Solving, Estimating and Testing Nonlinear Stochastic Equilibrium Models

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Disciplines
Economic History | Economic Theory | Statistical Models | Taxation

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SOLVING, ESTIMATING AND TESTING
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Bong-Soo Lee

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August 1987
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of Chris Sims, Barry Falk, and two anonymous referees. All errors are my
own responsibility.
1. Introduction

Recently, many economists have tried to explain empirical regularities of interest using various kinds of stochastic equilibrium frameworks. When the model is formulated as a linear quadratic framework, closed form solutions can be derived by making use of the certainty equivalence principle. For nonlinear quadratic frameworks, other solution methods have been also suggested. One of the problems with the solution methods proposed for nonlinear quadratic models has been the development of a systematic econometric treatment of the estimation of the parameters of such models. Generalized method of moments (GMM) procedures (Hansen (1982)) have been applied by many economists [Hansen and Singleton (1982), Hansen and Sargent (1980), etc.]. GMM procedures are very attractive because they do not require either a complete, explicit representation of the economic environment or a complete system of equations generating solution paths. An economist, however, may want to use the solution paths to nonlinear equilibrium models to make forecasts or to organize a broad range of dynamics of the model and to treat a goodness of fit of the model in a more rigorous and systematic fashion. Several methods are now available to generate complete solution paths to the nonlinear equilibrium models at a relatively low cost. In this context, estimation of the parameters of the complete stochastic equilibrium model by simulation was suggested by Lee (1986) [Lee and Ingram (1986)] by extending GMM ideas.

Its basic idea can be explained as follows. Suppose an economist wishes to explain some empirical regularities (i.e., stylized facts) organized by some statistics by proposing a stochastic equilibrium model. The theoretical economic model can be used to generate solutions for the equilibrium time paths of the variables of interest by, for example, simulation methods.
Then the same kind of statistics can be calculated for the data generated from the model. The statistics of the model depend on various parameter values of the model. Estimation by simulation suggests finding a set of parameter values which minimizes an appropriately defined distance between the two statistics: one from the real data and the other from the simulated data of the model. One important feature of these estimators is that they have a limiting normal distribution under some regularity assumptions about stochastic processes generating the observed time series. Also, when more statistics are to be calculated for estimation than there are parameters to be estimated, the overidentifying statistics can be used to test the implications of the model.

However, an economic example which implements and discusses the various aspects of solving, estimating, and testing of nonlinear stochastic equilibrium models using this procedure is not available in the literature. On the other hand, there have been many attempts to explain the observed negative correlation between inflation and asset returns (stock returns and interest rates) using other kinds of theoretical frameworks [Fama (1981,1983), Geske and Roll (1983), Ram and Spencer (1983), Stulz (1986)]. The purpose of this paper is to describe and implement an econometric strategy taking account of a broad range of dynamics of the model in solving, estimating and testing a nonlinear stochastic equilibrium model with an example of the asset returns and inflation relationship.

In section 2 of this paper, empirical regularities concerning the relationship between inflation and asset returns are reported by using cross correlations and a vector autoregression (VAR) analysis. In section 3, a stochastic nonlinear equilibrium model is suggested, which purports to explain the empirical regularities in section 2. In section 4, solution paths are generated for this nonlinear equilibrium model through a backwards
mapping method that does not require any quadratic approximation procedures. In section 5, parameter values of the model are estimated by simulation and overidentifying restrictions of the model are tested by extending GMM ideas. The simulation results are examined to provide some suggestions for further improvements of the model. Section 6 concludes the paper.

2. Empirical relationships

Empirical relationships between inflation and asset returns are organized based on a VAR analysis and cross correlations to avoid unreliable restrictions from the structural model and to capture a broad range of dynamics of the relationships. With a VAR representation, forecasts of the time series variables (e.g., expected inflation rate) can be easily calculated, and causal relations and dynamic interactions among the variables can be investigated as by-products of the analysis. In section 5, the VAR coefficients will be used as a criterion of the comparison between the real data and the simulated data from the model.

The notations used are (for the description and source of the data, see Appendix A);

IR : rate of interest (nominal), IRR : real rate of interest (ex ante),
SR : return on the common stocks (nominal), SRR : real return on the common stocks,
SP : nominal stock price index, SPR : real stock price index,

We assume agents' expectations of future inflation are rational and identify the projection of future inflation on current observables with the agents' expectations. Thus the expected price levels are computed by
\[ E_t P_{t+1} \approx \text{Proj.} (CPI_{t+1}, CPI_{t-s}, MB_{t-s}, IR_{t-s}, IP_{t-s}; s=0,\ldots,5) \]

where \( E_t \) denotes a mathematical expectation conditional on information available at time \( t \), Proj. is the linear least squares projection operator on the linear space spanned by current and past variables, and the projection is based on the four variable (CPI, MB, IR, IP) VAR system with a constant and six lags. These four variables are chosen because they are conjectured to help predict future prices [see Litterman and Weiss (1985)].

Using the expected price level, the expected inflation rate \( E\text{INF}_t = (E_t P_{t+1}/P_t)-1 \) is computed, and by subtracting this \( E\text{INF}_t \) from \( IR_t \) and \( SR_t \), the ex ante real interest rate, \( IRR_t \), and the real rate of return on the common stock, \( SRR_t \), are calculated.

Some of the statistics and cross correlations that were computed from quarterly data (47,1-84,4) are reported in Tables 1 and 2. From these tables, the following is observed:

1. \( IRR \) and \( INF \) are negatively correlated for all the lags and the leads;
2. \( SRR \) and \( INF \) are negatively correlated for most of the lags and the leads;
3. \( IRR \) and \( SRR \) are (weakly) positively correlated;
4. \( SPR \) and \( INF \) are negatively correlated; and,
5. \( SRR \) is on average much higher than \( IRR \), and is much more volatile in its variations compared with \( IRR \).

In order to further investigate statistical regularities such as causal realtionships and dynamic interactions among the variables of interest in the sample period 47,1-84,12, a VAR analysis- in particular, impulse responses and error decompositions- is implemented. The innovation accounting, which provides a basis for the error decompositions and impulse responses is well explained in Sims (1980, 1978). In this section, a four variable
Table 1. Summary Statistics of Interest Rates, Stock returns, and Inflation (Quarterly data: 55,1-84,4)

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRR</td>
<td>.2902486</td>
<td>.5961346</td>
</tr>
<tr>
<td>SRR</td>
<td>1.461653</td>
<td>8.182738</td>
</tr>
<tr>
<td>INF</td>
<td>1.166300</td>
<td>.9327624</td>
</tr>
</tbody>
</table>

Note: IRR = real rate of interest, SRR = real return on the common stocks, INF = rate of inflation.

Table 2. Cross Correlations (Quarterly data: 55,1-84,4)

<table>
<thead>
<tr>
<th>x(t) y(t-s)</th>
<th>s=-6</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRR INF</td>
<td>-.37</td>
<td>-.39</td>
<td>-.47</td>
<td>-.51</td>
<td>-.46</td>
<td>-.48</td>
<td>-.50</td>
<td>-.39</td>
<td>-.42</td>
<td>-.32</td>
<td>-.14</td>
<td>-.13</td>
<td>-.09</td>
</tr>
<tr>
<td>SRR INF</td>
<td>-.10</td>
<td>-.12</td>
<td>-.14</td>
<td>-.11</td>
<td>-.08</td>
<td>-.30</td>
<td>-.31</td>
<td>-.12</td>
<td>-.10</td>
<td>-.09</td>
<td>.04</td>
<td>.03</td>
<td>.00</td>
</tr>
<tr>
<td>IRR SRR</td>
<td>.03</td>
<td>.10</td>
<td>.06</td>
<td>.13</td>
<td>.01</td>
<td>.01</td>
<td>.11</td>
<td>.12</td>
<td>.16</td>
<td>.20</td>
<td>.06</td>
<td>.04</td>
<td>.12</td>
</tr>
<tr>
<td>SPR INF</td>
<td>-.04</td>
<td>-.08</td>
<td>-.12</td>
<td>-.14</td>
<td>-.17</td>
<td>-.23</td>
<td>-.30</td>
<td>-.38</td>
<td>-.42</td>
<td>-.46</td>
<td>-.48</td>
<td>-.48</td>
<td>-.49</td>
</tr>
</tbody>
</table>

Note: SPR = real stock price index (common stock price index / CPI).

Table 3. Four variable Innovation Accounting

—Percentage of 4 (8) quarter forecast error variance—

<table>
<thead>
<tr>
<th>Variables</th>
<th>Standard Error $^2$</th>
<th>SRR</th>
<th>IRR</th>
<th>INF</th>
<th>IPL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explained</td>
<td>By Innovations in</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SRR</td>
<td>7.6(7.9)</td>
<td>86.3(81.7)</td>
<td>3.6 (4.3)</td>
<td>6.2 (8.1)</td>
<td>3.9 (5.9)</td>
</tr>
<tr>
<td>IRR</td>
<td>.4(.5)</td>
<td>8.0 (10.3)</td>
<td>81.6(76.5)</td>
<td>6.9 (7.7)</td>
<td>3.6 (5.4)</td>
</tr>
<tr>
<td>INF</td>
<td>.5(.6)</td>
<td>10.6 (8.5)</td>
<td>15.5(26.4)</td>
<td>71.0(56.2)</td>
<td>2.9 (9.0)</td>
</tr>
<tr>
<td>IPL</td>
<td>.0(.1)</td>
<td>38.1(27.2)</td>
<td>.8 (1.7)</td>
<td>7.3(50.5)</td>
<td>53.8(20.6)</td>
</tr>
</tbody>
</table>

Note: IPL = ln(IP), where IP is industrial production.

1. Since the time period 47,1-54,4 is used for the computation of EINF $^t$, statistics are computed for the period 55,1-84,4.
2. Standard error reports the forecast standard error over two different forecasting horizons when sampling error in the estimated coefficients is ignored.
Figure 1. Dynamics of the four variable system (Quarterly data)

REsPOnSES TO SHOCK IN SNA

RESPOnSES TO SHOCK IN IAB

RESPOnSES TO SHOCK IN IHF

RESPOnSES TO SHOCK IN IP
VAR system - SRR, IRR, INF, IPL - with a constant and six lags is employed for the analysis. The four variable VAR was initially estimated with lag lengths of twelve and six, and then the former was tested as a restriction of the latter. The \( \chi^2(96) = 97.8851 \) and the significance level was .4273. Therefore, the shorter lag length was accepted and used. In addition to INF and two asset returns (IRR and SRR), IP is included as a measure of the general real economic activity.

From Table 3 and Figure 1, the following is noted.

1. SRR appears to behave as if it is Granger-causally prior in the sense that most of its forecast error variance is accounted for by its own innovations. SRR appears to explain 38 (27) percent of 4 (8) quarter forecast error variance in IP. The response of IP to innovations (or disturbances) in SRR is consistently positive.

2. IRR also appears to behave as if it is Granger-causally prior in the same sense as described above (even though it is not that strong as SRR). IRR explains 16 (26) percent of 4 (8) quarter forecast error variance in INF. The response of INF to innovations in IRR are consistently negative.

3. A stochastic equilibrium model

In order to explain some of the important empirical regularities concerning asset returns and inflation relation in section 2, suppose that the following stochastic equilibrium model with production in which money is introduced through a cash-in-advance constraint is suggested. The model consists of a single consumer, a firm and the government. The representative consumer maximizes the expected utility subject to his budget constraint. The firm rents capital from the consumer in each period to maximize profit and distributes the profit (dividends) to the shareholder (i.e.,
The government's role is to finance a given path of its spending, $G_t$, by collecting a lump sum tax, $T_t$, by issuing one-period real bonds, $B_{t+1}$, and by collecting seignorage. The shocks to technology, $Z_t$, government expenditure, $G_t$, and the growth rate of the money supply, $v_t$, are assumed to be exogenous stochastic processes. In order to enforce cash holdings by agents, the following two restrictions are imposed. First, dividends and rentals are assumed to be paid in cash at the end of each period so that they are carried over to the next period to be spent. Second, the government bonds are assumed to be indexed bonds that can not be used to purchase either consumption goods or investment goods directly.

The consumer's decision is to allocate his budget among various assets subject to a cash-in-advance constraint to maximize expected utility

$$\max_{\mathcal{B}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t U(C_t), \quad U' > 0, \quad U'' < 0, \quad (3.1)$$

subject to

$$\left( \frac{M_{ct}}{P_t} \right) + T_t + SPR_t S_t + \left( \frac{B_{t+1}}{1 + IRR_t} \right) \leq S_{t-1} \left[ SPR_t + \left( \frac{P_{t-1}}{P_t} \right) D_{t-1} \right] +$$

$$\left( \frac{P_{t-1}}{P_t} \right) K_{t-1} u_{t-1} + B_t,$$

and

$$P_t \left( C_t + IV_t \right) \leq M_{ct}, \quad (3.3)$$

where $\beta$ is the subjective time discount factor and $0 < \beta < 1$, $C_t$ is per capita real consumption, $P_t$ is the price level at $t$, $M_{ct}$ is the amount of currency held by the consumer. $T_t$ can be paid in cash, government bonds, $B_t$, or private shares $S_t$. $B_{t+1}$ is the amount of government issued one-period indexed bonds paying $B_{t+1}$ sure units of consumption goods in period $t+1$. $S_t$ is the amount of shares purchased at $t$, $SPR_t$ is the price of one share of the firm, $D_{t-1}$ is the dividend (or profit) made in period $t-1$, $u_{t-1}$ is
the rental rate on an unit of capital leased to the firm and received by the consumer in period $t-1$. The first restriction mentioned above explains why $P_{t-1}D_{t-1}$ and $P_{t-1}K_{t-1}u_{t-1}$ are divided by $P_t$. $K_t$ is the capital stock owned by the consumer and leased to the firm at the beginning of period $t$, and IRR$_t$ is the real interest rate used to discount the government bonds. Investment at time $t$, $IV_t$, is given as

$$IV_t = K_{t+1} - K_t.$$  \hfill (3.4)

The firm is assumed to hire $K_t$ in period $t$ to maximize dividend (or profit), which is defined as

$$D_t = Y_t - u_tK_t,$$  \hfill (3.5)

where $f(K_t,Z_t)$ is the production function, $Z_t$ is a technology shock which realizes at the beginning of the period $t$, and is generated by a stochastic process. The government has budget constraint

$$G_t - T_t = B_{t+1}/(1+IRR_t) - B_t + (M_t-M_{t-1})/P_t,$$  \hfill (3.7)

where $M_{t-1}$ is the total supply of currency which the consumer has carried over from $t-1$. The government increases (or reduces) $M_t - M_{t-1}$ units of currency to finance its expenditure which is not covered by tax and bond issues. The government is also subject to the cash-in-advance constraint

$$G_t, P_t < M_{gt}.$$  \hfill (3.8)

The equilibrium condition in the currency market and goods market will be

$$M_t = M_{ct} + M_{gt},$$  \hfill (3.9)
$$Y_t = f(K_t,Z_t) = C_t + IV_t + G_t,$$  \hfill (3.10)
$$M_{t+1} = (1+\omega_t)M_t.$$  \hfill (3.11)

The equilibrium price level $P_t$ is determined from the cash-in-advance constraints (3.3), (3.8), (3.9), and (3.10)

$$M_t/P_t = (M_{ct} + M_{gt})/P_t = C_t + IV_t + G_t = Y_t,$$  \hfill (3.12)
or

\[ P_t = M_t / Y_t, \quad (3.13) \]

which gives the unit income velocity of the quantity theory of money.

The first order necessary conditions (FOC) for the equilibrium of this model can be obtained by the stochastic Lagrangian multiplier method:

\[ f'(K_t, Z_t) = u_t, \quad (3.14) \]

\[ D_t = Y_t - u_t K_t, \quad (3.15) \]

\[ U'(C_t) = \lambda_t, \quad (3.16) \]

where \( \lambda_t \) is a random lagrangian multiplier.

\[ \lambda_t = \beta E_t \lambda_{t+1} + \beta^2 E_t \left[ \lambda_{t+2}(P_{t+1}/P_{t+2})u_{t+1} \right], \quad (3.17) \]

\[ \lambda_t \text{ SPR}_t = \beta E_t \left[ \lambda_{t+1} (\text{SPR}_{t+1} + (P_t D_t)/P_{t+1}) \right], \quad (3.18) \]

\[ \beta E_t [\lambda_{t+1}/\lambda_t] = (1 + \text{IRR}_t)^{-1} < E_t [P_{t+1}/P_t], \quad (3.19) \]

\[ T_t + C_t + IV_t + \text{SPR}_t S_t + B_{t+1}/(1 + \text{IRR}_t) = S_{t-1} [\text{SPR}_t + (P_{t-1} D_{t-1})/P_t] + \]

\[ (P_{t-1} K_{t-1} u_{t-1})/P_t + B_t, \quad (3.20) \]

\[ G_t - T_t = B_{t+1}/(1 + \text{IRR}_t) - B_t + (M_t - M_{t-1})/P_t, \quad (3.21) \]

\[ K_{t+1} = K_t + IV_t, \quad (3.22) \]

\[ f(K_t, Z_t) = Y_t = C_t + IV_t + G_t, \quad (3.23) \]

\[ M_t / P_t = f(K_t, Z_t) = Y_t, \quad (3.24) \]

At this point, it is noted only that the functional form of the utility function \( U(.) \) and the production function \( f(.) \) need not be quadratic, and will be parameterized later.
4. Solving the model by simulation

4.1 Non-quadratic property

If the model took the form of a linear quadratic set-up in which economic agents are assumed to solve a quadratic objective function subject to linear constraints, then linear optimal decision rules can be derived, and a complete characterization of the equilibrium solution paths in the form of linear difference equations would be available. This is due to the certainty equivalence principle (or separation principle) which allows the problem to be separated into a non-stochastic optimization problem and a forecasting problem [Simon (1956), Theil (1964). For recent examples, see Sargent (1979), Hansen and Sargent (1980)].

For nonlinear quadratic models, the certainty equivalence does not apply, and an alternative, computationally practical method which provides equilibrium solution paths needs to be found. One way to handle this problem is to use the quadratic approximation method suggested by Kydland and Prescott (1982). This method approximates the nonquadratic objective function by a quadratic function in the neighborhood of the model’s steady state at the cost of possible approximation errors. Then the model reduces to a linear-quadratic framework, and a certainty equivalence principle applies [See Hansen (1985). Fair and Taylor (1980) also imposed certainty equivalence on the nonlinear rational expectations model]. A second approach is to approximate the nonlinear Euler equations successively based on a contraction mapping theorem [Labadie (1984) and Sargent (1984)]. A third approach to the nonlinear-quadratic model is the so-called backwards mapping method suggested by Novales (1983) and Sims (1985). According to Sims (1985), an economist can take a more symmetric view of endogenous variables
and exogenous (or forcing) variables than engineers. Therefore, one can take arbitrary stochastic processes for some of the endogenous variables to determine the corresponding stochastic processes for the exogenous variables and remaining endogenous variables from the model as opposed to the standard engineering solution method which only allows for the arbitrary specification of exogenous processes and works in the reverse direction. The restriction in this case is that the number of variables whose stochastic processes are to be assumed should be equal to the number of exogenous variables of the model. Allowing for a backwards mapping from controlled processes to forcing processes often turns out to be much easier in computing the conditional expectations of the variables in the model than the traditional approach, and it is sensible because an economist may observe data on both processes and may be interested in the mapping between them rather than in the direction of the mapping. In particular, this method generates mutually consistent joint processes for exogenous disturbances and choice variables without going through any approximation procedure.

Specifically, its implementation, in general, takes the following steps. From the dynamic optimization problem, a set of nonlinear Euler equations is derived. Next, assume some form of the stochastic processes (e.g., AR form or ARMA form) for the variables whose conditional expectations are to be taken so that the conditional expectations can be computed using a prediction formula (e.g., the Wiener-Kolmogorov prediction formula). Then the remaining variables, either control variables or forcing variables, are expressed in terms of the variables whose processes are assumed by making use of a set of Euler equations along with budget constraints and other constraints. Through simulations, we can generate realizations of the entire set of variables as a joint stochastic process which is consistent with the optimal solution of the model taking the processes of exogenous
disturbances as given. Because this method does not require any approximation procedures, it seems more appropriate to take this approach when a VAR analysis is used as a criterion of a goodness of fit of the model. In addition, considering that the VAR approach does not impose strict exogeneity restrictions on the variables in the system, employing a backwards mapping method taking a symmetric view of endogenous and exogenous variables appears to be appropriate.

4.2. Generating data by a backwards mapping method

We take a backwards mapping method assuming some stochastic processes for the variables whose conditional expectations are included in the first order conditions (FOC) of an equilibrium. Because there are three exogenous processes (i.e., \( Z_t \), \( G_t \) and \( W_t \)) in the model, we introduce three auxiliary equations specifying stochastic processes for three of the model's variables. \( U'(C_t) = \lambda_t = X_1,t' \), \( \lambda_t SFR = X_2,t' \) and \( \lambda_t P_{t-1}/P_t = X_3,t' \) are chosen for the specification of the processes. Let \( X_t = [X_1,t', X_2,t', X_3,t']' \) where ' denotes transpose, and assume that this vector stochastic process has the logarithmic representation:

\[
x_t = A(0) + A(1)x_{t-1} + A(2)x_{t-2} + A(3)x_{t-3} + A(4)x_{t-4} + u_t,
\]

where lower-case letters represent the natural logarithms of their upper-case counterparts; \( A(0) \) is a 3x1 vector of constants; \( A(i) \) for \( i=1,2,3,4 \) is a 3x3 matrix of coefficients; and \( u_t \) is a serially uncorrelated disturbance term that is distributed as a normal \( N(\mu, \Sigma) \) with \( \mu = [\mu_1, \mu_2, \mu_3]' \) and \( \Sigma \) has elements \( \sigma_{ij} \) for \( i,j = 1,2,3 \). When \( x_t \) is specified like this (i.e., AR(4) with a constant term), using random numbers generated from \( N(\mu, \Sigma) \) distribution and initial values of \( x_t \) (i.e., \( x_1', x_2', x_3', x_4' \)), paths can be generated through a simulation method. From the \( x_t \) process,
conditional expectations can be calculated as follows. Let

\[ E_t X_{t+1} = [E_t X_{1,t+1}, E_t X_{2,t+1}, E_t X_{3,t+1}]'. \]

Because,

\[ \ln X_{i,t} = A_i(0) + \sum_{s=1}^{3} A_{ij}(s) \ln X_{j,t-s} + u_{i,t} \quad \text{for } i=1,2,3, \]

\[ X_{i,t} = e^{A_i(0) + \sum_{s=1}^{3} A_{ij}(s) X_{j,t-s} u_{i,t}}. \]

By making use of the log normal property

\[ E_t u_{i,t+1} = e^{\mu_{i+\sigma_{ii}/2}}, \]

\[ E_t X_{i,t+1} \text{ is computed as} \]

\[ E_t X_{i,t+1} = e^{A_i(0) + \sum_{s=1}^{3} A_{ij}(s) X_{j,t+1-s} u_{i,t}}. \]

In this manner, \( [E_t X_{1,t+1}, E_t X_{2,t+1}, E_t X_{3,t+1}]' \) can be calculated in an exact manner without going through any approximation process. Once these paths - \( X_{1,t}, X_{2,t}, X_{3,t}, E_t X_{1,t+1}, E_t X_{2,t+1}, E_t X_{3,t+1} \) - are generated, the paths of the other variables can be also generated from the FOC and the budget constraints along with initial values. In order to make the analysis more tractable, we restrict the preference and technology of our model economy as follows.

\[ U(C_t) = C_t^{\gamma} / \gamma, \]

\[ Y_t = f(K_t, Z_t) = Y_0 + Y_1, t = Y_0 + K_t Z_t, \quad 0 < \alpha < 1, \quad Y_0 < 0. \]

With this parameterization, solution paths to the model are generated in appendix B.
5. Estimating parameters of the complete stochastic equilibrium model by simulation

5.1 Estimation by simulation and chi-square test

Complete solution paths (i.e., realizations) to the model depend on parameter values of the model that need to be estimated. We have noted above that GMM procedures have been proposed for the estimation of nonlinear rational expectations models without requiring a complete specification of the model and explicitly solving for the stochastic equilibrium model. The GMM procedure can be interpreted as a limited information estimation. However, some theoretical model may not provide any convenient population orthogonality conditions [i.e., \( E_t(h(x_{t+n}, b_0)) = 0 \)] which GMM procedure works with. One may also want to use a complete structure of the model to organize a broad range of dynamics of the model and compare the fit with another model. In other words, we would like the model to mimic the empirical regularities quantitatively as well as qualitatively. Therefore, we can check whether the model is capable of replicating empirical regularities organized by some statistics (e.g., auto-covariances, cross correlations, VAR coefficients). As a way of handling these problems, estimation by simulation has been proposed by Lee (1986) [Lee and Ingram (1986)] to take account of a complete, explicit representation of the stochastic equilibrium model so that it allows us to test whether the underlying theory is quantitatively consistent with observed empirical regularities.

Consider a model with an \( \lambda \times 1 \) vector of parameters. A \( k \times 1 \) vector of data \( y(j,b) \) for \( j = 1,2,...,N \) are generated from the model by one of the methods for solving stochastic equilibrium models in section 4.1 [here, \( b \) may include parameters of auxiliary equations]. From \( y(j,b) \), an \( s \times 1 \)
The vector of statistics $H_N(b)$ is computed as averages of $h(y(j,b))$, where $h: \mathbb{R}^k \rightarrow \mathbb{R}^s$, 

$$H_N(b) = N^{-1} \sum_{j=1}^{N} h(y(j,b)).$$

Similarly, from the real data $y(1), \ldots, y(T)$, an $s \times 1$ vector of statistics is also computed according to

$$H_T = T^{-1} \sum_{t=1}^{T} h(y(t)).$$

Under ergodicity of $y(t)$ and $y(j,b)$,

$$a.s. \quad H_N(b) \xrightarrow{a.s.} E[h(y(j,b))], \quad H_T \xrightarrow{a.s.} E[h(y(t))].$$

Furthermore, if the model is true, then at the true $b$, $b_0$, the ergodicity implies that

$$E[h(y(j,b_0))] = E[h(y(t))].$$

Now, define

$$g(y(t), y(j,t,b)) = h(y(t)) - n^{-1} \sum_{j=1}^{n} h(y((t-1)n+j,b)),$$

where $n = N/T$. Then we are again in a GMM framework with

$$E[g(y(t), y(j,t,b_0))] = E[h(y(t)) - n^{-1} \sum_{j=1}^{n} h(y((t-1)n+j,b_0))] = 0.$$

In case $s > l$, we need to introduce a sequence of random weighting matrices $a_N$, $l \times s$, which converges in probability to a constant matrix $a^*$ and

$$a E[g(y(t), y(j,t,b_0))] = a E[h(y(t)) - n^{-1} \sum_{j=1}^{n} h(y((t-1)n+j,b_0))] = 0$$

is obtained. If the model is correctly specified, then the sample analogue of $aE[g(y(t), y(j,t,b))], a_T g_T(b)$, evaluated at $b=b_0$ should be close to zero for a large value of sample size $T$. The estimator by simulation $b_{TN}$ is chosen to minimize the function $J_T$ given by
\[ J_T(b) = [a_T g_T(b)]^2 = g_T(b)' a_T a_T g_T(b) = g_T(b)' W_T g_T(b), \]

where \( g_T(b) = T^{-1} \sum_{t=1}^{T} g(y(t), y(j, t, b)) = H_T - H_N(b), \) and \( W_T = a_T' a_T \)
is an \( s \times s \) symmetric weighting matrix which depends on sample information, and is given as
\[
W_T = \left( \Sigma_T + n^{-1} \Omega_N \right)^{-1},
\]
where \( \Sigma_T \) and \( \Omega_N \) are covariance matrix of \( h(y(t)) \), and \( h(y(j, t, b)) \), respectively. Under the null hypothesis that the model is a true description of the real data, all the moments should be the same so that \( \Sigma_T = \Omega_N \) and
\[
W_T = (1+n^{-1}) \Sigma_T.
\]

Because it fits into a \( G\)M framework, the estimator by simulation can be constructed to be consistent, asymptotically normal and to have an asymptotic covariance matrix that can be estimated consistently following \( G\)M procedures in Hansen (1982) under some regularity conditions. Specifically, the estimator by simulation \( b_{TN} \) is defined as a sequence of random vectors that converges in probability to \( b_0 \) for which
\[
\sqrt{T} a_T (H_T - H_N(b_{TN})) \text{ converges in probability to zero.}
\]
Under the regularity assumptions, it can be shown that
\[
\sqrt{T} (b_{TN} - b_0) \overset{D}{\longrightarrow} N(0, (aB)^{-1} a (\Sigma + n^{-1} \Sigma) a' ((aB)^{-1})),
\]
where \( B = E \{ \partial h(y(j, b_0)) / \partial b \} \). When \( 'a' \) is chosen optimally in the sense that it yields the smallest asymptotic covariance matrix for the estimator \( b_{TN} \), it is given as \( a^* = B' (\Sigma (1+n^{-1}))^{-1} \). With the optimal \( a^* \),
\[
\sqrt{T} (b_{TN} - b_0) \longrightarrow N(0, [B' (\Sigma (1+n^{-1})^{-1} B)]^{-1}).
\]
[see, theorem 3.2 in Hansen (1982)]. The estimation procedure sets 4 linear combinations of the \( s \) statistics \( g_T(b) \) equal to zero asymptotically.
by using random weighting matrices \( a_T \). Therefore, when \( s \gg 1 \), there are \( s - 1 \) remaining statistics \( g_T(b) \) that ought to be close to zero if the model is correctly specified. These can be used to obtain test statistics of the stochastic rational expectations model since \( T \) times the minimized value of \( J_T(b) = g_T(b)'^W_T g_T(b) \) for an optimal choice of \( W_T \) can be shown to be asymptotically distributed as a chi-square with \( s - 1 \) degrees of freedom [see lemma 4.2 in Hansen (1982)].

Because the empirical regularities in section 2 were organized mostly based on the VAR analysis, VAR coefficients are chosen as statistics \( H \) for the estimation. To be more precise, VAR coefficients do not fit into \( H_T \) defined earlier, because \( H_T \) is supposed to be a form of average of the data. A minor modification, however, suffices to show that using VAR coefficients as \( H_T \) does not cause any serious problem, still all the results being valid, as will be seen below.

Let's denote the estimated OLS coefficients as \( b_T \) and the true coefficients as \( b_0 \), then

\[
\begin{align*}
b_T - b_0 &= (X'X)^{-1} X'Y - b_0 \\
 &= (X'X/T)^{-1} X'Y/T - b_0 \\
 &= (X'X/T)^{-1} X'u/T \\
 &= M_T^{-1} H_T = M_T^{-1} \sum_{t=1}^{T} h_t /T,
\end{align*}
\]

where \( M_T = (X'X/T) \), and \( h_t = x_t' u_t \). Similarly,

\[
\begin{align*}
b_N - b_0 &= M_N^{-1} H_N = M_N^{-1} \sum_{j=1}^{N} h_j /N,
\end{align*}
\]

where \( M_N = (X'X/N) \), and \( h_j = x_j' u_j \) (\( X'X \) is from the simulated data, hence it is different from \( X'X \) above, which is from the real data)

\[
b_T - b_N = M_T^{-1} H_T - M_N^{-1} H_N
\]
\[ M_T^{-1} (H_T - H_N) + (M_T^{-1} - M_N^{-1}) H_N \]
\[ = M_T^{-1} T^{-1} \sum_{t=1}^T [h_t - \Sigma h_j/n] + (M_T^{-1} - M_N^{-1}) H_N. \]

Since \( M_T \mathop \longrightarrow \limits^p M_\infty \), \( M_N \mathop \longrightarrow \limits^p M_\infty \),
\[ V \sim N(0, V), \]
and \( V \) is given as \( V_T (1-1/n) \) where \( V_T \) is a covariance matrix of \( h_t \).
\[ \sqrt{T} [ M_T (b_T - b_N) ] \mathop \longrightarrow \limits^D N(0, V), \]
Hence,
\[ \sqrt{T} [ b_T - b_N ] \mathop \longrightarrow \limits^D N(0, V \cdot M_\infty^{-1} V^{-1} \cdot M_\infty^{-1}) \]
\[ = N(0, VCV), \]
where \( VCV \) is given as \( \Sigma (1-1/n) \) where \( \Sigma \) is a covariance matrix of \( b_T \) (or \( M_T^{-1} h_t \)). Therefore, \( W_T \) corresponds to the inverse of \( VCV \) here.

5.2. Empirical results and discussion of the simulation results

With a given parameter set \( 'b' = [\alpha, \beta, \gamma, \theta_0; A(0), A(1), A(2), A(3), A(4), \mu, \Sigma] \), a backwards simulation method generates the data \( y(j,b) \), \( j=1,2,\ldots,N \). From these realizations, VAR coefficients are calculated and are treated as the statistics \( H_N(b) \). These are compared with statistics \( H_T \) from the real data (i.e., the unrestricted VAR coefficients).

Among the parameters \( b \), \( [A(0), A(1), A(2), A(3), A(4), \mu, \Sigma] \) are used only to generate \( x_t \) from the auxiliary equations. Although, in theory, an arbitrary process for the auxiliary equation \( x_t \) may be assumed, we want the data generated from this process to be as close as possible to the real data as summarized by VAR coefficients. In this spirit, also by noting that \( C_t, \text{SPR}_t \) and \( P_t \) are all observable, the estimation by simulation idea is applied for the estimation of the parameters of the auxiliary equation as
well as the parameters of the original model \([\alpha, \beta, \gamma, Y_0]\). This amounts to estimating the parameters \([A(0), A(1), A(2), A(3), A(4), \mu, \Sigma]\) from the transformed real data \(C_t, P_t, \text{SPR}_t\) by forming a three variable VAR \(x_t\) with a constant and four lags. Specifically, quarterly data on per capita real consumption expenditures on nondurables and services, the consumer price index (CPI), and the NYSE composite common stock price index divided by the CPI are used as the data for \(C_t, P_t, \text{SPR}_t\). All the data are normalized to one by being divided by their sample means. The estimated coefficients values are reported in Table 4.2 along with the initial values of \(x_t = [x_{1t}, x_{2t}, x_{3t}]\), which are used in the simulations.

From the estimated process for \(x_t\) together with (3.14)'-(3.24)', all the other processes are generated by a backwards mapping method as discussed in section 4.2. As in section 2, we will focus on the behavior of \(\text{SRR}_t, \text{IRR}_t, \text{INF}_t, \text{Y}_t\). Among these variables, \(\text{IRR}_t\) and \(\text{INF}_t\) can be easily calculated. While \(\text{IRR}_t\) and \(\text{INF}_t\) are measured as the ratios of the variables (i.e., unit-free), \(\text{Y}_t\), which corresponds to \(\text{IP}_t\) in the real data, could be sensitive to the unit measured and \(\text{SRR}_t\) is also affected by the scale of \(\text{Y}_t\) because \(D_t = (1-\alpha)Y_t + \alpha Y_0\) is included in \(\text{SRR}_t\) equation. Recalling that the real data \(\text{IP}_t\) is indexed data with 1967 = 100, the following scaling for \(\text{Y}_t\) is employed for the simulated data:

\[
Y_t = (Y_t/\bar{Y}) \times 100.0, \quad \text{and} \quad y_t = \ln(Y_t),
\]

where \(\bar{Y}\) is sample mean of \(Y_t\). Because the theory doesn't give any clue to the relative size of \(\text{SPR}_t\) and \(D_t\) in the \(\text{SRR}_t\) formula [e.g. (4.13)], and \(\text{SPR}_t\) is indexed data (by normalization), the following scaling is done by introducing one additional parameter \(\eta\):

\[
D_t = (D_t/\bar{D}) \times \eta,
\]

where \(\bar{D}\) is sample mean of \(D_t\). Analogous to the way the real data \(\text{SRR}\) is
computed, $SRR_t$ for the simulated data is computed as

$$SRR_t = (SPR_{t+1} + D_t)/SRR_t - E_t(P_{t+1}/P_t).$$

Then, the four variable - $SRR_t$, $IRR_t$, $INF_t$, $y_t$ - VAR system with a constant and six lags is formed for the simulated data, which was also employed to organize empirical regularities for the real data in section 2.

$$X_t = C(0) + \sum_{s=1}^{6} C(s) X_{t-s} + \epsilon_t,$$

where $X_t = [SRR_t, IRR_t, INF_t, y_t]'$ and $C(s)$ is a 4x4 coefficients matrix. Each equation contains a constant and 6 lags so that the statistics $H_N(b)$ will form a 100x1 vector. From the VAR systems (the real system and the simulated system),

$$J_T = (H_T-H_N(b))' W_T (H_T-H_N(b))$$

is defined where $W_T$ is an estimate of the inverse of the 100x100 covariance matrix of $(H_T-H_N(b))$, i.e., $W_T = [\Sigma (1+1/n)]^{-1}$. It is computed from the real data as $S^{-1} X'(X'X/T)$, where $S$ is the covariance matrix of the residuals in the four variable VAR system and $X$ is a 1 x 25 row vector of all the right hand side variables in the system ($X = [1, X_{t-1}, \ldots, X_{t-6}]$).

A set of parameter values $[\alpha, \beta, \gamma, Y_0, \eta]$ which minimizes $J_T$ is sought, which gives us an estimator by simulation $b_{TN}$. The range of the parameter values being searched and the values that yield what we considered to generate the best fit based on this criteria are reported in Table 4.1.

<table>
<thead>
<tr>
<th>range</th>
<th>estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>[.80, 1.00]</td>
<td>.9893</td>
</tr>
<tr>
<td>[.85, 1.00]</td>
<td>.9907</td>
</tr>
<tr>
<td>[.06, .30]</td>
<td>.1201</td>
</tr>
<tr>
<td>[-1.0, 32.0]</td>
<td>-15.974</td>
</tr>
<tr>
<td>[.0001, .200]</td>
<td>.0062</td>
</tr>
</tbody>
</table>

From the realizations of the model with these parameter values, cross
Table 4.2 The Estimates of Initial Values and the Auxiliary Equations Used for the Simulation.

<table>
<thead>
<tr>
<th>ENTRY(t)</th>
<th>$x_{1t}$</th>
<th>$x_{2t}$</th>
<th>$x_{3t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.316372</td>
<td>-.543129</td>
<td>.282168</td>
</tr>
<tr>
<td>2</td>
<td>.329663</td>
<td>-.567999</td>
<td>.305113</td>
</tr>
<tr>
<td>3</td>
<td>.326466</td>
<td>-.644425</td>
<td>.322195</td>
</tr>
<tr>
<td>4</td>
<td>.319881</td>
<td>-.561884</td>
<td>.291868</td>
</tr>
</tbody>
</table>

$A(0) = \begin{bmatrix} -6029383E-02 \\ .2592595E-01 \\ -.8063668E-02 \end{bmatrix}$

$A(1) = \begin{bmatrix} 1.268174 & -.2535295E-01 & -.2080735 \\ -.8914861 & 1.311327 & 1.282894 \\ .9096715 & -.1956905E-01 & .2508547 \end{bmatrix}$

$A(2) = \begin{bmatrix} -.2449279 & .2829041E-01 & .1633383 \\ .5395172 & -.4407849 & -.4171239 \\ -.2255908 & .2165485E-01 & .2326680 \end{bmatrix}$

$A(3) = \begin{bmatrix} .1228754 & -.1509965E-01 & -.1085585 \\ -1.307501 & .9893326E-02 & .7323825 \\ -.4379583 & -.2445519E-01 & -.3069951 \end{bmatrix}$

$A(4) = \begin{bmatrix} .7090215E-03 & .1002130E-01 & .1550297E-01 \\ -.4348801 & .8572688E-01 & .4903198 \\ .1928802 & .2119463E-01 & -.2121849 \end{bmatrix}$

$\Sigma = \begin{bmatrix} .1849845E-04 \\ -.2129423E-04 & .2665152E-02 \\ .1970533E-04 & .2135941E-04 & .5558252E-04 \end{bmatrix}$

$\mu = \begin{bmatrix} .1206037E-04 \\ .4535108E-03 \\ -.7015394E-04 \end{bmatrix}$

Note: $x_{1t} = \ln(x_t)$, where $x_t = U'(C_t)$,
$x_{2t} = \ln(x_{SPR_t})$, $x_{3t} = \ln(x_t^{P_{t-1}/P_t})$. 

19.1
### Table 4.3 Summary Statistics of the Simulated Data

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRR</td>
<td>1.320897</td>
<td>.3058821</td>
</tr>
<tr>
<td>SRR</td>
<td>2.304822</td>
<td>5.592057</td>
</tr>
<tr>
<td>INF</td>
<td>1.456857</td>
<td>1.143568</td>
</tr>
</tbody>
</table>

Note: IRR = real rate of interest, SRR = real return on the common stock, INF = rate of interest.

### Table 4.4 Cross Correlations (from simulated data)

| x(t) | y(t-s) | s=-6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|------|--------|------|----|----|----|----|----|---|---|---|---|---|---|---|---|
| IRR  | INF    | -.05 | .04| -.08| -.23| -.18| -.36| -.59| -.27| -.38| -.40| -.17| -.05| -.07|
| SRR  | INF    | -.01 | .08| -.01| -.07| -.03| -.02| -.17| -.06| -.05| -.13| -.02| .07| -.07|
| IRR  | SRR    | -.04 | -.11| .06| -.02| -.14| .06| .07| .39| .09| .19| -.03| -.10| .05|
| SPR  | INF    | .03  | .03| .04| .04| .02| .01| -.01| -.07| -.10| -.12| -.16| -.16| -.15|

Note: SPR = real stock price index.

### Table 4.5 Four variable Innovation Accounting (from simulated data)

<table>
<thead>
<tr>
<th>Variables By Innovations in</th>
<th>Explained</th>
<th>SRR</th>
<th>IRR</th>
<th>INF</th>
<th>YL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SRR</td>
<td>.96(93.2)</td>
<td>2.4(4.5)</td>
<td>0.5(0.6)</td>
<td>0.8(1.8)</td>
</tr>
<tr>
<td></td>
<td>IRR</td>
<td>32.0(30.7)</td>
<td>66.5(67.4)</td>
<td>1.0(1.3)</td>
<td>0.5(0.6)</td>
</tr>
<tr>
<td></td>
<td>INF</td>
<td>1.1(1.8)</td>
<td>95.0(92.5)</td>
<td>3.3(4.2)</td>
<td>0.6(1.5)</td>
</tr>
<tr>
<td></td>
<td>YL</td>
<td>50.7(51.9)</td>
<td>30.5(26.8)</td>
<td>3.0(4.1)</td>
<td>15.8(17.2)</td>
</tr>
</tbody>
</table>

Note: YL = log(Y), where Y is income.
Figure 2. Dynamics of the four variable system from the simulation.
correlations are computed and the VAR analysis is performed in the same manner as in section 2. The results are reported in Tables 4.3 - 4.5 and Figure 2. The model generates (see Table 4.4)

1. negative cross correlations between IRR and INF;
2. negative cross correlations between SRR and INF;
3. positive correlations between SRR and IRR for the small lags and leads; and,
4. negative cross correlations between current INF and subsequent SRR.

Table 4.5 and Figure 2 illustrate that

1. SRR appears to be Granger-causally prior and explains 51 (52) percent of 4 (8) quarter forecast error variations in Y, and Y responds positively to the shocks in SRR after second quarter; and,
2. IRR explains 95 (93) percent of 4 (8) quarter forecast error variance in INF, and INF responds negatively to shocks in IRR.

The implications of the model seem to be fairly close to the those of the real data in light of the simplicity of the model and small number of free parameters. In particular, the model generates a negative correlation between inflation and asset returns, and important dynamics summarized by the VAR analysis. Also, the average simulated value of SRR is much higher than that of IRR, and SRR appears to be much volatile than IRR (Table 4.3). These results appear to be quite robust to repeated simulations with given parameters and to small variations in the parameter values. If the test of the theory is interpreted as checking whether there are reasonable values of parameters for which the model's equilibrium solutions are quantitatively consistent with the observed behavior of the corresponding series for the economy during the period being examined, the model seems to perform pretty well.
From the asymptotic normality of the estimator by simulation, \( T J_T(b_{TN}) \) is shown to be asymptotically \( \chi^2(s-J) \) distributed. In the current example, \( s-J = 100 - 5 = 95 \) since \( H_T(sx1) \) is a 100x1 vector, and \( b_{TN}(Jlx1) \) is a 5x1 vector. The minimized value of \( T J_T(b_{TN}) = 121.4793 \) when \( \alpha = .9893, \beta = .9907, \gamma = .1201, Y_0 = -15.974, \) and \( \eta = .0062. \) (\( T = 117 \) since quarterly data from 55,2-84,2 are employed)

In view of the large value of \( T J_T(b_{TN}) \), for most significance levels, the \( \chi^2 \) test statistic appears to be evidence against the model, even though it mimics some of the important dynamics and cross correlations we are interested in. Indeed, the test result shows that at this value of the \( \chi^2 \) statistic, the marginal significance level is .03481924. This implies that the model does not generate all the dynamics of the real data as reflected in the coefficients of the four variable VAR system. Considering the simplicity of the model and the small number of free parameters, the overall rejection of the model based on the formal \( \chi^2 \) fit test may have been expected if the asymptotic distribution theory were taken seriously. However, it is noted that the \( \chi^2 \) statistic is much smaller than twice of the degrees of freedom. According to the information criterion for model selection suggested by Akaike (1974), which purports to reject restrictions false enough to increase expected squared prediction error of a future observation, this would imply that a small model (i.e., a misspecified model) such as the one in the present paper may lead to a better prediction even though the restrictions are not true because it reduces the variance matrix of estimation errors. In this sense, the model still appears to be pretty good. As is always the case in such statistical tests, the statistical inference in this section is based on an asymptotic distribution (or large sample) theory so that it is possible that a small sample bias may change some of the empirical results reported here.
The simulation results help to understand the model's limitations and may provide useful suggestions for further improvement of the model by asking where and why the simulated data fails to match the real data. One of the significant differences between the simulated data and the real data is that SRR accounts for much of IRR's variation in simulated data. It is suspected that the formula generating IRR from the model [e.g., equation (3.19)] may not allow enough variation in IRR compared to SRR, which suggests that the specification of preferences may be too simple to capture more volatile IRR. This is also reflected in a small standard deviation of simulated IRR. Under this circumstance, more volatile SRR is believed to contain more information than it should have. In the real data, IRR explains a small fraction of the variance in INF. In the simulated data, most of the variance in INF is explained by innovations in IRR, and INF responds negatively to the shocks in IRR. This discrepancy in explanatory power may be viewed as a matter of degree and is thought to result from the simplicity of the model.

Looking at the impulse responses of simulated data, the response of Y to shocks in SRR appears to be quite different from that of real data. In real data, in response to shocks in SRR, IP gradually rises and this positive effect is persistent. In simulation, Y declines sharply for the first two quarters and then responds positively. It is suspected that in the SRR formula, \( D_t = (1-\alpha)Y_t + \alpha Y_0 \) may not fully reflect real dividend and output processes. This can be partly ascribed to the naive specification of the production function which does not allow any adjustment process of capital stock or a time-to-build type of dynamics [Kydland and Prescott (1982)].

The negative response of INF to shocks in IRR is consistent with the real data. The response of IRR to shocks in IRR does not fit real data.
well. A similar explanation given above concerning error decomposition may be of some help. The response of $Y$ to shocks in IRR reveals some departures from the real data, which may be attributed to either a too simple specification of the production process or the naive treatment of the index problem associated with $\eta$.

To summarize, much of the discrepancy between simulated data and real data seems to be ascribed to either a simple specification of the preference and production function or a naive treatment of the index problems; it is hoped that a more sophisticated and realistic treatment of all these would improve the performance of the model. In this context, to allow for a dynamic structure in production process following Kydland and Prescott (1982) or for a more flexible dynamic preference could be quite revealing and suggestive.

6. Concluding remarks

This paper discusses a procedure for solving, estimating and testing a nonlinear stochastic equilibrium model taking into account a broad range of dynamics of the model in the context of an asset pricing model with production and with money. The empirical regularities concerning asset returns and inflation are organized by using cross correlations and a VAR analysis. A simple general equilibrium model is suggested as a vehicle for the discussion of various issues associated with solving, estimating and testing the model. The solution for the model was generated by using a backwards mapping method to avoid approximation errors due to a linear-quadratic approximation. Parameters of the model were estimated in a more systematic way by extending GMM procedures and by taking into account a broad range of dynamics of the model. Furthermore, overidentifying restrictions of the
model were used to test the overall performance of the model.

Even though the model manages to generate some important dynamic relations among the variables we are interested in, an investigation of the simulated data from the model strongly suggests that there is still much room for further improvement of the model. For example, a more realistic specification of technology and preference as well as a more sophisticated treatment of the price indices should improve the performance of the model.

The method described in this paper can be easily applied to many problems that can be reduced to a nonlinear stochastic equilibrium framework in which agents are assumed to solve nonquadratic objective functions subject to nonlinear restrictions.

Appendix A. Data

CPI (or P): consumer price index, all urban consumers, seasonally adjusted. units: 1967 = 100, source: U.S. Dept. of Labor, Bureau of Labor Statistics (from citibase file)

MB: monetary base, seasonally adjusted. source: Federal Reserve Bank of St. Louis (from citibase file)

IR: 3 month Treasury Bills, Auction average discount rate. source: Board of Governors of Fed. Res. System (citibase file)


SR: The return on a value-weighted index of NYSE stocks (from CRSP file)

SPR: Common stock prices/CPI, NYSE composite, monthly average of daily closing prices (from citibase file)

C: personal consumption expenditures on nondurables and services, seasonally adjusted (from citibase file)

POP: population (all ages) (from citibase file)
Appendix B

With the parameterization in section 4.2, the FOC and budget constraints (3.14)–(3.24) can be rewritten as follows.

\[ u_t = f'(K_t, Z_t) = \alpha K_t Z_t = \alpha Y_{1,t} / K_t, \]  
(3.14)'

\[ D_t = Y_t - u_t K_t = (1 - \alpha) Y_t + \alpha Y_0 = Y_0 + (1 - \alpha) Y_{1,t}, \]
(3.15)'

\[ U'(C_t) = \lambda_t = C_t = X_{1,t}, \]
(3.16)'

\[ X_{1,t} = \beta E_{t,1,t} + \beta^2 \alpha E_t [X_{3,t+2} Y_{1,t+1} / K_{t+1}], \]  
(3.17)'

\[ X_{2,t} = \beta E_{t,2,t+1} + \beta (1 - \alpha) E_{t,3,t+1} Y_t + \alpha \beta Y_0 E_{t,3,t+1}, \]  
(3.18)'

\[ \beta E_{t,1,t+1} = \frac{X_{1,t}}{1 + \text{IRR}_t}, \]
(3.19)'

\[ T_t + C_t + IV_t + SPR_t S_t + B_{t+1} / (1 + \text{IRR}_t) = S_{t-1} [\text{SPR}_t + \frac{(P_{t-1} / P_t) D_{t-1}}{1 + \text{IRR}_t} + (P_{t-1} / P_t) u_{t-1} K_{t-1} + B_t], \]
(3.20)'

\[ G_t - T_t = B_{t+1} / (1 + \text{IRR}_t) - B_t + (M_t - M_{t-1}) / P_t, \]
(3.21)'

\[ K_{t+1} = K_t + IV_t, \]
(3.22)'

\[ C_t + IV_t + G_t = Y_t, \]
(3.23)'

\[ Y_t = \frac{[X_{2,t} - \beta E_{t,2,t+1} - \alpha \beta Y_0 E_{t,3,t+1}]}{\beta (1 - \alpha) E_{t,3,t+1}}, \]  
[from (3.18)']
(4.4)

\[ D_t = \alpha Y_0 + (1 - \alpha) Y_t, \]
[from (3.15)']
(4.5)

\[ \text{IRR}_t = \frac{X_{1,t}}{\beta E_{t,1,t+1}} - 1, \]
[from (3.19)']
(4.6)

Now, we want to compute \( K_{t+1} \). By letting

\[ q_{t+1} = X_{3,t+1} Y_{1,t} / \]

(3.18)' can be rewritten as

\[ q_{t+1} = X_{3,t+1} Y_{1,t}, \]
In order to compute \( K_{t+1} \) from (3.17)', we need to know

\[
E_t(X_3,t+2,Y_{1,t+1}) = E_t q_{t+2}.
\]

From (3.18)',

\[
[\beta(1-\alpha)]^{-1}[X_2,t - \beta E_t X_2,t+1 - \beta Y_0 E_t X_3,t+1] = E_t q_{t+1} + \epsilon_{t+1},
\]

with \( E_t \epsilon_{t+1} = 0 \).

Hence,

\[
[\beta(1-\alpha)]^{-1}[X_2,t+1 - \beta E_t X_2,t+2 - \beta Y_0 E_t X_3,t+2] = E_t q_{t+2}.
\]

By taking \( E_t \) of both sides and applying the iterated expectations rule,

\[
[\beta(1-\alpha)]^{-1}[E_t X_2,t+1 - \beta E_t X_2,t+2 - \beta Y_0 E_t X_3,t+2] = E_t q_{t+2}.
\]

Since the \([X_2,t] \) and \([X_3,t] \) processes are already assumed, the LHS of the (3.17)' can be easily computed. Then, from (3.17)'

\[
K_{t+1} = \beta^2 q_{t+2}/[X_1,t - \beta E_t X_1,t+1].
\]

Therefore, \( K_{t+1} \) process can be generated since all the processes in the RHS are known paths in the equation. Once \( K_{t+1} \) path is generated, we obtain

\[
IV_t = K_{t+1} - K_t, \quad \text{[from (3.22)']} \tag{4.8}
\]

\[
Q_t = Y_t - C_t - IV_t, \quad \text{[from (3.23)']} \tag{4.9}
\]

\[
\omega_t = M_t/M_{t-1} - 1 = P_t Y_t/(P_{t-1} Y_{t-1}) - 1 = \text{INF}_t(Y_t/Y_{t-1}) - 1, \tag{4.10}
\]

\[
u_t = \alpha Y_{1,t}/K_t, \quad \text{[from (3.14)']} \tag{4.11}
\]

\[
Z_t = (Y_t - Y_0)/K^\alpha_t. \tag{4.12}
\]

In this manner, all the exogenous processes \( Z_t, \dot{G}_t, \) and \( \omega_t \) can be generated from the model by the backwards mapping method. Since we generated \([X_1,t] \) and \([E_t X_1,t+1] \) from (4.6)
\[ \text{IRR}_t = \left[ x_{1,t} / \beta E_t x_{1,t+1} \right] - 1. \]

On the other hand, SRR\textsubscript{t} in this model is defined as

\[ \text{SRR}_t = E_t [\text{SPR}_{t+1}] + D_t E_t (P_t / P_{t+1}) - 1, \quad \text{SPR}_{t+1} \]

Here, \( E_t [\text{SPR}_{t+1}] \) and \( E_t (P_t / P_{t+1}) \) can be computed as follows. Recall that

\[ x_{1,t} = \lambda_t, \quad x_{2,t} = \lambda_t \text{SPR}_t, \quad x_{3,t} = \lambda_t (P_{t-1} / P_t). \]

Hence,

\[ \ln \text{SPR}_t = \ln x_{2,t} - \ln x_{1,t} \]

\[ = A_2(0) + \sum_{s=1}^{\ell} \sum_{j=1}^{3} A_{2j}(s) \ln x_{j,t-s} + u_{2,t} \]

\[ - \left[ A_1(0) + \sum_{s=1}^{\ell} \sum_{j=1}^{3} A_{1j}(s) \ln x_{j,t-s} + u_{1,t} \right]. \]

From this,

\[ E_t \text{SPR}_{t+1} = e^{\sum_{s=1}^{\ell} \sum_{j=1}^{3} A_{2j}(s) - A_{1j}(s) x_{j,t+1-s}} e_{m_1}, \]

where \( m_1 = (\mu_2 - \mu_1) + (\sigma_{22} + \sigma_{11} - 2\sigma_{12})/2. \)

Similarly,

\[ \ln (P_{t-1} / P_t) = \ln x_{3,t} - \ln x_{1,t} \]

\[ E_t (P_t / P_{t+1}) = e^{\sum_{s=1}^{\ell} \sum_{j=1}^{3} A_{3j}(s) - A_{1j}(s) x_{j,t+1-s}} e_{m_2}, \]

where \( m_2 = (\mu_3 - \mu_1) + (\sigma_{33} + \sigma_{11} - 2\sigma_{13})/2. \) In this way, SRR\textsubscript{t} can be computed exactly without any approximation process.
Footnotes

1. The cross correlations reported in Table 2 agree with other studies on this issue, which employ different ways of computing expected inflations. The qualitative results of the VAR analysis appear to be robust to different orderings of the variables and the number of the lags in the system [See Lee (1986)]. When the number of lags is increased to 8 or 12, there is little, if any, change in the forecasts of inflation and the implications of the empirical results in this section. When GNP data are used instead of IP for a measure of the general economic activity, all the implications in this section remain unchanged.

The cross correlations between SPR and INF are included because the inverse relation between higher inflation and lower share prices has been an important issue [see Feldstein (1980), Carmichael (1985)]. Lee (1986) also discusses the inverse relation between INF and SPR based on a steady state equilibrium analysis of a production economy.

An economic interpretation of the empirical result is contained in Lee (1986). The observed negative correlation between SRR and INF is interpreted as proxying for the contemporaneous positive correlation between SRR and IRR combined with a negative dynamic association between IRR, and INF for j>0. Because the focus of the paper is to discuss the implementation of an econometric strategy of solving, estimating and testing nonlinear models, the details are not reproduced in this paper.

2. This example economy is taken from Lee (1986). Lee (1986) discusses a possible trading scenario for this economy and the implications of the model in a steady state: for example, a negative inflation-stock price relationship, and the fiscal and monetary linkage between stock returns and inflation. A positive correlation between real stock returns and real interest rates is explained in terms of an arbitrage condition between two financial assets - bonds and stocks - in the model. A possible negative dynamic association between real interest rates and inflation is discussed in terms of the changes in money supply (v,) and production shocks (Z) with ad-hoc assumptions. However, due to the nonlinear-quadratic nature of the model, the explicit solution paths (or realizations) could not be derived, which is a major concern of the present paper.

3. Closed form solutions for the equilibrium time paths of the variables of interest could be obtained after imposing strong assumptions on the stochastic properties of the forcing variables. McFadden (1986) and Pakes and Pollard (1986) have recently proposed simulation estimators for the discrete response model.

4. The selection of the variables whose stochastic processes are assumed needs a 'shrewd guess' as Sims (1985) puts it to compute the conditional expectations of the variables in FOC without much difficulty. In general, there could be more than one way to choose the assumed processes, and the choice seems to depend on the other considerations (e.g., stable paths).

5. A possible problem with a backwards mapping solution method is the so-called invertibility problem. This occurs when the assumed (endogenous) processes (e.g., x(t)) contain more information than is contained in current and past exogenous variables (e.g., v(t)). However, as Sims (1985) pointed out, the assumption that economic agents may have access to more information than current and past disturbances does not invalidates the
solutions. Let \( I(x(t)) \) denotes the information set consisting of current and past \( x(t) \), and \( I(v(t)) \) is the information set consisting of current and past \( v(t) \). Then,
\[
E[(x(t+1)|I(x(t))|I(v(t))] = E[x(t+1)|I(v(t))],
\]
when \( I(v(t)) \subseteq I(x(t)) \). Therefore, the backwards mapping solution is still the solution of the forwards mapping in this case. However, the rationality assumption prevents economic agents from having a perfect foresight, which is a contradiction to the whole solution procedure and invalidates the solution procedures.

6. A negative \( Y_q \) is included in the production function in order to reconcile very volatile stock prices (\( SPR_t \)) and relatively smooth output \( (Y_t) \) stream. As seen in (4.4), without \( Y_0 \) (or \( Y_0 = 0 \))
\[
Y_t = \frac{[X_{2,t} - \beta E_{t-2}X_{2,t+1}]}{[\beta(1-\alpha)E_{t-3}X_{2,t}+1]}
\]
and \( Y_t \) is directly related to very volatile \( SPR_t \). (Recall \( X_{2,t} = \lambda_t SPR_t \)). When \( Y_0 \) is included in the production function, as given in (4.4), \( Y_t \) becomes smoother than the case without \( Y_0 \) by making \( Y_0 \) relatively large compared to \( [X_{2,t} - \beta E_{t-2}X_{2,t+1}] \) term.

7. See Labadie (1984). In case of time aggregation, censoring and seasonal adjustment, it would also be hard to find an appropriate disturbance term for the estimation (Sims (1985)).

Another important case which DSM procedure does not apply is that the first order conditions for an equilibrium (i.e., Euler equations) involve unobservable forcing variables. In relation to the present paper, this can be the case as seen in (3.14) and (3.17) in section 3, since the model involves a production function of the form \( Y_t = f(K_t, Z_t) \) with a technology shock \( Z_t \). To quote Hansen and Singleton (1982), "More generally, our approach to estimation is appropriate for any model that yields implications of the form (2.1) \( E_t h(x_{t+n}, b_t) = 0 \) with \( x \) observed. This latter qualification does rule out some models in which the implied Euler equations involve unobservable forcing variables." (p. 1271)

8. \( g_m(b) \) and \( a_m g_m(b) \) in the estimator by simulation correspond to \( g_m(\beta) \) and \( b_N(\beta) \) of the DSM framework in Hansen (1982) respectively. Some regularity assumptions include: \( x(t), y(j) \) are independent, stationary and ergodic; \( \text{dh}(y(j,b))/db \) is continuous in the mean at \( b \); \( E[\text{dh}(y(j,b))/db] = B \) exists and is finite and has full rank; \( a_m \) converges to \( a \) in probability; \( b_m \) converges to \( b \) in probability; \( E[g(t)g(t')] \) exists and is finite, \( E[g(t)|g(t-i), g(t-i-1), ...] \) converges in the mean square to zero, where \( g(t) = g(y(t), y(j, t, b)) \).

9. The VAR coefficients, in general, have no meaningful economic interpretations. An equivalent information is, however, preserved in the corresponding moving average coefficients that provide the basis for impulse responses and error decompositions. VAR coefficients were used as statistics \( II \) for the estimation because an important issue in this paper is to investigate causal relations and dynamic interactions among the variables of interest as well as measures of association.

10. This method using the real data for the estimation of the parameters of the auxiliary equation \( x_t \) is clearly motivated to reduce the number of the free parameters to be estimated so that only five free parameters \([\alpha, \beta, \gamma, Y_0, \eta] \) need to be estimated.
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