APPLICATIONS OF THE VOLUME INTEGRAL TECHNIQUE TO MODELING IMPEDANCE CHANGES DUE TO SURFACE CRACKS

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INTRODUCTION
The volume integral technique solves the electromagnetic diffusion equation for the electric field vector $\mathbf{E}$ in the presence of a flaw of conductivity $\sigma_f$ in a host medium of conductivity $\sigma_h$. Given the electric field, the impedance change in a probe coil may be computed. The technique has been used to model electromagnetic data collected in geophysical exploration programs [1].

THEORETICAL BACKGROUND
Using Maxwell's equations and ignoring displacement currents, the partial differential equation for the electric field $\mathbf{E}$ due to a source (probe coil) current density $\mathbf{J}^0$ is found to be

$$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = i\omega \mu \mathbf{J}^0$$

(1)

where $\gamma^2 = i\omega \mu \sigma$, $\omega$ is the frequency of the source, $\mu$ is the magnetic permeability, $\sigma$ is the conductivity and $i = (-1)^{1/2}$.

For a source $\mathbf{J}^0$ distributed over a volume $V$, the solution to Equation 1 is

$$\mathbf{E}^0(\mathbf{r}) = \int_V \mathbf{G}(\mathbf{r},\mathbf{r'}) \cdot \mathbf{J}^0(\mathbf{r'})dV$$

(2)

where $\mathbf{G}(\mathbf{r},\mathbf{r'})$ is the Green's tensor for Equation 1; i.e., the solution of the equation

$$\nabla^2 \mathbf{G} - \gamma^2 \mathbf{G} = i\omega \mu \mathbf{I}\delta(\mathbf{r},\mathbf{r'})$$

(3)

where $\mathbf{I}$ is the identity matrix and $\delta(\mathbf{r},\mathbf{r'})$ is the Dirac delta function. In three dimensions, Green's tensor contains nine components $G_{ij}(\mathbf{r},\mathbf{r'})$. $G_{ij}(\mathbf{r},\mathbf{r'})$ is the electric field component $E_i$ at $\mathbf{r}$ due to a point source of current density of unit magnitude oriented in the $j$ direction at the point $\mathbf{r'}$. Green's tensor may be derived for homogeneous and layered half spaces [1, 2].
To model flaws of conductivity \( \sigma_f \) in a host medium of conductivity \( \sigma_h \), the conductivity \( \sigma \) is written as \( \sigma = \sigma_h + \delta \sigma \) where

\[
\delta \sigma = \begin{cases} 
0 & \text{outside flaw} \\
\sigma_f - \sigma_h & \text{inside flaw}
\end{cases}
\]  

(4)

The constitutive equation relating current density \( J \) to electric field \( E \) therefore becomes

\[
J = \sigma E + J^0 = (\sigma_h + \delta \sigma)E + J^0.
\]  

(5)

When the above equation is used to re-derive Equation 1, the result is

\[
V^2 E - \gamma_h^2 E = i\omega \mu (J^0 + \delta \sigma E)
\]  

(6)

for which the solution is

\[
E(r) = E^0(r) + \int_V \delta \sigma G(r,r') \cdot E(r')dV
\]  

(7)

where Equation 2 has been used. Equation 6 shows that the flaw is modeled as an equivalent distribution of dipole current sources of strength \( \delta \sigma E \). The volume integration in Equation 4 extends only over the region where \( \delta \sigma \) is non-zero i.e., the flaw volume.

An integral equation may be derived to compute the electric field due to the dipole distribution (the scattered field). Assuming \( \delta \sigma = \sigma_f - \sigma_h \) is constant in \( V \) (this is not a necessary assumption). Equation 6 may be rearranged to give

\[
E^0(r) = E(r) - (\sigma_f - \sigma_h) \int_V G(r,r') \cdot E(r')dV.
\]  

(8)

Thus, knowing the incident electric field \( E^0 \), the scattered electric field \( E \) which is equivalent to the presence of the defect may be computed. Given the scattered field, the corresponding impedance change in the probe coil may be computed, as discussed in the next section.

The discretization and solution of Equation 5 begins by dividing the volume of the flaw into \( N \) volume elements, within which the scattered electric field is assumed constant. (More complicated variations of \( E \) could be assumed). Equation 8 is then written

\[
E^0(r_j) = E(r_j) - (\sigma_f - \sigma_h) \sum_{k=1}^{N} [\int_{V_k} G(r_j,r')dV_k] \cdot E_k
\]  

(9)

where \( r_j \) is a position vector and \( E_k \) denotes the electric field vector at the centroid of the Kth element. A system of equations of order \( 3N \) in \( E_k \) is formed by letting \( r_j \) be the centroid of each of the volume elements \( V_j \), \( 1 \leq j \leq N \):

\[
E^0 = (I-C)E = AE
\]  

(10)

where \( E^0 \) is a vector of length \( 3N \) composed of incident field values, \( I \) is the identity matrix and

\[
C_{jk} = (\sigma_f - \sigma_h) \int_{V_k} G(r_j,r')dV_k
\]  

(11)

is a 3 by 3 matrix formed by integrating the Green's tensor over the volume element \( V_k \).

The details of the discretization of the flaw, the integration of the Green's tensor and the solution of the system of equations are described by Dunbar [3]. Generally the matrix \( A \) is asymmetric and fully populated.
CALCULATION OF THE IMPEDANCE CHANGE

The change in coil impedance due to the scattered electric field $E$ is given by Faraday's law

$$\Delta Z = -\frac{i\omega}{1} \int_S B \cdot ndS$$

where $B$ is the magnetic field in the coil due to the scattered current density in the flaw volume and $I$ is the current in the probe coil. The vector $n$ is the unit normal to the coil surface $S$. For a flat-lying coil whose axis lies along the $Z$ coordinate axis directed into the host, $n = (0, 0, -1)$ and

$$\Delta Z = \frac{i\omega}{1} \int_S B_z dS$$

The above equation was used to compute impedance changes.

An even simpler method of computing impedance changes would be to use the formula for $\Delta Z$ derived by means of the reciprocity theorem:

$$\Delta Z = \frac{1}{1} \int_{V_f} (\sigma_f - \sigma_h) E^* \cdot EdV$$

where $V_f$ is the flaw volume [4]. The author is grateful to Dr. S. Burke of the Aeronautical Research Laboratories in Australia for pointing this out.

EXAMPLES

Two examples are presented:

1) Rectangular surface slot in aluminum plate.
2) 'Realistic' surface crack in aluminum plate.

In each example, the flaw was assumed to occur in a semi-infinite half space. This assumption is valid since the skin depth in each example is shallow; i.e. the coil frequencies are sufficiently high that the host may be assumed to be a half space.

The geometry and discretization of the slot of Example 1 are shown in Figure 1. Table 1 lists the model and coil parameters for this example. The coil is of rectangular cross section with an air core so that the incident electric field may be calculated by the Dodd and Deeds [5] formulation. The impedance change profile along the positive $y$ axis is shown in Figure 2. This profile is similar in form and magnitude to that calculated by Auld et al [6] by another method for a similar geometry. This is regarded as partial verification of the program.

The geometry and discretization of the 'realistic' surface crack of Example 2 is shown in Figure 3. Table 1 lists the model parameters for this example. The same coil as in Example 1 was used. The impedance change profile along the positive $y$ axis is shown in Figure 4. As expected, it is similar to the impedance profile of Example 1, but lower in magnitude.

In both Examples 1 and 2, the impedance change profile is symmetric about the $x$ axis. The results in Figures 2 and 4 were found to be sensitive to the amount of discretization of the flaw, particularly along the $y$ axis. This is to be expected in these cases but implies that, in general, different discretizations should be attempted so that confidence in the results can be established.
TABLE 1

Model Parameters - Examples 1 and 2

Coil

Inner radius: 2.0 mm  
Outer radius: 10.0 mm  
Height: 5.0 mm  
Lift-off: 0.1 mm  
Frequency: 200 kHz  
Number of turns: 1000

Test Specimen

Conductivity: 2.6 x 10^7 S m⁻¹

Defect

Example 1
Conductivity: 0.0 S m⁻¹  
Length: 4.0 mm  
Depth: 1.0 mm  
Width: 0.2 mm

Example 2
Conductivity: 0.0 S m⁻¹  
Length: 4.0 mm  
Depth: 1.0 mm  
Width: 0.0 mm at ends; 0.2 mm maximum

FIGURE 1. Example 1. Plan view, cross-section and discretization
FIGURE 2. Example 1. Impedance change profile along y axis

FIGURE 3. Example 2. Discretization
Very little time was required to set up the data for each model. Although the method is economical with respect to computer storage, computing times are long. Most of the computer time is spent evaluating Hankel transforms associated with the incident field, $E^0$, and the magnetic field, $B$, in the coil. Use of Equation 14 for the impedance calculation would undoubtedly lower computing time requirements.

**VOLUME INTEGRAL–FINITE ELEMENT HYBRID**

For some geometries (e.g., near a corner) it is impossible to derive an analytical solution for the incident electric field or for Green's tensor. An alternative is the finite element method. However, in order to model the boundary conditions at infinity, the mesh of a finite element model must be extended out to large distances where the electric field may be assumed negligible. A hybrid finite element technique, which uses the Green's tensor of the volume integral method to satisfy the boundary conditions at infinity, is described in this section.

The system of equations associated with a finite element model is

$$K E = S$$

(15)

where $K$ is a 'diffusion matrix', $E$ is a vector of electric field components at the nodes of the finite element mesh and $S$ is a source vector. The matrix $K$ is sparse, banded and symmetric. The vector $S$ is zero everywhere except in the elements comprising the source (probe coil).

Equation 15 may be partitioned as follows:

$$
\begin{bmatrix}
K_{vv} & K_{vb}
\end{bmatrix}
\begin{bmatrix}
E_v \\
E_b
\end{bmatrix} =
\begin{bmatrix}
S_v \\
S_b
\end{bmatrix}
$$

(16)

where the subscripts $v$ and $b$ denote the interior and boundary respectively of the finite element mesh as shown in Figure 5. The first equation in the system 16 may be written

$$K_{vv} E_v = S_v - K_{vb} E_b .$$

(17)
Within the finite element mesh and on its boundary, the volume integral method may be used to compute $E_b$ by means of Equation 8 (using the notation of Equation 10).

$$E_b = E_b^* + C E_v .$$  

(18)

Substituting Equation 18 into Equation 17 gives

$$(K_{vv} + K_{vb} C) E_v = S_v - K_{vb} E_b^*$$  

(19)

which may be solved for $E_v$. Equation 18 may then be used to compute the electric field in the probe if it lies outside the finite element mesh.

Owing to the presence of the matrix $C$, the matrix $K_{vv} + K_{vb} C$ is asymmetric and tends to be fully populated. However, it is likely to be of relatively small dimension compared to $K$.

FIGURE 5. Exterior and interior regions of a hybrid mode.

Lee et al [7] proposed an iterative scheme that avoids the inversion or decomposition of asymmetric matrices. The iteration proceeds as follows:

1) Assume a value for $E_b$, say $E_b^*$,
2) Solve for $E_v$ from Equation 17,
3) Compute new $E_b$ from Equation 18,
4) If the difference between $E_b$ and $E_b^*$ is greater than a prescribed tolerance, set $E_b^* = E_b$ and go to 2. Otherwise quit.
Lee et al [7] found that when the conductivity contrast between the interior and exterior regions is greater than 1000, the rate of convergence is extremely slow or divergence can occur. This might pose a problem in some applications. However, a similar iterative algorithm is used in the stress analysis of solids. The iteration proceeds by adjusting displacements at the boundary between the exterior and interior regions. When the ratio of the elastic modulus of the interior and exterior regions exceeds a critical value, slow convergence or divergence occurs. The remedy is to reformulate the problem in terms of stresses at the boundary between the interior and exterior regions. Thus, if one analogizes the electric field $E$ with displacements, derivatives of $E$ (the magnetic field) with stresses and conductivity with elastic modulus, the problem should be reformulated in terms of the magnetic field when slow convergence occurs. This is somewhat speculative and needs to be tested.

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