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On deterministic and Markovian production systems

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On deterministic and Markovian production systems

Woo, Hoon-Shik, Ph.D.

Iowa State University, 1993

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On deterministic and Markovian production systems

by

Hoon-Shik Woo

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CHAPTER 1. INTRODUCTION

A common model for production systems with blocking is that of a sequence of servers connected by intervening "buffers" of finite capacity. It is this finiteness which in effect couples successive servers, and places the analysis beyond the scope of Jackson's pioneering work in queueing networks with infinite capacity buffers [15, 22].

There are two commonly treated types of blocking [3, 19]. Transfer blocking, also called production system blocking, occurs when a full buffer prevents exit from the preceding service stage. Communication system blocking, pertinent for example to Kanban systems, occurs when a full buffer prevents entry into the preceding stage. According to Berkley [3], two buffers, one with capacity $C$, subject to transfer blocking, the other with capacity $C + 1$, subject to communication blocking, are equivalent. Moreover, Muth [17, 18] noted that there is no need to distinguish the system whether or not it has buffers, since these buffers can be treated as zero service time stages.

Exact stochastic analyses of production systems with blocking can be classified into two approaches. One approach is to model the systems as a classical continuous time Markov process (e.g., Hunt [11], Hillier and Boling [8, 9], Patterson [20], Hatcher [7], Knott [14], Siha and David [25]). This approach is hampered by the rapid increase in number of states with increasing number of stages. Recent attempts to overcome
this difficulty include the work of Gershwin and Schick [4], and of Schweitzer and Altiok [24]. Gershwin and Schick suggest a decomposition method which separates states into certain “internal” states and “boundary” states. Schweizer and Altiok propose a decomposition method which divides the state space into “server aggregate” states.

The second approach, not restricted to exponential service time distributions, has been introduced by Muth [18] (see also Leisten [16] in a scheduling context). Muth analyzed item flow through a certain “activity network” and developed a recursive relation. Based on this relation, he derived closed form solutions by considering item holding times, for less than four stages, and Erlang service time distributions. For the exponential specialization of the Erlang, he verified early results of Hunt [11].

Analyses of the case of constant service times (here called the deterministic case) include the work of Altiok and Kao [1], on production systems, where constant processing times are considered in conjunction with exponential interarrival times. Also, Muth [17] has proposed certain performance measure expressions, to which we shall return in Chapter 2.

Here we shall consider the transfer blocking model without buffers considered by many of the above-cited authors. According to Berkley’s remark cited in the second paragraph, this model can equivalently be interpreted as a communication blocking model with intermediate buffers of unit capacity, pertinent to Kanban systems.

As pointed out in Hillier and Boling [8], the realistic analysis of production systems lies somewhere between the deterministic and exponential service times; these two extreme cases occupy, respectively, the succeeding two chapters.

In Chapter 2 a deterministic analysis of production lines is described. Muth [17]
proposed the form $\frac{1}{\tau_M}$ and $M \tau_M + \sum_{j=M+1}^{N} \tau_j$ where $\tau_M = \max(\tau_1, \tau_2, \ldots, \tau_N)$, respectively, for flow rate and flow time. We interpret these forms as applicable to steady state, and study the transient behavior of these two performance measures.

Chapter 3 is devoted to deriving asymptotic flow rate and flow time in the Markov case of production lines, utilizing the periodicity of the pertinent embedded chain.

Chapter 4 deals with the deterministic analysis of confluent production systems where the items successively are made and fed into sub-assembly line, and then assembly. We consider the above production systems, with two sub-assembly lines feeding into one assembly line. We then analyzed the steady state performance measures as well as transient behavior.

Chapter 5 discusses the Markovian analysis of confluent production systems. We confirm the periodicity of the pertinent embedded chain and present the form of performance measures for certain special assembly situations.

Chapter 6 presents conclusions and recommendations for further research in related topics.
CHAPTER 2. DETERMINISTIC PRODUCTION LINES

2.1 Introduction

One purpose of this chapter is to investigate the behavior of blocking times for the production lines. This is done in section 2.3, where these blocking times typically are shown to be subject to an initial "transient" phase, followed by a "steady state" phase. In the transient phase, there typically is an initial (finite) constant portion, followed by a decreasing, piecewise linear, and concave portion.

Section 2.4 considers a certain "lock step" phenomenon, which, contrary to expectation, always occurs throughout the transient phase of blocking times, and not in steady state.

Section 2.5 considers flow times in terms of summed blocking times. It is found, again contrary to expectation, that the transient phase of flow time is convex, despite the near-concavity of the transient phases of its blocking time addends.

It is also pointed out that Muth's flow time expression [17] for deterministic production lines may be interpreted as flow time in steady state. When that interpretation is carefully adapted to the case when maximal stage service times coincide, a certain discontinuity phenomenon is seen to arise. That discontinuity is in the steady state flow time $F(\tau)$ expressed as a function of the vector $\tau = (\tau_1, \cdots, \tau_N)$ of stage service times $\tau_j$. In particular, one finds, for certain $\tau^0$ featuring coincident
maximal stage service times, and certain sequences \( \tau \) approaching \( \tau^0 \), that

\[
\lim_{\tau \to \tau^0} F(\tau) \neq F(\tau^0).
\]

Such discontinuity will occur only when the transient phases for the members of the sequence \( \tau \) involve ever-increasing numbers of items, tending to infinity, with the discontinuity easily understood in the light of an interchange of limits in \( \tau \) and the index \( i \) of successive items adjusted as for the equation (2.6) below.

### 2.2 Explicit Starting Time Solution

Our deterministic analysis of production lines borrows from the recursion relation developed by Muth [18] and Leisten [16]. Their model assumes a lineal production system of \( N \) stages. It further assumes that there is unlimited demand from the market and a continual supply of material at the initial stage. Also, no down times, set up times, or transportation times are allowed (see Figure 2.1).

Using the notation of Leisten [16], the recursion relation for \( S(i',j) \), the starting time of item \( i' \) at stage \( j \), is

\[
S(i',j) = \max\{T(i',j - 1), T(i' - 1, j), S(i' - 1, j + 1)\} \tag{2.1}
\]

where,

\( i' = 1, 2, \ldots \) is the index for items,
\[ j = 1, 2, \ldots, N \] is the index for stages, and

\[ T(i', j), \text{ the completion time of item } i' \text{ at stage } j, \text{ satisfies} \]

\[ T(i', j) = r_j + S(i', j). \quad (2.2) \]

To solve (2.1), we assume the fully occupied line as initial condition. This assumption does not materially affect the point of this paper.

We compute the doubly indexed sequence \( S(i', j) \) recursively in the order of \( S(1, N), S(2, N - 1), \ldots, S(N, 1), S(2, N), \ldots, S(N + 1, 1), S(3, N), \ldots \). The reader will note that, in this sequence, the \( S \)'s are ordered in groups of \( N \) increasing in \( (i' + j) \), with ordering within the groups by increasing \( i' \).

Since we assume the initial condition of all items ready for service, the starting times \( S(i', j) \) satisfy

\[ S(i', j) = 0, \quad (2.3) \]

for all \( i' = 1, 2, \ldots, N \) and \( j = N, N - 1, \ldots, 1 \) with \( i' + j = N + 1 \).

The starting time \( S(2, N) \), for the second item \( (i' = 2) \) at stage \( N \), satisfies

\[ S(2, N) = \max\{T(1, N), T(2, N - 1)\}, \]

\[ = \max\{r_N, r_{N-1}\}, \]

where the first and second equalities are due, respectively, to (2.1) and (2.2), and (2.3).

Analogously,

\[ S(i', j) = \max\{T(i', j - 1), T(i' - 1, j), S(i' - 1, j + 1)\}, \]

\[ = \max\{r_N, r_{N-1}, \ldots, r_{j-1}\} \]
for all \( i = 2, 3, \ldots, N + 1 \) and \( j = N, N - 1, \ldots, 1 \) with \( i' + j = N + 2 \).

Generally (that is, without restriction on \( i' + j \)),

\[
S(i', j) = \max_{\substack{b=1,2,\ldots,j-2, \\
i'-N+b>0}} \{ \max_{i'-N+b>0} \{(i' - N + b)\tau_b + \tau_{b+1} + \cdots + \tau_{j-1}\}, \max_{b=j-1,j,\ldots,N>0} \{(i' + j) - (N + 1)\tau_b\} \} 
\]

Relation (2.4) is verified by inductively verifying

\[
\mathbb{H}_k : (2.4) \text{ is true for } i' + j = N + k \text{ and all } j = 1, 2, \ldots, N. 
\]

Therefore, we need to establish the validity of (2.4) for \( \mathbb{H}_1 \), and also to derive the validity of (2.4) for \( \mathbb{H}_k \) from its validity for \( \mathbb{H}_{k-1} \).

\( \mathbb{H}_1 \) : relation (2.4) for \( k = 1 \) is equivalent to (2.3).

\( \mathbb{H}_{k-1} \rightarrow \mathbb{H}_k \) : We show that, if (2.4) holds for \( i' + j = N + k - 1 \), then \( i' + j = N + k \) implies (2.4).

This is done by first looking at the particular case \( i' = k, j = N \); in other words, by verifying that if (2.4) holds for \( i' + j = N + k - 1 \), then (2.4) holds for \( i' = k, j = N \). This requires an infinite induction on \( k \).

We then use a finite induction to "fill in" the remaining \( N - 1 \) \( (i', j) \) pairs satisfying \( i' + j = N + k \); in other words, we show that, if (2.4) holds for \( i' + j = N + k - 1 \), then (2.4) holds for the pair \( (N + k - j + 1, j - 1) \) when (2.4) holds for the pair \( (N + k - j, j), j = N, N - 1, \ldots, 2. \)
2.3 Transient Blocking Times

While certain features of blocking time transiency, such as piecewise linear declivity, apply generally, a more refined description requires identifying the sequence $M_1, M_2, \ldots, M_L$ of the indices of the following successive maxima of service times

\[
\tau_{M_1} = \max_{1 \leq j \leq N} \tau_j, \\
\tau_{M_2} = \max_{M_1 + 1 \leq j \leq N} \tau_j, \\
\tau_{M_3} = \max_{M_2 + 2 \leq j \leq N} \tau_j, \\
\vdots
\]

with the last index of this sequence denoted by $M_L$, with $M_L$ = either $N - 1$ or $N$. Given a stage $j$, $1 \leq j \leq N$, let $\tau_{M_R(j)} = \max\{\tau_j, \tau_{j+1}, \ldots, \tau_N\}$ and set $i = i' - N$; then the starting time solution (2.4) becomes

\[
S(i, j) = \max \left\{ \begin{array}{c}
(i + M_1)\tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_{j-1}, \\
(i + M_2)\tau_{M_2} + \tau_{M_2+1} + \cdots + \tau_{j-1}, \\
\vdots \\
(i + j - 1)\tau_{M_R(j)}
\end{array} \right\}
\]

Let $B(i, j)$ be the time that item $i$ spends blocked at stage $j$ given by $B(i, j) = S(i, j + 1) - S(i, j) - \tau_j$. The behavior of $B(i, j)$, as a function of $i$, is naturally described in terms of the indices $M_1, M_2, \ldots, M_L$ and $M_R(j)$.

\((1)\): For $j < M_1$ (implying $M_R(j) = M_1$),
\[ B(i, j) = \tau_{M_1} - \tau_j \quad \text{for all } i \]

(2): For \( j = M_1 \) (implying \( M_{R(j)} = M_1 \)),

\[ B(i, j) = 0 \quad \text{for all } i \]

(3): For \( M_1 < j < M_2 \) (implying \( M_{R(j)} = M_2 \)), we have

\[
B(i, j) = \max \left\{ \frac{1}{2} (i + M_1)\tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_j, \quad \frac{1}{2} (i + j)\tau_{M_2} \right\} \\
- \max \left\{ \frac{1}{2} (i + M_1)\tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_j, \quad \frac{1}{2} (i + j)\tau_{M_2} + (\tau_j - \tau_{M_2}) \right\}
\]

There are three possibilities, depending on the relative magnitudes of \( \tau_1, \tau_2, \ldots, \tau_{M_2} \) as shown Figure 2.2.

(3a) \( B(i, j) = 0 \) for all \( i \)

(3b) \( B(i, j) = \frac{j\tau_{M_2} - \tau_{M_1}}{\tau_{M_1} - \tau_{M_2}} \cdot \left( M_1 \tau_{M_1} + \cdots + \tau_j \right) \)
for \( i < \frac{j\tau_{M_2} - \tau_{M_1}}{\tau_{M_1} - \tau_{M_2}} \cdot \left( M_1 \tau_{M_1} + \cdots + \tau_j \right) \)

\[ = 0 \]
for \( i \geq \frac{j\tau_{M_2} - \tau_{M_1}}{\tau_{M_1} - \tau_{M_2}} \cdot \left( M_1 \tau_{M_1} + \cdots + \tau_j \right) \)

(3c) \( B(i, j) = \tau_{M_2} - \tau_j \)
for \( i < \frac{(j - 1)\tau_{M_2} - (\tau_{M_1} + \cdots + \tau_j - 1)}{\tau_{M_1} - \tau_{M_2}} \)
\[ = \frac{i(\tau_{M_2} - \tau_{M_1}) + j\tau_{M_2} - (M_1 \tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_j)}{\tau_{M_1} - \tau_{M_2}} \]
for \( \frac{(j - 1)\tau_{M_2} - (\tau_{M_1} + \cdots + \tau_j - 1)}{\tau_{M_1} - \tau_{M_2}} \)}
Figure 2.2: Possible solutions for $M_1 \leq j \leq M_2$. 
\[ - \frac{j \tau M_2 - (M_1 \tau M_1 + \cdots + \tau_j)}{\tau M_1 - \tau M_2} \leq i < \frac{j \tau M_2 - (M_1 \tau M_1 + \cdots + \tau_j)}{\tau M_1 - \tau M_2} = 0 \]

for \( i \geq \frac{j \tau M_2 - (M_1 \tau M_1 + \cdots + \tau_j)}{\tau M_1 - \tau M_2} \)

It is further possible to jointly describe the possibilities \((3a), (3b),\) or \((3c)\) for the successive stages \(j, M_1 < j < M_2\): If case \((3a)\) pertains for such a \(j\), then either case \((3a)\) or case \((3b)\) will pertain at \(j + 1\). Further, if either case \((3b)\) or case \((3c)\) pertains for such a \(j\), then case \((3c)\) will pertain at \(j + 1\).

(4): For \(j = M_2\) (implying \(M_{R(j)} = M_2\)),

\[ B(i, j) = 0 \quad \text{for all } i \]

(5): For \(M_2 < j < M_3\) (implying \(M_{R(j)} = M_3\)), we have

\[ B(i, j) = \max \left\{ \begin{array}{l}
(i + M_1)\tau M_1 + \tau M_1 + \cdots + \tau_j,
(i + M_2)\tau M_2 + \tau M_2 + \cdots + \tau_j,
(i + j)\tau M_3
\end{array} \right\} - \max \left\{ \begin{array}{l}
(i + M_1)\tau M_1 + \tau M_1 + \cdots + \tau_j,
(i + M_2)\tau M_2 + \tau M_2 + \cdots + \tau_j,
(i + j)\tau M_3 + (\tau_j - \tau_{M_3})
\end{array} \right\} \]

There are seven possibilities, depending on the relative magnitudes of \(\tau_1, \tau_2, \cdots, \tau_{M_3}\):

\[(5a) \ B(i, j) = 0 \quad \text{for all } i \]
(5b) $B(i,j) = i(\tau_{M_3} - \tau_{M_1}) + j \tau_{M_3} - (M_1 \tau_{M_1} + \tau_{M_1} + \cdots + \tau_j)$
for $i < \frac{j \tau_{M_3} - (M_1 \tau_{M_1} + \cdots + \tau_j)}{\tau_{M_1} - \tau_{M_3}}$
$= 0$
for $i \geq \frac{j \tau_{M_3} - (M_1 \tau_{M_1} + \cdots + \tau_j)}{\tau_{M_1} - \tau_{M_3}}$

(5c) $B(i,j) = \tau_{M_3} - \tau_j$
for $i < \frac{(j-1) \tau_{M_3} - (M_1 \tau_{M_1} + \cdots + \tau_{j-1})}{\tau_{M_1} - \tau_{M_3}}$
$= i(\tau_{M_3} - \tau_{M_1}) + j \tau_{M_3} - (M_1 \tau_{M_1} + \tau_{M_1} + \cdots + \tau_j)$
for $\frac{(j-1) \tau_{M_3} - (M_1 \tau_{M_1} + \cdots + \tau_{j-1})}{\tau_{M_1} - \tau_{M_3}} \leq i \leq \frac{j \tau_{M_3} - (M_1 \tau_{M_1} + \cdots + \tau_j)}{\tau_{M_1} - \tau_{M_3}}$
$= 0$
for $i \geq \frac{j \tau_{M_3} - (M_1 \tau_{M_1} + \cdots + \tau_j)}{\tau_{M_1} - \tau_{M_3}}$

(5d) $B(i,j) = i(\tau_{M_3} - \tau_{M_2}) + j \tau_{M_3} - (M_2 \tau_{M_2} + \tau_{M_2} + \cdots + \tau_j)$
for $i < \frac{j \tau_{M_3} - (M_2 \tau_{M_2} + \cdots + \tau_j)}{\tau_{M_2} - \tau_{M_3}}$
$= 0$
for $i \geq \frac{j \tau_{M_3} - (M_2 \tau_{M_2} + \cdots + \tau_j)}{\tau_{M_2} - \tau_{M_3}}$

(5e) $B(i,j) = \tau_{M_3} - \tau_j$
for $i < \frac{(j-1) \tau_{M_3} - (M_2 \tau_{M_2} + \cdots + \tau_{j-1})}{\tau_{M_2} - \tau_{M_3}}$
$= i(\tau_{M_3} - \tau_{M_2}) + j \tau_{M_3} - (M_2 \tau_{M_2} + \tau_{M_2} + \cdots + \tau_j)$
for $\frac{(j-1) \tau_{M_3} - (M_2 \tau_{M_2} + \cdots + \tau_{j-1})}{\tau_{M_2} - \tau_{M_3}} \leq i \leq \frac{j \tau_{M_3} - (M_2 \tau_{M_2} + \cdots + \tau_j)}{\tau_{M_2} - \tau_{M_3}}$
$= 0$
for $i \geq \frac{j \tau_{M_3} - (M_2 \tau_{M_2} + \cdots + \tau_j)}{\tau_{M_2} - \tau_{M_3}}$
\begin{align*}
\leq i &< \frac{j^2 \tau M_3 - (M_2^2 \tau M_2 + \cdots + \tau_j)}{\tau M_2 - \tau M_3} \\
&= 0 \\
\text{for } i &\geq \frac{j^2 \tau M_3 - (M_2^2 \tau M_2 + \cdots + \tau_j)}{\tau M_2 - \tau M_3} \\
(5f) \ B(i,j) &= i(\tau M_3 - \tau M_2) + j^2 \tau M_3 - (M_2^2 \tau M_2 + \tau M_2 + 1 + \cdots + \tau_j) \\
\text{for } i &< \frac{M_2^2 \tau M_2 - (M_1 \tau M_1 + \cdots + \tau M_2)}{\tau M_1 - \tau M_2} \\
&= i(\tau M_3 - \tau M_1) + j^2 \tau M_3 - (M_1 \tau M_1 + \tau M_1 + 1 + \cdots + \tau_j) \\
\text{for } i &< \frac{M_2^2 \tau M_2 - (M_1 \tau M_1 + \cdots + \tau M_2)}{\tau M_1 - \tau M_2} \\
&= 0 \\
\text{for } i &\geq \frac{j^2 \tau M_3 - (M_1 \tau M_1 + \cdots + \tau_j)}{\tau M_1 - \tau M_3} \\
(5g) \ B(i,j) &= \tau M_3 - \tau j \\
\text{for } i &< \frac{(j - 1) \tau M_3 - (M_2^2 \tau M_2 + \cdots + \tau_{j - 1})}{\tau M_2 - \tau M_3} \\
&= i(\tau M_3 - \tau M_2) + (j - 1) \tau M_3 - (M_2^2 \tau M_2 + \tau M_2 + 1 + \cdots + \tau_j) \\
\text{for } i &< \frac{M_2^2 \tau M_2 - (M_1 \tau M_1 + \cdots + \tau M_2)}{\tau M_1 - \tau M_2} \\
&= i(\tau M_3 - \tau M_1) + (j - 1) \tau M_3 - (M_1 \tau M_1 + \tau M_1 + 1 + \cdots + \tau_j) \\
\text{for } i &< \frac{M_2^2 \tau M_2 - (M_1 \tau M_1 + \cdots + \tau M_2)}{\tau M_1 - \tau M_2} \\
&= 0 \\
\text{for } i &\geq \frac{j^2 \tau M_3 - (M_1 \tau M_1 + \cdots + \tau_j)}{\tau M_1 - \tau M_3}
\end{align*}
\[= 0 \quad \text{for } i \geq \frac{j^r M_3 - (M_1^r M_1 + \cdots + r_j)}{\tau M_1 - \tau M_3}\]

It is again possible to describe the possibilities ((5a), (5b), \cdots, (5g)) for the successive stages \(j, M_2 < j < M_3\), but this description is complex and provides little insight, and will not be reported here.

(6): \quad \cdots

(7): \quad \cdots

\vdots

(L + 2): For \(j \geq M_L\) (implying \(M_{R(j)} = M_L\)),

\[B(i, j) = 0 \quad \text{for all } i\]

Concerning ranges \(M_{k-1} < j < M_k\) for \(k > 3\) (i.e., (6) to (L + 2) above), one simple unifying thread does emerge: All the possible cases for given \(k\) involve functions \(B(\cdot, j)\) composed of some or all of the following three kinds of portions: an initial (finite) portion, where \(B(\cdot, j)\) is constant at the value \(\tau M_k - \tau j\); followed by a (finite) portion, where \(B(\cdot, j)\) is decreasing, piecewise linear, and concave, involving some or all of the \((k-1)\) successive slopes \(\tau M_k - \tau M_{k-1}, \tau M_k - \tau M_{k-2}, \ldots, \tau M_k - \tau 1\); followed by an infinite portion where \(B(i, j)\) is 0.

2.4 Lock-step and Blocking Time Transiency

Lock step is said to occur for the stage pair \((j, j + 1)\) if item \(i\) starts processing at stage \(j\) at the same time that item \(i - 1\) starts processing at stage \(j + 1\). From the
expression for \( S(i,j) \) in (2.6), the counter-intuitive fact can be demonstrated that
lock step can only occur at stage pair \((j, j + 1)\) during the transient phase of \( B(i,j) \),
and indeed must occur throughout the transient phase.

Let \( D(i,j) \) be the difference between \( S(i,j) \) and \( S(i-1,j+1) \), given by
\[
D(i,j) = S(i,j) - S(i-1,j+1).
\]
Thus, if \( D(i,j) = 0 \), then we have the lock-step.

To demonstrate this fact, we choose an example of case (3) considered in section
2.3.

(3): For \( M_1 < j < M_2 \) (implying \( M_R(j) = M_2 \) ), we have

\[
D(i,j) = \max \left\{ \frac{1}{2} (i + 1) \tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_{j-1}, \right. \\
\left. \frac{1}{2} (i + j) \tau_{M_2} \right\} \\
- \max \left\{ \frac{1}{2} (i + M_1) \tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_j, \right. \\
\left. \frac{1}{2} (i + j) \tau_{M_2} \right\}
\]

There are three possibilities, depending on the relative magnitudes of \( \tau_1, \tau_2, \ldots, \tau_{M_2} \) as shown Figure 2.3.

\[
(3a) \quad B(i,j) = 0 \quad \text{for all } i
\]

\[
(3b) \quad B(i,j) = i(\tau_{M_1} - \tau_{M_2}) + (M_1 + 1)\tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_{j-1} \quad \text{for } i \leq \frac{j\tau_{M_2} - (M_1 \tau_{M_1} + \cdots + \tau_j)}{\tau_{M_1} - \tau_{M_2}} \\
\]

\[
= \tau_{M_1} - \tau_j \quad \text{for } i \geq \frac{j\tau_{M_2} - (M_1 \tau_{M_1} + \cdots + \tau_j)}{\tau_{M_1} - \tau_{M_2}}
\]
Figure 2.3: Possible solutions for $M_1 \leq j \leq M_2$ in lock-step phenomenon.
(3c) \( B(i,j) = 0 \)
\[
\begin{align*}
\text{for } i &< \frac{(j - 1)\tau_{M_2} - (M_1\tau_{M_1} + \cdots + \tau_{j-1})}{\tau_{M_1} - \tau_{M_2}} \\
&= i(\tau_{M_1} - \tau_{M_2}) + (M_1 + 1)\tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_{j-1} - i\tau_{M_2} \\
\text{for } &\frac{(j - 1)\tau_{M_2} - (M_1\tau_{M_1} + \cdots + \tau_{j-1})}{\tau_{M_1} - \tau_{M_2}} \\
&\leq i < \frac{j\tau_{M_2} - (M_1\tau_{M_1} + \cdots + \tau_{j})}{\tau_{M_1} - \tau_{M_2}} \\
&= \tau_{M_1} - \tau_{j} \\
\text{for } &i \geq \frac{j\tau_{M_2} - (M_1\tau_{M_1} + \cdots + \tau_{j})}{\tau_{M_1} - \tau_{M_2}}
\end{align*}
\]

2.5 Transient and Steady State Flow Times

The flow time \( F_i(\tau) \) of item \( i \), defined as the time required for transiting item \( i \) through the system, is given by

\[
F_i(\tau) = S(i, N) + \tau_N - S(i, 1) \tag{2.7}
\]

for \( i = 0, 1, 2, \cdots \). In other words, \( F_i(\tau) \) is the sum of stage blocking times and service times. Note that \( M_{R(N)} \), required in (2.6) to compute \( S(i, N) \), equals \( N \), while \( M_{R(1)} \), required in (2.6) to compute \( S(i, 1) \), equals \( M_1 \).

In view of (2.5), (2.6) and (2.7), we have

\[
F_i(\tau) = \max \left\{ \begin{array}{l}
M_1\tau_{M_1} + \tau_{M_1+1} + \cdots + \tau_N, \\
i(\tau_{M_2} - \tau_{M_1}) + M_2\tau_{M_2} + \cdots + \tau_N, \\
i(\tau_{M_3} - \tau_{M_1}) + M_3\tau_{M_3} + \cdots + \tau_N, \\
\vdots \\
i(\tau_N - \tau_{M_1}) + N\tau_N
\end{array} \right\} \tag{2.8}
\]
As in the case of blocking times, there is typically an initial "transient" phase during which $F_i(\tau)$ varies, followed by a "steady state" phase with $F_i(\tau)$ constant.

We now embark on two tasks. We first consider, in (i) and (ii), the steady state (i.e., limiting) value of flow time; we then consider its transient behavior in (iii) and (iv).

(i) Muth's flow time expression for untied cases.

Relation (2.8) leads to Muth's expression [17] for flow time, as the limit

$$
\lim_{i \to \infty} F_i(\tau) = M_1 r_{M_1} + \sum_{j=M_1+1}^{N} r_j \equiv F(\tau)
$$

(2.9)

where $M_1$ is the index $j$ of the maximum $r_j$ when that maximum is unique.

(ii) Modified Muth flow time expression for tied cases.

Relation (2.8) also implies that the $F_i(\tau)$ tends to the right hand side of (2.9) when the above maximizing index is not unique, where $M_1$ now is defined to be the largest of the indices corresponding to the tied maxima.

Relation (2.9) differs slightly from Muth's conclusion for tied cases, in that Muth defined $M_1$, in such cases, as the smallest, rather than the largest, of the indices corresponding to the tied maxima.

(iii) Transient behavior in the untied case: existence and structure

In general, flow time $F_i(\tau)$ is decreasing, piecewise linear, and convex, with the initial (finite) transient portion, if any, involving some or all of the successive slopes $\tau_{M_L - M_1}, \tau_{M_{L-1} - M_1}, \ldots, \tau_{M_2 - M_1}$. It is striking that flow time, with convex transient phase, is the sum of blocking times, all with near-concave transient phases.
(iv) Transient behaviors in the tied case: existence and structure

It is straightforward (though a bit tedious) to verify that (iii) holds for tied cases, as long as one adheres to the definition of $M_1$ given in (ii) and deletes the corresponding rows in expression (2.8); i.e., if $M_{k-1} < M_1 < M_k$, then the form of $F_i(\tau)$ is expression (2.8) without the first $k-1$ rows.

2.6 A Discontinuity Phenomenon and Transiency

It may happen, for certain service time sequences $\tau : (\tau_1, \ldots, \tau_N)$ approaching a limit $\tau^0 : (\tau^0_1, \ldots, \tau^0_N)$, that the expression $F(\tau)$ in (2.9), as computed in terms of $\tau$, does not converge to the expression $F(\tau^0)$ in (2.9) as computed in terms of $\tau^0$.

This will happen only when $\tau^0$ contains tied maximal components, and the $\tau$'s approaching $\tau^0$ involve transient phase lengths growing to infinity.

In such circumstances, the discontinuity is explained by the fact that the $F(\tau)$ for $\tau$ approaching $\tau^0$ are attained only after ever-increasing transient phases have elapsed.

As an example, consider the case $N = 4$. Let $\tau_{M_1} = \tau_2$, $\tau^0 = (\tau_1, \tau_2, \tau_3, \tau_2)$, $\tau = (\tau_1, \tau_2, \tau_3, \tau_2 - \epsilon)$, where $0 \leq \epsilon < \tau_2$. Consider separately (a) $\epsilon = 0$, and (b) $0 < \epsilon < \tau_2$.

(a) $\epsilon = 0$; by (ii) of section 2.5,

$$F_i(\tau) = F(\tau) = 4\tau_2.$$

(b) $0 < \epsilon < \tau_2$; here, $\tau_2$ is the unique maximum, so that, by (2.9),

$$F(\tau) = 3\tau_2 + \tau_3 - \epsilon.$$
It is of course clear that our discontinuity phenomenon is at work here, since

\[
\lim_{\tau \to \tau_0} F(\tau) = \lim_{\epsilon \to 0} (3\tau_2 + \tau_3 - \epsilon)
\]
\[
= 3\tau_2 + \tau_3
\]
\[
\neq 4\tau_2
\]
\[
= F(\tau_0).
\]

What we have here are parametric situations, involving tied maximal service times for limiting vectors \(\tau^0\), and ever-increasing transient phase lengths for vectors \(\tau\) approaching \(\tau^0\), for which,

\[
\lim_{\tau \to \tau^0} \lim_{i \to \infty} F_i(\tau) \neq \lim_{i \to \infty} F_i(\tau^0).
\]

On the other hand, if it is always true that

\[
\lim_{i \to \infty} \lim_{\tau \to \tau^0} F_i(\tau) = \lim_{i \to \infty} F_i(\tau^0)
\]

which follows from the fact that flow times at fixed elapsed times (i.e., fixed \(i\)) is continuous in \(\tau\); i.e.,

\[
\lim_{\tau \to \tau^0} F_i(\tau) = F_i(\tau^0).
\]

That ever-increasing transient phase length must be involved in the discontinuity phenomenon now follows from the fact that the order in which limits are taken is immaterial when finite index sets are involved. In any event, thinking of Muth's expressions for flow time in terms of limits explains the discontinuity.

### 2.7 The Initially Empty Lines

As another type of initial condition, an empty line is considered which has no items when it starts to produce. We compute the starting time \(S(i, j)\) recursively
in the order of $S(1,1)$, $S(1,2)$, $\cdots$, $S(1,N)$, $S(2,1)$, $\cdots$, $S(2,N)$, $S(3,1)$, $\cdots$. The reader will note that the $S$'s are ordered in groups of $N$ increasing in $i$, with ordering within the group by increasing $j$.

Steps analogous to those in section 2.2 give the following induction verified relation for $S(i,j)$, $1 \leq j \leq i = 1, 2, \cdots$.

$$S(i,j) = \max_{l=1,2,\ldots,j} \max_{K_l=I[j \geq l+1]} \left\{ \begin{array}{c}
\tau_1 + \cdots + \tau_{a-1} + (i - a + \sum_{l=1}^{a} K_l)\tau_a \\
+ K_{a+1} \tau_{a+1} + \cdots + K_{N-1} \tau_{N-1} \\
\tau_1 + \cdots + \tau_N + (i - N + \sum_{l=1}^{j} K_l)\tau_N
\end{array} \right. \right.$$

where expression $\tau_1 + \tau_2 + \cdots + \tau_{a-1}$ is to be equal to zero when $a \leq 1$, and $I[j \geq l+1]$ is the "indicator" of the event $[j \geq l+1]$, which equals 1 when $j \geq l+1$, and equals zero otherwise.

Based on this induction verified starting time solution, it is possible to similar analysis done in previous sections and the results are the same.
CHAPTER 3. MARKOVIAN PRODUCTION LINES

3.1 Introduction

The purpose of this chapter is to derive asymptotic flow rate and flow time in the Markov case, utilizing the periodicity of the pertinent embedded chain. This periodicity appears to have been first noted by Patterson [20]. The periodicity provides an especially tractable form of the aggregation principle of Schassberger [23] and Kemeny, Snell and Knapp [13], and thus allows an alternative approach to the problem of rapid increase of number of states. Beyond exploiting periodicity of the embedded chain, the analysis also involves two different methods of studying individual stage blocking times, one "system- oriented" and the other "customer-oriented", that are cross-checked against each other. The first of these is based on extending a familiar utilization formula, plus conservation of flow; the second is based on a listing of "blocking scenarios".

3.2 Periodicity of Production Lines

Consider the production line of $N$ stages. It assumes that there is unlimited demand from the market and a continual supply of material at the initial stage. It further assumes that no down times, set up times, or transportation times are allowed. This system is viewed as capable of being in one of $S$ possible states. As
shown in Figure 3.1, for the case $N = 3$, each one of $S = 8$ possible system states is denoted by a 3-vector $(x_1, x_2, x_3)$, where, for $x_j$, the state of stage $j$

$$x_j = \begin{cases} 
0 & \text{if stage } j \text{ is idle}, \\
1 & \text{if stage } j \text{ is busy}, \\
b & \text{if stage } j \text{ is blocked}.
\end{cases}$$

The system undergoes a change of state whenever a stage finishes service, with the non-zero transition probabilities indicate in Figure 3.1. The $8 \times 8$ matrix $Q(\alpha, \beta)$ of transition probabilities is sparse, with only 14 non-zero elements.

As mentioned in Patterson [20], systems such as the one considered here are not aperiodic (for a discussion of periodicity, see Isaacson and Madsen [12]). In fact, the production line considered here generally does have periodic character, with period

![Diagram](image.png)

Figure 3.1: Periodic structure and transition probabilities of the embedded Markov chain, for the 3-stage production line.
This periodicity is important both in structuring the analysis of the production line (as in sections 3.3 and 3.4 below), and also in computation. With regard to the latter, we remark that the kind of computational simplification generally made possible by periodicity is exactly of the sort talked about in more general terms by Kemeny, Snell and Knapp [13] and Schassberger [23]. These authors talked about a \( k \)-dimensional subset \( S' \), of the set \( S \) of all \( K \) states of an arbitrary Markov chain with transition matrix \( Q \), on which it is possible to define a sub-transition matrix \( Q' \), by

\[
Q'(\alpha, \beta) = Q(\alpha, \beta) + \sum_{\gamma \in S - S'} Q(\alpha, \gamma)q(\gamma, \beta), \quad \alpha, \beta \in S'
\]  

(3.1)

where \( q(\gamma, \beta) \) is the probability, starting from state \( \gamma \), of first visiting \( S' \) at state \( \beta \). They also point out that the stationary probabilities \( \pi'_k, k \in S' \) are proportional to the stationary probabilities of the states \( k \) in \( S \):

\[
\pi'_i = c\pi_i.
\]  

(3.2)

The focus of our remark here is that, when, in the case of a periodic chain, \( S' \) is taken as a periodic subset, then the constant of proportionality \( c \) in (3.2) simply is the period \( d \) of the chain, specifically \( N = d = 3 \), in the case of our 3-stage lineal production systems, and \( Q'(\alpha, \beta) \) simply is a diagonal block of \( Q^d \). When \( S' \) is taken to be the set \( \{4, 8\} \) in our example, the task of computing the eight stationary probabilities \( \pi_\alpha \) is reduced to the that of computing the two stationary probabilities of the transition matrix \( Q' \) for \( S' = \{4, 8\} \), followed by application of \( Q \).
3.3 Computation of Performance Measures

Given the stationary probabilities $\pi_\alpha$ of the embedded Markov chain, plus the expected residence times $\psi_\alpha$ in state $\alpha$, the limiting process probabilities $P_\alpha$ are computed as

$$P_\alpha = \frac{\pi_\alpha \psi_\alpha}{\sum_{i=1}^{S} \pi_i \psi_i} \quad (3.3)$$

3.3.1 Flow rates

Most authors have focused on the study of flow rate. From these studies (e.g., see Muth [17]), we have, for all stage $j$, that steady state flow rate $r$ satisfies

$$r_j = (E(\tau_j) + E(B_j) + E(I_j))^{-1} \quad (3.4)$$

where

- $\tau_j$ : the service time of stage $j$,
- $B_j$ : the blocking time of stage $j$,
- $I_j$ : the idle time of stage $j$.

Relation (3.4) leads to another well-known relation:

$$r_j = \frac{E(\tau_j)^{-1}}{\sum_{\alpha \in A_1} P_\alpha} = (E(\tau_j))^{-1} \cdot \text{Prob} \{ \text{Stage } j \text{ is busy} \}$$

where, $A_1$ is the set of vector states with 1 in position $j$. 

3.3.2 Expected blocking times and flow time

The expected flow time, taken into account for the fact that there is no blocking at stage $N$, clearly is given by

$$E(F) = \sum_{j=1}^{N} E(\tau_j) + \sum_{j=1}^{N-1} E(B_j).$$  \hspace{1cm} (3.6)$$

Further, extending the idea behind (3.5), it must equally be true that

$$\frac{1}{r} = \frac{[E(\tau_j) + E(B_j)]/ Pr[ stage j is busy or blocked]}{\sum_{\alpha \in A_j} P_\alpha} \hspace{1cm} (3.7)$$

where $A_j$ is the set of vector states showing stage $j$ to be either busy or blocked.

Expressions (3.5) and (3.7) lead to the relation for expected blocking time at stage $j$

$$E(B_j) = E(\tau_j) \cdot \frac{\sum_{\alpha \in A_j} P_\alpha}{\sum_{\alpha \in A_1} P_\alpha}. \hspace{1cm} (3.8)$$

Thus, substituting (3.8) into (3.6), with $E(\tau_j) = \frac{1}{\lambda_j}$, gives

$$E(F) = \sum_{j=1}^{N} \frac{1}{\lambda_j} + \sum_{j=1}^{N-1} \frac{1}{\lambda_j} \cdot \sum_{\alpha \in A_j} P_\alpha \hspace{1cm} (3.9)$$

for expected flow time.
3.4 Verification Using Blocking Scenarios

An alternative approach to computing the expected blocking times and flow time is based on a listing of so-called “blocking scenarios”, and is now illustrated for the case $N = 3$.

3.4.1 Listing the possible blocking scenarios

The procedure for listing all possible blocking scenarios is as follows:

1) Scan all the vector system states, conveniently given as in Figure reffig31, for the presence of blocking at stage $j$ (status $b$ at the $j$th location of the vector system state). For stage 1, such a state, for example, is $3(b11)$.

2) By scanning both preceding and succeeding vector system states, find all the possible “blocking scenarios”, i.e., strings of $b$'s bordered on either side by 0 or 1, that include the given vector system state.

For our example with $j = 1$, $N = 3$, and state $3(b11)$, the possible scenarios are:

$$8(111) \rightarrow 3(b11) \rightarrow 1(b11) \rightarrow 8(111)$$

$$8(111) \rightarrow 3(b11) \rightarrow 2(b10) \rightarrow 8(111)$$

3) After 1) and 2) have exhaustively been completed, delete all duplicate scenarios. There will remain a listing of the possible blocking scenarios that a customer may encounter at stage $j$.

For our example, with $N = 3$, the full list of scenarios is follows:

1. For stage 1;
(6, 1, 8) : 6(1b1) \rightarrow 1(bb1) \rightarrow 8(111)
(7, 2, 8) : 7(110) \rightarrow 2(b10) \rightarrow 8(111)
(8, 3, 1, 8) : 8(111) \rightarrow 3(b11) \rightarrow 1(bb1) \rightarrow 8(111)
(8, 3, 2, 8) : 8(111) \rightarrow 3(b11) \rightarrow 2(b10) \rightarrow 8(111)

2. For stage 2;

(3, 1, 8) : 3(b11) \rightarrow 1(bb1) \rightarrow 8(111)
(8, 6, 1, 8) : 8(111) \rightarrow 6(1b1) \rightarrow 1(bb1) \rightarrow 8(111)
(8, 6, 5) : 8(111) \rightarrow 6(1b1) \rightarrow 5(101)

3.4.2 Expected blocking times

Given stationary probabilities $\Pi'$ for the stationary probabilities of the Markov chain on subset \{4, 8\}, and the transition matrix $Q'$, compute stationary probabilities for the blocking scenarios, say

$$Pr\{(8, 3, 1, 8)\} = \pi_8 Q(8, 3)Q(3, 1)Q(1, 8) \quad (3.10)$$

Moreover, the expected duration of a given blocking scenario involves the expectations of the durations of the successive states of the blocking scenario. Thus, for this example,

$$E[ \text{Duration of } (8, 3, 1, 8)] = E[ \text{Duration of transition from 3 to 1}] + E[ \text{Duration of transition from 1 to 8}] = \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_3}. \quad (3.11)$$
The last equality is based on considering how service completions trigger transitions.

Expected blocking time at say stage 1, is then simply computed as the sum, over the (in this case, four) different blocking scenarios for stage 1, of the four products of expressions such as (3.10) and (3.11).

3.4.3 Verification for \( N = 3 \)

Cross-validating the analyses of subsection 3.3.2 and subsection 3.4.2, in the case of \( N = 3 \), we first derive the stationary probabilities that are required for both approaches. For the illustrative \( S' = \{4, 8\} \), with

\[
Q' = \begin{bmatrix}
Q'(4,4) & Q'(4,8) \\
Q'(8,4) & Q'(8,8)
\end{bmatrix}
\]

Relation (3.1) gives

\[
Q'(4,4) = \frac{\lambda_2 \lambda_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)},
\]
\[
Q'(4,8) = \frac{\lambda_1 (\lambda_1 + \lambda_2 + \lambda_3)}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)},
\]
\[
Q'(8,4) = \frac{\lambda_2 \lambda_3^2 (2\lambda_1 + \lambda_2 + \lambda_3)}{(\lambda_1 + \lambda_3)^2 (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2)},
\]
\[
Q'(8,8) = 1 - \frac{\lambda_2 \lambda_3^2 (2\lambda_1 + \lambda_2 + \lambda_3)}{(\lambda_1 + \lambda_3)^2 (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2)},
\]

and \( \Pi' \), satisfying of \( \Pi' = \Pi' \cdot Q' \) and \( \pi'_4 + \pi'_8 = 1 \), is

\[
\pi'_4 = \frac{X_1}{X_1 + X_2},
\]
\[
\pi'_8 = \frac{X_2}{X_1 + X_2}
\]
where

\[
X_1 = \lambda_2 \lambda_3^2 (2 \lambda_1 + \lambda_2 + \lambda_3),
\]

\[
X_2 = \lambda_1 \lambda_3^3 + (2 \lambda_2 + \lambda_2 + 3 \lambda_3) \lambda_1^2 + (\lambda_2^2 + 4 \lambda_2 \lambda_3 + 3 \lambda_3^2) \lambda_1 + (\lambda_2 + \lambda_3)^2 \lambda_3.
\]

By (3.2) and using \(Q\), we have all stationary probabilities \(\Pi = (\pi_1, \pi_2, \ldots, \pi_8)\), for the embedded chain, as follows:

\[
\pi_1 = X_2 X_3 [Q(8,3)Q(3,1) + Q(8,6)Q(6,1)],
\]

\[
\pi_2 = X_3 \{X_2 [Q(8,3)Q(3,2) + Q(8,7)Q(7,2)] + X_1 Q(7,2)\},
\]

\[
\pi_3 = X_2 X_3 Q(8,3),
\]

\[
\pi_4 = X_1 X_3,
\]

\[
\pi_5 = X_3 \{X_2 [Q(8,6)Q(6,5) + Q(8,7)Q(7,5)] + X_1 Q(7,5)\},
\]

\[
\pi_6 = X_2 X_3 Q(8,6),
\]

\[
\pi_7 = X_3 [X_1 + X_2 Q(8,7)] ,
\]

\[
\pi_8 = X_2 X_3 ,
\]

(3.12)

where, \(\frac{1}{X_3} = 3(X_1 + X_2)\).

Additionally, for the approach of subsection 3.3.2, we need the expected waiting times \(\Psi_\alpha\) in the eight vector states, computed as in the previous subsection, by considering how service completions trigger vector state transitions. We find

\[
\Psi = [\psi_1, \psi_2, \ldots, \psi_8]
\]

\[
= \left[ \frac{1}{\lambda_3} \frac{1}{\lambda_2} \lambda_2 + \frac{1}{\lambda_3} \frac{1}{\lambda_1} \lambda_1 + \frac{1}{\lambda_3} \frac{1}{\lambda_2} \lambda_2 + \frac{1}{\lambda_3} \frac{1}{\lambda_2} \lambda_2 + \frac{1}{\lambda_3} \frac{1}{\lambda_2} \lambda_2 + \frac{1}{\lambda_3} \frac{1}{\lambda_2} \lambda_2 + \frac{1}{\lambda_3} \frac{1}{\lambda_2} \lambda_2 \right] (3.13)
\]

The limiting process probabilities are then computed, according to (3.3), using (3.12) and (3.13).
Now, we are in position to check the equivalence of the approaches in subsection 3.3.2 and subsection 3.4.2. The key ingredients of both approaches are the expected blocking times at stages 1 and 2. It is possible to verify that analytic expressions for these, in terms of $\lambda_1$, $\lambda_2$, and $\lambda_3$, are precisely the same.

3.5 Implementation for $N = 4$

In the case of $N = 4$ (and hence $d = 4$), we have, as shown in Figure 3.2, a small subset $S' = \{4, 11, 12, 21\}$.

From relation (3.1), it is possible to obtain the 4x4 submatrix $Q'$, and vector $\Pi'$. The analysis does allow "closed form" algebraic treatment of the case $N = 4$; the resulting expressions, while cumbersome, have been derived. Numerically, for the example of Hillier and Boling [9], with parameters $E(\tau_1) = E(\tau_4) = 1.137$ and $E(\tau_2) = E(\tau_3) = .863$, one finds, using steps analogous to those of in the section 3.3,

$$Q' = \begin{bmatrix}
.15364 & .13899 & .16761 & .53976 \\
.24018 & .10263 & .28869 & .36850 \\
.26289 & .10481 & .31666 & .31564 \\
.09876 & .10768 & .10981 & .68376
\end{bmatrix},$$

and

$$\Pi' = \begin{bmatrix}
.15148 & .11136 & .17460 & .56256
\end{bmatrix}.$$

Analogously to subsection 3.4.2, the expected blocking times $E(B_j)$, $j = 1, 2, 3$,.
Figure 3.2: Periodic structure of the embedded Markov chain, for the 4-stage production line.

are

\[ E(B_1) = .78744 \]
\[ E(B_2) = .64020 \]
\[ E(B_3) = .49025, \]

and the expected flow rate \( r \), using (3.5), is

\[ r = .51963. \]  

(3.14)
Further, the expected flow time $E(F)$, using (3.6), is

$$E(F) = \sum_{j=1}^{4} E(\tau_j) + \sum_{j=1}^{3} E(B_j)$$

$$= 5.91789.$$

Finally, the average number of customers $\bar{L}$ in the system is

$$\bar{L} = \bar{r} \cdot F$$

$$= 3.07512. \quad (3.15)$$

The results (3.14) and (3.15) agree with those of Hillier and Boling [10].

### 3.6 Impact of Variability on Performance Measures

Analyses such as those in Chapter 2 and Chapter 3 allow assessment of the impact of variability on performance measures. Table 3.1 gives, for the case of three stages, the three performance measures for both the deterministic (D) and Markovian (M) case, as functions of service time ($\tau_j$ and $\frac{1}{\lambda_j}$, respectively, for the deterministic and Markov analysis). Table 3.1 shows that flow rate is substantially more sensitive to random service time variations than is flow time. Table 3.1 also shows that the well-known phenomenon of line reversibility for flow rate does not apply to flow time.

<table>
<thead>
<tr>
<th>Service Times</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
<th>$\bar{r}$</th>
<th>$F$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>D</td>
<td>M</td>
<td>D</td>
</tr>
<tr>
<td>1/2 1/3 1/4</td>
<td>2</td>
<td>1.4998</td>
<td>1.0833</td>
<td>1.3195</td>
<td>2.1667</td>
<td>1.9790</td>
</tr>
<tr>
<td>1/3 1/4 1/2</td>
<td>2</td>
<td>1.5348</td>
<td>1.5</td>
<td>1.6894</td>
<td>3.</td>
<td>2.5929</td>
</tr>
<tr>
<td>1/4 1/2 1/3</td>
<td>2</td>
<td>1.4806</td>
<td>1.3333</td>
<td>1.6296</td>
<td>2.6667</td>
<td>2.4127</td>
</tr>
<tr>
<td>1/3 1/2 1/4</td>
<td>2</td>
<td>1.4806</td>
<td>1.25</td>
<td>1.4934</td>
<td>2.5</td>
<td>2.2111</td>
</tr>
<tr>
<td>1/2 1/4 1/3</td>
<td>2</td>
<td>1.5348</td>
<td>1.0833</td>
<td>1.3628</td>
<td>2.1667</td>
<td>2.0916</td>
</tr>
<tr>
<td>1/4 1/3 1/2</td>
<td>2</td>
<td>1.4998</td>
<td>1.5</td>
<td>1.7778</td>
<td>3.</td>
<td>2.6663</td>
</tr>
</tbody>
</table>
It is also of interest, based on section 3, to compare optimality conclusions for flow rate and flow time. Hillier and Boling [10] show, for \( N = 4 \), that maximizing the flow rate for \( \sum_{j=1}^{4} \frac{1}{\lambda_j} = 4 \), optimal \( \left( \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_4} \right) = (1.137, 0.863, 0.863, 1.137) \). While a complete analysis for flow time has not been carried out, we note that, among the six permutations given in Table 3.2, the optimal permutation for flow time \( F \) and average number of customers \( L \) in the system is \((1.137, 1.137, 0.863, 0.863)\). This is of interest not only because it indicates that optimal allocation of service capacity may differ from one performance measure to the other. Also, it appears that small \( L \) is associated with low expected blocking times at later stages.

<table>
<thead>
<tr>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
<th>( \tau_3 )</th>
<th>( \tau_4 )</th>
<th>( B_1 )</th>
<th>( B_2 )</th>
<th>( B_3 )</th>
<th>( r )</th>
<th>( F )</th>
<th>( L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.137</td>
<td>1.137</td>
<td>.863</td>
<td>.863</td>
<td>.83499</td>
<td>.42663</td>
<td>.27454</td>
<td>.50710</td>
<td>5.53616</td>
<td>2.80740</td>
</tr>
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<td>.863</td>
<td>1.137</td>
<td>.863</td>
<td>.82172</td>
<td>.68496</td>
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<td>.51054</td>
<td>5.78148</td>
<td>2.95167</td>
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<td>.863</td>
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<td>.78744</td>
<td>.64020</td>
<td>.49025</td>
<td>.51963</td>
<td>5.91789</td>
<td>3.07512</td>
</tr>
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<td>.863</td>
<td>1.13459</td>
<td>.65476</td>
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<td>.50060</td>
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<td>3.03250</td>
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<tr>
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<td>1.137</td>
<td>.863</td>
<td>1.137</td>
<td>.109572</td>
<td>.60902</td>
<td>.48013</td>
<td>.51054</td>
<td>6.18486</td>
<td>3.15761</td>
</tr>
<tr>
<td>.863</td>
<td>.863</td>
<td>1.137</td>
<td>1.137</td>
<td>1.10899</td>
<td>.89999</td>
<td>.47301</td>
<td>.50710</td>
<td>6.48200</td>
<td>3.28703</td>
</tr>
</tbody>
</table>
CHAPTER 4. DETERMINISTIC CONFLUENT PRODUCTION SYSTEMS

4.1 Introduction

The previous two chapters were devoted to lineal production systems. In general, lineal systems are usually employed to model transfer and assembly lines, where processes take place in sequence.

However, there also are situations where processes are confluent tree-structures. In this situation, the items successively are made and fed into sub-assembly, and then to assembly. In this chapter, we consider this situation, with two sub-assembly lines feeding into one assembly line (see Figure 4.1).

4.2 Explicit Starting Time Solution

This deterministic analysis of a confluent tree-shaped structure is an extension of the analysis in Chapter 2. This structure is modeled under the same basic assumptions as those for lineal production lines (see section 2.2). Additional notation concerning the tree structure is as follows;

1. The assembly system is thought to be composed of a "long" sub-assembly line composed of $N_1$ stages, labeled $1, 2, \cdots, N_1$; a "short" sub-assembly line of
Figure 4.1: Two sub-assembly lines feeding one assembly line.
$N_1 - N_2' + 1$ stages, $N_2' \geq 1$, labeled $N_2', (N_2 + 1)', \ldots, N_1'$; and an assembly line of $N - N_1$ stages, labeled $N_1 + 1, N_1 + 2, \ldots, N$.

It is helpful, in the analysis below, to equivalently think of the system as composed of an $N$-stage lineal production line, with a further $(N_1 - N_2' + 1)$-stage lineal production line “attached” to the first at stage $N_1 + 1$.

2. The item indices at stage $N_1$ and $N_1'$ are chosen to coincide, allowing for unambiguous designation of assembled items by a single index.

Using essentially the notation of section 2.2, the recursion relation for $S(i, j)$, the starting time of item $i$ at stage $j$, is modified here to

$$S(i, j) = \max\{T(i, j - 1), S(i - 1, j + 1)\}$$

(4.1)

for all stages except stage $j = N_1 + 1$, for which

$$S(i, N_1 + 1) = \max\{T(i, N_1), T(i, N_1'), S(i - 1, N_1 + 1)\}$$

(4.2)

where,

$i = 1, 2, \ldots$ is the index for items,

$j = 1, 2, \ldots, N, N_2', \ldots, N_1'$ is the index for stages,

and $T(i, j)$, the completion time of item $i$ at stage $j$, satisfies

$$T(i, j) = \tau_j + S(i, j).$$

(4.3)

To solve (4.1) and (4.2), we assume the empty line situation as an initial condition. This assumption does not materially affect the point of this chapter.
We compute the doubly indexed sequence $S(i,j)$ recursively in the order of $S(1,1), S(1,2), \cdots, S(1,N_1), S(1,N_2^{'},), \cdots, S(1,N_1^{'},), S(1,N_1 + 1),$ $\cdots, S(1,N), S(2,1), S(2,2), \cdots$.

Since we assume the initial condition of empty lines, the starting time $S(1,1)$ (for the long sub-assembly line) satisfies

$$S(1,1) = 0.$$  \hfill (4.4)

The starting time $S(1,2)$, for the first item ($i = 1$) at stage 2, satisfies

$$S(1,2) = \max\{T(1,1), S(0,3)\},$$

$$= \tau_1$$  \hfill (4.5)

where the first and second equalities are due, respectively, to (4.1) and (4.3).

Analogously, $S(1,3)$, the starting time of item 1 at stage 3, satisfies

$$S(1,3) = \max\{T(1,2), S(0,4)\},$$

$$= \tau_1 + \tau_2$$  \hfill (4.6)

Similarly, the starting time $S(1,N_1)$ satisfies

$$S(1,N_1) = \max\{T(1,N_1 - 1), S(0,N_1 + 1)\},$$

$$= \tau_1 + \tau_2 + \cdots + \tau_{N_1 - 1}$$  \hfill (4.7)

For the starting time $S(1,N_2^{'})$ (for the short sub-assembly line), with the assumption of empty assembly system, one has

$$S(1,N_2^{'}) = 0.$$  \hfill (4.8)
The starting time $S(1, [N_2 + 1])$, for the first item ($i = 1$) at stage $[N_2 + 1]'$, satisfies

$$S(1, [N_2 + 1]) = \max\{T(1, N_2'), S(0, [N_2 + 2])\},$$

$$= \tau_{N_2}$$

where the first and second equalities are due, respectively, to (4.1) and (4.3).

This step in (4.9) is similarly repeated until it reaches stage $N_1'$, where the starting time $S(1, N_1')$ satisfies

$$S(1, N_1') = \max\{T(1, [N_1 - 1]'), S(0, N_1 + 1)\},$$

$$= \tau_{N_1} + \tau_{N_2} + \cdots + \tau_{N_1 - 1}$$

(4.10)

At the (first) assembly stage $N_1 + 1$, the starting time of item 1 satisfies

$$S(1, N_1 + 1) = \{T(1, N_1), T(1, N_1'), S(0, N_1 + 2)\},$$

$$= \max \left\{ \tau_1 + \tau_2 + \cdots + \tau_{N_1 - 1}, \right. \right.$$

$$\left. \left. \tau_{N_2} + \tau_{N_2 + 1}' + \cdots + \tau_{N_1 - 1}' \right\}$$

where the first and second equalities are due, respectively, to (4.2), and (4.7) and (4.10). Analogously, this step is repeated until it reaches to stage $N$.

Generally (that is, without restriction on item index $i$), we have three solutions, respectively, $S^L(i, j)$ for the "long" sub-assembly line, $S^S(i, j)$ for the "short" sub-
assembly line, and $S^A(i, j)$ for the assembly line; namely:

$$S^L(i, j) = \max \left\{ \begin{array}{l}
\max_{a=1, 2, \ldots, j-1} \{ \tau_1 + \cdots + \tau_{a-1} + i\tau_a + \cdots + \tau_{j-1} \} \\
\max_{b=1, j+1, \ldots, N-1} \{ \tau_1 + \cdots + \tau_b + (i + j - b - 2)\tau_{b+1} \} \\
\tau_1 + \cdots + \tau_N \\
\max_{c=N_2, \ldots, N_1} \{ (i + j - 2N_1 + c - 2)\tau_c + \cdots + \tau'_{N_1} \} \\
\max_{d=N_2, \ldots, N_1} \{ \tau'_{N_1} + \cdots + (i + j - N_1 - 1)\tau_d + \cdots + \tau'_{N_1} \} \\
\max_{e=N_1+1, \ldots, N_2} \{ \tau'_{N_1} + \cdots + (i + j - e - 1)\tau_e \} \\
\tau'_{N_2} + \cdots + \tau'_{N_1} + \max_{f=1, \ldots, j-1} \{ (i - N_1 + f - 1)\tau_f + \cdots + \tau_{j-1} \} \\
\tau'_{N_2} + \cdots + \tau'_{N_1} + \max_{g=j+1, \ldots, N_1} \{ (i + j - N_1 - 2)\tau_g \}
\end{array} \right\}$$

$$S^S(i, j) = \max \left\{ \begin{array}{l}
\max_{a=1, \ldots, N_1} \{ \tau_1 + \cdots + (i + j - N_1 - 1)\tau_a + \cdots + \tau_{N_1} \} \\
\max_{b=N_1+1, \ldots, N} \{ \tau_1 + \cdots + \tau_{N_1} + (i + j - b - 1)\tau_b \} \\
\tau_1 + \cdots + \tau_{N_1} + \max_{c=N_2, \ldots, [j-1]' \ldots, N_1} \{ (i - N_1 + c - 1)\tau_c + \cdots + \tau_{j-1} \} \\
\tau_1 + \tau_2 + \cdots + \tau_{N_1} + \max_{d=[j+1]', \ldots, N_1} \{ (i + j - N_1 - 2)\tau_d \} \\
\max_{e=N_2, \ldots, j-1} \{ \tau'_{[e-1]' + i\tau_e} + \cdots + \tau_{[j-1]'} \} \\
\max_{f=j-1, j, \ldots, N-1} \{ \tau'_{N_2} + \cdots + \tau'_{f} + (i + j - b - 2)\tau_{f+1} \} \\
\tau'_{N_2} + \cdots + \tau'_{N_1} \\
\max_{g=N_2, \ldots, N_1} \{ (i + j - 2N_1 + g - 2)\tau_g + \cdots + \tau_{N_1} \}
\end{array} \right\}$$
These three expressions (4.11) now are verified by a four-stage induction, as follows:

A) Given $S_{i-1}^L$ and $S^L(i, j - 1)$,

we get $S^L(i, j)$.

B) Given $S_{i-1}^S$ and $S^S(i, j - 1)$,

we get $S^S(i, j)$.

C) Given $S_{i-1}^A$ and $S^A(i, j - 1)$,

we get $S^A(i, j)$.

D) Given $S_{i-1}^A$, $S^L(i, N_1)$, and $S^S(i, N_1')$,
we get $S^A(i, N_1 + 1)$.

4.3 Steady State Blocking Times and Flow Times

Relation (4.11) can be written according to the position of maximum processing time $\tau_M$.

When $\tau_M \in \{\tau_1, \tau_2, \ldots, \tau_{N_1}\}$,

$$S^L(i, j) = \tau_1 + \tau_2 + \cdots + \tau_{M-1} + i\tau_M + \cdots + \tau_{j-1}$$

for $M < j$

$$= \tau_1 + \tau_2 + \cdots + \tau_{M-1} + (i + j - M - 1)\tau_M$$

for $M \geq j$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} > \tau_{N_2} + \cdots + \tau_{N_1}'$

$$= \tau_{N_2}' + \cdots + \tau_{N_1}' + (i + j - N_1 - 2)\tau_M$$

for $M > j$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} < \tau_{N_2} + \cdots + \tau_{N_1}'$

$$S^S(i, j) = \tau_1 + \tau_2 + \cdots + \tau_{M-1} + (i + j - N_1 - 1)\tau_M$$

for $M \geq j$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} > \tau_{N_2} + \cdots + \tau_{N_1}'$

$$= \tau_1 + \tau_2 + \cdots + \tau_{M-1} + (i + j - N_1 - 1)\tau_M + \cdots + \tau_{N_1}$$

for $M \neq N_1$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} < \tau_{N_2} + \cdots + \tau_{N_1}'$

$$= \tau_{N_2}' + \cdots + \tau_{N_1}' + (i + j - M - 2)\tau_M$$

for $M = N_1$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} < \tau_{N_2} + \cdots + \tau_{N_1}'$

$$S^A(i, j) = \tau_1 + \tau_2 + \cdots + i\tau_M + \cdots + \tau_{j-1}$$
for all $j$

When $\tau_M \in \{\tau_{N_2}, \tau_{(N_2+1)}, \ldots, \tau_{N_1}\}$,

$$S^E(i,j) = \tau_{N_2} + \tau_{(N_2+1)} + \cdots + \tau_{N_1} + (i + j - N_1 - 1)\tau_M$$

for $M \neq N_1'$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} > \tau_{N_2} + \cdots + \tau_{N_1}$

$$= \tau_1 + \tau_2 + \cdots + \tau_{N_1} + (i + j - N_1 - 2)\tau_{N_1}$$

for $M = N_1'$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} > \tau_{N_2} + \cdots + \tau_{N_1}$

$$= \tau_{N_2} + \tau_{(N_2+1)} + \cdots + (i + j - N_1 - 1)\tau_M + \cdots + \tau_{N_1}$$

for $\tau_1 + \tau_2 + \cdots + \tau_{N_1} < \tau_{N_2} + \cdots + \tau_{N_1}$

$$S^S(i,j) = \tau_{N_2} + \tau_{(N_2+1)} + \cdots + i\tau_M + \cdots + \tau_{N_1}$$

for $M < j$

$$= \tau_1 + \tau_2 + \cdots + \tau_{N_1} + (i + j - N_1 - 2)\tau_M$$

for $M \geq j$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} > \tau_{N_2} + \cdots + \tau_{N_1}$

$$= \tau_{N_2} + \tau_{(N_2+1)} + \cdots + (i + j - M - 1)\tau_M$$

for $M \geq j$ and $\tau_1 + \tau_2 + \cdots + \tau_{N_1} < \tau_{N_2} + \cdots + \tau_{N_1}$

$$S^A(i,j) = \tau_{N_2} + \tau_{(N_2+1)} + \cdots + i\tau_M + \cdots + \tau_{j-1}$$

for all $j$

When $\tau_M \in \{\tau_{N_1+1}, \tau_{N_1+2}, \ldots, \tau_N\}$,
\[ S^L(i,j) = \tau_1 + \tau_2 + \cdots + \tau_{M-1} + (i + j - M - 1)\tau_M \]

for \( \tau_1 + \tau_2 + \cdots + \tau_N_1 > \tau_{N_2} + \cdots + \tau_{N_1} \)

\[ = \tau_{N_2} + \cdots + \tau_{N_1} + (i + j - M - 1)\tau_M \]

for \( \tau_1 + \tau_2 + \cdots + \tau_{N_1} < \tau_{N_2} + \cdots + \tau_{N_1} \)

\[ S^S(i,j) = \tau_1 + \tau_2 + \cdots + \tau_{N_1} + \cdots + \tau_{M-1} + (i + j - M - 1)\tau_M \]

for \( \tau_1 + \tau_2 + \cdots + \tau_{N_1} > \tau_{N_2} + \cdots + \tau_{N_1} \)

\[ = \tau_{N_2} + \tau_{(N_2+1)} + \cdots + (i + j - M - 1)\tau_M \]

for \( \tau_1 + \tau_2 + \cdots + \tau_{N_1} < \tau_{N_2} + \cdots + \tau_{N_1} \)

\[ S^A(i,j) = \tau_1 + \tau_2 + \cdots + (i+1)\tau_M + \cdots + \tau_{j-1} \]

for \( M < j \) and \( \tau_1 + \tau_2 + \cdots + \tau_{N_1} > \tau_{N_2} + \cdots + \tau_{N_1} \)

\[ = \tau_1 + \tau_2 + \cdots + \tau_{M-1} + (i + j - M - 1)\tau_M \]

for \( M \geq j \) and \( \tau_1 + \tau_2 + \cdots + \tau_{N_1} > \tau_{N_2} + \cdots + \tau_{N_1} \)

\[ = \tau_{N_2} + \tau_{(N_2+1)} + \cdots + (i+1)\tau_M + \cdots + \tau_{j-1} \]

for \( \tau_1 + \tau_2 + \cdots + \tau_{N_1} < \tau_{N_2} + \cdots + \tau_{N_1} \)

Let \( B(i,j) \) be the time that item \( i \) spends blocked at stage \( j \). Then we have three blocking times according to the starting time solutions above:

The blocking time \( B^L(i,j), j = 1, 2, \cdots, N_1 \), of the long sub-assembly line is

\[
B^L(i,j) = \begin{cases} 
0 & \text{where } \tau_M \in \{\tau_1, \tau_2, \cdots, \tau_{j-1}\} \\
\tau_M - \tau_j & \text{elsewhere}
\end{cases}
\]
For the blocking time $B^S(i,j)$, $j = N_2, (N_2 + 1), \ldots, N_1$, of the short sub-assembly line, $B^S(i,j)$ satisfies

$$B^S(i,j) = \begin{cases} 0 & \text{where } \tau_M \in \{\tau_{N_2}, \tau_{N_2+1}, \ldots, \tau_{N_1}\} \\ \tau_M - \tau_j & \text{elsewhere} \end{cases}$$

Finally, the blocking times $B^A(i,j)$, $j = N_1 + 1, N_1 + 2, \ldots, N$, of the assembly line satisfies

$$B^A(i,j) = \begin{cases} \tau_M - \tau_j & \text{where } \tau_M \in \{\tau_j, \tau_{j+1}, \ldots, \tau_N\} \\ 0 & \text{elsewhere} \end{cases}$$

While the sum of processing times and blocking times is equal to flow times, in the present confluent assembly situation, one needs to individually consider the two sub-assembly flow times given by

$$F^1 = \sum_{j=1}^{N} \{\tau_j + B^L(i,j) + B^A(i,j)\}$$

for the long sub-assembly line, and

$$F^2 = \sum_{j=N_2}^{N_1} \{\tau_j + B^S(i,j)\} + \sum_{j=N_1+1}^{N} \{\tau_j + B^A(i,j)\}$$

for the short sub-assembly line.

Then we have flow times according to the blocking times considered above:

When $\tau_M \in \{\tau_1, \tau_2, \ldots, \tau_{N_1}\}$,

$$F^1 = M \tau_M + \sum_{j=M+1}^{N} \tau_j$$

$$F^2 = (N_1 - N_2) \tau_M + \sum_{j=N_1+1}^{N} \tau_j$$
When $\tau_M \in \{\tau_{N_2}^1, \tau_{(N_2+1)}^1, \ldots, \tau_{N_1}^1\}$,

$$F^1 = N_1\tau_M + \sum_{j=N_1+1}^{N} \tau_j$$
$$F^2 = M\tau_M + \sum_{j=M+1}^{N} \tau_j$$

Finally, when $\tau_M \in \{\tau_{N_1+1}, \tau_{N_1+2}, \ldots, \tau_N\}$,

$$F^1 = M\tau_M + \sum_{j=M+1}^{N} \tau_j$$
$$F^2 = (M-N_2)\tau_M + \sum_{j=M+1}^{N} \tau_j$$

### 4.4 Transient Blocking Times

In general, it is possible to compute the transient blocking times for the general model discussed in the previous section 4.3. However, for clarity and simplicity, we consider the simple example of Figure 4.2. This specialization does not materially affect the point of this chapter.

Let $l_j$ be the level of stage $j$, such that $l_1 = 0$, $l_2 = l_3 = 1$, $l_4 = 2$. Then, in view of the general forms of the $S(i,j)$ in the section 4.3, we have starting times $S(i,j)$, $j = 1, 2, 3, 4$, satisfying

$$S(i,j) = \max \begin{cases} 
(i + l_j - 2)\tau_1 + \tau_j - l_j, & \tau_1 + (i + l_j - 2)\tau_2 \\
\tau_1 + \tau_2 + (i + l_j - 3)\tau_3, & \tau_1 + \tau_2 + (i + l_j - 3)\tau_4, \\
(i + l_j - 2)\tau_3, & \tau_3 + (i + l_j - 3)\tau_4
\end{cases}$$
Now, $B(i, j)$, the time that item $i$ spends blocked at stage $j$, is given by

$$B(i, 1) = S(i, 2) - S(i, 1) - \tau_1,$$

$$B(i, 2) = S(i, 4) - S(i, 2) - \tau_2,$$

$$B(i, 3) = S(i, 4) - S(i, 3) - \tau_3 \quad (4.12)$$

We first consider $B(i, 1)$, and then consider $B(i, 2)$ and $B(i, 3)$.

**$B(i, 1)$**

By (4.12), $B(i, 1)$ satisfies

$$B(i, 1) = \max \left\{ i\tau_1, \tau_1 + (i-1)\tau_2, \tau_1 + \tau_2 + (i-2)\tau_3, \tau_1 + \tau_2 + (i-2)\tau_4, (i-1)\tau_3, \tau_3 + (i-2)\tau_4 \right\}$$
\begin{align*}
\text{(4.13)}
\begin{cases}
\frac{i\tau_1}{2} + (i-2)\tau_2 \\
\text{max}
\begin{align*}
2\tau_1 + \tau_2 + (i-3)\tau_3, & 2\tau_1 + \tau_2 + (i-3)\tau_4, \\
\tau_1 + (i-2)\tau_3, & \tau_1 + \tau_3 + (i-3)\tau_4
\end{align*}
\end{cases}
\end{align*}

We here consider separately the four cases \(\tau_M = \tau_1, \tau_2, \tau_3,\) and \(\tau_4,\) where \(\tau_M\) is the maximum of the processing times.

When \(\tau_M = \tau_1\), by (4.13), \(B(i,1)\) satisfies

\[B(i,1) = 0\quad\text{for all } i\]

When \(\tau_M = \tau_2\), by (4.13), \(B(i,1)\) satisfies

\[B(i,1) = \tau_2 - \tau_1\quad\text{for all } i\]

When \(\tau_M = \tau_3\), by (4.13), \(B(i,1)\) satisfies

\[B(i,1) = \max \left\{ \frac{i\tau_1}{\tau_1 + \tau_2 + (i-2)\tau_3}, \frac{i\tau_1}{(i-1)\tau_3} \right\} - \max \left\{ \frac{2\tau_1 + \tau_2 + (i-3)\tau_3}{\tau_1 + (i-2)\tau_3} \right\} \quad (4.14)\]

Further, for the case of \(\tau_1 + \tau_2 < \tau_3\), implying that the sum of processing times of the long sub-assembly line is less than that of the short sub-assembly line, by (4.14), \(B(i,1)\) satisfies

\[B(i,1) = \max \left\{ \frac{i\tau_1}{(i-1)\tau_3} \right\} - \max \left\{ \frac{i\tau_1}{(i-2)\tau_3 + \tau_1} \right\} \]
and there are three possibilities, depending on the relative magnitudes of \( \tau_1 \) and \( \tau_3 \).

(a) \[ B(i, 1) = 0 \quad \text{for} \quad i < \frac{\tau_3}{\tau_3 - \tau_1} \]

\[ = -\tau_3 + i(\tau_3 - \tau_1) \quad \text{for} \quad \frac{\tau_3}{\tau_3 - \tau_1} \leq i < 1 + \frac{\tau_3}{\tau_3 - \tau_1} \]

\[ = \tau_3 - \tau_1 \quad \text{for} \quad i \geq 1 + \frac{\tau_3}{\tau_3 - \tau_1} \]

(b) \[ B(i, 1) = -\tau_3 + i(\tau_3 - \tau_1) \quad \text{for} \quad i < 1 + \frac{\tau_3}{\tau_3 - \tau_1} \]

\[ = \tau_3 - \tau_1 \quad \text{for} \quad i \geq 1 + \frac{\tau_3}{\tau_3 - \tau_1} \]

(c) \[ B(i, 1) = \tau_3 - \tau_1 \quad \text{for all} \quad i \]

For the case of \( \tau_1 + \tau_2 > \tau_3 \), implying that the sum of processing times of the long sub-assembly line is greater than that of the short sub-assembly line, by (4.14), \( B(i, 1) \) satisfies

\[ B(i, 1) = \max \left\{ \frac{i\tau_1}{\tau_1 + \tau_2 + (i - 2)\tau_3} \right\} \quad \text{and} \quad -\max \left\{ \frac{i\tau_1}{2\tau_1 + \tau_2 + (i - 3)\tau_3} \right\} \]

and there again are three possibilities, depending on the relative magnitudes of \( \tau_1 \), \( \tau_2 \), and \( \tau_3 \).

(a) \[ B(i, 1) = 0 \quad \text{for} \quad i < 1 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1} \]

\[ = \tau_1 + \tau_2 + i(\tau_3 - \tau_1) \quad \text{for} \quad 1 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1} \leq i < 2 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1} \]

\[ = \tau_3 - \tau_1 \quad \text{for} \quad i \geq 2 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1} \]

(b) \[ B(i, 1) = \tau_1 + \tau_2 + i(\tau_3 - \tau_1) \quad \text{for} \quad i < 2 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1} \]

\[ = \tau_3 - \tau_1 \quad \text{for} \quad i \geq 2 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1} \]

(c) \[ B(i, 1) = \tau_3 - \tau_1 \quad \text{for all} \quad i \]
Similarly, when $\tau_M = \tau_4$, by (4.12), $B(i, 1)$ satisfies

$$B(i, 1) = \max \left\{ \begin{array}{l} i \tau_1 \\ \tau_1 + \tau_2 + (i - 2)\tau_4 \\ \tau_3 + (i - 2)\tau_4 \end{array} \right\} - \max \left\{ \begin{array}{l} i \tau_1 \\ 2\tau_1 + \tau_2 + (i - 3)\tau_4 \\ \tau_1 + \tau_3 + (i - 2)\tau_4 \end{array} \right\} \quad (4.15)$$

Further, for the case of $\tau_1 + \tau_2 < \tau_3$, by (4.15), $B(i, 1)$ satisfies

$$B(i, 1) = \max \left\{ \begin{array}{l} i \tau_1, \\ \tau_3 + (i - 2)\tau_4 \end{array} \right\} - \max \left\{ \begin{array}{l} i \tau_1, \\ (i - 2)\tau_4 + \tau_1 + \tau_3 \end{array} \right\}$$

and there again are three possibilities, depending on the relative magnitudes of $\tau_1$, $\tau_3$ and $\tau_4$.

(a) $B(i, 1) = 0$ for $i < \frac{2\tau_4 - \tau_3}{\tau_4 - \tau_1}$

(b) $B(i, 1) = \tau_3 - 2\tau_4 + i(\tau_4 - \tau_1)$ for $\frac{2\tau_4 - \tau_3}{\tau_4 - \tau_1} \leq i < 1 + \frac{2\tau_4 - \tau_3}{\tau_4 - \tau_1}$

(c) $B(i, 1) = \tau_4 - \tau_1$ for $i \geq 1 + \frac{2\tau_4 - \tau_3}{\tau_4 - \tau_1}$

For the case of $\tau_1 + \tau_2 > \tau_3$, by (4.15), $B(i, 1)$ satisfies

$$B(i, 1) = \max \left\{ \begin{array}{l} i \tau_1, \\ \tau_1 + \tau_2 + (i - 2)\tau_4 \end{array} \right\} - \max \left\{ \begin{array}{l} i \tau_1, \\ 2\tau_1 + \tau_2 + (i - 3)\tau_4 \end{array} \right\}$$

and there again are three possibilities, depending on the relative magnitudes of $\tau_1$, $\tau_2$, and $\tau_4$.

(a) $B(i, 1) = 0$ for $i < 1 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1}$
\begin{align*}
\tau_1 + \tau_2 + i(\tau_4 - \tau_1) & \quad \text{for } 1 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \leq i < 2 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \\
\tau_4 - \tau_1 & \quad \text{for } i \geq 2 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \\
(b) \ B(i, 1) & = \tau_1 + \tau_2 - 2\tau_4 + i(\tau_4 - \tau_1) \quad \text{for } i < 2 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \\
& = \tau_4 - \tau_1 \quad \text{for } i \geq 2 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \\
(c) \ B(i, 1) & = \tau_4 - \tau_1 \quad \text{for all } i
\end{align*}

\textbf{B(i, 2)}

By (4.12), $B(i, 2)$ satisfies

\begin{align*}
B(i, 2) &= \max \left\{ \begin{array}{c}
i\tau_1 + \tau_2, \ \tau_1 + i\tau_2 \\
\tau_1 + \tau_2 + (i-1)\tau_3, \ \tau_1 + \tau_2 + (i-1)\tau_4, \\
\tau_1, \ \tau_1 + (i-1)\tau_2 \\
\tau_1 + \tau_2 + (i-2)\tau_3, \ \tau_1 + \tau_2 + (i-2)\tau_4, \\
(i-1)\tau_3, \ \tau_3 + (i-2)\tau_4
\end{array} \right\} \\
&= \tau_2
\end{align*}

(4.16)

We consider separately the four cases $\tau_M = \tau_1, \tau_2, \tau_3,$ and $\tau_4$.

When $\tau_M = \tau_1$, by (4.16), $B(i, 2)$ satisfies

$$B(i, 1) = 0 \quad \text{for all } i$$

When $\tau_M = \tau_2$, by (4.16), $B(i, 2)$ satisfies

$$B(i, 1) = 0 \quad \text{for all } i$$
When $\tau_M = \tau_3$, by (4.16), $B(i,2)$ satisfies

$$B(i,2) = \max \left\{ \frac{\tau_1 + \tau_2 + (i-1)\tau_3}{i\tau_3} \right\} \cdot \max \left\{ \frac{i\tau_1 + \tau_2}{\tau_1 - 2\tau_2 + (i-2)\tau_3} \right\} + \frac{\tau_2 + (i-1)\tau_3}{\tau_2 + (i-1)\tau_3} \right\}$$

Further, for the case of $\tau_1 + \tau_2 < \tau_3$, by (4.17), $B(i,2)$ satisfies

$$B(i,2) = \max \{i(\tau_3 - \tau_1) - \tau_2, \tau_3 - \tau_2\}$$

and also satisfies

$$B(i,2) = -\tau_2 + i(\tau_3 - \tau_1) \quad \text{for} \quad i < \frac{\tau_3}{\tau_3 - \tau_1}$$

$$= \tau_3 - \tau_2 \quad \text{for} \quad i \geq \frac{\tau_3}{\tau_3 - \tau_1}$$

depending on the relative magnitudes of $\tau_1$ and $\tau_3$.

For the case of $\tau_1 + \tau_2 > \tau_3$, by (4.17), $B(i,2)$ satisfies

$$B(i,2) = \max \{i(\tau_3 - \tau_1) + \tau_1 - \tau_3, \tau_3 - \tau_2\}$$

and also satisfies

$$B(i,2) = (\tau_1 - \tau_3) + i(\tau_3 - \tau_1) \quad \text{for} \quad i < 1 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1}$$

$$= \tau_3 - \tau_2 \quad \text{for} \quad i \geq 1 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1}$$

depending on the relative magnitudes of $\tau_1$, $\tau_2$, and $\tau_3$.

Similarly, when $\tau_M = \tau_4$, by (4.16), $B(i,2)$ satisfies
\[ B(i, 2) = \max \left\{ \frac{\tau_1 + \tau_2 + (i-1)\tau_4}{\tau_3 + (i-1)\tau_4} \right\} - \max \left\{ \begin{array}{c} i\tau_1 + \tau_2 \\ \tau_1 + 2\tau_2 + (i-2)\tau_4 \\ \tau_2 + \tau_3 + (i-2)\tau_4 \end{array} \right\} \] (4.18)

Further, for the case of \( \tau_1 + \tau_2 < \tau_3 \), by (4.18), \( B(i, 2) \) satisfies

\[ B(i, 2) = \max\{i(\tau_4 - \tau_1) - \tau_2 + \tau_3 - \tau_4, \ tau_4 - \tau_2\} \]

and also satisfies

\[ B(i, 2) = (-\tau_2 + \tau_3 - \tau_4) + i(\tau_4 - \tau_1) \quad \text{for} \quad i < \frac{\tau_4 - \tau_3}{\tau_4 - \tau_1} \]

\[ = \tau_4 - \tau_2 \quad \text{for} \quad i \geq 1 + \frac{\tau_4 - \tau_3}{\tau_4 - \tau_1} \]

depending on the relative magnitudes of \( \tau_1, \tau_3 \) and \( \tau_4 \).

For the case of \( \tau_1 + \tau_2 > \tau_3 \), by (4.18), \( B(i, 2) \) satisfies

\[ B(i, 2) = \max\{i(\tau_4 - \tau_1) + \tau_1 - \tau_4, \ tau_4 - \tau_2\} \]

and its solution is

\[ B(i, 2) = (\tau_1 - \tau_4) + i(\tau_4 - \tau_1) \quad \text{for} \quad i < 1 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \]

\[ = \tau_4 - \tau_2 \quad \text{for} \quad i \geq 1 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \]

\( B(i, 3) \)

By (4.12), \( B(i, 3) \) satisfies

\[ B(i, 3) = \max \left\{ \begin{array}{c} i\tau_1 + \tau_2, \ \tau_1 + i\tau_2 \\ \tau_1 + \tau_2 + (i-1)\tau_3, \ \tau_1 + \tau_2 + (i-1)\tau_4, \\ i\tau_3, \ \tau_3 + (i-1)\tau_4 \end{array} \right\} \]
\[ \begin{align*}
- \max & \left\{ (i-1)\tau_1, \tau_1 + (i-1)\tau_2, \right. \\
& \left. \tau_1 + \tau_2 + (i-2)\tau_3, \tau_1 + \tau_2 + (i-2)\tau_4, \\ & (i-1)\tau_3, \tau_3 + (i-2)\tau_4 \right\} \right. \\
- \tau_3
\end{align*} \] (4.19)

When \( \tau_M = \tau_1 \), by (4.19), \( B(i, 3) \) satisfies

\[ B(i, 3) = \tau_1 - \tau_3 \quad \text{for all } i \]

When \( \tau_M = \tau_2 \), by (4.19), \( B(i, 3) \) satisfies

\[ B(i, 3) = \tau_2 - \tau_3 \quad \text{for all } i \]

When \( \tau_M = \tau_3 \), by (4.19), \( B(i, 3) \) satisfies

\[ B(i, 3) = 0 \quad \text{for all } i \]

Finally, when \( \tau_M = \tau_4 \), by (4.19), \( B(i, 3) \) satisfies

\[ B(i, 3) = \tau_4 - \tau_3 \quad \text{for all } i \]

4.5 Transient and Steady State Flow Times

The flow time \( F_i \) of item \( i \) was defined in Chapter 2 as the time required for transiting item \( i \) through the system (see section 2.5). However, in the present confluent assembly situation, one needs to individually consider the two sub-assembly flow times, respectively, given respectively by

\[ F_i^1 = S(i, 4) + \tau_4 - S(i, 1), \quad \text{for all } i \] (4.20)
for the long sub-assembly line, and

\[ F_i^2 = S(i, 4) + \tau_4 - S(i, 3), \quad \text{for all } i \]  \hspace{1cm} (4.21)

for the short sub-assembly line.

We consider \( F_i^1 \) and \( F_i^2 \) in terms:

\[ F_i^1 \]

By (4.20), \( F_i^1 \) satisfies

\[ F_i^1 = \max \left\{ \begin{array}{l}
i\tau_1 + \tau_2, \quad \tau_1 + i\tau_2, \\
\tau_1 + \tau_2 + (i-1)\tau_3, \quad \tau_1 + \tau_2 + (i-1)\tau_4,
\end{array} \right\} + \tau_4 \]

\[ - \max \left\{ \begin{array}{l}
i\tau_3, \quad \tau_3 + (i-1)\tau_4 \\
(i-1)\tau_1, \quad \tau_1 + (i-2)\tau_2,
\end{array} \right\} \]  \hspace{1cm} (4.22)

We now evaluate separately the four cases \( \tau_M = \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \).

When \( \tau_M = \tau_1 \), by (4.22), \( F_i^1 \) satisfies

\[ F_i^1 = \tau_1 + \tau_2 + \tau_4, \quad \text{for all } i \]

When \( \tau_M = \tau_2 \), by (4.22), \( F_i^1 \) satisfies

\[ F_i^1 = 2\tau_2 + \tau_4, \quad \text{for all } i \]

When \( \tau_M = \tau_3 \), by (4.22), \( F_i^1 \) satisfies
Further, for the case of \( \tau_1 + \tau_2 < \tau_3 \), by (4.23), \( F_i^1 \) satisfies

\[
F_i^1 = \max\{i(\tau_3 - \tau_1) + \tau_1 + \tau_4, \ 2\tau_3 + \tau_4\}
\]

and also satisfies

\[
F_i^1 = \tau_1 + \tau_4 + i(\tau_3 - \tau_1) \quad \text{for} \quad i \leq 1 + \frac{\tau_3}{\tau_3 - \tau_1}
\]
\[
= 2\tau_3 + \tau_4 \quad \text{for} \quad i > 1 + \frac{\tau_3}{\tau_3 - \tau_1}
\]

depending on the relative magnitude of \( \tau_1 \) and \( \tau_3 \).

For the case of \( \tau_1 + \tau_2 > \tau_3 \), \( F_i^1 \) satisfies

\[
F_i^1 = \max\{i(\tau_3 - \tau_1) - \tau_3 + 2\tau_1 + \tau_2 + \tau_4, \ 2\tau_3 + \tau_4\}
\]

and also satisfies

\[
F_i^1 = i(\tau_3 - \tau_1) - \tau_3 + 2\tau_1 + \tau_2 + \tau_4 \quad \text{for} \quad i \leq 2 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1}
\]
\[
= 2\tau_3 + \tau_4 \quad \text{for} \quad i > 1 + \frac{\tau_3 - \tau_2}{\tau_3 - \tau_1}
\]

Finally, when \( \tau_M = \tau_4 \), by (4.22), \( F_i^1 \) satisfies

\[
F_i^1 = \max\{\tau_1 + \tau_2, \ \tau_3\} + 3\tau_4
\]
\[
- \max\{(3\tau_4 - \tau_1) - i(\tau_4 - \tau_1), \ \tau_1 + \tau_2, \ \tau_3\}
\]

Further, for the case of \( \tau_1 + \tau_2 < \tau_3 \), by (4.24), \( F_i^1 \) satisfies
\[ F_i^1 = \max\{i(\tau_4 - \tau_1) + \tau_1 + \tau_3, \ 3\tau_4\} \]

and also satisfies
\[ F_i^1 = \tau_1 + \tau_3 + i(\tau_4 - \tau_1) \quad \text{for} \quad i \leq 1 + \frac{2\tau_4 - \tau_3}{\tau_4 - \tau_1} \]
\[ = 3\tau_4 \quad \text{for} \quad i > 1 + \frac{2\tau_4 - \tau_3}{\tau_4 - \tau_1} \]

For the case of \( \tau_1 + \tau_2 > \tau_3 \), by (4.24), \( F_i^1 \) satisfies
\[ F_i^1 = \max\{i(\tau_4 - \tau_1) + 2\tau_1 + \tau_2, \ 3\tau_4\} \]

and also satisfies
\[ F_i^1 = 2\tau_1 + \tau_2 + i(\tau_4 - \tau_1) \quad \text{for} \quad i \leq 2 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \]
\[ = 3\tau_4 \quad \text{for} \quad i > 2 + \frac{\tau_4 - \tau_2}{\tau_4 - \tau_1} \]

In general, the steady state flow time of \( F_i^1 \) is given by
\[ F_i^1 = i\tau_M + \sum_{j=M+1}^{N} \tau_j, \quad \text{for all} \quad i \]

By (4.21), \( F_i^2 \) satisfies
\[
F_i^2 = \max \left\{ \begin{array}{l}
i\tau_1 + \tau_2, \ \tau_1 + i\tau_2, \\
\tau_1 + \tau_2 + (i-1)\tau_3, \ \tau_1 + \tau_2 + (i-1)\tau_4, \\
i\tau_3, \ \tau_3 + (i-1)\tau_4 \\
(i-1)\tau_1 + \tau_2, \ \tau_1 + (i-1)\tau_2 \\
\end{array} \right\} + \tau_4
\]
\[
- \max \left\{ \begin{array}{l}
i\tau_1 + \tau_2 + (i-2)\tau_3, \ \tau_1 + \tau_2 + (i-2)\tau_4, \\
(i-1)\tau_3, \ \tau_3 + (i-2)\tau_4 \\
\end{array} \right\} \tag{4.25}
\]

We now evaluate separately the four cases \( \tau_M = \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \).
When $\tau_M = \tau_1$, by (4.25), $F_i^2$ satisfies

$$F_i^2 = \tau_1 + \tau_4, \quad \text{for all } i$$

When $\tau_M = \tau_2$, by (4.25), $F_i^2$ satisfies

$$F_i^2 = \tau_2 + \tau_4, \quad \text{for all } i$$

When $\tau_M = \tau_3$, by (4.25), $F_i^2$ satisfies

$$F_i^2 = \tau_3 + \tau_4, \quad \text{for all } i$$

Finally, when $\tau_M = \tau_4$, by (4.25), $F_i^2$ satisfies

$$F_i^2 = 2\tau_4, \quad \text{for all } i$$

In general, $F_i^2 = \tau_M + \tau_4$ for all $i$. 
CHAPTER 5. MARKOVIAN CONFLUENT PRODUCTION SYSTEMS

5.1 Introduction

In the previous chapter, we analyzed a certain deterministic confluent production systems consisting of two confluent sub-assembly lines feeding into one assembly line. In this chapter, the same system is analyzed, employing the Markovian concepts treated in Chapter 3.

The main point of this chapter is to identify and exploit the periodicity of the embedded Markov chain in the case of the above sort of assembly production system. This periodicity provides an especially tractable analysis, even when the number of stages is large, in the case of certain special assembly situations. Two of these are presented; one involves a general number of "parallel" stages feeding into an assembly stage; the second substitutes a two stage line for one of the above parallel feeding stages.

5.2 Periodicity of Confluent Production Systems

Consider a certain assembly situation, involving a general number \( N - 1 \) of "parallel" stages feeding into an assembly stage \( N \) (see Figure 5.1). It assumes that there is unlimited demand from the market (demand of stage \( N = \infty \)) and a
continual supply of material at \((N - 1)\) feeding stages.

To solve this sort of system, we first consider the two-stage tandem type production line (i.e., \(N = 2\)).

Utilizing the vector states in section 3.2, we have three possible vector states \((x_1, x_2)\), where \(x_j\) is the state of stage \(j\):

\[
(x_1, x_2) = \{(b, 1), (1, 0), (1, 1)\}
\]

and their transition diagram is shown in Figure 5.2.

Now, we define the vector states \((y_1, y_2)\), where

\[
y_1 : \text{state of the assembly stage}
\]

\[
y_2 : \text{total number of working stages for } j = 1, 2, \ldots, N - 1
\]

Then, the diagram in Figure 5.2 is changed to that of Figure 5.3. The reader will note that this system undergoes a change of state whenever a stage finishes service, and has period of \(N = 2\), as does the lineal system of Chapter 3.
Figure 5.2: Two-stage production line and transition diagram.

Figure 5.3: New transition diagram of two-stage production line.

Let a "parallel" feeding stage be attached to above two-stage line. Then, employing the vector states in (5.1), we have five states interconnected as shown in Figure 5.4.

Further, adding another a feeding stage to the above system, we have seven states and the corresponding transition diagram is shown in Figure 5.5.

Finally, the production system of an assembly stage and \((N - 1)\) feeding stages has \((2N - 1)\) states and its transition diagram is shown in Figure 5.6.

In general, this kind of system has the periodic character, with period \(d\) equal to the number \(N\) of stages.
Another type of assembly situation is considered by substituting a two stage line for one of the previous parallel feeding stages (see Figure 5.7).

To solve this kind of system, we first consider the three-stage tandem type production line discussed in Chapter 3.

Now, we define the vector states \((z_1, z_2)\), where

\[
\begin{align*}
  z_1 & : \text{state of the stage 2} \\
  z_2 & : \text{total number of working stages except stage 2}
\end{align*}
\]

Then, the previous Figure 3.1 is changed to that of Figure 5.8.

Let a “parallel” feeding stage be attached to the above three-stage line. Then, employing vector state \((z_1, z_2)\), we have ten states, and their transition diagram is
Figure 5.6: Transition diagram of an assembly stage and \((N - 1)\) feeding stages.

Further, adding another feeding stage to the above system, we have thirteen states and its transition diagram is shown in Figure 5.10.

In general, the above type of production system of a three-stage line and \((N - 3)\) parallel feeding stages has \((3N - 2)\) states and the transition diagram is shown in Figure 5.11. Note that the period is equal to the number \(N\) of stages.

So, systems such as the one considered here are periodic. In fact, the confluent production system generally does have periodic character, with period \(d\) equals to the number \(N\) of stages. This periodicity of the system thus allow an alternative approach to the problem of rapid increase of number of states.

5.3 Numerical Example

In section 5.2, we exploited the periodic nature for certain types of assembly structures involving a general number of feeding stages and an assembly stage. In this section, utilizing the performance measure expressions in section 3.3, we present numerical example of corresponding to Figure 5.9.

In general, it is impossible to obtain the transition matrix of an abstract diagram as shown in Figure 5.9. Thus, here we present the transition diagram Figure 5.12,
Figure 5.7: Production systems of 3-stage line and \((N - 3)\) feeding stages.

with the ordinary vector state definitions of section 3.2.

By Figure 5.12, we have a smallest subset \(S' = \{9, 10, 18\}\). From relation (3.1), it is possible to obtain the 3X3 submatrix \(Q'(\alpha, \beta)\), and vector \(\Pi'\). For the numerical example, with parameters \(E(\tau_1) = E(\tau_4) = 1.137\) and \(E(\tau_2) = E(\tau_3) = .863\), one finds, using steps analogous to those in subsection 3.4.3,

\[
Q' = \begin{bmatrix}
.1863 & .1414 & .6722 \\
.1990 & .1510 & .6500 \\
.1188 & .0902 & .7910
\end{bmatrix}.
\]
Figure 5.8: New transition diagram of 3-stage production line.

Figure 5.9: Production systems of 3-stage line and a parallel feeding stage, and the transition diagram.
Figure 5.10: Production systems of 3-stage line and two parallel feeding stages, and the transition diagram.

Figure 5.11: Transition diagram of the system involving 3-stage line and \((N - 3)\) parallel feeding stage.
Figure 5.12: Transition diagram of 3-stage line and a parallel feeding stage.

and

\[ \Pi' = \begin{bmatrix} .1363 & .1035 & .7602 \end{bmatrix}. \]

Analogously to subsection 3.2, the expected blocking times \( E(B_j), j = 1, 2, 3, \) are

\[ E(B_1) = .8639 \]
\[ E(B_2) = .7476 \]
\[ E(B_3) = .8639, \]

and the expected flow rate \( r, \) using (3.5), is

\[ r = .50. \]
Further, the expected flow time $E(F)$, using (3.6), is

$$E(F) = \sum_{j=1}^{4} E(\tau_j) + \sum_{j=1}^{3} E(B_j)$$

$$= 6.4753.$$

Finally, the average number of customers $\bar{L}$ in the system is

$$\bar{L} = r \cdot F$$

$$= 3.24.$$
CHAPTER 6. CONCLUSION

Certain types of production systems with blocking are analyzed both by deterministic and Markovian models. We first consider deterministic production lines in Chapter 2. Our study identifies three counter-intuitive aspects of the transient behavior of deterministic production lines. To begin with, flow time is found to be convex in item serial number, while stage blocking times which are essentially its addends, are concave in item serial number, in their (initial) transient phase. Secondly, items will move in “lock step” at any given stage throughout its transient phase, but never in steady state. Finally, steady state flow times, as a function of the vector of successive stage service times, can be discontinuous in a certain sense, when such vectors $\tau$ tend to a limiting vector $\tau^0$ with tied maximal service times, and the $\tau$'s approaching $\tau^0$ have flow times with ever-increasing transient phase lengths.

In Chapter 3, the production line is analyzed utilizing the Markovian concept. Our study identifies the periodicity of the pertinent embedded chain. This periodicity provides an especially tractable analysis, and thus allows an alternative approach to the problem of rapid increase of number of stages, and also is utilized to efficiently obtain asymptotic flow rate, stage blocking times, and flow time. Results are used to study the consequences of random service time variability and it is also suggested that optimality conclusions will depend on the performance measure considered.
In Chapter 4, we consider the deterministic analysis of confluent tree-structured production systems, i.e., production systems consisting of two sub-assembly lines feeding into an assembly line. In this model, utilizing the recursion relation approach discussed in Chapter 2, we deduce the steady state behavior of general such systems, and the transient behavior of specific such system, particularly with respect to the transiency of blocking times and flow times.

Finally, in Chapter 5, the assembly type production systems is analyzed employing the Markovian approach discussed in Chapter 3. For certain assembly structures, we confirm the periodicity of the pertinent embedded chain, and obtain asymptotic flow rate, stage blocking times, and flow times.
BIBLIOGRAPHY


