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Limit theorems for branching Markov processes

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Limit theorems for branching Markov processes

by

Hye-Jeong Kang

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
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Iowa State University
Ames, Iowa
1995

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1. PRELIMINARIES

1.1 Introduction

Branching processes form one of the classical fields of applied probability and deal with a mathematical representation of the development of a population whose members reproduce and die, subject to laws of chance. The particles may be of different types, depending on their age, energy, position, or other factors. However, they must not interfere with one another. This assumption, which unifies the mathematical theory, seems justified for some populations of physical particles such as neutrons or cosmic rays, but only under very restricted circumstances for biological populations. There is a natural classification of branching processes according to their criticality condition, their time parameter, the single or multi-type particle cases, the Markovian or non-Markovian character of the processes, etc.

In the rest of this chapter we review basic definitions and some results in branching processes such as extinction probability, growth rates of population, additive property of branching processes, and convergence of age distribution. We also give in chapter 1 a brief resume of some results in renewal theory which play an important role in chapter 3. In this thesis we present results for three different models.

In the second chapter we consider a supercritical Markov branching process in which particles move according to a process with stationary independent increments.
We present conditions for convergence of the normalized empirical distribution of the positions of the particles at time $t$, as $t$ goes to infinity.

In chapter 3 we prove central limit theorems for functionals of the empirical age distribution of Bellman-Harris processes. In other words, for a real valued function $f$ on the nonnegative reals that integrates to zero with respect to the stable age distribution in a supercritical Bellman-Harris process with no extinction we present sufficient conditions for the asymptotic normality of the mean of $f$ with respect to the empirical age distribution at time $t$.

The last model in chapter 4 is named positive recurrent branching Markov chains. In this case we superimpose a Markovian movement structure on a simple Galton-Watson process. That is, a particle reproduce its offsprings in the usual way but the offsprings choose their positions according to a Markov process. The Markov process is assumed positive recurrent for the discrete state space case and Harris-recurrent for the continuous case. We prove first the law of large numbers for the empirical position distribution. Then we discuss the large deviation aspects of these convergences under some finite moment assumption of offspring distribution.

1.2 Preliminary Results in Branching Processes

This section contains a short survey of the major problems of branching processes and some results which are needed in this thesis. The results are fundamental and may be found in any book on branching processes. We mention four books which together cover the classical theory, Harris(1963), Athreya and Ney(1972), Mode (1971), and Jagers(1975).
1.2.1 The Extinction and Growth of Populations

The simplest type of a branching process is the Galton-Watson process which is a Markov chain. The process can be thought of as representing an evolving population of particles. It starts at time 0 with $Z_0$ particles, each of which (after one unit of time) splits independently of the others into a random number, $\xi$ say, of offspring according to the probability law $\{p_k\}$. The total number $Z_1$ of particles thus produced is the sum of $Z_0$ independent random variables, each with probability law $\{p_k\}$. That is,

$$Z_1 = \sum_{j=1}^{Z_0} \xi_{0,j},$$

where $\{\xi_{0,j}, j = 1, 2, \ldots\}$ are i.i.d. copies of $\xi$. It constitutes the first generation. These go on to produce a second generation of $Z_2$ particles, and so on. The number of particles in the $(n+1)$th generation is a random variable $Z_{n+1}$ which is the sum of $Z_n$ independent random variables with probability law $\{p_k\}$, in other words,

$$Z_{n+1} = \sum_{j=1}^{Z_n} \xi_{n,j},$$

where $\{\xi_{n,j}, j = 1, 2, \ldots, n = 1, 2, \ldots\}$ are i.i.d. copies of $\xi$. The question of eventual extinction of such a process is answered with the aid of the generating function. Let

$$h(s) = \sum_{k=0}^{\infty} p_k s^k, \quad m = h'(1) = \sum_{k=1}^{\infty} kp_k < \infty.$$

We assume throughout that $Z_0 = 1$ a.s.

**Theorem 1.1** Let $q = P(Z_n = 0$ for some $n \geq 1)$. Then

(a) $q$ is the smallest non-negative solution to the equation $s = h(s)$.

$b) P(Z_n \to \infty) = 1 - q.$
The cases \( m < 1, m = 1, \) and \( m > 1 \) are called the subcritical, the critical, and the supercritical cases respectively. Part (b) states that if a Galton-Watson population does not die out, its size will tend to infinity. Hence, no independently reproducing population ever stabilizes in size; it either dies out or grows beyond all bounds.

One extension is to assume that an individual lives for a random amount of time, say \( \lambda \), with distribution function \( G \), then dies and at time of death gives birth to a random number of offspring, \( \xi \), with probability law \( \{ p_k \} \). Individuals still reproduce independently and the Galton-Watson process can be viewed as the special case, \( \lambda \equiv 1 \). Rather than \( Z_n \), the number of individuals in the \( n \)th generation, the number of individuals alive at time \( t \), \( Z(t) \), is studied. In this case, we have the following analog of Theorem 1.1.

**Theorem 1.2** Let \( q = P(Z(t) = 0 \text{ for some } t > 0) \), then

(a) \( q \) is the smallest non-negative solution to the equation \( s = h(s) \).

\[ q \text{ is equal to } 1 \text{ if } m \leq 1 \text{ and } < 1 \text{ if } m > 1. \]

(b) \( P(Z_t \to \infty) = 1 - q \).

This process is in general not Markovian, unless the lifetimes are independent, exponentially distributed random variables. The latter process is called continuous time Markov branching process and the general non-Markovian case is called age-dependent branching process or Bellman-Harris process.

In the supercritical case, the growth of population is described by a real number \( \alpha \), called Malthusian parameter.
Definition 1.1 The Malthusian parameter $\alpha$ for $m$ and $G$ is the root, provided it exists, of the equation

$$m\hat{G}(\alpha) = 1, \quad (1.1)$$

where hat denotes the Laplace transform, i.e., $\hat{G}(\alpha) = \int_0^\infty e^{-\alpha y}G(dy)$.

Due to the strict monotonicity and continuity of $\hat{G}(\alpha)$, such a root, when it exists, is always unique. We note that when $m \geq 1$ such a Malthusian parameter $\alpha$ always exists and is necessarily nonnegative. For $m < 1$ a Malthusian parameter $\alpha$ satisfying (1.1) need not exist. The next two theorems are concerned with asymptotic behavior of the critical and supercritical case, respectively.

Theorem 1.3 Let $m = 1$. If $h''(1) = \sigma^2 < \infty$, $\int_0^\infty t^2 dG(t) = \mu < \infty$, and $t^2(1 - G(t)) \to 0$ as $t \to \infty$, then

(a) $\lim_{t \to \infty} P\{Z(t) \leq x | Z(t) > 0\} = 1 - e^{-(2\mu/\sigma^2)x}, \quad x \geq 0.$

(b) $P(Z(t) > 0) \sim \frac{2\mu 1}{\sigma^2 t}.$

Theorem 1.4 Assume that $m > 1$. Let $W(t) = e^{-\alpha t}Z(t)$.

(a) If $\sum p_j j \log j = \infty$ then $W(t) \to 0$ w.p.1.

(b) If $\sum p_j j \log j < \infty$ then $W(t)$ converges w.p.1 to a nonnegative random variable $W$ having the following properties (assume $Z_0 = 1$);

(i) $P(W = 0) = q.$

(ii) The distribution of $W$ is absolutely continuous on $(0, \infty)$.

(iii) $\varphi(u) = Ee^{-uW}, u \geq 0$ is the unique solution of the equation

$$\varphi(u) = \int_0^\infty f(\varphi(ue^{-\alpha y}))dG(y),$$

subject to $\varphi(0+) = 1$ and $\varphi'(0) = -1.$
The process can be made more general by also allowing individuals to be of different types. The population consists of \( p \) types of particles, whose reproductive behavior is governed by a \( p \)-dimensional generating function \( h(s) \) where

\[
h(s) = (h_1(s), \ldots, h_p(s)),
\]

\[
h_i(s) = h_i(s_1, \ldots, s_p) = \sum_{j_1, \ldots, j_p \geq 0} p_i(j_1, \ldots, j_p) s_1^{j_1} \ldots s_p^{j_p},
\]

\[0 \leq s_k \leq 1, \quad 1 \leq k \leq p,
\]

\[
p_i(j_1, \ldots, j_p) = \text{the probability that a type } i \text{ parent produces } j_1 \text{ particles of type 1, } j_2 \text{ particles of type 2}, \ldots, j_p \text{ particles of type } p.
\]

Let \( Z(t) = (Z_1(t), \ldots, Z_p(t)) \) denote the number of particles of the various types existing at time \( t \). The lifetime \( \lambda_i \) of a type \( i \) particle is a random variable with distribution \( G_i(\cdot), i = 1, \ldots, p \) and a type \( i \) particle reproduces \( \xi_{ij} \) particles of type \( j \) on its death. Let \( m_{ij} = E(\xi_{ij}) \) and let \( M = ((m_{ij}))_{i,j=1}^p \) be the particle production mean matrix associated with \( h(s) \). We assume that \( M \) is positively regular and nonsingular throughout this chapter. If we write \( \rho(M) \) for its Perron-Frobenius root which is the maximal eigenvalue of \( M \), the cases \( \rho(M) < 1 \), \( \rho(M) = 1 \) and \( \rho(M) > 1 \) are called the subcritical, the critical and the supercritical case, respectively. Here again is an analog to the concept of a Malthusian parameter for Bellman-Harris process.

**Definition 1.2** Let \( \bar{M}(\alpha) = ((m_{ij} \bar{G}_i(\alpha)))_{i,j=1}^p \). The Malthusian parameter \( \alpha \) for \( M \) and \((G_1, \ldots, G_p)\) is defined to be the number \( \alpha \) which satisfied the equation \( \rho(\bar{M}(\alpha)) = 1 \), provided it exists. In the critical and supercritical cases, the Malthusian parameter \( \alpha \) always exists and is nonnegative.
Growth rate is again related to this Malthusian parameter in the supercritical case. Let $u$ and $v$ be the left- and right-eigenvector of $\tilde{M}(\alpha)$ corresponding to eigenvalue 1 such that $1 \cdot v = 1, u \cdot v = 1$.

**Theorem 1.5** Suppose that $\rho(M) > 1$, and $d_{ij} = E(\xi_{ij}^2) < \infty, i, j = 1, \ldots, p$.

Let $W(t) = e^{-\alpha t}Z(t)$, then

(a) $\lim_{t \to \infty} W(t) = W$ exists a.s.

(b) there exists a scalar random variable $W$ such that $W = \eta W$ a.s., where

$$\eta = (\eta_1, \ldots, \eta_p), \eta_i = u_i(1 - \tilde{G}_i(\alpha)).$$

### 1.2.2 Additive Property of Branching Processes

The Kolmogorov consistency theorem for stochastic processes (Chow and Teicher (1988)) assures us that we can construct a probability space $(\Omega, \mathcal{F}, P)$, on which the process lives. In this construction each point $\omega$ in the sample space $\Omega$ represents a complete "family tree" (or "family history") specifying the time of birth, life length, ancestors and descendants of each particle. The $\sigma$-algebra $\mathcal{F}$ is to be taken large enough so that $\{Z(t, \omega), t \geq 0\}$ (and other similar random variables which will be studied) are measurable functions on $(\Omega, \mathcal{F})$ and an appropriate probability measure is constructed on the Borel extension of the cylinder sets of this space. Such a construction has been carried out by T. Harris in chapter 6 of his book (1963). So when we need to discuss such matters as a.s. convergence, equivalence of various statements about the process, we will not hesitate to refer to the space $(\Omega, \mathcal{F}, P)$.

One of the basic assumptions of most of the branching process models is that of the independence of lines of descent. This independence of lines of descent leads to what may be called the additive property in branching process. More precisely, for
the single type Galton-Watson process it states that for any \( n \), the stochastic process \( \{Z_{n+m}(\omega); m = 1, 2, \ldots, \} \) when conditioned on \( \{Z_0, Z_1, \ldots, Z_n\} \) equivalent to

\[
\sum_{j=1}^{Z_n(\omega)} Z_{m,j}(\omega), \quad m = 1, \ldots,
\]

where \( \{Z_{m,j}(\omega); m = 1, 2, \ldots\} \) for \( j = 1, 2, \ldots \), are independent copies of \( \{Z_m; m = 1, 2, \ldots\} \) with \( Z_{m,0}(\omega) = 1 \) w.p.1, and independent of \( \{Z_0, Z_1, \ldots, Z_n\} \). This is referred to as the additive property. We use this in conjunction with classical results from the theory of sums of independent random variables to obtain many of the limit properties of branching models.

For a Bellman-Harris process of a single type the additive property can be stated in the same way. In fact, for a measurable function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \), define

\[
Z_f^\alpha(t, \omega) = \sum_{j=1}^{Z(t, \omega)} f(a_j(t, \omega)),
\]

where \( \{a_j(t, \omega), j = 1, \ldots, Z(t, \omega)\} \) is the age-chart at time \( t \) and superscript \( \alpha \) indicates that \( \mathbb{P} \) is supported by those \( \omega \)'s which start with one particle of age \( \alpha > 0 \). We simply write \( Z_f(t, \omega) \) for \( Z_f^\alpha(t, \omega) \). By the additive property of branching processes, we can write

\[
Z_f(t + s, \omega) = \sum_{j=1}^{Z(t, \omega)} Z_f^{a_j(t, \omega)}(s, \omega),
\]

where

\[
Z_f^{a_j(t, \omega)}(s, \omega) = \sum_{k=1}^{Z(s, \omega)} f(a_{jk}(s + t, \omega)),
\]

\( a_{jk}(s + t, \omega) \) = the age at time \( t + s \) of \( k \)th particle in a line of descent initiated by a particle of age \( a_j(t, \omega) \) living at time \( t \).

It is clear that conditioned on the age chart at time \( t \), \( \{Z_f^{a_j(t, \omega)}(s, \omega); j = 1, \ldots\} \) are independently distributed and further if \( a_j(t, \omega) = \alpha \), then the conditional distribution
of $Z_j^{a(t,\omega)}(s,\omega)$ is the same as $Z_j^0(s,\omega)$.

In the multitype Bellman-Harris process, we define

$$Z_j^0(t,\omega) = \sum_{k=1}^{p} \sum_{j=1}^{Z_k(t,\omega)} f(a_{kj}(t,\omega)),$$

where $\{a_{kj}(t,\omega); j = 1, \cdots, Z_k(t,\omega)\}$ is the age-chart at time $t$ of type $k$ particles, $k = 1, \ldots, p$, and superscript $a$ indicate that the ancestor is of age $a$ at time 0. In particular, if $f \equiv 1$,

$$Z_j^a(t,\omega) = \sum_{k=1}^{p} Z_k^a(t,\omega) = |Z^a(t,\omega)|$$

denotes the number of living particles at time $t$ when the ancestor is of age $a$ at time 0. A moment's reflection is enough to arrive at the following representation of $Z_j(t+s,\omega)$ as

$$Z_j(t+s,\omega) = \sum_{k=1}^{p} \sum_{j=1}^{Z_k(t,\omega)} Z_j^{a_{kj}(t,\omega)}(s,\omega)$$

(1.4)

where $\{a_{kj}(t,\omega); j = 1, \cdots, Z_k(t)\}, k = 1, \cdots, p$, is the age chart of type $k$ particles at time $t$,

$$Z_j^{a_{kj}(t,\omega)}(s,\omega) = \sum_{i=1}^{p} \sum_{i=1}^{Z_i(s,\omega)} f(a_{kj}^i(t+s,\omega)),$$

and $\{a_{kj}^i(t+s,\omega); i = 1, \cdots, Z_i(s,\omega)\}$ is the age chart at time $t+s$ of type $l$ particles in a line of descent initiated by the $j$th particle of type $k$ of age $a_{kj}(t,\omega)$ living at time $t$. Furthermore, $\{Z_j^{a_{kj}(t,\omega)}(s,\omega), k = 1, \cdots, p, j = 1, \cdots, Z_k(t)\}$ are independently distributed when conditioned on the age chart at time $t$. If $a_{kj}(t,\omega) = a$, then the conditional distribution of $Z_j^{a_{kj}(t,\omega)}(s,\omega)$ is the same as $Z_j^a(s,\omega)$ which starts with one type $k$ particle whose initial age is $a$.

The usefulness of these decompositions are well illustrated by the proof of convergence of age distributions (see Athreya and Kaplan(1976) for example). We apply
this technique in chapter 2 on branching Markov processes (see Chapter 2 for definition and decomposition) to prove the convergence of the joint empirical distribution of the age and the scaled position of particles and in chapter 3 to prove central limit theorems for the age distribution in Bellman-Harris processes.

1.2.3 Age Distribution in the Bellman-Harris Processes

An important and useful aspect of Bellman-Harris processes is the limiting behavior of the age distribution. There has been considerable interest shown in the past in the limiting behavior of the age distribution.

Single type case We consider a Bellman-Harris process with offspring distribution \{p_j\} and lifetime distribution \(G(\cdot)\). We make the following standard assumptions throughout in this subsection; \(G\) is non-lattice and \(G(0+) = 0\).

Now, introduce the following notation. For any family history \(\omega\) let:

\[
Z(t, \omega) = \text{the number of particles living at time } t,
\]

\[
Z(t, x, \omega) = \text{the number of particles living at time } t \text{ whose age } \leq x,
\]

\[
A(t, x, \omega) = \frac{Z(t, x, \omega)}{Z(t, \omega)}, \quad \text{if } Z(t, \omega) > 0,
\]

\[
A(x) = \frac{\int_0^x e^{-\alpha u}[1 - G(u)]du}{\int_0^\infty e^{-\alpha u}[1 - G(u)]du},
\]

\[
G^x(t) = \frac{G(x + t) - G(x)}{1 - G(x)}.
\]

**Theorem 1.6** Assume \(1 < m < \infty, p_0 = 0\). If \(\sum_{j=1}^\infty p_j(j \log j) < \infty\), then

\[
\lim_{t \to \infty} \sup_{x \geq 0} |A(t, x, \omega) - A(x)| = 0 \quad a.s.
\]
Remark 1.1 The assumption $p_0 = 0$ is primarily for the convenience of exposition. Otherwise one has to qualify “on the set of explosion.” We have the immediate

Corollary 1.1 Under the hypothesis of Theorem 1.6

$$\int_0^\infty h(x)A(t,dx,\omega) \xrightarrow{a.s.} \int_0^\infty h(x)dA(x) \quad \text{as} \quad t \to \infty,$$

for any $h(\cdot)$ which is bounded and continuous a.e. (w.r.t. Lebesgue measure) on the support of $G$.

For the critical case we have

Theorem 1.7 Let $m = 1$. Assume also that $\limsup\limits_{t \to \infty} [1 - G^2(t)] = 0$, then for any $\epsilon > 0$,

$$\lim_{t \to \infty} P(\sup_{x \geq 0} |A(t,x,\omega) - \lambda(x)| > \epsilon |Z(t) > 0| = 0.$$

Corollary 1.2 Under the assumptions of Theorem 1.7, for any $\epsilon > 0$,

$$\lim_{t \to \infty} P(|\int_0^\infty g(x)A(t,dx) - \int_0^\infty g(x)A(dx)| > \epsilon |Z(t) > 0| = 0,$$

where $g(\cdot)$ is bounded and continuous a.e. (w.r.t. Lebesgue measure) on the support of $G$.

Multitype case There are analogs of Theorem 1.6 and Corollary 1.1 for the multitype Bellman-Harris processes. We assume that for each $i = 1, \ldots, p$, $G_i$ is non-lattice, and $G_i(0+) = 0$. For any family history $\omega$ let:

$$Z_k(t,\omega) = \text{the number of type } k \text{ particles at time } t,$$

$$Z(t,\omega) = (Z_1(t,\omega),\ldots,Z_p(t,\omega)), \quad |Z(t,\omega)| = \sum_{k=1}^p Z_k(t,\omega),$$
\[ Z_k(t, x, \omega) = \text{the number of type } k \text{ particles at time } t \text{ whose age } \leq x, \]

\[ A_k(t, x, \omega) = \frac{Z_k(t, x, \omega)}{Z(t, \omega)} \text{ for } Z_k(t, \omega) > 0, \]

\[ A_k(x) = \frac{\int_0^x e^{-xu}(1 - G_k(u))du}{\int_0^x e^{-xu}(1 - G_k(u))du}. \]

Recall that Malthusian parameter \( \alpha \) for \( M \) and \((G_1, \ldots, G_p)\) is that number such that

\[ \rho(\bar{M}(\alpha)) = 1, \]

where \( M = ((m_{ij}))_{i,j=1}^p, \bar{M}(\alpha) = ((m_{ij}\bar{g}_i(\alpha)))_{i,j=1}^p \) and that \( u = (u_1, \ldots, u_p) \), and \( v = (v_1, \ldots, v_p) \) denotes the left- and right-eigenvector of \( \bar{M}(\alpha) \) corresponding to \( 1 \), respectively.

**Theorem 1.8** Assume the process is supercritical, that is, \( \rho(M) > 1 \). Assume the 'j log j' condition on the offspring distributions; \( E(\xi_{ij} \log^+ \xi_{ij}) < \infty, \quad i,j = 1, \ldots, p. \)

Then, for \( k = 1, \ldots, p, \) on the set of nonextinction,

\[ \sup \frac{Z_k(t, x, \omega)}{|Z(t, \omega)|} - c_0 u_k \int_0^x e^{-xu}(1 - G_k(u))du \xrightarrow{a.s.} 0 \quad \text{as } t \to \infty, \]

where

\[ c_0 = \left[ \sum_{j=1}^p u_j \int_0^\infty e^{-xu}(1 - G_j(u))du \right]^{-1}. \]

With \( x = \infty \), we have for each \( k = 1, \ldots, p \)

\[ \frac{Z_k(t, \omega)}{|Z(t, \omega)|} \xrightarrow{a.s.} c_0 u_k \int_0^\infty e^{-xu}(1 - G_k(u))du \quad \text{as } t \to \infty. \]

Since \( u_k > 0, k = 1, \ldots, p \), we know that \( Z_k(t, \omega) \xrightarrow{a.s.} \infty \) as \( t \to \infty \) on the set of nonextinction. So we have the following
Corollary 1.3 Under the hypothesis of Theorem 1.8, on the set of nonextinction,
\[ \sup_{x \geq 0} \left| \frac{Z_k(t, x, \omega)}{Z_k(t, \omega)} - A_k(x) \right| \xrightarrow{a.s.} 0 \quad \text{as} \quad t \to \infty. \]

Remark 1.2

1. The proof of Theorem 1.8 as well as the more general case of the multitype Crump-Mode-Jagers process can be found in Ramamurthy (1976).

2. Even though the hypothesis of Theorems in this subsection are not minimal, these are good enough for many cases. For the finer results, refer to Kuczek (1982) and Ramamurthy (1976).

1.3 Renewal Theory and Its Applications to Branching Processes

Renewal theory is one of the very fundamental and important areas of probability theory in the sense of the applicability to, and strong implications for, a number of other fields. As we can see in the subsection 1.3.2, asymptotics of the expected population size are also analyzed by means of the renewal theorem. So we give in the next subsection a brief resume of some results needed for this purpose, following Asmussen (1987) and Athreya and Ney (1978). There is also a new result in subsection 1.3.3 on the convergence rate of the renewal function for special cases.

1.3.1 Renewal Theory

Let \( X_1, X_2, \ldots \), be i.i.d. random variables with common distribution \( G \) which is supported by \([0, \infty)\). Define \( S_0 = 0, S_n = \sum_{j=1}^{n} X_j \) for \( n \geq 1 \). Then \( \{S_n; n \geq 0\} \) is called a pure renewal process and the distribution of \( S_n \) is simply \( G^n \), the \( n \)-fold
convolution of $G$ with itself, and $G^*(t) = 1$ for $t \geq 0$. We define the renewal measure by

$$U(dx) = \sum_{n=0}^{\infty} G^{*n}(dx)$$

and the renewal function $U$ by

$$U(t) = \sum_{n=0}^{\infty} G^{*n}(t).$$

The renewal equation is the convolution equation $R = r + G * R$, i.e.,

$$R(t) = r(t) + \int_{[0,t]} R(t-u)G(du), \quad t \geq 0, \quad (1.5)$$

where $R$ is an unknown function on $[0, \infty)$, $r$ a known function on $[0, \infty)$ and $G$ a known nonnegative measure on $[0, \infty)$. In this section it is assumed that $G$ is a probability distribution on $[0, \infty)$. The next theorem deals with existence and uniqueness of solutions to equation (1.5).

**Theorem 1.9** Assume $G(0^+) = 0$.

(a) The renewal function $U(t)$ is finite for all $t < \infty$, and $U(t) = EN_t$, where $N_t = \sup_{n \geq 0} \{ n : S_n \leq t \}$, the number of renewals up to time $t$ in a pure renewal process with interarrival distribution $G$.

(b) If the function $r$ in the renewal equation (1.5) is Borel-measurable and bounded on finite intervals, then $R = U * r$ (i.e. $R(t) = \int_{[0,t]} r(t-x)U(dx)$) is a well-defined solution to (1.5) and is the unique solution to (1.5) which is bounded on finite intervals.

Before we state some versions of the renewal theorem, we need a definition.
Definition 1.3 A function $r \geq 0$ is directly Riemann integrable (in short, d.R.i.) if

(i) $h \sum_{n=1}^{\infty} \bar{r}_h(n) < \infty$ for some $h > 0$,

(ii) $h \left( \sum_{n=1}^{\infty} \bar{r}_h(n) - \sum_{n=1}^{\infty} r_h(n) \right) \to 0$ as $h \to 0$,

where $\bar{r}_h(n) = \sup_{n h \leq x \leq (n+1) h} r(x)$ and $r_h(n) = \inf_{n h \leq x \leq (n+1) h} r(x)$.

For general $r$, we say that $r$ is d.R.i. if both $r^+$ and $r^-$ are so.

For functions with compact support this concept is the same as Riemann integrability, whereas in the general case it is somewhat stronger than Lebesgue integrability.

Some sufficient conditions for d.R.i. of $r$ are:

(a) $r \geq 0$, bounded continuous and $\sum \bar{r}_1(n) < \infty$;

(b) $r \geq 0$, nonincreasing, and Riemann integrable;

(c) $r$ is continuous a.e. and bounded by a d.R.i. function.

Definition 1.4 A random variable $X$ is lattice if there exists $d > 0$ and $a > 0$ such that $\frac{X-a}{d}$ is integer-valued with probability 1. A c.d.f. $G(\cdot)$ is lattice if the random variable $X$ with c.d.f. $G(\cdot)$ is lattice.

Theorem 1.10 Let $G$ be non-lattice, $G(0+) = 0$ and let $\mu = \int_0^{\infty} uG(du) < \infty$.

(a) Blackwell's Renewal Theorem Let $U(t) = \sum_{n=0}^{\infty} G^n(t)$ be the renewal function with $G$ a probability distribution. Then for all $a > 0$,

$$\lim_{t \to \infty} \{U(t+a) - U(t)\} = \frac{a}{\mu}. \tag{1.6}$$

(b) Key Renewal Theorem Suppose that the function $r$ in the renewal equation

$$(1.5)$$

is d.R.i. Then

$$\lim_{t \to \infty} R(t) = \lim_{t \to \infty} (U * r)(t) = \frac{1}{\mu} \int_0^{\infty} r(x)dx. \tag{1.7}$$
Remark 1.3 If \( \mu = \infty \) then the limits in (1.6) and (1.7) are defined as zero.

Now consider the system of renewal equations

\[
m_i(t) = \zeta_i(t) + \sum_{j=1}^{\infty} \int_{[0, t]} m_j(t - u) F_{ij}(du), \quad i = 1, 2, \ldots,
\]

where \( F_{ij} \) is a matrix of nondecreasing right continuous nonnegative functions on \([0, \infty)\), and \( \{\zeta_i(\cdot)\} \) is a vector of measurable functions on \([0, \infty)\) that are bounded on finite intervals. We shall call the system (1.8) semi-Markov if \( P \equiv (p_{ij}) \), \( p_{ij} = F_{ij}(\infty) = \lim_{t \to \infty} F_{ij}(t) \), is the transition probability matrix of a Markov chain. A semi-Markov system is easily studied via an associated semi-Markov process. Let \( \{X_n\} \) be a Markov chain with \( P \) as its transition probability matrix. Conditioned on a realization \( \{X_n = x_n\} \), generate nonnegative random variables \( \{L_n\} \) such that

(i) the \( L_n \)'s are independent, and

(ii) \( P(L_n \leq l | \{X_n = x_n\}) = \frac{F_{x_i,x_{i+1}}(l)}{F_{x_i,x_{i+1}}(\infty)} \).

We assume that \( \{X_n\} \) is irreducible and recurrent. Fix \( i_0 \) and let

\[
N = \inf\{n; n \geq 1, X_n = i_0\}, \quad T = \sum_{i=0}^{N-1} L_i.
\]

Theorem 1.11 Assume

(i) \( P_{i_0}(T \leq u) \) is nonlattice in \( u \),

(ii) For some nontrivial stationary distribution \( \{\pi_j\} \) for \( P \)

\[
c_0 \equiv \sum_j \pi_j \int_0^\infty P_j(L_0 > u)du = \sum_j \pi_j \left( \sum_k \int_0^\infty (F_{jk}(\infty) - F_{jk}(u))du \right) < \infty,
\]

(iii) \( \zeta_i(\cdot) \) is continuous a.e. for each \( i \),

(iv) \( \sup_{i,t} \zeta_i(t)(1 - F_{ij}(t))^{-1} < \infty \).
Then, the solution \(\{m_i(\cdot)\}\) to (1.8) exists, is unique, and satisfies

\[
\lim_{t \to \infty} m_i(t) = \frac{1}{c_0} \sum_j \pi_j \int_0^\infty \zeta_j(u) du.
\]

Now let \((S, \mathcal{S})\) be a measurable space and \(\{\mu(x, \cdot); x \in S\}\) be a family of measures on \((S \times [0, \infty), \mathcal{S} \times \mathcal{B}[0, \infty))\), \(\mathcal{B}\) being the Borel sets. Consider the system of renewal equations

\[
m(x, t) = \zeta(x, t) + \int \int_S m(x', t-u) \mu(x, d(x' \times u)),
\]

where \(\zeta(\cdot, \cdot)\) is a given measurable function. We call the system (1.9) semi-Markov if \(\mu(x, A \times [0, \infty)) \overset{\text{def}}{=} P(x, A)\) is a transition probability function on \(S \times S\). In this case let \(\{X_n; n \geq 0\}\) be the Markov chain associated with \(P(\cdot, \cdot)\). To state an analog of the Key Renewal Theorem for this system we need a notion of recurrence for \(\{X_n\}\).

**Definition 1.5** \(\{X_n\}\) is \((A, \varepsilon, \varphi, n_0)\)-recurrent if there exists a set \(A \in \mathcal{S}\), a probability measure \(\varphi\) on \(A\), a constant \(\varepsilon > 0\) and an integer \(n_0\) such that

(i) \(P_x(X_n \in A \text{ for some } n \geq 1) = 1 \text{ for all } x \in S\),

(ii) \(P_x(X_{n_0} \in B) \geq \varepsilon \varphi(B) \text{ for all } x \in A \text{ and } B \subset A\).

This notion of recurrence is equivalent to the more standard definition of Harris recurrence (see Athreya and Ney (1978)). Under this condition it can be shown that there exists a stationary measure for \(P\), say \(\pi(\cdot)\), such that

\[
\pi(\cdot) = \int_S P(x, \cdot) \pi(dx).
\]

We assume \(\mathcal{S}\) to be countably generated. We can then find a function \(G(x, x', t)\) which is jointly measurable in \((x, x', t)\), is a probability measure in \(t\) for fixed \((x, x')\),
and in terms of which (1.9) can be rewritten as

\[ m(x, t) = \zeta(x, t) + \int_0^t \int m(x', s - u)G(x, x', du)P(x, dx'). \]  

(1.10)

If we further assume that \( G(x, x', t) = G(x, t) \) is independent of \( x' \), then we have the following result (see Athreya, McDonald, and Ney(1978)).

**Theorem 1.12** Assume that \( \{X_n\} \) is \((\varepsilon, \varphi, 1)\)-recurrent, \( G(x, t) \) is non-lattice in \( t \) for all \( x \), and that \( \theta = \int_0^\infty (1 - G(x, t)) dt \pi(dx) < \infty \). If \( \zeta \) satisfies

(i) \( \pi(x; \zeta(x, t) \) is discontinuous for some \( t) = 0 \),

(ii) \( \int_0^\infty \pi(dx) \sum_{n=0}^\infty \sup\{ |\zeta(x, t)|; nh \leq t < (n + 1)h \} < \infty \) for some \( h > 0 \),

then for all \( x \in S \), the solution of (1.10) satisfies

\[ \lim_{t \to \infty} m(x, t) = \frac{1}{\theta} \int_S \pi(dx) \int_0^\infty \zeta(x, t)dt. \]

**1.3.2 Application to Branching Processes**

We will now see how renewal theory is applied for Bellman-Harris processes. Let \( f : \mathbb{R}^+ \to \mathbb{R} \) be a measurable function. Let \( \{Z(t); t \geq 0\} \) be a one-dimensional supercritical Bellman-Harris process evolving from 1 particle of age 0 at time \( t = 0 \).

By an abuse of notation we shall rewrite (1.2) as

\[ Z_f(t) = \sum_{j=1}^{Z(t)} f(a_j) \]

suppressing \( \omega \) and \((t, \omega)\). Define \( m_f(t) = E(Z_f(t)) \). Note that

\[ Z_f(t) = I(\lambda_0 > t)f(t) + \sum_{j=1}^\xi Z_{f,j}(t - \lambda_0), \]  

(1.11)
where \( \lambda_0 \) and \( \xi \) are the lifetime random variable of the ancestor and the number of offspring produced by the ancestor, respectively, and \( \{Z_f,j(u); u \geq 0\} \) are independent copies of \( \{Z_f(u); u \geq 0\} \). Since we assumed the independence of \( \lambda_0 \) and \( \xi \),

\[
m_f(t) = f(t)(1 - G(t)) + m \int_0^t m_f(t - u)G(du).
\]

(1.12)

To apply Renewal Theorems, we multiply \( e^{-\alpha t} \) both sides of (1.12), then we get a renewal equation

\[
e^{-\alpha t}m_f(t) = e^{-\alpha t}f(t)(1 - G(t)) + \int_0^t e^{-\alpha (t-u)}m_f(t-u)\mu_\alpha(du)
\]

where \( \alpha \) is the Malthusian parameter for \( m \) and \( \mu_\alpha(du) = me^{-\alpha u}G(du) \). Note that \( \mu_\alpha \) is a probability distribution. So (Theorem 1.9),

\[
e^{-\alpha t}m_f(t) = \int_0^t e^{-\alpha(t-u)}f(t-u)(1 - G(t-u))U_\alpha(du),
\]

where \( U_\alpha(t) = \sum_{n=0}^{\infty} \mu_\alpha^n(t) \). Furthermore if \( e^{-\alpha t}f(t)(1 - G(t)) \) is d.R.i.,

\[
\lim_{t \to \infty} e^{-\alpha t}m_f(t) = \frac{1}{\beta} \int_0^\infty e^{-\alpha u}f(u)(1 - G(u))du,
\]

where \( \beta = \int_0^\infty u\mu_\alpha(du) = m \int_0^\infty u e^{-\alpha u}G(du) \). In particular, if we take \( f \equiv 1 \),

\[
\lim_{t \to \infty} e^{-\alpha t}E(Z(t)) = \frac{1}{\beta} \int_0^\infty e^{-\alpha u}(1 - G(u))du,
\]

that is, the population size grows exponentially fast with rate \( \alpha \).

Now consider the multitype supercritical case. Define \( m_f(t) = E(Z_f(t)|Z(0) = e_i) \). Given \( Z(0) = e_i \),

\[
Z_f(t) = I(\lambda_i > t)f(t) + \sum_{k=1}^{p} \sum_{j=1}^{\xi_{ik}} Z_{f,j}(t - \lambda_i),
\]

(1.13)
where \( \{Z_{fj}(u); u \geq 0\} \) are i.i.d. copies of \( \{Z_f(u); u \geq 0\} \) which is initiated by an ancestor of type \( k \). Taking expectation we get the following system of renewal equations,
\[
m_f(t) = (1 - G_i(t))f(t) + \sum_{k=1}^{p} m_{ik} \int_0^t k m_f(t-u)G_i(du), i = 1, \ldots, p
\]  
(1.14)

Multiply \( e^{-\alpha t} \) both sides of (1.14) to get
\[
e^{-\alpha t}m_f(t) = e^{-\alpha u}f(u)(1 - G_i(u)) + \sum_{k=1}^{p} \int_0^t e^{-\alpha (t-u)} k m_f(t-u) F_{ik}(du),
\]
where \( \nu = (\nu_1, \ldots, \nu_p) \) is the right-eigenvector of \( \bar{M}(\alpha) \) corresponding to 1, and
\[
F_{ik}(t) = \frac{\nu_k}{\nu_i} m_{ik} \int_0^t e^{-\alpha u} G_i(du). 
\]
For each \( i, i = 1, \ldots, p, \)
\[
\sum_{k=1}^{p} F_{ik}(\infty) = \frac{1}{\nu_i} \sum_{k=1}^{p} \bar{M}_{ik}(\alpha) \cdot \nu_k = \frac{1}{\nu_i} \cdot \nu_i = 1,
\]
since \( \nu = (\nu_1, \ldots, \nu_p) \) is the right eigenvector of \( \bar{M}(\alpha) \) corresponding to 1. That is, \( \nu = (\nu_1, \ldots, \nu_p) \) is the right eigenvector of \( \bar{M}(\alpha) \) corresponding to 1. That is, \( F = ((F_{ij}(\infty))) \) is a semi-Markov kernel and its stationary measure \( \pi = (\pi_1, \ldots, \pi_p) \) satisfies
\[
\sum_i \pi_i \frac{\nu_k}{\nu_i} \bar{M}_{ik}(\alpha) = \pi_k, \quad \text{or equivalently,} \quad \sum_i \frac{\pi_i}{\nu_i} \bar{M}_{ik}(\alpha) = \frac{\pi_k}{\nu_k}. 
\]  
(1.15)

We deduce from (1.15) that \( \left( \frac{\pi_1}{\nu_1}, \ldots, \frac{\pi_p}{\nu_p} \right) \) is the left eigenvector of \( \bar{M}(\alpha) \) corresponding to 1, so, \( \pi_i = u_i \nu_i, i = 1, \ldots, p \). So if \( \{e^{-\alpha t} f_j(t)(1 - G_j(t))\}_{j=1}^{p} \) are d.R.i.,
\[
e^{-\alpha t}m_f(t) \to c_0 \sum_{k=1}^{p} u_k \int_0^\infty E^{-\alpha u} f(u)(1 - G_k(u))du,
\]
i.e.,
\[
\lim_{t \to \infty} e^{-\alpha t}m_f(t) = c_0 \nu_i \sum_{k=1}^{p} u_k \int_0^\infty e^{-\alpha u} f(u)(1 - G_k(u))du,
\]
where
\[ c_0 = \left( \sum_{m=1}^{P} \sum_{n=1}^{P} u_m v_n m_{mn} \int_{0}^{\infty} u e^{-au} G_m(du) \right)^{-1}. \]

The second moments \( E(Z^2(t)) \) in both cases can be analyzed in a similar way and we'll present the details later.

### 1.3.3 Convergence Rates in Renewal Function and Equation; Examples

Many papers have been devoted to the rates of convergence to 0 in (1.6) and (1.7) when \( G \) is assumed to satisfy a variety of further conditions. For example, Stone (1965) proved

**Theorem 1.13** Let \( G \) have finite first and second moments \( \mu, \mu_2 > 0 \), respectively. If for some \( r_1 > 0, 1 - G(x) = O(e^{-r_1x}) \) as \( x \to \infty \), and if \( F \) is strongly non-lattice then for some \( r > 0 \)

\[ U(x) = \frac{x}{\mu} + \frac{\mu^2}{2\mu^2} + o(e^{-rx}) \quad \text{as} \quad x \to \infty. \]

More recently, the idea of a coupling was introduced which gave an elegant proof of renewal theorem (see Lindvall (1977)). The idea was also applied to find convergence rates in renewal theory. However, it is not easy to compute \( U \) explicitly, in general. The following is the simplest case.

**Example 1.1** Suppose \( G \) is the exponential distribution with density

\[ g(x) = ae^{-ax}, \quad x \geq 0. \]

It is a well-known fact that \( G^n \) is the gamma distribution with density

\[ g_n(x) = \frac{a}{\Gamma(n)}(ax)^{n-1}e^{-ax} = \frac{a}{(n-1)!}(ax)^{n-1}e^{-ax}. \]
\[
So \quad U(x) = 1 + \sum_{n=1}^{\infty} G^n(x)
\]
\[
= 1 + \sum_{n=1}^{\infty} \frac{a}{(n-1)!} (ay)^{n-1} e^{-av} dy
\]
\[
= 1 + a \int_{0}^{\infty} e^{-av} \sum_{n=0}^{\infty} \frac{(ay)^n}{n!} dy
\]
\[
= 1 + ax.
\]

In the next example, we'll present two approaches to find the convergence rate; one is probabilistic and the other analytic.

**Example 1.2** Let \( G \) be a gamma distribution with parameters \((a, k), a > 0, k \geq 2 \) integer and let \( g \) be the density.

**A. Probabilistic approach:** Let \( \{Y_n\}_{n=1}^{\infty} \) be i.i.d. with common distribution \( G \).

Define renewal process \( S_0 = 0, \quad S_n = \sum_{j=1}^{n} Y_j, \quad n \geq 1 \). Let \( B_t \) denote the forward recurrence time, i.e., the waiting time until the next renewal after \( t \). For a bounded measurable function \( h : \mathbb{R} \to \mathbb{R} \), define \( H(t) = E[h(B_t)] \). Then we get a renewal equation

\[
H(t) = (1 - G(t))E[h(Y_1)|Y_1 > t] + (H * G)(t).
\]

Now, let \( f \) be a function which is bounded on bounded sets and consider the following renewal equation

\[
(f * U)(t) = f(t) + [(f * U) * G](t),
\]

where \( U \) is the renewal function with interarrival distribution \( G \). If \( f(t) = (1 - G(t))E[h(Y_1)|Y_1 > t] \), we conclude that \( H(t) = (f * U)(t) \) by the uniqueness of bounded solution to the renewal equation. Furthermore, if \( f \) is d.R.i.

\[
|/(f * U)(t) - \frac{1}{\mu} \int_{0}^{\infty} f(u)du| = |E[h(B_t)] - E[h(B_{\infty})]|
\]
where \( \| \cdot \|_\infty, \| \cdot \| \) denote the supremum norm and the total variation norm, respectively. Now, for each \( n \geq 1 \), we may write

\[
Y_n = Y_{n,1} + \ldots + Y_{n,k},
\]

where \( \{Y_{n,j}; j = 1, \ldots, k\}_{n=1}^{\infty} \) are i.i.d. with \( P(Y_{1,1} > x) = e^{-ax} \).

Define a Markov process \( X(t) \) on state space \( S = \{1, 2, \ldots, k\} \) by

\[
X(t) = j \quad \text{if} \quad S_{m-1} + \sum_{i=1}^{j} Y_{m,i} \leq t < S_{m-1} + \sum_{i=1}^{j+1} Y_{m,i}, \quad \text{for some} \ m \geq 1.
\]

Clearly, the process is irreducible and so positive recurrent and for each \( t \geq 0 \),

\[
P^t = e^{At}
\]

where \( \{P^t\}_{t \geq 0} \) is the transition semigroup; i.e., \( P^t(i,j) = P(X(t) = j | X(0) = i) \), and \( \Lambda \) is its intensity matrix, i.e.;

\[
\Lambda = \begin{bmatrix}
-a & a & 0 & \cdots & 0 \\
0 & -a & a & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a & 0 & 0 & \cdots & -a
\end{bmatrix}
\]

The eigenvalues of \( \Lambda \) are \( d_j = a(\exp \{ \frac{2\pi j i}{k} \} - 1), j = 1, \ldots, k \). Hence, we have

\[
\|P^t(\cdot) - \pi(\cdot)\| = O(e^{-c_1 t}) \quad \text{as} \ t \to \infty.
\]

where \( \pi = (\pi_1, \ldots, \pi_k) \) is the stationary measure of \( X(t) \) and \( c_1 = -Re d_1 \). Since

\[
P(B_t \in \cdot) = \sum_{j=1}^{k} P(X(t) = j)P(T_{n-j} \in \cdot) \quad \text{and} \quad P(B_\infty \in \cdot) = \sum_{j=1}^{k} \pi_j P(T_{n-j} \in \cdot),
\]

where \( T_{n-j} \) is a gamma random variable with parameters \( (a, n-j) \), we conclude that

\[
\|P(B_t \in \cdot) - P(B_\infty \in \cdot)\| = O(e^{-c_1 t}). \quad (1.17)
\]
Now, consider the equation
\[ f(t) = E[h(Y_1)|Y_1 > t] = \frac{1}{1 - G(t)} \int_s^\infty h(u)G(du). \]
If \( f \) is differentiable a.e., then \( f'(t)[1 - G(t)] - f(t)g(t) = h(t)g(t) \) a.e. This implies that if \( f \) is differentiable a.e. and \( ||f'||_\infty \) finite, then \( ||h||_\infty \) is also finite. Combining this fact with (1.16) and (1.17) we get the following

**Theorem 1.14** If \( f \) is d.R.i., differentiable a.e., and if \( ||f'||_\infty \) is finite, then
\[ \|(f * U)(t) - \frac{1}{\mu} \int_0^\infty f(u)du\| = O(e^{-c_1 t}) \]
where \( \mu = \int_0^\infty uG(du) = k/a \), and \( c_1 = a(1 - \cos \frac{2\pi t}{k}) \).

The following is an immediate result with \( f = I_{[0,A]} \).

**Corollary 1.4** With \( \mu \) and \( c_1 \) as in Theorem 1.14, \( U(t) - U(t - h) = \frac{h}{\mu} + O(e^{-c_1 t}) \).

**B. Analytic approach:** We generalize the method in Resnick (1992) (Example 3.3.2, p. 188). By the definition of convolution and Laplace transform, we have
\[ \left( \sum_{n=1}^{\infty} G^{*n}(\lambda) \right) = \sum_{n=1}^{\infty} (\hat{G}^{*n})(\lambda) = \sum_{n=1}^{\infty} [\hat{G}(\lambda)]^n \]
\[ = \sum_{n=1}^{\infty} \left[ \left( \frac{a}{a + \lambda} \right)^n \right] = \frac{a^k}{(a + \lambda)^k - a^k} \]
\[ = \sum_{j=1}^{k} \frac{b_j}{\lambda - d_j} \]
where hat denotes the Laplace transform, \( d_j = a(exp(\frac{2\pi j}{k}i) - 1), j = 1, \ldots, k \) and \( b's \) are suitably chosen constants. So
\[ \left( \sum_{n=1}^{\infty} G^{*n}(\lambda) \right) = \sum_{j=1}^{k} Re[b_j \int_0^\infty e^{-\lambda x}e^{d_j x} dx] \]
\[ = \int_0^\infty e^{-\lambda x} Re(\sum_{j=1}^{k} b_j e^{d_j x}) dx \]
Since the Laplace transform determines the measure uniquely and $b_k = a/k = 1/\mu$

$$\sum_{n=1}^{\infty} (G^n)(x) = \int_0^x \text{Re}(\sum_{j=1}^{k} b_j e^{d_j y}) dy$$

$$= C + \frac{x}{\mu} + \sum_{j=1}^{k-1} b'_j e^{-c_j x}$$

where $c_j = -Red_j = a(1 - \cos \frac{2\pi j}{k}), C, b'$'s are suitably chosen constants. So

$$U(x + h) - U(x) = \frac{h}{\mu} + O(e^{-c_1 x}).$$
2. LAW OF LARGE NUMBERS FOR BRANCHING LEVY PROCESSES

2.1 Introduction

Let \( \{Z(t); t \geq 0\} \) be a supercritical Bellman-Harris process evolving from one particle at \( t = 0 \) whose lifetime distribution is \( G \) and offspring distribution is \( \{p_k\} \). That is, the process starts at time \( 0 \) with one particle of age \( 0 \) and it dies at time \( \lambda \) and produces \( \xi \) offsprings where \( \lambda \) and \( \xi \) are independent random variables with distribution \( G \) and \( \{p_k\} \) respectively. Then each particle dies and reproduces independent of each other in the same way as its parent, and so on. We superimpose on the process the additional structure of movement. A particle whose parent was at \( x \) at its time of birth moves until it dies according to a Markov process starting at \( x \). The motions of different particles are assumed independent. If the movement process is a Brownian motion the process is called a branching Brownian motion, whereas we call it branching Levy process for a Levy movement process.

For any family tree \( \omega \), let \( Z(t, a, b, \omega) \) be the number of particles living which are of age at most \( a \) with position \( \leq b \) and let \( A(t, a, \omega) = Z(t, a, \infty, \omega)/Z(t, \infty, \infty, \omega) \). Then under \( 'j \log j' \) condition \( A(t, a, \omega) \) converges to \( A(a) \) the stable age distribution with probability 1 (see Athreya and Kaplan (1976)). If the underlying movement process is Brownian then it is known (see Asmussen and Kaplan (1976b)) that under
finite second moment condition on the offspring law $Z(t, \infty, \sqrt[3]{b}, \omega) / Z(t, \omega) \xrightarrow{a.s.} \Phi(b)$
where $\Phi(b) = (2\pi)^{-1/2} \int_{-\infty}^{b} e^{-x^2/2} dx$. Thus one would expect the proportion of particles with position $\leq \sqrt[3]{b}$ who are younger than or equal to $a$ tends to $A(a)\Phi(b)$. Indeed this essentially turns out to be the case here under ‘$j \log j$’ condition. Furthermore we can extend the result to the branching Levy process.

2.2 Definitions, Assumptions, and Statement of Results

We adopt the following notations throughout this chapter. For any family history $\omega$:

(N 1) $\{x_j(t, \omega), j = 1, \cdots, Z(t, \omega)\} =$ the position-chart at time $t$.

(N 2) $\{a_j(t, \omega), j = 1, \cdots, Z(t, \omega)\} =$ the age-chart at time $t$.

(N 3) For $a \in R^+$ and $b \in R$

$Z(t, a, b, \omega) = \sum_{j=1}^{Z(t, \omega)} I(a_j(t, \omega) \leq a)I(x_j(t, \omega) \leq b)$

= the number of particles at time $t$ with age $\leq a$ and position $\leq b$

$Z(t, a, \omega) = Z(t, a, \infty, \omega)$

$Z(t, \omega) = Z(t, \infty, \infty, \omega) =$ the number of particles at time $t$

(N 4) $m(t, a, b) = E(Z(t, a, b, \omega))$ $m(t, a) = E(Z(t, a, \omega))$ $m(t) = E(Z(t, \omega))$

(N 5) We add superscript $y$ and subscript $x$ to random variables and their moments to indicate the case when $P$ is supported by those $\omega$’s which start with one particle of age $y \geq 0$ at position $x \in R$. 

\( (N\ 6) \quad A(a) = \frac{\int_0^\infty e^{-\alpha t} (1 - G(u)) du}{\int_0^\infty e^{-\alpha u} (1 - G(u)) du} \)

\( (N\ 7) \) We define two \( \sigma \)-algebras
\[ F_t = \sigma(Z(s,\omega), \{a_j(s,\omega), x_j(s,\omega), j = 1, \ldots, Z(s,\omega)\}; s \leq t), \]
\[ G_t = \sigma(Z(s,\omega), s \leq t). \]

We make the following assumptions throughout. Sometimes they will appear in lemmas and theorems explicitly and sometimes not, but they will always be in force.

(A 1) \( p_0 = 0, \)

(A 2) \( 1 < m = \sum_{j=0}^{\infty} j p_j < \infty. \)

(A 3) \( \sum_{j=1}^{\infty} (j \log j) p_j < \infty. \)

The assumption \( p_0 = 0 \) is primarily for convenience of exposition. Otherwise one has to keep qualifying "on the set of explosion". (A 3) along with (A 1) guarantees the existence of random variable \( W \) such that
\[ \lim_{t \to \infty} e^{-\alpha t} Z(t) = W \quad \text{a.s.} \quad \text{and} \quad P(W > 0) = 1, \quad (2.1) \]
where \( \alpha = \alpha(m, G) \) is the Malthusian parameter for \( m \) and \( G \) defined by \( m \int_0^\infty e^{-\alpha t} dG(t) = 1. \)

**Theorem 2.1** Let the underlying movement process be standard Brownian motion.

Then for \( a \in R^+, b \in R, \)
\[ H_t(a, \sqrt{t} b, \omega) \overset{\text{def}}{=} \frac{Z(t, a, \sqrt{t} b, \omega)}{Z(t, \omega)} \overset{\text{a.s.}}{\longrightarrow} A(a) \Phi(b), \quad \text{as} \quad t \to \infty, \]
where \( \Phi(b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{x^2}{2}} dx, \) and \( A(a) \) is as in \((N\ 7)\).
Now consider a branching Markov process. Let \( \{X(t); t \geq 0\} \) be a underlying Markov process such that \( X(0) = 0 \) a.s. Suppose that \( \{X(t); t \geq 0\} \) is stationary with independent increments and that for some measurable functions \( a(t) \) and \( m(t) \)

\[
Y(t) \stackrel{d}{\sim} \frac{X(t) - m(t)}{a(t)}, \quad Y \quad \text{as} \quad t \to \infty,
\]

where \( P(Y \leq x) = F(x) \), a nondegenerate and continuous distribution. Then we have

**Theorem 2.2** Suppose (2.2) holds with \( a(t) = t^c L_1(t) \) and \( m(t) = t^d L_2(t) \), where \( c > 0, c \geq d \) and \( L_1, L_2 \) are slowly varying functions at infinity such that

\[
\limsup_{t \to \infty} \left| \frac{L_2(t)}{L_1(t)} \right| < \infty.
\]

If \( E(|Y|^u) < \infty \) for some \( u > 1/c \), then for any \( a \in \mathbb{R}^+ \), \( b \in C_F \),

\[
H_t(a,a(t)b + m(t),\omega) \stackrel{d}{\rightarrow} A(a)F(b) \quad \text{as} \quad t \to \infty
\]

where \( C_F \) is the set of continuity points of \( F \).

### 2.3 Preliminary Results

In the proofs to come we make use of the following Lemmas. The first one can be found in Nerman(1981).

**Lemma 2.1** Let \( M = \sup_{t \geq 0} \{e^{-\alpha t}Z(t)\} \) with \( \alpha \) as the Malthusian parameter.

If \( \sum_{j=1}^{\infty} (j \log j) p_j < \infty \) then \( E(M) < \infty \).

**Corollary 2.1** Put \( M = \sup_{s \geq 0, \alpha \geq 0} \{e^{-\alpha s}Z^\alpha(s)\} \). If \( \sum_{j=1}^{\infty} (j \log j) p_j < \infty \), then \( E(M) < \infty \).
PROOF. We first note that

\[ Z^\alpha(s) = I(\lambda^\alpha > s) + \sum_{j=1}^{\xi} Z_j(s - \lambda^\alpha) \quad (2.3) \]

where \( \{Z_j(s), s \geq 0\}, j = 1, 2, \ldots, \) are i.i.d. with \( \{Z(s), s \geq 0\} \). So we have

\[
e^{-\alpha s} Z^\alpha(s) = e^{-\alpha s} I(\lambda^\alpha > s) + \sum_{j=1}^{\xi} e^{-\alpha(s-\lambda^\alpha)} Z_j(s - \lambda^\alpha) e^{-\alpha \lambda^\alpha}
\]

\[
\leq 1 + \sum_{j=1}^{\xi} e^{-\alpha(s-\lambda^\alpha)} Z_j(s - \lambda^\alpha)
\]

\[
\leq 1 + \sum_{j=1}^{\xi} M_j
\]

where \( M_j = \sup_{s \geq 0} e^{-\alpha s} Z_j(s) \). Thus \( M \leq 1 + \sum_{j=1}^{\xi} M_j \). Since \( \sum_{j=1}^{\infty} (j \log j)p_j < \infty \), \( E(M_j) < \infty \)(Lemma 2.1) and hence by the independence of \( \{M_j\} \) and \( \xi \), \( E(M) \leq 1 + mE(M_1) < \infty \). \( \square \)

The following is due to T. Kurtz (1972).

**Lemma 2.2** Let \( X_1, \ldots, X_n \) be independent random variables with mean 0. Assume that

\[
P(|X_i| > t) \leq c_1 \int_{t}^{\infty} dQ(x), \quad i = 1, \ldots, n,
\]

for a probability distribution \( Q \) on \([0, \infty)\) with finite mean and a constant \( c_1 \) in \((0, \infty)\). Then for any \( \delta > 0 \), there exists a constant \( c_2 \) in \((0, \infty)\) such that

\[
P(|\overline{X}_n| > \delta) \leq c_2 (n \int_{0}^{\infty} dQ(x) + \frac{1}{n} \int_{0}^{\infty} x^2 dQ(x)),
\]

where \( \overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \).
Lemma 2.3 Let $V$ be a nonnegative random variable with $E(V) < \infty$. Then for any $\rho > 1$,
\[ \sum_{n=0}^{\infty} (\rho^n P(V > \rho^n) + \frac{1}{\rho^n} E(V^2; V \leq \rho^n)) < \infty. \]

PROOF. 
\[ \infty > E(V) = \int_0^\infty P(V > t) dt \]
\[ \geq \sum_{n=0}^{\infty} \int_{\rho^{n-1}}^{\rho^n} P(V > t) dt \]
\[ \geq \sum_{n=0}^{\infty} P(V > \rho^n)(\rho^n - \rho^{n-1}) \]
\[ = (1 - \frac{1}{\rho}) \sum_{n=0}^{\infty} \rho^n P(V > \rho^n) \]

So $\sum_{n=0}^{\infty} \rho^n P(V > \rho^n) < \infty$. For the second term, put $q_j = P(j \leq V < j + 1)$. Then
\[ \sum_{n=0}^{\infty} \frac{1}{\rho^n} E(V^2; V \leq \rho^n) \leq \sum_{n=0}^{\infty} \frac{1}{\rho^n} \sum_{j=0}^{[\rho^n]} E(V^2; j \leq V < j + 1) \]
\[ \leq 1 + \sum_{n=0}^{\infty} \frac{1}{\rho^n} \sum_{j=1}^{[\rho^n]} (j + 1)^2 q_j \]
\[ = 1 + \sum_{j=1}^{\infty} (j + 1)^2 q_j \sum_{\rho^n \geq j} \frac{1}{\rho^n} \]
\[ \leq 1 + \frac{\rho}{\rho - 1} \sum_{j=1}^{\infty} \frac{(j + 1)^2}{j} q_j < \infty. \quad \square \]

The following two lemmas are in Athreya and Kaplan (1976).

Lemma 2.4 Let $V(y) = m \int_0^{\infty} e^{-\alpha t} G_{\alpha}(dt)$ and let $n_1 = \frac{\int_0^{\infty} e^{-\alpha t}(1 - G(t))dt}{m \int_0^{\infty} t e^{-\alpha t} G(dt)}$.

Then for any $a \geq 0$
\[ \sup_{y \geq 0} (|m^y(s, a)e^{-\alpha s} - n_1 V(y) A(a)|, |m^y(s)e^{-\alpha s} - n_1 V(y)|) \to 0 \quad \text{as} \quad s \to \infty. \]
Lemma 2.5 Let $V_i(\omega) = \sum_{j=1}^{Z(t,\omega)} V(a_j(t,\omega))$. Suppose $\sum_{j=1}^{\infty} (j \log j) p_j < \infty$. Then for every $\delta > 0$,

$$\lim_{n \to \infty} \frac{V_{nf}(\omega)}{Z(n\delta,\omega)} = n_1^{-1} \quad a.s.$$ 

2.4 Proof of Theorem 2.1

We begin with the following representation appealing to the additive property of branching processes.

$$Z(t + s, a, \sqrt{t + s} b, \omega) = \sum_{j=1}^{Z(t,\omega)} Z^{a_j(t,\omega)}(s, a, \sqrt{t + s} b, \omega) \quad (2.4)$$

where $Z^{a_j(t,\omega)}(s, a, x, \omega)$ is the number of particles at time $t + s$ whose age is $\leq a$ and whose position is $\leq x$ in the line of descent initiated by a particle of age $a_j(t,\omega)$ and position $x_j(t,\omega)$ at time $t$. With abuse of notation we rewrite (2.4) as (suppressing $\omega$ and $(t,\omega)$),

$$Z(t + s, a, \sqrt{t + s} b) = \sum_{j=1}^{Z(t)} Z^{a_j(t,\omega)}(s, a, \sqrt{t + s} b) \quad (2.5)$$

Noting that (see Assmussen and Kaplan (1976b))

$$E(Z^{a_j(t,\omega)}(s, a, \sqrt{t + s} b)|\mathcal{F}_t) = m^{a_j}(s, a) \Phi(\frac{\sqrt{t + s} b - x_j}{\sqrt{s}}), \quad (2.6)$$

we decompose (2.5) as follows:

$$Z(t + s, a, \sqrt{t + s} b) = \sum_{j=1}^{Z(t)} \left\{ Z^{a_j(t,\omega)}(s, a, \sqrt{t + s} b) - m^{a_j}(s, a) \Phi(\frac{\sqrt{t + s} b - x_j}{\sqrt{s}}) \right\}$$

$$+ \sum_{j=1}^{Z(t)} \left\{ m^{a_j}(s, a) \Phi(\frac{\sqrt{t + s} b - x_j}{\sqrt{s}}) - n_1 e^{as} V(a_j) A(a) \Phi(b) \right\}$$

$$+ n_1 A(a) \Phi(b) e^{as} V_i.$$
In particular, with \( a = \infty \) and \( b = \infty \) we have

\[
Z(t + s) = \sum_{j=1}^{Z(t)} (Z_{x_j}^{(s)} - m^{x_j}(s)) + \sum_{j=1}^{Z(t)} \{m^{x_j}(s) - n_1 e^{\alpha x} V(a_j)\} + n_1 V_t.
\]

So

\[
H_{t+s}(a, \sqrt{t+s} b) \overset{\text{def}}{=} \frac{Z(t+s, a, \sqrt{t+s} b)}{Z(t+s)} = \frac{a_t(s, a, b) + b_t(s, a, b) + c_t A(a) \Phi(b)}{a_t(s, \infty, \infty) + b_t(s, \infty, \infty) + c_t},
\]

where

\[
a_t(s, a, b) = \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} \{e^{-\alpha s} Z_{x_j}^{(s)}(s, a, \sqrt{t+s} b) - e^{-\alpha s} m^{x_j}(s, a) \Phi(\sqrt{t+s} b - x_j)\},
\]

\[
b_t(s, a, b) = \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} \{e^{-\alpha s} m^{x_j}(s, a) \Phi(\sqrt{t+s} b - x_j) - n_1 V(a_j) A(a) \Phi(b)\},
\]

\[
c_t = \frac{n_1}{Z(t)} V_t.
\]

Note that \( c_t \xrightarrow{n \to \infty} 1 \) as \( t \to \infty \)(Lemma 2.5). Following Athreya and Kaplan(1978) we first discretize the process, i.e., let \( t_n = n \delta \) and \( s_n = s(t_n) \) and consider

\[
H_{n \delta + s_n}(a, \sqrt{n \delta + s_n} b) = \frac{a_n(s_n, a, b) + b_n(s_n, a, b) + c_n A(a) \Phi(b)}{a_n(s_n, \infty, \infty) + b_n(s_n, \infty, \infty) + c_n}.
\]

We'll first show that \( a_n(s_n, a, b) \xrightarrow{n \to \infty} 0 \) as \( n \to \infty \) for any choice of sequence \( s_n \) (Lemma 2.6) and then prove that \( b_n(s_n, a, b) \xrightarrow{n \to \infty} 0 \) as \( n \to \infty \) for some choice of \( s_n \) (Lemma 2.7).

**Lemma 2.6** Fix \( \delta > 0, a \in R^+ \), \( b \in R \), then for any choice of sequence \( s_n \)

\[
a_{n \delta}(s_n, a, b) \xrightarrow{n \to \infty} 0 \quad \text{as} \quad n \to \infty.
\]
PROOF. Due to the conditional Borel-Cantelli lemma, it is enough to show that for any \( \varepsilon > 0 \)

\[
\sum_{n=0}^{\infty} P(|a_{n\delta}(s_n, a, b)| > \varepsilon |F_{n\delta}) < \infty \quad \text{a.s.}
\]

Since \( Z_{n\delta}(s, a, \sqrt{t + \delta} b) \leq Z_{n\delta}(s) \) for any \( x \in \mathbb{R} \)

\[
e^{-\alpha n} Z_{n\delta}^{a_j}(s_n, a, \sqrt{n\delta} + s_n b) - e^{-\alpha n} m_{n\delta}^{a_j}(s_n, a) \Phi\left( \frac{\sqrt{n\delta} + s_n b - x_j}{\sqrt{s_n}} \right) \leq M + E(M)
\]

where \( M \) is as in Corollary 2.1. So by Lemma 2.2, we have

\[
P(|a_{n\delta}(s_n, a, b)| > \varepsilon |F_{n\delta}) \leq c \{Z(n\delta)h(Z(n\delta)) + \frac{1}{Z(n\delta)} k(Z(n\delta))\},
\]

where \( h(t) = P(M > t) \), \( k(t) = E(M^2; M < t) \), and \( c \) is a constant.

Since \( Z_{n\delta} \sim e^{n\delta W} \) a.s. and \( W > 0 \) a.s.,

\[
\sum_{n=0}^{\infty} P(|a(n\delta, s_n, b)| > \varepsilon |F_{n\delta}) \leq c \sum_{n=0}^{\infty} \{Z(n\delta)h(Z(n\delta)) + \frac{1}{Z(n\delta)} k(Z(n\delta))\} < \infty \quad \text{a.s. by Lemma 2.3.}
\]

\( \Box \)

Lemma 2.7 For a fixed \( \delta > 0 \), let \( s_n = (n\delta)^3 - n\delta \). Then

\[
b_{n\delta}(s_n, a, b) \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty
\]

PROOF. For \( j = 1, \cdots, Z(t) \), put \( I_{n_j} = I(|\frac{x_j}{\sqrt{n\delta}}| \leq \sqrt{n\delta}) \), \( J_{n_j} = I_{n_j}^{c} \). Then

\[
b_{n\delta}(s_n, a, b) = b_{n\delta}^{1}(s_n, a, b) + b_{n\delta}^{2}(s_n, a, b) + b_{n\delta}^{3}(s_n, a, b).
\]

where

\[
b_{n\delta}^{1}(s_n, a, b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} \{\Phi\left( \sqrt{1 + \frac{n\delta}{s_n} b - \frac{x_j}{\sqrt{n\delta} \sqrt{s_n}} \right) - \Phi(b)\} e^{-\alpha n} m_{n\delta}^{a_j}(s_n) I_{n_j}
\]
$b_{n\delta}^{2}(s_{n},a,b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} \left\{ \Phi\left( \sqrt{1 + \frac{n\delta}{s_{n}} b - \frac{x_{j}}{\sqrt{n\delta}} \sqrt{\frac{n\delta}{s_{n}}} \right) - \Phi(b) \right\} e^{-\alpha_{n} m_{\alpha_{j}}(s_{n}) J_{nj}}$

$- b_{n\delta}^{3}(s_{n},a,b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} \left\{ e^{-\alpha_{n} m_{\alpha_{j}}(s_{n},a)} - n_{1} V(a_{j}) A(a) \right\} \Phi(b)$

By the continuity of $\Phi$ and Corollary 2.1, it is easy to see that $b_{n\delta}^{1}(s_{n},a,b) \xrightarrow{a.s.} 0$ as $n \to \infty$. On the other hand, for any $\varepsilon > 0$,

$$P(\|b_{n\delta}^{2}(s_{n},a,b)\| > \varepsilon | G_{n\delta}) \leq \frac{E(M)}{\varepsilon Z(n\delta)} \sum_{j=1}^{Z(n\delta)} 2E(J_{nj})$$

$$\leq \frac{2E(M)}{\varepsilon} 2(1 - \Phi(\sqrt{n\delta}))$$

$$\leq \frac{4E(M)}{\varepsilon \sqrt{2\pi}} e^{-\frac{n\delta}{2}}.$$

So

$$\sum_{n=0}^{\infty} P(\|b_{n\delta}^{2}(s_{n},a,b)\| > \varepsilon | G_{n\delta}) < \infty$$

and by the conditional Borel-Cantelli lemma

$$b_{n\delta}^{2}(s_{n},a,b) \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty.$$

Finally we have

$$b_{n\delta}^{3}(s_{n},a,b) \to 0 \quad \text{as} \quad n \to \infty$$

directly from Lemma 2.4

Lemma 2.5, Lemma 2.6 and Lemma 2.7 with (2.7) imply together that

$$H_{(n\delta)^{3}}(a,(n\delta)^{3/2}b) \xrightarrow{a.s.} A(a) \Phi(b) \quad \text{as} \quad n \to \infty. \quad (2.9)$$

To prove $H_{n\delta}(a,\sqrt{n\delta} b) \xrightarrow{a.s.} A(a) \Phi(b) \quad \text{as} \quad n \to \infty$, we adopt the method used in Athreya and Kaplan (1978).
Lemma 2.8 For a fixed $\delta > 0$ and $a \in R^+$, $b \in R$,

$$H_n(\sqrt{n} b) \xrightarrow{a.s.} A(a)\Phi(b) \text{ as } n \to \infty.$$  

PROOF. Let $\delta_0 = \delta^{1/3}$. For any $n > 0$, there exists an integer $m_n \geq 0$ such that $m_n^3 \leq n < (m_n + 1)^3$. Put $k_n = (m_n - 1)^3$ then $3(m_n - 1)^2 \leq n - k_n < 6(m_n + 1)^2$.

So as $n \to \infty$, $k_n \to \infty$ as well and further

\begin{align*}
\frac{\sqrt{n} - \sqrt{k_n}}{\sqrt{n} - k_n} &= \frac{\sqrt{n} - k_n}{\sqrt{n} + \sqrt{k_n}} \leq \frac{6(m_n + 1)}{m_n^{3/2}} \to 0 \text{ as } n \to \infty, \quad (2.10) \\
\frac{\sqrt{k_n}}{\sqrt{n} - k_n} &\geq \frac{(m_n - 1)^{3/2}}{6(m_n + 1)} \to \infty \text{ as } n \to \infty. \quad (2.11)
\end{align*}  

Fix $\epsilon > 0$ and define $B_1 = (-\infty, b - \epsilon]$, $B_2 = [b + \epsilon, \infty)$, and $B_3 = (b - \epsilon, b + \epsilon)$.

Then

$$H_n(\sqrt{n} b) = \frac{a_{k_n\delta}((n - k_n)\delta, a, b) + \sum_{i=1}^{3} d_{k_n\delta}^i((n - k_n)\delta, a, b)}{a_{k_n\delta}((n - k_n)\delta, \infty, \infty) + b_{k_n\delta}((n - k_n)\delta, \infty, \infty) + c_{k_n\delta}},$$

where

$$d_{k_n\delta}^i((n - k_n)\delta, a, b) = \frac{1}{Z(k_n\delta)} \sum_{j=1}^{Z(k_n\delta)} e^{-a(n-k_n)\delta} m_{a_j}((n - k_n)\delta, a)\Phi_b\left(\frac{\sqrt{k_n\delta}}{\sqrt{n-k_n}} b - x_j\right)I_{k_n\delta B_i}(x_j).$$

We have shown that

1. $a_{k_n\delta}((n - k_n)\delta, a, b) \xrightarrow{a.s.} 0$ as $n \to \infty$ (Lemma 2.6)

2. $b_{k_n\delta}((n - k_n)\delta, \infty, \infty) \xrightarrow{a.s.} 0$ as $n \to \infty$ (Lemma 2.4)

3. $c_{k_n\delta} \xrightarrow{a.s.} 1$ as $n \to \infty$ (Lemma 2.5)
Furthermore, since $k_n \delta = ((m_n - 1)\delta_0)^3$, (2.9) with $a = \infty$ implies

$$d_{k_n \delta}^3((n - k_n) \delta, a, b) \leq \frac{E(M)}{Z(k_n \delta)} \sum_{j=1}^{Z(k_n \delta)} I_{\sqrt{k_n \delta} B_1}(x_j) \xrightarrow{a.s.} E(M) \Phi(B_3) \quad \text{as} \quad n \to \infty,$$  

(2.12)

where $\Phi(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-y^2/2} \, dy$.

Note that if $x_j \in \sqrt{k_n \delta} B_1$, from (2.10) and (2.11)

$$\frac{\sqrt{n \delta} b - x_j}{\sqrt{(n - k_n) \delta}} = \frac{\sqrt{n \delta} - \sqrt{k_n \delta}}{\sqrt{(n - k_n) \delta}} \frac{b + \sqrt{k_n \delta} b - x_j}{\sqrt{(n - k_n) \delta}} \geq \frac{\sqrt{n} - \sqrt{k_n \delta}}{\sqrt{(n - k_n) \delta}} b + \frac{\sqrt{k_n \delta} \varepsilon}{\sqrt{(n - k_n) \delta}} \to \infty \quad \text{as} \quad n \to \infty.$$

So

$$\limsup_{n \to \infty} \left| \frac{1}{Z(k_n \delta)} \sum_{j=1}^{Z(k_n \delta)} e^{-\alpha(n-k_n)\delta^3} m^{-1}((n-k_n)\delta, a) I_{\sqrt{k_n \delta} B_1}(x_j) - d_{k_n \delta}^3((n-k_n)\delta, a, b) \right|$$

$$= \limsup_{n \to \infty} \left| \frac{1}{Z(k_n \delta)} \sum_{j=1}^{Z(k_n \delta)} e^{-\alpha(n-k_n)\delta^3} m^{-1}((n-k_n)\delta, a)(1 - \Phi(\frac{\sqrt{k_n \delta} b - x_j}{\sqrt{(n - k_n) \delta}})) I_{\sqrt{k_n \delta} B_1}(x_j) \right|$$

$$\leq E(M)(1 - \Phi(\frac{\sqrt{k_n \delta} \varepsilon}{\sqrt{(n - k_n) \delta}}))$$

$$\to 0 \quad \text{a.s.} \quad n \to \infty \quad \text{by} \quad (2.11).$$

Hence

$$\lim_{n \to \infty} d_{k_n \delta}^1((n-k_n)\delta, a, b)$$

$$= \lim_{n \to \infty} \frac{1}{Z(k_n \delta)} \sum_{j=1}^{Z(k_n \delta)} e^{-\alpha(n-k_n)\delta^3} m^{-1}((n-k_n)\delta, a) I_{\sqrt{k_n \delta} B_1}(x_j)$$

$$= \int_{R^R B_1} e^{-\alpha(n-k_n)\delta^3} m^v((n-k_n)\delta, a) I_{B_1}(x) dH_{k_n \delta}(y, \sqrt{k_n \delta} x)$$

$$\xrightarrow{a.s.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n_1 V(y) A(a) dA(y) I_{B_1}(x) d\Phi(x) \quad \text{by} \quad 2.4 \text{and} \quad (2.9)$$

$$= n_1 A(a) \Phi(b - \varepsilon) \int_{-\infty}^{\infty} V(y) dA(y)$$

$$= A(a) \Phi(b - \varepsilon)$$  

(2.13)
On the other hand, if \( x_j \in \sqrt{k_n \delta} B_2 \),

\[
\frac{\sqrt{n \delta - x_j}}{(n - k_n) \delta} = \frac{\sqrt{n \delta - k_n \delta} \ b - \sqrt{k_n \delta} \ b - x_j}{(n - k_n) \delta} \leq \frac{\sqrt{n - k_n} \ b - \sqrt{k_n \ v}}{(n - k_n) \delta} \to -\infty \quad \text{as} \quad n \to \infty.
\]

Hence

\[
\limsup_{n \to \infty} \sup_{x_j \in \sqrt{n \delta} B_2} \frac{\sqrt{n \delta - x_j}}{(n - k_n) \delta} = -\infty,
\]

and so

\[
d^2_{k_n}(n - k_n) \delta, a, b) \to 0 \quad \text{as} \quad n \to \infty. \quad (2.14)
\]

So recalling Lemma 2.4 and Lemma 2.5

\[
\limsup_{n \to \infty} |H_{n \delta}(a, \sqrt{n \delta} b) - A(a) \Phi(b - \varepsilon)| \\
\leq \limsup_{n \to \infty} \left| d^2_{k_n}(n - k_n) \delta, a, b) - A(a) \Phi(b - \varepsilon) \right| + \limsup_{n \to \infty} |d^3_{k_n}(n - k_n) \delta, a, b) + \limsup_{n \to \infty} |d^4_{k_n}(n - k_n) \delta, a, b)| \\
\leq E(M) \Phi(B) \quad \text{by} \ (2.12) \ (2.13) \ \text{and} \ (2.14)
\]

Letting \( \varepsilon \downarrow 0 \), we get \( \limsup_{n \to \infty} |H_{n \delta}(a, \sqrt{n \delta} b) - A(a) \Phi(b)| = 0 \quad \text{a.s.} \) \( \Box \)

Now we prove that \( \lim_{t \to \infty} H_t(a, \sqrt{t} b) = A(a) \Phi(b) \) a.s. Let \( \varepsilon > 0 \) and \( \delta > 0 \) be fixed.

Let \( n \delta \leq t < (n + 1) \delta \) and define

\[
\delta_j = \begin{cases} 
1 & \text{if } j \text{th particle at time } n \delta \text{ doesn't split until } (n + 1) \delta \\
0 & \text{and the particle doesn't cover a distance } > \sqrt{n \delta} \varepsilon,
\end{cases}
\]

Let \( \{(a_j, x_j), j = 1, \cdots, Z(n \delta)\} \) be the (age, position)-chart at time \( n \delta \). Since the lifetime and the movement of a particle are independent,

\[
E(\delta_j | F_{n \delta}) = P(\delta_j = 1 | F_{n \delta}) = P(\lambda_1 > \delta)P(\xi(\delta) \leq \sqrt{n \delta} \varepsilon)
\]
\[ = (1 - G^{a_j}(\delta))P(\xi(\delta) \leq \sqrt{n\delta} \varepsilon), \]  
(2.15)

where
\[ \bar{\xi}(\delta) = \sup_{0 \leq t \leq \delta} |B_0(t)|, \]

\[ B_0(t) = \text{standard Brownian motion starting at 0.} \]

It is easy to see the following inequality from the definition of \( \delta_j \),

\[ Z(t, a, \sqrt{b}) \geq \sum_{j=1}^{Z(n\delta)} I(a_j + \delta \leq a)I(x_j \leq \sqrt{n\delta} (b - \varepsilon))\delta_j. \]

So
\[ \frac{Z(t, a, \sqrt{b})}{Z(t)} \geq \frac{Z(n\delta)}{Z(t)} \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta)I(x_j \leq \sqrt{n\delta} (b - \varepsilon))\delta_j \]
\[ = \frac{Z(n\delta)}{Z(t)} \{ A(n\delta, a, b) + P(\bar{\xi}(\delta) \leq \sqrt{n\delta} \varepsilon)B(n\delta, a, b) \}, \]

where
\[ A(n\delta, a, b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta)I(x_j \leq \sqrt{n\delta} (b - \varepsilon)) \]
\[ \cdot \{ \delta_j - (1 - G^{a_j}(\delta))P(\bar{\xi}(\delta) \leq \sqrt{n\delta} \varepsilon) \} \]
\[ B(n\delta, a, b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta)I(x_j \leq \sqrt{n\delta}(b - \varepsilon))(1 - G^{a_j}(\delta)). \]

Since
\[ E(I(a_j \leq a - \delta)I(x_j \leq \sqrt{n\delta} (b - \varepsilon))\{ \delta_j - (1 - G^{a_j}(\delta))P(\bar{\xi}(\delta) \leq \sqrt{n\delta} \varepsilon) \}) = 0 \]
we apply Lemma 2.2 and Lemma 2.3 to get
\[ A(n\delta, a, b) \overset{a.s.}{\to} 0 \text{ as } n \to \infty. \]

On the other hand
\[ B(n\delta, a, b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta)I(x_j \leq \sqrt{n\delta}(b - \varepsilon)) \]
\[ - \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta)I(x_j \leq \sqrt{n\delta}(b - \varepsilon))G^{a_j}(\delta) \]
\[ \geq \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta)I(x_j \leq \sqrt{n\delta}(b - \varepsilon)) - \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} G^{a_j}(\delta) \]
40

Note that $G_\delta(a) \equiv G^a(\delta)$ is bounded and continuous except on a countable set. So

$$
\frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} G^{n\delta j}(\delta) = \int_{0}^{\infty} G_\delta(a) A(da, n\delta) \xrightarrow{n,\delta \to \infty} \int_{0}^{\infty} G_\delta(a) A(da) \quad \text{as} \quad n \to \infty \quad (2.16)
$$

by Corollary 1.1(p11, Chapter1). Since $P(\bar{\xi}(\delta) \leq \sqrt{n\delta \epsilon}) \to 1$ as $n \to \infty$, (2.16) and Lemma 2.7 imply

$$
\liminf_{n \to \infty} B(n\delta, a, b) \geq A(a - \delta) \Phi(b - \epsilon) - \int_{0}^{\infty} G_\delta(a) A(da).
$$

Further from (2.1) we have

$$
\liminf_{t \to \infty} \frac{Z(n\delta)}{Z(t)} \geq \liminf_{n \to \infty} \frac{Z(n\delta)}{Z((n+1)\delta)} = \liminf_{n \to \infty} \frac{Z(n\delta)}{Z((n+1)\delta)} e^{-o n \delta} e^{-a \delta} = e^{-a \delta}
$$

Hence

$$
\liminf_{t \to \infty} \frac{Z(t, a, \sqrt{t} b)}{Z(t)} \geq e^{-a \delta} (A(a - \delta) \Phi(b - \epsilon) - \int_{0}^{\infty} G_\delta(a) A(da)). \quad (2.17)
$$

Since $G_\delta(a) \to 0$ a.e. as $\delta \to 0$, we see $\int_{0}^{\infty} G_\delta(a) A(da) \to 0$ as $\delta \to 0$ by the dominated convergence theorem. Letting $\delta \downarrow 0$ and then letting $\epsilon \downarrow 0$, we get from (2.17) that

$$
\liminf_{t \to \infty} \frac{Z(t, a, \sqrt{t} b)}{Z(t)} \geq A(a) \Phi(b).
$$

For the other direction we have the following inequality

$$
Z(t) - Z(t, a, \sqrt{t} b) \geq \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta) I(x_j > \sqrt{(n+1)\delta (b + \epsilon)}) \delta_j
$$

and so

$$
1 - \frac{Z(t, a, \sqrt{t} b)}{Z(t)} \geq \frac{Z(n\delta)}{Z(t)} \{A'(n\delta, a, b) + P(\bar{\xi}(\delta) \leq \sqrt{n\delta \epsilon}) B'(n\delta, a, b)\},
$$
where
\[ A'(n\delta, a, b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta)I(x_j > \sqrt{(n + 1)\delta (b + \varepsilon)}) \]
\[ \cdot \left\{ \delta_j - (1 - G_{a_j}(\delta))P(\xi(\delta) \leq \sqrt{n\delta\varepsilon}) \right\} \]
\[ B'(n\delta, a, b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a - \delta)I(x_j > \sqrt{(n + 1)\delta (b + \varepsilon)})(1 - G_{a_j}(\delta)). \]

The same argument as above establishes
\[ A'(n\delta, a, b) \xrightarrow{n \to \infty} 0 \]
and
\[ \lim_{n \to \infty} \inf B'(n\delta, a, b) \geq (1 - A(a - \delta)\Phi(b + \varepsilon)) - \int_0^\infty G_\varepsilon(a)A(da). \]
So
\[ \lim_{t \to -\infty} \inf (1 - H_t(a, \sqrt{t} b)) \geq e^{-a\varepsilon}(1 - A(a - \delta)\Phi(b + \varepsilon)) - \int_0^\infty G_\varepsilon(a)A(da)). \]
Letting \( \delta \downarrow 0 \) and then letting \( \varepsilon \downarrow 0 \), we get
\[ \lim_{t \to -\infty} \inf (1 - H_t(a, \sqrt{t} b)) \geq 1 - A(a)\Phi(b). \]
So we have completed the proof of Theorem 2.1.

### 2.5 Proof of Theorem 2.2

In this case we have the following representation.
\[ Z(t + s, a, a(t + s)b + m(t + s)) = \sum_{j=1}^{Z(t)} Z^{2j}_{2j}(s, a, a(t + s)b + m(t + s)) \tag{2.18} \]
where \( Z^{2j}_{2j}(s, a, x) \) is as defined in section 2.4.

Now put
\[ y_j(t, s, b) = \frac{a(t + s)b + m(t + s) - x_j - m(s)}{a(s)}, \quad j = 1, \ldots, Z(t), \tag{2.19} \]
then we have the following decomposition

\[
Z(t + s, a, a(t + s)b + m(t + s)) = \\
\sum_{j=1}^{\mathcal{Z}(t)} \left\{ Z_{\alpha_j}^{a_j}(s, a, a(t + s)b + m(t + s)) - m_{\alpha_j}(s, a) P(Y(s) \leq y_j(t, s, b)) \right\} + \\
\sum_{j=1}^{\mathcal{Z}(t)} m_{\alpha_j}(s, a) \{ P(Y(s) \leq y_j(t, s, b)) - F(y_j(t, s, b)) \} + \\
\sum_{j=1}^{\mathcal{Z}(t)} \{ m_{\alpha_j}(s, a) F(y_j(t, s, b)) - e^{\alpha_n_1} V(a_j) A(a) F(b) \}.
\]

So we can write

\[
H_{t+s}(a, a(t + s)b + m(t + s)) = \frac{a_t(s, a, b) + b_t(s, a, b) + c_t(s, a, b) + d_t A(a) F(b)}{a_t(s, \infty, \infty) + c_t(s, \infty, \infty) + d_t},
\]

where

\[
a_t(s, a, b) = \frac{1}{\mathcal{Z}(t)} \sum_{j=1}^{\mathcal{Z}(t)} \left\{ e^{-\alpha_n} Z_{\alpha_j}^{a_j}(s, a, a(t + s)b + m(t + s)) - e^{-\alpha_n} m_{\alpha_j}(s, a) P(Y(s) \leq y_j(t, s, b)) \right\},
\]

\[
b_t(s, a, b) = \frac{1}{\mathcal{Z}(t)} \sum_{j=1}^{\mathcal{Z}(t)} e^{-\alpha_n} m_{\alpha_j}(s, a) \{ P(Y(s) \leq y_j(t, s, b)) - F(y_j(t, s, b)) \},
\]

\[
c_t(s, a, b) = \frac{1}{\mathcal{Z}(t)} \sum_{j=1}^{\mathcal{Z}(t)} \left\{ e^{-\alpha_n} m_{\alpha_j}(s, a) F(y_j(t, s, b)) - n_1 A(a) F(b) V(a_j) \right\},
\]

\[
d_t = \frac{n_1}{\mathcal{Z}(t)} V_t
\]

Arguing in exactly the same way as Lemma 2.6 we can show that for any sequence \( s_n \)

\[
a_{n_0}(s_n, a, b) \xrightarrow{a.s.} 0, \quad a_{n_0}(s_n, a, b) \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty
\]
and it is immediate from Polya's Theorem and (2.2) that for any choice of sequence $s_n$,

$$b_{n\delta}(s_n, a, b) \leq \sup_{x} |P(Y(s) \leq x) - F(x)|$$

$$\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

**Lemma 2.9** Fix $\delta > 0$ and let $s_n = (n\delta)^3 - n\delta$, then

$$c_{n\delta}(s_n, a, b) \xrightarrow{a.s.} 0 \quad \text{as} \quad n \rightarrow \infty.$$

**Proof.** Note that

$$\frac{|m(n\delta + s_n) - m(s_n) - m(n\delta)|}{a(s_n)} = \frac{|(n\delta)^3 L_2((n\delta)^3) - ((n\delta)^3 - n\delta)^3 L_2((n\delta)^3 - n\delta) - (n\delta)^d L_2(n\delta)|}{((n\delta)^3 - n\delta)^c L_1((n\delta)^3 - n\delta)}$$

$$= \frac{L_2((n\delta)^3) - (1 - \frac{1}{(n\delta)^3})d L_2((n\delta)^3 - n\delta) - \frac{1}{(n\delta)^3} L_2(n\delta)}{(1 - \frac{1}{(n\delta)^3})d((n\delta)^3 - n\delta)^{c-d}}$$

$$\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad (2.20)$$

and that

$$\psi_j(n\delta, s_n, b) = \frac{a((n\delta)^3)}{a(s_n)} b + \frac{m((n\delta)^3) - m(s_n) - m(n\delta)}{a(s_n)}$$

$$- \frac{x_j - m(n\delta)}{a(n\delta)} \cdot \frac{a(n\delta)}{a(s_n)} \quad (2.21)$$

Let $I_{nj} = I\{\frac{x_j - m(n\delta)}{a(n\delta)} \leq a(n\delta)\}$ and let $J_{nj} = I_{nj}^c$ then

$$c_{n\delta}(s_n, a, b) = c_{n\delta}^1(s_n, a, b) + c_{n\delta}^2(s_n, a, b) + c_{n\delta}^3(s_n, a, b)$$

where
From (2.20) and (2.21) we conclude that
\[ c_{n\delta}^1(s_n, a, b) \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty. \]

On the other hand, for any \( \varepsilon > 0 \),
\[
P(|c_{n\delta}^2(s_n, a, b)| > \varepsilon |G_{n\delta}) \leq \frac{2E(M)}{\varepsilon Z(n\delta)} E(\sum_{j=1}^{Z(n\delta)} J_n |G_{n\delta}) \]
\[
= \frac{2E(M)}{\varepsilon} P\left(\frac{x_j - m(n\delta)}{a(n\delta)} \geq a(n\delta)\right) \]
\[
\leq \frac{2E(M)}{\varepsilon a(n\delta)} E\left(\left|\frac{x_j - m(n\delta)}{a(n\delta)}\right|^u\right) \]

Since \( a(t) = t^c L_1(t), u > 1/c \) and \( L_1(n\delta) \to L_1(\delta) \) and since \( E\left(\left|\frac{x_j - m(n\delta)}{a(n\delta)}\right|^u\right) \to E(|Y^u|) \) as \( n \to \infty \),
\[
\sum_{n=0}^{\infty} P(|c_{n\delta}^2(s_n, a, b)| > \varepsilon |G_{n\delta}) < \infty. \]

Hence
\[ c_{n\delta}^2(s_n, a, b) \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty, \]
by the conditional Borel-Cantelli lemma. Finally it can be easily shown from Lemma 2.4 that \( c_{n\delta}^3(s_n, a, b) \xrightarrow{\text{a.s.}} 0 \) as \( n \to \infty. \)

Since \( d_{n\delta} \xrightarrow{\text{a.s.}} 1 \) as \( n \to \infty \)(Lemma 2.5) we have shown that for \( a \in R^+, b \in C_F \),
\[ H_{(n\delta)^3}(a, a((n\delta)^3)b + m((n\delta)^3)) \xrightarrow{\text{a.s.}} A(a)F(b) \quad \text{as} \quad n \to \infty. \]
The techniques used in the proof of Lemma 2.8 can be applied to prove

$$H_{n\delta}(a, a(n\delta)b + m(n\delta)) \xrightarrow{\text{as}} A(a)F(b) \quad \text{as} \quad n \to \infty$$

with some modification.

**Lemma 2.10** For a fixed $\delta > 0$, $a \in R^+$, $b \in R$,

$$H_{n\delta}(a, a(n\delta)b + m(n\delta)) \xrightarrow{\text{as}} A(a)F(b) \quad \text{as} \quad n \to \infty.$$  

**PROOF.** We use the notations $m_n, k_n, B_i, i = 1, 2, 3$ which are defined in the proof of Lemma 2.8 without any change, but we define

$$d_{k_n \delta}^i((n - k_n)\delta, a, b) = \frac{1}{Z(k_n \delta)} \sum_{j=1}^{Z(k_n \delta)} e^{-\alpha(n-k_n)\delta} m^{a_j}((n - k_n)\delta, a)
\begin{array}{c}
\int F(y_j(k_n \delta, (n - k_n)\delta, b)) I_{a(n\delta)B_i+m(n\delta)}(x_j),
\end{array}$$

where $\{(a_j, x_j); j = 1, \ldots, Z(k_n \delta)\}$ is the (age, position)-chart at time $k_n \delta$ and $\{y_j; j = 1, \ldots, Z(k_n \delta)\}$ is given by (2.19). So we have

$$H_{n\delta}(a, a(n\delta)b + m(n\delta)) = a_{k_n \delta}((n - k_n)\delta, a, b) + b_{k_n \delta}((n - k_n)\delta, a, b) + \sum_{i=1}^{3} d_{k_n \delta}^i((n - k_n)\delta, a, b)
\begin{array}{c}
a_{k_n \delta}((n - k_n)\delta, \infty, \infty) + c_{k_n \delta}((n - k_n)\delta, \infty, \infty) + d_{k_n \delta}.
\end{array}$$

We already know that for any $\delta > 0$ and for any $a \in R^+ \cup \infty$, $b \in R \cup \{\infty\}$

$$a_{k_n \delta}((n - k_n)\delta, a, b) \xrightarrow{\text{as}} 0 \quad \text{as} \quad n \to \infty,$$

$$b_{k_n \delta}((n - k_n)\delta, a, b) \xrightarrow{\text{as}} 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, $c_{k_n \delta}((n - k_n)\delta, \infty, \infty) \xrightarrow{\text{as}} 0$(Lemma 2.4), and $d_{k_n \delta} \to 1$(Lemma 2.5) as $n \to \infty$. With some notational change the arguments in the proof of Lemma 2.8 give us
\[
\limsup_{n \to \infty} |d_{kn}^2((n - kn)\delta, a, b)| \leq E(M)(F(b + \varepsilon) - F(b - \varepsilon)),
\]

\[
\limsup_{n \to \infty} |d_{kn}^1((n - kn)\delta, a, b) - A(a)F(b - \varepsilon)| = 0,
\]

\[
\limsup_{n \to \infty} |d_{kn}^2((n - kn)\delta, a, b)| = 0.
\]

Letting \(\varepsilon \downarrow 0\) for \(a \in R^+, \ b \in R\) we have

\[
H_{n\delta}(a, a(n\delta)b + m(n\delta)) \xrightarrow{\Delta_n} A(a)F(b) \quad \text{as} \quad n \to \infty. \quad \Box
\]

Now the proof of Theorem 2.2 can be completed with the exactly same lines as that of Theorem 2.1 and so it is omitted.
3. CENTRAL LIMIT THEOREMS FOR BELLMAN-HARRIS PROCESSES

3.1 One Dimensional Case

Let \( \{Z(t); t \geq 0\} \) be a one-dimensional nonsubcritical Bellman-Harris process evolving from 1 particle of age 0 at time 0 with lifetime distribution \( G \) and offspring law \( \{p_k\} \). For any family history \( \omega \), let \( \{a_j(t, \omega); j = 1, \ldots, Z(t, \omega)\} \) be the age-chart at time \( t \) and let \( f : R^+ \to R \) be a Borel-measurable function. Define

\[
Z_f(t, \omega) = \sum_{j=1}^{Z(t, \omega)} f(a_j(t, \omega)) \\
m_f(t) = E(Z_f(t, \omega))
\]

Since \( \lim_{t \to \infty} e^{-\alpha t} m_f(t) = \frac{1}{\beta} \int_0^\infty e^{-\alpha u} (1 - G(u)) f(u) du \) (see subsection 1.3.2), if we assume that \( \int_0^\infty e^{-\alpha u} (1 - G(u)) f(u) du = 0 \), then \( e^{-\alpha t} m_f(t) \to 0 \) as \( t \to \infty \). In this section we develop limit theorems for this class of stochastic processes \( \{Z_f(t); t \geq 0\} \). That is, we want to determine the asymptotic behavior of the random variable \( Z_f(t) \) as \( t \) goes to infinity.

3.1.1 The Definitions, Assumptions, and Statement of Results

Even though some of the followings are already introduced in Chapter 1, we list them again here for easy reference. For any family history \( \omega \),
1. \( \{a_j(t, \omega); j = 1, \ldots, Z(t, \omega)\} \) = age-chart at time \( t \).

2. \( Z_f(t, \omega) = \sum_{j=1}^{Z(t, \omega)} f(a_j(t, \omega)) \).

3. \( m_f(t) = E(Z_f(t, \omega)), \ D_f(t) = E(Z_f^2(t, \omega)) \).

4. We add superscript \( a \) to the random variable \( Z_f(t) \) and its moments \( m_f(t) \) and \( D_f(t) \) to indicate the case when \( P \) is supported by those \( \omega \)'s which start with one particle of age \( a \geq 0 \).

5. \( \mathcal{F}_t \) = the \( \sigma \)-algebra containing all the informations of the family histories up to time \( t \).

6. \( Z(t, \omega) \) = the number of particles living at time \( t \).

7. \( Z(t, a, \omega) \) = the number of particles living at time \( t \) whose age \( \leq a \).

8. \( A(t, a, \omega) = \frac{Z(t, a, \omega)}{Z(t, \omega)} \) if \( Z(t, \omega) > 0 \).

9. \( N(a, b) \) = a normal distribution with mean \( a \) and variance \( b \).

10. \( m = \sum_{j=0}^{\infty} j p_j, \ m_2 = \sum_{j=0}^{\infty} j^2 p_j \).

11. \( \alpha \) = the Malthusian parameter for \( m \) and \( G \).

12. \( \mu_\alpha(t) = m \int_0^t e^{-\alpha u} G(du), \ U_\alpha(t) = \sum_{j=0}^{\infty} \mu_\alpha^j(t) \).

13. \( W(t, \omega) = e^{-\alpha t} Z(t, \omega), \ W(\omega) = \lim_{t \to \infty} W(t) \) a.s.

14. \( D_\alpha^q = \lim_{t \to \infty} e^{-\alpha t} D_f(t) \).

15. \( G^a(t) = \frac{G(t + a) - G(a)}{1 - G(a)} \).
16. $\beta = \int_0^\infty u \mu_u(du) = m \int_0^\infty ue^{-au}G(du)$.

17. $A(x) = \frac{\int_0^x e^{-au}(1 - G(u)) du}{\int_0^\infty e^{-au}(1 - G(u)) du}, \quad n_1 = \frac{\int_0^\infty e^{-au}(1 - G(u)) du}{m \int_0^\infty ue^{-au}G(du)}$.

18. $A_t = \{\omega; Z(t, \omega) > 0\}$.

19. $\bar{f}(t_0) = \sup\{|f(t)|; 0 \leq t \leq t_0\}$.

Throughout this section we make the following assumptions.

(A 1) $p_0 = 0$ (supercritical case only).

(A 2) $m_2 < \infty$.

(A 3) $G(0+) = 0$, $G$ is non-lattice.

(A 4) $\int_0^\infty u G(du) < \infty$.

The assumption (A 1) is primarily of convenience of exposition. Otherwise, one has to keep qualifying “on the set of explosion”. For example, under (A 1) $A(a, t, \omega)$ is well-defined a.s. With (A 1) and (A 2) we know that there exists a random variable $W$ such that

$$\lim_{t \to \infty} e^{-at}Z(t) = W \quad \text{a.s. and} \quad P(W > 0) = 1.$$ 

The assumption (A 3) guarantees that $U(t)$ is finite for any finite $t$ (see Theorem 1.9).

We impose the following assumptions on a measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ which are not all valid at all times.

(F 1) $f$ is continuous a.e.(w.r.t. Lebesgue measure) on the support of $G$.

(F 2) $e^{-at}(1 - G(t)) f(t)$ is d.R.i. and $\int_0^\infty e^{-at}(1 - G(t)) f(t) dt = 0$.

(F 3) $e^{-at}f^2(t) \to 0$ as $t \to \infty$. 
(F 4) $e^{-at}(m^2 * G)(t)$ is d.R.i.

(F 5) There exists $s_0 > 0$ such that for $s \geq s_0$,
\[
\sup_{a \geq 0} |f(a + s)(1 - G^a(s))| < \infty,
\]
\[
\sup_{a \geq 0} |f^2(a + s)(1 - G^a(s))G^a(s)| < \infty.
\]

(F 6) $\frac{f^2(t)}{t} \to 0$ as $t \to \infty$.

(F 7) $f^2(t)(1 - G(t))$ is d.R.i.

Remark 3.1

1. (F 3) with (A 4) implies (F 3); $e^{-at}f^2(t)(1 - G(t))$ is d.R.i.

2. (F 3) implies (F 3)"; $e^{-at}f(t) \to 0$ as $t \to \infty$.

3. (F 4) is not directly in terms of $f$ and is difficult to verify. We do provide some examples later when (F 4) is verified.

Now we are ready to state the results.

**Theorem 3.1** Let $m > 1$. Assume (F 1) -- (F 5). Then
\[
\frac{Z_f(t)}{\sqrt{Z(t)}} \xrightarrow{d} N(0, \sigma_f^2) \quad \text{as} \quad t \to \infty,
\]
where $\sigma_f^2 \overset{\text{def}}{=} n_1^{-1}D_f^2$.

**Remark 3.2** In the next section, we’ll show that $D_f^2$ exists and is finite.

**Theorem 3.2** Let $m = 1$. Assume (F 1) -- (F 2) and (F 4) -- (F 7) hold. If $t^2(1 - G(t)) \to 0$ as $t \to \infty$, then
\[
\left\{ \frac{Z_f(t)}{\sqrt{Z(t)}} | A_t \right\} \xrightarrow{d} N(0, \sigma_f^2) \quad \text{as} \quad t \to \infty,
\]
where $\sigma_f^2 \overset{\text{def}}{=} \lim_{t \to \infty} D_f(t)$.
3.1.2 The First and Second Moments

We assume $m \geq 1$ throughout this subsection.

**Proposition 3.1** Let $m > 1$.

(a) Assume that $e^{-\alpha t}f(t)(1 - G(t))$ is Borel measurable and bounded on bounded intervals. Then

$$e^{-\alpha t}m_f(t) = \int_0^t e^{-\alpha(t-u)}f(t-u)(1 - G(t-u))U_\alpha(du)$$

(b) Let $m_2 < \infty$ and $e^{-\alpha t}f^2(t)(1 - G(t))$ is Borel measurable and bounded on bounded intervals. Then

$$e^{-\alpha t}D_f(t) = \int_0^t e^{-\alpha(t-u)}f^2(t-u)(1 - G(t-u)) + (m_2 - m)e^{-\alpha(t-u)}(m_2 G(t-u))U_\alpha(du).$$

**Proof.** We prove (b) only (see subsection 1.3.2 for the proof of part (a)). Recall that

$$Z_f(t) = I(\lambda_0 > t)f(t) + \sum_{j=1}^\xi Z_{f,j}(t - \lambda_0)$$

(3.1)

where $\lambda_0$ is the lifetime of the ancestor and $\xi$ is the number of offsprings produced by it. So,

$$Z_f^2(t) = I(\lambda_0 > t)f^2(t) + \sum_{i \neq j}^\xi Z_{f,i}(t - \lambda_0)Z_{f,j}(t - \lambda_0) + \sum_{j=1}^\xi Z_{f,j}^2(t - \lambda_0)$$

Since $\lambda_0$ and $\xi$ are independent, and $Z_f(t - \lambda_0)$'s are independent given $\lambda_0$, taking expectation we get

$$D_f(t) = (1 - G(t))f^2(t) + (m_2 - m) \int_0^t m_2^2(t - u) G(du) + m \int_0^t D_f(t - u) G(du)$$
Multiplying $e^{-at}$ both sides we arrive at the following renewal equation,

$$e^{-at} D_f(t) = e^{-at} f^2(t)(1 - G(t)) + (m_2 - m)e^{-at}(m_f^2 * G)(t) + \int_0^t e^{-a(t-u)} D_f(t-u) \mu_\alpha(du) \quad (3.2)$$

Hence (see Theorem 1.9)

$$e^{-at} D_f(t) = \int_0^t (e^{-a(t-u)} f^2(t-u)(1-G(t-u)) + (m_2-m)e^{-a(t-u})(m_f^2*G)(t-u)) U_\alpha(du).$$

In the critical case we assume $(F7)$ instead of $(F3)$ then the same lines of the proof of Proposition 3.1 give us

**Proposition 3.2** Let $m = 1$.

(a) Assume that $f(t)(1-G(t))$ is Borel measurable and bounded on finite intervals, then

$$m_f(t) = \int_0^t f(t-u)(1-G(t-u)) U(du)$$

(b) Let $m_2 < \infty$. If $f^2(t)(1-G(t))$ is Borel measurable and bounded on bounded intervals then

$$D_f(t) = \int_0^t (f^2(t-u)(1-G(t-u)) + (m_2-m)(m_f^2*G)(t-u)) U(du).$$

The following is an immediate consequence of the Key Renewal Theorem and Remark 3.1.

**Proposition 3.3** Let $f$ satisfy $(F3)$, and $(F4)$ if $m > 1$ and $(F4)$ and $(F7)$ if $m = 1$. Then $e^{-at} f^2(t)(1-G(t))$ is d.R.i. and $D_f^\alpha = \lim_{t \to \infty} e^{-at} D_f(t)$ exists and is
given by

\[ D_j^g = \frac{1}{\beta} \int_0^\infty (e^{-\alpha u}f^2(u)(1 - G(u))) + (m_2 - m)e^{-\alpha u}(m_f^2 * G)(u))du < \infty. \]

Now define \( M(s) \); \( f \mapsto M(s)f \) by \( (M(s)f)(t) = m_f^s(t) \).

**Proposition 3.4** Let \( m > 1 \). Then

(a) \( m_{M(s)f}(t) = m_f(t + s) \)

(b) Further, assume that \((F3)\) and \((F4)\) hold, then

\[ \lim_{s \to \infty} e^{-\alpha s} \lim_{t \to \infty} e^{-\alpha t}D_{M(s)f}(t) = 0 \]

**Proof.**

(a) \( m_{M(s)f}(t) = E(Z_{M(s)f}(t)) \)

\[ = E(\sum_{j=1}^{\mathbb{Z}(t)}(M(s)f)(a_j(t))) \]

\[ = E(Z_f(t + s)) = m_f(t + s) \]

(b) From equation (3.2) above with \( M(s)f \) in the place of \( f \)

\[ e^{-\alpha t}D_{M(s)f}(t) = e^{-\alpha t}(M(s)f)^2(t)(1 - G(t)) + (m_2 - m)e^{-\alpha t}(m_f^2 * G)(t) \]

\[ + \int_0^t e^{-\alpha(t-u)}D_{M(s)f}(t-u)\mu_\alpha(du) \]

First, we'll show that \( e^{-\alpha t}(M(s)f)^2(t)(1 - G(t)) \) and \( e^{-\alpha t}(m_f^2 * G)(t) \) are d.R.i. for fixed \( s \). Beginning with an ancestor of age \( t \) at time 0, we have the following identity

\[ Z_j^t(s) = I(\lambda^f > s)f(t + s) + \sum_{j=1}^{\xi} Z_{f,j}(s - \lambda^f) \quad (3.3) \]

where \( \lambda^f \) and \( \xi \) are the lifetime and the number of children of the ancestor respectively and \( \{Z_{f,j}(s), s \geq 0\} \) is the \( Z_f(\cdot) \) process initiated by the \( j \)th child of the ancestor.
Conditioned on \( \lambda \), \( \{Z_{j,i}(s - \lambda^i) ; j = 1, \cdots, \xi \} \) are i.i.d. and further if \( \lambda^i = u \), then the conditional distribution of \( Z_{j,i}(s - \lambda^i) \) is the same as \( Z_{j,s}(s - u) \). So we have

\[
\{M(s)f\}(t) = f(t + s)(1 - G^t(s)) + m \int_0^s m_f(s - u) G^t(du)
\]  \hspace{1cm} (3.4)

and so,

\[
\{M(s)f\}^2(t) \leq 2f^2(t + s)(1 - G^t(s))^2 + 2m^2 \int_0^s m_f(s - u) G^t(du)^2. \hspace{1cm} (3.5)
\]

Using Cauchy-Schwarz inequality,

\[
\left( \int_0^s m_f(s - u) G^t(du) \right)^2 \leq \left( \int_0^s m_f^2(s - u) G^t(du) \right) G^t(s) \leq \frac{(m_f^2 \ast G)(t + s)}{1 - G(t)}.
\]

Combining this with inequality (3.5), we have

\[
e^{-at}(M(s)f)^2(t)(1 - G(t)) \leq 2e^{as}(e^{-a(t+s)}f^2(t + s)(1 - G(t + s)) + m^2e^{-a(t+s)}(m_f^2 \ast G)(t + s)).
\]

(3F) and (F4) along with this inequality implies that \( e^{-at}(M(s)f)^2(t)(1 - G(t)) \) is d.R.i. for fixed \( s \geq 0 \). On the other hand,

\[
e^{-at} \int_0^t m_{M(s)f}^2(t - u) G(du) = e^{-at} \int_0^t m_f^2(t + s - u) G(du) \leq e^{-at} \int_0^{t+s} m_f^2(t + s - u) G(du) = e^{as}e^{-a(t+s)}(m_f^2 \ast G)(t + s).
\]

So \( e^{-at}(m_{M(s)f}^2 \ast G)(t) \) is d.R. by (F4). Hence we can apply the Key Renewal Theorem to get

\[
\lim_{t \to \infty} e^{-at} D_{M(s)f}(t)
\]
\[
\frac{1}{\beta} \int_0^\infty \left( e^{-\alpha u} (M(s)f)^2(u)(1 - G(u)) + (m_2 - m) e^{-\alpha u} (m_2^2_{M(s)} * G)(u) \right) du \\
\leq \frac{2e^{\alpha s}}{\beta} \int_s^\infty e^{-\alpha u} f^2(u)(1 - G(u)) + m^2 e^{-\alpha u} (m_2^2 * G)(u) + (m_2 - m) e^{-\alpha u} (m_2^2 * G)(u) du
\]

and since \((F3)'\) and \((F4)\) hold, we conclude that

\[
\lim_{t \to \infty} e^{-\alpha s} \lim_{t \to \infty} e^{-\alpha t} D_{M(s)f}(t) = 0 \quad \square
\]

In the critical case we have

**Proposition 3.5** Let \( m = 1. \) Then

(a) \( m_{M(s)f}(t) = m_f(t + s) \)

(b) Further, assume that \((F4)\) and \((F7)\) hold, then

\[
\lim_{s \to \infty} \lim_{t \to \infty} D_{M(s)f}(t) = 0
\]

### 3.1.3 Proof of Theorem 3.1

Referring to the additive property mentioned in subsection 1.2.2 we can write

(suppressing \( \omega \) and \( (t, \omega) \))

\[
Z_f(t + s) = \sum_{j=1}^{Z(t)} Z_f^{a_j}(s), \quad (3.6)
\]

where \( \{Z_f^{a_j}(s), s \geq 0\} \) is the process \( \{Z_f(s) : s \geq 0\} \) initiated by the ancestor of age \( a_j \) at time \( t \). It is obvious that conditioned on the age chart at time \( t \), \( \{Z_f^{a_j}(s); j = 1, \cdots, Z(t)\} \) are independently distributed. Furthermore, if \( a_j = a \) then the conditional distribution of \( Z_f^{a_j}(s) \) is the same as \( Z_f(s) \). Starting from equation (3.6) we have the following identity

\[
Z_f(t + s) = \sum_{j=1}^{Z(t)} (Z_f^{a_j}(s) - m_f^{a_j}(s)) + Z_{M(s)f}(t). \quad (3.7)
\]
Dividing equation (3.7) by \(\sqrt{Z(t + s)}\) we get

\[
\frac{Z_j(t + s)}{\sqrt{Z(t + s)}} = \sqrt{\frac{Z(t)}{Z(t + s)}} \frac{1}{\sqrt{Z(t)}} \sum_{j=1}^{Z(t)} (\hat{Z}_j - m_j) + \frac{Z_M(t)}{\sqrt{Z(t + s)}}
\]

Here is the basic idea of the proof; we first choose \(s\) large enough to make \(A_2(t, s)\) small in probability and then with this large but fixed \(s\), we show that \(A_1(t, s)\) converges to the desired normal distribution as \(t \rightarrow \infty\) using the Lindberg-Feller theorem. We carry this out in a series of lemmas below where we assume that (F 1)-(F 5) hold.

**Lemma 3.1** For any \(\eta > 0\), \(\delta > 0\), there exists \(s_0(\eta, \delta)\) such that

\[
\lim_{t \rightarrow \infty} P(|A_2(t, s)| > \eta) < \delta, \quad \text{for all } s \geq s_0(\eta, \delta).
\]

**Proof.** Recall that there exists \(W = \lim_{t \rightarrow \infty} W(t)\) a.s. and \(P(W > 0) = 1\) if \(p_0 = 0\). Choose \(x\) such that \(P(W \leq x) \leq \delta/3\) and let \(\varepsilon > 0\) be such that \(\varepsilon < x/2\). Since \(W(t)\) converges to \(W\) a.s., it does so in probability and hence we can choose \(s_0' = s_0'(\delta)\) such that

\[
P(|W(t + s_0') - W| > \varepsilon) < \delta/3, \quad \text{for all } t \geq 0.
\]

So for \(s \geq s_0'\) and for all \(t \geq 0,

\[
P(|A_2(t, s)| > \eta) \leq P(|A_2(t, s)| > \eta, |W(t + s) - W| < \varepsilon, W > x)
\]

\[+ P(|W(t + s) - W| \geq \varepsilon) + P(W \leq x)
\]

\[\leq P(|A_2(t, s)| > \eta, |W(t + s) - W| < \varepsilon, W > x) + \frac{2\delta}{3} (3.8)
\]
Now,

\[ P(|A_2(t, s)| > \eta, |W(t + s) - W| < \varepsilon, W > x) \]
\[ \leq P\left(\left|Z_{M(s)}f(t)\right| > \eta, Z(t + s) > (x - \varepsilon) e^{\alpha(t+s)}\right) \]
\[ \leq P\left(\left|Z_{M(s)}f(t)\right| > \eta \sqrt{x - \varepsilon} e^{\alpha(t+s)}\right) \]
\[ \leq \frac{e^{-\alpha(t+s)}}{\eta^2(x - \varepsilon)} E(Z^2_{M(s)}f(t)), \tag{3.9} \]

by Markov's inequality. We can choose \( s_0'' \) by Proposition 3.4 (b) such that \( s \geq s_0'' \)
implies

\[ e^{-\alpha x} \lim_{t \to \infty} E(Z^2_{M(s)}f(t)) e^{-\alpha t} \leq \frac{\delta}{3} (x - \varepsilon) \eta^2. \tag{3.10} \]

Let \( s_0 = \max(s_0', s_0'') \), then from inequalities (3.8), (3.9) and (3.10) we have

\[ \lim_{t \to \infty} P(A_2(t, s) > \eta) < \delta \quad \text{for} \quad s \geq s_0(\eta, \delta). \]

Next we find the conditional variance of \( A_1(t, s_0) \) given \( \mathcal{F}_t \) and its limit.

**Lemma 3.2** Fix \( s_0 > 0 \), then \( \lim_{t \to \infty} \text{Var}(A_1(t, s_0)|\mathcal{F}_t) = \sigma^2_j(s_0), \) a.s. where

\[
\begin{align*}
\sigma^2_j(s_0) &= e^{-\alpha s_0} \int_0^\infty \sum_{i=1}^{5} V_i(a, s_0) A(da) \\
V_1(a, s_0) &= f^2(a + s_0)G^a(s_0)(1 - G^a(s_0)) \\
V_2(a, s_0) &= m(D_1 * G^a)(s_0) \\
V_3(a, s_0) &= (m_2 - m)(m_2^2 * G^a)(s_0) \\
V_4(a, s_0) &= -m^2(m_1 * G^a)^2(s_0) \\
V_5(a, s_0) &= 2mf(a + s_0)(1 - G^a(s_0))(m_1 * G^a)(s_0).
\end{align*}
\]

**Proof.** Write \( Y^{a_j}(s_0) = [Z_j^a(s_0) - m_j^a(s_0)] e^{-\frac{s_0}{2}}, \) then

\[ A_1(t, s_0) = \frac{1}{\sqrt{Z(t)}} \sum_{j=1}^{Z(t)} Y^{a_j}(s_0). \]
Since \( \{Y_t^{a_j}(s_0); j = 1, \cdots, Z(t)\} \) are mutually independent conditioned on \( \mathcal{F}_t \) and also independent of \( Z(t) \),

\[
\text{Var}(A_1(t, s_0)|\mathcal{F}_t) = \frac{1}{Z(t)} \sum_{i=1}^{Z(t)} \text{Var}(Y_t^{a_i}(s_0)|\mathcal{F}_t).
\]

Recalling equations (3.3) and (3.4) we have

\[
Z_f^{a_j}(s_0) = I(\lambda^{a_j} > s_0)f(a_j + s_0) + \sum_{i=1}^{\xi} Z_{f,i}(s_0 - \lambda^{a_j}),
\]

\[
m_f^{a_j}(s_0) = (1 - G^{a_j}(s_0))f(a_j + s_0) + m \int_0^s m_f(s_0 - u)G^{a_j}(du).
\]

So \( E(\{Y_t^{a_i}(s_0)\}) = e^{-a_0} \sum_{i=1}^5 V_i(a_j, s_0) \) and

\[
\text{Var}(A_1(t, s_0)|\mathcal{F}_t) = e^{-a_0} \sum_{i=1}^5 \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} V_i(a_j, s_0)
\]

\[
= e^{-a_0} \int_0^\infty V_i(a, s_0) A(t, da).
\]

Note that since \( m_f(s_0) \) and \( D_f(s_0) \) are finite \( V_2(\cdot, s_0), V_3(\cdot, s_0) \) and \( V_4(\cdot, s_0) \) are bounded. The boundedness of \( V_1(\cdot, s_0) \) and \( V_5(\cdot, s_0) \) is direct from \( F5 \). Also it can be verified that \( V_i(\cdot, s_0), i = 1, \cdots, 5 \) are continuous a.e. Now we apply Corollary 1.1, i.e.;

\[
\text{Var}(A_1(t, s_0)|\mathcal{F}_t) = e^{-a_0} \sum_{i=1}^5 \int_0^{\infty} V_i(a, s_0) A(t, da)
\]

\[
\overset{a.s.}{\rightarrow} e^{-a_0} \sum_{i=1}^5 \int_0^{\infty} V_i(a, s_0) A(da) \quad \text{as} \quad t \rightarrow \infty.
\]

Lemma 3.3 For a fixed \( s_0 > 0 \) and \( \eta > 0 \)

\[
\sup_{0 \leq y \leq t} E(\{Y_t^{a}(s_0)\}^2; |Y_t^{a}(s_0)| > \eta e^{2t}) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]
PROOF. First we see from (F5) that
\[
\sup_{0 \leq a \leq t} |m_f(t, s_0)| \leq \sup_{0 \leq a \leq t} |(1 - G_a(s_0))f(a + s_0)| + m \bar{m}_f(s_0) < \infty,
\]
and
\[
E(\left| Y^a_t(s_0) \right|^2; |Y^a_t(s_0)| > \eta e^{2t}) \leq 2E(e^{-\alpha t} \left| Z^a_t(s_0) \right|^2; |Z^a_t(s_0)| > \eta e^{2(t+s_0)}) + 2E(e^{-\alpha t} \left| m_f(s_0) \right|^2; |Z^a_t(s_0)| > \eta e^{2(t+s_0)}).
\]
So it is enough to show that
\begin{enumerate}
  \item \(\sup_{0 \leq a \leq t} P(\left| Z^a_t(s_0) \right| > \eta e^{2(t+s_0)}) \to 0 \quad \text{as} \quad t \to \infty,\)
  \item \(\sup_{0 \leq a \leq t} e^{-\alpha t} E(\left| Z^a_t(s_0) \right|^2; |Z^a_t(s_0)| > \eta e^{2(t+s_0)}) \to 0 \quad \text{as} \quad t \to \infty.\)
\end{enumerate}
Note that
\[
\left| \sum_{i=1}^{\xi} Z_{f,i}(s_0 - \lambda^a) \right| \leq \sum_{i=1}^{\xi} Z_i(s_0 - \lambda^a)\overline{f}(s_0) = X\overline{f}(s_0), \quad \text{say.} \quad (3.11)
\]
where \(\preceq\) denote the stochastic order; \(X \preceq Y\) implies \(P(X \geq x) \leq P(Y \geq x),\) for all \(x\) and \(\overline{f}\) is as in (19) on p43. Combining the equation (3.3) with the inequality (3.11) we get
\[
\left| Z^a_t(s_0) \right| \preceq |f(a + s_0)|I(\lambda^a > s_0) + X\overline{f}(s_0) \quad (3.12)
\]
So
\[
\sup_{0 \leq a \leq t} \left| Z^a_t(s_0) \right| \preceq \overline{f}(t + s_0) + X\overline{f}(s_0). \quad (3.13)
\]
Now we observe from (3.12) that
\[
\sup_{0 \leq a \leq t} P(\left| Z^a_t(s_0) \right| > \eta e^{2(t+s_0)}) \leq P(\sup_{0 \leq a \leq t} \left| Z^a_t(s_0) \right| > \eta e^{2(t+s_0)}) \leq P(\overline{f}(t + s_0) > (1/2)\eta e^{2(t+s_0)}) + P(X\overline{f}(s_0) > (1/2)\eta e^{2(t+s_0)})
\]
then since \((F 3)''\) holds the first term is zero for large \(t\), and since \(\bar{f}(s_0)\) is finite a.s., the second term goes to zero as \(t \to \infty\) so \((i)\) is proved.

Turning to \((ii)\) we note first from inequality (3.12) that

\[
|Z_f^2(s_0)|^2 \leq f(a + s_0)^2 I(\lambda^a > s_0) + X^2\bar{f}^2(s_0) \tag{3.14}
\]

Note that \(Z_f^2(s_0) = f(a + s_0)\) on \(\{\lambda^a > s_0\}\) and so,

\[
\sup_{0 \leq s \leq t} E[f(a + s_0)^2 I(\lambda^a > s_0) I(|Z_f^2(s_0)| > \eta e^{\delta(t+s_0)})] \\
= \sup_{0 \leq s \leq t} f^2(a + s_0) E[I(\lambda^a > s_0) I(|f(a + s_0)| > \eta e^{\delta(t+s_0)})] \\
\leq \sup_{0 \leq s \leq t} f^2(a + s_0)(1 - G^a(s_0)) I(|f(t + s_0)| > \eta e^{\delta(t+s_0)})
\]

Again \((F 3)''\) implies \(I(|f(t + s_0)| > \eta e^{\delta(t+s_0)}) = 0\) for \(t\) large enough. So

\[
\lim_{t \to \infty} \sup_{0 \leq s \leq t} E[f(a + s_0)^2 I(\lambda^a > s_0) I(|Z_f^2(s_0)| > \eta e^{\delta(t+s_0)})] = 0 \tag{3.15}
\]

On the other hand, by (3.13) we have

\[
\sup_{0 \leq s \leq t} E[X^2\bar{f}^2(s_0) I(|Z_f^2(s_0)| > \eta e^{\delta(t+s_0)})] \\
\leq \bar{f}^2(s_0) E[X^2 \{I(\bar{f}(t + s_0) > \eta e^{\delta(t+s_0)}) + I(X\bar{f}(s_0) > \eta e^{\delta(t+s_0)})\}]
\]

Note that \(I(\bar{f}(t + s_0) > \eta e^{\delta(t+s_0)}) = 0\) for large \(t\) and \(I(X\bar{f}(s_0) > \eta e^{\delta(t+s_0)}) \xrightarrow{\Delta_a} 0\) as \(t \to \infty\). Since \(E(X^2) < \infty\) we conclude that

\[
\lim_{t \to \infty} \sup_{0 \leq s \leq t} E[X^2\bar{f}^2(s_0) I(|Z_f^2(s_0)| > \eta e^{\delta(t+s_0)})] = 0 \tag{3.16}
\]

by the dominated convergence theorem. Now, (3.14) and (3.15), along with (3.16) prove \((ii)\).

The following lemma concerns the conditional Lindeberg-Feller condition.
Lemma 3.4  Fix $s_0 > 0$, $\eta > 0$, then
\[
\sum_{j=1}^{Z(t)} E\left( \frac{\{Y_{i}^{s_j}(s_0)\}^2}{Z(t)} \right) \frac{|Y_{i}^{s_j}(s_0)| > \eta |F_t|}{\sqrt{Z(t)}} \xrightarrow{pr} 0 \quad \text{as} \quad t \to \infty.
\]

PROOF. Write $S_t(s_0, \eta) \overset{\text{def}}{=} \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} E\left( \{Y_{i}^{s_j}(s_0)\}^2 \right) |Y_{i}^{s_j}(s_0)| > \eta \sqrt{Z(t)} |F_t|$. Given $\delta_1 > 0$, $\delta_2 > 0$, there exist $t_0 > 0$ and a set $A$ such that
\begin{align*}
(i) & \quad P(A) > 1 - \delta_1, \quad (3.17) \\
(ii) & \quad t > t_0 \quad \text{and} \quad \omega \in A \quad \text{imply together} \quad Z(t, \omega) \geq \delta_2 e^{nt}. \quad (3.18)
\end{align*}

So for any $\varepsilon > 0$, from (3.17) we get
\[
P(S_t(s_0, \eta) > \varepsilon) = P(S_t(s_0, \eta) > \varepsilon; A) + P(S_t(s_0, \eta) > \varepsilon; A^c) \\
\leq P(S_t(s_0, \eta) > \varepsilon; A) + \delta_1
\]
and for $t \geq t_0$, we have the following from (3.18)
\[
P(S_t(s_0, \eta) > \varepsilon; A) \leq P\left( \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} E\left( \{Y_{i}^{s_j}(s_0)\}^2 \right) |Y_{i}^{s_j}(s_0)| > \eta \sqrt{Z(t)} |F_t| > \varepsilon \right) \\
\leq P\left( \sup_{0 \leq s \leq t} E\left( \{Y_{i}^{s_j}(s_0)\}^2 \right) |Y_{i}^{s_j}(s_0)| > \eta \sqrt{Z(t)} |F_t| > \varepsilon \right)
\]
which is zero for large $t$ by Lemma 3.3. So we conclude that $\lim_{t \to \infty} P(S_t(s_0, \eta) > \varepsilon) < \delta_1$. Letting $\delta_1 \downarrow 0$ we get the result. $\square$

Lemma 3.5  For a fixed $s_0$, $A_1(t, s_0) \overset{d}{\rightarrow} N(0, \sigma_f^2(s_0))$ as $t \to \infty$.

PROOF. $E(\exp(i\theta A_1(t, s_0)) | F_t) = \prod_{j=1}^{Z(t)} E(\exp(i\theta \frac{Y_{i}^{s_j}(s_0)}{\sqrt{Z(t)}}) | F_t)$
\[
def \prod_{j=1}^{Z(t)} \phi_{\theta}^{s_j}(s_0, \theta).
\]
As in the proof of the usual Lindberg-Feller central limit theorem (see Durrett p 98) it is possible to show that

\[
\prod_{j=1}^{Z(t)} \phi_{i,j}^{s,(s_0, \theta)} \to \exp(-\frac{t^2}{2} \sigma_f^2(s_0)) \quad \text{as} \quad t \to \infty \quad (3.19)
\]

with aid of Lemma 3.2 and Lemma 3.4. Therefore, the dominated convergence theorem completes the proof. That is, since (3.19) holds and since

\[
\left| \prod_{j=1}^{Z(t)} \phi_{i,j}^{s,(s_0, \theta)} - \exp(-\frac{\theta^2}{2} \sigma_f^2(s_0)) \right| \leq 2,
\]

\[
E(\exp(i\theta A_1(t, s_0))) = E(E(\exp(i\theta A_1(t, s_0)) | F_t))
\]

\[
= E(E(\exp(i\theta A_1(t, s_0)) - \exp(-\frac{\theta^2}{2} \sigma_f^2(s_0)) | F_t))
\]

\[
+ \exp(-\frac{\theta^2}{2} \sigma_f^2(s_0))
\]

\[
\to \exp(-\frac{\theta^2}{2} \sigma_f^2(s_0)) \quad \text{as} \quad t \to \infty.
\]

So \( A_1(t, s_0) \) has the desired limit distribution. \( \square \)

Lemma 3.6 \( \sigma_f^2(s) \to \sigma_f^2 = n_1^{-1} D_f^2 \) as \( s \to \infty. \)

PROOF. Let \( c_1 = (\int_0^{\infty} e^{-\alpha u} (1 - G(u)) du)^{-1} \), then \( A(da) = c_1 e^{-\alpha a} (1 - G(a)) da. \)

\[
e^{-\alpha a} \int_0^{\infty} V_1(a, s) A(da) = c_1 e^{-\alpha a} \int_0^{\infty} f^2(a + s) G^2(s) e^{-\alpha a} (1 - G(a + s)) da
\]

\[
\leq c_1 \int_0^{\infty} e^{-\alpha (a+s)} f^2(a+s) (1 - G(a + s)) da
\]

\[
= c_1 \int_0^{\infty} e^{-\alpha a} f^2(a) (1 - G(a)) da.
\]
The last term goes to 0 as \( s \to \infty \) since \( e^{-\alpha s} f^2(a)(1 - G(a)) \) is integrable \(((F'3)'\).

\[
e^{-\alpha s} \int_0^\infty V_2(a, s) A(da) = c_1 e^{-\alpha s} \int_0^\infty D_f(s - u)G^a(du)e^{-\alpha a}(1 - G(a)) da
\]

\[
= c_1 m \int_0^\infty e^{-\alpha(s + u)} D_f(s + a - u)e^{-\alpha u} G(du) da
\]

\[
\to c_1 m \int_0^\infty D_f^\alpha e^{-\alpha u} G(du) da
\]

by Lebesgue dominated convergence theorem

\[
= c_1 m D_f^\alpha \int_0^\infty du e^{-\alpha u} G(du)
\]

\[
= c_1 m D_f^\alpha \int_0^\infty u e^{-\alpha u} G(du) = n_1^{-1} D_f^\alpha.
\]

\[
e^{-\alpha s} \int_0^\infty V_3(a, s) A(da) = c_1 e^{-\alpha s} \int_0^\infty (m_2 - m)(m_2^2 * G^a)(s)e^{-\alpha a}(1 - G(a)) da
\]

\[
\leq c_1 (m_2 - m) \int_0^\infty e^{-\alpha(a + s)}(m_2^2 * G)(a + s) da
\]

\[
= c_1 (m_2 - m) \int_0^\infty e^{-\alpha a}(m_2^2 * G)(a) da
\]

\[
\to 0 \quad \text{by} \quad (F'4).
\]

Since \((m_f * G^a)^2(s) \leq (m_f^2 * G^a)(s)\), we have from above that

\[
e^{-\alpha s} \int_0^\infty V_4(a, s) A(da) \to 0 \quad \text{as} \quad s \to \infty.
\]

Finally,

\[
|e^{-\alpha s} \int_0^\infty V_5(a, s) A(da)|^2
\]

\[
= 4m^2 c_1^2 \int_0^\infty e^{-\alpha(a + s)} f(a + s)(1 - G(a + s))(m_f * G^a)(s) da|^2
\]
by the fact that $e^{-\alpha f^2(a)(1 - G(a))}$ is integrable and by the assumption (F4).
Hence
\[ \sigma^2(s) = \sum_{i=1}^{5} e^{-\alpha s} \int_{0}^{\infty} V_i(a, s) A(da) \to n_1^{-1} D_f^2 \quad \text{as} \quad s \to \infty. \]

Now we complete the proof of Theorem 3.1 by assembling all the Lemmas together. Let $\varepsilon > 0$ be arbitrary and $y$ fixed. Choose $\eta_\varepsilon > 0$ such that
\[ |\Phi(\frac{y + \eta_\varepsilon}{\sigma_f}) - \Phi(\frac{y - \eta_\varepsilon}{\sigma_f})| < \frac{\varepsilon}{2}. \quad (3.20) \]

Since $\lim_{s \to \infty} \sigma_f^2(s) = \sigma_f^2$, there exists $s_1(\varepsilon)$ such that $s \geq s_1(\varepsilon)$ implies
\[ |\Phi(\frac{y + r\eta_\varepsilon}{\sigma_f}) - \Phi(\frac{y - r\eta_\varepsilon}{\sigma_f(s)})| < \frac{\varepsilon}{2} \quad \text{for} \quad r = \pm 1. \quad (3.21) \]

Let $\delta = \varepsilon/2$ and let $s^* = \max\{s_0(\eta_\varepsilon, \delta), s_1(\varepsilon)\}$ where $s_0(\eta_\varepsilon, \delta)$ is defined by Lemma 3.1, then
\[
\limsup_{t \to \infty} P(A_1(t, s^*) + A_2(t, s^*) \leq y)
\leq \limsup_{t \to \infty} P(A_1(t, s^*) \leq y + \eta_\varepsilon) + \limsup_{t \to \infty} P(|A_2(t, s^*)| \geq \eta_\varepsilon)
\leq \Phi\left(\frac{y + \eta_\varepsilon}{\sigma_f(s^*)}\right) + \frac{\varepsilon}{2} \quad (3.22)
\]

and
\[
\liminf_{t \to \infty} P(A_1(t, s^*) + A_2(t, s^*) \leq y)
\geq \liminf_{t \to \infty} P(A_1(t, s^*) \leq y - \eta_\varepsilon) - \liminf_{t \to \infty} P(|A_2(t, s^*)| \geq \eta_\varepsilon)
\geq \Phi\left(\frac{y - \eta_\varepsilon}{\sigma_f(s^*)}\right) - \frac{\varepsilon}{2} \quad (3.23)
\]
(3.20), (3.21), (3.22) and (3.23) and the fact
\[
\frac{Z(t)e^{-\alpha t}}{Z(t + s^*)e^{-\alpha(t+s^*)}} \xrightarrow{a.s.} 1 \quad \text{as} \quad t \to \infty
\]
imply together that
\[
\Phi\left(\frac{y}{\sigma_f}\right) - \frac{3}{2} \varepsilon \leq \lim_{t \to \infty} P\left(\frac{Z_f(t + s^*)}{\sqrt{Z(t + s^*)}} \leq y\right) \leq \Phi\left(\frac{y}{\sigma_f}\right) + \frac{3}{2} \varepsilon.
\]
Since \(\varepsilon > 0\) is arbitrary, the proof is completed.

3.1.4 Proof of Theorem 3.2

In this section we put \(\mu = \int_0^\infty uG(du), \sigma^2 = h''(1) = m_2 - 1\) and assume that (F 1)-(F 2), and (F 4)-(F 7) hold. We begin with the following decomposition on the set \(A_{t+s};\)
\[
\frac{Z_f(t + s)}{\sqrt{Z(t + s)}} = \sqrt{\frac{Z(t)}{Z(t + s)}} \frac{1}{\sqrt{Z(t)}} \sum_{j=1}^{Z(t)} (Z_f^{a_j}(s) - m_f^{a_j}(s)) + \frac{Z_M(s,f)(t)}{\sqrt{Z(t + s)}}.
\]

**Lemma 3.7** For any \(\eta > 0, \delta > 0,\) there exists \(s_0(\eta, \delta)\) such that \(s > s_0(\eta, \delta)\) implies
\[
\lim_{t \to \infty} P\left(\frac{|Z_M(s,f)(t)|}{\sqrt{t + s}} > \eta \mid A_{t+s}\right) < \delta.
\]

**Proof.** By Chebyshev's inequality
\[
P\left(|Z_M(s,f)(t)| > \eta \sqrt{t + s} \mid A_{t+s}\right) \leq \frac{E[(Z_M(s,f)(t))^2]}{\eta^2(t + s)P(A_{t+s})}
\]
Since \(P(A_t) \sim \frac{2\mu}{\sigma^2} \frac{1}{t}\) (Theorem 1.3),
\[
\lim_{t \to \infty} P\left(|Z_M(s,f)(t)| > \eta \sqrt{t + s} \mid A_{t+s}\right) \leq \frac{\sigma^2}{2\mu \eta^2} \lim_{t \to \infty} E[(Z_M(s,f)(t))^2]
\]
Now we can choose \(s_0 > 0\) (Proposition 3.5) such that \(s \geq s_0\) implies
\[
\lim_{t \to \infty} D_M(s,f)(t) = \lim_{t \to \infty} E[(Z_M(s,f)(t))^2] < \frac{2\mu \eta^2}{\sigma^2} \delta. \quad \square
\]
Lemma 3.8 For any $\eta > 0$, $\delta > 0$, we can choose $s_1(\eta, \delta)$ such that $s \geq s_1(\eta, \delta)$ implies
\[
\lim_{t \to \infty} P\left(\frac{\left|Z_{M(t)}(t)\right|}{\sqrt{Z(t + s)}} > \eta \mid A_{t+s}\right) < \delta.
\]

PROOF. Recall that (Theorem 1.3) for $x \geq 0$, $P\left(\frac{Z(t)}{t} > x \mid A_t\right) \to e^{-(2\mu/\sigma^2)x}$ as $t \to \infty$. Choose $x > 0$ such that $1 - e^{-(2\mu/\sigma^2)x} < \delta/4$ and choose $s_0' > 0$ such that for all $t \geq 0$ and for $s \geq s_0'$
\[
P\left(\frac{Z(t + s)}{t + s} \leq x \mid A_{t+s}\right) \leq 2(1 - e^{-(2\mu/\sigma^2)x}) \leq \frac{\delta}{2}.
\]
So if $s \geq s_0'$, then for all $t > 0$
\[
P\left(\frac{\left|Z_{M(t)}(t)\right|}{\sqrt{Z(t + s)}} > \eta \mid A_{t+s}\right) = P\left(\frac{\left|Z_{M(t)}(t)\right|}{\sqrt{Z(t + s)}} > \eta, \frac{Z(t + s)}{t + s} > x \mid A_{t+s}\right) + P\left(\frac{Z(t + s)}{t + s} \leq x \mid A_{t+s}\right) \leq P\left(\frac{Z_{M(t)}(t)}{\sqrt{t + s}} > \eta \sqrt{x} \mid A_{t+s}\right) + \frac{\delta}{2}.
\]
Let $s''_0 = s_0(\eta \sqrt{x}, \delta/2)$ which is defined in Lemma 3.7 and then let $s_1(\eta, \delta) = \max\{s_0', s''_0\}$.

Lemma 3.9 For a fixed $s_0$, \[\left\{\frac{Z(t)}{Z(t + s_0)} \mid A_{t+s_0}\right\} \xrightarrow{pr} 1 \text{ as } t \to \infty.\]

PROOF. Because of additive property we have
\[
\left\{\frac{Z(t + s_0)}{Z(t)} \mid A_t\right\} = \left\{\frac{1}{Z(t)} \sum_{j=1}^{Z(t)} Z_j(s_0) \mid A_t\right\}
\]
where \{\(Z_j(s_0); j = 1, \cdots, Z(t)\)\} are i.i.d. with $E(Z_j(s_0)) = 1$. So by weak law of large numbers and the fact that conditioned on $A_t$, $Z(t) \xrightarrow{pr} \infty$, we get
\[
\left\{\frac{Z(t + s_0)}{Z(t)} \mid A_t\right\} \xrightarrow{pr} 1, \text{ as } t \to \infty.
\]
Since \( P(A_{t+s_0} | A_t) = \frac{P(A_{t+s_0})}{P(A_t)} \rightarrow 1 \) as \( t \rightarrow \infty \), we conclude that

\[
\left\{ \frac{Z(t)}{Z(t+s_0)} \bigg| A_{t+s_0} \right\} \xrightarrow{\text{pr}} 1 \quad \text{as} \quad t \rightarrow \infty. \quad \square
\]

The following Lemma is a version of Lemma 3.2 for critical case and its proof is exactly same with that of Lemma 3.2 except that it converges in probability and so its proof is omitted.

**Lemma 3.10** For a fixed \( s_0 > 0 \)

\[
\text{Var}(\frac{1}{\sqrt{Z(t)}} \sum_{j=1}^{Z(t)} [Z_j^{\alpha}(s) - m_j^{\alpha}(s)] | \mathcal{F}_t) \xrightarrow{\text{pr}} \sigma_j^2(s_0) \quad \text{as} \quad t \rightarrow \infty,
\]

where \( \sigma_j^2(s_0) \) is defined as in Lemma 3.2.

The next two lemmas can be proved in the exactly same way with Lemma 3.3 and Lemma 3.4.

**Lemma 3.11** Fix \( s_0 > 0, \eta > 0 \), then

\[
\sup_{0 \leq s \leq t} E\left([Y_t^{\alpha}(s_0)]^2; |Y_t^{\alpha}(s_0)| > \eta t \right) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]

**Lemma 3.12** For any \( s_0 > 0, \eta > 0 \),

\[
\sum_{j=1}^{Z(t)} E\left[\frac{[Y_t^{\alpha}(s_0)]^2}{Z(t)}; \frac{|Y_t^{\alpha}(s_0)|}{\sqrt{Z(t)}} > \eta |A_t| \right] \xrightarrow{\text{pr}} 0 \quad \text{as} \quad t \rightarrow \infty.
\]

**Lemma 3.13** For a fixed \( s_0 > 0 \),

\[
\{ \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} Y_t^{\alpha}(s_0) | A_t \} \xrightarrow{d} N(0, \sigma_j^2(s_0)) \quad \text{as} \quad t \rightarrow \infty,
\]

where \( \sigma_j^2(s_0) \) is defined as in Lemma 3.2.
PROOF. It is an immediate result of Lemma 3.10, Lemma 3.12, and Lindeberg-Feller Theorem. □

Lemma 3.14 \( \sigma_f^2(s) \to D_f \) as \( s \to \infty \).

PROOF. See the proof of Lemma 3.6. □

Proof of Theorem 3.2 Let \( \varepsilon > 0 \) be arbitrary and \( y \) fixed. Choose \( \eta_\varepsilon > 0 \) such that

\[
|\Phi\left(\frac{y + \eta_\varepsilon}{\sigma_f}\right) - \Phi\left(\frac{y - \eta_\varepsilon}{\sigma_f}\right)| < \frac{\varepsilon}{2}
\]

Since \( \lim_{s \to \infty} \sigma_f^2(s) = \sigma_f^2 \), there exists \( s_1(\varepsilon) \) such that \( s \geq s_1(\varepsilon) \) implies

\[
|\Phi\left(\frac{y + r\eta_\varepsilon}{\sigma_f}\right) - \Phi\left(\frac{y - r\eta_\varepsilon}{\sigma_f(s)}\right)| < \frac{\varepsilon}{2} \quad \text{for} \quad r = \pm 1.
\]

Let \( \delta = \varepsilon/2 \) and let \( s^* = \max\{s_0(\eta_\varepsilon, \delta), s_1(\varepsilon)\} \) where \( s_0(\eta_\varepsilon, \delta) \) is defined by Lemma 3.8. Then

\[
\limsup_{t \to \infty} P\left(\frac{Z_f(t + s^*)}{\sqrt{Z(t + s^*)}} \leq y \mid A_{t+s^*}\right)
\]

\[
\leq \limsup_{t \to \infty} P\left(\frac{|Z(t)|}{\sqrt{Z(t + s^*)}} \sum_{j=1}^{Z(t)} Y_i^{\alpha_j}(s^*) \leq y + \eta_\varepsilon \mid A_{t+s^*}\right)
\]

\[
+ \limsup_{t \to \infty} P\left(\frac{|Z_M(s^*)|}{\sqrt{Z(t + s^*)}} > \eta_\varepsilon \mid A_{t+s^*}\right)
\]

\[
\leq \limsup_{t \to \infty} P\left(\frac{1}{\sqrt{Z(t)}} \sum_{j=1}^{Z(t)} Y_i^{\alpha_j}(s^*) \leq y + \eta_\varepsilon \mid A_t\right) P(A_t \mid A_{t+s^*}) + \delta
\]

\[
= \Phi\left(\frac{y + \eta_\varepsilon}{\sigma_f(s^*)}\right) + \frac{\varepsilon}{2}
\]

On the other hand,

\[
\liminf_{t \to \infty} P\left(\frac{Z_f(t + s^*)}{\sqrt{Z(t + s^*)}} \leq y \mid A_{t+s^*}\right)
\]
\[ \geq \liminf_{t \to \infty} P\left( \frac{Z(t)}{Z(t + s^*)} \frac{1}{\sqrt{Z(t)}} \sum_{j=1}^{Z(t)} Y_t^{q_j}(s^*) \leq y - \eta \mid A_{t+s^*} \right) - \frac{\varepsilon}{2} \]

So

\[ \Phi\left( \frac{y - \eta}{\sigma_f(s^*)} \right) - \frac{3\varepsilon}{2} \leq \Phi\left( \frac{y - \eta}{\sigma_f(s^*)} \right) - \varepsilon \leq \Phi\left( \frac{y - \eta}{\sigma_f(s^*)} \right) - \frac{1}{2}\varepsilon \]

Since \( \varepsilon > 0 \) is arbitrary, the proof is completed.

### 3.2 Multitype Case

Let \( Z(t) = (Z_1(t), \ldots, Z_p(t)) \) be a multitype Bellman-Harris process defined on the probability space \((\Omega, \mathcal{F}, P)\). A type \( i \) particle dies at time \( \lambda_i \) which has distribution \( G_i \) and on death it creates offspring according to a distribution whose generating function is given by \( h_i(s_1, \ldots, s_p) \). We consider a stochastic processes \( \{Z_f(t) \mid t \geq 0\} \) defined by

\[ Z_f(t) = \sum_{k=1}^{p} \sum_{j=1}^{Z_k(t)} f(a_{kj}(t)) \]

and develop a central limit theorem for it with \( f \) satisfying some conditions.
3.2.1 The Definitions, Assumptions, and Statement of Result

Throughout this section, we adopt the following conventions.

1. \([0,1] \times \cdots \times [0,1]\) is the unit square in \(\mathbb{R}^p\), the \(p\)-dimensional Euclidean space.

2. \(1 = (1, \cdots , 1)\), \(e_i = (0, \cdots , 0, 1, 0, \cdots , 0)\) with the 1 in the \(i\)th component.

3. \(h(s) = (h_1(s), \cdots , h_p(s))\) = the generating function of offspring distribution.

For any family history \(\omega\),

4. \(Z_k(t, \omega) = \) the number of type \(k\) particles living at time \(t\),
   \[Z(t, \omega) = (Z_1(t, \omega), \cdots , Z_p(t, \omega)), \quad |Z(t, \omega)| = \sum_{k=1}^{p} Z_k(t, \omega).\]

5. \(Z_k(t, a, \omega) = \) the number of type \(k\) particles living at time \(t\) whose age \(\leq a\).

6. \(\{a_{kj}(t, \omega); j = 1, \cdots , Z_k(t, \omega)\}\) = the age-chart of type \(k\) particles at time \(t\).

7. \(Z_f(t, \omega) = \sum_{k=1}^{p} \sum_{j=1}^{Z_k(t, \omega)} f(a_{kj}(t, \omega)).\)

8. \(A_k(t, a, \omega) = \frac{Z_k(t, a, \omega)}{Z_k(t, \omega)}\) if \(Z_k(t, \omega) > 0\).

9. \(i m_f(t) = E_i(Z_f(t)) \overset{\text{def}}{=} E(Z_f(t)|Z(0) = e_i).\)

10. \(i D_f(t) = E_i[(Z_f(t))^2] \overset{\text{def}}{=} E[(Z_f(t))^2|Z(0) = e_i].\)

11. We add superscript \(a\) to the random variable \(Z_f(t)\) and its moments to indicate the case when \(P\) is supported by those \(\omega\)'s which start with one particle of age \(a \geq 0\).

12. \(\mathcal{F}_t = \) the \(\sigma\)-algebra containing all the informations of the family histories up to time \(t\).
\[ \xi_{ij} = \text{the number of type-} j \text{ particles produced by a type-} i \text{ parent.} \]

\[ m_{ij} = E(\xi_{ij}), \quad M = \left\langle (m_{ij})^2 \right\rangle_{i,j=1}^p, \quad d_{ij} = E((\xi_{ij})^2), \quad D = \left\langle (d_{ij})^2 \right\rangle_{i,j=1}^p. \]

\[ \alpha = \text{the Malthusian parameter for } M \text{ and } G = (G_1, \cdots, G_p). \]

\[ \mu_k = \int_0^\infty e^{-\alpha u}(1 - G_k(u)) \, du. \]

\[ A_k(\alpha) = \frac{1}{\mu_k} \int_0^\alpha e^{-\alpha u}(1 - G_k(u)) \, du. \]

\[ \tilde{G}_i(\alpha) = \int_0^\infty e^{-\alpha u}G_i(du), \quad \tilde{M}_{ij}(\alpha) = m_{ij}\tilde{G}_i(\alpha), \quad \tilde{M}(\alpha) = \left\langle (\tilde{M}_{ij}(\alpha))_{i,j=1}^p \right\rangle. \]

For strictly positive matrix \( M \), \( \rho(M) \) denotes the Perron-Frobenius root of \( M \). (see Karlin and Taylor(1975) for definitions of strict positiveness and the Perron-Frobenius root).

\[ u = (u_1, \cdots, u_p), \quad v = (v_1, \cdots, v_p) \text{ are left-, right-eigenvectors of } \tilde{M}(\alpha) \text{ corresponding to } \rho(\tilde{M}(\alpha)) = 1 \text{ such that } u \cdot v = 1, \quad 1 \cdot v = 1. \]

\[ \eta_i = u_i(1 - \tilde{G}_i(\alpha)), \quad \eta = (\eta_1, \cdots, \eta_p) \]

Throughout this section we make the following assumptions without any further mention.

\[ (A 1) \quad M \text{ is strictly positive and } \rho(M) > 1. \]

\[ (A 2) \quad d_{ij} < \infty, \quad i, j = 1, \cdots, p. \]

\[ (A 3) \quad G_j \text{ is non-lattice, } G_j(0^+) = 0, \quad j = 1, \cdots, p. \]

\[ (A 4) \quad \int_0^\infty u G_j(du) < \infty, \quad j = 1, \cdots, p. \]

We impose the following assumptions on a measurable function \( f : R^+ \to R \) which are not all valid at all times.
(G 1) For each $j = 1, \ldots, p$, $f$ is continuous a.e. (w.r.t. Lebesgue measure) on the support of $G_j$.

(G 2) For each $j = 1, \ldots, p$, $e^{-\alpha t}(1 - G_j(t))f(t)$ is d.R.i. and
\[ \int_0^\infty e^{-\alpha t}(1 - G_j(t))\, f(t)\, dt = 0. \]

(G 3) $e^{-\alpha t}f^2(t) \to 0$ as $t \to \infty$.

(G 4) $e^{-\alpha t}((k m_j \cdot m_j) \ast G_j)(t)$ is d.R.i. for all $j, k, l = 1, \ldots, p$.

(G 5) There exists $s_0 > 0$ such that for $s \geq s_0$
\[
\sup_{1 \leq k \leq p} \sup_{a \geq 0} |f(a + s)(1 - G_k^a(s))| < \infty
\]
\[
\sup_{1 \leq k \leq p} \sup_{a \geq 0} |f^2(a + s)(1 - G_k^a(s))G_k^a(s)| < \infty.
\]

Here is the theorem of this section.

**Theorem 3.3** Assume that $f$ and $G_j$ satisfy (G 1) – (G 5), then for $0 < x_1 < x_2 < \infty$
\[
\lim_{t \to \infty} P_t(x_1 \leq W \leq x_2, \frac{Z_f(t)}{\sqrt{\nu} \cdot Z(t)} \leq y) = P_t(x_1 \leq W \leq x_2)\Phi\left(\frac{y}{\sigma_f}\right),
\]
where $W$ is the scalar random variable defined in Theorem 1.5, and
\[
\sigma_f^2 = \sum_{k=1}^p \frac{\theta_k^2}{\mu_k} \sum_{j=1}^p m_{kj} jD_f^\alpha \int_0^\infty \frac{ue^{-\alpha u}G_j(du)}{
\]
\[
\theta_k^2 = \frac{\eta_k}{(\sum_{j=1}^p \eta_i u_j)},
\]
\[
jD_f^\alpha = \lim_{t \to \infty} e^{-\alpha t} jD_f(t).
\]

**Remark 3.3** It will be proved in the next section that $jD_f^\alpha$ exists and is finite.
3.2.2 The First and Second Moments

Put $F_{ij}(t) = m_{ij}^{v_i} \frac{v_i}{v_i} \int_{0}^{t} e^{-\alpha u} G_i(du)$. The next Proposition has been proved in subsection 1.3.2.

**Proposition 3.6** Assume $(G 1), (G 2), \text{ and } (G 5)$ hold. If $\{e^{-\alpha t} f(t)(1 - G_k(t))\}_{k=1}^{p}$ are d.R.i., then for $i = 1, \cdots, p,$

$$\lim_{t \to \infty} e^{-\alpha t} m_f(t) = c_0 \sum_{k=1}^{p} u_k \int_{0}^{\infty} e^{-\alpha u} f(u)(1 - G_k(u))du,$$

where $c_0 = \left\{ \sum_{k=1}^{p} \sum_{j=1}^{p} u_k v_k \int_{0}^{t} F_{kj}(dt) \right\}^{-1}$.

The following Proposition concerns the second moment.

**Proposition 3.7** Assume $(G 2) - (G 4)$ hold then with $c_0$ given in Proposition 3.6,

$$\lim_{t \to \infty} e^{-\alpha t} D_f(t) = c_0 \sum_{j=1}^{p} u_j \int_{0}^{\infty} g_{f,j}^2(u)du$$

where

$$g_{f,j}^2(t) = e^{-\alpha t} \{F^2(t)(1 - G_j(t)) + \sum_{k=1}^{p} (d_{jk} - m_{jk})(k m_j \ast G_j)(t)$$

$$+ \sum_{k \neq j}^{p} m_{jk} m_{jl}[(k m_j \cdot m_f) \ast G_j](t) \}.$$ 

**Proof.** Given $Z(0) = e_i$, $Z_f(t) = I(\lambda_i > t)f(t) + \sum_{k=1}^{p} \sum_{j=1}^{p} Z_{f,j}(t - \lambda_i)$. So we have the following equation

$$Z_f^2(t) = I(\lambda_i > t)f^2(t) + \sum_{k=1}^{p} \sum_{j=1}^{p} Z_{f,j}(t - \lambda_i)Z_{f,l}(t - \lambda_l)$$

$$+ \sum_{k \neq l}^{p} \sum_{j=1}^{p} \sum_{l=1}^{p} Z_{f,j}(t - \lambda_i)Z_{f,l}(t - \lambda_l) + \sum_{k=1}^{p} \sum_{j=1}^{p} [Z_{f,j}(t - \lambda_i)]^2.$$
Since $Z^i_{f,j}(t - \lambda_i)'s$ are conditionally independent given $\lambda_i$,

$$iD_f(t) = (1 - G_i(t))f^2(t) + \sum_{k=1}^{p}(d_{ik} - m_{ik})(k \cdot m_f \cdot G_i)(t) + \sum_{k=1}^{p}m_{ik}(k \cdot D_f \cdot G_i)(t).$$

Multiplying $\frac{1}{v_i}e^{-\alpha t}$ both sides we get the following system of renewal equations

$$\frac{e^{-\alpha t}D_f(t)}{v_i} = \sum_{k=1}^{p} \frac{e^{-\alpha(t-u)}kD_f(t-u)F_{ik}(du)}{v_k} ,$$

where $F_{ik}(du) = \frac{v_k}{v_i}m_{ik}e^{-\alpha u}G_i(du)$. Note that $(G3)$ with assumption $(A4)$ implies that $e^{-\alpha t}f^2(t)(1 - G_j(t))$ is d.R.i. Hence $\{g^*_{i,j}; i = 1, \cdots, p\}$ are d.R.i. (see $(G5)$) and so

$$e^{-\alpha t}D_f(t) \to \sum_{j=1}^{p} \int_{0}^{\infty} g^*_{i,j}(u)du \quad \text{as} \quad t \to \infty,$$

by Theorem 1.11. \qed

Define $(M(s)f)(t) = \iota m_f(t)$ if $t$ is the age of type $i$ particle.

**Proposition 3.8** For $i = 1, \cdots, p$

(a) $\iota m_{M(s)f}(t) = \iota m_f(t + s)$

(b) Assume $(G3)$ and $(G4)$ hold, then $\lim_{s \to \infty} e^{-\alpha s} \lim_{t \to \infty} e^{-\alpha t}iD_{M(s)f}(t) = 0$

**Proof.** (a) $\iota m_{M(s)f}(t) = E_i(Z_{M(s)f}(t))$

$$= E_i(\sum_{k=1}^{p} \sum_{j=1}^{p} (M(s)f)(a_{kj}(t)))$$

$$= E_i(\sum_{k=1}^{p} \sum_{j=1}^{p} k^2 \cdot m_f^2(s))$$

$$= E_i(E_i(\sum_{k=1}^{p} \sum_{j=1}^{p} kZ^2_f(s) | F_i))$$

$$= E_i(Z_f(t + s)) = \iota m_f(t + s)$$. 
(b) From (3.24), we have
\[
\frac{e^{-\alpha t}D_M(t)}{v_i} = \frac{g^{(s)}_{M}f_{ij}(t)}{v_i} + \sum_{k=1}^{p} \int_{0}^{t} e^{-\alpha(t-u)}kD_M(t-u)F_{ik}(du)
\]
First, we'll show that $g^{(s)}_{M}f_{ij}$ is d.r.i. for fixed $s$ and for $i = 1, \ldots, p$, so that
\[
\frac{e^{-\alpha t}D_M(t)}{v_i} \rightarrow c_0 v_i \sum_{j=1}^{p} u_j \int_{0}^{\infty} g^{(s)}_{M}f_{ij}(u)du \quad \text{as} \quad t \rightarrow \infty,
\]
then we'll prove that for each $j = 1, \ldots, p$
\[
e^{-\alpha s} \int_{0}^{\infty} g^{(s)}_{M}f_{ij}(u)du \rightarrow 0 \quad \text{as} \quad s \rightarrow \infty.
\]
If $t$ is the initial age of type $i$ particle, we have the following representation of $Z_j(t)$;
\[
Z_j(t) = I(\lambda^i > s)f(t+s) + \sum_{k=1}^{P} \sum_{j=1}^{p} Z_{f,j}(t - \lambda^i),
\]
where $\lambda^i$ is the lifetime random variable of type $i$ particle whose initial age is $t$. Due to the independence of $Z_{f,j}(s - \lambda^i)$'s given $\lambda^i$, taking expectation we get
\[
m_f^j(s) = f(t+s)(1-G^i_{f}(s)) + \sum_{k=1}^{P} \int_{0}^{s} k m_f(s-u) G^i_{f}(du),
\]
and so,
\[
[m_f^j(s)]^2 \leq C[f^2(t+s)(1-G^i_{f}(s))^2 + \sum_{k=1}^{P} (km_f * G^i_{f})^2(s)] \quad (3.25)
\]
where $C$ is a generic constant. By Cauchy-Schwarz inequality and since $G^i_{f}(s) \leq 1$,
\[
(km_f * G^i_{f}(s))^2 \leq (km_f)^2 * G^i_{f}(s)
\]
but
\[
[km_f]^2 * G^i_{f}(s) \leq \int_{0}^{s} [km_f(s-u)]^2 G^i_{f}(du)
\]
\[
= \frac{1}{1-G^i(t)} \int_{t}^{t+s} [km_f(t+s-u)]^2 G^i_{f}(du)
\]
\[
\leq \frac{(km_f^2 * G^i)(t+s)}{1-G^i(t)}
\]
Combining this with (3.25), we have
\[
e^{-\alpha t}(1 - G_i(t))\{M(s)f\}^2(t)
\leq Ce^{\alpha s}\left\{e^{-\alpha(t+s)}(1 - G_i(t+s))f^2(t+s) + \sum_{k=1}^{p} e^{-\alpha(t+s)}(km_f^2 * G_i)(t+s)\right\}
\]

On the other hand,
\[
e^{-\alpha t}|km_f(t + s - u) \cdot mf(s + t - u)| G_i(du)
= e^{-\alpha t} \int _0^t |km_f(t + s - u) \cdot mf(s + t - u)| G_i(du)
\leq e^{\alpha s}e^{-\alpha(t+s)}|km_f \cdot mf| G_i(t+s),
\]
where the last one is d.R.i. for fixed $s$ by assumption. Since
\[
g_M(t) \leq Ce^{-\alpha t}\{(1 - G_i(t))\{M(s)f\}^2(t) + \sum_{k=1}^{P} \sum_{l=0}^{p} |km_f \cdot mf| * G_i)(t)\},
\]
we conclude that $g_M(t)$ is d.R.i. for fixed $s$. Furthermore,
\[
\int \limits_0^\infty g_M^2(u)du \leq C e^{\alpha t} \int \limits_0^\infty e^{-\alpha(t+s)}f^2(t)(1 - G_i(t+s))
\leq C e^{\alpha s} \int \limits_0^\infty e^{-\alpha(t+s)}(1 - G_i(t+s)) + \sum_{k=1}^{P} \sum_{l=1}^{p} e^{-\alpha(t+s)}|km_f \cdot mf| G_i(t)dt.
\]
So
\[
\lim_{s \to \infty} e^{-\alpha s} \lim_{t \to \infty} e^{-\alpha t} D_M(t)
= e^{\alpha s} \sum_{j=1}^{p} u_j \lim_{s \to \infty} e^{-\alpha s} \int \limits_0^\infty g_M^2(u)du
\leq K v_i \sum_{j=1}^{p} u_j \lim_{s \to \infty} e^{-\alpha t} f^2(t)(1 - G_i(t)) + \sum_{k=1}^{p} \sum_{l=1}^{p} e^{-\alpha(t+s)}|km_f \cdot mf| G_i(t)dt
= 0,
\]
where the last equality comes from (G 3) and (G 4).
3.2.3 Proof of Theorem 3.3

We begin with the representation (1.4) (suppressing $\omega$ and $(t, \omega)$)

$$Z_f(t + s) = \sum_{k=1}^{p} \sum_{j=1}^{Z_k(t)} Z_f^{akj}(s),$$

We rewrite (1.4) as

$$Z_f(t + s) = \sum_{k=1}^{p} \sum_{j=1}^{Z_k(t)} [Z_f^{akj}(s) - km_f^{akj}(s)] + Z_{M(s)}f(t). \quad (3.26)$$

Dividing (3.26) by $\sqrt{v \cdot Z(t + s)}$ and introducing $X_k^{akj}(s) \overset{\text{def}}{=} (Z_f^{akj}(s) - km_f^{akj}(s)) e^{-\frac{s}{2}}$

we get

$$\frac{Z_f(t + s)}{\sqrt{v \cdot Z(t + s)}} = \sum_{k=1}^{p} \frac{\theta_k}{\sqrt{Z_k(t)}} \sum_{j=1}^{Z_k(t)} X_k^{akj}(s)$$

$$\quad + \sum_{k=1}^{p} \left( \frac{Z_k(t)e^{-at}}{v \cdot Z(t + s)e^{-at(s)}} - \theta_k \right) \frac{1}{\sqrt{Z_k(t)}} \sum_{j=1}^{Z_k(t)} X_k^{akj}(s)$$

$$\quad + \frac{Z_{M(s)}f(t)}{\sqrt{v \cdot Z(t + s)}} \overset{\text{def}}{=} A_1(t, s) + A_2(t, s) + A_3(t, s).$$

We first show that $A_3(t, s)$ can be made small in probability uniformly in $t$, by choosing $s$ large, and then with this large but fixed $s$, we use the Lindberg-Feller theorem to prove that as $t \to \infty$, $A_1(t, s)$ converges to the desired distribution.

Finally, for this fixed $s$, we show that $A_2(t, s) \overset{\text{pr}}{\to} 0$ as $t \to \infty$. Let $I = [x_1, x_2]$ be fixed with $0 < x_1 < x_2 < \infty$.

**Lemma 3.15** Given $\varepsilon > 0$, $\delta > 0$, we can find $s_0(\varepsilon, \delta)$ such that $s \geq s_0(\varepsilon, \delta)$ implies

$$\lim_{t \to \infty} P_i(W \in I, |A_3(t, s)| > \varepsilon) < \delta, \quad i = 1, \cdots, p.$$
PROOF. Fix \( i \) and recall that
\[
e^{-\alpha t}v \cdot Z(t) \xrightarrow{t \to \infty} (v \cdot \eta)W,
\]
as \( t \to \infty \).

For \( \varepsilon_1 = \frac{-\alpha}{2}(v \cdot \eta) \) choose \( s_0 = s_0(\delta) \), such that for \( s \geq s_0 \),
\[
\sup_{t \geq 0} P_i(\{e^{-\alpha(t+s)}v \cdot Z(t+s) - (v \cdot \eta)W| > \varepsilon_1\} < \frac{\delta}{2}
\]
So for \( s \geq s_0 \) and for all \( t \geq 0 \),
\[
P_i(W \in I, |A_3(t,s)| > \varepsilon) \leq P_i(W \in I, |A_3(t,s)| > \varepsilon, |e^{-\alpha(t+s)}v \cdot Z(t+s) - (v \cdot \eta)W| < \varepsilon_1)
\]
\[
+ P_i(W \in I, |e^{-\alpha(t+s)}v \cdot Z(t+s) - (v \cdot \eta)W| \geq \varepsilon_1)
\]
\[
\leq P_i\left(\frac{|Z_M(t+\delta)|}{\sqrt{(v \cdot \eta)^2} - \varepsilon_1} e^{-\frac{\delta}{2}(t+s)} > \varepsilon\right) + \frac{\delta}{2}
\]
\[
\leq \frac{2e^{-\alpha(t+s)}}{\varepsilon^2(v \cdot \eta)} E_i[(Z_{M(t)}(s))^2] + \frac{\delta}{2}.
\]

The last inequality is from Chebyshev's inequality. Now, choose \( s''_0 \) such that for \( s \geq s''_0 \)
(Proposition 3.8)
\[
e^{-\alpha s} \lim_{t \to \infty} e^{-\alpha t}iD_M(t) \leq \frac{1}{4}e^{-2s}(v \cdot \eta).
\]

Let \( s_0 = \max\{s_0', s''_0\} \), then for \( s \geq s_0 \)
\[
\lim_{t \to \infty} P_i(W \in I, |A_3(t,s)| > \varepsilon) < \delta \quad \square
\]

\textbf{Lemma 3.16} Fix \( s > 0 \), then for any \( \varepsilon \geq 0 \),
\[
\lim_{t \to \infty} P_i(W \in I, |A_2(t,s)| > \varepsilon) < \delta, \quad i = 1, \cdots, p.
\]
PROOF. Put
\[ U_k(t, s) = \frac{Z_k(t)e^{-\alpha s}}{\sqrt{Z(t + s)e^{-\alpha(t + s)}}} - \theta_k, \quad V_k(t, s) = \frac{1}{\sqrt{Z_k(t)}} \sum_{j=1}^{Z_k(t)} X_{s,t}^j(s) \]
so that \( A_2(t, s) = \sum_{j=1}^{Z_k(t)} U_k(t, s)V_k(t, s) \).
Hence it is enough to show that for each \( k = 1, \ldots, p \),
\[
\lim_{t \to \infty} P(W \in I, |U_k(t, s)V_k(t, s)| > \varepsilon) = 0. \tag{3.27}
\]
Since \( d_{ij} < \infty \) for all \( i, j = 1, \ldots, p \), we know (by Theorem 7.1, Chapter V, Athreya and Ney (1972)) that
\[ K \overset{\text{def}}{=} \sup_{t,s,k} E(V_k^2(t, s)) < \infty. \]
Given \( \delta > 0 \), choose \( M \) such that \( K M^2 < \delta \), then
\[
P(W \in I, |U_k(t, s)V_k(t, s)| > \varepsilon)
\leq P(W \in I, |U_k(t, s)V_k(t, s)| > \varepsilon, |V_k(t, s)| \leq M) + P(|V_k(t, s)| > M)
\leq P(W \in I, |U_k(t, s)| > \frac{\varepsilon}{M}) + \delta.
\]
Since \( U_k(t, s) \overset{a.s.}{\to} 0 \) on \( \{ W \in I \} \) as \( t \to \infty \),
\[
\lim_{t \to \infty} P(W \in I, |U_k(t, s)| > \frac{\varepsilon}{M}) = 0
\]
Being \( \delta > 0 \) arbitrary, we have proved (3.27). \( \square \)

The following three lemmas are the multitype versions of Lemma 3.2, Lemma 3.3, and Lemma 3.4, respectively. The proofs can be carried out in the exactly same way as in the previous ones. The complication comes not from the idea but from the notation, so we omit the proofs.
Lemma 3.17 For a fixed $s_0 > 0$,

$$\lim_{t \to \infty} \text{Var}(A_1(t, s_0)|\mathcal{F}_t) = \sigma^2_f(s_0) \quad \text{in probability},$$

where

$$\sigma^2_f(s_0) = \sum_{k=1}^{p} \theta_k^2 e^{-\alpha s_0} \sum_{i=1}^{5} V_{k,i}(a, s_0),$$

$$V_{k,1}(a, s_0) = f^2(a + s_0)G_k^a(s_0)(1 - G_k^a(s_0)),$$

$$V_{k,2}(a, s_0) = -2 \sum_{j=1}^{p} m_{kj} \int_0^\infty f(a + s)(1 - G_k^a(s))(jm_f * G_k^a)(s)A_k(da),$$

$$V_{k,3}(a, s_0) = \sum_{j=1}^{p} m_{kj} \int_0^\infty (jD_f * G_k^a)(s)A_k(da),$$

$$V_{k,4}(a, s_0) = \sum_{j=1}^{p} \sum_{k=1}^{p} m_{kj}m_{kl} \int_0^\infty (jm_f * G_k^a)(s)(jm_f * G_k^a)(s)A_k(da),$$

$$V_{k,5}(a, s_0) = \sum_{j=1}^{p} \sum_{l=1}^{p} \epsilon_{k,j,l} \int_0^\infty \frac{(jm_f * G_k^a)(s)}{jm_f * G_k^a}(s)A_k(da),$$

$$\epsilon_{k,j,l} = \begin{cases} d_{kj} - m_{kl} & \text{if } j = l, \\ m_{kj}m_{kl} & \text{if } j \neq l. \end{cases}$$

Lemma 3.18 For a fixed $s_0 > 0$ and $\varepsilon > 0$

$$\sup_{0 \leq s \leq t} E_k([X_k^2(s_0)]^2; |X_k^2(s_0)| > \varepsilon e^{s_0} \xrightarrow{\text{pr}} 0 \quad \text{as } t \to \infty,$$

where $X_k^2(s_0) = (Z_k^2(s_0) - \kappa m_f(s_0))e^{-\alpha s_0}$.

The following lemma concerns the conditional Lindeberg-Feller condition.

Lemma 3.19 Fix $s_0 > 0$, $\varepsilon > 0$, then for each $k = 1, \ldots, p$

$$\sum_{j=1}^{Z_k(t)} \frac{E_k\{X_{kj}^2(s_0)^2\}}{Z_k(t)} \frac{|X_{kj}^2(s_0)|}{\sqrt{Z_k(t)}} > \varepsilon |\mathcal{F}_t| \xrightarrow{\text{pr}} 0 \quad \text{as } t \to \infty.$$
Now we examine the limiting behavior of $P(W \in I, A_1(t, s) \leq y)$ for a fixed $s$. For any $\delta > 0$, there exists $t(\delta)$ such that $t \geq t(\delta)$ implies

$$P(|d(Y_1(t)e^{-\alpha t}, I) - d(W, I)| > \delta) \leq \varepsilon$$

where $Y_1(t) \overset{\text{def}}{=} \nu \cdot \mathbb{E}(t)/\nu \cdot \eta$, and $d(x, I) = \inf_{y \in I} d(x, y)$. Let $B_t = \{|d(Y_1(t)e^{-\alpha t}, I) - d(W, I)| \leq \delta\}$, then

$$\{W \in I\} \cap B_t = \{d(Y_1(t)e^{-\alpha t}, I) \leq \delta\} = \{x_1 - \delta \leq Y_1(t)e^{-\alpha t} \leq x_2 + \delta\} \overset{\text{def}}{=} B'_t$$

So

$$0 \leq P(W \in I, A_1(t, s) \leq y) - P(B'_t, A_1(t, s) \leq y) \leq P(B'_t) \leq \varepsilon$$

Recall that

$$A_1(t, s) = \sum_{k=1}^{p} \frac{\theta_k}{\sqrt{Z_k(t)}} \sum_{j=1}^{T} X_{k_j}^a(s)$$

where $X_{k_j}^a(s) \overset{\text{def}}{=} (Z_{k_j}^a(s) - \bar{m}_{k_j}^a(s))e^{-\frac{j^2}{2}}$, and $X_{k_j}^a(s)$ are mutually independent conditioned on $\mathcal{F}_j$ and also independent of $Z_k(t)$. Further, for each $k$, $X_{k_j}^a(s)$, $j = 1, \cdots, Z_k(t)$ satisfy the conditions of Lindeberg-Feller theorem (Lemma 3.19). Hence

$$\lim_{t \to \infty} P(B'_t, A_1(t, s) \leq y) = \lim_{t \to \infty} P(B'_t)P(A_1(t, s) \leq y) = P(x_1 - \delta < W < x_2 + \delta)\Phi\left(\frac{y}{\sigma_I(s)}\right)$$

Since $\delta > 0$ is arbitrary, we have proved

**Lemma 3.20** For a fixed $s_0 > 0$,

$$\lim_{t \to \infty} P_t(W \in I, A_1(t, s_0) \leq y) = P_t(W \in I)\Phi\left(\frac{y}{\sigma_I(s_0)}\right), \quad i = 1, \cdots, p.$$
Proof of Theorem 3.3

Let \( \varepsilon > 0 \) be arbitrary and \( y \) fixed. Choose \( \eta_\varepsilon > 0 \) such that

\[
|\Phi\left(\frac{y + \eta_\varepsilon}{\sigma_f}\right) - \Phi\left(\frac{y - \eta_\varepsilon}{\sigma_f}\right)| < \frac{\varepsilon}{4} \tag{3.28}
\]

Since \( \lim_{s \to \infty} \sigma_f^2(s) = \sigma_f^2 \), there exists \( s_1(\varepsilon) \) such that \( s \geq s_1(\varepsilon) \) implies

\[
|\Phi\left(\frac{y + r\eta_\varepsilon}{\sigma_f}\right) - \Phi\left(\frac{y - r\eta_\varepsilon}{\sigma_f(s)}\right)| < \frac{\varepsilon}{4} \text{ for } r = \pm 1. \tag{3.29}
\]

Let \( \delta = \varepsilon/4 \) and let \( s^* = \max\{s_0(\eta_\varepsilon/2, \delta), s_1(\varepsilon)\} \) where \( s_0(\eta_\varepsilon/2, \delta) \) is defined by Lemma 3.15. Then,

\[
\limsup_{t \to \infty} P_t(W \in I, A_1(t, s^*) + A_2(t, s^*) + A_3(t, s^*) \leq y) \\
\leq \limsup_{t \to \infty} P_t(W \in I, A_1(t, s^*) \leq y + \eta_\varepsilon) \\
+ \limsup_{t \to \infty} P_t(W \in I, |A_2(t, s^*)| \geq \frac{\eta_\varepsilon}{2}) + \limsup_{t \to \infty} P_t(W \in I, |A_3(t, s^*)| \geq \frac{\eta_\varepsilon}{2}) \\
\leq P_t(W \in I)\Phi\left(\frac{y + \eta_\varepsilon}{\sigma_f(s^*)}\right) + \frac{\varepsilon}{2} \text{ by Lemma 3.20, Lemma 3.16 and Lemma 3.15} \\
\leq P_t(W \in I)\Phi\left(\frac{y}{\sigma_f}\right) + \varepsilon \text{ by (3.28) and (3.29)} \tag{3.30}
\]

On the other hand,

\[
\liminf_{t \to \infty} P_t(W \in I, A_1(t, s^*) + A_2(t, s^*) + A_3(t, s^*) \leq y) \\
\geq \liminf_{t \to \infty} P_t(W \in I, A_1(t, s^*) \leq y - \eta_\varepsilon) \\
- \liminf_{t \to \infty} P_t(W \in I, |A_2(t, s^*)| \geq \frac{\eta_\varepsilon}{2}) - \liminf_{t \to \infty} P_t(W \in I, |A_3(t, s^*)| \geq \frac{\eta_\varepsilon}{2}) \\
\geq P_t(W \in I)\Phi\left(\frac{y - \eta_\varepsilon}{\sigma_f(s^*)}\right) - \frac{\varepsilon}{2} \text{ by Lemma 3.20, lemma 3.16 and Lemma 3.15} \\
\geq P_t(W \in I)\Phi\left(\frac{y}{\sigma_f}\right) - \varepsilon \text{ by (3.28) and (3.29)}
\]

Letting \( \varepsilon \downarrow 0 \) we get the result.
3.3 Examples: Single Type Case

We have analyzed the convergence rates in renewal theorems for some cases (Example 1.1, 1.2). So we can reduce the assumption (F4) in Theorem 3.1 in those cases to more reasonable ones.

**Theorem 3.4** Consider a Markov branching processes with offspring mean \( m > 1 \) and exponential life time distribution \( G \) with mean \( 1/b \). Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) be bounded and continuous a.e. If

\[
(F2) \quad e^{-mt}f(t) \text{ is d.R.i. and } \int_0^\infty e^{-mt}f(t)dt = 0.
\]

then

\[
\frac{Z(t)}{\sqrt{Z(t)}} \xrightarrow{d} N(0, \sigma_f^2) \quad \text{as } t \to \infty,
\]

where

\[
\sigma_f^2 = mb \int_0^\infty e^{-mt}f(t)dt + \frac{m^2b^2(m_2 - m)}{m - 1} \int_0^\infty e^{-\alpha t} \left( \int_0^t f(u) e^{-bu}du \right)^2dt.
\]

**Proof.** Clearly boundedness of \( f \) implies (F3) and (F5). It remains to check the condition (F4). In exponential case, the Malthusian parameter \( \alpha \) can be found easily, i.e., \( \alpha = b(m - 1) \). So we have

\[
\mu_\alpha(dt) = me^{-\alpha t}e^{-bt}dt = bm e^{-mt}dt, \quad U_\alpha(dt) = bm dt
\]

Hence

\[
e^{-\alpha t}m_f(t) = \int_0^t e^{-\alpha(t-u)}f(t-u)(1 - G(t-u))bm du
\]

\[
= bm \int_0^t e^{-\alpha u}f(u)e^{-bu}du
\]

\[
= -bm \int_t^\infty e^{-bu}f(u)du
\]
where the third equation comes from (F 2) and \( \alpha = b(m - 1) \). Since \( f \) is bounded

\[
|m_f(t)| \leq bm\|f\|e^{\alpha t}e^{-bt} = bm\|f\|e^{-bt}.
\]

So \( e^{-\alpha t}(m_f^2 + G)(t) = O(e^{-\alpha t}) \) and it is d.R.i. Furthermore,

\[
\beta = m \int_0^\infty te^{-\alpha t}G(dt) = m \int_0^\infty bte^{-bt}dt = \frac{1}{mb}
\]

\[
n_1 = \frac{\int_0^\infty e^{-\alpha t}(1 - G(t))dt}{m \int_0^\infty te^{-\alpha t}G(dt)} = 1
\]

\[
m_f(t) = bme^{\alpha t} \int_0^t e^{-bu}f(u)du.
\]

\[
D_t^2 = \frac{1}{\beta} \int_0^\infty (e^{-\alpha t}f^2(t)e^{-bt} + (m_2 - m)e^{-\alpha t}(m_f^2 + G)(t))dt
\]

\[
= mb(\int_0^\infty e^{-mbt}f^2(t)dt + (m_2 - m) \int_0^\infty e^{-\alpha t}m_f^2(t)dt \int_0^\infty e^{-\alpha t}(1 - G(t))dt)
\]

\[
= mb(\int_0^\infty e^{-mbt}f^2(t)dt + \frac{mb(m_2 - m)}{m - 1} \int_0^\infty e^{-\alpha t}(\int_0^t e^{-mbu}f(u)du)^2dt)
\]

So

\[
\sigma_t^2 = n_1^{-1}D_t^2 = mb \int_0^\infty e^{-mbt}f^2(t)dt + \frac{mb^2(m_2 - m)}{m - 1} \int_0^\infty e^{\alpha t}(\int_0^t e^{-mbu}f(u)du)^2dt.
\]

Next example is a continuation of Example 1.2.

**Example 3.1** Let \( \{Z(t), t \geq 0\} \) be a Bellman-Harris process with Gamma lifetime distribution with parameter \((b,k)\), where \( b > 0, k \geq 2 \) integer. In this case the Malthusian parameter \( \alpha \) is \( b(m^{1/k} - 1) \). Assume that (F 1) - (F 2) hold. Suppose
that \( f \) is d.R.i., differentiable a.e., and that \( ||f'||_\infty \) is finite. Then

\[
\left| (f_\alpha \ast U_\alpha)(t) - \frac{1}{\mu} \int_0^\infty f_\alpha(u)du \right| = |(f_\alpha \ast U_\alpha)(t)|
\]

\[= O(e^{-c_1 t})\]

where \( c_1 = bm^{1/k}(1 - \cos \frac{2\pi}{k}) \) (see Theorem 1.14).

Simple calculations give us that \( \alpha < 2c_1 \) if and only if \( \cos \frac{2\pi}{k} < \frac{1}{2}(1 + m^{1/k}) \). So if \( k \geq 2 \) is such that \( \cos \frac{2\pi}{k} < \frac{1}{2}(1 + m^{-1/k}) \), then \( e^{-\alpha t}(m_2^* \ast G)(t) \) is d.R.i. Note that the conditions (F 3) and (F 5) are satisfied trivially because \( f \) is d.R.i. Hence we conclude that

\[
\frac{Z_f(t)}{\sqrt{Z(t)}} \xrightarrow{d} N(0, \sigma_f^2) \quad \text{as} \quad t \to \infty \quad \Box
\]
4. SOME LIMIT THEOREMS FOR POSITIVE RECURRENT BRANCHING MARKOV CHAINS

Let \( \{Z_n, n \geq 0\} \) be a supercritical Galton-Watson process evolving from 1 particle at time 0 with offspring law \( \{p_k\} \). We superimpose on this process the additional structure of movement. That is, each particle moves according to a Markov chain. The Markov chain is assumed positive recurrent in the discrete state space case and Harris recurrent in the general state space case respectively. We prove first a law of large numbers for the empirical distribution of the position of particles and then discuss the large deviation aspects of this convergence.

4.1 Some Preliminary Results

Let \( h(s) \overset{\text{def}}{=} E(s^{Z_1} | Z_0 = 1) = \sum_0^{\infty} p_j s^j \) and let \( h_n(s) \) be the nth iterate of \( h \) for \( n \geq 1 \). Then

Proposition 4.1 \( h_n(s) = E(s^{Z_n} | Z_0 = 1) \) and \( h_n(s) \to q, \) for \( 0 \leq s < 1 \) where \( q \) is the smallest root in \([0, 1]\) of \( h(s) = s \).

We have the following rate of convergence of \( h_n(s) \).
Proposition 4.2 Let \( p_0 = 0, p_1 > 0 \). Then \( q = 0 \) and there exists \( 0 \leq q_j < \infty \) such that

\[
\lim_{n \to \infty} \frac{h_n(s)}{p_1} = \sum_{j=0}^{\infty} q_j s^j \overset{\text{def}}{=} Q(s) < \infty, \text{ for } 0 \leq s < 1.
\] (4.1)

Further, \( Q(s) \) is the unique solution of the functional equation

\[
Q(h(s)) = p_1 Q(s), \quad 0 \leq s < 1,
\]

subject to

\[
Q(0) = 0, \quad Q'(0) = 1.
\] (4.2)

For the proofs of Proposition 4.1 and 4.2, see Athreya and Ney(1972).

Let \( \{X_n, n = 0, 1, 2, \ldots\} \) be a Markov chain on \((E, \mathcal{E})\) with transition function \( P(\cdot, \cdot): E \times \mathcal{E} \to [0,1] \). The hitting time of a set \( A \in \mathcal{E} \) is defined by \( \tau_A = \inf\{n; n \geq 1, X_n \in A\} \). A nonnegative measure \( \nu \) on \((E, \mathcal{E})\) is called stationary for \( P \) if \( \nu \) is \( \sigma \)-finite and \( \nu P = \nu \), i.e.,

\[
\int P(x, A) \nu(dx) = \nu(A) \quad \text{for all} \quad A \in \mathcal{E}.
\]

If the state space is discrete, say, \( E = Z^+ = \{0, 1, 2, \ldots\} \), we say that a state \( j \) is recurrent if \( P_j(\tau_j < \infty) = 1 \) and positive recurrent if \( E_j(\tau_j) < \infty \). The period \( d = d(j) \) is the period of the recurrent time distribution, i.e., the greatest integer \( d \) such that \( P_j(\tau_j \in L_d) = 1 \), where \( L_d = \{d, 2d, 3d, \ldots\} \). If \( d = 1 \), \( j \) is called aperiodic.

It is known that if the chain is irreducible then all the states have the same period. The following three propositions about positive recurrent Markov chain are parts of the standard textbook literature(see Hoel, Port and Stone(1972), for example).
Proposition 4.3 If a Markov chain \( \{X_n, n = 0, 1, 2, \cdots \} \) is irreducible, and positive recurrent, there exists a unique stationary distribution \( \pi \) given by

\[
\pi_j = (E_j(\tau_j))^{-1}.
\]

Proposition 4.4 Suppose that a Markov chain \( \{X_n, n = 0, 1, 2, \cdots \} \) is irreducible positive recurrent and aperiodic. Let \( \pi = (\pi_0, \pi_1, \cdots) \) be the stationary distribution. Then for each \( i \)

\[
\sum_{j=0}^{\infty} |p_{ij}^{(n)} - \pi_j| \to 0,
\]

where \( p_{ij}^{(n)} = P_i(X_n = j) \). In particular, \( p_{ij} \to \pi_j \) as \( n \to \infty \).

The case \( d > 1 \) can be reduced to the case \( d = 1 \) with the following

Proposition 4.5 Consider an irreducible chain with period \( d > 1 \). Let \( i \) be some arbitrary but fixed state and define for \( r = 0, 1, \cdots, d - 1 \)

\[
E_r = \{ j \in \mathbb{Z}^+; p_{ij}^{(nd+r)} > 0 \text{ for some } n \}.
\]

Then \( E_0, \cdots, E_{d-1} \) partition \( E = \mathbb{Z}^+ \) into non-empty disjoint sets and if \( j \in E_k \) then \( P_j(X_1 \in E_{k+1}) = 1 \), where \( E_d = E_0 \).

In the irreducible positive recurrent case, it is clear that \( \{X_{nd} \} \) is aperiodic positive recurrent on each \( E_r \), so it admits a unique stationary distribution \( \pi^{(r)} \) concentrated on \( E_r \). If \( \pi \) is stationary for \( \{X_n \} \), by uniqueness of the stationary distribution we have

\[
\pi = \frac{1}{d} \sum_{r=0}^{d-1} \pi^{(r)}.
\]

Also the limiting behavior of \( p_{jk}^{(n)} \) can be easily seen from \( p_{ji}^{(nd)} \to \pi^{(r)}_i \) if \( j, l \in E_r \).

Indeed, supposing that \( j \in E_r \), then \( p_{jk}^{(nd+s)} = 0 \) for all \( n \) if \( k \notin E_{r+s} \), whereas if \( k \in E_{r+s} \), \( p_{jk}^{(nd+s)} \to \pi^{(r+s)}_k \).
When the state space is not discrete, we have the corresponding definitions and results.

**Definition 4.1** A Markov chain \( \{X_n; n = 0, 1, 2, \cdots\} \) is called Harris recurrent if there exists a \( \sigma \)-finite measure \( \varphi \) on the state space \( (E, \mathcal{E}) \) such that \( \varphi(A_0) > 0 \) implies

\[
P_x(\tau_{A_0} < \infty) = 1 \quad \text{for all } x \in E.
\]

Note that any irreducible and recurrent countable state space Markov chain is Harris recurrent since \( \varphi \) may be chosen to be the delta measure on some state \( j \). The following is an equivalent definition given by Athreya and Ney(1978).

**Definition 4.2** A Markov chain \( \{X_n; n = 0, 1, 2, \cdots\} \) is called \( (A_0, \varepsilon, \varphi, n_0) \)-recurrent if there exist a set \( A_0 \in \mathcal{E} \), a probability measure \( \varphi \) on \( A_0 \), a real number \( \varepsilon > 0 \), and an integer \( n_0 > 0 \) such that

\[
P_x(\tau_{A_0} < \infty) = P_x(X_n \in A \text{ for some } n \geq 1) = 1, \quad \text{for all } x \in E
\]

\[
P_x(X_{n_0} \in E) = P^{n_0}(x, B) \geq \varepsilon \varphi(B), \quad \text{for all } x \in E \text{ and for all } B \subset A.
\]

The set \( A_0 \) is called a regeneration set.

The following lemma proved by Athreya and Ney(1978) makes the limit theory for Harris chains easy via renewal theory.

**Lemma 4.1 (Regeneration Lemma)**

If \( \{X_n; n = 0, 1, 2, \cdots\} \) is \( (A_0, \varepsilon, \varphi, n_0) \)-recurrent then there exists a random time \( N \) which is called the first regeneration time such that \( P_x(N < \infty) = 1 \) for all \( x \) and

\[
a(x, k, n) \overset{\text{def}}{=} P_x(X_n \in A_0, X_{n+1} \in A_1, \ldots, X_{n+k} \in A_k, N = n)
\]

\[
= P_x(N = n) \int_{A_0} P_y(X_1 \in A_1, \ldots, X_k \in A_k) \varphi(dy).
\]
The Harris recurrent chain is called \textit{positive recurrent} if $E_\nu(N) < \infty$. The regeneration lemma can be used to show the existence of a stationary measure for Harris recurrent chains.

**Proposition 4.6** For Harris recurrent chain $\{X_n; n = 0, 1, 2, \cdots\}$ define

$$\nu(A) = E_\nu(\sum_{i=0}^{N-1} I(X_i \in A))$$

where $N$ is the first regeneration time as in Lemma 4.1. Then $\nu$ is the unique (up to multiplicative constant) stationary measure for the chain.

Since $\nu(E) = E_\nu(N)$, we have the following immediate

**Corollary 4.1** A stationary probability distribution $\pi(\cdot)$ for Harris recurrent chain $\{X_n; n = 0, 1, 2, \cdots\}$ exists if and if only $E_\nu(N) < \infty$, and in this case,

$$\pi(A) = \frac{\nu(A)}{\nu(E)}.$$

Finally, we have the following ergodic theorem for a positive recurrent Harris chain whose proof can be found in Asmussen(1987).

**Proposition 4.7** Let $\{X_n; n = 0, 1, 2, \cdots\}$ be $(A, \epsilon, \varphi, 1)$-recurrent with a finite stationary $\pi$ distribution. Then the $P_\epsilon$-distribution of $X_n$ converges to $\pi$ in total variation norm. In particular, for each $x$, $P^n(x, A) \to \pi(A)$, as $n \to \infty$ for $A \in \mathcal{E}$.

There is an extension of Proposition 4.7 to the $(A_0, \epsilon, \varphi, n_0)$-recurrent case for $n_0 > 1$ in both the periodic and aperiodic cases. The proofs are in Athreya and Ney(1978).

We close this section by proving some analytic results which are useful in later sections.
Lemma 4.2 Let $a_n(s) = \sum_{j=0}^{\infty} a_{nj}s^j$, $a_{nj} \geq 0$, $s \geq 0$. Suppose that for some $0 < s_0 < 1$, $a_n(s_0) < \infty$ for all $n \geq 1$ and that $\lim_{n \to \infty} a_n(s) = a(s) < \infty$ for $0 \leq s < s_0$. Then there exist $a_j \geq 0$, $j = 0, 1, 2, \ldots$ such that for each $j = 0, 1, 2, \ldots$, $\lim_{n \to \infty} a_{nj} = a_j$ and

$$a(s) = \sum_{j=0}^{\infty} a_j s^j \quad \text{for } 0 \leq s < s_0.$$  

Proof. Fix $0 < t < s_0$. Since $a_{nj} \geq 0$ and $a_n(t) \to a(t) < \infty$, there exists a constant $0 < C < \infty$, such that

$$|a_n(z)| \leq a_n(t) \leq C$$

for all $|z| \leq t$ and all $n$. Thus $\{a_n(z); n = 1, 2, \ldots\}$ is a family of functions that are analytic and uniformly bounded on $\{z; |z| < t\}$. So by a theorem on normal families (see Rudin (1987) p 282) $a_n(z)$ converges to $a(z)$ uniformly on $\{z; |z| < t'\}$ for each $0 < t' < t$ and so $a(z)$ is analytic on $\{z; |z| < t\}$ and hence on $\{|z| < s_0\}$. Hence we may write $a(z)$ as power series, i.e., there exist $a_j, j = 0, 1, 2, \ldots$ such that $a(z) = \sum_{j=0}^{\infty} a_j z^j$ for $|z| < s_0$. Furthermore,

$$\frac{d}{dz}a_n(z) \to \frac{d}{dz}a(z) \quad \text{uniformly on } \{z; |z| < t\} \quad \text{for } 0 < t < s_0.$$  

Hence for any $j \geq 1$,  

$$\frac{d^j}{dz^j}a_n(z) \to \frac{d^j}{dz^j}a(z) \quad \text{uniformly on } \{z; |z| < t\} \quad \text{for } 0 < t < s_0.$$  

In particular, for any $j \geq 1$

$$a_{nj} = \frac{d^j}{dz^j}a_n(z)|_{z=0} \to \frac{d^j}{dz^j}a(z)|_{z=0} = a_j.$$  

Now fix an integer $k \geq 2$.  

Corollary 4.2 Let \( g_n(s_k) = \sum_{j \in \mathbb{Z}_k^+} a_n(j) s_k^j \), \( a_n(j) \geq 0 \), \( s_k \in \mathcal{C}_k' \), where \( s_k = (s_1, \ldots, s_k) \), \( \mathcal{C}_k' \) is defined as \( \mathcal{C}_k' = \{ s_k \in [0,1]^k; \| s_k \| < 1 \} \), \( \| s_k \| = \sup_{1 \leq i \leq k} s_i \). Assume
\[
g_n(s_k) \to g(s_k) < \infty \quad \text{as} \quad n \to \infty \quad \text{for all} \quad s_k \in \mathcal{C}_k'.
\]
Then there exist \( a(j) \) for \( j \in \mathbb{Z}_k^+ \) such that
\[
g(s_k) = \sum_{j \in \mathbb{Z}_k^+} a(j) s_k^j \quad \text{for} \quad s_k \in \mathcal{C}_k',
\]
and
\[
a_n(j) \to a(j) \quad \text{as} \quad n \to \infty.
\]

**Proof.** It is enough to show for \( k = 2 \). Fix \( 0 < s_2 < 1 \) and put \( h_{n,s_2}(s_1) = g_n(s_1, s_2), h_2(s_1) = g(s_1, s_2) \). Then for \( 0 \leq s_1 < 1 \)
\[
h_{n,s_2}(s_1) = \sum_{j_1=0}^\infty b_{n,s_2}(j_1) s_1^{j_1} \to h_2(s_1) \quad \text{as} \quad n \to \infty.
\]
where \( b_{n,s_2}(j_1) = \sum_{j_2=0}^{\infty} a_n(j_1, j_2) s_2^{j_2} \). So by Lemma 4.2, there exist \( b_2(j_1) \geq 0, j_1 = 0, 1, 2, \cdots \) such that for each \( j_1 \)
\[
b_{n,s_2}(j_1) \to b_2(j_1) \quad \text{as} \quad n \to \infty
\]
i.e.,
\[
\sum_{j_2=0}^{\infty} a_n(j_1, j_2) s_2^{j_2} \to b_2(j_1)
\]
and
\[
g(s_1, s_2) = h_2(s_1) = \sum_{j_1=0}^{\infty} b_2(j_1) s_1^{j_1} \quad \text{(4.3)}
\]
Since \( c_j(s_2) \) is defined as \( b_2(j_1) < \infty \) for \( 0 \leq s_2 < 1 \), Lemma 4.2 guarantees again the existence of \( a(j_1, j_2) \geq 0, j_2 = 0, 1, 2, \cdots \) such that for each \( j_1 = 0, 1, 2, \cdots \)
\[
a_n(j_1, j_2) \to a(j_1, j_2) \quad \text{as} \quad n \to \infty
\]
and
\[
c_{j_1}(s_2) = \sum_{j_2=0}^{\infty} a(j_1, j_2) s_2^{j_2}. \quad (4.4)
\]
Combining (4.3) and (4.4) we get
\[
g(s_1, s_2) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} a(j_1, j_2) s_1^{j_1} s_2^{j_2}.
\]

**Corollary 4.3** Let \( A_\infty \overset{\text{def}}{=} \{ j \in \mathbb{Z}_+^+; j_i = 0 \text{ except for finite } i \} \), \( C_\infty \overset{\text{def}}{=} \{ s \in \mathbb{R}_\infty; 0 \leq s_i \leq 1, \text{ for all } i = 1, 2, \cdots \} \) and \( C'_\infty \overset{\text{def}}{=} \{ s \in C_\infty; ||s|| < 1 \} \) where \( ||s|| = \sup_{i > 1} s_i \).

Suppose that \( a_n(j) \geq 0 \), for all \( j \in A_\infty, n \geq 0 \). If \( g_n(s) = \sum_{j \in A_\infty} a_n(j) s^j \rightarrow g(s) \) for \( s \in C'_\infty \), then there exist \( a(j), j \in A_\infty \) such that
\[
\lim_{n \to \infty} a_n(j) = a(j).
\]
Furthermore if for some \( 0 < s_0 < 1 \) \( g_n(s) = \sum_{j \in A_\infty} a_n(j) s^j \rightarrow g(s) \) uniformly on \( \{ s \in C_\infty; ||s|| \leq s_0 \} \) then
\[
g(s) = \sum_{j \in A_\infty} a(j) s^j \text{ for } ||s|| \leq s_0.
\]

**Proof.** Let \( A_0 = \{0\} \), and for \( r \geq 1 \), let \( A_r = \{ j \in A_\infty; j_r \neq 0 j_s = 0 \text{ for } s > r \} \).

Then \( A_r, r = 0, 1, 2, \cdots \) partition \( A_\infty \). That is, \( A_r \cap A_s = \emptyset \) if \( r \neq s \) and \( \cup_{r \geq 0} A_r = A_\infty \).

Let \( s_r = (s_1, \cdots, s_r, 0, \cdots) \), \( 0 \leq s_i < 1, i = 1, \cdots, r \), then for \( 0 \leq t \leq 1 \)
\[
g_n(t s_r) = \sum_{l=0}^{\infty} \left( \sum_{k=1}^{r} a_n(j) s_1^{j_1} \cdots s_k^{j_k} \right) t^l 
\rightarrow g(t s_r).
\]

So Lemma 4.2 says that there exists \( b(l, s_r) \) such that
\[
g(t s_r) = \sum_{l=0}^{\infty} b(l, s_r) t^l \quad (4.5)
\]
and for each \( l \geq 0 \)
\[
\sum_{k=1}^{r} \sum_{j \in A_k, |j| = l} a_n(j) s_1^{j_1} \cdots s_k^{j_k} \to b(l, s_r) \quad \text{as} \quad n \to \infty. \tag{4.6}
\]

Since the lefthand side of (4.6) is a multinomial in \( s_1, \cdots, s_k \), so is the limit, i.e., there exists \( a(j) \) for \( j \in A_k \), with \( |j| = l \), \( 0 \leq k \leq r \) such that \( a_n(j) \to a(j) \) and
\[
b(l, s_r) = \sum_{k=1}^{r} \sum_{j \in A_k, |j| = l} a(j) s_1^{j_1} \cdots s_k^{j_k}.
\]

So we have by (4.5)
\[
g(ts_r) = \sum_{l=0}^{\infty} \left( \sum_{k=1}^{r} \sum_{j \in A_k, |j| = l} a(j) s_1^{j_1} \cdots s_k^{j_k} \right) t^l.
\]

Now assume that there is \( 0 < s_0 < 1 \) such that \( g_n(s) \) converges to \( g(s) \) uniformly on \( \{ s \in C_\infty; ||s|| \leq s_0 \} \). We want to show \( g(s) = \sum_{j \in A_\infty} a(j) s^j \). Let \( \varepsilon > 0 \) be given, then by hypothesis there exists \( n_0 = n_0(s, \varepsilon) \) such that for \( n \geq n_0 \) and for all \( r \geq 1 \) and \( 0 \leq t \leq 1 \)
\[
g_n(ts) - \varepsilon \leq g(ts) \leq g_n(ts) + \varepsilon
\]
\[
g_n(ts_r) - \varepsilon \leq g(ts_r) \leq g_n(ts_r) + \varepsilon.
\]

So by the monotone convergence theorem
\[
\lim_{r \to \infty} |g(ts) - g(ts_r)| \leq \lim_{r \to \infty} |g_n(ts) - g_n(ts_r)| + 2\varepsilon = 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we conclude that
\[
g(ts) = \lim_{r \to \infty} g(ts_r) = \sum_{l=0}^{\infty} \left( \sum_{k=1}^{r} \sum_{j \in A_k, |j| = l} a(j) s^j \right) t^l.
\]

Letting \( t \uparrow 1 \), we get
\[
g(s) = \sum_{j \in A_\infty} a(j) s^j.
\]
4.2 Discrete State Space Case

Consider a supercritical Galton-Watson process evolving from 1 particle at time 0 with reproduction law \( \{p_j\} \). Suppose that for each \( i \) the offspring of a particle located at site \( i \) choose their positions independently of each other according to the probability distribution \( \{P_{ij}\} \). Thus the matrix \( P = ((P_{ij})) \) is a stochastic matrix that governs the motion.

4.2.1 Notations, Definitions, and Assumptions

Notations

(N 1) \( \mathbb{C}_\infty = \{s = (s_0, s_1, s_2, \cdots); 0 \leq s_i \leq 1, \quad i = 0, 1, 2, \cdots\} \)

\[ A_\infty = \{i = (i_0, i_1, i_2, \cdots); i_j \in \mathbb{Z}^+, \quad i_j = 0 \quad \text{except for finite } j\} \]

(N 2) \( 0 = (0, 0, \cdots), \quad 1 = (1, 1, \cdots) \)

\[ e_j = (0, \cdots, 0, 1, 0, \cdots) \quad 1 \text{ on the } (j + 1)\text{th coordinate}. \]

(N 3) \( ||s|| = \sup_{j \geq 0} |s_j|, \quad |i| = i_0 + i_1 + i_2 + \cdots. \)

(N 4) \( \mathbb{C}'_\infty = \{s \in \mathbb{C}_\infty; ||s|| < 1\} \)

(N 5) For \( s \in \mathbb{C}_\infty \) and for \( i \in A_\infty \), \( si = s_0^i s_1^i s_2^i \cdots \).

(N 6) For any matrix \( A \) its transpose will be denoted by \( A' \).

Definitions

(D 1) \( Z_{n,j} \) = the number of particles at site \( j \) in the \( n \)th generation.

(D 2) \( Z_n = (Z_{n,0}, Z_{n,1}, Z_{n,2}, \cdots) \) the population vector in the \( n \)th generation.
(D3) \( F_n = \sigma(Z_0, Z_1, \ldots, Z_n) \) the \( \sigma \)-algebra generated by the population vector of the first \( n \) generation.

(D4) \( Z_n = |Z_n| = \) the total population in the \( n \)th generation

(D5) \( G_{n,j} = \frac{Z_{n,j}}{Z_n} = \) proportion of particles in site \( j \) at the \( n \)th generation.

(D6) \( h(s) = E(s^{Z_1}|Z_0 = 1) = \sum_{j=0}^{\infty} p_j s^j \), \( h_n(s) = \) the \( n \)th iterate of \( h \).

(D7) \( \mu = E(Z_1|Z_0 = 1) = \sum_{j=0}^{\infty} j p_j \).

(D8) \( W_n = \frac{Z_n}{\mu^n}, \ W = \lim_{n \to \infty} W_n \) a.s.

(D9) For each \( i = 0, 1, 2, \ldots, P_i(\cdot) = P(\cdot|Z_0 = e_i) \) the probability measure for the process with \( Z_0 = e_i \) and \( E_i(\cdot) = E(\cdot|Z_0 = e_i) \) the corresponding expectation.

(D10) \( p_i(j) = P_i(Z_1 = j) \)

(D11) For \( s \in C_\infty, f_{n,i}(s) = E_i(s^{Z_n}), \ f_n(s) = (f_{n,0}(s), f_{n,1}(s), f_{n,2}(s), \ldots), \ n \geq 1 \)

(\( f_0(s) \equiv s, \ f(s) = f_1(s) \).

(D12) \( D_{ij}(s) = \frac{\partial f_i(s)}{\partial s_j}, \ a_{ij} = D_{ij}(0), \ m_{ij} = D_{ij}(1) \)

(\( A = ((a_{ij}))_{i,j=0}^{\infty}, \ M = ((m_{ij}))_{i,j=0}^{\infty} \).

(D13) \( \{X_n; n = 0, 1, 2, \ldots\} \) a Markov chain with transition matrix \( P \).

Remark 4.1

1. \( Z_n \) is the simple Galton-Watson process with offspring probability generating function \( h \).
2. \( h_n(s) = E(s^{Z_n} \mid Z_0 = 1), \quad f_{n+1}(s) = f(f_n(s)). \)

3. Since \( a_{ij} = P_i(Z_1 = e_j) = p_i P_{ij}, \quad m_{ij} = E_i(Z_{1,j}) = \mu P_{ij}, \) we have
   \[ A = p_i P, \quad M = \mu P. \]

### 4.2.2 Law of Large Numbers

We make the following assumptions throughout this chapter,

(i) \( p_0 = 0, \quad 1 < \mu < \infty. \)

(ii) \( \sum_{j=1}^{\infty} (j \log j) p_j < \infty. \)

and we assume throughout this section that \( P \) is irreducible and positive recurrent.

Since \( P \) is irreducible and positive recurrent, there exists a unique stationary probability measure \( \pi = (\pi_0, \pi_1, \pi_2, \ldots) \) for \( P \),

\[ \sum_{i=0}^{\infty} \pi_i P_{ij} = \pi_j, \quad \text{for} \quad j = 0, 1, 2, \ldots. \]

**Theorem 4.1** Suppose that \( P \) is aperiodic, then for each \( j = 0, 1, 2, \ldots \)

\[ G_{n,j} \xrightarrow{\text{pr}} \pi_j \quad \text{as} \quad n \to \infty. \]

**PROOF.** By the additive property of branching process, we have

\[ Z_{n+m,j} = \sum_{i=0}^{\infty} \sum_{l=1}^{Z_{n,i}^{(i)}} Z_{m,j}^{(i)} \quad \text{(4.7)} \]

where \( Z_{m,j}^{(i)} \) is the number of particles in the \((n+m)\)th generation that are on the site \( j \) who are descendents of \( l \)th particle on the site \( i \) in the \( n \)th generation. Note that conditioned on \( \mathcal{F}_n, \{ Z_{m,j}^{(i)} ; i = 0, 1, 2, \ldots \; l = 1, 2, \ldots \} \) is a family of independent random variables which is independent of \( Z_n \), and

\[ E(Z_{m,j}^{(i)}) = E(Z_{m,j} \mid Z_0 = e_i) = \mu^m P_{ij}^{(m)}. \]
We decompose $Z_{n+m,j}$ as follows

$$Z_{n+m,j} = \sum_{i=0}^{\infty} \sum_{l=1}^{\infty} (Z_{m,i}^{(ii)} - \mu^m P_{ij}^{(m)}) + \sum_{i=0}^{\infty} \mu^m Z_{n,i}(P_{ij}^{(m)} - \pi_j) + \mu^m Z_n \pi_j.$$ 

So

$$G_{n+m,j} = \frac{1}{Z_n} \sum_{i=0}^{\infty} \sum_{l=1}^{\infty} (\mu^{-m} Z_{m,i}^{(ii)} - P_{ij}^{(m)}) + \sum_{i=0}^{\infty} G_{n,i}(P_{ij}^{(m)} - \pi_j) + \pi_j$$

where $Z_m^{(ii)} = \sum_{j=0}^{\infty} Z_m^{(ii)}$. By Lemma 2.2 and Lemma 2.3 (see Chapter 2 p29) we have for any $m > 1$,

$$\frac{1}{Z_n} \sum_{i=0}^{\infty} \sum_{l=1}^{\infty} (\mu^{-m} Z_{m,i}^{(ii)} - P_{ij}^{(m)}) \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty \quad \text{(4.8)}$$

and

$$\frac{1}{Z_n} \sum_{i=0}^{\infty} \sum_{l=1}^{\infty} (\mu^{-m} Z_m^{(ii)} - 1) \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty \quad \text{(4.9)}$$

So it is enough to show that for any $\varepsilon > 0$

$$\limsup_{m \to \infty} \limsup_{n \to \infty} P(|\sum_{i=0}^{\infty} G_{n,i}(P_{ij}^{(m)} - \pi_j)| > \varepsilon) = 0$$

Let $\varepsilon > 0$, $\delta > 0$ be given. Choose $\eta(\delta) > 0$ such that

$$P(W < \eta) < \delta/2 \quad \text{(4.10)}$$

and choose $K = K(\varepsilon, \delta, \eta)$ such that

$$\sum_{i=K+1}^{\infty} \pi_i \leq (\varepsilon \delta \eta)/4. \quad \text{(4.11)}$$

Then

$$P(\sum_{i=0}^{\infty} G_{n,i}(P_{ij}^{(m)} - \pi_j)| > \varepsilon) \leq P(\sum_{i=0}^{K} G_{n,i}(P_{ij}^{(m)} - \pi_j)| > \varepsilon/2)$$

$$+ P(\sum_{i=K+1}^{\infty} G_{n,i}(P_{ij}^{(m)} - \pi_j)| > \varepsilon/2)$$

$$= a_{nm} + b_{nm}, \quad \text{say.}$$
Choose $m_0$ such that for $m \geq m_0$

$$\sup_{0 \leq i \leq K} |P_{ij}^{(m)} - \pi_j| < \varepsilon/2$$

then

$$|\sum_{i=0}^{K} G_{n,i}(P_{ij}^{(m)} - \pi_j)| < \sum_{i=0}^{K} G_{n,i} \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}.$$ 

So $a_{nm} = 0$ for all $n \geq 1$ if $m \geq m_0$. On the other hand,

$$b_{nm} = P(\sum_{i=K+1}^{\infty} G_{n,i}(P_{ij}^{(m)} - \pi_j) > \varepsilon/2)$$

$$\leq P(\sum_{i=K+1}^{\infty} G_{n,i}(P_{ij}^{(m)} - \pi_j) > \varepsilon/2, W_n > \eta) + P(W_n \leq \eta)$$

$$\leq P(\sum_{i=K+1}^{\infty} Z_{n,i} > \frac{\varepsilon}{2\eta \mu^n}) + P(W_n \leq \eta)$$

$$\leq \frac{2}{\varepsilon \eta \mu^n} E(\sum_{i=K+1}^{\infty} Z_{n,i}) + P(W_n \leq \eta)$$

Taking limit we see that from (4.10) and (4.11) that for $m \geq m_0$

$$\limsup_{n \to \infty} b_{nm} \leq \frac{2}{\varepsilon \eta} \sum_{i=K+1}^{\infty} \pi_i + P(W \leq \eta) \leq \frac{2 \varepsilon \eta \delta}{4} + \frac{\delta}{2} = \delta$$

Hence for $m \geq m_0$

$$\limsup_{n \to \infty} P(|(G_n P^m)_j - \pi_j| > \varepsilon) < \delta.$$ 

Since $\delta > 0$ is arbitrary, the proof is completed.

\[ \Box \]

**Theorem 4.2** Let $P$ be aperiodic. Suppose

(A 1) \( \sup_i |P^{(m)}_{ij} - \pi_j| \to 0 \) as \( m \to \infty \).

Then for each \( j = 0, 1, \ldots \)

$$G_{n,j} \converges \text{a.s.} \pi_j \text{ as } n \to \infty.$$
PROOF. Due to (4.8) and (4.9) it suffices to show that

\[ \lim_{m \to \infty} \limsup_{n \to \infty} \left| \sum_{i=0}^{\infty} G_{n,i}(P_{ij}^{(m)} - \pi_j) \right| = 0 \quad \text{a.s.} \quad (4.12) \]

Given \( \varepsilon > 0 \) choose \( m_0 \) such that for \( m \geq m_0 \)

\[ \sup_{i} |P_{ij}^{(m)} - \pi_j| < \varepsilon. \]

Then for any \( n \)

\[ \left| \sum_{i=0}^{\infty} G_{n,i}(P_{ij}^{(m_0)} - \pi_j) \right| \leq \varepsilon \sum_{i=0}^{\infty} G_{n,i} = \varepsilon. \]

Being \( \varepsilon > 0 \) arbitrary we have (4.12). \( \Box \)

If \( P \) is periodic with period \( d \geq 2 \), there is a partition \( E_0, E_1, \ldots, E_{d-1} \) of \( Z^+ \)
and on each \( E_r \) there is a stationary measure \( \pi^{(r)} \) of \( \{X_{nd}; n = 0, 1, \ldots\} \). Further if \( j \in E_r, k \in E_{r+s}, \)

\[ P_{jk}^{(nd+s)} \to \pi_k^{(r+s)} \quad \text{as} \quad n \to \infty, \]

and we have the following theorem which can be proved in a similar way with Theorem 4.2.

**Theorem 4.3** Let \( P \) be periodic with period \( d \geq 2 \). Suppose that for each \( j \in E_r, r = 0, 1, \ldots, d-1 \)
sup \( E_r \) \( |P_{ij}^{(nd)} - \pi_j| \to 0 \) as \( n \to \infty \), and the process starts
with one particle in state \( k \in E_0 \). Then for \( j \in E_r \)

\[ G_{nd+r,j} \xrightarrow{a.s.} \pi_j^{(r)} \quad \text{as} \quad n \to \infty. \]

### 4.2.3 Large Deviation

Now we study the decay rate of \( P(|G_{n,j} - \pi_j| > \varepsilon) \). We begin with the following simple
Lemma 4.3 Assume $0 < p_1 < 1$. Then for any $s \in C'_\infty$ there exists a constant $C = C(||s||)$ such that

$$||f_n(s)|| \leq C p_1^n.$$ 

**Proof.** Note that $f_{n,i}(s) = E_i(s^{Z_n}) \leq h_n(||s||)$. So

$$\frac{||f_n(s)||}{p_1^n} \leq \frac{h_n(||s||)}{p_1^n} \to Q(||s||),$$

where $Q$ is the limit function given by Proposition 4.1. Since $Q(||s||)$ is finite for $||s|| < 1$, there exists $C(||s||)$, such that

$$\frac{||f_n(s)||}{p_1^n} \leq C(||s||).$$

\[\Box\]

Theorem 4.4 Assume $0 < p_1 < 1$. Then there exists $Q = (Q_0, Q_1, \cdots); C'_\infty \to R_\infty$ such that

$$\frac{f_n(s)}{p_1^n} \to Q(s) < \infty \text{ for } s \in C'_\infty$$

and $Q$ is the unique solution to the vector equation

$$Q(f(s)) = p_1 Q(s)$$

subject to

$$Q(0) = 0, \quad Q'(0) = \Pi^t,$$

where $\Pi = ((\Pi_{ij}))$, $\Pi_{ij} = \pi_j$ for $i, j = 0, 1, 2, \cdots$.

**Proof.** Fix $s \in C'_\infty$. Since $p_0 = 0$, $P_i(0) = 0$. So we can write for $i = 0, 1, 2, \cdots$,

$$f_i(s) = \sum_{j \in A_\infty} P_i(j)s^j$$

$$= (a_{i0}s_0 + a_{i1}s_1 + \cdots) + \sum_{j \in A_\infty, |j| \geq 2} P_i(j)s^j.$$
Equivalently, we have the following vector equation

$$f(s) = sB + g(s) \quad (4.15)$$

where $B = A^t$, $g(s) = (g_0(s), g_1(s), \cdots)$, and $g_i(s) = \sum_{j \in A, \|j\| \geq 2} P_i(j)s^j$. Iterating (4.15) we get

$$f_n(s) = sB^n + \sum_{k=0}^{n-1} g(f_{n-k-1}(s))B^k$$

Hence

$$\frac{f_n(s)}{p^n_l} = s\frac{B^n}{p^n_l} + \frac{1}{p_l} \sum_{k=0}^{n-1} \frac{g(f_k(s))}{p^k_l} \left(\frac{B}{p_l}\right)^{n-k}.$$ 

Since $\|g(s)\| \leq (1 - p_1)\|s\|^2$

$$\sum_{k=0}^{\infty} \frac{\|g(f_k(s))\|}{p^k_l} \leq (1 - p_1) \sum_{k=0}^{\infty} \frac{\|f_k(s)\|}{p^k_l} \|f_k(s)\|$$

$$\leq (1 - p_1)C(\|s\|)^2 \sum_{k=0}^{\infty} p^k_l$$

by Lemma 4.3

$$< \infty$$

So we apply the dominated convergence theorem to $Z^+$ with counting measure to get

$$\lim_{n \to \infty} \frac{f(s)}{p^n_l} = s\Pi^t + \frac{1}{p_l} \sum_{k=0}^{\infty} \frac{g(f_k(s))}{p^k_l} \Pi^t$$

$$\equiv Q(s), \text{ say.}$$

From the definition of $Q$, it is easy to see that $Q(s) < \infty$ for $\|s\| < 1$, $Q(0) = 0$, and $Q'(0) = \Pi^t$. Furthermore

$$Q(f(s)) = f(s)\Pi^t + \frac{1}{p_l} \sum_{k=0}^{\infty} \frac{g(f_{k+1}(s))}{p^k_l} \Pi^t$$

$$= (sB + g(s))\Pi^t + \sum_{k=1}^{\infty} \frac{g(f_k(s))}{p^k_l} \Pi^t \quad \text{by (4.15)}$$

$$= p_l(s\Pi^t + \frac{1}{p_l} \sum_{k=0}^{\infty} \frac{g(f_k(s))}{p^k_l} \Pi^t)$$

$$= p_lQ(s).$$
As for uniqueness, let $Q^1(s)$ and $Q^2(s)$ be two solutions of (4.13) satisfying (4.14). Then for each $i = 0, 1, 2, \cdots$,

$$|Q_i^1(s) - Q_i^2(s)| = \frac{1}{p_i} |Q_i^1(f(s)) - Q_i^2(f(s))|$$

$$= \frac{1}{p_i} |Q_i^1(f_n(s)) - Q_i^2(f_n(s))|$$

$$\leq \frac{f_{n,j}(s)}{p_i} \{ |\pi_i - \frac{Q_i^1(f_n(s))}{f_{n,j}(s)}| + |\pi_i - \frac{Q_i^2(f_n(s))}{f_{n,j}(s)}| \}$$

$$\leq C(|s|) \{ |\pi_i - \frac{Q_i^1(f_n(s))}{f_{n,j}(s)}| + |\pi_i - \frac{Q_i^2(f_n(s))}{f_{n,j}(s)}| \}$$

(4.16)

Since $f_{n,j}(s) \to 0$ as $n \to \infty$, for $k = 1, 2$,

$$\lim_{n \to \infty} \frac{Q_i^k(f_n(s))}{f_{n,j}(s)} = \lim_{s_j \to 0} \frac{\partial Q_i^k(s)}{\partial s_j} = \Pi_i^j = \pi_i.$$

Thus from (4.16) we conclude for each $i = 0, 1, 2, \cdots$, $|Q_i^1(s) - Q_i^2(s)| = 0$. \hfill \Box

**Theorem 4.5** Assume $0 < p_1 < 1$. Then there exist $q_i(j)$, for $i \in Z^+$, $j \in A_\infty$ such that for each $i = 0, 1, 2, \cdots$,

(i) $Q_i(j) = \sum_{j \in A_\infty} q_i(j)s^j$, $s \in C'_\infty$

(ii) $\lim_{n \to \infty} \frac{P_i(Z_n = j)}{p_i^n} = q_i(j)$, $j \in A_\infty$.

**Proof.** Since $\sum_{k=0}^{\infty} ||g(f_k(s))|| < \infty$, $f_n(s)$ converges uniformly to $Q(s)$ for $||s|| \leq s_0$, $0 \leq s_0 < 1$. Since $\frac{f_n(s)}{p_i^n} = \sum_{j \in A_\infty} \frac{P_i(Z_n = j)}{p_i^n} s^j$, we have the theorem by Corollary 4.3. \hfill \Box

Now let $\ell = (\ell_0, \ell_1, \cdots)$ be a bounded sequence of positive real numbers. Then for an aperiodic $P$ we have that

$$\frac{\ell \cdot Z_n}{Z_n} - \ell \cdot \pi \quad pr \to 0 \quad \text{as} \quad n \to \infty.$$
Theorem 4.6 Let $0 < p_1 < 1$ and assume $P$ is aperiodic. Further assume

$$\sup_{i} \sum_{j=0}^{\infty} |P_{ij}^{(m)} - \pi_j| \to 0 \text{ as } m \to \infty$$

(A 2) If

$$\sum_{j=0}^{\infty} j^{2r} p_j < \infty \text{ and } \mu^r p_1 > 1.$$ 

Then for each $\varepsilon > 0$, there exists $m_0$ such that for all $m \geq m_0$

$$\lim_{n \to \infty} \frac{1}{p_1^n} P_i(\frac{\ell \cdot Z_{n+m}}{Z_{n+m}} - \ell \cdot \pi > \varepsilon) = \frac{1}{p_1^n} \sum_{j \in A_{\infty}} \phi(j, m, \varepsilon) q_i(j),$$

which is finite and positive and where $\phi(j, m, \varepsilon) = P(\frac{\ell \cdot Z_{m}}{Z_{m}} - \ell \cdot \pi > \varepsilon | Z_0 = j)$ and $q_i(j) = \lim_{n \to \infty} \frac{P_i(Z_n = j)}{p_1^n}$.

PROOF. Without loss of generality $\ell$ is not a multiple of the vector 1. By conditioning on $Z_n$ we have

$$P_i(\frac{\ell \cdot Z_{n+m}}{Z_{n+m}} - \ell \cdot \pi > \varepsilon) = E_i(P(\frac{\ell \cdot Z_{n+m}}{Z_{n+m}} - \ell \cdot \pi > \varepsilon | Z_n))$$

$$= \sum_{j \in A_{\infty}} P(\frac{\ell \cdot Z_{n+m}}{Z_{n+m}} - \ell \cdot \pi > \varepsilon | Z_n = j) P_i(Z_n = j)$$

Consider the event $\{\frac{\ell \cdot Z_{n+m}}{Z_{n+m}} - \ell \cdot \pi > \varepsilon\}$ conditioned on $\{Z_n = j\}$. Recall

$$Z_{n+m} = \sum_{i=0}^{\infty} \sum_{l=1}^{Z_{n,i}} Z_{n,i}^{(l)}.$$ 

Now

$$\frac{\ell \cdot Z_{n+m}}{Z_{n+m}} > \ell \cdot \pi + \varepsilon \iff \ell \cdot Z_{n+m} > (\ell \cdot \pi + \varepsilon) Z_{n+m}$$

$$\iff \ell \cdot Z_{n+m} - \ell \cdot (jM^m)$$

$$> (\ell \cdot \pi + \varepsilon)(Z_{n+m} - 1 \cdot (jM^m)) + (\ell \cdot \pi + \varepsilon)(1 \cdot (jM^m)) - \ell \cdot (jM^m)$$

$$\iff \ell - (\ell \cdot \pi + \varepsilon) 1 \frac{Z_{n+m} - (jM^m)}{\mu^m[j]} > ((\ell \cdot \pi + \varepsilon) 1 - \ell) \cdot j \frac{P^m}{|j|} \quad (4.17)$$
Choose $m_0$ such that for $m \geq m_0$

$$\sup_i \sum_{j=0}^{\infty} |P_{ij}^{(m)} - \pi_j| \leq \frac{\varepsilon}{2(2||\ell|| + \varepsilon)}.$$  \hfill (4.18)

Then

$$((\ell \cdot \pi + \varepsilon)1 - \ell) \cdot \frac{P^m}{||j||} = \sum_{i=0}^{\infty} (\ell \cdot \pi + \varepsilon - \ell_i) \sum_{r=0}^{\infty} \frac{j_r}{||j||} P_{ri}^{(m)}$$

$$= \sum_{i=0}^{\infty} (\ell \cdot \pi + \varepsilon - \ell_i) \pi_i + \sum_{i=0}^{\infty} (\ell \cdot \pi + \varepsilon - \ell_i) \sum_{r=0}^{\infty} \frac{j_r}{||j||} (P_{ri}^{(m)} - \pi_i)$$

$$\geq \varepsilon - (2||\ell|| + \varepsilon) \sum_{i=0}^{\infty} \frac{j_r}{||j||} \sum_{i=0}^{\infty} |P_{ri}^{(m)} - \pi_i|$$

$$\geq \frac{\varepsilon}{2} \quad \text{by (4.18)}. \hfill (4.19)$$

Combining (4.17) and (4.19) we see that for $m \geq m_0$, $\frac{\ell \cdot Z_n^{m+n}}{Z_n^{m+n}} - \ell \cdot \pi > \varepsilon$ and $Z_n = j$

imply together

$$(\ell - (\ell \cdot \pi + \varepsilon)1) \frac{Z_n^{m+n} - (jM^m)}{\mu^m ||j||} > \frac{\varepsilon}{2} \hfill (4.20)$$

or equivalently,

$$\frac{1}{||j||} \sum_{i=0}^{\infty} \sum_{l=1}^{\infty} \sum_{r=0}^{\infty} (\ell_r - \ell \cdot \pi - \varepsilon) \frac{(S_{mr}^{(i)})}{\mu^m} - P_{ir}^{(m)} > \frac{\varepsilon}{2}.$$  \hfill (4.21)

Put

$$Y_{m_0}^{(ii)} = \sum_{r=0}^{\infty} (\ell_r - \ell \cdot \pi - \varepsilon) \frac{(Z_{mr}^{(i)})}{\mu^{m_0}} - P_{ir}^{(m_0)}$$

$$\sigma_i^2 = E(Y_{m_0}^{(ii)})^2,$$

$$S_n^{(i)} = \sum_{i=1}^{n} Y_{m_0}^{(ii)} \hfill (4.21)$$

Since

$$|Y_{m_0}^{(ii)}| \leq (2||\ell|| + \varepsilon)(W_{m_0} + 1),$$
we see from the assumption (A 3) that

$$\sup_i E(Y_{m_0}^{(ii)})^{2r} < \infty. \quad (4.22)$$

It is also easy to see from (4.22) by Hölder’s inequality that

$$\sigma^2 = \sup_i \sigma_i^2 < \infty \quad (4.23)$$

Since \(\{Y_{m_0}^{(ii)}, l = 1, 2, \ldots\}\) are i.i.d. we can find finite constants \(B\) such that for any \(i = 1, 2, \ldots, n = 1, 2, \ldots\),

$$E\left(\frac{S_n^{(i)}}{\sqrt{n}}\right)^{2r} \leq \sigma_i^{2r} B + \frac{B}{r^{r-1}} E|Y_{m_0}^{(ii)}|^{2r} \quad (4.24)$$

due to the following Proposition which is a special form of Burkholder-Davis-Gundy inequality (Chow and Teicher (1988) p409).

**Proposition 4.8** Let \(\{X_j; j \geq 1\}\) be independent with \(EX_j = 0\) and \(\sigma_j^2 = EX_j^2 < \infty\) for all \(j \geq 1\). Let \(s_n^2 = \sum_{j=1}^n \sigma_j^2\) and \(S_n = \sum_{j=1}^n X_j\). If there exists \(r \geq 1\) such that \(EX_j^{2r} < \infty\) for all \(j = 1, 2, \ldots\) then there exist constant \(B\) depending only on \(r\) such that

$$E|S_n|^{2r} \leq B s_n^{2r} + B \sum_{j=1}^n E|X_j|^{2r}. \quad (4.25)$$

(4.24) along with (4.22) and (4.23) establishes that

$$K = \sup_{i, n} E\left(\frac{S_n^{(i)}}{\sqrt{n}}\right)^{2r} < \infty \quad (4.25)$$

where \(S_n^{(i)}\) is as in (4.21). So

$$P\left(\frac{\ell \cdot Z_{n+m_0}}{Z_{n+m_0}} - \ell \cdot \pi > \epsilon|Z_n = j\right)$$

$$\leq P\left((\ell - (\ell \cdot \pi + \epsilon)1)\frac{Z_{n+m_0} - (jM_{m_0})}{\mu_{m_0}^j/j} > \frac{\epsilon}{2}\right) \quad \text{by (4.20)}$$
Now let

\[ S_i = S_{j,i}^{(i)}, \quad \text{and} \quad T_n = \sum_{i=1}^{n} S_i, \]

then \( t_n \equiv Var(T_n) = \sum_{i=1}^{n} j_i \sigma_i^2 \). By the Proposition 4.8 there exists a constant \( C \) (depending only on \( r \)) such that

\[ E(T_{n^r}) \leq C(\sum_{i=1}^{n} j_i \sigma_i^2)^r + C \sum_{i=1}^{n} E(|S_i|^2) \]

\[ = C(\sum_{i=1}^{n} j_i \sigma_i^2)^r + C \sum_{i=1}^{n} j_i E\left(\frac{|S_i|}{\sqrt{j_i}}\right)^{2r} \]

\[ \leq C\sigma^{2r} \epsilon^r + CK \sum_{i=1}^{n} j_i^r \quad \text{by (4.25)} \]

Note that \( \sum_{i=1}^{n} j_i^r = \sum_{i=1}^{n} j_i^{r-1} j_i \leq |j|^r - 1 \sum_{i=1}^{n} j_i = |j|^r \). So we get

\[ E(T_{n^r}) \leq C(\sigma^{2r} + K)|j|^r. \tag{4.27} \]

Hence from (4.26) and (4.27) we arrive at

\[ P\left(\frac{\ell \cdot Z_{n+m_0}}{Z_{n+m_0}} - \ell \cdot \pi > \epsilon|Z_n = j\right) \leq C_0 \frac{1}{|j|^r} \]

for some constant \( C_0 \) in \((0, \infty)\). So

\[ \frac{1}{p_{n+m_0}^i} P_i \left(\frac{\ell \cdot Z_{n+m_0}}{Z_{n+m_0}} - \ell \cdot \pi > \epsilon\right) \]

\[ = \frac{1}{p_{i}^{m_0}} \sum_{j \in A_{\infty}} P_i \left(\frac{\ell \cdot Z_{n+m_0}}{Z_{n+m_0}} - \ell \cdot \pi > \epsilon|Z_n = j\right) \frac{P_i(Z_n = j)}{p_i^n} \]

\[ \leq \frac{C_0}{p_{i}^{m_0}} \sum_{j=1}^{\infty} \frac{1}{j^r} \frac{P_i(Z_n = j)}{p_i^n} \]

\[ = \frac{C_0}{p_{i}^{m_0}} E(Z_n^{-r}) \]
Since
\[
\frac{1}{p_t} E(Z^{-r}) \uparrow \frac{1}{\Gamma(r)} \int_0^1 Q(s)k(s)ds < \infty
\]
where \( Q \) is as in Proposition 4.1 and \( k(s) = |\log s|^{-1}/s \), (see Athreya(1994) for proof) by the generalized Lebesgue dominated convergence theorem we have
\[
\frac{1}{p_t^{n+m_0}} P_t(\ell \cdot \mathbf{Z}_{n+m_0} - \ell \cdot \pi > \varepsilon) = \frac{1}{p_1} \sum_{j \in A_\infty} P(\ell \cdot \mathbf{Z}_{n+m_0} - \ell \cdot \pi > \varepsilon | Z_n = j) \frac{P(Z_n = j)}{p_t} \to \frac{1}{p_1} \sum_{j \in A_\infty} \phi_1(j, m_0, \varepsilon) q_t(j) \tag{4.28}
\]
where \( \phi_1(j, m_0, \varepsilon) = P(\ell \cdot \mathbf{Z}_{m_0} - \ell \cdot \pi > \varepsilon | Z_0 = j) \).

Similar calculation prevails for the other part, i.e.,
\[
\frac{1}{p_t^{n+m_0}} P_t(\ell \cdot \mathbf{Z}_{n+m_0} - \ell \cdot \pi < -\varepsilon) \to \frac{1}{p_1} \sum_{j \in A_\infty} \phi_2(j, m_0, \varepsilon) q_t(j) \tag{4.29}
\]
where \( \phi_2(j, m_0, \varepsilon) = P(\ell \cdot \mathbf{Z}_{m_0} - \ell \cdot \pi < -\varepsilon | Z_0 = j) \). So (4.28) and (4.29) imply together
\[
\frac{1}{p_t^{n+m_0}} P_t(|\ell \cdot \mathbf{Z}_{n+m_0} - \ell \cdot \pi| > \varepsilon) \to \frac{1}{p_1} \sum_{j \in A_\infty} \phi(j, m_0, \varepsilon) q_t(j)
\]
where \( \phi(j, m_0, \varepsilon) = P(|\ell \cdot \mathbf{Z}_{m_0} - \ell \cdot \pi| > \varepsilon | Z_0 = j) \). \( \square \)

4.3 Continuous State Space Case

Suppose the particles are in \( R \). We form a branching Markov process in the following way. Assume an initial ancestor, who forms the zeroth generation, is at \( x \).

It produces offspring according to offspring generating function \( h(s) = \sum p_k s^k \) and the
offsprings choose their positions with probability distribution \( P(x, \cdot) \) independently of each other. The particles in the \( n \)th generation give birth independently of the one another and of the preceding generations to form the \((n + 1)\)th generation and a particle whose parent is at \( y \) on its death moves according to \( P(y, \cdot) \). Let \( Z_n^x \) be the point process describing the positions of particles alive in the \( n \)th generation where superscript \( x \) indicate that the ancestor is on position \( x \) at time \( 0 \). Thus \( Z_n^x \) is a random locally finite counting measure on \( \mathbb{R} \) and \( Z_n^x(A) \) denotes the number of particles in the \( n \)th generation which are in \( A \). We write \( Z_n \) for \( Z_n^0 \) and \( |Z_n| \) for \( Z_n(\mathbb{R}) \).

4.3.1 Notations and Definitions

Notations

(N 1) For a Borel measurable function \( s; \mathbb{R} \rightarrow \mathbb{R} \), \( ||s|| = \sup_x |s(x)| \), whereas \( ||\nu|| \) is the total variation of \( \nu \) for a signed measure \( \nu \) on \( \mathbb{R} \).

(N 2) \( \mathcal{S} = \{s| s; \mathbb{R} \rightarrow [0,1] \ \text{Borel measurable}\} \), \( \mathcal{S}' = \{s \in \mathcal{S}; ||s|| < 1\} \)

(N 3) For a measure \( \nu \) on \( \mathbb{R} \) and for a measurable function \( f; \mathbb{R} \rightarrow \mathbb{R} \), \( \nu(f) = \int f \, d\nu \).

(N 4) \( C_K^+(\mathbb{R}) = \{f; f \text{ is a continuous function from } \mathbb{R} \text{ to } \mathbb{R}^+ \text{ with compact support}\} \).

(N 5) \( B(y, \delta) = \{x \in \mathbb{R}; |x - y| < \delta\} \).

(N 6) \( \tilde{x}_k = (x_1, \cdots, x_k) \in \mathbb{R}^k \).
Definitions

(D1) \( \{x_1, \cdots, x_{\left|Z_n\right|}\} \) is the enumeration of the positions of the particles in the \( n \)th generation.

(D2) For \( \tilde{x}_k \in R^k \), \( m_{\tilde{x}_k} \) is the point process \( \sum_{j=1}^{k} I_{x_j} \).

(D3) For a Borel set \( B \), \( G_n(B) = \frac{Z_n(B)}{|Z_n|} \).

(D4) For a measurable function \( f; R \rightarrow R \),

\[
M^x_n(f) = \sum_{j=1}^{\left|Z_n\right|} f(x_j), \quad M^x_n(f) = E(x(Z_n(f)).
\]

(D5) Laplace functional \( \Psi_n = \Psi_{Z_n} \) of the point process \( Z_n \) is defined as

\[
\Psi_n(f)(x) = E_x(\exp(-Z_n(f))),
\]

where \( f; R \rightarrow R^+ \) is a Borel measurable function.

(D6) Probability generating functional \( \Phi_n = \Phi_{Z_n} \) of the point process \( Z_n \) is defined as

\[
\Phi_n(s)(x) = \Psi_n(-\log s)(x) = E_x(\exp(\int \log s(y)Z_n(dy))),
\]

for \( s; R \rightarrow [0,1] \) Borel measurable.

(D7) \( \{X_n; n = 0, 1, \cdots\} \) is a Markov chain on \( R \) with transition distribution \( P(\cdot, \cdot) \).

(D8) \( P^{n+1}(x, B) = \int P^n(y, B)P(x, dy) \) for \( n \geq 1 \).

We recall the following proposition from Harris(1963).
Proposition 4.9 For any Borel measurable $f : \mathbb{R} \to \mathbb{R}^+$

$$\Psi_{n+m}(f)(x) = \Psi_n(-\log \Psi_m(f))(x), \quad n, m = 0, 1, \cdots.$$ 

Remark 4.2 From this proposition it is easy to see that the probability generating functional $\Phi_n$ form a semigroup in $n$, i.e.,

$$\Phi_{n+m}(s) = \Phi_n(\Phi_m(s)) \quad \text{for} \quad n, m = 0, 1, \cdots.$$ 

4.3.2 Law of Large Numbers

We assume throughout this section that the underlying movement chain $\{X_n; n = 0, 1, \cdots\}$ on $\mathbb{R}$ is $(A_0, \varepsilon, \lambda_1, 1)$-recurrent with stationary measure $\pi(\cdot)$ for some $A_0$.

Theorem 4.7 Assume for each compact set $K$,

$$\sup_{x \in K} \|P^m(x, \cdot) - \pi(\cdot)\| \to 0 \quad \text{as} \quad m \to \infty.$$ 

Then for any Borel set $A$

$$G_n(A) \xrightarrow{\text{pr}} \pi(A) \quad \text{as} \quad n \to \infty.$$ 

Proof. By the branching property we may write

$$Z_{n+m}(A) = \sum_{j=1}^{Z_n} Z_m^{x_j}(A) \quad \text{(4.30)}$$

where $Z_m^{x_j}(A)$ is the number of particles in $A$ in the $(n + m)$th generation which are in the line of descent initiated by a particle of position $x_j$ in the $n$th generation. It is
well-known that conditioned on \{x_j; j = 1, \cdots, x_{|Z_n|}\}, \{Z^x_j(A); j = 1, \cdots, x_{|Z_n|}\} are independently distributed. Starting from (4.30) we have the identity
\[
\frac{\mu^{-m}}{|Z_n|} Z_{n+m}(A) = \frac{1}{|Z_n|} \sum_{j=1}^{|Z_n|} \{\mu^{-m} Z^x_j(A) - P^m(x_j, A)\}
+ \frac{1}{|Z_n|} \sum_{j=1}^{|Z_n|} \{P^m(x_j, A) - \pi(A)\} + \pi(A)
= a_n(m, A) + b_n(m, A) + \pi(A), \quad \text{say.}
\]
Trivially,
\[
G_{n+m}(A) = \frac{a_n(m, A) + b_n(m, A) + \pi(A)}{a_n(m, R) + 1}.
\]
Since \(\sup_{x_j} |\mu^{-m} Z^x_j(A) - P^m(x_j, A)| \leq \mu^{-m}|Z_m| + 1\), we see from Lemma 2.2 and Lemma 2.3 that for any \(m \geq 0\)
\[
\sup_A a_n(m, A) \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty.
\]
So it is enough to show that given \(\varepsilon > 0\)
\[
\limsup_{m \to \infty} \limsup_{n \to \infty} P\left(\left|\frac{1}{|Z_n|} \sum_{j=1}^{|Z_n|} \{P^m(x_j, A) - \pi(A)\}\right| > \varepsilon\right) = 0
\]
Let \(\varepsilon > 0, \delta > 0\) be given. Choose \(\eta(\delta) > 0\) such that
\[
P(W < \eta) < \delta/2
\]
and choose \(K = K(\varepsilon, \delta, \eta)\) such that
\[
\pi(I_K^c) \leq \frac{\varepsilon \delta \eta}{4}
\]
where \(I_K = [-K, K]\). Then
\[
P\left(\left|\frac{1}{|Z_n|} \sum_{j=1}^{|Z_n|} (P^m(x_j, A) - \pi(A))\right| > \varepsilon\right) \leq P\left(\left|\frac{1}{|Z_n|} \sum_{x_j \in I_K} (P^m(x_j, A) - \pi(A))\right| > \varepsilon/2\right)
+ P\left(\left|\frac{1}{|Z_n|} \sum_{x_j \notin I_K} (P^m(x_j, A) - \pi(A))\right| > \varepsilon/2\right)
= c_{nm} + d_{nm}, \quad \text{say.}
Now choose \( m_0 \) such that for \( m \geq m_0 \)
\[
\sup_{x \in I_K} |P^m(x, A) - \pi(A)| \leq \varepsilon/2. \tag{4.36}
\]
Then for \( m \geq m_0 \) it is immediate that \( c_{nm} = 0 \). On the other hand
\[
d_{nm} = P\left(\frac{1}{|Z_n|} \sum_{x_j \in I_K} (P^m(x_j, A) - \pi(A)) > \varepsilon/2\right)
\leq P\left(\frac{Z_n(I_K^*)}{|Z_n|} > \frac{\varepsilon}{2}, W_n > \eta\right) + P(W_n \leq \eta)
\leq \frac{2}{\varepsilon \eta \mu^n} E(Z_n(I_K^*)) + P(W_n \leq \eta)
\]
Taking limit we have from (4.34) and (4.35) that for \( m \geq m_0 \)
\[
\limsup_{n \to \infty} d_{nm} \leq \frac{2}{\varepsilon \eta} \pi(I_K^*) + P(W < \eta) \leq \frac{2}{\varepsilon \eta} \frac{\varepsilon \eta \delta}{4} + \frac{\delta}{2} = \delta \tag{4.37}
\]
Hence for \( m \geq m_0 \)
\[
\limsup_{n \to \infty} P\left(\frac{1}{|Z_n|} \sum_{j=1}^{|Z_n|} (P^m(x_j, A) - \pi(A)) > \varepsilon\right) < \delta.
\]
Being \( \delta > 0 \) arbitrary we have shown (4.33) and so the proof is completed. \( \square \)

**Corollary 4.4** Suppose for each compact set \( K \) \{\( P_m(x, A), m = 0, 1, \ldots \)\} is equicontinuous in \( x \) on \( K \). Then
\[
G_n(A) \overset{pr}{\to} \pi(A) \quad \text{as} \quad n \to \infty.
\]

**Proof.** Given \( \varepsilon > 0, \delta > 0 \) choose \( \eta(\delta) > 0 \) and \( K = K(\varepsilon, \delta, \eta) \) satisfying (4.34) and (4.35) respectively. Then we have seen that for any \( m \geq 1 \) \( \limsup_{n \to \infty} d_{nm} \leq \delta \).

Now we show that \( c_{nm} = 0 \) for large \( m \). Since \( P^m(x, A) \) is uniformly equicontinuous on the compact set \( I_K \), there exists \( \delta' > 0 \) such that for any \( m \geq 1 \)
\[
|P^m(x, A) - P^m(y, A)| < \varepsilon/4 \quad \text{if} \quad |x - y| < \delta' \quad \text{and} \quad x, y \in I_K. \tag{4.38}
\]
By the compactness of $I_K$ we can find $y_1, \cdots, y_l \in I_K$ such that

$$I_K \subset \bigcup_{i=1}^l B(y_i, \delta')$$

and for each $x \in I_K$ we can find $x' \in \{y_1, \cdots, y_l\}$ such that $|x' - x| < \delta'$. Now choose $m_0$ such that for $m \geq m_0$

$$\sup_{1 \leq j \leq l} |P^m(y_j, A) - \pi(A)| \leq \varepsilon/4. \quad (4.39)$$

Then for $m \geq m_0$

$$\left| \frac{1}{|Z_n|} \sum_{x_j \in I_K} \{P^m(x_j, A) - \pi(A)\} \right|$$

$$\leq \frac{1}{|Z_n|} \sum_{x_j \in I_K} \{|P^m(x_j, A) - P^m(x'_j, A)| + |P^m(x'_j, A) - \pi(A)|\}$$

$$\leq \frac{1}{|Z_n|} \sum_{x_j \in I_K} \left( \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \right) \quad \text{by (4.38) and (4.39)}$$

$$\leq \frac{\varepsilon}{2}$$

So $c_{nm} = 0$ for $m \geq m_0$. Hence for $m \geq m_0$

$$\limsup_{n \to \infty} P\left( \frac{1}{|Z_n|} \sum_{j=1}^{|Z_n|} |P^m(x_j, A) - \pi(A)| > \varepsilon \right) \leq \delta.$$ 

Being $\delta > 0$ arbitrary the proof is completed. \qed

**Theorem 4.8** Assume that for any Borel set $A$

(A 4) $\sup_{\varepsilon} |P^m(x, A) - \pi(A)| \to 0$ as $m \to \infty$.

Then for any Borel set $A$

$$G_n(A) \xrightarrow{a.s.} \pi(A) \quad \text{as} \quad n \to \infty.$$
PROOF. Recalling (4.31) and (4.32) it suffices to show that

\[
\lim_{m \to \infty} \sup_{n \to \infty} \frac{1}{|Z_n|} \sum_{j=1}^{Z_n} \{P^m(x_j, A) - \pi(A)\} = 0. \tag{4.40}
\]

Given \( \varepsilon > 0 \) choose \( m_0 \) such that for \( m \geq m_0 \)

\[
\sup_{x} |P^m(x, A) - \pi(A)| \leq \varepsilon
\]

So for \( m \geq m_0 \) and for all \( n \geq 1 \)

\[
\frac{1}{|Z_n|} \sum_{j=1}^{Z_n} \{P^m(x_j, A) - \pi(A)\} \leq \varepsilon,
\]

which establishes (4.40).

\[\square\]

4.3.3 Large Deviation

Lemma 4.4 Assume \( 0 < p_1 < 1 \). Then for \( s \in S' \), there exists a constant \( C = C(||s||) \) such that for all \( n \geq 1 \)

\[||\Phi_n(s)|| \leq C p_1^n.\]

PROOF. Note that

\[
\Phi_n(s)(x) = E_x(\exp(\int \log s(y)Z_n(dy))) \\
\leq E_x(||s||^{Z_n}) = h_n(||s||).
\]

So \( \frac{\Phi_n(s)(x)}{p_1^n} \leq \frac{h_n(||s||)}{p_1^n} \to Q(||s||) \) as \( n \to \infty \), where \( Q \) is given by Proposition 4.2. Now choose a constant \( c \) such that

\[
\frac{\Phi_n(s)(x)}{p_1^n} \leq cQ(||s||) \text{ for all } n \geq 1.
\]

Put \( C(||s||) = cQ(||s||). \) \[\square\]
The next theorem gives the rate of convergence of the probability generating functional $\Phi_n(s)$ to 0 and is the key to the main result of this subsection contained in Theorem 4.11.

**Theorem 4.9** Suppose $0 < p_1 < 1$. Then there exists $\Lambda; S' \to [0, \infty)$ such that for all $x \in \mathcal{R}$

$$\lim_{n \to \infty} \frac{\Phi_n(s)(x)}{p_1^n} = \Lambda(s).$$

**Proof.** Let $s \in S'$ be fixed. We write $\Phi_n(s)(x)$ as

$$\Phi_1(s)(x)$$

$$= E_x(\exp(\int \log s(y)Z_1(dy); |Z_1| = 1) + E_x(\exp(\int \log s(y)Z_1(dy)); |Z_1| \geq 2)$$

$$= p_1 \int s(y)P(x, dy) + g(s)(x)$$

where $g(s)(x) = E_x(\exp(\int \log s(y)Z_1(dy)); |Z_1| \geq 2)$. Iterating we get

$$\Phi_n(s)(x) = p_1^n \int s(y)P^n(x, dy) + \sum_{k=0}^{n-1} p_1^{n-1-k} \int g(\Phi_k(s))(y)P^{n-1-k}(x, dy).$$

So

$$\frac{\Phi_n(s)(x)}{p_1^n} = \int s(y)P^n(x, dy) + \frac{1}{p_1} \sum_{k=0}^{n-1} \int \frac{g(\Phi_k(s))(y)}{p_1^k}P^{n-1-k}(x, dy).$$

Since $||g(s)|| \leq E(||s||^{|Z_1|}; |Z_1| \geq 2) \leq (1 - p_1)||s||^2$ we have by Lemma 4.4.

$$\sum_{k=0}^{\infty} \frac{||g(\Phi_k(s))||}{p_1^k} \leq (1 - p_1)C(||s||^2) \sum_{k=0}^{\infty} p_1^k < \infty.$$

From this it follows that

$$\lim_{n \to \infty} \frac{\Phi_n(s)(x)}{p_1^n} = \int s(y)\pi(dy) + \frac{1}{p_1} \sum_{k=0}^{\infty} \int \frac{g(\Phi_k(s))(y)}{p_1^k}\pi(dy)$$

$$\overset{\text{def}}{=} \Lambda(s),$$
by the dominated convergence theorem.

Before we state the main theorem we prove the weak convergence of conditioned point process \( \{Z_n \mid |Z_n| = k\} \).

**Theorem 4.10** Define \( \xi_{n,k} = \{Z_n \mid |Z_n| = k\} \), then for each \( k = 1, 2, \cdots \), there exists a point process \( \xi_k \) such that

\[
\xi_{n,k} \xrightarrow{d} \xi_k \quad \text{as} \quad n \to \infty.
\]

**PROOF.** Let \( s \in S \) and \( 0 < t < 1 \). Then \( ts \in S' \) and so

\[
\frac{\Phi_n(ts)(x)}{p^n_l} = \sum_{k=0}^{\infty} E_x(\exp(\int \log s(y)\xi_{n,k}(dy))) \frac{P(|Z_n| = k)}{p^n_l} t^k \to \Lambda(ts).
\]

Appealing to Lemma 4.2 we know that for \( k = 0, 1, 2, \cdots \) there exists \( \lambda_k(s) \) such that

\[
\Lambda(ts) = \sum_{k=0}^{\infty} \lambda_k(s) t^k
\]

and

\[
\lim_{n \to \infty} E_x(\exp(\int \log s(y)\xi_{n,k}(dy))) \frac{P(|Z_n| = k)}{p^n_l} = \lambda_k(s).
\]

Note that (4.41) is equivalent to

\[
\lim_{n \to \infty} E_x(\exp(\int \log s(y)\xi_{n,k}(dy))) = \tilde{\lambda}_k(s) \overset{\text{def}}{=} \lambda_k(s) g_k^{-1}
\]

due to the Proposition 4.2. Now fix \( k \) and let \( f \in C^+_K(R) \). Since \( \xi_{n,k}(f) = \int f(x)\xi_{n,k}(dx) \leq k\|f\| < \infty \), \( \{\xi_{n,k}(f); n = 1, 2, \cdots\} \) is tight and so is \( \{\xi_{n,k}; n = 1, 2, \cdots\} \) (see Resnick(1987) p153 Lemma 3.20). So given any subsequence \( \{n''\} \subset \{n\} \), there is a further subsequence \( \{n'\} \subset \{n''\} \) and a point process \( \xi_k \) such that

\[
\xi_{n',k} \xrightarrow{d} \xi_k.
\]

(4.42)
Now (4.42) implies (Resnick(1987) p153 Proposition 3.19) that for any $f \in C_K^+(R)$

$$\Psi_{\xi_{n,k}}(f) \to \Psi_{\xi_k}(f).$$

(4.43)

Let $\psi_{n,k,f}$ be the Laplace transform of the random variable $\xi_{n,k}(f)$ then

$$\psi_{n,k,f}(\lambda) \to \psi_{k,f}(\lambda)$$

(4.44)

where $\psi_{k,f}(\lambda)$ is the Laplace transform of a random variable $\xi_{k,f}$, say. Noting that $\psi_{n,k,f}(\lambda) = \Psi_{\xi_n}(\lambda f)$ we conclude from (4.43) and (4.44)

$$\psi_{k,f}(\lambda) = \Psi_{\xi_k}(\lambda f) = E(\exp(-\lambda \xi_k(f))).$$

So $\xi_k(f) \overset{d}{=} \xi_{k,f}$ by the uniqueness of Laplace transform. If $\xi_k$ is a limit of another subsequence of $\{\xi_{n,k}\}$, then the same argument give us that $\xi_k(f) \overset{d}{=} \xi_{k,f}$. That is, $\xi_k(f) \overset{d}{=} \xi_{k,f}$ for any $f \in C_K^+(R)$. Hence $\Psi_{\xi_k}(f) = \Psi_{\xi_k}(f)$ for any $f \in C_K^+(R)$ or equivalently (see Resnick(1987) p153 Proposition 3.19) $\xi_k \overset{d}{=} \xi_{k,f}$. So $\xi_{n,k} \overset{d}{\to} \xi_k$ as $n \to \infty$. □

The next theorem is a large deviation result for functionals of the process under a moment hypothesis on the offspring distributions.

**Theorem 4.11** Let $0 < p_1 < 1$. Assume (A 3) and (A 4). Then given $\epsilon > 0$ and $f; R \to R$ bounded measurable there exists $m_0$ such that for $m \geq m_0$

$$\lim_{n \to \infty} P\left(\frac{Z_n(f)}{|Z_n|} - \pi(f) > \epsilon\right) = \frac{1}{p_1^m} \sum_{k=1}^{\infty} q_k \int_{R^k} \phi_{m,k}(\tilde{x}_k) P(\xi_k \in d\tilde{x}_k),$$

which is finite and positive and where $q_k = \lim_{n \to \infty} P(|Z_n| = k) / p_1^k$, $\xi_k = \lim_{n \to \infty} \{Z_n | Z_n = k\}$ and $\phi_{m,k}(\tilde{x}_k) = P(\sum_{j=1}^{k} (Z_j - \pi(f) - \epsilon) > 0)$. 
PROOF. With an abuse of notation we write \( \tilde{x}_k \) for the point process \( m_{\tilde{x}_k} = \sum_{j=1}^{k} I_{x_j} \) if it doesn’t cause any confusion. By the branching property we may write

\[
Z_{n+m}(f) = \sum_{j=1}^{\lfloor m \rfloor} Z_m^{x_j}(f).
\]

So given \( Z_n = \tilde{x}_k \), we have

\[
\frac{Z_{n+m}(f)}{|Z_{n+m}|} > \pi(f) + \varepsilon
\]

if

\[
\sum_{j=1}^{k} (Z_m^{x_j}(f) - M_m^{x_j}(f)) > (\pi(f) + \varepsilon) \sum_{j=1}^{k} (Z_m^{x_j}(1) - M_m^{x_j}(1)) + (\pi(f) + \varepsilon) \sum_{j=1}^{k} M_m^{x_j}(1) - \sum_{j=1}^{k} M_m^{x_j}(f)
\]

if

\[
\frac{1}{k} \sum_{j=1}^{k} Y_m(x_j) > y_m(\tilde{x}_k)
\]

where

\[
Y_m(x) = \frac{1}{\mu_m} (Z_m^{x}(f) - \pi(f) - \varepsilon) - M_m^{x}(f - \pi(f) - \varepsilon)
\]

\[
y_m(\tilde{x}_k) = \frac{1}{k} \sum_{j=1}^{k} \frac{1}{\mu_m} M_m^{x_j}(\pi(f) + \varepsilon - f).
\]

So

\[
\frac{1}{p_1^{n+m}} P\left( \frac{Z_{n+m}(f)}{|Z_{n+m}|} > \pi(f) + \varepsilon \right) = \frac{1}{p_1^{n+m}} E\left( P\left( \frac{Z_{n+m}(f)}{|Z_{n+m}|} > \pi(f) + \varepsilon \mid Z_n \right) \right)
\]

\[
= \frac{1}{p_1^n} \sum_{k=0}^{\infty} \int_{R^k} \phi_{m,k}(\tilde{x}_k) \frac{P(|Z_n| = k)}{p_1^n} P(\xi_{n,k} \in d\tilde{x}_k)
\]

where \( \phi_{m,k}(\tilde{x}_k) = P\left( \frac{1}{k} \sum_{j=1}^{k} Y_m(x_j) > y_m(\tilde{x}_k) \right) \). Suppose for a while that \( \phi_{m,k}(\tilde{x}_k) \) is a continuous function of \( \tilde{x}_k \) in the vague topology on the point process space and

\[
|\phi_{m,k}(\tilde{x}_k)| = O(\frac{1}{k^r}). \quad (4.45)
\]

Then from (4.45) we have

\[
\int_{R^k} \phi_{m,k}(\tilde{x}_k) \frac{P(|Z_n| = k)}{p_1^n} P(\xi_{n,k} \in d\tilde{x}_k) \leq C \frac{1}{k^r} \frac{P(|Z_n| = k)}{p_1^n}
\]
and since
\[ \sum_{k=1}^{\infty} \frac{1}{k^r} \frac{P(|Z_n| = k)}{P_1^n} = \frac{1}{p_1^n} E \left( \frac{1}{|Z_n|^r} \right) \rightarrow \frac{1}{\Gamma(r)} \int_0^1 Q(s)k(s)ds < \infty \text{ as } n \rightarrow \infty \]

where \( k(s) = \frac{\log s}{s} \) (see Athreya(1994) for proof), by the generalized dominated convergence theorem we get

\[
\lim_{n \to \infty} \frac{1}{p_1^{n+m_0}} P(\frac{Z_{n+m_0}(f)}{|Z_{n+m_0}|} > \pi(f) + \varepsilon) \geq \frac{1}{p_1^{m_0}} \lim_{n \to \infty} \sum_{k=1}^{\infty} \int_{R^k} \phi_{m_0,k}(\tilde{x}_k) P(|Z_n| = k) \frac{P(\xi_n,k \in d\tilde{x}_k)}{p_1^n} \\
= \frac{1}{p_1^{m_0}} \sum_{k=1}^{\infty} \int_{R^k} \phi_{m_0,k}(\tilde{x}_k) P(|Z_n| = k) \frac{P(\xi_n,k \in d\tilde{x}_k)}{p_1^n} \\
= \frac{1}{p_1^{m_0}} \int_{R^k} \phi_{m_0,k}(\tilde{x}_k) P(\xi_k \in d\tilde{x}_k) (4.46)
\]

Now we prove (4.45) in the next lemma.

**Lemma 4.5** Assume (A 3). Then there exists \( m_0 \) such that for \( m \geq m_0 \)

\[ |\phi_{m,k}(\tilde{x}_k)| = O(\frac{1}{k^r}). \]

**PROOF.** Choose \( m_0 \) such that for \( m \geq m_0 \)

\[ \sup_x |E_x(f(X_m)) - \pi(f)| < \varepsilon/2. \]  

(4.47)

First note that given \( Z_n = \tilde{x}_k, \{Y_{m_0}(x_j), j = 1, \ldots, k\} \) are independently distributed with mean 0. Since

\[ \sup_x |Y_{m_0}(x)| \leq (||f|| + \pi(f) + \varepsilon)(\frac{Z_{m_0}}{\mu_{m_0}} + 1). \]
(A 3) implies
\[ K \equiv \sup_x E(Y_{m_0}(x))^{2r} < \infty. \]
and so it follows from Hölder's inequality that
\[ \bar{\sigma}^2 \equiv \sup_x E(Y_{m_0}(x))^2 < \infty. \]

By (4.47) we see that \( y_{m_0}(\tilde{x}_k) \geq \varepsilon/2 \) and so that
\[
\phi_{m,k}^1(\tilde{x}_k) \leq P\left( \frac{1}{k} \sum_{j=1}^{k} Y_{m_0}(x_j) \geq \frac{\varepsilon}{2} \right) \leq \left( \frac{2}{\varepsilon} \right)^{2r} \left( \frac{1}{k} \sum_{j=1}^{k} Y_{m_0}(x_j) \right)^{2r} \tag{4.48}
\]

Via Proposition 4.8 we can find a constant \( C \) in \((0, \infty)\) such that
\[
E\left( \sum_{j=1}^{k} Y_{m_0}(x_j) \right)^{2r} \leq C\left( \sum_{j=1}^{k} E(Y_{m_0}(x_j))^{2r} \right) + C \sum_{j=1}^{k} E|Y_{m_0}(x_j)|^{2r} \leq C k r \bar{\sigma}^2 + C k K.
\]

So there exists a constant \( C_r < \infty \) such that
\[
\sup_{k \in \tilde{x}_k} \sup_{x_k} E\left( \frac{1}{k} \sum_{j=1}^{k} Y_{m_0}(x_j) \right)^{2r} \leq C_r \tag{4.49}
\]

(4.48) along with (4.49) proves lemma. \qed

Next we prove \( \phi_{m,k}^1(\tilde{x}_k) = P(\sum_{j=1}^{k} Z_{m}^{x_j}(f) > 0) \) is continuous in \( \tilde{x}_k \).

**Lemma 4.6** Let \( f \) be a Borel measurable function. Then for any \( m \geq 1 \) and for any Borel set \( B \), \( P(\sum_{j=1}^{k} Z_{m}^{x_j}(f) \in B) \) is a continuous function of \( \tilde{x}_k \) in the vague topology on the point process space.

**PROOF.** Fix a Borel measurable function \( f \) and Borel set \( B \). Since \( m_{\tilde{x}_k} \Rightarrow m_{\tilde{y}_k} \) if and only if \( \tilde{x}_k \to \tilde{y}_k \) coordinatewise it suffices to show that \( P(\sum_{j=1}^{k} Z_{m}^{x_j}(f) \in B) \) is
continuous in $\tilde{x}_k$ coordinatewise. Suppose that $P(Z_m^x(f) \in B)$ is continuous in $x$.
Then by the independence of $Z_m^{x_1}$ and $Z_m^{x_2}$

$$P(Z_m^{x_1} + Z_m^{x_2} \in B) = \int P(Z_m^{x_1} \in B - r)P(Z_m^{x_2} \in dr)$$

$$\rightarrow \int P(Z_m^{x_1} \in B - r)P(Z_m^{x_2} \in dr) \quad \text{as} \quad x_1 \to y_1 \text{ and } x_2 \to y_2$$

by the generalized Lebesgue dominated convergence theorem

$$= P(Z_m^{y_1} + Z_m^{y_2} \in B)$$

The same argument give us that for $k \geq 3$, $P(\sum_{j=1}^k Z_m^{x_j}(f) \in B)$ is continuous in
$\tilde{x}_k$ coordinatewise. In other words, to prove the lemma we need to show only that
$P(Z_m^x(f) \in B)$ is continuous in $x$ for $m \geq 1$. Now let $\{X_{ij}, j = 1, 2, \ldots\}$ be iid with
$P_x(X_{11} \in B) = P(x, B)$. Then by the independence and by the generalized Lebesgue
dominated convergence theorem we have

$$P_x(f(X_{11}) + f(X_{12}) \in B) = \int P_x(f(X_{11}) \in B - r)P_x(f(X_{12}) \in dr)$$

$$\rightarrow \int P_y(f(X_{11}) \in B - r)P_y(f(X_{12}) \in dr) \quad \text{as} \quad x \to y$$

$$= P_y(f(X_{11}) + f(X_{12}) \in B)$$

The continuity of $P_x(\sum_{j=1}^k f(X_{ij}) \in B)$ for $k \geq 3$ follows by induction. Since

$$P(Z_1^x(f) \in B) = \sum_{k=0}^{\infty} p_k P_x(\sum_{j=1}^k f(X_{ij}) \in B)$$

$P(Z_1^x(f) \in B)$ is continuous in $x$ by bounded convergence theorem. Now suppose
$P(Z_m^x(f) \in B)$ is continuous in $x$. Recall that this implies $P(\sum_{j=1}^k Z_m^{x_j}(f) \in B)$ is
continuous in $\tilde{x}_k$. Since

$$P(Z_{m+1}^x(f) \in B) = \sum_{k=0}^{\infty} \int_{R^k} P(\sum_{j=1}^k Z_m^{x_j}(f) \in B) P_x(x_1, \ldots, x_k \in d\tilde{x}_k)$$
we conclude that \( P(Z_{m+1}^\pi(f) \in B) \) is continuous in \( x \) by the generalized Lebesgue dominated convergence theorem and the lemma follows. \( \square \)

Arguing in exactly the same way as above we can show

\[
\frac{1}{p_{n+m_0}} P\left( \frac{Z_{n+m_0}(f)}{|Z_{n+m_0}|} < \pi(f) - \varepsilon \right) \\
\rightarrow \frac{1}{p_{n+m_0}^{m_0}} \sum_{k=1}^{\infty} q_k \int_{R^k} \phi_{m_k}^2(\tilde{x}_k) P(\xi_k \in d\tilde{x}_k) \quad \text{as} \quad n \rightarrow \infty \quad (4.50)
\]

where \( \phi_{m,k}(\tilde{x}_k) = \mathcal{I}(\frac{1}{k} \sum_{j=1}^{k} Y_m(x_j) < -y_m(\tilde{x}_k)) \). Combining (4.39) and (4.41) we arrive at

\[
\lim_{n \rightarrow \infty} \frac{1}{p_{n+m_0}^{n+m_0}} P\left( \frac{Z_{n+m_0}(f)}{|Z_{n+m_0}|} > \pi(f) - \varepsilon \right) = \frac{1}{p_{n+m_0}^{m_0}} \sum_{k=1}^{\infty} q_k \int_{R^k} \phi_{m_k}(\tilde{x}_k) P(\xi_k \in d\tilde{x}_k)
\]

where \( \phi_{m,k}(\tilde{x}_k) = P\left( \frac{1}{k} \sum_{j=1}^{k} Y_m(x_j) > y_m(\tilde{x}_k) \right) = P\left( \sum_{j=1}^{k} Z_m^\pi(f - \pi(f) - \varepsilon) > 0 \right) \). \( \square \)
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