Efficient inversion of Fourier and Laplace domain boundary element solutions for elastodynamic scattering

Joel Kevin Ness

Iowa State University

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Efficient inversion of Fourier and Laplace domain boundary element solutions for elastodynamic scattering

Ness, Joel Kevin, Ph.D.

Iowa State University, 1994
Efficient inversion of Fourier and Laplace domain boundary element solutions for elastodynamic scattering

by

Joel Kevin Ness

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY


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For the Major Department

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For the Graduate College

Iowa State University
Ames, Iowa

1994

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# TABLE OF CONTENTS

**LIST OF FIGURES** ................................................................. iv

**LIST OF TABLES** ............................................................... vii

**NOMENCLATURE** ................................................................ viii

**INTRODUCTION** .................................................................... 1

- Introduction ........................................................................ 1
- The Scattering Problem .................................................... 3
- Numerical Solutions .......................................................... 6
- The Transient Problem ....................................................... 9

**SCATTERING BY AN OBSTACLE IN AN ELASTIC MEDIUM** ........ 11

- Introduction ........................................................................ 11
- Governing Equations ....................................................... 12
- Incident Waves ................................................................... 16
- Boundary Integral Formulation and Regularization .............. 18
- Numerical Evaluation of the Boundary Integral Equation ...... 26

**FOURIER TRANSFORM INVERSION** ........................................ 35

- Introduction ....................................................................... 35
- The Fourier Transform .................................................... 36
- The Fast Fourier Transform ............................................... 37

**LAPLACE TRANSFORM INVERSIONS** ....................................... 41

- Introduction ...................................................................... 41
- The Laplace Transform .................................................... 41
- Durban's Method of the Laplace Transform Inversion ........... 44

**FOURIER DOMAIN WITH DAMPING EFFECTS** ....................... 50

- Fictitious Eigenfrequencies ............................................... 50
LIST OF FIGURES

Figure 1: Ultrasonic transducer ......................................................... 4
Figure 2: Flaw detection with a contact probe ................................. 5
Figure 3: Oscilloscope observation ..................................................... 5
Figure 4: Scattering of an elastic wave by a flaw .............................. 11
Figure 5: Gaussian incident wave ...................................................... 18
Figure 6: Infinite region with a flaw .................................................... 22
Figure 7: Local surface region about a boundary point ...................... 25
Figure 8: Nodal points on an octant of a sphere model ..................... 29
Figure 9: Linear triangular element discretization .............................. 29
Figure 10: Boundary elements and node numbering conventions .......... 30
Figure 11: Six node triangular and eight node quadrilateral elements .... 31
Figure 12: Fourier transform of the incident pulse via FFT ................. 39
Figure 13: Recovery of incident pulse from the inverse FFT ............... 40
Figure 14: Laplace transform of the incident wave ............................ 49
Figure 15: Durban Laplace inversion of the incident wave ................. 49
Figure 16: Fictitious eigenfrequencies ............................................... 51
Figure 17: Condition number damping .............................................. 55
Figure 18: Reflection-Transmission coefficients ............................... 59
Figure 19: Photo of the spherical void ............................................. 60
Figure 20: Experimental setup for the spherical void sample ............... 61
Figure 21: Buehler's transoptic scattering samples ............................ 61
Figure 22: Experimental scattering from a spherical void ................. 62
Figure 23: Incident Wave frequency domain .................................... 63
Figure 24: Time domain spherical void scattered signal .................... 64
Figure 25: Frequency domain spherical void scattered signal
Figure 26: Experimental scattering from a tin-lead spherical inclusion
Figure 27: Time domain tin-lead spherical inclusion signal
Figure 28: Frequency domain tin-lead spherical inclusion signal
Figure 29: Experimental scattering from a polystyrene spherical inclusion
Figure 30: Time domain polystyrene spherical inclusion signal
Figure 31: Frequency domain polystyrene spherical inclusion signal
Figure 32: Time domain comparison of front surface reflected wave
Figure 33: Frequency domain comparison of front surface reflected wave
Figure 34: Spectrum of the incident wave for the spherical void
Figure 35: BEM-FFT time domain comparison of the spherical void
Figure 36: BEM-FFT comparison of the damping coefficient
Figure 37: BEM-FFT frequency domain comparison of the spherical void
Figure 38: BEM-FFT/LAP condition number comparison for spherical void
Figure 39: 18 node/8 quadratic triangular element sphere model comparison
Figure 40: 164 node/54 quadratic quadrilateral element sphere model comparison
Figure 41: BEM-FFT/LAP time domain comparison of the spherical void
Figure 42: Spectrum of the incident wave for the tin-lead spherical inclusion
Figure 43: BEM-FFT time domain comparison of the tin-lead sphere
Figure 44: BEM-FFT frequency domain comparison of the tin-lead sphere
Figure 45: BEM-FFT condition number comparison for the tin-lead sphere
Figure 46: Damping/no damping comparison
Figure 47: BEM-FFT 164 node/54 quadratic quadrilateral element model comparison
Figure 48: BEM-FFT frequency domain comparison for the 164 node/54 element model
Figure 49: BEM-LAP comparison for the tin-lead sphere
Figure 50: Spectrum of the incident wave for the polystyrene sphere
Figure 51: BEM-FFT time domain comparison of the polystyrene sphere .................. 101
Figure 52: BEM-FFT frequency domain comparison of the polystyrene sphere ............. 102
Figure 53: BEM-FFT 164 node/54 quadratic quadrilateral model comparison .............. 103
Figure 54: BEM-LAP comparison for the polystyrene sphere .................................. 104
Figure 55: BEM-FFT condition number comparison of the polystyrene sphere .......... 105
vii

LIST OF TABLES

Table 1: Material constants.................................................................59
Table 2: BEM total computation time comparison ..................................79
# NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^i$</td>
<td>incident wave displacement in vector notation</td>
</tr>
<tr>
<td>$u^s$</td>
<td>scattered wave in vector notation</td>
</tr>
<tr>
<td>$u$</td>
<td>total wave</td>
</tr>
<tr>
<td>$c_L$</td>
<td>longitudinal wave speed</td>
</tr>
<tr>
<td>$c_T$</td>
<td>shear wave speed</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>Lame’ constant for scattering medium</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>Lame’ constant for inclusion</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>Lame’ constant for scattering medium</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>Lame’ constant for inclusion</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>mass per unit volume of scattering medium</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>mass per unit volume of inclusion</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>$D$</td>
<td>domain of exterior elastic region</td>
</tr>
<tr>
<td>$S$</td>
<td>boundary of scatterer</td>
</tr>
<tr>
<td>$S_R$</td>
<td>boundary of sphere of radius $R$</td>
</tr>
<tr>
<td>$\sigma_{ij}$</td>
<td>stress tensor</td>
</tr>
<tr>
<td>$t$</td>
<td>traction</td>
</tr>
<tr>
<td>$F_{ij}^D$</td>
<td>elastodynamic fundamental solution</td>
</tr>
<tr>
<td>$G_{ij}^D$</td>
<td>normal derivative of the fundamental solution</td>
</tr>
<tr>
<td>$G_{ij}^S$</td>
<td>Stoke’s stress tensor</td>
</tr>
</tbody>
</table>
CHAPTER 1. INTRODUCTION

Introduction

The history of the study of vibration and wave phenomena dates to the sixth century B.C. when Pythagoras studied the origin of musical sounds and the vibration of strings [1]. Predating this to prehistoric times, early man placed careful blows to the edge of a flint to shape his stone tools. The fragments of rock were broken off in specific patterns corresponding to the resulting stress waves caused by the blows. The historical list of uses of vibration could continue almost to infinitum but most of these early studies were more of an observational nature and were concerned with musical tones or water waves. As in many of the other branches of science, major advances in the study of wave phenomenon have been preceded by empirical observation. The majority of the quantitative analysis did not begin until the advent of the mathematical and physical tools that were necessary to properly model such physical phenomena.

Motivation for the current level of interest in wave phenomena are the many practical applications in science and industry. In the area of structures, the interest is mainly in the response to impact or blast loads. The study of seismic phenomenon has produced an impressive knowledge about the interior of the earth. For example, waves artificially generated in the earth by a blast make it possible to locate possible oil and gas-bearing deposits by measuring the scattering of the blast wave by the underground discontinuities.

The most familiar acoustic phenomenon is that associated with the sensation and analysis of sound [2]. Many situations involving acoustics are as much psychological as physical. Music, speech, and noise, more often than not, are produced and controlled for the pleasure or displeasure of individuals. Hence, an important part of the understanding of the needs of an acoustic system is a knowledge of the way people react when subjected to the
stimulus of a particular sound. The defense industry has faced the difficulty of minimizing the acoustic output generated by submarine propellers through cavitation (partial vacuums) as well as increasing the sensitivity of their own instruments in detecting and analyzing the emissions generated by other vessels.

The field of ultrasonics represents another major area of application of wave phenomena. The general idea of this testing technique is to introduce a very low energy-level, high-frequency stress pulse into a material and then observe the subsequent propagation and reflection of this energy. In the medical community, this process has proved invaluable in obstetrics, where, through an imaging process, the health of a fetus can be easily be visually studied without any risks to the mother or baby. Another biomedical application involves focusing the ultrasonic signal into damaged connective and muscle tissue. This increases the blood flow into the area, decreasing the recovery time from the injury. In the field of nondestructive evaluation and elastodynamics, stress or displacement pulses are directed into an elastic material for the purpose of observing the reflected wave in the detection of flaws in the material.

Applications abound in the study of acoustics and noise control, acoustic emission, electromagnetic phenomenon, ultrasonics and elastodynamics. A unique feature of the study of propagation of disturbances in a medium is that many common features are shared. From a mathematical modeling perspective, one develops the governing partial differential equation [1,3,4] which describes the motion of the waves in the medium, either scalar or vector valued. A driving mechanism in the form of an incident wave (for the scattering problem) is prescribed, boundary conditions and initial conditions in the medium are imposed, and at remote distances, one utilizes the Sommerfeld radiation condition [5]. The similarities shared by scattering and wave propagation problems from the various areas allow a person versed in one area to understand much about the other areas. However, there are sufficient differences that make a completely general development of the subject impractical.
The Scattering Problem

In an unbounded homogeneous medium, waves propagate along a fixed path at a constant speed without interruption [3]. When an object is inserted into the medium, the path of the propagating wave is altered, and the obstacle, when excited by the otherwise undisturbed incident wave, acts as a secondary source, radially emitting waves outward from itself. Diffraction is the common term used for the deviation of the incident wave from its original path. Scattering is the radiation of secondary waves emitted from the obstacle. Such problems have been of long standing interest in acoustics, ultrasonics, and electro-magnetic theory. Other problems of interest in these fields include sonar detection, architectural acoustic design, radar applications, and antenna design. A more contemporary interest is that of elastodynamic scattering with applications in ultrasonic testing, dynamic stress concentrations, and blast effects on buried structures or anomalies.

This dissertation concentrates on the study of ultrasonics, specifically elastodynamic wave scattering. In general, a very low energy-level, high frequency stress or displacement pulse is introduced into a material and observations are made on the subsequent propagation and reflection of the pulse. In the realm of nondestructive evaluation (NDE), on-line monitoring of the integrity of structural elements is performed for the purpose of identifying and discriminating between inhomogeneities like cracks, pores, and inclusions. Ultrasonic transducers consisting of a piezoelectric ceramic element or crystal are the most common means of generating and receiving an ultrasonic pulse [1,7]. The crystal, when excited by an electric pulse, generates a mechanical pulse which can then be directed through mechanical contact with an elastic medium. The subsequent transmission of this pulse through the medium and the observations made of the scattering of the pulse can lead to the detection of undesirable cracks, impurities and various other flaws in the medium.
A schematic diagram of an ultrasonic transducer is illustrated in Figure 1. To keep the pulse duration short, which provides a better spatial resolution in detecting flaws, a backing material of epoxy loaded with tungsten powder is often used. The front surface of the element consists of a wear plate for protection against rubbing for contact transducers while immersion type transducers often use a quarter wave plate to improve the energy transfer from the solid piezoelectric element to the liquid medium.

In ultrasonic nondestructive testing, the actual time domain signature of the scattered response of the pulsed input is the most common observable (via oscilloscope) quantity. Thus, it is desirable to generate numerically, through mathematical modeling of the physical scattering problem, simulated solutions in the time domain. Figure 2 illustrates a common procedure performed in nondestructive evaluation - the utilization of a contact probe while Figure 3 is a sample of an oscilloscope observation of an inclusion imbedded in a material.
Figure 2: Flaw detection with a contact probe.

Figure 3: Oscilloscope observation.
Numerical Solutions

Since the advent of computers as a tool for solving problems, engineers and physical scientists have become very proficient with numerical techniques of analysis. These techniques are based on obtaining an approximate solution of an equation or set of equations describing a physical problem. The use of a truncated Taylor series in the form of finite differencing [8,9] was the first widely used method. This involved approximating the governing equations of the problem with local Taylor series expansions for the variables on a series of nodes with the region of interest.

Another approach in obtaining an approximate solution is to recast the governing differential or partial differential equation as the minimization of a functional. In this context, the finite element method (FEM) [9,10] has attracted the attention of analysts due to its property of dividing the domain into elements. One advantage of the finite-element method over finite-difference methods is the relative ease with which the boundary conditions of the problem are handled. Many physical problems have boundary conditions involving derivatives and, in general, the boundary of the region is irregularly shaped. Boundary conditions of this type are very difficult to handle using finite-difference techniques, since each boundary condition involving a derivative must be approximated by a difference quotient at each grid point near the boundary. Irregular boundary shapes increases the difficulty and makes grid point placement awkward. The finite-element method includes the boundary conditions as integrals in the functional being minimized, so the construction procedure is independent of the particular boundary conditions of the problem. Although the finite element method is a very powerful tool for solving difficult physical problems, it becomes cumbersome to develop the necessary grid for 3D problems. Furthermore, there is a difficulty in properly modeling the remote boundary for unbounded regions.
An integral equation method of solution, developed in the early 1960's, but often overshadowed by the finite-element method, is the boundary element method (BEM) [11,12,13,14,15]. The boundary element method utilizes Green's theorem or the vector equivalent, Betti's reciprocal work theorem, along with the appropriate fundamental solution to the original governing differential equation. The original governing equations are recast as singular integral equations over the boundary of the domain. Since the boundary integral equations represent an exact formulation of a problem, then by directly applying numerical quadrature techniques to the integrals, a highly accurate solution can be expected. The inherent advantage of the BEM is the reduction of dimensionality of the problem by one. For example, the analysis generates a one-dimensional boundary integral equation for two dimensional problems and for three-dimensional problems only two-dimensional surface boundary integral equations arise. This is of immense practical importance since in most cases it is possible to model the surface of the problem only.

In many physical problems, it is much more convenient to think of discretizing the boundary of the domain, which has known boundary conditions, than to attempt to discretize the entire domain which is required by the FEM. This is of profound importance when the domain of interest is unbounded.

The aforementioned "reduction of order," resulting from the boundary element method, leads to a much smaller system of simultaneous equations than any scheme of entire body discretization. Unfortunately, the system matrices generated by the BEM are fully populated for a homogeneous region and block banded when more than one region is involved. This contrasts with the much larger matrices generated by the FEM which are generally sparsely populated and banded. The clear advantages [16] of the BEM are:

a) reduction in dimensionality

b) boundary data is obtained directly
c) generation of an entire field solution is not required; one can focus on a specific area of interest.

d) good resolution of high gradient fields

e) unbounded regions are easily handled, assuming a radiation condition holds at infinitely remote distances.

Yet, there are some disadvantages of the BEM. These include:

a) limited to problems where a fundamental solution can be found

b) poor resolution of problems with a high surface area to volume ratio (cracks)

c) not readily adaptable to inhomogeneous problems.

There appear to be very few problems solvable by finite element methods which cannot be solved at least as efficiently by the BEM. An example where this would not be the case would be a problem in which the properties are nonhomogeneous. Then for the BEM, a large number of regions, each with constant properties, would be required for a good model. This scheme eventually begins to degenerate into an essentially full region subdivision, virtually indistinguishable from a finite element scheme.

From various studies, it has been concluded that comparable solution times between finite element and the boundary element method on three dimensional problems solved with similar precision generally show a time advantage of from four to ten in favor of the BEM. This difference is especially noticeable in certain classes of problems which are particularly amenable [14] to the BEM, one of which is problems on unbounded regions. The BEM solution scheme automatically satisfies admissible boundary conditions at infinity since the fundamental singular solutions used in the construction of the boundary integral equations obey the Sommerfeld radiation condition a priori. Hence, no subdivisions of these boundaries are necessary. Whereas with the finite element method, infinite boundaries have to be approximated by an appreciable number of distant elements or a fictitious boundary with boundary conditions constructed so as to have negligible effect on the local points of interest.
Since the scattering problem considered in this dissertation consists precisely of an unbounded homogeneous region with a finite number internal scattering obstacles, the choice of the boundary element method as a solution strategy is a natural one.

**The Transient Problem**

Many problems in engineering require the solutions of partial differential equations where the field variable is both time and space dependent. Of specific interest in this dissertation is the transient scattering problem. Specifically, the determination of the behavior of the field variable at a location in space with respect to time. In general, scattering problems of acoustics or elastodynamics are often formulated in the frequency domain through Fourier transforms with the studies made on generally unrealistic incident pulses such as the delta function, Heaviside step function or a continuous sinusoid. Furthermore, the inversion back to the time domain is seldom done. The scope of this dissertation is to simulate a realistic transducer pulse in an elastodynamic medium and to calculate the resultant scattered wave in the actual time domain.

With the ever increasing capabilities of computational facilities for fast numerical computation, it is becoming even more possible to solve complicated problems with difficult geometries and of a transient nature with good results [17,18]. These transient solutions can be obtained by working directly in the time domain or by inversions of integral transform solutions in the frequency (Fourier) or Laplace domains.

One approach is to formulate the boundary integral equation directly in the time domain. Unfortunately, one must then store all past values [19,20] of the potential (displacement) and the normal derivative (stress or tractions) in evaluating the current time step values. This can present a significant computational and storage problem. Furthermore, direct time solutions are suitable for short time intervals and generally deteriorate for longer times.
Another approach is to utilize either the Fourier or Laplace transform [17,18,21]. The transform techniques generally require solutions over a wide spectrum in the transform variable to provide accurate inversions back to the time domain. However, without special provisions, numerical implementations like boundary or finite elements are limited to lower frequencies or Laplace parameter values. For exterior problems, which are of primary interest herein, the boundary integral equation unfortunately breaks down and admits arbitrary solutions [22] at certain frequencies. However, the significance of this problem is diminished when considering the natural damping an elastic material has with respect to an ultrasonic signal or vibration. Therefore, ways to minimize the number of transform variable solutions for canonical problems are important as well as a proper incorporation of the damping term.

Comparing the efficiency of the Fourier and Laplace transform methods enables investigation into ways of minimizing the number of transformed variable calculations in the boundary element setting. To alleviate the problem of the spurious eigenfrequencies, which result in uncharacteristically large matrix condition numbers at those frequencies, a small complex component is added to the Fourier parameter to dampen the effect.

In elastodynamic wave scattering and its application to ultrasonics and NDE, the detection of flaws in material components is of paramount importance. The amplitude of the scattered wave is a function of the geometry, orientation, and size of the flaw. An amplitude of the scattered wave above a detection threshold is necessary for positive identification of an internal flaw. The question: by choosing as few points as possible in the frequency space, can the scattered wave characteristics be maintained? This is the fundamental question to be answered in this dissertation.
SCATTERING BY AN OBSTACLE IN AN ELASTIC MEDIUM

Introduction

This chapter develops the mathematical model necessary to solve three-dimensional scattering problems of an elastic wave by using the boundary element method. The governing differential equations of elastodynamics are the departure point. The total elastic displacement field illustrated in Figure 4 is represented by the sum of the incident field, as if the scatterer were absent, and a perturbation, or scattered field. Incident fields and their representations are discussed and the boundary integral formulations are given in both the Fourier and Laplace domains. Then a general numerical solution method of the BIE's are outlined. The two chapters that follow will formally develop the transform tools necessary to obtain time domain farfield responses through suitable transform techniques.

Figure 4: Scattering of an elastic wave by a flaw.
In the general problem illustrated, the exterior region $D$ is a homogeneous, isotropic, linearly elastic solid [23] of infinite extent in a three-dimensional space with global coordinates $x_i$. The flaw $F$ (void, inclusion or crack) assumably occupies the interior of domain $D$ with bounding surface $S$. The inward and outward normals to the surface $S$ of the flaw are denoted by $n$ and $v$ respectively.

When the flaw $F$ is excited by the incident wave $u^i$, a scattered wave $u^s$ is generated by the interaction of the incident wave with the flaw. The scattered field is defined such that it negates the incident field in $F$ since the incident field penetrates all space. As is common in scattering problems, the total wave field $u$ in $D$ taken to be a superposition of the incident wave and the scattered wave, i.e.,

$$u = u^i + u^s \tag{2.1}$$

**Governing Equations**

In a homogeneous, isotropic linearly elastic medium, the displacement equation of motion are given by (in the Einstein summation convention)

$$(\lambda+\mu)u_{ij,j} + \mu u_{ij,ij} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \ i=1,2,3 \tag{2.2}$$

where $\lambda$ and $\mu$ are the Lamé' constants for the medium, $\rho$ is the medium mass per unit volume, $f_i$ is the body force components and $u_i$ are the displacement components. Equations (2.2) are commonly known as the Navier-Cauchy equations of elasticity.

Now, the internal stresses are related to the displacements by the kinematical relationship and the constitutive law, respectively, i.e.,
\[ \varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji}) \]  
\[ \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \]  
(2.3)  
(2.4)

Combining (2.3) and (2.4) gives
\[ \sigma_{ij} = \lambda u_{kk} \delta_{ij} + \mu (u_{ij} + u_{ji}) \]  
(2.5)

where \( \delta_{ij} \) is the Kronecker delta defined such that \( \delta_{ij} = 1 \) if \( i=j \) and \( \delta_{ij} = 0 \) if \( i\neq j \).

For an elastic body occupying the volume \( V \) and bounded by the surface \( S = S_1 \cup S_2 \) and body forces \( f_i \) in \( V \), the displacements \( u_i \) are determined through (2.2) by imposing surface traction \( t_i \) boundary conditions on \( S_1 \) and displacement boundary conditions \( u_i \) on \( S_2 \), i.e.,
\[ t_i = \sigma_{ij} n_j, \quad x \in S_1 \]  
(2.6a)
\[ u_i = U_i, \quad x \in S_2 \]  
(2.6b)

With the explicit presence of time in (2.2), initial conditions at an initial time, say \( t = 0 \), must also be prescribed. These initial values are denoted by
\[ u_i(x,0) = u_i^0, \quad x \in V \]  
(2.7a)
\[ \frac{\partial u_i}{\partial t}(x,0) = \dot{u}_i^0, \quad x \in V \]  
(2.7b)

Then, equations (2.2), along with (2.5), (2.6), and (2.7), constitute the mathematical formulation of the initial boundary value problem (IBVP) governing dynamic problems of elasticity. It is assumed here that \( u_i \) and its derivatives up to second order are continuous except at propagating wave fronts, where only \( u_i \) is continuous. At a wave front propagating with velocity \( c \), the kinematic jump condition
\[ [u_i] = -cn_j [u_{ij}] \]  
(2.8)

and the momentum condition
\[ [t_i] = -c \partial [\dddot{u}_{i,j}] \]  \hspace{1cm} (2.9)

must be satisfied \cite{24} where the brackets denote the "jump" in the enclosed quantity. Since a Gaussian incident wave will ultimately be used, the right hand sides of both (2.8) and (2.9) are zero since the Gaussian and all its derivatives are continuously differentiable across the wave front.

Before proceeding to the boundary integral formulation and the numerical aspects of obtaining a solution to the elastodynamic equations, it is important to address a theoretical aspect. A fundamental question pertaining to the governing equations as well as certain boundary conditions and initial conditions is whether there is a unique solution.

**Theorem 2.1:**

The initial boundary value problem specified by equations (2.2) has a unique solution provided that at all points on the surface \( S \), any one member of the three products

\[ \sigma_{nn}u_n, \sigma_{ns}u_s, \sigma_{nb}u_b \]  \hspace{1cm} (2.10)

are specified as well as the initial values of these quantities, where \( n, s, \) and \( b \) indicate a mutually orthogonal set of directions with \( n \) as the unit outward normal to the surface. Also, at each point in \( D \), \( f_i \) as well as the initial values of \( u_i \) and \( \dddot{u}_i \), must also be specified.

The proof of Theorem 2.1 is given in Appendix A.

Returning to the model formulation, if body forces are not present, the elastodynamic field equation reduces to
Through the Helmholz decomposition, this vector field can be reduced (or decoupled) as a sum of a scalar field gradient and a vector field curl, i.e.,

$$u = \nabla \phi + \nabla \times \psi, \nabla \cdot \psi = 0$$ (2.12)

where $\phi$ and $\psi$ are called the scalar and vector displacement potentials. Substitution of (2.12) into (2.11) gives the decoupled system.

$$\nabla^2 \phi = \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2}, \quad c_L^2 = \frac{\lambda + 2\mu}{\rho}$$ (2.13)

$$\nabla^2 \psi = \frac{1}{c_T^2} \frac{\partial^2 \psi}{\partial t^2}, \quad c_T^2 = \frac{\mu}{\rho}$$ (2.14)

The Helmholz decomposition is not formally required in the subsequent boundary element formulation, but does illustrate the two types of wave types present in an elastic medium as well as the simple and recognizable equations that can be obtained from Navier’s equation. The above decomposition will be used in the determination of the analytical solution for the scattering from spherical obstacles.

Solutions for the displacements can be constructed from (2.13) and (2.14) along with the appropriate initial and boundary conditions. From the decomposition we are assured that there are only two types of waves in an elastic solid, one with wave speed $c_L$ determined by $\phi$ and the other with wave speed $c_T$ determined by $\psi$. In this context, $c_L > c_T$, and accordingly, the wave associated with $c_L$ is called the primary wave or more commonly the longitudinal wave, and that associated with $c_T$ is called the secondary wave or physically , the shear wave.
Incident Waves

In general, a plane wave [3] emanating from an infinitely remote position in an elastic solid takes the form

\[ u^i = \mathbf{d} F \left( \frac{-\mathbf{p} \cdot \mathbf{e}_i}{c} - t \right) \]  

(2.16)

where

\[ \mathbf{d} = A_i \mathbf{e}_i \]  

(2.17)

is the displacement direction with \( A_i \) being the associated amplitude in the direction \( \mathbf{e}_i \). Also, in equations (2.16) and (2.17), \( F \) is an arbitrary function; \( \mathbf{p} \) is the direction of propagation; \( \mathbf{e}_i \) corresponds to the orthonormal basis vectors; and \( c \) is the wave speed.

Upon substitution of (2.16) into (2.11), one notes that not every \( c \) and vector \( \mathbf{d} \) are feasible solutions. Since

\[ (\lambda + \mu) \mathbf{p} (\mathbf{d} \cdot \mathbf{p}) + \mu \mathbf{d} - \rho c^2 \mathbf{d} = 0 \]  

(2.18)

is satisfied only if \( \mathbf{p} = \frac{\mathbf{d}}{|\mathbf{d}|} \) or \( \mathbf{p} \cdot \frac{\mathbf{d}}{|\mathbf{d}|} = 1 \). Then

\[ c_L^2 = \frac{\lambda + 2\mu}{\rho} \]  

(2.19)

which indicates a longitudinal wave. Another possible solution is obtained if \( \mathbf{p} \neq \frac{\mathbf{d}}{|\mathbf{d}|} \) or \( \mathbf{p} \cdot \frac{\mathbf{d}}{|\mathbf{d}|} = 0 \). Then

\[ c_T^2 = \frac{\mu}{\rho} \]  

(2.20)

which indicates a transverse or shear wave.

When flaws are small relative to an ultrasonic transducer diameter, the quasi-plane wave assumption is appropriate for the incident wave. When such an incident wave impinges
upon a flaw, its wave front can be approximated by a plane wave. For large flaws, a more sophisticated modeling of the transducer pulse may be needed.

To simulate an actual transducer pulse, a fifth derivative of the Gaussian will be chosen as a generating function. This choice is appropriate when comparisons are made to the actual transducer signal as seen in Figure 30 on page 74. Specifically, the plane wave model will be of the form

\[ u^l = dF \left( t \cdot \frac{x^l + t_0}{c_L} \right) \] (2.21)

where \( x \) is the field point; \( p \) is the direction of propagation; \( d \) is the displacement direction; and \( c_L \) is the longitudinal wave speed. The starting point of the wave at \( t = 0 \) is represented by \( t_0 > a \) where \( a \) is the radius of the scatterer. The Gaussian incident pulse \( F \), is given by

\[ F(t,x) = \frac{8A}{D^3} \left( -4 \frac{(t - B)^5}{D^2} + 20 \frac{(t - B)^3}{D} - 15 (t - B) \right) e^{-(t-B)yD} \] (2.22)

where \( D \) is the center frequency of the incident wave and
\[ B = \frac{x^l + t_0}{c} \] (2.23)

Figure 5 illustrates the behavior of the function chosen to simulate the pulse generated by a transducer.

Differentiation of expression (2.22) for the purpose of locating the global maximum yields the location \( y_1 = 0.19016351 \). Substituting this result into equation (2.22) and equating the equation to unity normalizes the amplitude \( A \) so that it has an easily specified magnitude \( \tilde{A} \). One then obtains

\[ A = \frac{\tilde{A}}{\sqrt{y_1D} e^{y_1} \left[ -4y_1^2 + 20y_1 - 15 \right]} \] (2.24)

where \( \tilde{A} \) is now an easily specified driving amplitude of the incident wave. Figure 5 illustrates the model of the transducer pulse given in equations (2.22), (2.23), and (2.24).
Boundary Integral Formulation and Regularization

The starting point of the BIE formulation is the conversion of the differential equation along with its boundary conditions into an integral representation involving the field variable and the fundamental solution or free space Green's function. For the vector values governing equations of elastodynamics, this is done using Betti's reciprocal theorem. Before doing this, the governing differential equations will be first transformed into the appropriate Fourier or Laplace domain. The analysis is then greatly simplified since the time variable is thereby eliminated from the governing differential equations and the initial-boundary-value problem reduces to a boundary value problem only. Once the solution is obtained in the respective transform space, a suitable inverse transform or its numerical equivalent is utilized to transform the solution back into the time domain.
The exponential Fourier transformation is defined as
\[ \hat{f}(x, \omega) = \int_{-\infty}^{\infty} f(x, t) e^{i\omega t} \, dt \] (2.25)
where \( \omega \) is the circular frequency. Furthermore, the Laplace transformation with respect to time of a function \( f(x, t) \) is defined as
\[ \tilde{f}(x, s) = \int_{0}^{\infty} f(x, t) e^{-st} \, dt \] (2.26)
provided \( f \) is piecewise continuous in time and bounded at infinity. Also, the real part of the complex transform parameter \( s \) must be greater than some positive constant \( b \). Since the initial conditions are zero, Eringen and Suhubi [25] showed that the formulation of the problem in the Fourier domain is identical to formulation in the Laplace domain when \( s \) is replaced by \(-i\omega\).

In the Fourier domain the governing differential equation becomes, in absence of body forces,
\[ \begin{align*}
(\lambda+\mu)\ddot{u}_{i,ij} + \mu \dddot{u}_{j,ii} &= -\rho \omega^2 \ddot{u}_j \\
\end{align*} \] (2.27)
In the Laplace domain, it is represented by
\[ \begin{align*}
(\lambda+\mu)\ddot{u}_{i,ij} + \mu \dddot{u}_{j,ii} &= \rho s^2 \dddot{u}_j \\
\end{align*} \] (2.28)

It was earlier stated that the total field \( u_i \) can be expressed as a superposition of the incident field \( u^I_i \) and the scattered field \( u^S_i \). Originally posed as an initial value problem, \( u_i \) must satisfy inhomogeneous boundary data (in general) and homogeneous initial data. The displacements, \( u_i \), are prescribed for all time and all space in the absence of the scatterer. Here, the scattered field carries the burden of negating the incident field in regions where it is blocked by the scatterer. Without loss of generality, the transform notation for \( \tilde{u} \) or \( \hat{u} \) will be subsequently dropped with the assumption that all variables (or quantities) are in the respective transform domain. The BVP for the scattered field is then posed as
In a scalar field setting, one would obtain the boundary integral equation by the use of Green's Theorem to effectively integrate by parts the governing differential equation multiplied through by a suitable continuously differentiable function. In elastostatics or elastodynamics, Betti's reciprocal work theorem \cite{26,27,28} is the most convenient way of obtaining the direct formulation of the boundary integral equation.

**Theorem 2.2 (Betti's Reciprocal Work Theorem):**

Let $(V, t', u')$ and $(V, t, u)$ be two distinct elastic equilibrium states existing in a region $D$ bounded by a surface $S$ where $u$, $t$, and $\Psi$ denote the displacements, tractions, and body forces respectively. Then the work done by the forces of the first system on the displacements of the second is equal to the work done by the forces of the second system on the displacements of the first.

Hence, if $x$ is a point on $S$, and $\xi$ is a point in $D$,

$$
\int_{S} t_{i}(x, \omega) \ u_{i}^{*}(x, \omega) \ dS(x) + \int_{V} \Psi_{i}(x, \omega) \ u_{i}^{*}(x, \omega) \ dV(\xi)
$$

$$
= \int_{S} t_{i}^{*}(x, \omega) \ u_{i}(x, \omega) \ dS(x) + \int_{V} \Psi_{i}^{*}(x, \omega) \ u_{i}(x, \omega) \ dV(\xi)
$$

(2.30)

To utilize the above integral formulation, the ()* in equation (2.30) state is chosen to correspond to a unit force system in an infinite solid. More precisely, let $u_{i}^{*} = g_{ij}^{D}$ be the fundamental singular solution or free space Green's function \cite{11,29}. The fundamental solution for the tractions $t_{i}^{*}$ is given by

$$
G_{ij}^{D} = (\lambda F_{mk,m}^{D} + \mu (F_{ik,j}^{D} + F_{jk,i}^{D})) n_{j}
$$

(2.31)
where $n$ is the outward pointing unit normal from the surface.

The fundamental solution, $F_{ji}$, is a solution to the transformed differential operator (2.2) with the body force term $f_i = \Psi_i^*$ given by the Dirac delta function, i.e.,

\[
\begin{align*}
(\lambda + \mu)F_{i,ij}^{D} + \mu F_{i,ii}^{D} + \rho \omega^2 F_j^{D} &= \delta(x-\xi) \\
(\lambda + \mu)F_{i,ij}^{D} + \mu F_{i,ii}^{D} - \rho s^2 F_j^{D} &= \delta(x-\xi)
\end{align*}
\]  

The fundamental solution $F_{ij}^{D}$ and its "normal derivative" $G_{ij}^{D}$ are given in Appendix B and C.

In a physical sense, $F_{ij}^{D}(x,\xi)$ is characterized by a displacement component $i$ at a receiving point $x$ due to a concentrated pulse at a source point $\xi$ in a direction $j$ in the unbounded three-dimensional elastic solid.

An important characteristic of the Dirac delta function is its sifting property. In the mathematical theory of distributions, this property is given symbolically by

\[
u_i(\xi, \omega) = \int V \delta(x-\xi) \ u_i(x,\omega) \ dV(\xi)
\]

(2.33)

With body forces neglected in equation (2.22), one obtains, after substitution of (2.38) along with $F$ and $G$ into Betti's reciprocal theorem, the representation integral for the elastodynamic PDE, valid in the exterior region $D$. Thus,

\[
u_i^*(\xi) = \int_S \left[ F_{ij}^{D}(x,\xi) t_i^j(x) - G_{ij}^{D}(x,\xi) u_j^i(x) \right] \ dS
\]

(2.34)

where $x$ is on the boundary and $\xi$ is called the field point. The point $x$ is referred to as the source point and $r(x,\xi)$ is the distance vector between the field point and source point. Also, since the problem under consideration is an exterior problem, the boundary of the elastic region has been split into two parts, the boundary of the scatterer $S$ and the infinite boundary $S_\infty$. 

Since the domain in which the region defined is unbounded, certain physical considerations place restrictions on the behavior of the field quantities at infinity. Since the sources that cause the motion (the incident wave) are confined to the interior of D, then it is physically reasonable to postulate that there is no wave propagation toward the interior of the region from infinity. Therefore, the field quantities must behave at infinity in such a way that the infinitely remote surface integral, $S_\infty$, defined in equation (2.34) vanishes. Such a hypothesis, carefully derived by Eringen and Suhubi [25], is concerned with the behavior of the field variables at an infinitely remote distant surface and is referred to as regularity conditions or more specifically, the *Sommerfeld radiation condition*. This condition is now imposed above to ensure that $u^S$ exhibits the appropriate properties as well as decays in amplitude appropriately as $R \to \infty$. This will force the argument of the integral of equation (2.34) to zero. With this hypothesis in mind, let $R$ be the radius of sphere of the surface $S_R$ and centered at $\xi$, which encloses the cavity of the external problem depicted in Figure 6.

![Figure 6: Infinite region with a flaw.](image-url)
If the limiting case \( R \to \infty \) is considered, equation (2.34) can be expressed in terms of boundary integrals over \( S \) alone provided

\[
\int_{S} \left[ F_{ij}^{P}(x,\xi)t_{j}^{f}(x) - G_{ij}^{P}(x,\xi)u_{j}^{f}(x) \right] dS \to 0 \text{ as } R \to \infty \tag{2.35}
\]

For three-dimensional problems, one has for \( x \in S \),

\[
dS(x) = |J| \, d\varphi \, d\theta \text{ with } |J| = O\left( \frac{R^2}{R} \right) \tag{2.36}
\]

\[
F_{ij}^{P}(x,\xi) = O\left( \frac{1}{R} \right) \tag{2.36a}
\]

\[
G_{ij}^{P}(x,\xi) = O\left( \frac{1}{R^2} \right) \tag{2.36b}
\]

where \( O(\, ) \) represents the order of convergence as \( R \to \infty \).

Therefore, if \( u \) has the behavior \( O(R^{-1}) \) and \( t \) is \( O(R^{-2}) \) as \( R \to \infty \), the radiation conditions (2.35) are satisfied. In a physical sense, this means that the scattered wave must decay appropriately as it propagates away from the scattering source. Hence the equation (2.34) becomes

\[
u_{j}^{f}(\xi) = \int_{S} \left[ F_{ij}^{P}(x,\xi)t_{j}^{f}(x) - G_{ij}^{P}(x,\xi)u_{j}^{f}(x) \right] dS \tag{2.37}
\]

The incident wave must satisfy

\[
(\lambda + \mu)u_{i,j}^{j} + \mu u_{j,ii}^{j} + \rho \omega^{2}u_{j}^{j} = 0 \tag{2.38a}
\]

\[
(\lambda + \mu)u_{i,jj}^{j} + \mu u_{j,jii}^{j} - \rho s^{2}u_{j}^{j} = 0 \tag{2.38b}
\]

for all \( x \) in the respective domains. Invoking a radiation condition is unwarranted in the integral formulation for \( u^{1} \). Hence its domain is \( F \), the region occupied by the scatter, with its representation integral given by
\[ u_i(\xi) = \int_s \left[ F^D_{ij}(x,\xi) t_j(x) - G^D_{ij}(x,\xi) u_j(x) \right] dS \]  

(2.39)

Since the region of interest is \( \Omega \) and noting the superpositional nature of the incident and scattered wave, the general representation integral is given by

\[ u_i(\xi) = u_i(\xi) + \int_s \left[ F^D_{ij}(x,\xi) t_j(x) - G^D_{ij}(x,\xi) u_j(x) \right] dS \]  

(2.40)

The boundary integral equation is obtained by taking the field point \( \xi \) to the boundary. The fundamental solution is weakly singular of \( O(1/r) \) as \( r \to 0 \) but presents no problem in the integration process. During the process of formal integration, the integrations are performed in a polar coordinate system on a planar surface element. The product of the fundamental solution with \( dS \) becomes regular and is of the form \( O(1) \). However, \( G^D_{ij} \) is singular in the sense that it is \( O(1/r^2) \) as \( r \to 0 \). The product of \( G^D_{ij} \) with \( dS \) is \( O(1/r) \) so a regularization \([30]\) process must be undertaken to alleviate this problem.

The static fundamental solution, in particular the Stoke's stress tensor \( G^S_{ij} \) from the elastostatic case (given in Appendix C and D), will be utilized to remove the singularity. So reformulating (2.48) gives the equation

\[ u_i(\xi) = u_i(\xi) + \int_s F^D_{ij}(x,\xi) t_j(x) dS - \int_s \left[ G^D_{ij}(x,\xi) \cdot G^S_{ij}(x,\xi) \right] u_j(x) dS - \int_s G^S_{ij}(x,\xi) \left[ u_j(x) - u_j(\xi) \right] dS - \int_s G^S_{ij}(x,\xi) dS u_j(\xi) \]  

(2.41)

By the distributional properties of the static fundamental solution, the last term becomes

\[ \int_s G^S_{ij}(x,\xi) dS u_j(\xi) = u_i(\xi) \]  

(2.42)

Equation (2.41) can then be rewritten as
Now take $\xi \to \xi_0 \in S_0$ and divide the surface into a singular part and a non-singular part, with $S = \hat{S} + S_0$ as indicated in Figure 7. Typically, $S_0$ is a boundary element, with one of its nodes being a collocation point. This gives a regularized form of the BIE

$$u_j(\xi) + \int_S F^D_{ij}(x,\xi) t_j(x) \, dS = \int_S \left[ G^D_{ij}(x,\xi) - G^S_{ij}(x,\xi) \right] u_j(x) \, dS - \int_{S_0} G^S_{ij}(x,\xi) \left[ u_j(x) - u_j(\xi) \right] dS$$

The weakly singular integrals $\int_{S_0} \left[ G^D_{ij}(x,\xi) - G^S_{ij}(x,\xi) \right] u_j(x) \, dS$ and $\int_{S_0} G^S_{ij}(x,\xi) \left[ u_j(x) - u_j(\xi) \right] dS$ can be numerically evaluated by the polar coordinate transformation method implemented by Rizzo, Shippy and Rezayat [24].

![Figure 7: Local surface region about a boundary point.](image)
Once the unknown boundary values have been determined by solving the boundary integral equation, the scattered wave at an arbitrary point in the exterior region $D$ can be obtained by substituting the boundary values into representation equation (2.40). This advantage of the boundary element method is of immense practical importance since only the solution at selected farfield points of interest need to be determined.

**Numerical Evaluation of the Boundary Integral Equation**

Closed form solutions of the boundary integral equation are only attainable for simple geometry and boundary conditions [11,12,13]. Evaluation of the boundary integral equation can be done numerically by dividing the boundary of the scatterer into surface elements and approximating the field variables within each element by suitable shape functions. Hence the name given to this method of approach, the Boundary Element Method.

The procedure involves discretizing the boundary $S$ into a series of $N$ isoparametric elements over which the displacements and tractions are chosen to be piecewise interpolated between the element nodes. The boundary integral equation is applied in its discretized form to each nodal point $\xi$ of the boundary $S$ and the integrals are evaluated by a numerical quadrature scheme over each boundary element. The result is a system of $N$ linear algebraic equations involving the set of $N$ displacements and tractions. Boundary conditions are then imposed with $N$ nodal values of traction or displacement being specified. The resulting linear system, which is fully populated, can then be solved to obtain the remaining boundary data. Displacements at farfield points in $B$ can be obtained by substituting the resulting boundary solutions into the discretized form of the representation integral (2.40).

In the numerical scheme developed, a modular [31] approach is taken in the programming style. This makes it suitable for easy adaptability for a numerous array of problems (scalar or vector), boundary conditions, boundary elements, quadrature order and driving forces (incident wave or radiation). The boundary may be discretized into triangular or
quadrilateral elements. The triangular elements may be a 3 node triangle with linear shape functions or a 6 node triangle with quadratic shape functions. The quadrilateral may be a 4 node element with linear shape function, an 8 node element with quadratic shape functions, or a 9 node element with a Lagrangian shape function. In order to develop accurate BEM algorithms, it is essential that isoparametric representations of the geometry and the problem parameters based on polynomial shape functions be introduced. Although the ones described here are only up to quadratic in nature, we are by no means limited. Higher order shape functions may be introduced, increasing the accuracy of the approximations, but the tradeoff being significantly increased computation time.

Figures 8 and 9 illustrate a nodal configuration with a linear triangular element discretization on the surface of a sphere. Other element configurations consisting of quadratic triangular, linear quadrilateral, quadratic quadrilateral, and a 9 node Lagrangian quadrilateral quadric element along with node the numbering configurations are illustrated in Figure 10.

In general, the boundary S is divided into N elements. The Cartesian coordinates $x$ of a point on the surface of a boundary element is then given in terms of an element nodal coordinate system, $X_{\alpha i}$, defined locally over each element by the mapping

$$x_i = H_\alpha(\xi, \eta)X_{\alpha i} \quad (2.45)$$

where $i = 1, 2, 3$, $k = 1, 2$, and $\alpha = 1, 2, \ldots, M$ with $M$ being the number of nodes required for describing the element. In the above equation, $H_\alpha$ is the defined shape function in the local $\xi, \eta$ coordinate system as depicted in Figure 11.

For the three dimensional problems considered here, the shape functions for the three node triangular and six node triangular elements are given by
\[ H_1 = \xi \]
\[ H_2 = \eta \]
\[ H_3 = 1 - \xi - \eta \]

(2.46)

and

\[ H_1 = \xi (2\xi - 1) \]
\[ H_2 = \eta (2\eta - 1) \]
\[ H_3 = (1 - \xi - \eta)(1 - 2(\xi + \eta)) \]

(2.47)

In each case, \( \xi \) and \( \eta \) are the two independent elemental coordinates. The shape functions for the 4 node linear quadrilateral are given by

\[ H_\alpha = \frac{1}{4}(1 + \xi_0)(1 + \eta_0) \]
\[ \alpha = 1,2,3,4 \]

(2.48)

The 8 node quadratic quadrilateral shape functions are

\[ H_\alpha = \begin{cases} 
\frac{1}{4}(1 + \xi_0)(1 + \eta_0)(\xi_0 + \eta_0 - 1) & , \alpha = 1,2,3,4 \\
\frac{1}{2}(1 + \xi^2)(1 - \eta_0) & , \alpha = 6,8 \\
\frac{1}{2}(1 + \xi_0)(1 - \eta^2) & , \alpha = 5,7 
\end{cases} \]

(2.49)

where \( \xi_0 = \xi \xi_\alpha \) and \( \eta_0 = \eta \eta_\alpha \) with \( \xi \) and \( \eta \) being the two independent coordinates and \( (\xi_\alpha, \eta_\alpha) \) the coordinates of node \( \alpha \).

The transformation from the Cartesian coordinate system to an element intrinsic coordinate system is completed through the Jacobian matrix

\[ J_{ij} = \frac{\partial H_\alpha}{\partial x_i} X_{ix} \]

(2.50)
Figure 8: Nodal points on an octant of a sphere model.

Figure 9: Linear triangular element discretization.
Figure 10: Boundary elements and node numbering conventions.
Figure 11: Six node triangular and eight node quadrilateral elements.
The spacial variation of the field variables \( u_i(x) \) and \( t_i(x) \) are also expressed in terms of the same shape functions used for the geometry. With \( U_{i\alpha} \) and \( T_{i\alpha} \) being the nodal values of the displacements and tractions respectively in the local elemental coordinate system, one has

\[
\begin{align*}
  u_i &= H_{i\alpha} U_{i\alpha} \\
  t_i &= H_{i\alpha} T_{i\alpha} 
\end{align*}
\]  

(2.51)

One can also choose different shape functions for the field variables than were chosen for the geometry. Furthermore, one can use different shape functions for the displacements than those for the tractions. Certain characteristics in the solutions may be anticipated and a higher order approximation may seem appropriate to capture the required behavior.

Substituting the isoparametric representations for the geometry, equations (2.45) and (2.50), and the isoparametric representations of the field variables, equation (2.51), into the boundary integral equation gives the "almost" fully discretized equation

\[
\begin{align*}
  u_l(x) + \sum_{q=1}^{N} \int_{s_{q}} F_{ij}^{p}(x(\eta),\xi_{q}) H_{i\alpha}(\eta) \ dS(x(\eta)) T_{j\alpha} - \sum_{q=1}^{N} \int_{s_{q}} G_{ij}^{p}(x,\xi_{q}) \ dS_{j}(\xi_{q}) + \sum_{q=1}^{N} \int_{s_{q}} G_{ij}^{q}(x,\xi_{q}) H_{i\alpha}(\eta) \ dS(x(\eta)) U_{j\alpha} - \sum_{q=1}^{N} \int_{s_{q}} G_{ij}^{q}(x,\xi_{q}) H_{i\alpha}(\eta) U_{j\alpha} - u_j(x) dS_{q} 
\end{align*}
\]  

(2.60)

where \( S_q \) is the surface of the \( q \)th element and \( N \) is the total number of boundary elements needed to model the surfaces of the boundary. The integrals can now be integrated over each element using Gaussian quadrature. Some "bookkeeping" is needed to track the nodal values for \( U \) and \( T \) in the local elemental coordinate system relative to the global system values the displacements (\( u \)) and tractions (\( t \)).

Equation (2.50) becomes a finite linear system of complex valued algebraic equations. In matrix form, the algebraic equations are expressed as
Incorporation of the traction free boundary conditions for a void can now easily be imposed and the solution for the displacements obtained.

For the case of a multiregion problem, continuity of the displacements and tractions must be imposed at the boundary of the two regions. These boundary conditions are given by

\[
\begin{align*}
    u^1 &= u^2 \\
    t^1 &= -t^2
\end{align*}
\]  

(2.62)

where the superscripts 1 and 2 denote the exterior and interior regions, respectively. The solution on the boundary of the exterior region 1 with inward pointing normal is given by

\[
\left[ A^1 \right] u^1 + \left[ B^1 \right] t^1 = u^1
\]  

(2.63)

The solution on the boundary of the inclusion with outward pointing normal is

\[
\left[ A^2 \right] u^2 + \left[ B^2 \right] t^2 = 0
\]  

(2.64)

Rewriting equations (2.63) and (2.64) give the following:

\[
\begin{align*}
    \left[ B^1 \right]^{-1} \left[ A^1 \right] u^1 + t^1 &= \left[ B^1 \right]^{-1} u^1 \\
    \left[ B^2 \right]^{-1} \left[ A^2 \right] u^2 + t^2 &= 0
\end{align*}
\]  

(2.65)  

(2.66)

Adding equations (2.65) and (2.66) and applying the boundary conditions to (2.62) yield the system of algebraic equations for the displacements on the interface for the two scattering problems under consideration. For the inclusion, the displacements on the boundary are determined from the linear algebraic system of equations given by
\[
\left( [B^1]^{-1} [A^1] + [B^2]^{-1} [A^2] \right) u = [B^1]^{-1} u^i
\] (2.67)

Substitution of the resulting displacements into equations (2.65) or (2.66) give the tractions at
the interface.
FOURIER TRANSFORM INVERSION

Introduction

Most engineers and applied mathematicians are familiar with the idea of frequency or Fourier analysis by which a periodic function can be broken down into its harmonic components. They readily accept that a periodic function may be synthesized by adding together its harmonic components. For many years this fact was not fully agreed upon and it is said that the famous mathematicians, Euler, d'Alembert and Lagrange, held that an arbitrary function could not be represented by trigonometric series [32].

In actuality, the question of the existence of integral transforms may safely be ignored when the function to be transformed is an accurately specified description of a physical quantity. Physical possibility is a valid sufficient condition for the existence of a transform. Sometimes, however, it is convenient to substitute a simple mathematical expression for physical quantity. For example, it very common to consider the following wave forms:

\[ \sin(t) \text{ (harmonic wave, pure alternating current)}, \]
\[ H(t) \text{ (step function)} \]

Strictly speaking, neither of the above functions are physically possible and neither has a Fourier transform. In a physical sense, an impulse function, \( \delta(t) \), is often utilized but is characterized as having an infinitely large amplitude for an infinitely short time. A step function \( H(t) \) would have to be maintained steady for an infinite time, and a wave form, \( \sin(t) \), would have to have been switched on an infinite time ago. However, we can often achieve an approximation so close that any further improvement would be immaterial; and these simple expressions can be utilized.
In the context of this dissertation, the Fourier transform is used to temporally transform the governing partial differential equation, along with the driving incident wave, into a domain in which a solution is more plausible. Once solutions are obtained for a spectrum in the Fourier domain, the fast Fourier transform (FFT) will be utilized to numerically invert the solution into the time domain.

**The Fourier Transform**

The Fourier transform is actually a restatement of the Fourier integral formula [5,33], which, in complex exponential form, is given by

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega \tau} d\tau
\]

where equality holds almost "everywhere" [5] and is expected to satisfy the following criterion:

a) \( f(t) \) is piecewise smooth over \((-1,1)\) for every \( l \)

b) \( \int_{-\infty}^{\infty} |f(\tau)| d\tau < M \) for some finite constant \( M \)

Specifically, the Fourier transform pair is defined by

\[
\hat{f}(\omega) = c_1 \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt
\]

(3.2a)

\[
f(t) = c_2 \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega
\]

(3.2b)

where \( c_1 c_2 = 1/2\pi \), \( i = \sqrt{-1} \), \( t \) is time and \( \omega \) is the circular frequency. Calling \( \hat{f}(\omega) \) the Fourier transform of \( f(t) \), equation (3.2b) is the corresponding inversion formula since it recovers \( f(t) \) from \( f(\omega) \).
The Fast Fourier Transform

The fast Fourier transform (FFT) is a computational algorithm [34] for determining discrete Fourier transforms (DFT). The discrete Fourier transform pair of a finite sequence $f_k$, $n=0,1,2,...(N-1)$, is a new finite sequence $F_k$ defined as

$$F(\omega_k) = \frac{1}{N} \sum_{j=0}^{N-1} f(t_j) e^{i2\pi j k/N}, \quad k = 0,1,...(N-1)$$  \hspace{1cm} (3.3)

where

- $N$ = number of sample points
- $\Delta t$ = time subinterval
- $\Delta \omega$ = frequency subinterval
- $t = j \Delta t$
- $\omega = j \Delta \omega$.

Using the direct approach of calculating the values of $F$ or $f$, one would have to make $N$ multiplications of the form $f(t_j) = e^{i2\pi j k/N}$ for each of the $N$ values of $F$, and hence the total work of calculating the full sequence $F$ would require $N^2$ multiplications. Utilizing the FFT to perform the above calculations offers an enormous reduction in computer processing time with the added bonus of an increased accuracy. The number of operations is reduced to the order of $N \log(2N)$. Since fewer operations have to be performed computationally, round-off errors due to the truncations of products by the computer are reduced. The FFT, therefore, offers an enormous reduction in computer processing time. Due to this efficiency, the FFT algorithm is widely used and is readily available as a subroutine package in many computing libraries. Standard FORTRAN and C programs for this purpose can be readily found in various texts.
The FFT works by partitioning the full sequence $f_n$ into a number of shorter sequences. Instead of calculating the DFT of the original sequence, only the DFT's of the shorter sequences are worked out. The FFT then combines these together in an ingenious way to yield the full DFT of $f_n$.

Consider the original sequence $f_n$, $n=0,1,2,(N-1)$ where $N$ is even. Construction of the FFT is done by partitioning the sequence as follows. Let

$$\begin{cases}
y_j = f_{2j} \\
z_j = f_{2j+1}
\end{cases} \quad j = 0,1,2,\ldots,(N/2 - 1) \quad (3.4)$$

and

\[ Y_k = \frac{2}{N} \sum_{j=0}^{N/2-1} y_j e^{i\pi nk/N} \]
\[ Z_k = \frac{2}{N} \sum_{j=0}^{N/2-1} z_j e^{i\pi nk/N} \quad k = 1,2,3,\ldots,(N/2 - 1) \quad (3.5) \]
\[ W_k = e^{i\pi nk/N} \]

then the so-called computational "butterfly" which occurs in most FFT routines is

\[ X_k = \frac{1}{2}(Y_k + W_kZ_k) \quad k = 1,2,3,\ldots,(N/2 - 1) \quad (3.6) \]
\[ X_k + N/2 = \frac{1}{2}(Y_k - W_kZ_k) \]

One can see that the above algorithm, as well as that for the DFT, can be used to obtain the inverse Fourier transform with the stipulation that input frequency domain values must be given appropriately and the output from the algorithm must be multiplied by $\Delta f$. This is due to the complex conjugate nature of forward and inverse transforms. Furthermore, only one half of the frequency spectrum need be utilized. Due to the phenomenon called aliasing, one need only to let $f_{N-n} = f_n$ for $n=1,2,\ldots,N$. 

As a sample test of the NAG routines utilized in the wave scattering programs, the time domain incident wave given by equation (2.22) is transformed via the FFT into the Fourier domain. Figure 12 illustrates the magnitude of the real and imaginary parts of the transformed incident wave. Using the inverse FFT NAG routine on the data obtained with the FFT illustrated in Figure 12, the original time domain incident wave is recovered in Figure 13 as expected.

Figure 12: Fourier transform of the incident pulse via FFT.
Figure 13: Recovery of incident pulse from the inverse FFT.
LAPLACE TRANSFORM INVERSIONS

Introduction

In contrast to the Fourier transform approach for solving the wave scattering problem as well as numerous other physical problems, the time variable is often only semi-infinite. To illustrate, for the problem at hand, the system is in a state of rest until a certain instant, say t=0, when a disturbance is applied. It is then the response for t > 0 that is desired.

The essential advantage of the Fourier transform is its physical interpretability as a spectrum, a diffraction pattern, and so on. Laplace transforms are not so interpretable. Once a Laplace transform of an equation is taken, only a mathematical, and not a physical, grasp of its meaning is retained in contrast to extensive spectral or frequency analysis that can be performed in the Fourier domain. Although the Laplace transform is disadvantaged in this respect, it has its advantage in initial value problems, where only the transient nature of a dynamical system is sought.

The Laplace Transform

Suppose that f(t) satisfies the following conditions:

a) f(t) is piecewise smooth over (-l,l) for every l

b) f is of exponential order: that is there exist real constants K, c, and T such that |f(t)| < Ke^{ct} for all t > T.

The Laplace transform pair of a function f(t) is defined [33] by
where \( a > 0 \) is arbitrary, but must be chosen so that it is greater than the real parts of all the singularities of \( F(s) \). For this discussion, it is assumed that these integrals exist for \( \text{Re}(s) \geq a > 0 \).

One then has the freedom of choosing a contour over which the inverse Laplace transform may be evaluated. Any vertical line with \( a > 0 \) will provide a suitable contour. The freedom in assigning the parameter "a" provides the basis for a powerful computational method in determining the inverse transform. By utilizing a Fourier series, Dubner and Abate [35] showed that the inverse can be approximated arbitrarily close to the theoretical inverse with the error as small as desired by appropriately choosing the parameter \( a \). This contrasts the use of involved algorithms utilizing orthogonal functions which are encountered in the method developed by Papalous [36]. F. Durban [37] improved on Dubner and Abate's scheme by making the error bound on the inverse, \( f(t) \), independent of \( t \), instead of exponentially in \( t \) as in Dubner and Abate's method. As will be shown here for the sample incident wave, the error bound can be set arbitrarily small, and it is always possible to get good results even for other more difficult cases.

From the outset, \( f(t) \) will be taken to be a real function. Then (4.1) and (4.2) may be replaced by

\[
\text{Re}\left[ f(s) \right] = \int_{0}^{\infty} f(t)e^{-at}\cos(\omega t)dt \quad \text{(4.3)}
\]

\[
f(t) = \frac{2e^{at}}{\pi} \int_{0}^{\infty} \text{Re}\left[ f(s) \right] \cos(\omega t) d\omega \quad \text{(4.4)}
\]

where \( s = a + i\omega \). The above two equations represent an integral transform and its inversion.
formula, being a special case of the Laplace transform for real functions. Choosing $s=a+i\omega$ and $ds=id\omega$ in equations (4.1) gives

$$\mathcal{L}^{-1}f(s) = \int_0^\infty f(t)e^{-s}dt$$

Choosing $s=a+i\omega$ and $ds=du$ in equations (4.1) gives

$$\int_0^\infty f(t)e^{-a+i\omega}dt = Re|f(a+i\omega)| + Im|f(a+i\omega)|$$

The same substitution in equation (4.2) yields

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(s)e^{a+i\omega}d\omega$$

The imaginary part in (4.8) is even and cancels out. Also, the real part is odd so that equation (4.8) reduces to

$$f(t) = \frac{1}{\pi} \int_0^\infty \mathcal{F}(s) \cos(\omega t)d\omega - \int_0^\infty \mathcal{F}(s) \sin(\omega t)d\omega$$

Also, since $f(t) = 0$ for $t < 0$, one has

$$0 = \int_0^\infty \mathcal{F}(s) \cos(\omega t)d\omega - \int_0^\infty \mathcal{F}(s) \sin(\omega t)d\omega$$

Thus, three formulations for the Laplace inverse are obtained with

$$f(t) = \frac{1}{\pi} \int_0^\infty \mathcal{F}(s) \cos(\omega t)d\omega$$
and 

\[ f(t) = \frac{2e^{at}}{\pi} \int_{0}^{\infty} \text{Im}\{ \tilde{f}(\omega) \} \sin(\omega t) d\omega \]  

(4.12)

along with equation (4.8), which constitute formulas for the calculation of \( f(t) \) from \( F(s) \).

**Durban's Method of Laplace Transform Inversion**

Durban's method to perform the Laplace transform inversion is a refinement of Dubner and Abate's method. Dubner and Abate's method begins by constructing an infinite set of even \( 2T \)-periodic functions \( g_n(t) \). Letting \( h(t) = e^{at}f(t) \) for \( t > 0 \) and \( h(t) = 0 \) for \( t < 0 \), define \( g_n(t) \) by

\[
g_n(t) = \begin{cases} 
  h(nT - t), & -T \leq t \leq 0 \\
  h(nT + t), & 0 \leq t \leq T \\
  h((n+2)T - t), & T \leq t \leq 2T 
\end{cases}, \quad n=0,2,\ldots 
\]  

(4.13a)

(4.13b)

(4.13c)

\[
g_n(t) = \begin{cases} 
  h((n+1)T + t), & -T \leq t \leq 0 \\
  h((n+1)nT - t), & 0 \leq t \leq T \\
  h(((n-1))T + t), & T \leq t \leq 2T 
\end{cases}, \quad n=1,3,\ldots 
\]  

(4.14a)

(4.14b)

(4.14c)

The function \( g_n(t) \) is then developed into a Fourier cosine series as follows:

\[
g_n(t) = \frac{A_{n,k}}{2} + \sum_{k=0}^{\infty} A_{n,k} \cos\left(\frac{k\pi}{T} t\right) \]  

(4.15)

where

\[
A_{n,k} = \frac{2}{T} \int_{0}^{T} h(t) \cos\left(\frac{k\pi}{T} t\right) dt 
\]  

(4.16)

Summing over the \( A_{n,k} \) values gives the expression
Furthermore, summing over equation (4.15) gives

\[ \sum_{n=0}^{\infty} A_{n,k} = \frac{2}{T} \text{Re} \left[ f(a + ik\pi/T) \right] \]  

(4.17)

Also, using relations (4.13b), (4.13c), (4.14b), and (4.14c), with \( h(t) = f(t)e^{-at} \) in the summation process yields

\[ \sum_{n=0}^{\infty} e^{at} g_n(t) = f(t) + \sum_{k=1}^{\infty} e^{-2ak} \left[ f(2kT + t) + e^{2at}f(2kT - t) \right] \]  

(4.18)

Thus for any \( 0 \leq t \leq 2T \), equating equations (4.18) and (4.19) yields Dubner and Abate's formula, which is given by

\[ f(t) + \text{Error}_1(a,t;T) = \sum_{n=0}^{\infty} e^{at} g_n(t) = f(t) + \sum_{k=1}^{\infty} e^{-2ak} \left[ f(2kT + t) + e^{2at}f(2kT - t) \right] \]  

(4.19)

Unfortunately, the error term in (4.20), \( \text{Error}_1(a,t;T) \), is an exponentially increasing function of \( t \) and can present difficulties in the inversion process.

The continuation of Dubner and Abate's method proceeds by considering the same function, \( h(t) \), on the interval \( (nT, (n+1)T) \). It then proceeds by constructing an infinite set of odd functions \( k_n(t) \) that are 2T-periodic. This is done by defining
On the intervals (-T,0), (0,T) and (T,2T) equations (4.21) can be written as

\[
\begin{cases}
  h(t), & n=1,2,... \\
  -h(2nT - t), & n=1,2,...
\end{cases}
\]

Each odd function \( k_n(t) \) is now expressed in a Fourier representation by

\[
k_n(t) = \sum_{k=0}^{\infty} B_{n,k} \sin\left(\frac{k\pi t}{T}\right)
\]

where

\[
B_{n,k} = \int_{0}^{T} h(t) \sin\left(\frac{k\pi t}{T}\right) dt
\]

Summing over the \( B_{n,k} \) values gives the expression

\[
\sum_{n=0}^{\infty} B_{n,k} = \frac{2}{T} \text{Im}\left[ \tilde{f}(a+i\frac{k\pi}{T}) \right]
\]

Multiplying both sides of (4.26) by \( e^{at} \) and summing over \( n \), yeilds an expression similar to equation (4.18):
\[ \sum_{n=0}^{\infty} e^{a it} k_n(t) = \frac{2 e^{a t}}{T} \ln \left[ \frac{f(a+i k \pi / T)}{T} \sin \left( k \pi t / T \right) \right] \] (4.27)

Likewise on the interval (0,2T), using equations (4.22b), (4.22c), (4.23b), and (4.23c), gives the expression

\[ \sum_{n=0}^{\infty} e^{a it} k_n(t) = f(t) + \sum_{k=1}^{\infty} e^{2a k \pi} \left[ f(2kT + t) - e^{2a t} f(2kT - t) \right] \] (4.28)

Hence, another representation for \( f(t) \) is given by

\[ f(t) + \text{Error}^{2}(a,t,T) = \frac{2 e^{a t}}{T} \sum_{k=0}^{\infty} \text{Im} \left[ \frac{f(a+i k \pi / T)}{T} \sin \left( k \pi T / T \right) \right] \] (4.29)

Individually, the formulations (4.20) and (4.29) have no advantage over another. Each has an error term that is exponentially increasing in \( t \). However, summing half of both sides of the two formulations gives

\[ f(t) = \frac{e^{a t}}{2} \left[ \text{Re} \left( f(a) + \sum_{k=0}^{\infty} \text{Re} \left[ f(a+i k \pi / T) \cos \left( k \pi T / T \right) \right] - \sum_{k=0}^{\infty} \text{Im} \left[ f(a+i k \pi / T) \sin \left( k \pi T / T \right) \right] \right) \right] + \text{E}^{3}(a,t,T) \] (4.30)

where

\[ \text{E}^{3}(a,t,T) = \sum_{k=1}^{\infty} e^{-2a k T} f(2kT + t) \] (4.31)

This step is important since the error term is now bounded and no longer becomes unbounded as \( t \to T/2 \) as in Dubner and Abate’s method. This allows one to use the representation of \( f(t) \) on the interval (0,2T) instead of (0,T/2). Furthermore, if \( |f(t)| < C \) for all \( t \in (0,2T) \), \( T > 0 \), then the error bound is
This bound depends only on the product $aT$ and is no longer exponentially increasing in $t$.

Since the time interval of interest is $(0,T)$, one should $T = T/2$ in the formulation developed in equation (4.30). Also, since only a finite number of terms are computationally summable, the Laplace inversion scheme can be rewritten as

$$f(t_j) = \frac{2e^{j\Delta t}}{T} \left[ -\frac{1}{2} \text{Re}(\tilde{f}(a)) + \sum_{k=0}^{\infty} \tilde{f}(s_k) \left[ \cos\left(\frac{2k\pi}{N}\right) + i \sin\left(\frac{2k\pi}{N}\right) \right] - \right. $$

$$\left. + E_3(a,t,T) + E_T(a,t,T) \right]$$

(4.33)

where $t_j = j\Delta t$, $T = n\Delta t$, and $s_k = a + i \frac{2k\pi}{T}$.

Durban's method of Laplace transform inversion will now be tested on the Gaussian incident wave given earlier. The wave is first transformed into the Laplace domain using Simpson's composite rule of numerical integration. The Laplace domain "spectrum" of the wave is illustrated Figure 14 where the amplitude is the magnitude of the real and imaginary part. Durban's method is then used to perform the Laplace inversion back into the time domain. The original time domain incident wave is recovered as expected in Figure 15.
Figure 14: Laplace transform of the incident wave.

Figure 15: Durban Laplace inversion of the incident wave.
FOURIER DOMAIN WITH DAMPING EFFECTS

Fictitious Eigenfrequencies

As the frequency increases in the spectrum of solutions in the Fourier domain, so does the difficulty in obtaining a numerical solution. Higher frequency values cause a higher variation in the boundary variables from one node to the next. Hence a choice of shape functions for the boundary elements can only represent this variation up to a certain degree. A finer mesh or a rearrangement the nodes and elements so that they are more dense in regions with the greatest variation will decrease the node to node variational difficulties. This improvement will give a better representation of the variation in the boundary quantities and a more accurate integration of the BIE formulas involving oscillatory integrands.

Another important concern of integral equation methods for exterior scattering problems in acoustics and elastodynamics is that they are known to have uniqueness problems at certain frequencies [38,39,40,41]. The integral equation formulation discussed here for the scattering from spherical voids, inclusions, and rigid scatterers all will suffer similar non-uniqueness difficulties. The difficulty is not due to non-uniqueness of the physical problem but to a breakdown in the integral equations for the scatterer at certain frequencies which correspond to the zeros of the spherical Bessel functions of the first kind for the spherical void.

The problem manifests itself as ill-conditioning of the algebraic system of equations near the critical frequencies in the discretized form of the boundary integral equation. It is expected that at or near these frequencies, one could obtain spurious solutions as indicated by a sharp rise in condition number of the system matrix. So, the methods described above will have little effect on correcting this difficulty. Figure 16 illustrates the unwelcome behavior of the condition number of the solution matrices for a frequency range of $2\leq f \leq 4.5$ for a spherical void in an elastodynamic medium impinged upon by the aforementioned incident wave.
Figure 16: Fictitious eigenfrequencies.

The numerical difficulty associated with the uniqueness of the boundary integral representation has received considerable attention and several methods have been formulated in an attempt to overcome these difficulties. Two methods in particular have been extensively investigated in the literature. In the first method, proposed by Burton and Miller [38,39], a linear combination of the integral equation and the normal derivative of the integral equation through a complex coupling parameter is taken to guarantee the uniqueness of the representation. Another method designated as the CHIEF (Coupled Helmholtz Integral Equation Formulation) method by Schenk [40,41], the system of linear equations is overdetermined by collocating at additional points in the interior of the domain.

Much debate persists as to the best method to suppress the spurious eigenfrequencies
associated with the boundary integral equation. A disadvantage of the Burton-Miller method is that it requires the evaluation of two integral equations per collocation point making it more computationally intensive. Hence, the total computation time in obtaining Fourier or Laplace domain solutions are significantly increased. Also, the derivative of the boundary integral equation is hypersingular, although this difficulty can be properly addressed. The main problem with the CHIEF method is its lack of formalism in the selection of the interior collocation points. This problem may become severe for convoluted geometries. The representation will again be non-unique if the interior collocation points are chosen on nodal surfaces. However, it has been shown that choosing as few as 3 appropriate CHIEF points does reduce the fictitious eigenfrequency problem for the spherical void, yet additional computation time is required. Since the scope here is to minimize the computation time in obtaining time domain solutions, an alternative method will be used.

Parameter Damping

The method used here in the Fourier domain is to introduce a small "damping" term in the Fourier parameter \( \omega \). Hence adding a small complex component to the real value \( \omega \) so that \( \omega = \bar{\omega} + \alpha i \). This natural damping process can have its origins in the original governing equations where now one considers

\[
(\lambda + \mu)u_{ij,ij} + \mu u_{ij,ij} = \rho \frac{\partial^2 u_i}{\partial t^2} - 2\alpha \rho \frac{\partial u_i}{\partial t} + \alpha^2 \rho u_i
\]  

(5.1)

Transforming (5.1) into the frequency domain gives

\[
\frac{(\lambda + \mu)}{\rho} \ddot{u}_{i,ij} + \frac{\mu}{\rho} \ddot{u}_{j,ii} + (\omega + \alpha i)^2 \ddot{\omega}_i = 0
\]  

(5.2)
provided $\alpha << 1$, where the product $\alpha \rho$ is small so that the velocity term in (5.1) negligibly effects the physical nature of the problem. Similarly, the displacement term in (5.1) can be ignored for small $a$. The introduction of the damping term also has a physical significance. In any elastic material, the damping is present in what's called ultrasonic attenuation [7]. This is characterized by the observance of the decrease in the amplitudes of successive back-surface reflections.

An alternative viewpoint is to consider the mathematical foundations fundamental to the Fourier transform process and the utilization of the fast Fourier transform to perform the procedure numerically. The motivation behind this approach is the existence of the singularities or poles (eigenfrequencies) along the Re($\omega$) axis. By shifting off of the real axis by a small perturbation, one can avoid the numerical difficulties and the inherent errors involved due to the high condition numbers. Here, integrations are performed in the complex plane about the poles, by a shift off the real axis by a small perturbation, $\omega + ai$.

Define

$$\hat{f}(\omega + ai) = c_1 \int_{-\infty}^{\infty} f(t) e^{i(\omega + ai)t} dt$$

$$= c_1 \int_{-\infty}^{\infty} \left[ f(t) e^{i\omega t} \right] e^{i\omega t} d\omega$$

$$= F \left[ f(t) e^{i\omega t} \right]$$ (5.3)

The product $u(t)e^{\omega t}$ clearly satisfies the requirements for the existence of its Fourier transform $F$. Furthermore, transforming the original elastodynamic differential equation (in absence of body forces) using (5.3) yields equation (5.2).

Formal transformation back into the time domain is given by
\[ f(t) = c_2 \int_{-\infty}^{\infty} \hat{f}(\omega + \alpha i)e^{(\omega + \alpha i)t}d\omega \]

\[ = c_2 \int_{-\infty}^{\infty} \hat{f}(\omega + \alpha i)e^{-\alpha t}e^{i\omega t}d\omega \]

\[ = c_2 e^{-\alpha t} \int_{-\infty}^{\infty} \hat{f}(\omega + \alpha i) e^{i\omega t}d\omega \]

\[ = e^{-\alpha t} F^{-1}\left[ \hat{f}(\omega + \alpha i) \right] \]  

Equations (5.3) and (5.4) are recognized in principle as the Fourier shift formulas [33].

Utilization of the shifting process, or alternatively, the usage of the damping term, significantly improves the unwelcome behavior of the matrix condition numbers as seen in Figure 17. The choice of the optimum value of the damping parameter is very critical. If the damping introduced is too small, the fictitious eigenfrequency difficulty remains. On the other hand, if the damping is too large, the results are significantly altered due to the presence of the damping term in the original differential equation. Yet the tradeoff in having to choose an appropriate parameter outweighs the concern of increased computation time, resulting from various other numerical treatments of the eigenfrequency difficulty. The damping coefficients will be given as ultrasonic attenuation coefficients for the medium containing the spherical inclusion samples. The medium containing the spherical void has little to no ultrasonic attenuation but to circumvent the fictitious eigenfrequency difficulty that does arise, a small value for the parameter will be chosen to lessen its influence.
Figure 17: Condition number damping.
COMPUTATIONAL RESULTS

Introduction

In the results that follow, the boundary element method will be compared with the analytical solution for the scattering from spherical obstacles. This seemingly preoccupation with the spherical geometry is due only to the fact that comparison solutions for any other geometry are more difficult to find and are less practical, except for the elliptical scatterer. Two types of flaws are considered: an inclusion and a cavity. Scattered, farfield solutions from these flaws are calculated as a function of time and are compared with various transform domain parameters to capture the necessary characteristics of the scattered wave.

Experimental results from the scattering from a spherical void and two spherical inclusions will also be used for comparison of scattered wave characteristics. The experimental analysis was performed to test the validity of the boundary element results and the analytical solution. The two inclusion samples include a tin-lead sphere and a polystyrene bead; both of which are imbedded in a thermoplastic (Buehler’s transoptic) disk. The $ka = 2\pi f a/c_L$ ($a=$radius of sphere) values for the spherical void and the tin-lead sphere are within an acceptable range (<10) if the boundary element method is to be utilized. For the polystyrene sphere, the center $ka$ is near 10 indicating possible difficulties in obtaining a suitable solution via the boundary element method. This presents an opportunity to also test the BEM for higher $ka$ values for the purpose of obtaining time domain solutions.

It is important to realize that measurements of scattering in a nondestructive evaluation setting are always made with a transducer of finite aperture, which generates a pulse that spreads out from the transducer in a predetermined geometrical configuration. Furthermore, in the experimental procedure outlined below, the incident wave first encounters a liquid-solid interface. After a portion of the incident wave’s energy is transmitted into the solid medium, it
proceeds to make contact and interact in a complicated way with the scattering obstacle. The produced scattered wave is spherically spreading with both longitudinal and shear components and which also encounters the fluid-solid interface in its return to the transducer. Also, since a Newtonian fluid supports only a longitudinal wave type, the shear component is lost. Although in principle this leads to a complicated fluid-solid-obstacle model and can be modeled theoretically, one can simplify the model in numerous ways and yet attain good solutions.

The transducer signal can be modeled in terms of a Gaussian in the plane perpendicular to the direction of propagation but for all practical purposes, as the wave encounters the fluid-solid interface, it closely resembles a plane wave. This is assuming a farfield location of the ultrasonic transducer. Upon determining the transmission and reflection coefficients of the amplitude of the incident wave at the interface, the transmitted portion of the incident wave, modeled as a plane wave, will impinge upon the obstacle. Hence, throughout the boundary element routine and analytical model, the incident wave will be simulated in the form of a plane wave.

**Experimental Results**

The results presented here will show an absolute agreement between experimental and theoretical results for the farfield backscattering in an unbounded elastic medium for three samples. The scattering samples include a spherical void of diameter 685 μm imbedded in fused quartz, a polystyrene spherical inclusion of diameter 920 μm imbedded in a thermoplastic disk (Buehler's Transoptic), illustrating a so-called weak scatterer, and a stronger spherical scatter comprised of a tin-lead solder of diameter 338 μm also imbedded in a thermoplastic disk. The scatterer geometry was determined by optical measurement.

The experiments were performed in a water immersion tank using a 1/4 in diameter broad band planar transducer with a nominal center frequency of 10 MHz (Panametrics serial number: V3404 82599). All experiments utilized the pulse-echo (backscatter) mode in
measuring the longitudinal wave amplitudes at normal incidence to the flat surface of the mediums containing the scattering samples. The signals were driven by a Panametrics 5052PR pulser-receiver and were digitized and the signal was averaged using a Lecroy 9400A programmable digitizing oscilloscope controlled by an AST Premium 386sx desktop computer. The signals were selected from the oscilloscope's measurement window with the temporal location of the first peak in the front surface and flaw backscatter signal recorded. The wave signals were sampled at a time increment of .01x10^{-6} seconds. The experimental distances and sample geometry are illustrated in Figures 19 and 20. For the void sample, the distance between the transducer and the front surface of the fused quartz is 13.9 cm, the depth of the void is 1.93 cm and the total thickness of the fused quartz sample is 3.8 cm. In the inclusion samples, the transducer to front surface distance is 7 cm.

The ultrasonic attenuation of the Buehler's transoptic is also listed in the following table. It corresponds to the damping effects on a propagating wave through a material and is commonly determined by measuring the successive second back-surface reflections and using a logarithmic least squares fit to determine the coefficients. The ultrasonic attenuation is expressed as the complex part of the frequency with \( T = f + i \alpha \), where, \( \alpha = \alpha_0^n \). This will be incorporated into the boundary element method for the purpose of developing an accurate scattering model and to alleviate the fictitious eigenfrequency difficulty that would present if this property were ignored.

With the material properties [7] listed in Table 1, the reflection and transmission coefficients at the fluid-solid interface are given by

\[
C_R = \frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} \quad (6.1a)
\]

\[
C_T = \frac{2 \rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2} \quad (6.1b)
\]

and are determined for the fluid-solid interaction scenarios in Figure 18.
### Table 1: Material constants

<table>
<thead>
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<th>Material</th>
<th>$\rho$ g/cm$^3$</th>
<th>$c_L$ cm/µs</th>
<th>$c_T$ cm/µs</th>
<th>$\alpha_0$</th>
<th>$n$</th>
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<td>Water</td>
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<td>.147</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fused Quartz</td>
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<td>.597</td>
<td>.376</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buehler's Transoptic</td>
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<td>.272</td>
<td>.134</td>
<td>.16</td>
<td>.86</td>
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<td>Polystyrene</td>
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<td>.24</td>
<td>.128</td>
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<tr>
<td>Tin-Lead Solder</td>
<td>8.41</td>
<td>.301</td>
<td>.145</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ H_2O \quad A=1 \quad C_R = -.798 \]

\[ H_2O \quad A=1 \quad C_T = .2013 \]

\[ Fused\ Quartz \quad C_T = .2013 \]

\[ Fused\ Quartz \quad A=1 \quad C_R = .769 \]

\[ H_2O \quad A=1 \quad C_T = .628 \]

\[ Buehler's\ Transoptic \quad C_T = .628 \]

\[ Buehler's\ Transoptic \quad A=1 \quad C_R = .372 \]

\[ H_2O \quad A=1 \quad C_T = 1.372 \]

\[ Buehler's\ Transoptic \quad C_T = 1.372 \]

\[ H_2O \quad A=1 \quad C_R = .372 \]

**Figure 18:** Reflection/Transmission coefficients.
Figure 19: Photograph of the spherical void.

Figure 19 is a photo of the spherical void utilized in the experimental procedure. The magnification is at 64x with a scale of 1 cm = 257 μm. The void is imbedded in a fused quartz block at a distance of 1.93 cm below the top surface.

Figures 20 and 21 illustrate the experimental setup that is to be modeled along with the samples used in the experiment. The transducer is submersed in a water tank with the samples lying on a damping material. The geometrical distances between the transducer and the samples are given in the figures along with the flaw locations in both mediums.

The transducer is mounted in a way which allows horizontal and vertical motion as well as an angular motion relative to the vertical axis. On setup the transducer is adjusted so its signal propagates through the fluid perpendicular to the front surface of the medium containing the flaw. This is done by observation. Slight angular adjustments are made on the gimbal so that the amplitude of the front surface echo is maximized. The flaw was then located visually in the sample with the transducer location adjusted in the horizontal plane until the flaw signal was detected on the oscilloscope. The scattered signal was then time-averaged to eliminate
spurious noise due to the electronics of the system. The front surface echo and flaw signals were then digitized. Fourier transforms of the digitized signals were also performed.

The polystyrene bead was slightly off center in the thermoplastic disk and this presented difficulties in obtaining a scattered signal. The tin-lead scatterer and the void presented no difficulties in locating the respective flaws. Their scattered signals were easily observed.

![Diagram](image)

Figure 20: Experimental setup for the spherical void sample.

![Diagram](image)

Figure 21: Buehler’s transoptic scattering samples.
Figures 22-30 are the results obtained during the experimental analysis. Included are time domain plots for the spherical void, tin-lead, and polystyrene scatterer. Also, frequency domain plots are given for the front surface echo and the three scatterers. Figure 22 illustrates the front surface echo and scattered signal from the spherical void. In an NDE setting, the depth of the flaw from the surface is determined by measuring the travel time between the two respective signals. Figure 23 is the incident wave in the Fourier domain.

![Amplitude vs Time](image)

Figure 22: Experimental scattering from a spherical void.
Figure 23: Incident wave in the frequency domain.
Figures 24 and 25 illustrate the scattered signal from the spherical void in the time domain and frequency domain respectively. One notes the characteristics of the front surface echo from the front surface of the sphere. The creep or Rayleigh waves that propagate around the inner surface were also recovered, unfortunately, the plot below was digitized on the time interval observed below which excluded the Rayleigh wave signal. The analytical and boundary element results presented later will illustrate creep wave. The Rayleigh wave velocity is roughly \(0.5c_L\) and with the geometry associated with the void, and is found near \(t = 7.3 \, \mu s\).

**Scattering From a Spherical Void**

![Graph showing scattered signal from a spherical void](image)

**Figure 24:** Time domain spherical void scattered signal.
Figure 25: Frequency domain of spherical void scattered wave.
Figure 26 and 27 are the experimental signals captured for the tin/lead spherical inclusion. Figure 26 illustrates the time domain signature of the front surface echo along with the scattered signal for the tin-lead scatterer. Near $t = 1\mu s$ there is a presence of another anomaly in the thermoplastic disk. Closer examination through adjustment of the oscilloscope settings for the purpose of increasing the sensitivity didn't provide any further information due to the size of the second inclusion or void. This process is precisely the procedural technique performed in ultrasonic nondestructive evaluation when attempting to find and locate flaws in an elastic material. Figure 28 represents the frequency domain signature of the tin/lead spherical inclusion.

![Graph of experimental scattering from a tin/lead spherical inclusion.](image)

Figure 26: Experimental scattering from a tin/lead spherical inclusion.
Figure 27: Time domain tin/lead spherical inclusion signal.
Figure 28: frequency domain tin/lead spherical inclusion signal.
Figures 29, 30 are the signals obtained for the polystyrene bead imbedded in a thermoplastic disk. The combination of the weak scattering nature of the bead and its location in the disk presented some difficulties in flaw detection. After careful adjustment of the transducer position, the scattered signal was recovered. The resulting scattered signal amplitude was not much above the spurious noise generated by the pulser. The signal was then smoothened out with a time averaging option available on the oscilloscope. Figure 31 is the frequency domain signature of the scattered signal.

![Amplitude vs. Time Graph](image)

**Figure 29:** Experimental scattering from a weak scatterer.
Figure 30: Time domain polystyrene spherical inclusion signal
Figure 31: Frequency domain polystyrene spherical inclusion signal.
Analytical Solution

Analytical solutions for the scattering of plane waves off of a spherical obstacle are well documented in the literature [1,4]. The referenced analytical formulations utilized a time harmonic incident plane wave with frequency domain solutions obtained. It is my intention to generalize the analytical results toward the ability to input an arbitrary time dependent incident wave. The formulation that follows will also alleviate the difficulty of indeterminate values occurring at the zeroes of the spherical Bessel functions of the first kind that arise in the series coefficients.

From the decoupled and transformed Navier's equation presented in equations (2.13) and (2.14), one can in spherical coordinates utilize separation of variables on any of the scalar wave equations. This is performed after transformation into the Fourier or Laplace domain. The general solution in the Fourier domain is in the form of a Legendre-spherical Bessel [4,42,43] series

$$\psi = \zeta_n(kr) P_n^m(\cos\theta) e^{i\omega t}$$  \hspace{1cm} (6.2)

where \( \zeta_n(kr) \) are the spherical functions \( j_n, y_n, \text{ or } h_n \); and \( P_n^m \) is the Legendre polynomial. Since there is a spherical symmetry in \( \psi \) about the z-axis, conveniently chosen as the incident wave direction of propagation, the spherical wave functions are independent of \( \phi \). It follows that \( m = 0 \) in \( P_n^m \). In the Laplace domain, the spherical Bessel functions are replaced by the modified spherical Bessel function. Since the two solutions are completely analogous, only the Fourier domain results are presented.

For a plane wave propagating in an infinite medium, when the wave impinges on the surface of the elastic inclusion, both compressional and shear waves are refracted into the inclusion and reflected back into the medium. For convenience in the following discussion, the
infinite solid is designated as medium 1 and the spherical inclusion as medium 2. The potentials, displacements, and stresses will be designated by the superscripts \((i), (f), \) and \((r)\) denoting the incident, reflected and refracted waves.

The two reflected waves, which are outward propagating, can be represented by

\[
q^{(r)}(r) = \sum_{n=0}^{\infty} A_n h_n^{(1)}(\alpha_s r) P_n(\cos\theta) \\
\chi^{(r)}(r) = \sum_{n=0}^{\infty} B_n h_n^{(1)}(\beta_s r) P_n(\cos\theta)
\]

The choice of Hankel functions in the above equation is due to the positive exponential in its series expansion. The product with \(e^{i\omega t}\) represents an outward propagating wave.

The refracted waves, being confined to the spherical scatterer are given by

\[
q^{(r)}(r) = \sum_{n=0}^{\infty} C_n j_n(\alpha_r r) P_n(\cos\theta) \\
\chi^{(r)}(r) = \sum_{n=0}^{\infty} D_n j_n(\beta_r r) P_n(\cos\theta)
\]

with \(j_n\), the spherical Bessel function of the first kind, chosen since its product with \(e^{i\omega t}\) represents a standing wave.

In equations (6.3) and (6.4), \(\alpha\) and \(\beta\) represent the compressional and shear wave numbers \((\omega/c)\) in mediums 1 and 2. A, B, C, and D are the expansion coefficients to be determined by the boundary conditions.

The displacements and stresses that are of concern in the application of the boundary conditions are given in spherical coordinates by
The incident wave is given by equations (3.33)-(3.35). In order to conveniently match the boundary conditions, it is necessary to express the incident potential in the form of a Legendre series. Such a function \( \varphi \) satisfies

\[
\mathbf{u}^{(i)}(t,\mathbf{x}) = \nabla \varphi^{(i)} = \frac{\partial}{\partial z} \varphi^{(i)}(t,\mathbf{x}) \hat{e}_3
\]

(6.6)

Formal determination of \( \varphi \) is as follows:

\[
\varphi^{(i)} = -\frac{A \omega D}{2} \left[ 4 \frac{(t-B)^4}{D^2} - 12 \frac{(t-B)^2}{D} + 3 \right] \exp \left( -\frac{(t-B)^2}{D} \right)
\]

\[
B = \frac{z+t_0}{c_L}
\]

(6.7)

In order to establish the correct frequency for the Gaussian wave model, one needs to find the zeros of \( u^i \). They are at \( t - B = \pm 2.0202 \sqrt{D} \) and \( \pm 0.95858 \sqrt{D} \). With the transducer having an actual nominal frequency of around 8.7 MHz = \( 1/\Delta t \), then \( D = \Delta t^2/4.0812 = 3.2405 \times 10^{-15} \) in the above expression as well as in the expression for the incident wave driving the boundary element solution.
Figures 32 and 33 compare the time and frequency domain signatures of the front surface echo for the Gaussian model to that of the pulse generated by the transducer used in the experiment. Further refinement the model by utilizing a $7^{th}$ derivative of a Gaussian is possible although the choice made did perform very well and did not warrant any further improvement.

Figure 32: Time domain comparison of front surface reflected wave.
For the plane wave, the incident potentials are symmetric about the z axis. Hence, they are independent of the spherical coordinate $\phi$. In spherical coordinates, one should consider the transformation $z = r\cos(\theta)$. Then expressing the incident potential as a Legendre-Bessel series gives

$$\phi^{(i)} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\theta)$$

$$a_n = \frac{2n+1}{2j_n(kr)} \int_0^{\pi} \phi^{(i)}(r,\theta) P_n(\cos\theta) \sin\theta \, d\theta$$

(6.8)

The coefficients are determined by the orthogonality condition of the Legendre polynomials on $(0,\pi)$. Simplifying,
The Fourier transform via the FFT or the Laplace transform is then performed on $\Phi$ in order to remove the time dependence.

For the case of the spherical void, at $r = a$, the boundary of the scatterer, the stress free boundary conditions are imposed

$$
\sigma_{rr}^{(i)} = \sigma_{rr}^{(f)}
$$

$$
\sigma_{r\theta}^{(i)} = \sigma_{r\theta}^{(f)}
$$

which yield a system for the unknown series coefficients

$$
\begin{align*}
2\mu \left[ \begin{array}{c}
E_{31}^{(3)} \\
E_{31}^{(3)}
\end{array} \right] \left[ \begin{array}{c}
A_n \\
B_n
\end{array} \right] &= - \left[ \begin{array}{c}
-\frac{\lambda k^2}{4} \Phi_n + 2\mu \Phi''_n \\
\frac{2\mu}{a} \Phi'_n - \frac{1}{a} \Phi_n
\end{array} \right] \\
\end{align*}
$$

Similarly for the spherical inclusion, the following boundary conditions at $r = a$

$$
\begin{align*}
\psi^{(i)} - \psi^{(f)} &= \psi^{(i)} \\
\psi^{(i)} - \psi^{(f)} &= \psi^{(i)} \\
\sigma_{rr}^{(i)} - \sigma_{rr}^{(f)} &= \sigma_{rr}^{(i)} \\
\sigma_{r\theta}^{(i)} - \sigma_{r\theta}^{(f)} &= \sigma_{r\theta}^{(i)}
\end{align*}
$$

give the system for the unknown series coefficients
The coefficients for the above matrices are found in Appendix E.

**Boundary Element Results**

The boundary element technique will be applied to the three scatterers using different sphere models. The scattered wave form obtained via the boundary element method will be compared to the analytical solution and experimental data for the purpose of verifying the acceptability of the results. Furthermore, the computation time required to obtain a Fourier and Laplace domain boundary element solution, for each model discretization, will be made.

The boundary element sphere models chosen were a 24 quadratic-quadrilateral element, 74 node model, an 8 element 18 node model comprised of quadratic triangular elements, and a 16 element 29 node quadratic-triangular element model. Furthermore, a 54 element 164 node model comprised of quadratic quadrilateral elements was also tested. The computation time comparison for the three models is listed in Table 2. All numerical runs were performed on a Personal DECstation 5000. Performance was inhibited at times due to other users.
Table 2: BEM total computation time comparison.

<table>
<thead>
<tr>
<th>Model</th>
<th>Void</th>
<th>Tin-Lead</th>
<th>Weak</th>
</tr>
</thead>
<tbody>
<tr>
<td>FFT 18 n 8 el</td>
<td>1657</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>FFT 74 n 24 el</td>
<td>10832</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>FFT 74 n 24 el damped</td>
<td>12495</td>
<td>39588</td>
<td>39098</td>
</tr>
<tr>
<td>FFT 164 n 54el</td>
<td>86921</td>
<td>258183</td>
<td>265537</td>
</tr>
<tr>
<td>Laplace 74 n 24 el</td>
<td>15565</td>
<td>29941</td>
<td>31271</td>
</tr>
</tbody>
</table>

A time interval of (0,9\mu s) was used for each boundary element model. This will adequately account for the travel time of the incident wave from the top surface of the medium to the scatterer and the subsequent return of the scattered wave from the inclusion or void. The number of steps in the time variable was chosen to minimally capture the behavior of the incident wave. With the incident wave having a center frequency of near 8.5 MHz, a sample rate of \Delta t = 1/(8f) will capture the incident wave as well as the scattered wave from the obstacle. A higher rate will certainly refine the signal but the result is a linear increase in computation time with respect to the number of time steps.

With the time interval and stepsize used above, the corresponding frequency interval is (0,N/T) or (0,66.7MHz). Since the frequency domain amplitude of the incident wave becomes appreciably small above 15 MHz, it is unnecessary to perform boundary element computations above this value. Rather, one can equate all displacement and traction values on the boundary to that of the incident wave which is effectively zero. This alleviates some of the difficulties associated with higher ka values, specifically when ka>10, as well as significantly decrease the time involved in obtaining a solution.
Spherical Void Imbedded in Fused Quartz

The set of plots that follow pertain to a spherical void imbedded in fused quartz. The center ka of the spherical void is near 3, indicating the boundary element formulation should produce acceptable results. Also included in the results is a comparison with the experimental values obtained from measurements. Usage of the transmission and reflection coefficients were made on the experimental scattering values. As crude as it may seem, the size of the front surface of the fused quartz sample was large enough so that contributions caused from secondary scattering from the edges will not significantly effect the scattered signal from the void. The results obtained show a good correlation with theory, computation, and experiment. Contributing to this agreement was that the void is very close to being a true spheroid as evidenced in the previous photo.

Since the fused quartz medium has little to no ultrasonic attenuation, a test was performed on the damping term in the BEM-FFT model. Damping was completely neglected and then compared with a small damping term of 0.2. This had the effect of lowering the condition numbers at the locations of the eigenfrequencies. However, the eigenfrequency contribution to the solution was not as significant as expected since the range of ka values was much less than 10 (Figure 34). This observation is of paramount importance in the usage of numerical methods for solving scattering problems. In particular, the higher ka values will admit poor solutions in less a finer mesh size is utilized. For the condition number comparison made earlier, the center ka for the incident wave was near 7. Since a portion of the ka spectrum was greater than 10 in that case, the contributions the fictitious eigenfrequencies made on the solution was greatly enhanced.

Figure 35 is a direct comparison between the 74 node 24 element quadratic quadrilateral boundary element model with the analytical solution. The damping value \( \alpha \) was 0.2. The farfield observation point for the analytic and boundary element model is at the fluid-solid
interface and will be so for all boundary element and analytic solutions. The experimental data from the spherical void is also included where its amplitude was adjusted in the crude manner of using the reflection-transmission coefficients determined earlier. The behavior of the scattered wave obtained by the BEM are agreeable with the analytical solution and also surprisingly with the experimental data.

The original intent was to directly compare the boundary element results with the analytical solution only. Since the experimental measurements are dependent upon the positioning of the transducer, only a general wave form characteristic comparison was intended. Fortunately, during the experimental setup, transducer positioning and flaw location allowed for an ideal opportunity for obtaining an excellent scattered signal.

Figure 36 compares the boundary element model with the damping term present to that of the absence of damping. As mentioned earlier, the $ka$ values are in a desirable range which does not allow for a significant contribution of the fictitious eigenfrequencies. When the transient behavior is observed, the effects were negligible.

![Figure 34: Spectrum of the incident wave for the spherical void.](image)
Figure 35: BEM-FFT time domain comparison of the spherical void.
The time domain plot also shows the presence of the Rayleigh or creep wave described previously. It also was present in the analytic solution with its location and amplitudes very closely comparing to the BEM results.

Figure 37 is a frequency domain comparison of the damping and no damping cases. In the absence of the damping term, the higher frequency domain solutions are clearly affected by the higher condition numbers illustrated in Figure 38. Although the condition numbers in general were not lowered significantly with the addition of the damping term in this case, their unwelcome contribution was significantly diminished. From a physical point of view, elastic materials have an inherent damping to some effect, and by excluding it, a proper scattering model is not accurately achieved. This will become more apparent when observing the two inclusion models.

Although the time domain scattered wave characteristics are of primary interest, one notes the general agreement of the boundary element solutions in the frequency domain relative to the analytical solution illustrated. This agreement should be expected for the frequency domain plots for the spherical void due the low ka values associated with the geometry and wave speed in the medium. With the inclusion of the damping term, the higher frequency contribution due to the fictitious eigenfrequencies are lessened although the no damping case did recover the scattered wave.

Figure 38 is the matrix condition number comparison for the 74 node 24 quadratic quadrilateral BEM models for the frequency domain and Laplace domain. The damping term did lessen the effect of the eigenfrequency difficulty although observations made of the time domain farfield signals did show that the effect was minimal due to the low ka range for the void.

Figure 39 illustrates an 18 node, 8 element quadratic triangular element model utilized for the purpose of quickly obtaining a BEM solution. The resulting node to node variation in the frequency domain displacements were too significant for a reasonable solution to be
obtained. Although, the travel time of the wave matches comparably between the two models compared below, the overall farfield signal for the “small” model contained too much spurious noise to be utilized effectively.

A 164 node, 54 element quadratic quadrilateral element model was also utilized to observe any improvement in the scattered signal. The results are shown in Figure 40. The improved model verified convergence as the number of nodes and elements was increased beyond the 24 element model. However, the significant increase in computation time necessary for obtaining the time domain solution in “large” model in comparison to the 24 element model is of a practical concern. The 24 element BEM model was clearly adequate for the purpose and yielded excellent results.

Illustrated in figure 41 is the time domain results obtained utilizing Durban’s method of Laplace transform inversion from the Laplace domain calculations performed by the BEM routine. The value “a” utilized in the Laplace inversion scheme is analogous to the damping term $\alpha$ used in the Fourier analysis. Durban found through trial and error that if the product $aT$ was between 5 and 10, good results were obtained for numerous function inversions. In this case, the product $aT$ was equal to 2 with excellent results being obtained.
Figure 36: BEM-FFT comparison of the damping coefficient.
Figure 37: BEM-FFT frequency domain comparison of the spherical void.
Figure 38: BEM-FFT/LAP condition number comparison for the spherical void.
Figure 39: 18 node/8 quadratic triangular element sphere model comparison.
Figure 40: 164 node/ 54 quadratic quadrilateral element sphere model comparison.
Figure 41: BEM-LAP time domain comparison of the spherical void.
Tin/lead Solder Imbedded in Buehler’s Transoptic

The following analysis pertains to a spherical tin/lead strong scatterer imbedded in a thermoplastic disk. Figure 42 shows that the center $k_a$ of the inclusion is near 4, indicating the boundary element formulation should again produce acceptable results. Also included in the results is a direct amplitude comparison with the experimental results that were obtained through adjustment of the proper transmission coefficient. Comparison with the actual wave forms produced a general agreement between experimental and BEM results, although the actual experimental inclusion was slightly oblate and contained numerous surface anomalies. Also, the size of the front surface of the thermoplastic disk sample was small enough so that edge effect contributions caused from secondary scattering from the edges could possibly effect the scattered signal from the tin/lead sphere.

The thermoplastic medium has a measurable ultrasonic attenuation, hence the BEM model included the damping term given in Table 1. As a comparison, the damping was completely neglected and then compared to the solution where the attenuative damping term was included. This had a significant effect of raising the matrix condition numbers. The results from the omission of the damping term produced a seemingly useless result.

In Figure 43, the boundary element solution is compared directly with the analytical solution and the experimental results adjusted via the transmission coefficients. The first echo from the leading edge of the inclusion is recovered very well by the boundary element method. Furthermore the transmitted secondary waves also are detected. From the geometry, it should arrive at roughly $t = \frac{2d}{c_1} = .22 \mu s$ after the first echo. What is probably observed is the second and third transmitted waves. The first may have been negated since the transmitted wave has undergone a mode conversion with respect to the scattered signal from the front surface of the sphere. Since this is typically called a strong scatterer, the complicated nature of the interaction more than likely includes shear wave components being converted to
longitudinal waves through the interface as well as the inclusion of creep waves analogous to what is observed in the void. Analysis of what appears to be the second and third transmitted echos reveal the diameter of the inclusion as expected.

Figure 44 is a frequency domain comparison of the the 74 node 24 element BEM model with the analytic solution. An excellent agreement with the center frequency is attained along with the general characteristics of the lower and higher frequency values. The exclusion of the damping term resulted in an agreement with the center frequency along with no usable comparison of the BEM and analytic amplitudes due to the high condition numbers associated with the solution matrices.

In the absence of the damping term, the higher condition numbers are apparent in Figure 45 are clearly affecting the time domain solution seen in Figure 46. Without proper material damping or the presence of ultrasonic attenuation, the result is a constant ringing of the tin-lead inclusion.

![Figure 42: Spectrum of the incident wave for the tin-lead spherical inclusion.](image-url)
Figure 43: BEM-FFT time domain comparison of the tin-lead sphere.
Figure 44: BEM-FFT frequency domain comparison of the tin/lead sphere.
Figure 45: BEM-FFT condition number comparison for the tin/lead sphere.
Figure 46: Damping/no damping comparison.
Figure 47 compares the results from the 164 node 54 element quadratic quadrilateral model with the analytic solution. A close agreement to the 74 node 24 element model verifies a convergence with the increase in discretization. However, the computation time necessary for the "large" model was on average 4 times that of the 74 node 24 element model. The smaller model delivered excellent results and clearly the increase in CPU time is of practical concern.

Figure 48 represents the frequency domain of the scattered wave from the large model indicating a closer agreement to the analytical solution compared to that of the smaller model.

Laplace domain solutions were also obtained using Durban's method of Laplace inversion to obtain time domain BEM results for the tin-lead inclusion and are illustrated in Figure 49. The scattered signal is virtually indistinguishable from that obtained for the same model in the Fourier domain with the noise preceding the scattered wave being reduced.

![Figure 47: BEM-FFT 164 node/54 quadratic quadrilateral element model comparison.](image)
Figure 48: BEM-FFT frequency domain comparison for the 164 node/54 element model.
Figure 49: BEM-LAP comparison for the tin-lead sphere.
Polystyrene Sphere Imbedded in Buehler's Transoptic

The following analysis represents the boundary element solutions for the spherical polystyrene weak scatterer imbedded in a thermoplastic disk. In Figure 50 illustrates that the center $ka$ of the inclusion is near 10 indicating that the boundary element formulation may pose difficulties in the attainment of a numerical solution and indeed this was the case. There will be no direct comparison with the experimental results as with the void and tin/lead inclusion due to the difficulties in attaining the experimental results. The resulting amplitudes differ significantly although the general wave form characteristics between the analytical and experimental results agree for the most part.

The BEM model included the ultrasonic attenuation term as it did with the tin/lead scatterer. The high $ka$ values posed a significant difficulty in producing a reasonable comparison with the analytical solution, although detection of the front and back surface echoes of the bead are detectable. This is yet a valuable result with respect to flaw sizing capabilities and the determination of the spherical flaw diameter. Figure 51 compares the 74 node 24 element quadratic quadrilateral element BEM model with the analytic solution. Figure 52 is the respective frequency domain comparison of the BEM model with the analytical results. The large sphere model is also addressed in figure 53 with the BEM solution compared with the analytical solution. Finally, the Laplace domain/Durbans method results for the 74 node 24 element model are illustrated in Figure 54. As with the previous Laplace and Fourier domain BEM results, there is a slight advantage in choosing the Laplace transform space for the intermediate calculations since the spurrious noise preceding the scattered wave is significantly decreased. Figure 55 compares the solution matrix condition numbers for the 74 node 24 element models evaluated in the Fourier and Laplace domains.
Figure 50: Spectrum of the incident wave for the polystyrene sphere.

Figure 51: BEM-FFT time domain comparison of the polystyrene sphere.
Figure 52: BEM-FFT frequency domain comparison of the polystyrene sphere.
Figure 53: BEM-FFT 164 node/54 element quadratic quadrilateral model comparison.
Figure 54: BEM-LAP comparison for the polystyrene sphere.
Figure 55: BEM-FFT condition number comparison of the polystyrene sphere.
DISCUSSIONS AND CONCLUSIONS

The objective of this research was to minimize the time required to obtain time domain responses of ultrasonic scattering from various flaw types utilizing the boundary element method. Presented was a numerical formalism to evaluate the scattered elastodynamic field from a void and two spherical inclusion types driven by a Gaussian incident transducer pulse. A mathematical model was developed to approximate the domain around the scattering obstacles as an infinite exterior region. The integral equation formulation of the elastic field is simplified by using the Sommerfeld radiation condition at the infinite boundary. Also, assumptions were made on the incident wave by approximating its behavior at an appreciable distance from the driving transducer by treating it as a plane wave. Analytical solutions were also developed for the spherical void and spherical inclusion in order to obtain time domain solutions for a Gaussian incident wave. Furthermore, experiments were performed to test the validity of the boundary element results.

The early mathematicians were motivated to mathematically explain the physical world around them. Newton, Leibniz, Gauss, Navier, etc., all developed mathematical premises based on, and correlated, with experimental procedures. Like them, I feel it is important to also have knowledge of the experimental procedures that the mathematics has modeled. Hence, experimental results were also obtained for the three scattering obstacles.

The sphere model which gave the overall best results was the 24 element quadratic quadrilateral, 74 node sphere utilizing a seventh order Gaussian quadrature on the boundary elements. It captured the behavior of the scattered waves appreciably better than two other "smaller" models tested, the 8 element 18 node model comprised of quadratic triangular elements and a 16 element 29 node quadratic-triangular element model. The highly oscillatory behavior of the real and imaginary parts of the displacement and tractions on the boundary
creates a high node to node variation in the solution in which the smaller models have difficulty capturing.

Furthermore, a 54 element 164 node model also comprised of quadratic quadrilateral elements was also tested. The added computation time necessary for this model did not admit a significant improvement in the solution, although it signified a relative convergence to the solution, most noticeably in the void and tin/lead scatterers. A moderate improvement was attained for the polystyrene sphere, although the high ka numbers associated with the scatterer still present a difficulty. Comparing the Fourier and Laplace domain results, the Laplace domain did eliminate some of the spurrious noise that preceded the arrival of the scattered wave although in general the scattered signals were identical.

The weak scatterer presented the most difficulty in obtaining a reasonable correlation between the analytical and boundary element solution. A mesh refinement beyond the 164 node 54 element model would seem appropriate, but the more logical solution would be to choose a transducer with a smaller center driving frequency. This does illustrate one of the disadvantages of the boundary element method: poor resolution at high ka values for scattering problems in the frequency domain.

There exists a concern over the direct comparison of the scattered wave amplitudes obtained theoretically verses those obtained experimentally. Since the planar fluid-solid interface was handled by reflection/transmission coefficients, much of the complicated interaction that occurs when the reflected wave from the scattering obstacle passes through the interface is ignored. It may be coincidence that the actual scattering amplitudes compare well for the void and tin/lead scatterer. This coincidence becomes apparent when a comparison between the scattering amplitudes is made for the polystyrene sphere. Nonetheless, the important observation is the comparison of the general wave form characteristics between the BEM and the experiment. Further refinement of the BEM model toward a submerged flaw
with the incorporation of a fluid-solid interface is warranted for a greater confidence and accuracy.

The boundary element code developed for this research is modular in nature and easily adaptable to a wide variety of applications. One needs only to change the fundamental solution in order to solve to effectively solve two dimensional elastodynamic scattering or a variety of scalar valued acoustic scattering problems. Furthermore, the original source code could be adapted to handle the fluid-solid interface, in particular, a curved interface which is a topic of current research interests by many in the boundary element field. Analogous refinements are also easily made to allow the code to solve a wide variety of stress analysis applications.

Further applications of the BEM modeling of ultrasonic scattering may be to develop neural network algorithms in conjunction with the BEM for the purpose of flaw identification and sizing.
REFERENCES


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Finally, I would like to acknowledge the love and support I have received from my parents, Jerome and Joan, over the years. They have provided me a cornerstone in which to turn when many changes and new experiences have taken place. Most importantly, I wish to dedicate this work to Becky, who has been there to support me throughout this final hurdle.
APPENDIX A
PROOF OF THEOREM 2.1

Proof of Theorem 2.1:

The proof of uniqueness of solution will be by contradiction. Hence, suppose that two solutions exist, given by \( u'_i, u''_i, t'_i, t''_i \) having the same initial and boundary conditions. Then, linear combinations of the two solutions are also solutions and one can write:

\[
\begin{align*}
  u_i &= u'_i - u''_i, \\
  t_i &= t'_i - t''_i, \\
  \sigma_i &= \sigma'_i - \sigma''_i, \\
  \varepsilon_i &= \varepsilon'_i - \varepsilon''_i, \\
  f_i &= f'_i - f''_i
\end{align*}
\]  

Since all of (A.1) are solutions to (2.2) conservation of energy holds for the system. In other words, the time rate of change of the kinetic and potential energy is equal to the work done upon the body by the external forces per unit time and all other energies per unit time[44]. This can be written as

\[
K + \dot{E} = W
\]  

where \( K \) is the kinetic energy defined by

\[
K = \int_V \frac{1}{2} \rho u_i^2 \, dV
\]  

\( E \) is the internal energy defined by

\[
E = \int_V \varepsilon \rho \, dV
\]

with \( \varepsilon \) the internal energy per unit mass. The internal energy function \( \varepsilon \) is a function of the strain and since the essence of the formulation is that the body is a perfectly linear elastic
isotropic solid and hence has no dissipative mechanisms, one has in absence of thermal strains, 

$$W = \frac{\varepsilon}{\rho} = \varepsilon(\varepsilon_{ij}) = \frac{1}{2} \lambda (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + \frac{\mu(3\lambda + 2\mu)}{2(\lambda + \mu)(1+\mu)} (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$$  \hspace{1cm} (A.5)$$

through the constitutive equation derivation with $\varepsilon_i$, $i=1,2,3$ denoting the principal strains. The work done by the external forces is

$$W = \int_S t_i \dot{u}_i \, dS + \int_V \rho f_i \dot{u}_i \, dV$$  \hspace{1cm} (A.6)$$

This gives a statement that the rate of change of kinetic and potential energy equals the work done per unit time by the system:

$$\frac{D}{Dt} \int_V \left[ \frac{1}{2} \rho \dot{u}_i \dot{u}_i + U \right] \, dV = \int_S t_i \dot{u}_i \, dS + \int_V \rho f_i \dot{u}_i \, dV$$  \hspace{1cm} (A.7)$$

Now integrating the time variable in (A.7)

$$\int_V \left[ \frac{1}{2} \rho \dot{u}_i^2 + U \right] \, dV = \int_{t_0}^t \int_S t_i \dot{u}_i \, dS \, dt + \int_{t_0}^t \int_V \rho f_i \dot{u}_i \, dV \, dt$$  \hspace{1cm} (A.8)$$

In the above equation, one notes that specification of the surface tractions $t_i$ and the time derivatives $u_i$ of displacement at an initial time $t_0$ is implied by the lower limits of integration in the first integral on the right hand side. Furthermore, the second integral contains time derivatives of displacement and body forces $f_i$. 

starting from identical conditions on S and in V. From the assumptions made in (A.1) regarding the existence of two solutions, it follows that the right hand side of (A.8) is zero if the boundary conditions are the same. So

\[
\int_V \left[ \frac{1}{2} \rho \dot{u}_i^2 + U \right]_{t=0}^t \, dV = 0 \quad (A.9)
\]

or expanding the integral into kinetic and internal energy,

\[
K + E = K_0 + E_0 \quad (A.10)
\]

But \(K_0\) and \(E_0\) are zero since they are based on the initial velocities and displacements of the difference system. Thus \(K + E = 0\). Clearly \(K\) is positive definite and \(E\) is by virtue of the fact that the elastic constants \(\mu > 0\) and \(3\lambda + 2\mu > 0\). Hence, \(\dot{u}_i = 0\). Since \(u_i(x,0) = 0\), then \(u_i = 0\) and \(u_i' = u_i''\) contradicting the assumption.

Sufficient conditions for uniqueness of solution are the properly prescribed boundary conditions. Disappearance of the left surface integral below

\[
\int_{S_0}^t \int_S t_i \mathrm{d}S \mathrm{d}t \quad \int_{S_0}^t \int_V \rho f_i \dot{u}_i \, dV \mathrm{d}t \quad (A.11)
\]

is based on specifying the surface conditions and from

\[
t_i \dot{u}_i = t_n u_n + t_s u_s + t_b u_b \\
= \sigma_{nn} u_n + \sigma_{ns} u_s + \sigma_{nb} u_b \\
\]

(A.12)

these conditions must be \(\sigma_{nn}, \sigma_{ns}, \sigma_{nb}, \) or \(u_i\) or a proper mix of these values. Furthermore, \(\sigma_{ij}\) or \(u_i\) at \(t = t_0\) must also be specified. For the right integral in (A.11) to disappear, specification of \(f_i\) in \(D\) is required as well as the initial conditions of \(u_i\) and \(\dot{u}_i\) at \(t = t_0\).
APPENDIX B
ELASTODYNAMIC FUNDAMENTAL SOLUTION: FOURIER DOMAIN

\[ F^{D}_{km}(\omega, x, \xi) = \frac{1}{4\pi \mu K^2_T} \left( \frac{\delta_{km}}{r^2} \left[ (1 - iK_T) e^{iK_T r} - (1 - iK_T - K^2_T r^2) e^{iK_T r} \right] \right) \]

\[ \left\{ - \frac{r_{ik} r_{jm}}{r^2} \left[ (3 - i3K_T - K^2_T r^2) e^{iK_T r} \right] \right\} \]

The F kernel as it appears in the subroutine SKERN:

\[ F_{ij} = \frac{1}{4\pi \mu r} \left( \delta_{ij} F^{(1)} + r_{ij} F^{(2)} \right) \]

\[ F^{(1)} = e^{iK_T r} + \frac{1}{K^2_T} \left[ \frac{1}{r^2} \left( e^{iK_T r} - e^{iK_L r} \right) + \frac{1}{r} \left( K_T e^{iK_T r} - K_L e^{iK_L r} \right) \right] \]

\[ F^{(2)} = \frac{1}{K^2_T} \left[ \frac{3}{r^2} \left( e^{iK_T r} - e^{iK_L r} \right) - \frac{3}{r} \left( K_T e^{iK_T r} - K_L e^{iK_L r} \right) \right] \]

The Dynamic T-Kernel is

\[ G^{D}_{ij}(\omega, x, \xi) = \frac{1}{4\pi} \left( \delta_{km} \left[ a_1(K_T, r) + \frac{2}{K^2_T} \left( a_2(K_T, r) - a_1(K_L, r) \right) \right]_{r_{ik} n_k} \right) \]

\[ \left[ \frac{\lambda K^2_T}{\mu K^2_T} a_1(K_L, r) + \frac{2}{K^2_T} \left( a_2(K_T, r) - a_1(K_L, r) \right) \right]_{r_{ij} n_j} \]

\[ + \left[ a_1(K_T, r) + \frac{2}{K^2_T} \left( a_2(K_T, r) - a_1(K_L, r) \right) \right]_{r_{ij} n_j} \]

\[ \left\{ - \frac{2r^2}{K^2_T} (a_3(K_T, r) - a_3(K_L, r)) r_{ij} r_{ik} n_i \right\} \]

where \( r = |x - \xi| \), and
\[ a_1(K,r) = \left( \frac{iK}{r} - \frac{1}{r^2} \right) e^{iKr} \]
\[ a_2(K,r) = \left( \frac{K^2}{r^2} - \frac{3iK}{r^3} + \frac{3}{r^4} \right) e^{iKr} \]
\[ a_3(K,r) = \left( \frac{iK^2}{r^3} + \frac{6K^2}{r^4} + \frac{15iK}{r^5} - \frac{15}{r^6} \right) e^{iKr} \]

for \( K = K_L, K_T \) with \( K_L = \frac{\omega}{c_L}, K_T = \frac{\omega}{c_T} \).
APPENDIX C
ELASTODYNAMIC FUNDAMENTAL SOLUTION: LAPLACE DOMAIN

\[ F_{k,m}(\omega, x, \xi) = \frac{1}{4\pi \mu K_T^2} \left[ \frac{\delta_{km}}{r^2} \left( 1 + K_J e^{\frac{r}{r^2}} - (1 + K_T + K_T^2 e^{\frac{r}{r^2}}) \right) \right] \]

\[ - \frac{r_k r_m}{r^2} \left[ (3 + 3K_J + K_J^2 e^{\frac{r}{r^2}}) e^{\frac{r}{r^2}} \right] \]

\[ \left\{ - \left( 3 + 3K_T + K_T^2 e^{\frac{r}{r^2}} \right) e^{\frac{r}{r^2}} \right\} \]

\[ G_{ij}(\omega, x, \xi) = \frac{1}{4\pi} \delta_{km} \left[ \left( a_1(K_T, r) + \frac{2}{K_T^2} (a_2(K_T, r) - a_1(K_L, r)) \right) r_k \right] \]

\[ + \left[ \frac{\lambda K_T^2 a_1(K_L, r)}{\mu K_T^2} + \frac{2}{K_T^2} \left( a_2(K_T, r) - a_1(K_L, r) \right) \right] \]

\[ \left\{ a_1(K_T, r) + \frac{2}{K_T^2} (a_2(K_T, r) - a_1(K_L, r)) \right\} r_j n_i \]

\[ + \frac{2r^2}{K_T^2} \left( a_3(K_T, r) - a_3(K_L, r) \right) r_j r_k n_k \]

where \( r = |x - \xi| \), and

\[ a_1(K_T, r) = \left( \frac{K_T}{r} + \frac{1}{r^2} \right) e^{\frac{r}{r^2}} \]

\[ a_2(K_T, r) = \left( \frac{K_T^2 + 3K_T}{r^2} + \frac{3}{r^4} \right) e^{\frac{r}{r^2}} \]

\[ a_3(K_T, r) = \left( \frac{K_T^3 - 6K_T^2 + 15K_T}{r^4} \right) e^{\frac{r}{r^2}} \]

for \( K = K_L, K_T \) with \( K_L = \frac{s_L}{c_L}, K_T = \frac{s_T}{c_T} \).
Expanding the exponentials in each of the F and G dynamic kernels via Taylor series gives

\[ F_{k,m}^{D}(\omega, x; \xi) = \frac{1}{4\pi \mu} \left\{ \delta_{km} \left[ \frac{1}{2r} (1+K^2) + \frac{1}{3} (K^2 K_L + 2K_T) \right] ight. \]
\[ + \left. \frac{1}{2r} (1 - K^2)r_k r_m + O(r) \right\} \]

As \( r \to 0 \) where \( K = \frac{K_L}{K_T} \). This is precisely the static fundamental solution along with a constant term

\[ F_{k,m}^{D}(\omega, x; \xi) \to F_{k,m}^{S}(\omega, x; \xi) + F_{k,m}^{C}(\omega, x; \xi) \]

where

\[ F_{k,m}^{S}(\omega, x; \xi) = \frac{1}{8\pi \mu r} \left\{ \delta_{km} \left( 1 + K^2 \right) + (1 - K^2)r_k r_m \right\} \]

\[ F_{k,m}^{C}(\omega, x; \xi) = \frac{1}{12\pi \mu} (K^2 K_L + 2K_T) \]

Proceeding in a similar manner with the T-Kernel where \( K = \frac{K_T}{K_L} \)

\[ G_{ij}^{D}(\omega, x; \xi) \to G_{ij}^{S}(\omega, x; \xi) + G_{ij}^{C}(\omega, x; \xi) \]

where

\[ G_{ij}^{S}(\omega, x; \xi) = \frac{-1}{4\pi r^2} \left\{ \delta_{km} K^2 + 3(1 - K^2) r_i r_j \right\} r_k n_k + K^2(r_i n_j - r_j n_i) \]
\[ G_{ij}(\omega, \mathbf{x}, \xi) = \frac{1}{16\pi} \left\{ \delta_{km}(K_i^2 + K_j^2) + (K_i^2 - K_j^2) r_{i,j} r_{j,i} \right\}_{r_i, n_i} \]

\[ \left\{ (K_i^2 - 2K^2) r_{i,j} n_j + (K_j^2 - 2K^2) r_{j,i} n_i \right\} \]
APPENDIX D
ASYMPTOTICS: LAPLACE DOMAIN

Expanding the exponentials in each of the F and G dynamic kernels via Taylor series gives

\[ F^{D}_{k,m}(\omega, x; \xi) = \frac{1}{4\pi \mu} \left[ \delta_{km} \left( \frac{1}{2r} (1 + K^2) + \frac{1}{3} (K^2 K_L + 2K_T) \right) \right] + \frac{1}{2r} (1 - K^2) r_k r_m + O(r) \]

As \( r \to 0 \) where \( K = \frac{K_L}{K_T} \). This is precisely the static fundamental solution along with a constant term

\[ F^{D}_{k,m}(\omega, x; \xi) \to F^{S}_{k,m}(\omega, x; \xi) + F^{C}_{k,m}(\omega, x; \xi) \]

where

\[ F^{S}_{k,m}(\omega, x; \xi) = \frac{1}{8\pi \mu r} \left\{ \delta_{km} (1 + K^2) + (1 - K^2) r_k r_m \right\} \]
\[ F^{C}_{k,m}(\omega, x; \xi) = \frac{1}{12\pi \mu} (K^2 K_L + 2K_T) \]

Proceeding in a similar manner with the T-Kernel where \( K = \frac{K^2}{K_T} \),

\[ G^{D}_{ij}(\omega, x; \xi) \to \frac{1}{4\pi r^2} \left[ \delta_{km} K^2 + 3(1 - K^2) r_i r_j \left( r_k n_k - K^2 r_i n_j + K^2 r_j n_i \right) \right] + \frac{1}{8\pi r} \left\{ \delta_{km} (K^2 + K^2) + (K^2 + K^2) r_i r_j \right\} r_k n_k \]
\[ + \frac{1}{16\pi} \left( [K^2 - 2K^2] r_i n_j + (K^2 - 2K^2) r_j n_i \right) \]

\[ G^{D}_{k,m}(\omega, x; \xi) \to G^{S}_{k,m}(\omega, x; \xi) + G^{C}_{k,m}(\omega, x; \xi) \]

where
\[ G_{ij}^S(\omega, x, \xi) = -\frac{1}{4\pi r^2} \left[ \delta_{km} K^2 + 3(1 - K^2) r_i r_j r_k n_k + K^2 (r_i n_j - r_j n_i) \right] \]

\[ G_{ij}^C(\omega, x, \xi) = -\frac{1}{16\pi} \left\{ \left[ \delta_{km} (K^2 + \bar{K}^2) + (K^2 - \bar{K}^2) r_i r_j r_k n_k \right] \right. \]
\[ \left. + \left[ (K^2 - 2K_2^2) + 3\bar{K}^2 \right] r_i n_j + (K^2 - 2\bar{K}^2) r_j n_i \right\} \]
APPENDIX E  
MATRIX COEFFICIENTS

\[ E_{11}^3 = n \, h_n(\alpha_{1r}) - \alpha_{1r} h_{n+1}(\alpha_{1r}) \]

\[ E_{11}^1 = n \, j_n(\alpha_{2r}) - \alpha_{2r} j_{n+1}(\alpha_{2r}) \]

\[ E_{12}^1 = n(n+1) h_n(\beta_{1r}) \]

\[ E_{12}^1 = n(n+1) j_n(\beta_{2r}) \]

\[ E_{21}^3 = h_n(\alpha_{1r}) \]

\[ E_{21}^1 = j_n(\alpha_{2r}) \]

\[ E_{22}^3 = -(n+1) \, h_n(\beta_{1r}) + \beta_{1r} h_{n+1}(\beta_{1r}) \]

\[ E_{22}^1 = -(n+1) \, j_n(\beta_{2r}) + \beta_{2r} j_{n+1}(\beta_{2r}) \]

\[ E_{31}^3 = -(n^2 - n \, \frac{\beta_{1r}^2}{2}) \, h_n(\alpha_{1r}) + 2 \alpha_{1r} h_{n+1}(\alpha_{1r}) \]

\[ E_{31}^1 = -(n^2 - n \, \frac{\beta_{2r}^2}{2}) \, j_n(\alpha_{2r}) + 2 \alpha_{2r} j_{n+1}(\alpha_{2r}) \]

\[ E_{32}^3 = -n(n+1) \left[ (n-1) \, h_n(\beta_{1r}) - \beta_{1r} h_{n+1}(\beta_{1r}) \right] \]

\[ E_{32}^1 = -n(n+1) \left[ (n-1) \, j_n(\beta_{2r}) - \beta_{2r} j_{n+1}(\beta_{2r}) \right] \]

\[ E_{41}^3 = (n-1) \, h_n(\alpha_{1r}) - \alpha_{1r} h_{n+1}(\alpha_{1r}) \]

\[ E_{41}^1 = (n-1) \, j_n(\alpha_{2r}) - \alpha_{2r} j_{n+1}(\alpha_{2r}) \]

\[ E_{42}^3 = -(n^2 - 1 - \frac{\beta_{1r}^2}{2}) \, h_n(\beta_{1r}) - \beta_{1r} h_{n+1}(\beta_{1r}) \]
$$E_{42} = -\langle n^2 - 1 \cdot \frac{\beta_{2t}^2}{2} \rangle j_n(\beta_{2t}) - \beta_{2t} j_{n+1}(\beta_{2t})$$