\max_{1 \leq i \leq k} L_{n,i,k} \xrightarrow{d} \max_{1 \leq i \leq k} E_i,$$

(2.27)

where $E_1, E_2, \ldots$, are independently and identically random variables with a common distribution function $F_E(t)$ and

$$F_E(t) = \int_{-\infty}^{\infty} \exp(-e^{(s-t)}) d \left[ \exp \left( -e^{-s} \right) \right].$$

In this section, we explore the heuristic hypothesis that the distribution of $\max_{1 \leq i \leq k} E_i$ can validly be used in place of the distribution of $\max_{1 \leq i \leq k} L_{n,i,k}$, when $k = k_n$ and $k_n \to \infty$ as $n \to \infty$. This problem is similar to one treated by Weissman (1988) for studying the $k$ largest sample extremes.
The aim of this section is to answer the question of how fast $k_n$ can grow with $n$ and we still have the same relation which states in Equation 2.27, in the sense that

$$\lim_{n \to \infty} \left| P \left( \max_{1 \leq i \leq k_n} L_{n,i,k_n} \leq c(k_n) + d(k_n)t \right) - P \left( \max_{1 \leq i \leq k_n} E_i \leq c(k_n) + d(k_n)t \right) \right| = 0,$$

(2.28)

for all $t$, for some constant sequences $c(k_n)$ and $d(k_n) > 0$. Since $E_i$'s are iid, we have

$$P \left( \max_{1 \leq i \leq k_n} E_i \leq x \right) = [P(E_1 \leq x)]^{k_n},$$

where $c(k_n)$ and $d(k_n)$ are the location and scale constants for the extreme value of $E_i$. By Theorem 2.5, $M_{n,1,k}, M_{n,2,k}, \ldots, M_{n,2,k,k}$ are asymptotically independent implies that $R_{n,1,k}, R_{n,2,k}, \ldots, R_{n,k,k}$ are asymptotically independent. Let

$$Z_{n,i,k_n} = \frac{M_{n,i,k_n} - a_n}{b_n} \quad i = 1, \ldots, 2k_n,$$

and $D_i$'s are iid with distribution function $F_D(t) = H_3(x)$. Hence we hypothesize that the order of $k_n$ such that, for all $t_i \in \mathcal{R}$,

$$\left| P \left( Z_{n,i,k_n} \leq a(n,k_n) + b(n,k_n)t_i; \forall i = 1, \ldots, 2k_n \right) - P \left( D_i \leq a(n,k_n) + b(n,k_n)t_i; \forall i = 1, \ldots, 2k_n \right) \right| \longrightarrow 0,$$

(2.29)

as $n \to \infty$, for some constant sequences $a(n,k_n)$ and $b(n,k_n) > 0$, will also satisfy Equation 2.28.

In Theorem 2.6, we search for the limiting distribution function of $\max_{1 \leq i \leq m} E_i$, as well as location and scale constants $c(m)$ and $d(m)$. Then Theorem 2.7 shows that every $k_n$ such that $\frac{k_n}{\sqrt{\log n}} \to 0$ as $n \to \infty$ will satisfy Equation 2.29, and hopefully it will work for Equation 2.29 too.
Theorem 2.6 Suppose $E_1, E_2, \ldots$ are independently and identically distributed random variables with common distribution function $F_E(t)$, and

$$F_E(t) = \int_{-\infty}^{\infty} \exp(-e^{(s-t)}) d\left[\exp\left(-e^{-s}\right)\right].$$

Then

$$\lim_{m \to \infty} P \left( \max_{1 \leq i \leq m} E_i \leq c(m) + d(m)t \right) = \exp \left( -e^{-t} \right), \quad (2.30)$$

where

$$\begin{cases} c(m) = \inf \{x : 1 - F_E(x) \leq \frac{1}{m} \} \\ d(m) = 1. \end{cases}$$

[Proof] There are two sequential steps to show Equation 2.30. First, we need to show that $F_E(t)$ is in the domain of attraction of $\exp(-e^{-t})$. Then searching for the location and scale constants.

Note that

$$1 - F_E(x) = \int_{-\infty}^{\infty} (1 - \exp(-e^{(s-x)})) d\left[\exp\left(-e^{-s}\right)\right]$$

(let $y = e^s$)

$$= \int_{0}^{\infty} e^{-\frac{1}{y^2}} \left(1 - e^{-y}(e^{-x})\right) dy.$$

Define $f_k(y) = e^{-yk}$, then by Taylor expansion we have

$$f_k(y) = 1 - ky + \frac{k^2}{2!} y^2 - \ldots,$$

and

$$\begin{cases} 1 - f_{e-x}(y) = -\sum_{i=1}^{n} \frac{(-ye-x)^i}{i!} \\ 1 - f_{e-x-t}(y) = -e^{-t} \sum_{i=1}^{n} \frac{(-ye-x)^i (e-t)^i-1}{i!}. \end{cases}$$
Let \( g(x) = 1 \), for all real \( x \), then

\[
\frac{1 - F_E(x + t g(x))}{1 - F_E(x)} = \frac{1 - F_E(x + t)}{1 - F_E(x)}
\]

\[
\int_0^\infty \left[ - \sum_{i=1}^{\infty} \frac{(-ye^{-x})^i}{i!} \left( e^{-t} \right)^{i-1} \frac{1}{y^2 e^{-\frac{1}{y}}} \right] dy
\]

\[
= e^{-t} \int_0^\infty \left[ - \sum_{i=1}^{\infty} \frac{(-ye^{-x})^i}{i!} \frac{1}{y^2 e^{-\frac{1}{y}}} \right] dy
\]

\[
= e^{-t} h_t(x).
\]

Note that

\[
h_t(x) = \frac{a_1(x) + e^{-t}a_2(x) + e^{-2t}a_3(x) + \ldots}{a_1(x) + a_2(x) + a_3 + \ldots}
\]

\[
= \frac{1 + e^{-t}a_2(x) + e^{-2t}a_3(x) + \ldots}{1 + a_2(x) + a_3(x) + \ldots},
\]

where \( a_i(x) = \int_0^\infty \frac{(-ye^{-x})^i}{i!} \frac{1}{y^2 e^{-\frac{1}{y}}} dy \)

\[
= e^{-ix} \int_0^\infty \frac{(-y)^i}{i!} \frac{1}{y^2 e^{-\frac{1}{y}}} dy,
\]

and then, \( \frac{a_i(x)}{a_1(x)} \rightarrow 0, \quad x \rightarrow \infty, \quad \forall i = 2, 3, \ldots \),

then, \( h_t(x) \rightarrow 1, \quad x \rightarrow \infty \).

Therefore, as \( x \rightarrow \infty \)

\[
\frac{1 - F_E(x + t g(x))}{1 - F_E(x)} \rightarrow e^{-t},
\]
and $F_E(t)$ is in the domain of attraction of $\exp(-e^{-t})$ by using Lemma 1.3. Consequently,

\[
\begin{align*}
  c(m) &= \inf\{x : 1 - F_E(x) \leq \frac{1}{m}\} \\
  d(m) &= g(c(m)) \\
  &= 1.
\end{align*}
\]

It is not easy to express $c(m)$ in a simple function, so we use the idea of Lemma 1.5 to search for $c(m)$ which satisfies

\[ m(1 - F_E(c(m))) \to 1, \quad \text{as} \quad m \to \infty, \]

by using numerical software.

\[
1 - F_E(c(m)) = 1 - \int_{-\infty}^{\infty} \exp(-e^{s-c(m)}) \left[ 1 - \exp(-e^{-s}) \right] \, ds = 1 - \int_{0}^{\infty} \exp\left[ - \left( x + \frac{1}{xe^{c(m)}} \right) \right] \, dx
\]

(let $x = e^{-s}$)

\[
= \int_{0}^{\infty} \left( e^{-x} - e^{-x} \left( x + \frac{1}{xe^{c(m)}} \right) \right) \, dx.
\]

Figure 2.3 shows the relation between $e^{-x}$ and $\exp[-(x + 1/(xe^{c(m)}))]$. Since $e^{c(m)} \to \infty$ as $m \to \infty$,

\[
e^{-\left( x + \frac{1}{xe^{c(m)}} \right)} \to e^{-x}, \quad \forall x > 0.
\]

There is no doubt about that $1 - F_E(c(m)) \to 0$, for all $c(m)$ satisfying $e^{c(m)} \to \infty$, as $n \to \infty$, but those $c(m)$'s may not satisfy

\[ m(1 - F_E(c(m))) \to 1, \quad \text{as} \quad m \to \infty. \]
Figure 2.3: Illustration of $1 - F_E(c(m))$

In the initial step, we set $c(m) = \log m$ and $c(m) = \log m$; computation strongly suggest that

$$m(1 - F_E(\log m)) \to \infty;$$

$$m(1 - F_E(\log 2m)) \to 0. \tag{2.31}$$

That shows that the true $c(m)$ should fall between $\log m$ and $\log 2m$. After a lot of try-and-error tests, we find that

$$m(1 - F_E(\log m + \log \log m))$$

seems to approach to 1 very slowly as $m$ increases. From Equation 2.31, we at least know that

$$m < e^{c(m)} < m^2, \tag{2.32}$$
or \[ \frac{c(m)}{m^2} \to 0, \quad \text{and} \quad \frac{m}{e^{c(m)}} \to 0 \quad \text{as} \quad m \to \infty. \] (2.33)

The following lemma represents an inequality about normal distribution functions, which we will need in the proof of Theorem 2.7.

**Lemma 2.2** Let \( F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \). Then for all \( x > 0 \), we have

\[
\frac{1}{x} e^{-\frac{x^2}{2}} \left( 1 - \frac{1}{x^2} \right) < \sqrt{2\pi}(1 - F(x)) < \frac{1}{x} e^{-\frac{x^2}{2}},
\]
or

\[
\sqrt{2\pi}(1 - F(x)) = \frac{1}{x} e^{-\frac{x^2}{2}} - \Delta x
\]
where \( 0 < \Delta x < \frac{1}{x^3} e^{-\frac{x^2}{2}} \).

**[Proof]**

\[
\sqrt{2\pi}(1 - F(x)) = \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \, dt
\]

\[
= -\int_{-\infty}^{x} \frac{1}{t} \left[ -te^{-\frac{t^2}{2}} \right] dt
\]

\[
= -\left\{ \frac{1}{t} e^{-\frac{t^2}{2}} \right\}_{-\infty}^{x} - \int_{-\infty}^{x} \frac{1}{t^2} e^{-\frac{t^2}{2}} \, dt
\]

\[
= \frac{1}{x} e^{-\frac{x^2}{2}} + \int_{-\infty}^{x} \frac{1}{t^3} \left[ -te^{-\frac{t^2}{2}} \right] dt
\]

\[
= \frac{1}{x} e^{-\frac{x^2}{2}} - \frac{1}{x^3} e^{-\frac{x^2}{2}} - \int_{-\infty}^{x} \frac{3}{t^4} e^{-\frac{t^2}{2}} \, dt
\]

\[
= \frac{1}{x} e^{-\frac{x^2}{2}} - \frac{1}{x^3} e^{-\frac{x^2}{2}} - e^{-\frac{t^2}{2}} \int_{-\infty}^{x} \frac{3}{t^4} dt
\]

(for some \( t_0 \) such that \( x < t_0 < \infty \))
Note that $x^2 < t_0^2 < \infty$ implies $0 < e^{-\frac{x^2}{2}} < e^{-\frac{t_0^2}{2}}$, then for all $x$,

\[
\frac{1}{x} e^{-\frac{x^2}{2}} \left( 1 - \frac{1}{x^2} \right) < \sqrt{2\pi}(1 - F(x)) < \frac{1}{x} e^{-\frac{x^2}{2}}.
\]

\[\textbf{Theorem 2.7} \text{ Suppose the assumption of Theorem 2.1 holds here. Let } \theta_i = \pi / k_n, \text{ } i = 1, \ldots, 2k_n, \text{ where } k_n \text{ is such that }
\]

\[
\frac{k_n}{\sqrt{\log n}} \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Then as $n \rightarrow \infty$, for all $t_i \in \mathcal{R}$ such that $\sum_{i=1}^{m} e^{-t_i/m}$ will converge to a constant,

\[
P \left( M_{n,i,k_n} \leq a'_n(k_n) + b_n t_i; i = 1, \ldots, 2k_n \right) - \prod_{i=1}^{2k_n} \exp \left( -\frac{e^{-t_i}}{2k_n} \right) \longrightarrow 0,
\]

where $a'_n(k_n) = \sqrt{2\log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2\log n}} + \frac{\log 2k_n}{\sqrt{2\log n}}$.

\[
b_n = \frac{1}{\sqrt{2\log n}}.
\]

\[\textbf{[Proof]} \text{ Define }
\]

\[
A_n(t) = \{(\epsilon, \eta) : \epsilon \cos \theta_i + \eta \sin \theta_i \leq a'_n(k_n) + b_n t_i, \ \forall i = 1, \ldots, 2k_n\}
\]

and the figure of $A_n(t)$ is similar as Figure 2.2 by using $k_n$ instead of $k$ and setting
where $x = a_n'(k_n) + b_n t_i$ and $y = (a_n'(k_n) + b_n t_i) \tan \left( \frac{\psi_i}{2k_n} \right)$, and $z = (a_n'(k_n) + b_n t_i) \tan \left( \frac{\phi_i}{2k_n} \right)$. Here $\psi_i + \phi_i + 1 = \pi$ and $\phi_i, \psi_i > 0$.

Hence,

\[
\left [ 1 - \sum_{i=1}^{2k_n} \left( F(x_i) - F(x_i)F(y_i) \right) \right ]^{n} \leq \left [ 1 - P(A_n(t)) \right ]^{n}
\]

Write $x_i, y_i$ and $\psi_i$ as $x, y$, and $\psi$ respectively in this moment. By Lemma 2.2, we have

\[
\overline{F}(x) \overline{F}(y) = \frac{1}{2\pi} \left[ \frac{1}{x} e^{-\frac{x^2}{2}} - \Delta x \right] \left[ \frac{1}{y} e^{-\frac{y^2}{2}} - \Delta y \right]
\]

where $0 < \Delta < \frac{1}{3} e^{-\frac{l^2}{2}}$, $l = x$ or $y$.
\[
\frac{1}{2\pi} \left( \frac{1}{xy} e^{-\frac{x^2+y^2}{2}} \right) - \Delta x \frac{1}{y} e^{-\frac{y^2}{2}} - \Delta y \frac{1}{x} e^{-\frac{x^2}{2}} + \Delta x \Delta y
\]

Then we have the following inequalities:

\[
- \left( \frac{1}{xy^3} + \frac{1}{x^3y} \right) e^{-\frac{x^2+y^2}{2}} \leq 2\pi F(x)F(y) - \frac{e^{-\frac{x^2+y^2}{2}}}{xy} \leq e^{-\frac{x^2+y^2}{2}} \quad (2.34)
\]

Since \(\tan \frac{\theta}{\theta} \to 1\) as \(\theta \to 0\), we can rewrite \(\tan \frac{\psi}{2kn} \) as \(\frac{\psi}{2kn} \delta_n\), where \(\delta_n \to 1\) as \(n \to \infty\). Hence, after some calculations

\[
\frac{e^{-\frac{x^2+y^2}{2}}}{2\pi xy} = \frac{\alpha_n}{\beta_n} = \frac{e^{-t}}{2n\psi\sqrt{\pi}\sqrt{\log n} \lambda_n \delta_n}
\]

\[
\rho_n = \exp \left( \frac{\log 2kn + t^2}{4\log n} + O_n \left( \frac{\log 2kn}{\log n} \right) + O_n \left( \frac{(\log \log n)(\log 2kn)}{\log n} \right) \right)
\]

\[
\lambda_n = 1 + \frac{\log 2kn + t^2}{4(\log n)^2} + \frac{\log 2kn}{\log n} - \frac{(\log \log n)(\log 2kn) + (\log 4\pi)(\log 2kn)}{8(\log n)^2}
\]

where

\[
\alpha_n = \frac{\sqrt{4\pi \log n} e^{-t}}{n2kn} \rho_n;
\]

\[
\beta_n = 2\pi\psi\lambda_n \log n \frac{n}{k_n} \delta_n.
\]

By assumption, \(k_n/\sqrt{\log n} \to 0\), we observe that \(\rho_n \to 0\) and \(\lambda_n \to 0\). Then,

\[
\frac{2kn}{2\pi x_i y_i} \sum_{i=1}^{2kn} e^{-\frac{x_i^2+y_i^2}{2}} = \frac{e^{-t}}{\psi\sqrt{\pi} \sqrt{\log n} \lambda_n \delta_n} \to 0, \quad \text{as} \quad n \to \infty,
\]

where \(e^{-t} = \frac{1}{2kn} \sum_{i=1}^{2kn} e^{-t_i} \) will converge to a constant. After some calculations,
we also observe

\[ n \sum_{i=1}^{2kn} \frac{1}{x_i^2 y_i} e^{-\frac{x_i^2 + y_i^2}{2}} \rightarrow 0 \]

and

\[ n \sum_{i=1}^{2kn} \left( \frac{1}{x_i^3 y_i^3} + \frac{1}{x_i^3 y_i} \right) e^{-\frac{x_i^2 + y_i^2}{2}} \rightarrow 0 \]

as \( n \rightarrow \infty \).

Similarly, \( F(x_i)F(z_i) \) has the same result as \( F(x_i)F(y_i) \) by using \( z_i \) instead of \( y_i \).

Then inequalities 2.34 becomes:

\[ \left[ 1 - \frac{1}{n} \left( e^{-t} + o_n(1) \right) \right]^n \leq \left[ 1 - P(A_n(t)) \right]^n \leq \left[ 1 - \frac{1}{n} \left( e^{-t} + o_n(1) \right) \right]^n, \]

then as \( n \rightarrow \infty \), for all \( t \in \mathcal{R} \),

\[ \left| \left[ 1 - P(A_n(t)) \right]^n - \Pi_{i=1}^{2kn} \exp \left( \frac{-t}{2kn} \right) \right| = \left| \left[ 1 - P(A_n(t)) \right]^n - \exp(-e^{-t}) \right| \rightarrow 0. \]

We complete this proof.

Based on the previous theorem, for all \( kn = o \left( \frac{1}{\sqrt{\log n}} \right) \), if \( \forall t \)

\[ \lim_{n \rightarrow \infty} \left| P \left( \max_{1 \leq i \leq kn} E_i \leq c(kn) + t \right) - \Pi_{i=1}^{kn} F_{E_i} \left( c(kn) + t \right) \right| = 0 \]

is true, then

\[ \lim_{n \rightarrow \infty} P \left( \max_{1 \leq i \leq kn} R_{n,i,kn} \leq (2a_n + c(kn)b_n) + b_nt \right) = \exp \left( -e^{-t} \right) \quad (2.35) \]

follows by Theorem 2.6. Note that \( a_n, b_n, \) and \( c(kn) \) are defined as before. In all, we hypothesize an approximate location constant for maximum range among
\( k_n = \sqrt{\log n} \) equal-angle directions, as \( c(k_n) = \log k_n + \log \log k_n \). Then half of the location constant of \( \max_{1 \leq i \leq k_n} R_{n,i,k_n} \) is:

\[
\frac{1}{2} (2a_n + c(k_n)b_n) = \sqrt{2 \log n} - \frac{\log \log n - \log \log \sqrt{\log n} + 2 \log 4\pi}{4 \sqrt{2 \log n}},
\]

which is about half-way between the location constant for a Rayleigh(\( \sqrt{2} \)) extreme values, and the location constant for a Standard Normal extreme value.

**Conclusion**

A. With respect to upper bounding of minmax interference, we have:

1. \( r_{[1]} \geq I_n \);
2. \( P \left( r_{[1]} \leq \sigma \sqrt{2 \log n} + \frac{\sigma t}{\sqrt{2 \log n}} \right) \rightarrow H_3(x) \).

There two facts can be interpreted as follows, in terms of a random variable \( D \) such that

\[
P(D \leq t) = H_3(x).
\]

Namely, when \( n \) is large we have

\[
I_n \leq r_{[1]} = \sigma \sqrt{2 \log n} + \frac{\sigma D}{\sqrt{2 \log n}}.
\]  

(2.36)

B. With respect to lower bounding of minmax interference, we have:

1. \( R_{n,1} \leq \max_{1 \leq i \leq k} R_{n,i,k} \leq \max_{1 \leq i \leq k_n} R_{n,i,k_n} \leq 2I_2, n \leq 2I_n \);
2. \( P(R_{n,1} \leq 2\sigma a_n + b_n \sigma t) \rightarrow \int_{-\infty}^{\infty} \exp \left( -e^{(s-t)} \right) d \left[ \exp \left( -e^{-s} \right) \right] \);
3. \( P(\max_{1 \leq i \leq k} R_{n,i,k} \leq 2\sigma a_n + b_n \sigma t) \rightarrow \left\{ \int_{-\infty}^{\infty} \exp \left( -e^{(s-t)} \right) d \left[ \exp \left( -e^{-s} \right) \right] \right\}^k \);
4. as heuristically derived,

\[
P(\max_{1 \leq i \leq k} R_{n,i,k_n} \leq \sigma (2a_n + c(k_n)b_n) + b_n \sigma t) \rightarrow H_3(x),
\]
where
\[
\begin{align*}
    a_n &= \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}}, \\
    b_n &= \frac{1}{\sqrt{2 \log n}}, \\
    k_n &= o(\sqrt{\log n}), \\
    c(m) &= \inf \left\{ t : 1 - \int_{-\infty}^\infty \exp \left(-e^{s-t}\right) d \left[\exp \left(-e^{-s}\right)\right] \leq \frac{1}{m} \right\}.
\end{align*}
\]

There various facts can be interpreted as follows, in terms of random variables $E$ and $Z$ such that
\[
P(E \leq t) = \int_{-\infty}^\infty \exp \left(-e^{s-t}\right) d \left[\exp \left(-e^{-s}\right)\right],
\]
and $P(Z \leq t) = \left\{ \int_{-\infty}^\infty \exp \left(-e^{s-t}\right) d \left[\exp \left(-e^{-s}\right)\right]\right\}^k$.

Namely, when $n$ is large we have
\[
\begin{align*}
    R_{n,1} &= 2\sigma a_n + b_n \sigma E, \\
    \max_{1 \leq i \leq k} R_{n,i,k_n} &= 2\sigma a_n + b_n \sigma Z,
\end{align*}
\]
and as heuristically derived,
\[
\max_{1 \leq i \leq k_n} R_{n,i,k_n} \doteq \sigma(2a_n + c(k_n)b_n) + b_n \sigma D.
\]

Consequently,
\[
2\sigma a_n + b_n \sigma E \leq 2\sigma a_n + b_n \sigma Z \leq \sigma(2a_n + c(k_n)b_n + b_n D) \leq 2I_{2,n} \leq 2I_n. \tag{2.37}
\]

C. Combining the results of 2.36 and 2.37 we have, to within a level of approximate $O\left(\frac{1}{\sqrt{\log n}}\right)$,
\[
\sigma \left(\sqrt{2 \log n} - \frac{\log \log n}{2\sqrt{2 \log n}}\right) \doteq I_n \doteq \sigma \sqrt{2 \log n},
\]
and as heuristically derived,

\[
\sigma \left( \sqrt{2 \log n} - \frac{\log \log n - c(k_n)}{2\sqrt{\log n}} \right) \leq I_n \leq \sigma \sqrt{2 \log n}.
\]
CHAPTER 3. LEAST SQUARES POINT-TO-POINT INTERFERENCE

Introductory

As in Chapter 2, we consider n circular pairs of hole and peg. Again, suppose the design centers of holes and pegs are located at \((h_i, v_i)\) for all \(i = 1, \ldots, n\), and the centers of holes are:

\[(u_i + \epsilon_i, v_i + \eta_i)\]

The notations \(\epsilon_i\) and \(\eta_i\), \(i = 1, \ldots, n\), are defined as horizontal and vertical displacements for each hole respectively. We assume that \(\epsilon_i\)'s are independently distributed random variables with \(E(\epsilon_i) = 0\) and \(V(\epsilon_i) = \sigma^2\), and \(\eta_i\)'s are independently distributed random variables with \(E(\eta_i) = 0\) and \(V(\eta_i) = \sigma^2\), for all \(i = 1, \ldots, n\). Additionally, we assume \(\epsilon_i\)'s and \(\eta_i\)'s are independent and normal.

In this chapter, the optimal positioning adjustment (with horizontal and vertical movements and rotation) which minimizes the sum of squares of point-to-point interference between peg centers and their corresponding hole centers is discussed. There are two ways to adjust the position of the peg-plane. The first way is rotating the peg-plane with respect to the origin, then shifting the peg-plane "up-down" or "left-right" to the best location. The other one is moving the peg-plane "up-down" or "left-right" to a new location first, then rotating the peg-plane with respect to the origin, then shifting the peg-plane "up-down" or "left-right" to the best location.
new origin. The optimal adjustments for those two ways are discussed in the first section. Although the optimal solutions of location parameters of those two ways are different, they provide the same positioning adjustment. In the second section, the asymptotic joint distribution of location parameters are found, and it shows that if we use the center of the design centers of pegs and holes as the origin, then the location parameters are asymptotically mutually independent one another. In the third section, the partitions of the total sum of squares of interference, $\sum(\epsilon_i^2 + \eta_i^2)$, with respect to location parameters are discussed. Additionally, those partitions are all (pre-asymptotically or asymptotically) chi-square distributed and mutually independent one another.

The Optimal Positioning Adjustment for Least Squares Interference

The way of adjustment by rotating first then shifting with respect to horizontal and vertical axes is discussed first. Let $x$ and $y$ be the horizontal and vertical parameters respectively, and $\theta$ be the rotation parameter with $0 \leq \theta < 2\pi$. Then the new location of the center of the $i$th peg after rotating $\theta$ and horizontal and vertical shifting, $(u_i'(x,y,\theta), v_i'(x,y,\theta))$, can be expressed as

$$(u_i'(x,y,\theta), v_i'(x,y,\theta)) = (u_i \cos \theta - v_i \sin \theta + x, u_i \cos \theta + v_i \sin \theta + y),$$

$\forall i = 1, \ldots, n.$

Recall that the point-to-point interference equals to the distance between the center of a peg and the center of its corresponding hole. Then the sum of squares of point-to-point interference, $SS(x,y,\theta)$, can be expressed as

$$SS(x,y,\theta) = \sum_{i=1}^{n} \left\{ \left[(u_i + \epsilon_i) - u_i'(x,y,\theta)\right]^2 + \left[(v_i + \eta_i) - v_i'(x,y,\theta)\right]^2 \right\}. \quad (3.1)$$
The following theorem provides simple formulas for the optimal location parameters, $x^*, y^*$, and $\theta^*$, which achieve the least squares interference.

**Theorem 3.1** Suppose $SS(x, y, \theta)$ is defined as in Equation 3.1, and $(x^*, y^*, \theta^*)$ is such that

$$SS(x^*, y^*, \theta^*) = \min_{x,y,\theta} SS(x, y, \theta).$$

Then

$$\tan \theta^* = \frac{\sum_{i=1}^{n} \eta_i (u_i - \bar{u}) - \sum_{i=1}^{n} \epsilon_i (v_i - \bar{v})}{nK_n + \sum_{i=1}^{n} \eta_i (v_i - \bar{v}) + \sum_{i=1}^{n} \epsilon_i (u_i - \bar{u})},$$

$$x^* = \bar{\epsilon} + \bar{v} \sin \theta^* + \bar{u} (1 - \cos \theta^*),$$

$$y^* = \bar{\eta} - \bar{u} \sin \theta^* + \bar{v} (1 - \cos \theta^*),$$

where $\bar{u} = \sum_{i=1}^{n} u_i / n$, $\bar{v} = \sum_{i=1}^{n} v_i / n$, $\bar{\epsilon} = \sum_{i=1}^{n} \epsilon_i / n$, $\bar{\eta} = \sum_{i=1}^{n} \eta_i / n$, and

$$K_n = \left[ \sum_{i=1}^{n} (u_i - \bar{u})^2 + \sum_{i=1}^{n} (v_i - \bar{v})^2 \right] / n.$$

Here the minimum with respect to $x, y, \theta$ is the minimum over $\{(x, y) : (x, y) \in \mathcal{R}^2\}$ and $0 \leq \theta < 2\pi$.

**[Proof]** Let $l$ be a row vector such that $l = (x, y, \theta)$. If $l^* = (x^*, y^*, \theta^*)$ is the optimal solution to minimize $SS(x, y, \theta)$, it should satisfy the following equations:

$$\left. \frac{\partial SS(x, y, \theta)}{\partial \theta} \right|_{l=l^*} = 0,$$

$$\left. \frac{\partial SS(x, y, \theta)}{\partial x} \right|_{l=l^*} = 0,$$

and

$$\left. \frac{\partial SS(x, y, \theta)}{\partial y} \right|_{l=l^*} = 0.$$
After some calculations, we get

\[
\tan \theta^* = \frac{-\sum_{i=1}^{n} e_i v_i + \sum_{i=1}^{n} \eta_i u_i - n y^* \tilde{u} + n x^* \tilde{v}}{\sum_{i=1}^{n} (u_i^2 + v_i^2) - n x^* \tilde{u} - n y^* \tilde{v} + \sum_{i=1}^{n} \epsilon_i u_i + \sum_{i=1}^{n} \eta_i v_i},
\]

(3.2)

\[x^* = \tilde{c} + \tilde{v} \sin \theta^* + \tilde{u}(1 - \cos \theta^*),\]

(3.3)

and

\[y^* = \tilde{\eta} - \tilde{u} \sin \theta^* + \tilde{v}(1 - \cos \theta^*).\]

(3.4)

Since \(x^*\) and \(y^*\) involve in the right hand side of Equation 3.2, we need to substitute \(x^*\) and \(y^*\) by the right hand sides of Equations 3.3 and 3.4, then Equation 3.2 can be rewritten as:

\[
\tan \theta^* = \frac{n(\tilde{u}^2 + \tilde{v}^2) \sin \theta^* + \sum_{i=1}^{n} \eta_i (u_i - \tilde{u}) - \sum_{i=1}^{n} \epsilon_i (v_i - \tilde{v})}{n K_n + n (\tilde{u}^2 + \tilde{v}^2) \cos \theta^* + \sum_{i=1}^{n} \epsilon_i (u_i - \tilde{u}) + \sum_{i=1}^{n} \eta_i (v_i - \tilde{v})}.
\]

(3.5)

According to the appearance of \(\theta^*\) in the right hand side of Equation 3.5, we need to use \((\sin \theta^*/\cos \theta^*)\) instead of \(\tan \theta^*\) and reorganize Equation 3.5, and then finally we have

\[
\tan \theta^* = \frac{\sum_{i=1}^{n} \eta_i (u_i - \tilde{u}) - \sum_{i=1}^{n} \epsilon_i (v_i - \tilde{v})}{n K_n + \sum_{i=1}^{n} \eta_i (v_i - \tilde{u}) + \sum_{i=1}^{n} \epsilon_i (u_i - \tilde{u})}.
\]

In order to know whether \(SS(x^*, y^*, \theta^*)\) is minimum or maximum, we need to check the Jacobean matrix \(J\), the second partial differential equation of sum of squares of interference. Since \(|J| \leq 0\) as \((x, y, \theta) = (x^*, y^*, \theta^*)\), \(SS(x^*, y^*, \theta^*)\) is minimum and we are done.

Now consider the other way of adjustment — shifting then rotation. The new location of the center of the \(i\)th peg, \((\hat{u}_i(s, t, \phi), \hat{v}_i(s, t, \phi))\), can be represented by

\[
\begin{align*}
\hat{u}_i(s, t, \phi) &= (u_i + s) \cos \phi - (v_i + t) \sin \phi \\
\hat{v}_i(s, t, \phi) &= (v_i + t) \cos \phi + (u_i + s) \sin \phi.
\end{align*}
\]
After repeating the procedure we used in Theorem 3.1, we have the optimal solution \((s^*, t^*, \phi^*)\) for this kind of adjustment, and

\[
\tan \phi^* = \frac{\sum_{i=1}^{n} \eta_i (u_i - \bar{u}) - \sum_{i=1}^{n} \varepsilon_i (v_i - \bar{v})}{n K_n + \sum_{i=1}^{n} \eta_i v_i + \sum_{i=1}^{n} \varepsilon_i u_i}
\]

\[
s^* = \bar{v} + (\bar{v} + \bar{u}) \sin \phi^* + (\bar{u} + \bar{v})(1 - \cos \phi^*)
\]

\[
t^* = \bar{v} - (\bar{u} + \bar{v}) \sin \phi^* + (\bar{v} + \bar{u})(1 - \cos \phi^*)
\]

Note that \(\tan \phi^* = \tan \theta^*\), and the difference between \((x^*, y^*)\) and \((s^*, t^*)\) causes by the different definitions of parameters. These two ways of positioning adjustments are equivalent, since for all \(i\),

\[
\begin{align*}
\begin{cases}
u_i'(x^*, y^*, \theta^*) &= \tilde{u}_i(s^*, t^*, \phi^*) \quad (3.6) \\
u_i'(x^*, y^*, \theta^*) &= \tilde{v}_i(s^*, t^*, \phi^*)
\end{cases}
\end{align*}
\]

Usually, the origin of the design centers of pegs and holes can be any point, so we can choose the center of the design centers (design-pattern) as the origin; that is \(\bar{u} = 0\) and \(\bar{v} = 0\). Then the optimal solution in Theorem 3.1 becomes:

\[
\begin{align*}
\begin{cases}
\tan \theta^* = \frac{\sum_{i=1}^{n} \eta_i u_i - \sum_{i=1}^{n} \varepsilon_i v_i}{n K_n + \sum_{i=1}^{n} \eta_i v_i + \sum_{i=1}^{n} \varepsilon_i u_i} \quad (3.6) \\
x^* = \bar{v} \\
y^* = \bar{v}
\end{cases}
\end{align*}
\]

and note that \((\bar{v}, \bar{v})\) is the optimal adjustment without rotation (See Lemma 3.7).

Consider a special positioning adjustment which only allows the rotation of axes with respect to the origin. Then the location of the \(i\)th peg center after rotation becomes:

\[
(u_i'(\psi), v_i'(\psi)) = (u_i \cos \psi - v_i \sin \psi, v_i \cos \psi + u_i \sin \psi),
\]

and by using Equation 3.2 with \(x^* = 0\) and \(y^* = 0\), we have

\[
\tan \psi^* = \frac{\sum_{i=1}^{n} \eta_i u_i - \sum_{i=1}^{n} \varepsilon_i v_i}{n K_n + \sum_{i=1}^{n} \eta_i v_i + \sum_{i=1}^{n} \varepsilon_i u_i}.
\]
Note that \( \tan \psi^* \) is same as \( \tan \theta^* \) in Equation 3.6, or we may say that the optimal rotation adjustments are all the same, no matter what kind of positioning adjustment is used when we choose the center of gravity of the design pattern as the origin.

The Joint Asymptotic Distribution of the Optimal Positioning Adjustment

Before discussing the joint asymptotical distribution of \( \sqrt{n} \tan \theta^* \), \( \sqrt{n} x^* \), and \( \sqrt{n} y^* \), let's introduce Taylor's expansion theorem and Slutsky theorem first.

**Lemma 3.1 (Taylor's expansion theorem)**

If \( f^{(n+1)}(x) \) is continuous in \( (a, b) \), then for any \( x_0 \in (a, b) \),

\[
f(x) = \sum_{i=1}^{n} \frac{f(i)(x_0)}{j!}(x - x_0)^i + \frac{f(n+1)(\xi(x_0))}{(n+1)!}(x - x_0)^{n+1}
\]

where \( \xi(x_0) \) is some number between \( x \) and \( x_0 \).

**Lemma 3.2 (Slutsky's theorem)**

If \( X_n \) and \( Y_n \) are two sequences of random variables such that

\[
X_n \xrightarrow{p} c \quad \text{and} \quad Y_n \xrightarrow{d} Y,
\]

then,

1. \( X_n + Y_n \xrightarrow{d} c + Y \);
2. \( X_n Y_n \xrightarrow{d} cY \);
3. \( Y_n/X_n \xrightarrow{d} Y_n/c \), if \( c \neq 0 \).
From Theorem 3.1, it is clearly that once we know \( \theta^* \), \( x^* \) and \( y^* \) can be got by using Equations 3.3 and 3.4. But even we know the distribution of \( \tan \theta^* \), it may not so easy to find out what the distribution of \( x^* \) and \( y^* \) are, since only \( \sin \theta^* \) and \( (1 - \cos \theta^*) \) involve in \( x^* \) and \( y^* \). In order to solve this problem we first need to know what distributions of \( \sin \theta^* \) and \( (1 - \cos \theta^*) \) are.

The following Lemma provides the asymptotic distributions of \( \sqrt{n} \sin \theta^* \) and \( \sqrt{n}(1 - \cos \theta^*) \) by assuming that \( \sqrt{n} \tan \theta^* \) is asymptotically normal with mean 0 and variance \( \sigma^2 \).

**Lemma 3.3** Suppose the asymptotic distribution of \( \sqrt{n} \tan \theta^* \) is \( N(0, \sigma^2) \), where \( \sigma^2 \) is constant. Then, as \( n \to \infty \),

\[
\sqrt{n} \sin \theta^* \xrightarrow{d} N(0, \sigma^2)
\]

and

\[
\sqrt{n}(1 - \cos \theta^*) \xrightarrow{p} 0.
\]

**Proof** (I) The relation between \( \sin \theta^* \) and \( \tan \theta^* \) is:

\[
\sin \theta^* = \frac{\tan \theta^*}{\sqrt{1 + \tan^2 \theta^*}}.
\]

Then

\[
\sqrt{n} \sin \theta^* = \sqrt{n} \tan \theta^* \frac{1}{\sqrt{1 + \tan^2 \theta^*}}.
\]

Suppose we can show that \( 1/\sqrt{1 + \tan^2 \theta^*} \to 1 \) in probability as \( n \to \infty \), then, following by Sclusky Theorem, the distribution of \( \sqrt{n} \sin \theta^* \) is the same as that of \( \sqrt{n} \tan \theta^* \) asymptotically. Consequently, the first part of Lemma can be proved.

When \( n \) is large, by assumption, we have

\[
\tan \theta^* \sim N(0, \sigma^2 / n).
\]
If $n$ goes larger and larger, the variance of $\tan \theta^*$ goes smaller and smaller. Finally, $\tan \theta^*$ is equal to its mean; that is, as $n \to \infty$,

$$\tan \theta^* \to 0, \quad \text{in probability.}$$

Hence,

$$\frac{1}{\sqrt{1 + \tan^2 \theta^*}} \to 1, \quad \text{in probability,}$$

and then

$$\sqrt{n} \sin \theta^* \xrightarrow{d} N(0, \sigma^2).$$

(II) The Taylor expansion of $1/\sqrt{1 + \tan^2 \theta^*}$ is needed here in order to find the asymptotic behavior of $\sqrt{n}(1 - \cos \theta^*)$, since

$$\sqrt{n}(1 - \cos \theta^*) = \sqrt{n} - \frac{\sqrt{n}}{\sqrt{1 + \tan^2 \theta^*}}.$$

Let $t = \tan \theta^*$ and $f(t) = 1/\sqrt{1 + t^2}$, then $f^{(k)}(t)$ exists for $k = 1, 2, \ldots$, and Taylor series for $f(t)$ at 0 is:

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k = 1 + f^{(1)}(0)t + f^{(2)}(0)t^2/2 + o(t^2) = 1 + 0 - t^2/2 + o(t^2) = 1 - \tan^2 \theta^*/2 + o\left(\tan^2 \theta^*\right).$$

Hence,

$$\sqrt{n}(1 - \cos \theta^*) = \sqrt{n} - \left(\sqrt{n} \sin \theta^* \frac{\tan \theta^*}{2} + \sqrt{n}o\left(\tan^2 \theta^*\right)\right) = (\sqrt{n} \tan \theta^*) \frac{\tan \theta^*}{2} - (\sqrt{n} \tan \theta^*) o(\tan \theta^*).$$
According to the assumptions, \( \sqrt{n} \tan \theta^* \xrightarrow{d} \mathcal{N}(0, \sigma^2) \) as \( n \to \infty \), and, again, \( \tan (\theta^*/2) \) and \( o(\tan \theta^*) \) both converge to 0 in probability, the following results follow by Slutsky’s Theorem. As \( n \to \infty \),

\[
\sqrt{n} \tan \theta^*(\tan \theta^*/2) \to 0, \quad \text{in probability}
\]

and

\[
\sqrt{n} \tan \theta^*o(\tan \theta^*) \to 0, \quad \text{in probability.}
\]

Therefore, as \( n \to \infty \)

\[
\sqrt{n}(1 - \cos \theta^*) \to 0, \quad \text{in probability.}
\]

Suppose the assumption of Lemma 3.3 holds, the optimal solution \( x^* \) and \( y^* \) can be expressed as functions of \( \tan \theta^* \):

\[
\sqrt{n}x^* = \sum_{i=1}^{n} \frac{\epsilon_i}{\sqrt{n}} + \bar{v} \sqrt{n} \tan \theta^*
\]

(3.7)

\[
\sqrt{ny^*} = \sum_{i=1}^{n} \frac{\eta_i}{\sqrt{n}} - \bar{u} \sqrt{n} \tan \theta^*
\]

(3.8)

when \( n \) is large. The following Theorem is shown by using Lemma 3.3 and Equations 3.7 and 3.8.

Now we have enough knowledge to discuss the joint distribution of \( \sqrt{n} \tan \theta^* \), \( \sqrt{n}x^* \), and \( \sqrt{ny^*} \).

**Theorem 3.2** Suppose the random displacements \( \epsilon_i \) and \( \eta_i \), \( i = 1, \ldots, n \), have common normal distribution, \( \mathcal{N}(0, \sigma^2) \) say, where \( \sigma^2 \) is constant and those 2n normal variables are independent, \( K_n = \sum_{i=1}^{n} [(u_i - \bar{u})^2 + (v_i - \bar{v})^2]/n \) converges to a finite
number \( K (> 0) \), and \((x^*, y^*, \theta^*)\) is the optimal solution to minimize Equation 3.2, then

\[
\begin{pmatrix}
\sqrt{n} \tan \theta^* \\
\sqrt{n} x^* \\
\sqrt{n} y^*
\end{pmatrix}
\xrightarrow{d} \mathcal{N}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \frac{\sigma^2}{K}
\begin{pmatrix}
1 & \bar{v} & -\bar{u} \\
\bar{v} & K + \bar{v}^2 & -\bar{u}\bar{v} \\
-\bar{u} & -\bar{u}\bar{v} & K + \bar{u}^2
\end{pmatrix}
\]

[Proof] Since \( x^* \) and \( y^* \) are functions of \( \theta^* \) (see Equations 3.3 and 3.4), the first step of the proof is to find the distribution of \( \tan \theta^* \).

Recall that

\[
\tan \theta^* = \frac{N_1}{K_n + N_2},
\]

where

\[
N_1 = \left[ \sum_{i=1}^{n} \eta_i(u_i - \bar{u}) - \sum_{i=1}^{n} \epsilon_i(v_i - \bar{v}) \right] / n
\]

and

\[
N_2 = \left[ \sum_{i=1}^{n} \eta_i(v_i - \bar{V}) + \sum_{i=1}^{n} \epsilon_i(h_i - \bar{H}) \right] / n.
\]

Note that \( N_1 \) and \( N_2 \) are normal distributed with mean zero and variance \( K_n \sigma^2 / n \), and \( N_1 \) and \( N_2 \) are independently distributed.

By assumption \( K_n \to K \), we have, as \( n \to \infty \)

\[
\sqrt{n} N_1 \xrightarrow{d} \mathcal{N}(0, K \sigma^2),
\]

and

\[
\sqrt{n} N_2 \xrightarrow{d} \mathcal{N}(0, K \sigma^2).
\]

Then, \( \sqrt{n} \tan \theta^* \) can be rewritten as:

\[
\sqrt{n} \tan \theta^* = \left( \frac{\sqrt{n} N_1}{K_n} \right) \left( \frac{K_n}{(K_n + N_2)} \right),
\]
where
\[ \frac{\sqrt{n}N_1}{K_n} \xrightarrow{d} \mathcal{N}(0, \sigma^2/K) \]
and
\[ \frac{K_n}{(K_n + N_2)} \xrightarrow{p} 1, \quad \text{in probability,} \]
since \(N_2 \xrightarrow{p} 1\) in probability. Hence, by Lemma 3.2,
\[ \sqrt{n} \tan \theta^* \xrightarrow{d} \mathcal{N}(0, \sigma^2/K). \quad (3.9) \]

Moreover, by Lemma 3.3, as \(n \to \infty\)
\[ \begin{cases} \sqrt{n} \sin \theta^* \xrightarrow{d} \mathcal{N}(0, \sigma^2/K); \\ \sqrt{n}(1 - \cos \theta^*) \xrightarrow{p} 0. \end{cases} \]

Suppose \(n\) is large, then by Equation 3.7 and Lemma 3.3 we have
\[ \sqrt{n}x^* = \sum_{i=1}^{n} \epsilon_i/\sqrt{n} + \bar{v} \sqrt{n}N_1/K \]
Since two terms on the right hand side are both normal distributions with mean 0, \(\sqrt{n}x^*\) is asymptotically normal with \(E(\sqrt{n}x^*) = 0\), and for large \(n\), we have
\[ \text{Var}(\sqrt{n}x^*) = \text{Var} \left( \sum_{i=1}^{n} \epsilon_i/\sqrt{n} \right) + \text{Var}(\bar{v} \sqrt{n} \tan \theta^*) + 2\text{Cov} \left( \sum_{i=1}^{n} \epsilon_i/\sqrt{n}, \bar{v} \sqrt{n} \tan \theta^* \right) \]
\[ = \sigma^2 + \bar{v}^2 \sigma^2/K + (2\bar{v}/(nK))\text{Cov} \left( \sum_{i=1}^{n} \epsilon_i, \sum_{i=1}^{n} \epsilon_i(v_i - \bar{v}) \right) \]
\[ = (K + \bar{v}^2)\sigma^2/K, \]
where
\[ \text{Cov} \left( \sum_{i=1}^{n} \epsilon_i/\sqrt{n}, \sqrt{n} \tan \theta^* \right) = 0. \quad (3.10) \]
Finally, we have
\[ \sqrt{n}x^* \xrightarrow{d} \mathcal{N}(0, (K + \bar{v}^2)\sigma^2/K). \quad (3.11) \]
Similarly, using Equation 3.8 and Lemma 3.3, we have

$$\sqrt{n}y^* \xrightarrow{d} \mathcal{N}(0, (K + \bar{u}^2)\sigma^2 / K)$$

(3.12)

and

$$\text{Cov} \left( \sum_{i=1}^{n} \eta_i / \sqrt{n}, \sqrt{n} \tan \theta^* \right) = 0.$$ 

(3.13)

Since $\sqrt{n} \tan \theta^*$, $\sqrt{n}x^*$, and $\sqrt{n}y^*$ are all asymptotically normal, the joint asymptotic distribution will be multiple normal distribution. As $n$ is quite large,

$$\text{Cov}(\sqrt{n} \tan \theta^*, \sqrt{n} x^*) \approx \text{Cov}(\sqrt{n} \tan \theta^*, \bar{v} \sqrt{n} \tan \theta^*) \rightarrow \bar{v} \sigma^2 / K,$$

$$\text{Cov}(\sqrt{n} \tan \theta^*, \sqrt{n} y^*) \approx \text{Cov}(\sqrt{n} \tan \theta^*, -\bar{u} \sqrt{n} \tan \theta^*) \rightarrow -\bar{u} \sigma^2 / K,$$

and

$$\text{Cov}(\sqrt{n} x^*, \sqrt{n} y^*) \approx \text{Cov}(\bar{v} \sqrt{n} \tan \theta^*, -\bar{u} \sqrt{n} \tan \theta^*) \rightarrow -\bar{u} \bar{v} \sigma^2 / K.$$ 

Therefore, we complete this proof.

Recall that if we use the center of gravity of the design centers as the origin, then the optimal positioning adjustment is $(\theta^*, \varepsilon, \eta)$. That is the center of gravity of the design centers will move “$\varepsilon$” units along with the horizontal axis and move “$\eta$” units along with the vertical axis. If we use other origin, then the center, $(\bar{u}, \bar{v})$, will first rotate $\theta^*$ with respect to the origin to

$$(\bar{u} \cos \theta^* - \bar{v} \sin \theta^*, \bar{v} \cos \theta^* + \bar{u} \sin \theta^*)$$

and then pull it back to $(\bar{u} + \bar{v}, \bar{v} + \bar{\eta})$. The vector to pull back the center of gravity of the design centers to the optimal position is $(x^*, y^*)$. Figure 3.1 illustrates that if $\theta^*$ is positive (i.e. $\tan \theta^*$ is positive), then $x^*$ is positive and $y^*$ is negative.
That explains the reason why Cov(\(\sqrt{n} \tan \theta^*\), \(\sqrt{n} \tan \theta^*\)) and Cov(\(\sqrt{n} \tan \theta^*\), \(\sqrt{n} y^*\)) have different signs. Furthermore, if we use a origin farther from the center of gravity of the design centers, the magnitudes of \(x^*\) and \(y^*\) will increase (see Figure 3.2). That shows that the dependence among \(\tan \theta^*, x^*,\) and \(y^*\) is only through the choice of the origin.

In order to simplify the optimal positioning adjustment, we can always use the center of gravity of the design centers as the origin. Then the those three location parameters are asymptotically mutually independent, and Theorem 3.2 becomes:

\[
\begin{pmatrix}
\sqrt{n} \tan \theta^* \\
\sqrt{n} x^* \\
\sqrt{n} y^*
\end{pmatrix} \xrightarrow{d} \mathcal{N}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
, \sigma^2
\begin{pmatrix}
1/K & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(3.14)

Asymptotically Independent Decomposition of the Interference Sum of Squares

The following Lemmas are very famous and useful in various fields and the their proofs can be found in many books.

**Lemma 3.4** If \(\epsilon_1, \epsilon_2, \ldots, \epsilon_n\) denotes a random sample from a normal distribution whose mean is zero and variance is a constant \(\sigma^2\), then

1. \(\bar{\epsilon}\) and the terms \(\epsilon_i - \bar{\epsilon}, i = 1, \ldots, n\), are independent;
2. \(\sum_{i=1}^{n} (\epsilon_i - \bar{\epsilon})^2 \sim \chi^2_{n-1}\).

**Lemma 3.5** Suppose \(X\) is an \(n \times 1\) random vector such that

\[X \sim \mathcal{N}(\mu, V)\]
Figure 3.1: Illustration of the relationship between horizontal, vertical and rotational adjustments
Figure 3.2: Illustration of the effect of origin choice on horizontal and vertical adjustments
and A and B are two matrices such that AVB is a null matrix. Then \( X'AX \) and \( X'BX \) are independently distributed.

**Lemma 3.6 (Cochran’s Theorem)**

Let \( \mathbf{z} \) represent an \( n \times 1 \) random vector whose distribution is \( \mathcal{N}(\mu, \sigma^2 I) \), where \( \sigma^2 \) is a positive scalar. Take \( A_1, A_2, \ldots, A_k \) to be symmetric matrices such that

\[
\mathbf{z}'A_1\mathbf{z} + \mathbf{z}'A_2\mathbf{z} + \ldots + \mathbf{z}'A_k\mathbf{z} = \mathbf{z}'\mathbf{z}
\]

for all values of \( \mathbf{z} \), and let \( r_i = \text{rank}(A_i), i = 1, \ldots, k \). If \( r_1 + r_2 + \ldots + r_k = n \), then

\[
\left( \frac{1}{\sigma^2} \right) \mathbf{z}'A_i\mathbf{z} \sim \chi^2_{r_i} \left( \frac{1}{\sigma^2} \mu'A_i\mu \right)
\]

and \( \mathbf{z}'A_1\mathbf{z}, \mathbf{z}'A_2\mathbf{z}, \ldots, \mathbf{z}'A_k\mathbf{z} \) are distributed independently.

Through this section, we use the center of gravity of the design centers as the origin such that \( \bar{u} = 0 \) and \( \bar{v} = 0 \). Let \( SS_X(x) \), \( SS_Y(y) \), and \( SS(x, y) \) be the residual sum of squares of interference after fitting horizontal adjustment, vertical adjustment, and horizontal and vertical adjustments, respectively. Then those residual sum of squares can be expressed as:

\[
SS_X(x) = \sum_{i=1}^{n} \left\{ [(u_i + \epsilon_i) - (u_i + x)]^2 + [(v_i + \eta_i) - v_i]^2 \right\}
\]

\[
= \sum_{i=1}^{n} \left[ (\epsilon_i - x)^2 + \eta_i^2 \right],
\]

\[
SS_Y(y) = \sum_{i=1}^{n} \left[ \epsilon_i^2 + (\eta_i - y)^2 \right],
\]

\[
SS(x, y) = \sum_{i=1}^{n} \left\{ [(u_i + \epsilon_i) - (u_i + x)]^2 + [(v_i + \eta_i) - (v_i + y)]^2 \right\}
\]

\[
= \sum_{i=1}^{n} \left[ (\epsilon_i - x)^2 + (\eta_i - y)^2 \right].
\]
Define
\[
\begin{align*}
    RSS(X) &= \sum_{i=1}^{n}(\epsilon_i^2 + \eta_i^2) - \min_x SS_X(x), \\
    RSS(Y) &= \sum_{i=1}^{n}(\epsilon_i^2 + \eta_i^2) - \min_y SS_Y(y), \\
    RSS(X, Y) &= \sum_{i=1}^{n}(\epsilon_i^2 + \eta_i^2) - \min_{x,y} SS(x,y), \\
    RSS(Y|X) &= RSS(X, Y) - RSS(X),
\end{align*}
\]
then the following lemma shows that the horizontal and vertical positioning adjustments are independent and the total sum of squares of interference can be expressed by three independent chi-square distributions.

**Lemma 3.7** Suppose that SS_X(x), SS_Y(y), and SS(x,y) are defined as before. Then, we have
\[
\begin{align*}
    SS_X(\bar{\epsilon}) &= \min_x SS_X(x), \\
    SS_Y(\bar{\eta}) &= \min_y SS_Y(y), \\
    SS(\bar{\epsilon}, \bar{\eta}) &= \min_{x,y} SS(x,y),
\end{align*}
\]
and
\[
\sum_{i=1}^{n} (\epsilon_i^2 + \eta_i^2) = RSS(X) + RSS(Y) + SS(\bar{\epsilon}, \bar{\eta})
\]
where
\[
\bar{\epsilon} = \frac{\sum_{i=1}^{n} \epsilon_i}{n} \quad \text{and} \quad \bar{\eta} = \frac{\sum_{i=1}^{n} \eta_i}{n}.
\]
Moreover, suppose \(\epsilon_i \sim i.i.d. N(0, \sigma^2)\), and \(\eta_i \sim i.i.d. N(0, \sigma^2)\), for all \(i = 1, \ldots, n\), and \(\epsilon_i\)'s and \(\eta_i\)'s are also independent. Then
\[
\begin{align*}
    RSS(X)/\sigma^2 &\sim \chi^2_1, \\
    RSS(Y)/\sigma^2 &\sim \chi^2_1, \\
    SS(\bar{\epsilon}, \bar{\eta})/\sigma^2 &\sim \chi^2_{2n-2}
\end{align*}
\]
and those three are independent.
[Proof] In order to find the optimal solutions to minimize \( SS_X(x) \), \( SS_Y(y) \), and \( SS(x, y) \), all we need to do is to set the partial differentiate functions equal to zero and then the optimal solutions can be found by solving those equations. If the determinants of the Jacobean matrices are negative, then the optimal solutions minimizing the residual sum of squares of interference. Hence, we find that

\[
\begin{align*}
SS_X(\varepsilon) &= \min_x SS_X(x) \\
SS_Y(\eta) &= \min_y SS_Y(y) \\
SS(\varepsilon, \eta) &= \min_{x,y} SS(x, y),
\end{align*}
\]

and

\[
\begin{align*}
RSS(X) &= n\varepsilon^2 \\
RSS(Y) &= n\eta^2 \\
RSS(X, Y) &= n\varepsilon^2 + n\eta^2.
\end{align*}
\]

The total sum of squares of interference can be written as:

\[
\sum_{i=1}^{n} (\varepsilon_i^2 + \eta_i^2) = RSS(X) + SS_X(\varepsilon) = RSS(x) + (SS_X(\varepsilon) - SS(\varepsilon, \eta)) + SS(\varepsilon, \eta) = RSS(x) + RSS(Y|X) + SS(\varepsilon, \eta)
\]

Note that

\[
RSS(Y|X) = SS_X(\varepsilon) - SS(\varepsilon, \eta) = n\eta^2 = RSS(Y),
\]

hence,

\[
\sum_{i=1}^{n} (\varepsilon_i^2 + \eta_i^2) = RSS(X) + RSS(Y) + SS(\varepsilon, \eta).
\]
By the assumption of $\epsilon_i$'s and $\eta_i$'s, we have
\[ \bar{\epsilon} \sim N \left( 0, \frac{\sigma^2}{n} \right) \quad \text{and} \quad \bar{\eta} \sim N \left( 0, \frac{\sigma^2}{n} \right), \]
then
\[ \begin{align*}
    &\frac{RSS(X)}{\sigma^2} \sim \chi_1^2 \\
    &\frac{RSS(Y)}{\sigma^2} \sim \chi_1^2 \\
    &\sum_{i=1}^n (\epsilon_i^2 + \eta_i^2) / \sigma^2 \sim \chi_{2n}^2
\end{align*} \]
Also, from Lemma 3.4, we have
\[ \begin{align*}
    &\sum_{i=1}^n (\epsilon_i - \bar{\epsilon})^2 / \sigma^2 \sim \chi_{n-1}^2 \\
    &\sum_{i=1}^n (\eta_i - \bar{\eta})^2 / \sigma^2 \sim \chi_{n-1}^2
\end{align*} \]
and those two are independent. Hence,
\[ SS(\bar{\epsilon}, \bar{\eta}) / \sigma^2 = \sum_{i=1}^n (\epsilon_i - \bar{\epsilon})^2 / \sigma^2 + \sum_{i=1}^n (\eta_i - \bar{\eta})^2 / \sigma^2 \sim \chi_{2n-2}^2, \]
and by Cochran's Theorem, $RSS(X)$, $RSS(Y)$, and $SS(\bar{\epsilon}, \bar{\eta})$ are mutually independent.

Before moving forward to three location parameters case, we need following lemmas.

**Lemma 3.8** Suppose the hypotheses of Theorem 3.2 hold here. Then, as $n \to \infty$,
\[ \frac{2nK_n(1/ \cos \theta^* - 1)}{\sigma^2} \xrightarrow{d} \chi_1^2. \] (3.15)
[Proof] Since $\frac{1}{\cos \theta^*} = \sqrt{1 + \tan^2 \theta^*}$, by Lemma 3.1 we have

$$\frac{1}{\cos \theta^*} - 1 = \tan^2 \theta^*/2 + o(\tan^3 \theta^*).$$

Hence,

$$2nK_n \left( \frac{1}{\cos \theta^*} - 1 \right) / \sigma^2 = \left( \frac{\sqrt{n} \tan \theta^*}{\sigma/K_n} \right)^2 (1 + o(\tan \theta^*)), $$

where (by Theorem 3.2)

$$\left( \frac{\sqrt{n} \tan \theta^*}{\sigma/K_n} \right)^2 \frac{d}{d} \chi_1^2$$

and

$$o(\tan \theta^*) \longrightarrow 0 \text{ in probability.}$$

Therefore, by Lemma 3.2 we have

$$2nK_n \left( \frac{1}{\cos \theta^*} - 1 \right) / \sigma^2 \overset{d}{\longrightarrow} \chi_1^2.$$

Lemma 3.9 Suppose the assumptions of Theorem 3.2 hold. Then as $n \to \infty$ we have

$$\left[ \sum_{i=1}^{n} (\epsilon_i u_i + \eta_i v_i) \right] (1/\cos \theta^* - 1) \longrightarrow 0 \text{ in probability.}$$

[Proof] By assumption, as $n \to \infty$

$$\left[ \sum_{i=1}^{n} u_i^2 + v_i^2 \right] / n \longrightarrow K.$$

Or, we can say that there exist real finite numbers $A_1$ and $A_2$ such that, as $n \to \infty$,

$$\frac{1}{n} \sum_{i=1}^{n} u_i^2 \longrightarrow A_1.$$
and
\[ \frac{1}{n} \sum_{i=1}^{n} v_i^2 \rightarrow A_2, \]
where \( A_1 + A_2 = K \). Now, define two asymptotically orthogonal standard normal random variables as
\[ N_1 = \sum_{i=1}^{n} \epsilon_i u_i / (\sqrt{n} \sigma \sqrt{A_1}) \]
and
\[ N_2 = \sum_{i=1}^{n} \eta_i v_i / (\sqrt{n} \sigma \sqrt{A_2}). \]
Therefore, the left hand side of Equation 3.15 becomes
\[ \sqrt{n} \sigma \left( \sqrt{A_1} N_1 + \sqrt{A_2} N_2 \right) \left( \tan^2 \theta^* / 2 + o \left( \tan^3 \theta^* \right) \right). \]
which is equal to
\[ \frac{\sigma^2}{2\sqrt{Kn}} \sum_{i=1,2} \sqrt{A_i} \tan \theta^* \left( N_i \frac{\sqrt{n} \tan \theta^*}{\sigma / Kn} \right) + \sum_{i=1,2} \sqrt{A_i} o \left( \tan^2 \theta^* \right) \left( N_i \frac{\sqrt{n} \tan \theta^*}{\sigma / Kn} \right) \]
where, for \( i = 1, \) or \( 2, \)
\[ N_i \frac{\sqrt{n} \tan \theta^*}{\sigma / Kn} \xrightarrow{d} A_i^2 \]
and
\[ o \left( \tan^2 \theta^* \right) \leq \tan \theta^* \rightarrow 0 \text{ in probability.} \]
Again, use Slutsky Theorem, we have Equation 3.15.

Now define
\[ \text{\textit{SS}}_\theta(\theta) = \sum_{i=1}^{n} \left\{ [(u_i + \epsilon_i) - (u_i \cos \theta - v_i \sin \theta)]^2 + [(v_i + \eta_i) - (v_i \cos \theta + u_i \sin \theta)]^2 \right\} \]
\[ \text{\textit{RSS}}(\Theta) = \sum_{i=1}^{n} (\epsilon_i^2 + \eta_i^2) - \min_{\theta} \text{\textit{SS}}_\theta(\theta) \]
Theorem 3.3 shows that $\sum (\epsilon_i^2 + \eta_i^2)/\sigma^2$, a random variable of chi-square distribution with $2n$ d.f., can be asymptotically expressed as a decomposition of 4 random variables of chi-square distribution with 1, 1, 1, and $(n - 3)$ d.f. respectively and mutually independently.

We now address the asymptotic independence of the sums of squares $RSS$. Asymptotic independence may be formulated in a variety of ways (see for example, Dorea, David, and Werner (1984) and Dorea, Sastrosoewtingjo, and David (1985)). In this dissertation we shall see that $X_n$ and $Y_n$ are asymptotic independent if

$$X_n = X + o_P(1),$$
$$Y_n = Y + o_P(1),$$

and $X$ and $Y$ are non-degenerate independent random variables.

**Theorem 3.3** Suppose the hypotheses of Theorem 3.2 hold, then

$$\sum_{i=1}^n \epsilon_i^2 + \sum_{i=1}^n \eta_i^2 = RSS(x) + RSS(Y) + RSS(\Theta) + SS(x^*, y^*, \theta^*),$$

(3.16)

where

$$RSS(X)/\sigma^2 \sim \chi_1^2,$$
$$RSS(Y)/\sigma^2 \sim \chi_1^2,$$
$$RSS(\Theta)/\sigma^2 \xrightarrow{d} \chi_1^2,$$

and

$$SS(x^*, y^*, \theta^*)/\sigma^2 \xrightarrow{d} \chi_{n-3}^2.$$
Moreover, those four random variables of chi-square distribution are asymptotically mutually independent one another.

[Proof] The total sum of squares, \( \sum \epsilon_i^2 + \sum \eta_i^2 \), can be represented as three different types of decomposition.

\[
\sum_{i=1}^{n} \epsilon_i^2 + \sum_{i=1}^{n} \eta_i^2 = RSS(X) + SS_X(\varepsilon)
\]
\[
= RSS(X, Y) + SS(\varepsilon, \eta)
\]
\[
= RSS(X, Y, \Theta) + SS(x^*, y^*, \theta^*),
\]

where \((x^*, y^*, \theta^*)\) remains the same as before. Hence, the total sum of squares can be broken into four parts as:

\[
\sum \epsilon_i^2 + \sum \eta_i^2 = RSS(X) + RSS(Y) + RSS(\Theta|X, Y) + SS(x^*, y^*, \theta^*). \tag{3.17}
\]

Now after using the result of Theorem 3.1 and doing some calculations, we have

\[
SS(x^*, y^*, \theta^*) = \sum_{i=1}^{n} (\epsilon_i - \bar{\varepsilon})^2 + \sum_{i=1}^{n} (\eta_i - \bar{\eta})^2
\]
\[
-2 \left( nK_n + \sum_{i=1}^{n} \epsilon_i u_i + \sum_{i=1}^{n} \eta_i v_i \right) (1/\cos \theta^* - 1),
\]

\[
SS(\Theta(\theta^*)) = \min_{0 \leq \theta < \pi} SS(\Theta),
\]

and

\[
RSS(\Theta) = 2 \left( nK_n + \sum_{i=1}^{n} \epsilon_i u_i + \sum_{i=1}^{n} \eta_i v_i \right) (1/\cos \theta^* - 1),
\]

hence,

\[
RSS(\Theta|X, Y) = RSS(\Theta).
\]

Then Equation 3.17 can be rewritten as:

\[
\sum \epsilon_i^2 + \sum \eta_i^2 = RSS(X) + RSS(Y) + RSS(\Theta) + SS(x^*, y^*, \theta^*).
\]
In Lemma 3.7, it has been shown that $RSS(X)/\sigma^2$ and $RSS(Y)/\sigma^2$ are chi-square distributed with 1 degree of freedom and independent. The asymptotic distribution of the third part of decomposition, $RSS(\Theta)$, can also be found by using Lemma 3.8 and Lemma 3.9. Let

$$X_n = 2nK_n(1/\cos \theta^* - 1)/\sigma^2$$

and

$$O_n = \left[ \sum_{i=1}^{n} \epsilon_i u_i + \sum_{i=1}^{n} \eta_i v_i \right] (1/\cos \theta^* - 1),$$

where $X_n^2 \to X_1^2$ asymptotically and $O_n \to 0$ in probability. Then, $RSS(\Theta)/\sigma^2$ asymptotically converges to a chi-square distribution with 1 degree of freedom follows by

$$RSS(\Theta) = \sigma^2 X_n^2 - 2O_n.$$

Then

$$RSS(\Theta)/\sigma^2 \overset{d}{\to} X_1^2.$$

The last part of the decomposition is most difficult to deal with. We need to show that $RSS(\Theta)$ and $SS(x^*, y^*, \theta^*)$ are asymptotically independent by using Lemma 3.5, and then we use Cochran’s Theorem to show that $SS(x^*, y^*, \theta^*)/\sigma^2$ is chi-square distributed with $(2n - 1)$ degree of freedom. Let

$$\bar{z}' = (\epsilon_1, \ldots, \epsilon_n, \eta_1, \ldots, \eta_n),$$

then $\bar{z}$ is a multiple normal distribution with $E(\bar{z}) = 0$ and $Var(\bar{z}) = \sigma^2I$. Note that $RSS(\Theta)$ and $SS(x^*, y^*, \theta^*)$ are linear combination of $\bar{z}$, so there exist matrices $C_n$ and $D_n$ such that they can be written as $\bar{z}'C_n\bar{z}$ and $\bar{z}'D_n\bar{z}$ respectively. In the
proof of Lemma 3.8, we observe that there exist $\delta_n$ and $\lambda_n$ such that

$$RSS(\Theta) = nK_n \tan^2 \theta^* + \delta_n$$

$$= \left( \frac{\sum_{i=1}^{n} \eta_i u_i - \sum_{i=1}^{n} e_i v_i}{(nK_n) + \lambda_n} \right)^2$$

where $\delta_n \to 0$ and $\lambda_n \to 0$ as $n \to \infty$.

According to previous equations, we can define two $2n \times 2n$ matrices, $A_n$ and $B_n$, such that

$$\hat{z}' C_n \hat{z} = \hat{z}' A_n \hat{z} + o_p(1)$$

and

$$\hat{z}' D_n \hat{z} = \hat{z}' B_n \hat{z} + o_p(1).$$

$$A_n = \frac{1}{Kn} \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} v_1^2 & v_1 v_2 & \ldots & v_1 v_n \\ v_2 v_1 & v_2^2 & \ldots & v_2 v_n \\ \vdots & \ddots & \ddots & \vdots \\ v_n v_1 & \ldots & \ldots & v_n^2 \end{bmatrix},$$

$$A_{12} = -\begin{bmatrix} v_1 u_1 & v_1 u_2 & \ldots & v_1 u_n \\ v_2 u_1 & v_2 u_2 & \ldots & v_2 u_n \\ \vdots & \ddots & \ddots & \vdots \\ v_n u_1 & \ldots & \ldots & v_n u_n \end{bmatrix}.$$
and

\[
A_{22} = \begin{bmatrix}
  u_1^2 & u_1u_2 & \ldots & u_1u_n \\
  u_2u_1 & u_2^2 & \ldots & u_2u_n \\
  \vdots & \ddots \\
  u_{n-2}u_1 & \ldots & u_{n-2}u_{n-1} & u_{n-1}^2 \\
  u_{n-1}u_1 & \ldots & u_{n-1}u_{n-2} & u_{n-1}u_{n-1} & u_n^2 \\
\end{bmatrix},
\]

and

\[
B_n = \begin{bmatrix}
  B_{11} & B_{12} \\
  B_{12} & B_{22} \\
\end{bmatrix}
\]

where

\[
B_{11} = I - \frac{1}{n}J - \frac{1}{K_n}A_{11},
\]

\[
B_{12} = -\frac{1}{K_n}A_{12},
\]

and

\[
B_{22} = I - \frac{1}{n}J - \frac{1}{K_n}A_{22}.
\]

Note that \(I\) is identical matrix and \(J\) is the matrix whose elements are all equal to 1.

Since \(A_n(\sigma I)B_n = 0\), by Lemma 3.5, we have that \(z' A_n z\) and \(z B_n z\) are independently distributed. In other words, \(R(\Theta|X,Y)\) and \(SS(x^*,y^*,\theta^*)\) are asymptotically independent. Recall that

\[
SS(\tilde{\epsilon},\tilde{\eta}) = RSS(\Theta) + SS(x^*,y^*,\theta^*)
\]

and

\[
\frac{SS(\tilde{\epsilon},\tilde{\eta})}{\sigma^2} \sim \chi^2_{2n-2},
\]
from Lemma 3.7. Suppose $M_{V1}(t)$, $M_{V2}(t)$, and $M_{V3}(t)$ represent the moment functions of $SS(\xi, \eta)/\sigma^2$, $RSS(\Theta)/\sigma^2$, and $SS(x^*, y^*, \theta^*)/\sigma^2$ respectively, then when $n$ is quite large, we have

$$M_{V1}(t) = M_{V2+V3}(t) \doteq M_{V2}M_{V3},$$

since $RSS(\Theta)/\sigma^2$ and $SS(x^*, y^*, \theta^*)/\sigma^2$ are asymptotically independent. Hence,

$$M_{V3} \doteq \frac{M_{V1}}{M_{V2}} = (1-2t)^{-\frac{2n-3}{2}};$$

that is

$$\frac{SS(x^*, y^*, \theta^*)}{\sigma^2} \doteq \chi^2_{2n-3}.$$

To show that those chi-square variables are asymptotically mutually independent, all we need to do is to repeat the technique which was shown in Lemma 3.5.

**Conclusion**

The sharpest conclusions are possible for the optimal position adjustments, when the center of gravity of the design pattern is the origin; then

$$\begin{pmatrix} \sqrt{n} \tan \theta^* \\ \sqrt{n} x^* \\ \sqrt{n} y^* \end{pmatrix} \xrightarrow{d} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1/K & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right),$$

where the variance of $\sqrt{n} \tan \theta^*$ is seen to depend on $K$, and, for example, $K = r^2$ when all of the design centers of pegs and holes are on a circle of radius $r$. $\theta^*$ is invariant in following three senses: (1) the order of rotation and translation in
adjustment transformation; (2) the choice of the origin; (3) including or excluding translation in adjustment transformation.

\[
\frac{\text{RSS}(X)}{\sigma^2}, \frac{\text{RSS}(Y)}{\sigma^2}, \text{ and } \frac{\text{RSS}(\Theta)}{\sigma^2}
\]

are asymptotically chi-square distributed and mutually independent, all with degrees of freedom 1. That means that horizontal, vertical, and rotational adjustment are equally important and each can reduce equally amount of misplacement. Thus adjustment reduces misplacement penalty by only \((3/2n)100\%).
CHAPTER 4. COMPENDIUM

Motivated by multiple tolerancing problems in manufacture, this dissertation has examined distributional problems of the following sort:

Let \((u_i, v_i), i = 1, \ldots, n,\) be points in the plane (e.g., design locations of peg-hole combination); let \(\phi(u, v)\) be a Euclidean transformation (either translation \((x, y)\), or rotation \(\theta\), or translation plus rotation); and let \(\epsilon_i\) and \(\eta_i\) be independent normal variables with mean zero and variance \(\sigma^2\). With \(\| \ldots \|\) indicating either maximizing ("\(L_\infty\)") or summing of squares ("\(L_2\)"), and \(| \ldots |\) indicating Euclidean distance, find the distribution of

\[
\min_{\phi} \| \phi(u_i, v_i) - (u_i + \epsilon_i, v_i + \eta_i) \|
\]

with emphasis on the case of large \(n\). In addition, study the distribution of the translation and/or rotation parameters corresponding to \(\phi^*\).

For the minmax fitting problem, define \((x^*, y^*)\) as the optimal translation when \(\theta\) (rotation) is not involved. Then the distribution of \((x^*, y^*)\) depends only on \(n\) (and not on \((u_i, v_i)\)), as can be seen by Equation 2.1 and also pointed out by David and McCann (private communication, 1991). Although the rate of convergence to 0 of \((x^*, y^*)\) is unknown, David and McCann hypothesized that

\[
(x^*, y^*) \xrightarrow{P} (0, 0),
\]
on the basis of simulation involving large $n$.

For the least squares fitting problem, with $(x^*, y^*)$ again defined as the optimal translation without rotation, $x^* = \bar{e}$ and $y^* = \bar{\eta}$, and

\[
\bar{e} \sim \mathcal{N} \left( 0, \frac{\sigma_e^2}{n} \right) \quad \text{and} \quad \bar{\eta} \sim \mathcal{N} \left( 0, \frac{\sigma_\eta^2}{n} \right). \tag{4.1}
\]

That is, $(x^*, y^*)$ tends to $(0, 0)$ with rate $\sqrt{\frac{1}{n}}$. If $(x^*, y^*)$ is the optimal translation with rotation involved, then the pattern locations, $(u_i, v_i)$, do affect $(x^*, y^*)$, unless we choose the center, $(\bar{u}, \bar{v})$, of design pattern as the origin, in which case Equation 4.1 applies. For arbitrary origin,

\[
\begin{align*}
\sqrt{n}x^* & = \bar{e} + \bar{v} \sin \theta^* + \bar{u}(1 - \cos \theta^*) \\
& \overset{d}{\to} \mathcal{N} \left( 0, \sigma^2 \left( 1 + \frac{\theta^2}{K} \right) \right) \\
\sqrt{n}y^* & = \bar{\eta} - \bar{u} \sin \theta^* + \bar{v}(1 - \cos \theta^*) \\
& \overset{d}{\to} \mathcal{N} \left( 0, \sigma^2 \left( 1 + \frac{\theta^2}{K} \right) \right)
\end{align*}
\]

where $\theta^*$ is the optimal rotation, and $K$ is the size of the design pattern as measured by

\[
K = \lim_{n \to \infty} \left[ \frac{1}{n} \sum_{i=1}^{n} (u_i - \bar{u})^2 + \frac{1}{n} \sum_{i=1}^{n} (v_i - \bar{v})^2 \right] / n.
\]

If $\theta^*$ is the rotation-part of joint $(x, y, \theta)$-fit for the minmax problem, then, David and McCann hypothesized on the basis of simulation that $\theta^*$ will tend to zero as pattern size increases.

If $(x^*, y^*, \theta^*)$ is the best fit for the least squares problem, then $\theta^*$ is invariant with respect to choice of origin. Furthermore, $\theta^*$ also is invariant to inclusion or exclusion of $(x, y)$ in the fitting process. The joint distribution of $(x^*, y^*, \theta^*)$ depends on the pattern size and the choice of origin, and those three positioning parameters are
mutually correlated, unless the pattern center is taken as the origin. The asymptotical joint distribution is given as follows:

\[
\begin{pmatrix}
\sqrt{n} \tan \theta^* \\
\sqrt{n} x^* \\
\sqrt{n} y^*
\end{pmatrix} \xrightarrow{d} \mathcal{N}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \frac{\sigma^2}{K}
\begin{pmatrix}
1 & v & -\bar{u} \\
v & K + \bar{v}^2 & -\bar{u}v \\
-\bar{u} & -\bar{u}v & K + \bar{v}^2
\end{pmatrix}.
\]

Note that all three exhibit the same rate \( \sqrt{\frac{1}{n}} \) of convergence to zero. If the center of gravity of the design center is used as the origin, then \( x^* \), \( y^* \), and \( \theta^* \) are asymptotically independent, and

\[
\sqrt{n} \tan \theta^* \xrightarrow{d} \mathcal{N} \left( 0, \frac{\sigma^2}{K} \right),
\]

which shows that \( \tan \theta^* \) is made small by large pattern size, as well as by large \( n \), in keeping with the hypothesis stated above for the minmax problem.

The objective for the least squares problem is to reduce the sum of squares of misplacements. Before any position adjustment the initial sum of squares of misplacements is

\[
\sum_{i=1}^{n} \left[ \epsilon_i^2 + \eta_i^2 \right],
\]

2\(n\)-df chi-square distributed, which can be broken into 4 parts: the first three parts are independently 1-df chi-square distributed which are contributed by \( x^* \), \( y^* \), and \( \theta^* \) respectively; the last part, \( SS(x^*, y^*, \theta^*) \), is due to "lack of fit," and is chi-square distributed with \( (2n - 3) \) degree of freedom. Although \( SS(x^*, y^*, \theta^*) \) is the smallest value we can get after Euclidean transformation, \( SS(x^*, y^*, \theta^*) \) still carries \( \frac{2n - 3}{2n} \) of the initial sum of squares of misplacements. In other words, as \( n \) becomes large,

\[
\frac{SS(x^*, y^*, \theta^*)}{\sum_{i=1}^{n} \left[ \epsilon_i^2 + \eta_i^2 \right]} \rightarrow 1,
\]
and the optimal Euclidean transformation \((x^*, y^*, \theta^*)\) does not play a big role, in this sense.

The minmax interference problem without rotation reduces to a problem with involving only points \((\epsilon_i, \eta_i), i = 1, \ldots, n\). The minmax interference, \(I_n\), has been bracketed by upper- and lower-bounding processes. Let \(r_{[1]}\) be the largest distance between the origin \((0,0)\) and \((\epsilon_i, \eta_i)\); let \(I_{2,n}\) be pairwise interference, which is half of the largest distance between any two points; and let \(R_{n,i,k}\) be the range of \(\{\epsilon_i \cos \theta_i + \eta_i \sin \theta_i, i = 1, \ldots, n\}\), where \(\theta_i = \frac{i \pi}{k}\). Then we have the following inequalities, where \(k_n\) is an increasing multiple of \(k\):

\[
\frac{1}{2} \max_{1 \leq i \leq k} R_{n,i,k} \leq \frac{1}{2} \max_{1 \leq i \leq k_n} R_{n,i,k_n} \leq I_{2,n} \leq I_n \leq r_{[1]}.
\]

Here \(r_{[1]}\) is the extreme value of the Rayleigh(\(\sqrt{2}\)) distribution, with

\[
P \left( r_{[1]} \leq \sigma \sqrt{2 \log n} + \frac{\sigma t}{\sqrt{2 \log n}} \right) \rightarrow H_3(t),
\]

\[
\max_{1 \leq i \leq k} R_{n,i,k} \text{ satisfies}
\]

\[
P \left( \max_{1 \leq i \leq k} R_{n,i,k} \leq 2\sigma \left( \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} \right) + \frac{\sigma t}{\sqrt{2 \log n}} \right) \rightarrow \left\{ \int_{-\infty}^{\infty} \exp \left( -e^{s-t} \right) d \left[ \exp \left( -e^{-s} \right) \right] \right\}^k,
\]

and \(\max_{1 \leq i \leq k_n} R_{n,i,k_n}\) with \(k_n = o(\sqrt{\log n})\) is hypothesized to satisfy

\[
P \left( \max_{1 \leq i \leq k_n} R_{n,i,k_n} \leq 2\sigma \left( \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} + \frac{c(k_n)}{2\sqrt{2 \log n}} \right) + \frac{\sigma t}{\sqrt{2 \log n}} \right) \rightarrow H_3(x),
\]

where \(c(m) = \inf \left\{ t : 1 - \int_{-\infty}^{\infty} \exp \left( -e^{s-t} \right) d \left[ \exp \left( -e^{-s} \right) \right] \leq \frac{1}{m} \right\} \),
which would allow the claim that

\[
\sigma \left( \sqrt{2 \log n} - \frac{\log \log n - c(k_n)}{2 \sqrt{2 \log n}} \right) \leq I_n \leq \sigma \sqrt{2 \log n}.
\]

This result suggests that as \( n \) increases, the standard deviation corresponding to manufacturing tolerance must decrease with rate at least \( \frac{1}{\sqrt{\log n}} \) for assembly to be possible.
BIBLIOGRAPHY


