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THE ANDRUNAKIEVICH LEMMA FOR ALTERNATIVE RINGS

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Abstract

Let $W$ be a subset of an alternative ring $A$, and let $I$ be the ideal of $A$ generated by $W$. We show that $I^2 \subseteq W_1 I + IW_1$, where $W_1 = W + AW + WA$. We use this result to prove that the Andrunakievich lemma for alternative rings has an index of at most 4. We also prove that the ideal generated by an absolute zero divisor has index of nilpotency at most 4. These last two indices are an improvement on the work of S. V. Pchelintsev. He established upper bounds of $4 \times 5^6$ and 13 respectively. Our work requires some assumptions on characteristic 3.

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The Andrunakievich lemma for associative rings says that whenever $V$ is an ideal of $I$, and $I$ is an ideal of $A$, then $(V\#)^3 \subseteq V$, where $V\#$ is the ideal of $A$ generated by the subring $V$. S. V. Pchelintsev [2] has shown that for alternative rings, $(V\#)^n \subseteq V$ for some $n$, but his upper bound of nilpotence is $n = 4 \cdot 5^6$. We lower that bound to 4. We shall use the notation and results given in [1] extensively. We make all references to that paper for background, even when it is not the original source of the identities.

An absolute zero divisor of an alternative algebra $A$ is an element $a \in A$ such that $(aA)a = 0$. We show that the ideal of $A$ generated by an absolute zero divisor is nilpotent of index at most 4.

For any subset $W$ of the alternative ring $A$ we are interested in the ideal of $A$ generated by $W$. This ideal is indicated by $W\#$. We define a sequence of additive subgroups of $A$. We let $W_0$ be the additive subgroup of $A$ spanned by $W$, and we define $W_{n+1} = W_n + W_nA + AW_n$.

It is clear that $W_n \subseteq W_{n+1}$ and that $W\# = \bigcup_{n=0}^{\infty} W_n$. In [1, Proposition 3.2] it is shown that for $n \geq 1$, $W_{n+1} = W_n + W_nA = W_n + AW_n$. Thus, for the first level, one has to use both right and left multiplications by $A$. But subsequently, the extensions can be obtained from multiplications from either side. This is particularly useful for us in induction arguments, since only one sided multiplications need to be considered.

By the notation $(a, b, c)$ for $a, b, c$ in $A$, we mean the associator $(a, b, c) = (ab)c - a(bc)$. The commutator $[a, b]$ is defined by $[a, b] = ab - ba$. To overcome a difficulty related to characteristic 3, we define a new set containment symbol with a new definition. We say that $W \subseteq V$ if for each $w$ in $W$ there exists an integer $i \geq 0$ which can depend on $w$ such that $3^iw \in V$. Clearly, $W \subseteq V$ implies $W \subseteq V$. Also, $W \subseteq V$ and $V \subseteq U$ implies $W \subseteq U$. 
LEMMA 1. In any alternative ring the following identities hold.

(a) \( a(bc + cb) = (ab)c + (ac)b \) and \( (ab + ba)c = a(bc) + b(ac) \)

(b) \( (a, b, c) = (b, c, a) = (c, a, b) = -(b, a, c) = -(a, c, b) = -(c, b, a) \)

(c) \( x(ax) = x(az) \)

(d) \( x(ab)x = (xa)(bx) \)

(e) \( (za)z = z(az) \)

(f) \( (za)(bx) = z(a, b) \) and \( b(za) = ((bx)a)z \)

(g) \( (a, (b, c, d)] = (a, b, cd) + (a, c, db) + (a, d, bc) \)

PROOF: An alternative ring is a nonassociative ring satisfying the identities 
\( (y, x, z) = (x, y, z) = 0 \). Part (a) is just the linearized form of the definition of 
alternative rings. Part (b) follows from (a). Part (c) follows from (b) and the 
definition of alternative rings. Part (d) is the middle Moufang identity. See [1, 
2.2(c)]. Part (e) contains the left and right Moufang identities. See [1, 2.2(e)] for 
the first listed. The other follows by symmetry. The first identity of Part (f) is 
given in [1, 2.2(d)]. The other follows by symmetry. Part (g) is [1, Lemma 2.3(b)].

For the proof of the Andrunakievich lemma, we will need these results.

LEMMA 2. If \( A \) is an alternative ring, \( I \) is an ideal of \( A \), \( V \) is an ideal of \( I \), and 
\( V \# = I \), then the following containments hold.

(a) \( IV_1 \subset V + VA \)

(b) \( VA \cdot I^2 \subset V \)

(c) \( (V_1, I, I^2) \subset V \)

(d) \( V_1 \cdot I^3 \subset V \)

(e) \( I^{r+} \subset I^r I^* + V \).

PROOF: The requirement that \( V \# = I \) is done for cosmetic reasons. It enables us 
to write \( I \) rather than the more complicated expression \( V \# \) in the statement of the
results. It also helps keep the terminology standard because in the rest of the paper I will be the ideal generated by the subset of \( A \) under discussion.

Part (a)(b)(c) are \([1, \text{Lemma 4.1 (c)(e)(f)}]\). Part (d) is \([1, \text{Proposition 4.8}]\). Notice that the use of "o" in \( I^3 \circ V_1 \subset V \) is explained in \([1, \text{page 244}]\); it means that \( I^3 V_1 \subset V \) and \( V_1 I^3 \subset V \). This result requires that \( I \) is actually the ideal generated by \( V \). Part (e) is \([1, \text{Proposition 6.6}]\). It is the most difficult result to prove. We had hoped for an independent and short proof in the one instance we need it for, that is \( I^4 \subset I^2 I^2 + V \). But we did not succeed in finding one.

This next lemma contains the key to the whole problem.

**LEMA 3.** In any alternative ring we have the following two identities.

\[
\begin{align*}
(a) \quad (i, b, [a, cj]) &= (i, c, a_j \cdot b + bj \cdot a) - [i, (a, b, cj) + (a, c, bj) + (b, c, aj)] \\
&- (i, ab, cj) - (i, ac, bj) - (i, bc, aj) \\
&+ (i, a, (bc + cb)j) + (i, b, (ca + ac)j) \\
(b) \quad (i, b, [a, jc]) &= -(i, c, a \cdot j b + b \cdot ja) + [i, (c, a, jb) + (c, b, ja) + (b, a, jc)] \\
&- (i, a, j(bc + cb)) - (i, b, j(ac + ca)) \\
&+ (i, ca, jib) + (i, cb, ja) + (i, ba, jc)
\end{align*}
\]

**PROOF OF PART (a):** Add the following identities. They are referenced at the left hand edge. The terms in the sum can be rearranged to give Part (a).

\[
\begin{align*}
\text{Lemma 1(g)} \quad [i, (a, b, cj)] &= (i, a, b \cdot cj) + (i, b, cj \cdot a) + (i, cj, ab) \\
\text{Lemma 1(g)} \quad [i, (a, c, bj)] &= (i, a, c \cdot bj) + (i, c, bj \cdot a) + (i, bj, ac)
\end{align*}
\]
Lemma 1(g) \[ [i, (b, c, aj)] = (i, b, c \cdot aj) + (i, c, aj \cdot b) + (i, aj, bc) \]

Lemma 1(a) \[ 0 = (i, a, (bc + cb)j - b \cdot cj - c \cdot bj) \]

Lemma 1(a) \[ 0 = (i, b, (ca + ac)j - c \cdot aj - a \cdot cj) \]

To prove Part (b), add the following identities and rearrange the terms.

Lemma 1(g) \[ [i, (c, a, jb)] = (i, c, a \cdot jb) + (i, a, jb \cdot c) + (i, jb, ca) \]

Lemma 1(g) \[ [i, (c, b, ja)] = (i, c, b \cdot ja) + (i, b, ja \cdot c) + (i, ja, cb) \]

Lemma 1(g) \[ [i, (b, a, jc)] = (i, b, a \cdot jc) + (i, a, jc \cdot b) + (i, jc, ba) \]

Lemma 1(a) \[ 0 = (i, b, j(ac + ca) - ja \cdot c - jc \cdot a) \]

Lemma 1(a) \[ 0 = (i, a, j(bc + cb) - jb \cdot c - jc \cdot b) \]

If \( W \) is a subset of \( A \), we will use the notation \( \widehat{W} \) to mean the additive subgroup generated by \( \{xwx \mid x \in A, w \in W\} \).

**Lemma 4.** If \( W \) is any subset of an alternative ring \( A \), and \( I \) is the ideal of \( A \) generated by \( W \), then

(a) \( \widehat{W}_n \subseteq W_{n+1} \) if \( n \geq 1 \)

(b) \( \widehat{W_0 \cdot I + I \cdot W_0} \subseteq W_1 \cdot I + I \cdot W_1 \).

**Proof of Part (a):** For \( n \geq 1 \) \( W_n = W_{n-1} + W_{n-1}A + AW_{n-1} \). Using Lemma 1(d), we compute \( x \cdot W_n \cdot x \subseteq x \cdot W_{n-1} \cdot x + x(W_{n-1} \cdot A) x + x(AW_{n-1}) x \subseteq x \cdot W_{n-1} \cdot x + (x \cdot W_{n-1}) (Ax) + (xA)(W_{n-1} \cdot x) \subseteq W_{n+1} \).

Part (b) is proved using Lemma 1(b)(e). \( I(xWx) \subseteq ((Ix)W)x \subseteq (Ix)(Wx) + (Ix, W, x) \subseteq (Ix)(Wx) + (Ix, x, W) \subseteq I \cdot W_1 \). The proof for the other term is similar.
The next lemma is the place in the paper where the problems with characteristic
arise. Our symbol $\subseteq$ is defined in the preliminaries.

Lemmas 5. If $W$ is a subset of an alternative ring $A$, then

(a) $(W_n, A, A) \subseteq W_{n+1} + [A, W_nA] + [A, AW_n]$ for all $n \geq 0$.

(b) $W_{n+2} \subseteq W_{n+1} + [A, W_nA] + [A, AW_n]$ for all $n \geq 0$.

Proof of Part (a): $(t, a, b) + (a, b, t) + (b, t, a) = ta \cdot b + ab \cdot t + bt \cdot a - t \cdot ab - a \cdot bt - b \cdot ta = [ta, b] + [ab, t] + [bt, a]$. Thus $3(t, a, b) = -[b, ta] + [ab, t] - [a, bt]$. This gives us that $3(W_n, A, A) \subseteq [A, W_nA] + [A, W_n] + [A, AW_n] \subseteq [A, W_nA] + W_{n+1} + [A, AW_n]$. Thus $(W_n, A, A) \subseteq W_{n+1} + [A, W_nA] + [A, AW_n]$.

We prove Part (b). For any $n \geq 0$

$$W_{n+2} \subseteq W_{n+1} + A(AW_n + W_nA) + (AW_n + W_nA)A$$

$$\subseteq W_{n+1} + A \cdot AW_n + W_nA \cdot A + [A, W_nA] + [A, AW_n]$$

$$\subseteq W_{n+1} + (W_n, A, A) + [A, W_nA] + [A, AW_n].$$

Using Part (a) we get $W_{n+2} \subseteq W_{n+1} + [A, W_nA] + [A, AW_n]$.

Lemmas 6. If $W$ is a subset of an alternative ring $A$, and $I$ is the ideal generated
by $W$, then

$$(i, A, I) \subseteq iI + Ii + (i, A, W_1) + (i, A, \widehat{W_0}).$$

Proof: First notice that the element $i$ is fixed. Eventually $i$ will be chosen to
be in the ideal $I$. But for now, it can be any element of $A$ at all. The proof is
by induction on the subscript $n$. Clearly $(i, A, W_1)$ is contained in the right hand
side. Any subscript greater than one can be written as $n + 2$ for $n \geq 0$. Assume
that the result is true for $(i, A, W_{n+1})$. We consider $(i, A, W_{n+2})$. Using Lemma
5(b) and Lemma 3 we get $(i, A, W_{n+2}) \subseteq (i, A, W_{n+1} + [A, W_nA] + [A, AW_n]) \subseteq [i, I] + (i, A, \widehat{W_n}) + (i, A, W_{n+1}) \subseteq iI + Ii + (i, A, \widehat{W_0}) + (i, A, W_{n+1})$. The last step
used the fact that if \( n = 0 \), then \( \widehat{W}_n = \widehat{W}_0 \), and if \( n > 0 \) then by Lemma 4(a) \( \widehat{W}_n \subset W_{n+1} \). The induction hypothesis now produces the required identity.

**Lemma 7.** If \( W \) is any subset of an alternative ring \( A \), and \( I \) is the ideal generated by \( W \), then

\[
I^2 \subseteq W_1 \cdot I + I \cdot W_1.
\]

**Proof:** We again use induction on the subscript \( n \) to show that \( W_nI + IW_n \subseteq W_1I + IW_1 \). The case \( n = 1 \) is trivial. Now assume that \( W_nI + IW_n \subseteq W_1I + IW_1 \) and that \( n \geq 1 \). We will use Lemma 6 twice, Lemma 1(b) and Lemma 4(b) as well as induction.

\[
W_{n+1}I + IW_{n+1} \subseteq (W_n + W_nA)I + I(W_n + AW_n)
\]

\[
\subseteq W_nI + IW_n + (W_n, A, I)
\]

\[
\subseteq W_nI + IW_n + (W_n, A, W_1) + (W_n, A, \widehat{W}_0)
\]

\[
\subseteq W_nI + IW_n + (W_1, A, W_n) + (\widehat{W}_0, A, W_n)
\]

\[
\subseteq W_nI + IW_n + (W_1, A, I) + (\widehat{W}_0, A, I)
\]

\[
\subseteq W_nI + IW_n + W_1I + IW_1 + (W_1, A, W_1) + (W_1, A, \widehat{W}_0)
\]

\[
\quad + \widehat{W}_0I + I\widehat{W}_0 + (\widehat{W}_0, A, W_1) + (\widehat{W}_0, A, \widehat{W}_0)
\]

\[
\subseteq W_nI + IW_n + W_1I + IW_1
\]

\[
\subseteq W_1I + IW_1.
\]
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THEOREM 8. Let \( A \) be an alternative ring. Suppose that \( V \) is an ideal of \( I \) and \( I \) is an ideal of \( A \). Furthermore, assume that \( V \neq I \). Then
\[
I^2 I^2 \subseteq V.
\]

PROOF: We use Lemma 7 and Lemma 2(a)(b)(c)(d).
\[
I^2 I^2 \subseteq (V_1 I + IV_1) \cdot I^2 \subseteq V_1 \cdot I^2 + (V_1, I, I^2) + (V + V A) \cdot I^2 \subseteq V.
\]

THEOREM 9. Let \( A \) be an alternative ring. Suppose that \( V \) is an ideal of \( I \) and \( I \) is an ideal of \( A \). Furthermore, assume that \( V \neq I \). Then
\[
I^4 \subseteq V.
\]

PROOF: From Lemma 2(e) and Theorem 8 we get
\[
I^4 \subseteq I^2 I^2 + V \subseteq V.
\]

REMARK: Unfortunately, we can get only that \( I^4 \subseteq V \) rather than the more desirable \( I^4 \subset V \). Certainly, if \( V \) had the property that for \( i \in I \), \( 3i \in V \) implies \( i \in V \), then \( I^4 \subset V \). For an example of such a situation, let \( A \) be an algebra over a commutative ring \( \phi \) containing \( 1/3 \) and assume that \( \phi V \subseteq V \).

ABSOLUTE ZERO DIVISORS

We now examine alternative rings with an element \( a \) which is an absolute zero divisor. We have to assume that our algebra has characteristic not 3 which means that \( 3x = 0 \) implies \( x = 0 \). With this assumption on characteristic, if \( I \subseteq 0 \), then \( I = 0 \).

LEMMA 10. Let \( A \) be an alternative ring such that \( 3x = 0 \) implies \( x = 0 \). Let \( a \) be an element of \( A \) such that \( aAa = 0 \). Let \( W = \{a\} \) and \( I \) be the ideal generated by \( W \). Then
\[
(a) \quad aW_1 + W_1 a \subset a^2 + a^2 A + Aa^2
\]
Proof: Parts (a)(b)(c) follow from the definition of alternative rings, absolute zero divisors, and Lemma 1(c)(e)(f). Part (d) uses Part (c), Lemma 6, and Lemma 1(c)(f). (a, A, I) \subseteq aI + Ia = 0. Part (e) uses Part (d), Lemma 7, and Lemma 1(e). We show only the first containment. The other is similar. \( I^2 \subseteq W_1I + IW_1 \subseteq (a + aA + Aa)I + I(a + aA + Aa) \subseteq aI + a(Al) + A(al) + I(aI) + I(aA) + (IA)a + (a, A, I). \) Left multiplication by \( a \) gives us \( aI^2 \subseteq a^2I. \) Part (f) follows from Part (e): \( a^2I^2 \subseteq a(aI^2) \subseteq a(a^2I) \subseteq a^3I = 0. \) The other half is similar. To prove part (g) we rewrite Lemma 1(g) to get \( (a, b, cd) = [a, (b, c, d)] - (a, c, db) - (a, d, bc). \) Then \( (a^2, I, W_nA) \subseteq [a^2, (I, W_n, A)] + (a^2, W_n, AI) + (a^2, A, I W_n) \subseteq [a^2, I^2] + (a^2, W_n, I) + (a^2, A, I^2) \subseteq (a^2, I, W_n) \) using the fact that \( I^2 \) is an ideal and Part (f). By induction we eventually get \( (a^2, I, I) \subseteq (a^2, I, W_1) \subseteq (a^2, W_1, I) = 0 \) by Parts (b)(f). For Part (h) we will show that \( I(a^2I) = 0. \) The rest follow from Parts (f)(g). \( W_1(a^2I) = 0 \) by Parts (b)(g). Now \( (AW_n)(a^2I) \subseteq A(W_n \cdot a^2I) + (A, W_n, a^2I) \subseteq A(W_n \cdot a^2I) + (a^2I, W_n, A) \subseteq A(W_n \cdot a^2I). \) The last containment follows from Parts (f)(g) and the proof finishes by induction.

**Theorem 11.** Let \( A \) be an alternative ring such that \( 3x = 0 \) implies \( x = 0. \) Let \( a \) be an element such that \( (aA)a = 0. \) Then the ideal of \( A \) generated by \( a \) is nilpotent of index at most 4.
PROOF: If $J$ and $K$ are any ideals of an alternative ring, then both $J^2$ and $Ann(J/K) = \{ x \in A \mid xJ + Jx \subseteq K \}$ are ideals. The proof follows from Lemma 1(b). Let $I$ be the ideal generated by $a$. Let $J$ be the ideal generated by $a^2 I + Ia^2$. From Lemma 10(e) $al^2 + P^2a \subseteq J$. Thus $a \in Ann(I^2/J)$ and so $I \subseteq Ann(I^2/J)$. Furthermore, by Lemma 10(h), $a^2 I + Ia^2 \subseteq Ann(I/0)$. Thus $J \subseteq Ann(I/0)$. We conclude that $I \subseteq Ann(I^2/J)$ and $J \subseteq Ann(I/0)$. Thus $I^3 I + I^3 = 0$. By Lemma 1(b) $I^2 I^2 = 0$ as well. Thus $I^4 = 0$.

S. V. Pchelintsev showed that if $a$ is a strong absolute zero divisor, that is: $a^2 = aAa = 0$, then the ideal generated by $a$ is nilpotent of index at most 3. This result is a consequence of Lemma 10(e). We show by the following example that 3 is the best possible index for the nilpotency index of the ideal generated by a strong absolute zero divisor.

The ring as 10 basis elements $a, b, c, ab, ac, ba, ca, t, u, y$. The element $a$ is an absolute zero divisor and $(ab)(ac) = y$. In this ring, the ideal generated by $a$ does not square to zero. The nonzero products of the algebra are: $a \cdot b = ab, a \cdot c = ac, a \cdot t = y, b \cdot a = ba, b \cdot ac = u, b \cdot ca = t - u, c \cdot a = ca, c \cdot ab = -u, c \cdot ba = -t + u, ab \cdot c = -t + u, ab \cdot ac = y, ac \cdot b = t - u, ac \cdot ab = -y, ba \cdot c = t, ba \cdot ac = -y, ba \cdot ca = y, ca \cdot b = -t, ca \cdot ab = y, ca \cdot ba = -y, u \cdot a = y$. There are 42 nonzero associators. We mention 7 of them. The rest are obtained by permuting the arguments of the associators given. $(a, b, c) = -t + u. (a, b, c)a = (a, ab, c) = (a, b, ac) = -a(a, b, c) = -(a, ba, c) = -(a, b, ca) = y$.

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