Direct photon production in a nuclear environment

Xiaofeng Guo
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Direct photon production in a nuclear environment

by

Xiaofeng Guo

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
Department: Physics and Astronomy
Major: Nuclear Physics

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For the Graduate College

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Ames, Iowa
1996

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To my parents
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1. INTRODUCTION

The high energy photon is a very good probe for short-distance physics in strong interactions, because photons couple directly to point-like quark constituents of nucleons and do not interact strongly once produced. Photon signals from high energy reactions provide very good tests of quantum chromodynamics (QCD) [1], which is a theory to describe strong interactions between quarks and gluons. In almost all high energy experiments, the photon is studied as one of the major signals for underlying processes.

Thermal photons are the possible signals for quark-gluon plasma in relativistic heavy ion collisions [2]. However, experimental data is a sum of both thermal photons and non-thermal photons from hard collisions. In order to extract the information on thermal photons, we need accurate predictions of non-thermal photon production from hard collisions. Non-thermal photon production in nucleus-nucleus collisions should include the photons produced from collisions of two nucleons, as well as those produced through multiple interaction involving more than two nucleons. This thesis addresses two issues associated with non-thermal photon productions: 1) making accurate predictions for photons produced in nucleon-nucleon collisions; and 2) calculating the contributions of photon productions through multiple scattering in heavy nucleus.
The major uncertainty on photon production in nucleon-nucleon collisions is photon fragmentation. In addition to short-distance production, photons can also be produced through long-distance fragmentation of the quarks and gluons, themselves produced in short-distance hard collisions. Perturbative QCD factorization theorem [3] provides a consistent method to separate the short-distance hard collision and long-distance fragmentation. The fragmentation functions, $D(z, \mu^2)$, are interpreted as the probability density to find a photon with momentum fraction $z$ from a quark or a gluon. These functions are nonperturbative quantities whose magnitude and dependence on fractional momentum $z$ must be measured by the experiments at a reference fragmentation scale $\mu_0^2$. The change of $D(z, \mu^2)$ with $\mu^2$ for large $\mu^2$ is specified by perturbative QCD evolution equations [4, 5].

The inclusive production of $e^+e^- \rightarrow \gamma X$ is an excellent process to extract such photon fragmentation functions. In $e^+e^-$ annihilations, the leading non-vanish contribution of high transverse momentum photon productions is from $e^+e^- \rightarrow q\bar{q}$ and $q$ (or $\bar{q}$) fragments into an energetic photon. Therefore, the fragmentation contributions in $e^+e^- \rightarrow \gamma X$ play a significantly greater role than they do in hadron-hadron collisions [6]. The dominant role of fragmentation contributions makes the inclusive process $e^+e^- \rightarrow \gamma X$ a potentially ideal source of information on $D(z, \mu^2)$. How to extract photon fragmentation functions from the inclusive photon cross section in $e^+e^- \rightarrow \gamma X$ is discussed in Chapter 2 [7]. Our calculations of the inclusive photon yields in $e^+e^- \rightarrow \gamma X$ are carried out through one-loop order. The results display the angular dependence of the cross sections, separated into longitudinal $\sin^2 \theta_\gamma$ and transverse components $(1 + \cos^2 \theta_\gamma)$, where $\theta_\gamma$ is the direction of the $\gamma$ with respect to the $e^+e^-$ collision axis. This work goes beyond that of previous
works [8, 9, 10, 11, 12]. The full angular dependence of the cross section including fragmentation terms is derived through $O(\alpha_s)$ for the first time.

Multiple scattering provides an extra mechanism to produce photons in nucleus-nucleus collisions. As a result, direct photon productions in nucleus-nucleus collisions may have strong dependence on nuclear size. As early as the 1970s, it was observed [13] that inclusive cross sections for single high transverse momentum particle produced in hadron-nucleus scattering show an "anomalous" nuclear dependence, in which the cross section at fixed transverse momentum grows approximately as $A^\alpha$ with $A$ as the atomic number of the nuclear target. The value of $\alpha$ is a function of transverse momentum, and can be as large as 4/3. This phenomenon has been known as the Cronin effect. The $A^{4/3}$ behavior signals a dependence on nuclear size. In a typical high transverse-momentum scattering process, energy exchange is so large that any single hard scattering should be very localized, and therefore, an almost linear $A$-dependence is expected for single scattering processes. Thus, the Cronin effect is often described as due to multiple scattering of partons in nuclear matter [14, 16, 17, 18], and the multiple scattering is primarily dominated by double scattering, due to the fact that $\alpha$ is approximately less or equal to 4/3.

Luo, Qiu, and Sterman have developed a consistent perturbative QCD treatment of double scattering [18]. This method can be applied to high transverse momentum direct photon production in hadron-nucleus collisions [19], where no quark-gluon plasma and thermal photons are expected. The derivations of the double scattering contribution in direct photon production are presented in Chapter 3. The double scattering contribution is factorized into calculable short-distance partonic parts and the multiparton correlation functions. Using the information on the multiparton cor-
relation functions, which were extracted from experiments on momentum imbalance of two-jet photoproduction on nuclear targets [18], the nuclear dependence of direct photon production is predicted without any free parameter. The numerical results are consistent with recent measurements of nuclear dependence in direct photon production from the Fermilab E706 experiment [20].

In general, double scattering in collisions with high momentum transfer should have at least one hard scattering to produce the high transverse momentum observables. In addition, there could be a soft scattering either before or after the hard scattering (called as a soft-hard process), or another hard scattering (called as a double hard process). Only the soft-hard processes contribute to the nuclear dependence of direct photon production at the order which is considered here. The fact that the photon does not interact strongly once produced at the hard collisions eliminates the final-state multiple scattering between the photon and nuclear matter. Therefore, direct photon production in hadron-nucleus scattering provides an excellent test for initial state multiple scatterings, while the jet or single particle production in photon-nucleus scattering provides an independent test for final state multiple scatterings. Final state multiple scattering in photoproduction has been discussed in Ref. [18]. This thesis provides the complementary information on the initial state multiple scattering.

Although the inclusive cross section of high energy photons is well-defined and may be calculated reliably within the context of QCD perturbation theory, it is very difficult to measure the inclusive high energy photons due to the tremendous background from $\pi^0 \rightarrow \gamma\gamma$ in high energy collisions. An experimental isolation cut is often imposed to minimize such background. However, for over six years, it has been
extensively debated whether or not the conventional perturbation theory still works for cross sections of isolated photons. The answer to this question has extremely important consequence for using the photon as a probe of short-distance physics within or beyond the Standard Model. The ability to evaluate precisely the QCD contribution to isolated photon production is essential if we are to detect any heavy neutral particles in or beyond the Standard Model, which always have a clean decay mode to a number of energetic photons.

The cross section for the isolated prompt photon production in $e^+e^-$ annihilations is presented in Chapter 4. Our calculations of the isolated photon cross section are carried out through one-loop order in QCD perturbation theory. The functional dependences on the isolation parameters are also derived. Through the derivations of the isolated photon cross section in $e^+e^-$ annihilation, we found that the conventional factorization theorem breaks down for the cross sections of isolated photons. This breakdown of factorization will have important impact on calculations of isolated photon productions at hadron colliders.

The summary and conclusions of this thesis are presented in Chapter 5.
2. PHOTON FRAGMENTATION FUNCTIONS

The inclusive process $e^+e^- \rightarrow \gamma X$ is potentially an ideal source of information on photon fragmentation functions $D_{q \rightarrow \gamma}(z, \mu^2)$ and $D_{g \rightarrow \gamma}(z, \mu^2)$. In lowest-order, the quark-to-photon and anti-quark-to-photon fragmentation processes dominate the inclusive reaction $e^+e^- \rightarrow \gamma X$ [6]. To extract the photon fragmentation functions from the inclusive process $e^+e^- \rightarrow \gamma X$, we need the analytic expressions of the inclusive photon cross section. We derived the analytic expressions for the inclusive photon yields through first order in the electromagnetic coupling strength, $\alpha_{em}$, and the quark-to-photon and gluon-to-photon fragmentation contributions through first order in the strong coupling strength $\alpha_s$. Dimensional regularization is used to handle infrared and collinear singularities. The final expressions for the inclusive photon cross section $E_{\gamma}d\sigma_{e^+e^-\rightarrow\gamma X}/d^3\ell$ have full $\theta_\gamma$ dependence, where $\theta_\gamma$ is the direction of the photon with respect to the $e^+e^-$ collision axis. Numerical results and suggestions for comparisons with $e^+e^-$ data at LEP, SLAC/SLC, TRISTAN, and CESR/CLEO energies are presented in Section 2.6 [7].
2.1 General Structure of the Cross Section in $e^+e^- \rightarrow \gamma X$

In $e^+e^- \rightarrow cX$, as sketched in Figure 2.1, the cross section for an $m$ parton final state is

$$d\sigma^{(m)} = \frac{1}{2s} |M_{e^+e^-\rightarrow c+\cdots}|^2 d\text{PS}^{(m)} \cdot dz_{c\rightarrow \gamma}(z), \quad (2.1)$$

with $c = \gamma, q, \bar{q}, g$ and $z = E_\gamma/E_c$. To obtain the inclusive photon cross section, we need to integrate over all phase space, $d\text{PS}^{(m)}$, except the momentum of parton "c".

For the scattering amplitude, $M_{\gamma^*Z^0\rightarrow c+\cdots}$, the vertex between the intermediate vector boson and the initial/final fermion pair is expressed as $ie\gamma_\mu (v_f + a_f \gamma_5)$. The absolute square of the matrix element $|\mathcal{M}|^2$, averaged over initial spins and summed over final spins and colors, may be expressed in terms of leptonic and
hadronic tensors, $L_{\mu\nu}$ and $H^{\mu\nu}$, as

$$|\mathcal{M}|^2 = e^2 C \left[ F^{PC}(q^2) L_{\mu\nu}^{PC} + F^{PV}(q^2) L_{\mu\nu}^{PV} \right] H^{\mu\nu};$$

(2.2)

where $e$ denotes the electric charge, and $C$ is the overall color factor. Since the physical observable, the energetic photon $\gamma$, does not distinguish between quarks and antiquarks, the parity violating (PV) term does not contribute. Equivalently, only the symmetric part of $H^{\mu\nu}$ contributes. Therefore,

$$|\mathcal{M}|^2 = e^2 C F^{PC}(q^2) L_{\mu\nu}^{PC} H^{\mu\nu} \equiv e^2 C F_q^{PC}(q^2) (H_1 + H_2).$$

(2.3)

$$H_1 = \left( -g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{q^2} \right) H^{\mu\nu} = -g_{\mu\nu} H^{\mu\nu}.$$ 

(2.4)

$$H_2 = -\frac{k_{\mu} k_{\nu}}{q^2} H^{\mu\nu}.$$ 

(2.5)

The four-momenta $q^\mu$ and $k^\mu$ are defined in terms of the four-momenta of the incident $e^+$ and $e^-$ ($k_1^\mu$ and $k_2^\mu$) as

$$q^\mu = k_1^\mu + k_2^\mu, \quad q^2 = (k_1 + k_2)^2 = s;$$

(2.6)

and

$$k^\mu = k_1^\mu - k_2^\mu, \quad k^2 = (k_1 - k_2)^2 = -s.$$ 

(2.7)

The normalization factor $F_q^{PC}(q^2)$ is expressed in terms of the vector ($v$) and axial-vector ($a$) couplings of the intermediate $\gamma^*$ and $Z^0$ to the leptons and quarks. At the $Z^0$ pole, neglecting $\gamma, Z^0$ interference, the normalization factor $F_q^{PC}(q^2)$ is

$$\frac{2}{s} F_q^{PC}(s) = \left( |v_e|^2 + |a_e|^2 \right) \left( |v_q|^2 + |a_q|^2 \right) \frac{1}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2}.$$ 

(2.8)
Table 2.1: Electroweak V-A coupling constants

<table>
<thead>
<tr>
<th>$v_e$</th>
<th>$\left(-1 + 4\sin^2 \theta_w\right) / (2\sin 2\theta_w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_e$</td>
<td>$1 / (2\sin 2\theta_w)$</td>
</tr>
<tr>
<td>$v_q$</td>
<td>$\left(t_3^q - 2\epsilon_q \sin^2 \theta_w\right) / (\sin 2\theta_w)$</td>
</tr>
<tr>
<td>$a_q$</td>
<td>$-t_3^q / \sin 2\theta_w$</td>
</tr>
</tbody>
</table>

Table 2.2: Isospin and fractional charges for quarks

<table>
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<tr>
<th></th>
<th>$t_3^q$</th>
<th>$\epsilon_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u, c, t$</td>
<td>$1/2$</td>
<td>$2/3$</td>
</tr>
<tr>
<td>$d, s, b$</td>
<td>$-1/2$</td>
<td>$-1/3$</td>
</tr>
</tbody>
</table>

The vector ($v$) and axial-vector ($a$) couplings in Eq. (2.8) are provided in Table 2.1 and Table 2.2. At modest energies where only the $\gamma^*$ intermediate state is relevant,

$$\frac{2}{s} F_q^{PC} (s) = \epsilon_q \frac{1}{s^2};$$

(2.9)

$\epsilon_q$ is the fractional quark charge ($\epsilon_u = 2/3; \epsilon_d = 1/3; \cdots$).

In terms of functions $H_1$ and $H_2$, defined through Eq. (2.3), we reexpress the cross section as

$$d\sigma^{(m)} = \sum_q \left[ \frac{2}{s} F_q^{PC} (s) \right] e^2 C \frac{1}{4} (H_1 + H_2) d\sigma^{(m)} dz D(z).$$

(2.10)

In the following sections, we calculate the functions $H_1$ and $H_2$ defined in Eqs. (2.4) and (2.5), respectively, at the lowest order and the first-order in $\alpha_s$. 
2.2 Factorization and Lowest Order Contribution

2.2.1 Factorized Cross Section

We are interested in the inclusive cross section for production of photons in association with hadrons, \( E_\gamma d^2\sigma^{incl}_{e^+e^-\rightarrow\gamma X}/d^3\ell \), where \( E_\gamma \) is the energy of the photon, and \( \ell \) is the momentum of the photon in the \( e^+e^- \) center-of-mass system. According to the pQCD factorization theorem [3], the cross section may be expressed as

\[
E_\gamma\frac{d^2\sigma^{incl}_{e^+e^-\rightarrow\gamma X}}{d^3\ell} = \sum_c E_c \frac{d^2\sigma^{incl}_{e^+e^-\rightarrow eX}}{d^3p_c} \otimes D_{c\rightarrow\gamma}(z). \tag{2.11}
\]

The intermediate partons \( c = \gamma, g, q, \) and \( \bar{q} \). The hard-scattering cross section \( E_c d^2\sigma^{incl}_{e^+e^-\rightarrow eX}/d^3p_c \) contains no infrared or collinear divergences. The fractional momentum \( z \) is defined as \( z = E_\gamma/E_c \); all intermediate partons \( c \) are assumed to be massless. The fragmentation functions \( D_{c\rightarrow\gamma}(z) \) represent all long-distance physics associated with the hadronic component of the photon. They are inherently non-perturbative quantities that must be measured experimentally. Models and phenomenological parameterizations [4] for \( D(z) \) have been published. In lowest-order, \( D_{\gamma\rightarrow\gamma}(z) = \delta(1 - z) \). The convolution expressed in Eq. (2.11) is sketched in Figure 2.2. In the figure, \( e \) denotes an intermediate photon or a gluon or a quark of any flavor. The convolution symbol \( \otimes \) in Eq. (2.11) is defined explicitly as follows:

\[
E_c \frac{d^2\sigma^{incl}_{e^+e^-\rightarrow eX}}{d^3p_c} \otimes D_{c\rightarrow\gamma}(z) = \int_{z_{\text{min}}}^{1} \frac{dz}{z^2} \left[ E_c \frac{d^2\sigma^{incl}_{e^+e^-\rightarrow eX}(E_c = \frac{E_\gamma}{z})}{d^3p_c} \right] D_{c\rightarrow\gamma}(z). \tag{2.12}
\]

Since \( z_{\text{min}} \) occurs when \( p_c \) has its maximum value, \( p_{c\text{max}} = \sqrt{s}/2 \), the lower limit of integration \( z_{\text{min}} = x_\gamma = 2E_\gamma/\sqrt{s} \); \( \sqrt{s} \) is the center of mass energy of the \( e^+e^- \).
2.2.2 The Lowest Order Contribution

The lowest order contribution to the inclusive photon yield is sketched in Figure 2.3. The differential inclusive cross section \( d\sigma_{e^+e^-\rightarrow\gamma X} \) is expressed as a product of the lowest order partonic cross section \( d\hat{\sigma}_{e^+e^-\rightarrow\gamma qq}^{(o)} \) and the \( q\rightarrow\gamma \) fragmentation function, \( D_{q\rightarrow\gamma}(z) \).

\[
d\sigma_{e^+e^-\rightarrow\gamma X} = \sum_q d\hat{\sigma}_{e^+e^-\rightarrow\gamma qq}^{(o)} \frac{dz}{\ell/p_q} D_{q\rightarrow\gamma}(z) + (q\rightarrow\bar{q}). \tag{2.13}
\]

In Eq. (2.13), \( p_q \) is the four-vector momentum of the quark \( q \), and \( z = \ell/p_q \). The partonic cross section is written in terms of the invariant matrix element and
Figure 2.3: Lowest order, $O(\alpha^2_{\text{em}} \alpha_s^0)$, photon production through quark fragmentation.

differential phase space factor,

$$
\frac{d\hat{\sigma}}{d^3p}^{(o)} e^+ e^- \rightarrow p q \bar{p} \bar{q} = \frac{1}{2s} |M_{e^+ e^- \rightarrow p q \bar{p} \bar{q}}|^2 dPS^{(2)}
$$

$$
= \left[ \frac{2}{s} F_{q}^{P C}(s) \right] e^2 N_c \frac{1}{4} (H_1 + H_2) dPS^{(2)} ,
$$

(2.14)

where $N_c = 3$ is the number of colors carried by the quarks, and Eq. (2.3) was used.

The symmetric part of the hadronic tensor $H^{\mu \nu}$, used to define functions $H_1$ and $H_2$, is particularly simple:

$$
H^{\mu \nu} = 4 (e \mu^\epsilon)^2 \left[ p_q^\mu p_{\bar{q}}^\nu + p_{\bar{q}}^\mu p_q^\nu - g^{\mu \nu} p_q \cdot p_{\bar{q}} \right].
$$

(2.15)

The factor $\mu^\epsilon$ in Eq. (2.15) accommodates the fact that we are working in $n$ dimensions. The dimensional scale $\mu$ will be specified further below. The functions $H_1$ and $H_2$, defined in Section 2.1, become

$$
H_1 = 4 (e \mu^\epsilon)^2 s (1 - \epsilon) ;
$$

(2.16a)
$$H_2 = -2 \left( e \mu^2 \right)^2 s \left( 1 - \cos^2 \theta \right). \quad (2.16b)$$

In Eq. (2.15), $\epsilon$ is defined through the number of space dimensions $n = 4 - 2\epsilon$, with $\epsilon \to 0$ at the end of the calculation. In the center of mass frame of the collision, $\theta$ is the angle of $\vec{p}_q$ with respect to the direction defined by the incident $e^+$. Combining $H_1$ and $H_2$, we obtain

$$\frac{1}{4} (H_1 + H_2) = \frac{1}{2} \left( e \mu^2 \right)^2 s \left[ (1 + \cos^2 \theta) - 2\epsilon \right]. \quad (2.17)$$

Combining Eq. (2.17) and the expression for two-particle phase space in $n$-dimensions, Eq. (A.4) of the Appendix, we can find that the lowest order partonic cross section, Eq. (2.14), is

$$\frac{d\sigma^{(0)}}{d^3pq} = \left[ 2 \frac{2F_q^{PC}(s)}{s} \right] \alpha_s^2 \alpha_e N_c \left( \frac{\Gamma(1 - \epsilon)}{(s/4) \sin^2 \theta} \right) \frac{1}{\Gamma(1 - \epsilon)} \times \left[ (1 + \cos^2 \theta) - 2\epsilon \right] \delta(x_q - 1), \quad (2.18)$$

with $x_q = 2E_q/\sqrt{s}$. At this order, the cross section is manifestly finite in the limit $\epsilon \to 0$, and we may set $\epsilon = 0$ directly in Eq. (2.18). Nevertheless, Eq. (2.18) expressed in $n$ dimensions is valuable for later comparison with the higher order cross section.

Noting that $\ell = zp_q$ implies $d^3pq/E_q = (1/z^2) d^3\ell/E_\gamma$, we obtain the lowest order inclusive cross section

$$\frac{d\sigma^{incl}}{d^3\ell} = 2 \sum_q \int_2^1 \frac{dz}{z^2} \left[ \frac{d\sigma^{(0)}}{d^3pq} \left( x_q = \frac{x_\gamma}{z} \right) \right] D_q \to \gamma(z, \mu_F)$$

$$= 2 \sum_q \left[ 2 \frac{2F_q^{PC}(s)}{s} \right] \alpha_s^2 \alpha_e N_c (1 + \cos^2 \theta) \frac{1}{x_\gamma} D_q \to \gamma(x_\gamma, \mu_F). \quad (2.19)$$
The angles $\theta_\gamma$ and $\theta$ are identical since we assume that the fragmentation from quark to photon is collinear. The overall factor of 2 in Eq. (2.19) accounts for the $\bar{q}$ contribution, with assumption $D_q \rightarrow \gamma(z) = D_{\bar{q}} \rightarrow \gamma(z)$. In Eq. (2.19), we have introduced a fragmentation scale $\mu_F$ in the specification of the fragmentation function.

### 2.3 Factorized Formula for First Order Contributions

There are three distinct contributions to $e^+e^- \rightarrow \gamma X$ in first order perturbation theory:

\[ e^+e^- \rightarrow \gamma, \quad O(\alpha_{em}) \]  
\[ e^+e^- \rightarrow q \text{ (or } \bar{q} \text{)} \rightarrow \gamma, \quad O(\alpha_s) \]  
\[ e^+e^- \rightarrow g \rightarrow \gamma, \quad O(\alpha_s) \]  

Eqs. (2.20b) and (2.20c) are contributions from quark and gluon fragmentation to photons in the three-parton final state process $e^+e^- \rightarrow q\bar{q}g$. The first contribution, Eq. (2.20a), arises from $e^+e^- \rightarrow q\bar{q}\gamma$ where the $\gamma$ is not collinear with either $\bar{q}$ or $q$.

Following the pQCD factorization theorem, and Eq. (2.11), in order to derive the explicit contributions to the inclusive yield $E_\gamma d\sigma^{incl}_{e^+e^- \rightarrow \gamma X} / d^3 \ell$ from each of the three processes in Eq. (2.20), we must calculate the short-distance hard-scattering cross sections, $E_c d\sigma^{incl}_{e^+e^- \rightarrow cX} / d^3 p_c$ for $c = \gamma, g, q$ and $\bar{q}$.

The Feynman graphs for $e^+e^- \rightarrow \gamma qq\bar{q}$ are sketched in Figure 2.4. Owing to the quark-photon collinear divergence, the cross section associated with these graphs is formally divergent. We denote this first order divergent cross-section $\sigma^{(1)}_{e^+e^- \rightarrow \gamma X}$, a short-hand notation for $E d\sigma^{(1)} / d^3 \ell$. To derive the corresponding short-distance
hard-scattering cross section, \( \hat{\sigma}_{e^+e^-\rightarrow \gamma X}^{(1)} \), we apply the factorized form, Eq. (2.11), perturbatively,

\[
\sigma_{e^+e^-\rightarrow \gamma X}^{(1)} = \hat{\sigma}_{e^+e^-\rightarrow \gamma X}^{(1)} \otimes D_{\gamma\rightarrow \gamma}^{(0)}(z) \\
+ \hat{\sigma}_{e^+e^-\rightarrow qX}^{(0)} \otimes D_{q\rightarrow \gamma}^{(1)}(z) \\
+ (q \rightarrow \bar{q}).
\] (2.21)

The convolution represented by \( \otimes \) is defined in Eq. (2.12). The superscripts (0) and (1) on the hard-scattering cross sections \( \hat{\sigma} \) and fragmentation functions \( D \) refer to lowest-order and first order, respectively. The collinear divergence resides in the first order fragmentation function \( D_{q\rightarrow \gamma}^{(1)}(z) \). The hard-scattering cross sections \( \hat{\sigma}_{e^+e^-\rightarrow \gamma X}^{(1)} \) and \( \hat{\sigma}_{e^+e^-\rightarrow qX}^{(0)} \) are finite. The expression for \( \hat{\sigma}_{e^+e^-\rightarrow \gamma X}^{(0)} \) was derived in Section 2.2. We can obtain \( \hat{\sigma}_{e^+e^-\rightarrow \gamma X}^{(1)} \):

\[
\hat{\sigma}_{e^+e^-\rightarrow \gamma X}^{(1)} = \sigma_{e^+e^-\rightarrow \gamma X}^{(1)} - \hat{\sigma}_{e^+e^-\rightarrow qX}^{(0)} \otimes D_{q\rightarrow \gamma}^{(1)}(z) - (q \rightarrow \bar{q}) \text{,} 
\] (2.22)

by calculating \( \sigma_{e^+e^-\rightarrow \gamma X}^{(1)} \) and \( D_{q\rightarrow \gamma}^{(1)}(z) \) in \( n \)-dimension. The derivation of \( \hat{\sigma}_{e^+e^-\rightarrow \gamma X}^{(1)} \) is given in Section 2.4.

The two Feynman graphs that provide the cross section for \( e^+e^- \rightarrow g \rightarrow \gamma \) in \( O(\alpha_s) \) are shown in Figure 2.5. In this case, the final gluon is effectively “observed”
through the fragmentation $g \to \gamma$; there are no virtual gluon exchange diagrams. The finite hard-scattering cross section $\hat{\sigma}^{(1)}_{e^+e^- \to gX}$ is derived from the difference

$$
\hat{\sigma}^{(1)}_{e^+e^- \to gX} = \sigma^{(1)}_{e^+e^- \to gX} - \sum_{q' = q} \hat{\sigma}^{(0)}_{e^+e^- \to q'X} \otimes D^{(1)}_{q' \to g}.
$$

In Eq. (2.23), the divergent cross section $\sigma^{(1)}_{e^+e^- \to gX}$ is evaluated from the Feynman graphs shown in Figure 2.5, and the quark-to-gluon collinear divergences are embedded in the first-order fragmentation function $D^{(1)}_{q' \to g}$. The derivation of $\hat{\sigma}^{(1)}_{e^+e^- \to gX}$ is presented in Section 2.5.

The Feynman graphs in $O(\alpha_s)$ that contribute to $e^+e^- \to q \to \gamma$ (Eq. (2.20b)) are sketched in Figure 2.6. A final state photon from quark fragmentation is observed. The complete $O(\alpha_s)$ result includes both real gluon emission and virtual gluon exchange graphs, as shown in Figure 2.6. Although infrared divergences associated with soft gluons cancel between the real and virtual graphs, the cross section $\sigma^{(1)}_{e^+e^- \to qX}$ obtained from the Feynman graphs is still divergent due to collinear singularities when the real gluon is emitted along the direction of its parent quark or antiquark. To obtain the corresponding hard-scattering cross section $\hat{\sigma}^{(1)}_{e^+e^- \to qX}$, we apply the factorized form, Eq. (2.11), perturbatively, to the production of a quark.
Figure 2.6: Contributions to the $O(\alpha_s)$ cross section
\[ \sigma^{(1)}_{e^+e^-\rightarrow qX} \] (a) real gluon emission diagrams ($e^+e^-\rightarrow q\bar{q}g$), (b) virtual gluon exchange diagrams that interfere with the lowest order tree diagram.

Instead of the photon,
\[ \sigma^{(1)}_{e^+e^-\rightarrow qX} = \sum_{q'=q} \sigma^{(0)}_{e^+e^-\rightarrow q'} \otimes D^{(1)}_{q'\rightarrow q} + \sigma^{(1)}_{e^+e^-\rightarrow qX} \otimes D^{(0)}_{q\rightarrow q}, \] (2.24)

with the collinear $q' \rightarrow q$ singularities in $O(\alpha_s)$ included in $D^{(1)}_{q'\rightarrow q}$. Note that $D^{(0)}_{q\rightarrow q}(z) = \delta(1-z)$. Correspondingly, the finite hard-scattering cross section $\hat{\sigma}^{(1)}_{e^+e^-\rightarrow qX}$ is
\[ \hat{\sigma}^{(1)}_{e^+e^-\rightarrow qX} = \sigma^{(1)}_{e^+e^-\rightarrow qX} - \sigma^{(0)}_{e^+e^-\rightarrow q} \otimes D^{(1)}_{q\rightarrow q}. \] (2.25)
A detailed derivation of $\sigma^{(1)}_{e^+e^-\rightarrow qX}$ is given in Section 2.5.

The two-loop direct contribution to $e^+e^-\rightarrow \gamma X$ is of $O(\alpha_s^4E)$ and can be derived as follows. First, apply the factorized form, Eq. (2.11), perturbatively, at two-loop level and sum over $c = \gamma, g, q$ and $\bar{q}$,

$$\sigma^{(2)}_{e^+e^-\rightarrow \gamma X} = \sigma^{(2)}_{e^+e^-\rightarrow \gamma X} \otimes D^{(0)}_{\gamma\rightarrow \gamma}(z) + \sigma^{(1)}_{e^+e^-\rightarrow \gamma X} \otimes D^{(1)}_{\gamma\rightarrow \gamma}(z)
+ \sigma^{(1)}_{e^+e^-\rightarrow gX} \otimes D^{(1)}_{g\rightarrow \gamma}(z) + \sigma^{(0)}_{e^+e^-\rightarrow gX} \otimes D^{(2)}_{g\rightarrow \gamma}(z)
+ \sigma^{(1)}_{e^+e^-\rightarrow qX} \otimes D^{(1)}_{q\rightarrow \gamma}(z) + \sigma^{(0)}_{e^+e^-\rightarrow qX} \otimes D^{(2)}_{q\rightarrow \gamma}(z)
+ (q \rightarrow \bar{q}).$$  \hspace{1cm} (2.26)

All first-order contributions, $\sigma^{(1)}$, in Eq. (2.26) are defined in Eqs. (2.22), (2.23) and (2.25), and will be calculated in the following sections. Since the first order fragmentation functions $D^{(1)}_{\gamma\rightarrow \gamma}(z)$ and $D^{(1)}_{g\rightarrow \gamma}(z)$ vanish, and the zeroth order hard-scattering cross section $\sigma^{(0)}_{e^+e^-\rightarrow \gamma X}$ vanishes, the two-loop hard-scattering cross sections $\sigma^{(2)}_{e^+e^-\rightarrow \gamma X}$ can be derived as:

$$\sigma^{(2)}_{e^+e^-\rightarrow \gamma X} = \sigma^{(2)}_{e^+e^-\rightarrow \gamma X}
- \sigma^{(1)}_{e^+e^-\rightarrow qX} \otimes D^{(1)}_{q\rightarrow \gamma}(z) - \sigma^{(0)}_{e^+e^-\rightarrow qX} \otimes D^{(2)}_{q\rightarrow \gamma}(z)
- (q \rightarrow \bar{q}).$$ \hspace{1cm} (2.27)

To complete the calculation of $\sigma^{(2)}_{e^+e^-\rightarrow \gamma X}$, it is necessary to calculate the two-loop parton-level cross section $\sigma^{(2)}_{e^+e^-\rightarrow \gamma X}$ and the two-loop quark-to-photon fragmentation function $D^{(2)}_{q\rightarrow \gamma}(z)$ in $n$-dimensions (implicitly, dimensional regularization is used), in addition to all the zeroth and first order contributions calculated in this chapter. The two-loop parton level cross section $\sigma^{(2)}_{e^+e^-\rightarrow \gamma X}$ is formally divergent.
As is true of the calculation of \( \hat{\sigma}^{(1)}_{e^+e^-\rightarrow qX} \) in Section 2.5, all infrared divergences associated with soft gluons cancel among the real emission and virtual exchange diagrams. All collinear divergences that appear when final-state quarks and/or gluons are parallel to the observed photon are cancelled by the subtraction terms given in Eq. (2.27). Consequently, the two-loop hard-scattering cross section \( \hat{\sigma}^{(2)}_{e^+e^-\rightarrow \gamma X} \) is finite if the pQCD factorization theorem holds.

As shown in Section 2.6, the leading order short-distance direct production contribution \( \hat{\sigma}^{(1)}_{e^+e^-\rightarrow \gamma X} \) is much smaller than the leading order fragmentation contribution \( \hat{\sigma}^{(0)}_{e^+e^-\rightarrow qX} \otimes D_{q\rightarrow \gamma}(z) + (q \leftrightarrow \bar{q}) \). The next-to-leading order direct contribution \( \hat{\sigma}^{(2)}_{e^+e^-\rightarrow \gamma X} \) will be much smaller than the next-to-leading order fragmentation contributions \( \hat{\sigma}^{(1)}_{e^+e^-\rightarrow cX} \otimes D_{c\rightarrow \gamma}(z) \) with \( c = g, q \) and \( \bar{q} \), which are completely derived in this chapter. The two-loop contributions are not calculated because their contributions to the overall cross section are much too small in comparison with those presented here.

### 2.4 Hard Parts for Leading Order Direct Contribution

To calculate the finite hard-scattering cross section \( \hat{\sigma}^{(1)}_{e^+e^-\rightarrow \gamma X} \) in \( O(\alpha_{em}) \), we first need to compute the functions \( H_1 \) and \( H_2 \), defined in Section 2.1, Eqs. (2.4) and (2.5). These will then be integrated over phase space to yield the cross section

\[
E\gamma d\sigma^{(1)}_{e^+e^-\rightarrow \gamma X} / d^3 \ell := \sum_q \left[ \frac{2}{s} F_q^{PC}(s) \right] e^2 N_c \frac{1}{4} (H_1 + H_2) dPS^{(3)},
\]

where three-particle phase space in \( n \)-dimensions is given in Eq. (A.34) of the Appendix.
Sketched in Figure 2.7 is the hadronic tensor $H_{\mu\nu}$ obtained from the two diagrams of Figure 2.4. Performing traces to sum over final spins, we may write the four contributions as

\begin{align*}
H_{\mu\nu}^{(a)} &= 2(1 - \epsilon) Tr \left[ \gamma \mu \gamma \cdot \ell \gamma \nu \gamma \cdot p_2 \right] \frac{1}{2p_1 \cdot \ell} \\
H_{\mu\nu}^{(b)} &= 2(1 - \epsilon) Tr \left[ \gamma \mu \gamma \cdot p_1 \gamma \nu \gamma \cdot \ell \right] \frac{1}{2p_2 \cdot \ell} \\
H_{\mu\nu}^{(c)} &= -2 Tr \left[ \gamma \mu \gamma \cdot p_1 \gamma \cdot p_2 \gamma \nu \gamma \cdot (p_1 + \ell) \gamma \cdot (p_2 + \ell) \right] \frac{1}{2p_1 \cdot \ell} \frac{1}{2p_2 \cdot \ell} \\
&+ 2\epsilon Tr \left[ \gamma \mu \gamma \cdot p_1 \gamma \cdot \ell \gamma \nu \gamma \cdot p_2 \gamma \cdot \ell \right] \frac{1}{2p_1 \cdot \ell} \frac{1}{2p_2 \cdot \ell} \\
H_{\mu\nu}^{(d)} &= -2 Tr \left[ \gamma \mu \gamma \cdot (p_1 + \ell) \gamma \cdot (p_2 + \ell) \gamma \nu \gamma \cdot p_1 \gamma \cdot p_2 \right] \frac{1}{2p_1 \cdot \ell} \frac{1}{2p_2 \cdot \ell} \\
&+ 2\epsilon Tr \left[ \gamma \mu \gamma \cdot \ell \gamma \cdot p_1 \gamma \nu \gamma \cdot \ell \gamma \cdot p_2 \right] \frac{1}{2p_1 \cdot \ell} \frac{1}{2p_2 \cdot \ell}.
\end{align*}

(2.29)

To avoid multiple repetition of a common factor, we temporarily omit the overall

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2-7.png}
\caption{Order $O(\alpha_{em}^2)$ contribution to the hadronic tensor $H_{\mu\nu}$.}
\end{figure}
coupling factor $\varepsilon^2 (e\mu^\epsilon)^4$ that appears in $H_{\mu\nu}$. We obtain

$$H_1 = -g_{\mu\nu} H^{\mu\nu} = -g_{\mu\nu} \sum_{i=a}^{d} H(i)^{\mu\nu}.$$  

We obtain

$$H_1 = 8(1 - \epsilon) \left\{ (1 - \epsilon) \left[ \frac{y_1 \ell}{y_2 \ell} + \frac{y_2 \ell}{y_1 \ell} \right] + \frac{2y_{12}}{y_1 \ell y_2 \ell} - 2\epsilon \right\}.$$  

(2.30)

The dimensionless quantities $y_1 \ell, y_2 \ell$, and $y_{12}$ are defined by

$$y_{i\ell} = \frac{2p_i \cdot \ell}{q^2} \quad (i = 1, 2);$$

$$y_{12} = \frac{2p_1 \cdot p_2}{q^2}.$$  

(2.31)

Here we have $y_{12} + y_1 \ell + y_2 \ell = 1$. In evaluating $H_2 = -\left( k \mu k_{\nu}/q^2 \right) H^{\mu\nu}$, we also make use of dimensionless quantities $y_{1k}, y_{2k},$ and $y_{k\ell}$:

$$y_{ik} = \frac{2p_i \cdot k}{q^2} \quad (i = 1, 2);$$

$$y_{k\ell} = \frac{2k \cdot \ell}{q^2}.$$  

(2.32)

Because $k \cdot q = 0$,

$$y_{1k} + y_{2k} + y_{k\ell} = 0.$$  

(2.33)

After some algebra we find

$$H_2 = -4 \left\{ (1 - \epsilon) \left[ \frac{y_1 \ell}{y_2 \ell} + \frac{y_2 \ell}{y_1 \ell} \right] + \frac{2y_{12}}{y_1 \ell y_2 \ell} - 2\epsilon \right\} + \frac{4}{y_1 \ell y_2 \ell} \left\{ y_{1k}^2 + y_{2k}^2 \right\} - \frac{4\epsilon}{y_1 \ell y_2 \ell} \left\{ y_{k\ell}^2 \right\}.$$  

(2.34)

The next task is to integrate $H_1$ and $H_2$ over three-body phase space in $n = 4 - 2\epsilon$ dimensions. Since the momentum of the photon ($\ell$) is an observable, and the momentum of either the quark ($p_1$) or antiquark ($p_2$) can be fixed by the overall momentum conservation $\delta$-function in the three-body phase space, we need to integrate over only
$p_1$ or $p_2$. In the following discussion, we let $p_2$ be fixed by the $\delta$-function, and we integrate over $p_1$. In the overall center of mass frame, as sketched in Figure 2.8, We choose the $z$-axis to be the direction of the observed photon. We take angle $\theta_\gamma$ to be the polar angle of the $\gamma$ with respect to the $e^+e^-$ collision axis and angle $\theta_{1\gamma}$ to be the angle between the $\gamma$'s momentum $\ell$ and the quark momentum $p_1$. The angle

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2_8}
\caption{Center of mass coordinate axes of an $e^+e^-$ collision.}
\end{figure}

$\theta_x$ is the $n$-dimensional generalization of the three-dimensional azimuthal angle $\phi$, defined through $p_1$ as

$$d\Omega_{n-2}(p_1) \equiv d\theta_1\gamma \sin^{n-3}\theta_{1\gamma} d\theta_x \sin^{n-4}\theta_x d\Omega_{n-4}(p_1).$$

Equation (2.35)

Having chosen the frame, we may reexpress the $y$ variables in terms of observables and integration angles as follows:

$$y_k \ell = -\gamma \cos \theta_\gamma,$$
\[ y_{1k} = -\left[\frac{y_{2k}y_{12} - y_{1k}}{x\gamma}\right] \cos \theta_\gamma - \left[\frac{2\sqrt{y_{12}y_{1k}y_{2k}}}{x\gamma}\right] \sin \theta_\gamma \cos \theta_x ; \]
\[ y_{2k} = -\left[\frac{y_{1k}y_{12} - y_{2k}}{x\gamma}\right] \cos \theta_\gamma + \left[\frac{2\sqrt{y_{12}y_{1k}y_{2k}}}{x\gamma}\right] \sin \theta_\gamma \cos \theta_x ; \quad (2.36) \]

where \( x_\gamma = 2E_\gamma/\sqrt{s} = (y_{1k} + y_{2k}) \). In deriving \( y_{1k} \) and \( y_{2k} \), we use the following identities

\[ \cos \theta_{1\gamma} = \frac{y_{2k}y_{12} - y_{1k}}{x_1 x_\gamma} ; \]
\[ \sin \theta_{1\gamma} = \frac{2\sqrt{y_{12}y_{1k}y_{2k}}}{x_1 x_\gamma} , \quad (2.37) \]

where \( x_1 = 2E_1/\sqrt{s} \).

In the integration of \( H_2 \) over phase space, the integral over \( d \cos \theta_x \) is done from \( \cos \theta_x = -1 \) to \(+1\). The expression for the three-body phase space, Eq. (A.28), is an even function of \( \cos \theta_x \). Correspondingly, terms in \( H_2 \) that are odd functions of \( \cos \theta_x \) do not survive. Because \( H_2 \) depends only on the square of the \( y_{1k} \) and \( y_{2k} \), after eliminating all terms linear in \( \cos \theta_x \), we find that the only \( \theta_x \) dependence in \( H_2 \) is \( \cos^2 \theta_x \). We can integrate over \( \theta_x \) independent of other variables, or we can effectively replace the \( \cos^2 \theta_x \) terms in \( H_2 \) by the average of \( \cos^2 \theta_x \) in \( n \)-dimensions and eliminate the \( \theta_x \) dependence in \( H_2 \) completely.

Given the average of \( \cos^2 \theta_x \) in \( n \)-dimensions, Eq. (A.31), we obtain, effectively,

\[ y_{1k}^2 = x_\gamma^2 \cos^2 \theta_\gamma ; \]
\[ y_{1k}^2 = \left[\frac{y_{2k}y_{12} - y_{1k}}{x\gamma}\right]^2 \cos^2 \theta_\gamma + \left(\frac{1}{1 - \epsilon}\right) \left[\frac{2(y_{12}y_{1k}y_{2k})}{x_\gamma^3}\right] \sin^2 \theta_\gamma ; \]
\[ y_{2k}^2 = \left[\frac{y_{1k}y_{12} - y_{2k}}{x\gamma}\right]^2 \cos^2 \theta_\gamma + \left(\frac{1}{1 - \epsilon}\right) \left[\frac{2(y_{12}y_{1k}y_{2k})}{x_\gamma^3}\right] \sin^2 \theta_\gamma ; \quad (2.38) \]

where the factor \( 1/(1 - \epsilon) \) is from the average of \( \cos^2 \theta_x \). Substituting the above
expressions into Eq. (2.34), and combining with \( H_1 \) in Eq. (2.30), we obtain,
\[
\frac{1}{4} (H_1 + H_2^{eff}) = \left(1 + \cos^2 \theta \gamma - 2\epsilon\right) \left[1 - \epsilon \left(\frac{y_{1\ell} y_{2\ell}}{y_{1\ell} + y_{2\ell}}\right) + 2 \left(\frac{y_{12}}{y_{1\ell} y_{2\ell} - \epsilon}\right)\right] \\
+ \left(1 - 3 \cos^2 \theta \gamma\right) \left[\frac{4 y_{12}}{x^2 \gamma}\right] \\
+ \left(\frac{\epsilon}{1 - \epsilon}\right) \left(1 - \cos^2 \theta \gamma\right) \left[\frac{4 y_{12}}{x^2 \gamma}\right],
\tag{2.39}
\]
where the superscript "eff" indicates that we have replaced \( \cos^2 \theta_x \) by its average in \( n \)-dimensions. The two \( \delta \)-functions in the three-particle phase space, \( dPS(3) \), provide the following identities
\[
y_{12} = 1 - x \gamma; \\
y_{2\ell} = x \gamma - y_{1\ell}.
\tag{2.40}
\]
Introducing \( \hat{y}_{1\ell} = y_{1\ell}/x \gamma \), and substituting these identities into Eq. (2.39), we derive
\[
\frac{1}{4} (H_1 + H_2^{eff}) = \left(1 + \cos^2 \theta \gamma - 2\epsilon\right) \left[1 + \frac{1 - x \gamma}{x^2 \gamma}\right] \left[\frac{1}{\hat{y}_{1\ell}} + \frac{1}{1 - \hat{y}_{1\ell}}\right] \\
+ \left(1 + \cos^2 \theta \gamma - 2\epsilon\right) \left[-2 - \epsilon \left(\frac{1}{\hat{y}_{1\ell}} + \frac{1}{1 - \hat{y}_{1\ell}}\right)\right] \\
+ \left(1 - 3 \cos^2 \theta \gamma\right) \left[\frac{4(1 - x \gamma)}{x^2 \gamma}\right] \\
+ \left(\frac{\epsilon}{1 - \epsilon}\right) \left(1 - \cos^2 \theta \gamma\right) \left[\frac{4(1 - x \gamma)}{x^2 \gamma}\right].
\tag{2.41}
\]
The last term vanishes as \( \epsilon \to 0 \).

Combining Eqs. (2.28) and (2.41), and integrating over \( d\hat{y}_{1\ell} \), we can derive the partonic cross section \( d\sigma^{(1)}_{e^+ e^- \to \gamma X} \). The limits of the \( d\hat{y}_{1\ell} \) integration are from 0 to 1. The integrals over \( d\hat{y}_{1\ell} \) for \( (H_1 + H_2^{eff})/4 \) may be expressed in terms of
\[
I_{n,m} = \int_0^1 d\hat{y}_{1\ell} \hat{y}_{1\ell}^{n-\epsilon} (1 - \hat{y}_{1\ell})^{m-\epsilon}.
\tag{2.42}
\]
Examining Eq. (2.41), we need only \( I_{0,0} \) and \( I_{-1,0} \) (= \( I_{0,-1} \)):

\[
I_{0,0} = B(1 - \epsilon, 1 - \epsilon) = \frac{(\Gamma(1 - \epsilon))^2}{\Gamma(2 - 2\epsilon)};
\]

\[
I_{-1,0} = B(-\epsilon, 1 - \epsilon) = \left(\frac{1}{-\epsilon}\right) \frac{(\Gamma(1 - \epsilon))^2}{\Gamma(1 - 2\epsilon)},
\]

where \( B(a, b) \) and \( \Gamma(a) \) are Beta function and Gamma function, respectively. For small \( \epsilon \), \( I_{0,0} = 1 + O(\epsilon) \), and \( I_{-1,0} = -\frac{1}{\epsilon} + O(\epsilon) \).

After performing the integration over \( d\gamma \), we expand the right-hand side of Eq. (2.28) in a power series in \( \epsilon \), keeping only the singular term proportional to \( 1/\epsilon \) and the terms independent of \( \epsilon \). (Terms of \( O(\epsilon^m) \), \( m > 1 \), vanish in the physical limit of four dimensions \( n = 4 - 2\epsilon \)). We obtain

\[
E\gamma \frac{d\sigma^{(1)}}{d^3\ell} \rightarrow \gamma X = 2 \sum_q \left[ \frac{2}{s} F_q^{PC}(s) \right] \left[ \alpha_{em}^2 N_c \left( \frac{4\pi\mu^2}{(s/4)\sin^2\theta_\gamma} \right) \frac{\epsilon}{\Gamma(1 - \epsilon)} \right]
\]

\[
\times \left( 1 + \cos^2\theta_\gamma - 2\epsilon \right) \frac{1}{x_\gamma} \left\{ e_q \left( \frac{\alpha_{em}}{2\pi} \left[ \frac{1 + (1 - x_\gamma)^2}{x_\gamma} \right] \right) \right\} \left( -\frac{1}{\epsilon} \right)
\]

\[
+ 2 \sum_q \left[ \frac{2}{s} F_q^{PC}(s) \right] \left[ \alpha_{em}^2 N_c \frac{1}{x_\gamma} \right] e_q \left( \frac{\alpha_{em}}{2\pi} \right)
\]

\[
\times \left\{ (1 + \cos^2\theta_\gamma) \left[ \frac{1 + (1 - x_\gamma)^2}{x_\gamma} \right] \left[ \ln \left( s/\mu_{\text{MS}}^2 \right) \right] 
\right\}
\]

\[
+ \ln \left( x_\gamma^2 \left( 1 - x_\gamma \right) \right)]
\]

\[
+ (1 - 3\cos^2\theta_\gamma) \left[ \frac{2(1 - x_\gamma)}{x_\gamma} \right].
\]

In deriving Eq. (2.45), we included the overall factor for coupling constants, \( e_q^2 (e\mu^4) \); and used the expansion \( \Gamma(1 - \epsilon) \simeq 1 + \epsilon\gamma_E \), where \( \gamma_E \) is Euler's constant, and the usual modified minimal subtraction scale

\[
\mu_{\text{MS}}^2 \equiv \mu^2 4\pi e^{-\gamma_E}.
\]
The $(1/\epsilon)$ singularity in Eq. (2.45) represents the quark-photon collinear singularity. This singular term is expected to be cancelled by subtraction terms defined in Eq. (2.22). By evaluating the diagram sketched in Figure 2.9, we obtain the one-loop quark-to-photon fragmentation function

$$D_{q \rightarrow \gamma}^{(1)}(z) = D_{q \rightarrow \gamma}^{(1)}(z) = e_q^2 \frac{\alpha_{em}}{2\pi} \left[ \frac{1 + (1 - z)^2}{z} \right] \left( \frac{1}{1 - \epsilon} \right),$$

(2.47)

where we keep only the $1/\epsilon$ pole term because we work in the $\overline{\text{MS}}$ factorization scheme. Using the fact that $D_{q \rightarrow \gamma}^{(1)}(z) = D_{q \rightarrow \gamma}^{(1)}(z)$, and comparing Eq. (2.45) with Eqs. (2.18) and (2.47), we observe that the divergent first term in Eq. (2.45) is cancelled exactly by the subtraction terms defined in Eq. (2.22), in accord with the pQCD factorization theorem. Using Eq. (2.22), we obtain the finite $O(\alpha_{em})$ hard-scattering cross section

$$\frac{d\hat{\sigma}^{(1)}}{d^3 \ell} \frac{e^+ e^- \rightarrow \gamma X}{E_{\gamma}} = 2 \sum_q \left[ \frac{2}{s} F_{qP}^C (s) \right] \left[ \frac{\alpha_{em}^2 N_c}{x_{\gamma}} \right] e_q^2 \left( \frac{\alpha_{em}}{2\pi} \right) \left\{ (1 + \cos^2 \theta_{\gamma}) \left[ \frac{1 + (1 - x_{\gamma})^2}{x_{\gamma}} \right] \left[ \ln \left( \frac{s}{\mu_{\overline{\text{MS}}}^2} \right) \right. \right.

+ \ln \left( \frac{x_{\gamma}^2 (1 - x_{\gamma})}{x_{\gamma}} \right) \right.

+ (1 - 3 \cos^2 \theta_{\gamma}) \left[ \frac{2(1 - x_{\gamma})}{x_{\gamma}} \right] \left\} \right),$$

(2.48)

We can see that the angular dependence of the $O(\alpha_{em})$ hard-scattering cross section has two components, one proportional to $(1 + \cos^2 \theta)$, familiar from the lowest order expression, and a second piece proportional to $(1 - 3 \cos^2 \theta_{\gamma})$. If one integrates over $\cos \theta_{\gamma}$, the second piece vanishes. However, the piece proportional to $(1 - 3 \cos^2 \theta_{\gamma})$ changes the predicted angular dependence from the often assumed form $(1 + \cos^2 \theta)$. The difference means that it would not be correct to assume a $(1 + \cos^2 \theta)$
dependence when attempting to correct an integrated cross section, such as $d\sigma/dE_\gamma$. In particular, such error reaches maximum when $\cos \theta_\gamma \sim \pm 1$.

2.5 Hard Parts for Next-to-leading Order Fragmentation Contribution

2.5.1 Hard Part For $O(\alpha_s)$ Order Gluon Fragmentation

The finite hard-scattering cross section $d\sigma^{(1)}_{e^+e^- \rightarrow g X}$ to first order in $\alpha_s$ may be obtained directly from Eq. (2.48) after three replacements: $x_\gamma \rightarrow x_g$; $N_c \rightarrow N_c C_F$; and $e^2 e_q^2$ of the final photon emission vertex by $g^2 = 4\pi \alpha_s$.

$$\frac{d\sigma^{(1)}_{e^+e^- \rightarrow g X}}{d^3 p_g} = 2 \sum_q \left[ \frac{2}{s} F_q^{PC} (s) \right] \left[ 2 e_{em}^2 N_c \frac{1}{x_g} \right] C_F \left( \frac{\alpha_s}{2\pi} \right) \times \left[ (1 + \cos^2 \theta_g) \mid \frac{1 + (1 - x_g)^2}{x_g} \right] \ln \left( \frac{s/\mu_M^2}{\mu_M^2} \right) + \ln \left( \frac{x_g (1 - x_g)}{x_g (1 - x_g)} \right) \right] + \left( 1 - 3 \cos^2 \theta_g \right) \frac{2(1 - x_g)}{x_g} \right). \tag{2.49}$$
In Eq. (2.49), \( x_g = 2E_g/\sqrt{s} \); \( C_F = \frac{4}{3} \), and \( N_c = 3 \).

The contribution \( O(\alpha_s) \) to the inclusive yield \( e^+e^- \rightarrow \gamma X \) via gluon fragmentation is therefore

\[
\frac{d\sigma^{(1)}_{e^+e^-\rightarrow gX \rightarrow \gamma X}}{E_\gamma d^3 \ell} = \int_{x_\gamma}^{1} \frac{dz}{z} \left[ \frac{d\hat{\sigma}^{(1)}_{e^+e^-\rightarrow gX}}{E_g} \left( x_g = \frac{x_\gamma}{z} \right) \right] \frac{D_{\gamma \rightarrow \gamma(z, p^2_{MS})}}{z} \tag{2.50}
\]

with \( x_\gamma = 2E_\gamma/\sqrt{s} \). Because the \( g \rightarrow \gamma \) fragmentation process is collinear, \( \theta_g = \theta_\gamma \).

### 2.5.2 Hard Part For \( O(\alpha_s) \) Order Quark Fragmentation

As sketched in Figure 2.6, both real gluon emission and virtual gluon exchange graphs contribute to the \( O(\alpha_s) \) order quark fragmentation processes. The real emission diagrams have both infrared and collinear divergences. The infrared divergence is cancelled by contributions from the virtual diagrams, while the collinear divergence is cancelled by the subtraction term defined in Eq. (2.25).

The real emission diagrams can be treated easily in the same way as \( d\sigma^{(1)}_{e^+e^-\rightarrow \gamma X}/d^3 \ell \) in Section 2.4. Except for the replacement of a photon by a gluon, the hadronic tensor \( H_{\mu\nu} \) obtained from the gluon emission diagrams in Figure 2.6a is identical to that computed in Section 2.4 for \( e^+e^- \rightarrow q\bar{q}\gamma \). Thus, we may employ our previous expressions for \( H_1 \) and \( H_2 \) again but with the replacement of subscript "\( \ell \)" in Eqs. (2.30) and (2.34) by subscript "3", since \( p_3 \) is now the momentum label for the gluon. Because the quark is now the fragmenting particle (i.e., effectively the "observed" particle), the \( y_{ik}^2 \) variables with \( i = 1, 2, 3 \) in \( H_2 \) are no longer those
in Eq. (2.38). Instead, they now are

\[
y_{1k}^2 = x_1^2 \cos^2 \theta_1 ;
\]

\[
y_{2k}^2 = \left[ \frac{y_{13}y_{23} - y_{12}}{x_1} \right]^2 \cos^2 \theta_1 + \left( \frac{1}{1 - \epsilon} \right) \left[ \frac{2(y_{12}y_{13}y_{23})}{x_1^2} \right] \sin^2 \theta_1 ;
\]

and

\[
y_{3k}^2 = \left[ \frac{y_{12}y_{23} - y_{13}}{x_1} \right]^2 \cos^2 \theta_1 + \left( \frac{1}{1 - \epsilon} \right) \left[ \frac{2(y_{12}y_{13}y_{23})}{x_1^2} \right] \sin^2 \theta_1 . \tag{2.51}
\]

In Eq. (2.51), \( \theta_1 \) is the scattering angle of the quark, and subscript “3” indicates the gluon of momentum \( p_3 \). All terms linear in \( \cos \theta_x \) have been dropped, and \( \cos^2 \theta_x \) was replaced by its average value in \( n \)-dimensions. Substituting these \( y_{ik}^2 \) with \( i = 1, 2, 3 \) into Eq. (2.34), we can obtain

\[
\frac{1}{4} \left( H_1 + H_2^{eff} \right) = (1 + \cos^2 \theta_1 - 2\epsilon) \left[ (1 - \epsilon) \left( \frac{y_{13}}{y_{23}} + \frac{y_{23}}{y_{13}} \right) \right. \\
+ 2 \left( \frac{y_{12}}{y_{13}y_{23}} - \epsilon \right) \\
+ \left. (1 - 3\cos^2 \theta_1) \left[ \frac{2y_{12}}{x_1^2} \right] \right] \\
+ \epsilon \cos^2 \theta_1 \left[ \frac{4y_{12}}{x_1^2} \right] , \tag{2.52}
\]

where the last term again vanishes as \( \epsilon \to 0 \). In analogy to Eq. (2.40), the useful identities here are

\[
y_{23} = 1 - x_1 ;
\]

\[
y_{12} = x_1 - y_{13} . \tag{2.53}
\]

By using these identities, Eq. (2.52) can be reexpressed in terms of \( x_1 \) and \( y_{13} \)

\[
\frac{1}{4} \left( H_1 + H_2^{eff} \right) = (1 + \cos^2 \theta_1 - 2\epsilon) \left\{ \left[ \frac{1 + x_1^2}{1 - x_1} \right] \frac{1}{y_{13}} + \frac{y_{13}}{1 - x_1} - \frac{2}{1 - x_1} \right\}
\]
where the last term in Eq. (2.52) was dropped. The contribution of real gluon emission can be obtained by combining Eq. (2.54) with the three particle final state phase space \( dP_S^{(3)} \), Eq. (A.35). The final result with the overall coupling factor \( (e\mu \epsilon)^2 (g\mu \epsilon)^2 \) and color factor \( N_c C_F \) is

\[
\frac{d\sigma^{(R)}}{E_1 d^3 p_1} = \left[ \frac{2}{s} FPC(s) \right] \left[ \frac{2}{\alpha_s^{em}} N_c \left( \frac{4\pi \mu^2}{(s/4) \sin^2 \theta_1} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \right]
\]

\[
\times \frac{1}{4} \left[ H_1 + H_2^{eff} \right] \frac{dy_{12}}{y_{12}^\epsilon} \frac{dy_{13}}{y_{13}^\epsilon} \frac{dy_{23}}{y_{23}^\epsilon}
\]

\[
\times \delta(1 - y_{12} - y_{13} - y_{23}) \, .
\]

(2.55)

where superscript \( (R) \) stands for the real emission. The two \( \delta \)-functions are used to fix \( y_{23} \) and \( y_{12} \), and \( dy_{13} \) is integrated from 0 to \( x_1 \). After inserting \( H_1 + H_2^{eff} \) from Eq. (2.54), we obtain

\[
\frac{d\sigma^{(R)}}{E_1 d^3 p_1} = \left[ \frac{2}{s} FPC(s) \right] \left[ \frac{2}{\alpha_s^{em}} N_c \left( \frac{4\pi \mu^2}{(s/4) \sin^2 \theta_1} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \right]
\]

\[
\times \frac{1}{4} \left[ H_1 + H_2^{eff} \right] \frac{dy_{12}}{y_{12}^\epsilon} \frac{dy_{13}}{y_{13}^\epsilon} \frac{dy_{23}}{y_{23}^\epsilon}
\]

\[
\times \left\{ (1 + \cos^2 \theta_1 - 2\epsilon) \left[ \left( \frac{1 + x_1^2}{(1 - x_1)^{+}} + \frac{3}{2} \delta(1 - x_1) \right) \left( \frac{1}{\epsilon} \right)
\right.
\]

\[
+ \left( \frac{1 + x_1^2}{1 - x_1} \right) \ln \left( x_1^2 \right) + (1 + x_1^2) \left( \frac{\ln(1 - x_1)}{1 - x_1} \right)^{+}
\]

(2.55)
The "+" prescription is defined as usual

\[ \frac{1}{1 - x_1} + \delta(1 - x_1) \left( \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{7}{2} \right) \]

\[ - \frac{1}{2} (3x_1 - 5) + \left( 1 - 3 \cos^2 \theta_1 \right) \]. \quad (2.56)

The right-hand-side of Eq. (2.56) is formally divergent as \( \epsilon \to 0 \). The \( 1/\epsilon \) poles in \( n \)-dimensions represent the infrared divergence, when the gluon momentum goes to zero, and/or a collinear divergence, when the gluon momentum is parallel to that of the fragmenting quark. As shown below, the infrared divergence is cancelled by the infrared divergence of the virtual diagrams, sketched in Figure 2.6b.

The contribution of the virtual diagrams results from the interference of the one-loop vertex and self-energy diagrams with the leading order tree diagram. The self-energy diagram does not have contributions in dimensional regularization. As for the leading order contribution, the virtual diagrams, sketched in Figure 2.6b, have a two-particle final state phase space. Therefore, the contribution from the virtual diagrams has the same kinematical structure and angular dependence as the leading order contribution, discussed in Section 2.2. It is proportional to \( \delta(1 - x_1) \), and, consequently, the virtual contribution cancels only the \( 1/\epsilon \) poles associated with the \( \delta(1 - x_1) \) terms in Eq. (2.56). The subtraction terms in Eq. (2.25) cancel the final state collinear poles that appear in the contribution of real gluon emission.
The virtual exchange diagrams are shown in Figure 2.6b. The one-loop vertex correction can be evaluated in n-dimension, and the first order virtual contribution can be obtained by combining the one-loop vertex correction with the lowest order tree diagram. The result is

\[
E_1 \frac{d\sigma^{(V)}}{d^3 p_1} = \left[ \frac{2}{s} F^{PC} (s) \right] \left[ \alpha^{em}_C N_C \left( \frac{4\pi \mu^2}{(s/4) \sin^2 \theta_1} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \right]
\times C_F \left( \frac{\alpha_s}{2\pi} \right) \left[ \left( \frac{4\pi \mu^2}{s} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \right] \left( \frac{1}{x_1} \right) \frac{\Gamma(1-\epsilon)^3 \Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}
\times \left\{ (1 + \cos^2 \theta_1 - 2\epsilon) \delta(1 - x_1) \left[ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \left( \pi^2 - 8 \right) \right] \right\}, \tag{2.59}
\]

where the superscript \((V)\) stands for the virtual contribution. After adding the real and virtual contributions, Eqs. (2.56) and (2.59), we can obtain the cross section for \(e^+e^- \rightarrow qX\) at order \(O(\alpha_s)\),

\[
E_1 \frac{d\sigma^{(1)}}{d^3 p_1} = \left[ \frac{2}{s} F^{PC} (s) \right] \left[ \alpha^{em}_C N_C \left( \frac{4\pi \mu^2}{(s/4) \sin^2 \theta_1} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \right] C_F \left( \frac{\alpha_s}{2\pi} \right)
\times \frac{1}{x_1} \left\{ (1 + \cos^2 \theta_1 - 2\epsilon) \left[ \frac{1 + x_1^2}{(1 - x_1)^+} + \frac{3}{2} \delta(1 - x_1) \right] \left( \frac{1}{-\epsilon} \right) \right\}
+ \left[ \frac{2}{s} F^{PC} (s) \right] \left[ \alpha^{em}_C N_C \frac{1}{x_1} \right] C_F \left( \frac{\alpha_s}{2\pi} \right)
\times \left\{ (1 + \cos^2 \theta_1) \left[ \left( \frac{1 + x_1^2}{(1 - x_1)^+} + \frac{3}{2} \delta(1 - x_1) \right) \ln \left( \frac{s}{\mu^2_{MS}} \right) \right.ight.
+ \left( \frac{1 + x_1^2}{1 - x_1} \right) \ln \left( x_1^2 \right) + (1 + x_1^2) \left( \frac{\ln(1 - x_1)}{1 - x_1} \right) +
\left. \left. - \frac{3}{2} \left( \frac{1}{1 - x_1} \right) + \delta(1 - x_1) \left( \frac{2\pi^2}{3} - \frac{9}{2} \right) - \frac{1}{2} (3x_1 - 5) \right] \right\}.
\]
As is evident from the $1/\epsilon$ terms, this cross section is divergent as $\epsilon \to 0$, a reflection of the fact that a cross section for producing a massless quark is an infrared sensitive quantity, not perturbatively calculable.

According to the pQCD factorization theorem, the short-distance hard-scattering cross sections, defined in Eq. (2.11), are infrared safe quantities. Beyond the Born level, the short-distance parts, $\hat{\sigma}_{e^+e^- \to cX}$, are not the same as the partonic cross sections $\sigma_{e^+e^- \to cX}$ for fragmenting parton $c$. Following Eq. (2.25), in order to derive the short-distance hard-scattering cross section $\hat{\sigma}_{e^+e^- \to qX}^{(1)}$, we must first calculate the one-loop perturbative fragmentation function $D_{q \to q}^{(1)}$. Feynman diagrams for $D_{q \to q}^{(1)}$ are sketched in Figure 2.10. These diagrams are evaluated in the same way as one evaluates parton-level parton distributions [21], and we obtain

$$D_{q \to q}^{(1)}(x_1) = C_F \left( \frac{\alpha_s}{2\pi} \right) \left[ \frac{1 + x_1^2}{(1 - x_1)_+} + \frac{3}{2} \delta(1 - x_1) \right] \left( \frac{1}{-\epsilon} \right),$$

where the "+" prescription is defined in Eq. (2.57).

By using Eq. (2.25), the lowest order cross section for $e^+e^- \to qX$, Eq. (2.18), and the one-loop quark fragmentation function $D_{q \to q}^{(1)}$, Eq. (2.61), we can obtain the short-distance hard-scattering cross section

$$E_1 \frac{d\hat{\sigma}_{e^+e^- \to qX}^{(1)}}{d^3p_1} = \left[ \frac{2}{s} F_q PC(s) \right] \left[ \alpha_{em}^2 N_c \frac{1}{x_1} \right] C_F \left( \frac{\alpha_s}{2\pi} \right)$$

$$\times \left\{ (1 + \cos^2 \theta) \left[ \left( \frac{1 + x_1^2}{(1 - x_1)_+} + \frac{3}{2} \delta(1 - x_1) \right) \ln \left( \frac{s}{\mu_{\overline{MS}}^2} \right) \right] \right.$$

$$+ \left( \frac{1 + x_1^2}{1 - x_1} \right) \ln (x_1^2) + (1 + x_1^2) \left( \frac{\ln(1 - x_1)}{1 - x_1} \right) \left. \right\}.$$
$\theta_\gamma$ is set to equal to $\theta_1$. It is based on the assumption of collinear fragmentation from quark to photon. As expected, the hard-scattering cross section is infrared insensitive. The $O(\alpha_s)$ quark fragmentation contribution to $e^+e^- \rightarrow \gamma X$ is

$$
\frac{d\sigma^{(1)}_{e^+e^- \rightarrow qX \rightarrow \gamma X}}{d^3 \ell} = \sum_q \int_{x_{\gamma}}^{1} \frac{dz}{z} \left[ \frac{d\hat{\sigma}^{(1)}_{e^+e^- \rightarrow qX}}{d^3 p_1} \left( x_1 = \frac{x_{\gamma}}{z} \right) D_{q \rightarrow \gamma(z, \mu^2_{MS})} \right].
$$

The derivation shows that the short-distance hard-scattering cross section for antiquark fragmentation to a photon is the same as that for quark fragmentation. Consequently, the $O(\alpha_s)$ antiquark fragmentation contribution to $e^+e^- \rightarrow \gamma X$ is the same as that given in Eq. (2.63).
2.6 Predictions and Suggestions for Experiments

The numerical values of the inclusive prompt photon cross sections can be obtained by using the analytic expressions derived in this chapter. The results are presented here for \( e^+e^- \) center-of-mass energies \( \sqrt{s} = 10 \) GeV, 58 GeV, and 91 GeV. They are appropriate for experimental investigations underway at Cornell, KEK, SLAC, and CERN. The figures show variation of the inclusive yield with photon energy \( E_\gamma \) and scattering angle \( \theta_\gamma \), where \( \theta_\gamma \) is the angle of the photon with respect to the \( e^+e^- \) collision axis. The dependence of cross sections on the choice of renormalization scale \( \mu \) is also presented.

The cross sections evaluated are those derived in the text: Eqs. (2.19), (2.48), (2.50), and (2.63). They are assembled here for convenience of comparison. The lowest order inclusive cross section is

\[
\frac{d\sigma^{incl}_{e^+e^-\to \gamma X}}{d\Omega} = 2 \sum_q \left[ \frac{2}{s} F^P C_q(s) \right] \alpha^2_{em}(s) N_c (1 + \cos^2 \theta_\gamma) \frac{1}{x_\gamma} D_{q\to \gamma}(x_\gamma, \mu^2_F). \tag{2.64}
\]

The finite \( O(\alpha_{em}) \) hard-scattering cross section is

\[
E_\gamma \frac{d\hat{\sigma}^{(1)}_{e^+e^-\to \gamma X}}{d^3 \ell} = 2 \sum_q \left[ \frac{2}{s} F^P C_q(s) \right] \left[ \alpha^2_{em}(s) N_c \frac{1}{x_\gamma} \right] \frac{e_q}{s} \left( \frac{\alpha_{em}(\mu^2_F)}{2\pi} \right)
\times \left\{ (1 + \cos^2 \theta_\gamma) \left[ \frac{1 + (1 - x_\gamma)^2}{x_\gamma} \ln \left( \frac{s}{\mu^2_F} \right) \right] + \ln \left( \frac{x_\gamma^2 (1 - x_\gamma)}{s} \right) \right\} + (1 - 3 \cos^2 \theta_\gamma) \left[ \frac{2(1 - x_\gamma)}{x_\gamma} \right]. \tag{2.65}
\]

The \( O(\alpha_s) \) contribution to the inclusive yield \( e^+e^- \to \gamma X \) via gluon fragmentation
is

\[
\frac{d\sigma^{(1)}_{e^+e^-\to gX\to\gamma X}}{E_\gamma \frac{d^3\ell}{d^3p_g}} = \int_x^1 \frac{dz}{z}
\left[ E_g \frac{d\hat{\sigma}^{(1)}_{e^+e^-\to gX}}{d^3p_g} \left( x_g = \frac{x_\gamma}{z} \right) \right]
\frac{D_{g\to\gamma}(z, \mu_F^2)}{z}
\]  

(2.66)

with

\[
E_g \frac{d\hat{\sigma}^{(1)}_{e^+e^-\to gX}}{d^3p_g} = 2 \sum_q \left[ \frac{2 F_q^{PC}(s)}{s} \right] \left[ \alpha_{em}^2(s) N_c \frac{1}{x_g} \right] C_F \left( \frac{\alpha_s(\mu_F^2)}{2\pi} \right)
\times \left\{ \left( 1 + \cos^2 \theta \right) \left[ \frac{1 + (1 - x_g)^2}{x_g} \right] \ln \left( \frac{s}{\mu_F^2} \right) 
+ \ln \left( \frac{x_g^2}{1 - x_g} \right) \right\}
+ \left( 1 - 3 \cos^2 \theta \right) \left[ \ln \left( \frac{2(1 - x_g)}{x_g} \right) \right].
\]

(2.67)

The renormalization scale \( \mu \) in \( \alpha_s(\mu^2) \) is chosen to be the same as the fragmentation scale \( \mu_F \) in \( D_{g\to\gamma}(z, \mu_F^2) \). The \( O(\alpha_s) \) contribution to the inclusive yield \( e^+e^- \to \gamma X \) via quark fragmentation is

\[
\frac{d\sigma^{(1)}_{e^+e^-\to qX\to\gamma X}}{E_\gamma \frac{d^3\ell}{d^3p_1}} = \sum_q \int_x^1 \frac{dz}{z}
\left[ E_1 \frac{d\hat{\sigma}^{(1)}_{e^+e^-\to qX}}{d^3p_1} \left( x_1 = \frac{x_\gamma}{z} \right) \right]
\frac{D_{q\to\gamma}(z, \mu_F^2)}{z}.
\]

(2.68)

with

\[
E_1 \frac{d\hat{\sigma}^{(1)}_{e^+e^-\to qX}}{d^3p_1} = \left[ \frac{2 F_q^{PC}(s)}{s} \right] \left[ \alpha_{em}^2(s) N_c \frac{1}{x_1} \right] C_F \left( \frac{\alpha_s(\mu_F^2)}{2\pi} \right).
\]
\begin{equation}
\times \left\{ (1 + \cos^2 \theta_\gamma) \left[ \left( \frac{1 + x_1^2}{(1 - x_1)^+} + \frac{3}{2} \delta(1 - x_1) \right) \ln \left( \frac{s}{\mu_F^2} \right) \right.ight.
\left. + \left( \frac{1 + x_1^2}{1 - x_1} \right) \ln \left( x_1^2 \right) + \left( 1 + x_1^2 \right) \left( \ln(1 - x_1) \right) \right. + \frac{3}{2} \left( \frac{1}{1 - x_1} \right) + \delta(1 - x_1) \left( \frac{2\pi^2}{3} - \frac{9}{2} \right) \right.
\left. - \frac{1}{2} \left( 3x_1 - 5 \right) \right] + (1 - 3\cos^2 \theta_\gamma) \right\}. \tag{2.69}
\end{equation}

For the common overall normalization function $F_0^{PC}(s)$, we use an expression that includes $\gamma$, $Z^0$ interference:

\begin{equation}
\frac{2}{s} F_0^{PC}(s) = \frac{1}{s^2} \left[ e_Q^2 + (|v_e|^2 + |a_e|^2) (|v_q|^2 + |a_q|^2) \frac{s^2}{(s - M_Z^2)^2 + 2M_Z^2 \Gamma_Z^2} 
\right. - 2e_Q v_e v_q \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \right]. \tag{2.70}
\end{equation}

The vector ($v$) and axial-vector ($a$) couplings are provided in Table 2.1 and Table 2.2. The mass and width of $Z^0$ particle are set to be $M_Z = 91.187$ GeV and $\Gamma_Z = 2.491$ GeV. These and other constants used here are taken from Ref. [22]. The weak mixing angle $\sin^2 \theta_W = 0.2319$. For the electromagnetic coupling strength $\alpha_{em}$, we use the solution of the first order QED renormalization group equation

\begin{equation}
\alpha_{em}(\mu^2) = \frac{\alpha_{em}(\mu_0^2)}{1 + \frac{\beta_0}{4\pi} \alpha_{em}(\mu_0^2) \ln(\mu^2/\mu_0^2)}. \tag{2.71}
\end{equation}

Here $\beta_0$ is the first order QED beta function,

\begin{equation}
\beta_0 = -\frac{4}{3} \sum_f N_{c}^f e_f^2, \tag{2.72}
\end{equation}

with $N_{c}^f$ the number of colors for flavor $f$ and $e_f$ the fractional charge of the fermions. The sum over $f$ extends over all fermions (leptons and quarks) with mass $m_f^2 < \mu^2$. 


For the energy region of interest here, we do not include the top quark in the sum in Eq. (2.72), and we obtain \( \beta_0 = -80/9 \). To fix the boundary condition in Eq. (2.71), we let \( \alpha_{em}(\frac{M_Z^2}{\mu^2}) = 1/128 \) and set \( \mu_0 = M_Z \).

In the \( O(\alpha_s) \) contributions, Eqs. (2.67) and (2.69), we employ a two-loop expression for \( \alpha_s(\mu^2) \) with quark threshold effects handled properly. \( \Lambda_{QCD}^{(4)} \) is set to be 0.231 GeV. At \( \sqrt{s} = M_Z \), this expression provides \( \alpha_s \left( \frac{M_Z^2}{\mu^2} \right) = 0.112 \).

At \( \sqrt{s} = 10 \) GeV, the sums in Eqs. (2.65), (2.67), and (2.69) run over 4 flavors of quarks \( (u, d, c, s) \), all assumed massless. At this energy, we do not include a \( b \) quark contribution in the calculation. For \( \sqrt{s} = 58 \) GeV and 91 GeV, we use 5 flavors, again assuming all quarks massless in the short-distance hard scattering cross sections. At these higher energies, non-zero mass effects for the \( c \) and \( b \) quarks are accommodated by our scale choice in the fragmentation functions, discussed below.

The quark-to-photon fragmentation function that appears in Eq. (2.64) and (2.69) is expressed as

\[
z D_{q \rightarrow \gamma}(z, \mu_F^2) = \frac{\alpha_{em}(\mu_F^2)}{2\pi} \left[ e_q^2 \frac{2.21 - 1.28 z + 1.29 z^2}{1 - 1.63 \ln(1 - z)} z^{-0.049} + 0.002 (1 - z)^2 z^{-1.54} \right] \ln \left( \frac{\mu_F^2}{\mu_0^2} \right). \tag{2.73}
\]

The gluon-to-photon fragmentation function in Eq. (2.66) is

\[
z D_{g \rightarrow \gamma}(z, \mu_F^2) = \frac{\alpha_{em}(\mu_F^2)}{2\pi} 0.0243 (1 - z) z^{-0.97} \ln \left( \frac{\mu_F^2}{\mu_0^2} \right). \tag{2.74}
\]

These expressions for \( D_{q \rightarrow \gamma} \) and \( D_{g \rightarrow \gamma} \), taken from Ref. [4], are used as a guideline for our estimates. The physical significance of scale \( \mu_0 \) is that the fragmentation function vanishes for energies less than \( \mu_0 \). For \( g \) and for the \( u, d, s, \) and \( c \) quarks, we set \( \mu_0 = \Lambda_{QCD}^{(4)} \), as in Ref. [4]. Eq. (2.73) is also used for the \( b \) quark fragmentation,
but \( \mu_0 \) is replaced by the mass of the quark, \( m_b = 5 \text{ GeV} \); \( D_{b\rightarrow \gamma}(z, \mu_F^2) = 0 \) for \( \mu_F < m_b \). we set the fragmentation scale \( \mu_F \) equal to the renormalization scale \( \mu \) for the inclusive cross sections. In the results presented below, we vary \( \mu \) to examine the sensitivity of the cross section to its choice.

In the figures, the inclusive cross sections are divided by an energy dependent cross section \( \sigma_0 \) that specifies the leading order total hadronic event rate at each value of \( \sqrt{s} \):

\[
\sigma_0 = \frac{4\pi s}{3} \sum_q \left[ \frac{1}{s} F_q^P \left( \frac{Q^2}{s} \right) \alpha_s^2 \rho_m(s) N_c \right]. \tag{2.75}
\]

By doing so, we can observe what fraction of the total hadronic rate is represented by inclusive prompt photon production.

The following figures show the predicted behavior of the inclusive yield as a function of \( E_\gamma \) and \( \theta_\gamma \), as well as the breakdown of the total yield into contributions from various components.

Figure 2.11, shows the inclusive yield as a function of \( E_\gamma \) at \( \sqrt{s} = 91 \text{ GeV} \) for photon scattering angle \( \theta_\gamma = 90^\circ \). Displayed are the total result and the four separate contributions from lowest-order fragmentation ("0th-frag"), \( O(\alpha_em) \) direct production, and the \( O(\alpha_s) \) quark and gluon fragmentation contributions. The renormalization/fragmentation scale is set to be \( \mu = E_\gamma \). The same results are displayed in Figure 2.12 as a function of scattering angle \( \theta_\gamma \) for the photon energy \( E_\gamma = 15 \text{ GeV} \). Dependence of the cross sections on \( \mu \) is examined in Figure 2.13 for fixed \( E_\gamma \). The figure shows the scale dependence of the cross section at \( \sqrt{s} = 91 \text{ GeV} \) for \( \theta_\gamma = 90^\circ \) and \( E_\gamma = 15 \text{ GeV} \). The total result shows little \( \mu \) dependence, whereas the component contributions display considerable compensating variation with \( \mu \).

Figure 2.14 and Figure 2.15 show the inclusive yield as a function of \( E_\gamma \) at \( \sqrt{s} = \)
Figure 2.11: Normalized invariant cross section for the inclusive process $e^+e^- \rightarrow \gamma X$ at $\sqrt{s} = 91 \text{ GeV}$ shown as a function of the photon energy $E_{\gamma}$ for $\theta_{\gamma} = 90^\circ$.

Figure 2.16 demonstrates a comparison of the predictions at the three energies by showing the cross section $\sigma^{-1}_0 d\sigma/dx_{\gamma}\,d\Omega_{\gamma}$ as a function of the scaling variable $x_{\gamma} = 2E_{\gamma}/\sqrt{s}$. Shown are curves for $\sqrt{s} = 10$, 58, and 91 GeV at $\theta_{\gamma} = 90^\circ$.

Evident in Figures 2.11–2.15 is the dominance of the lowest-order contribution to the inclusive yield, Eq. (2.64), at all values of $\sqrt{s}$, except at small values of $E_{\gamma}/\sqrt{s}$.
or at small values of $\mu$ where the $O(\alpha_{em})$ "direct" contribution, Eq. (2.65), becomes larger. Following the lowest-order contribution in importance at modest values of $E_\gamma/\sqrt{s}$ (or $\mu$) is the $O(\alpha_{em})$ direct contribution. The direct contribution falls away more rapidly with increasing $E_\gamma$ (or $\mu$) than the $O(\alpha_s)$ quark-to-photon fragmentation term, Eq. (2.69). Therefore, at large values of $E_\gamma/\sqrt{s}$ (or $\mu$), it is the $O(\alpha_s)$ fragmentation term, that is secondary in importance to the lowest-order term. The gluon-to-photon fragmentation contribution, Eq. (2.66) plays an insignificant role
except at very small $E_\gamma$.

Figure 2.12 shows the predicted $\theta_\gamma$ dependence of the cross sections. Presented with a linear scale, the figure demonstrates, perhaps more clearly, the importance of the roles of the $O(\alpha_{em})$ direct and $O(\alpha_s)$ fragmentation contributions. The lowest-order contribution, Eq. (2.64), is proportional to $(1 + \cos^2 \theta_\gamma)$. However, there are significant $\sin^2 \theta_\gamma$ components in the next-to-leading order direct term, Eq. (2.65), and the next-to-leading order fragmentation terms, Eqs. (2.67) and (2.69). The
Figure 2.14: Photon energy dependence, as in Figure 2.11, but for center-of-mass energy $\sqrt{s} = 58$ GeV.

The net result is that the predicted total yield in Figures 2.12, is not proportional to $(1 + \cos^2 \theta_\gamma)$. As illustrated in the figures, the deviation of the total yield from the $(1 + \cos^2 \theta_\gamma)$ form becomes greater at smaller values of $E_\gamma$. (The results shown in Figure 2.12, all pertain to the scale choice $\mu = E_\gamma$.) One lesson from this examination of dependence on $\theta_\gamma$ is that it is inappropriate and potentially misleading to assume that the functional form $(1 + \cos^2 \theta_\gamma)$ describes the data when attempts are made to correct distributions in the region of small $\theta_\gamma$ (where initial state bremsstrahlung overwhelms the final state radiation in which one is interested).
Dependence on the renormalization/factorization scale $\mu$ in Figure 2.13, shows several interesting features. As is expected from the functional form of $D_{q\rightarrow \gamma} (z, \mu^2)$ in Eq. (2.73), the lowest-order contribution, Eq. (2.64), increases logarithmically as $\mu$ is increased. On the other hand, the $\ln (s/\mu^2)$ dependent term in Eq. (2.65) causes a decrease of the $O(\alpha_{em})$ direct contribution as $\mu$ is increased. Indeed, the $(1 + \cos^2 \theta_{\gamma})$ part of the direct contribution becomes negative when $sx_{\gamma}^2 (1 - x_{\gamma}) / \mu^2 < 1$. The physical cross section, represented as a solid line in Figure 2.13, is of course always positive.
An especially noteworthy feature of Figure 2.13, is that the total inclusive yield is nearly independent of $\mu$, in spite of the strong variation with $\mu$ of its components. This independence reflects the role of the fragmentation scale $\mu$. It is introduced to separate "soft" and "hard" contributions into "fragmentation" and "direct" pieces. As the scale $\mu$ is increased, more of the cross section is necessarily factored into the fragmentation contribution, and vice versa, such that the sum remains nearly constant.
Figure 2.16, shows the overall $\sqrt{s}$ dependence of the predictions. To facilitate comparison, we present these results in terms of the “scaling” distribution $\sigma^{-1}_0 d\sigma/dx\gamma d\Omega\gamma$. The case of $\sqrt{s} = 10$ GeV is somewhat special since the contribution from $b$ quark fragmentation is not included at this energy. Otherwise, the contribution of the lowest-order process, Eq. (2.64), decreases at fixed $x\gamma$ as $\sqrt{s}$ is increased. This decrease is explained easily. In computing $d\sigma/dx\gamma d\Omega\gamma$, Eq. (2.70) is multiplied by Eq. (2.73). The final result has a charge weighting factor of $e^2 F_{q}^{PC}(s)$, whereas in the denominator $\sigma_0$, the factor is $F_q^{PC}(s)$. Owing to the values of the $\nu$ and $a$ couplings in Table 1, the up-type quark contribution to $F_q^{PC}(s)$ decreases as $\sqrt{s}$ increases, and the down-type contribution increases. The $O(\alpha_{em})$ direct contribution to $\sigma^{-1}_0 d\sigma/dx\gamma d\Omega\gamma$ decreases at fixed $x\gamma$ as $\sqrt{s}$ is increased from 10 to 91 GeV. Again, the explanation may be found in the energy dependence of the ratio $\sum_q e_q^2 F_{q}^{PC}(s)/\sum_q F_{q}^{PC}(s)$. Taken together these statements explain the energy dependence displayed in Figure 2.16.

As remarked earlier, the particular expressions chosen for the fragmentation functions are not meant to be anything but illustrative expressions. It would be very valuable if these non-perturbative functions could be determined directly from data. Dominance of the $q \rightarrow \gamma$ fragmentation contribution in Figures 2.11–2.15 demonstrates the important role that data from $e^+e^- \rightarrow \gamma X$ may play in the extraction of $D_{q\rightarrow\gamma}(z, \mu^2)$ and study of its properties.
3. NUCLEAR DEPENDENCE IN DIRECT PHOTON PRODUCTION

Direct photon production in a nuclear environment may have additional nuclear dependence due to multiple scattering, which is often described as multiple scattering of partons in nuclear matter [14, 16, 17, 18]. The multiple scattering for a photon with large transverse momentum is dominated by double scattering.

In some of the previous work on this topic, an independent scattering picture was adopted [14]. In this picture, each scattering was treated independently, and a cross section for multiple scattering is proportional to a classical convolution of many Born cross sections. For example, the cross section for double scattering was assumed to be proportional to a product of two Born cross sections. It is clear in this picture that the double scattering cross section is not infrared safe. This problem arises since the kinematics of the single particle inclusive cross section can only provide a constraint on the total momentum from the target, which leaves the possibility that one of the Born cross sections becomes infinity when the momentum from the target to this Born cross section approaches to zero. Therefore, theoretical predictions from this independent scattering picture are sensitive to the infrared cutoff introduced in the calculations.

Luo, Qiu and Sterman (LQS) have shown that the anomalous nuclear enhancement can be described naturally in perturbative QCD, in terms of a nonleading power,
or "high twist" formalism [18]. In their treatment, quantum interference between different scatterings was taken into account. In their generalized factorization theorem, the contribution from double scattering can be factorized into short-distance hard parts convoluted with corresponding multi-parton matrix elements in nuclei, also called multiparton correlation functions in nuclei. The short-distance partonic parts are calculable in perturbative QCD, and all infrared divergences associated with soft rescatterings in perturbation theory can systematically be absorbed into multiparton correlation functions. The multiparton correlation functions are nonperturbative, just like the parton distributions in the single scattering processes. These correlation functions in nuclei provide information about nuclear matter and its interaction with high energy probes. They can reveal information different from what the normal parton distributions in nuclei can provide; and, on the other hand, they are as fundamental as the parton distributions. The information of these new correlation functions can be extracted from some processes and applied to other processes.

We calculate the cross section of the high transverse momentum direct photon production in hadron-nucleus scattering, by using the perturbative QCD treatment of double scattering developed by LQS. The cross section of direct photon production is factorized into some calculable short-distance partonic parts times corresponding multiparton correlation functions, which are the same as those derived in Ref. [18]. The general formalism in our calculation is presented in the following section. Our detailed derivations are given in Section 3.2 and Section 3.3. The analytic results are summarized in Section 3.4. Numerical results on nuclear dependence are evaluated by using the information on the multiparton correlation functions, which were extracted from experiments on momentum imbalance of two-jet photoproduction on nuclear
targets [18]. They are presented in Section 3.5.

### 3.1 Process and General Formalism

The process we discuss in this chapter is the direct photon production in hadron-nucleus collisions,

$$ h(p') + A(p) \rightarrow \gamma(l) + X, $$

where $p$ is defined as the averaged momentum per nucleon. In general, the total cross section for the above process can be expressed as a sum of contributions from single scattering, double scattering and even higher multiple scattering,

$$ d\sigma_{hA\rightarrow\gamma}(l) = d\sigma_{hA\rightarrow\gamma}^{(S)}(l) + d\sigma_{hA\rightarrow\gamma}^{(D)}(l) + \ldots, $$

where the superscripts $(S)$ and $(D)$ represent the single and double scattering, respectively, and "..." represents other possible multiple scattering contributions. In this chapter, we consider only the double scattering, and its contribution to the nuclear dependence.

As a result of perturbative QCD factorization [3], the single scattering cross section can be expressed as

$$ d\sigma_{hA\rightarrow\gamma}^{(S)}(l) = A d\sigma_{hN\rightarrow\gamma}^{(S)}(l) $$

$$ = A \sum_{a,b} \int dx' f_{a/h}(x') \int dx f_{b/N}(x) d\sigma_{ab\rightarrow\gamma}(x', x, l). $$

In Eq. (3.3), $f_{a/h}(x')$ are the normal parton distributions in the beam hadron $h$, $f_{b/N}(x)$ are the effective nucleon parton distributions inside a nucleus, which should include the well-known EMC effect. In principle, the parton-parton scattering cross section, $d\sigma_{ab\rightarrow\gamma}$, should include both direct and fragmentation contributions, since
an energetic photon can be produced directly at short-distance, or produced from fragmentation of an energetic parton which itself was produced at short-distance [4, 23]. For example, the partonic scattering produces an energetic quark, and then the quark radiates a photon (as in Bremsstrahlung). Since we are most interested in fixed target experiments here, the fragmentation contribution is much smaller than the direct contribution in most of the phase space [23]. Therefore, in the rest of the discussion, we consider only the direct production of photons. For example, at the lowest order, we have contributions from \( q\bar{q} \rightarrow \gamma g \) "Annihilation" diagrams, sketched in Figure 3.1a; and \( gg(\text{or } \bar{q}) \rightarrow \gamma q(\text{or } \bar{q}) \) "Compton" diagrams, sketched in Figure 3.1b.

In terms of the generalized factorization theorem [24], the double scattering cross section can be written as:

\[
d\sigma^{(D)}_{hA \rightarrow \gamma}(l) = \sum_a \int dx' f_{a/H}(x') d\sigma^{(D)}_{aA \rightarrow \gamma}(x', p, l) ,
\]

where \( d\sigma^{(D)}_{aA \rightarrow \gamma}(x', p, l) \) can be viewed as the double scattering cross section between a parton and the nucleus. At the lowest order, it can be factorized as

\[
d\sigma^{(D)}_{aA \rightarrow \gamma}(x', p, l) = \int dx \int dx_k \int dx_{k'} \sum_{\{i\}} T_{\{i\}}(x, x_k, x_{k'}) H_{\{i\}}(x', x, x_k, x_{k'}, l) .
\]

In Eq. (3.5), \( T_{\{i\}}(x, x_k, x_{k'}) \) are the matrix elements of four-parton operators, characterized by the set of fields operators \( \{i\} \); and \( H_{\{i\}} \) are the corresponding parts of partonic scatterings. The \( x, x_k \) and \( x_{k'} \) are independent collinear momentum fractions carried by the partons from the nucleus. The graphical representation of Eq. (3.5) is shown in Figure 3.2.

At the lowest order, there are three types of partonic subprocesses that contribute to the double scatterings. Feynman diagrams of these partonic subprocesses are
Figure 3.1: Lowest order Feynman diagrams contribute to single scattering: a) "Annihilation", b) "Compton".

The invariant direct photon cross section in hadron-nucleus collision can be defined in terms of cross sections in hadron-nucleon collisions,

\[ E_l \frac{d\sigma_{hA \to \gamma(l)}}{d^3l} \equiv A \alpha(l) E_l \frac{d\sigma_{hN \to \gamma(l)}}{d^3l} \]
where Eq. (3.2) was used. Substituting Eq. (3.3) into Eq. (3.6), we can obtain the definition for the nuclear dependence parameter $\alpha(l)$,

$$
\alpha(l) = 1 + \frac{1}{\ln(A)} \ln \left( 1 + \frac{1}{A} \frac{E_l \frac{d\sigma^{(D)}}{d^3l}}{E_l \frac{d\sigma^{(S)}}{d^3l} + E_l \frac{d\sigma^{(D)}}{d^3l}} \right). \quad (3.7)
$$

From Eq. (3.7), $\alpha(l) > 1$ if $d\sigma^{(D)}_{hA \rightarrow \gamma}/d^3l$ is positive, which is true for the kinematical regime that we investigate here. However, in general, the double scattering contribution $\sigma^{(D)}$ may be negative, and $\alpha(l) < 1$ in certain parts of phase space. The fact that a cross section should be positive requires the sum of all possible multiple scattering contribution to be positive. The separation between single and double
Figure 3.3: Three types of leading order Feynman diagrams contribute to the double scattering. a) Type-1 subprocess; b) Type-2 subprocess; c) Type-3 subprocess.
scattering is not unique. For example, two scatterings can be very close to each other, and localized in one nucleon, in which case double scattering will not provide the anomalous nuclear dependence and may be classified as a single scattering.

In general, double scattering contributions are proportional to some four-parton matrix elements, for example, as shown in Figure 3.2. Such four-parton matrix elements, in principle, depend on three independent momentum fractions, $x$, $x_k$ and $x_{k'}$. After collinear expansion of parton momenta linking the matrix elements and partonic parts, all four-parton fields are put on the light-cone, and all three-momentum fractions become physical and limited from 0 to 1 [25]. Such four-parton matrix elements, depending on three independent momentum fractions, are very rich in information and are certainly difficult to extract from data. However, we argue [18] that under the leading pole approximation, $d\sigma^{(D)}_{hA \rightarrow \gamma}/d^3l$, depends on a sub-set of four-parton matrix elements, or correlation functions, which depend on only one-momentum fraction. With a reasonable assumption of nuclear structure in terms of color singlet nucleons, the double scattering contribution, $d\sigma^{(D)}_{hA \rightarrow \gamma}/d^3l$, is proportional to $A^{4/3}$. Consequently, the value of $\alpha(l)$ will be between 1 and $4/3$, depending on the relative size of contributions from the single and double scatterings. If the double scattering contribution is as large as the single scattering contribution in a certain part of the phase space, the value of $\alpha(l)$ in the same phase space can be as large as $4/3$.

The double scattering cross section are derived in the following sections. The method that is used here was first introduced in Ref. [18]. It can be summarized in the following technical steps: a) factorize the double scattering contribution into a convolution between the partonic hard parts and corresponding multiparton matrix
elements (e.g., see Eq. (3.5)); b) at the leading pole approximation, integrate over two of the three independent momentum fractions by contour integrations, and reexpress the multiparton matrix elements in terms of the $T_q(x, A)$ and $T_g(x, A)$. They are defined in Eq. (3.26) and Eq. (3.31); c) calculate the corresponding partonic hard parts.

### 3.2 Double Scattering and Factorization Beyond Leading Power

The double scattering cross section can be factorized into calculable hard parts and four-parton matrix elements. At the order that we are interested, only three types of partonic subprocesses, as sketched in Figure 3.3, contribute to the double scattering cross section $d\sigma^{(D)}_{aA\rightarrow \gamma}$ introduced in Eq. (3.5). These three types of subprocesses correspond to adding two gluons to the lowest order “Annihilation” and “Compton” subprocesses, shown in Figure 3.1.

Consider the subprocess shown in Figure 3.3a, There are four physical partons linking the matrix element $T$ and the partonic hard part $H$, as shown in Eq. 3.5. After taking into account of momentum conservation, there are still three independent four-momentum linking the partonic part and corresponding two-quark-two-gluon matrix element. In the center of mass frame of high energy collision, all partons inside the nucleus are moving almost parallel to each other, along the direction of the nucleus. Therefore, all three parton momenta can be approximately replaced by the components collinear to the hadron momentum, except gluon’s momenta, because we will integrate over momentum fractions, $x_k$ and $x_{k'}$, before putting these field operators on light-cone. After such collinear expansion, the double scattering contribution from the generalized “Annihilation” subprocess shown in Figure 3.3a
can be written as [18]

\[
\frac{d\sigma^{(D)}}{d^3l} = \frac{1}{2x's} \int dx \, dx_{k} \, dx_{k'} \int d^2k_T \, \bar{T}(x, x_{k}, x_{k'}, k_T, p) \\
\times \bar{H}(x'p', x, x_{k}, x_{k'}, k_T, p, l),
\]

(3.8)

where $2x's$ is the flux factor between the incoming beam quark and the nucleus, and $x' p'$ is the momentum carried by the beam quark. In Eq. (3.8), the two-quark-two-gluon matrix element, $\bar{T}$, is defined as

\[
\bar{T}(x, x_{k}, x_{k'}, k_T, p)
\]

\[
= \int \frac{dy_1^--dy_2^-}{2\pi} \frac{dy_1^+dy_2^-d^2y_T}{(2\pi)^2} \phi e^{i(k_p^+p'^+y_1^-+i(x_k-x_{k'})p^+y_2^-+i(k_T\cdot y_T)}
\]

\[
\times \frac{1}{2} (p_A|A^+(y_T^-0_T)\bar{\psi}_q(0)\gamma^+\psi_q(y_T^-)A^+(y_T^-0_T)|p_A).
\]

(3.9)

The corresponding partonic part $\bar{H}$ is given by the diagrams shown in Figure 3.4, with gluon lines contracted with $p^\rho p^\sigma$, quark lines from the target traced with $(\gamma \cdot p)/2$, and quark lines from the beam traced with $(\gamma \cdot (x' p'))/2$.

In deriving Eq. (3.9), we used Feynman gauge, and kept the leading contribution from the gluon field operators as $A^\rho \approx A^+(p^\rho/p^+)$.

By expanding the partonic part $\bar{H}$ introduced in Eq. (3.8) at $k_T = 0$, we have

\[
\bar{H}(x' p', x, x_{k}, x_{k'}, k_T, p, l) = \bar{H}(x' p', x, x_{k}, x_{k'}, k_T = 0, p, l)
\]

\[
+ \frac{\partial \bar{H}}{\partial k_T} \bigg|_{k_T=0} k_T^\alpha
\]
Figure 3.4: Feynman diagrams for the “annihilation” diagrams corresponding to the two-quark-two-gluon matrix element: a) real diagrams, b) and c): interference diagrams.
\[ + \frac{1}{2} \frac{\partial^2 \tilde{H}}{\partial k_T^\alpha \partial k_T^\beta} \bigg|_{k_T=0} k_T^\alpha k_T^\beta + \ldots \quad (3.10) \]

In the right-hand-side of Eq. (3.10), the first term is the leading twist eikonal contribution, which is not what we are interested in here. The second term vanishes after integrating over \( k_T \). The third term will give the finite contribution to the multiple scattering process. Substituting Eq. (3.10) into Eq. (3.8), and integrating over \( d^2k_T \), we obtain

\[
E_l \frac{d\sigma^{\gamma \rightarrow A^2}}{d^3l} = \frac{1}{2x's} \int dx \, dx_k \, dx_{k'} \, T(x, x_k, x_{k'}, A) \left( -\frac{1}{2} g^{\alpha\beta} \right) \times \left[ \frac{1}{2} \frac{\partial^2}{\partial k_T^\alpha \partial k_T^\beta} \tilde{H}(x' p', x, x_k, x_{k'}, k_T = 0, p, l) \right], \quad (3.11)
\]

where the modified matrix element \( T \) is given by

\[
T(x, x_k, x_{k'}, A) = \int \frac{dy_1}{2\pi} \, \frac{dy_2}{2\pi} \, e^{ixp^+ y_1^-} \, e^{ixk^p y^-} \, e^{-i(xk' - x_{k'}) p^+ y_2^-} \times \frac{1}{2} (p_A | F^+_{\alpha} (y_2^-) \bar{\psi}_q(0) \gamma^+ \psi_q(y_1^-) F^+_{\alpha} (y^-) | p_A) . \quad (3.12)
\]

In Eq. (3.12), \( F^+_{\alpha} = F^\beta_{\alpha} n_\beta \), and \( F^\beta_{\alpha} \) is the field strength, and vector \( n_\beta = \delta_\beta^+ \).

**3.3 Leading Pole Approximation and Final Factorized Form**

**3.3.1 Leading Pole Approximation**

The double scattering contribution defined in Eq. (3.11) depends on integrations over three independent partonic momentum fractions \( x, x_k, x_{k'} \). If all partons in Figure 3.4 carry some finite momentum fractions, the oscillations of the exponentials in the matrix element \( T \) defined in Eq. (3.12) will destroy any nuclear size enhancement that could come from the \( y \) integrations. However, even at the lowest order,
we find that there are some Feynman diagrams which have two poles corresponding to zero momentum fraction partons. Since these poles are not pinched, they do not correspond to any zero-momentum real gluons. Therefore, two of the three parton momentum fractions can be integrated explicitly by contour integration. These integrations will eliminate two exponentials, and thus, the corresponding \( y \) integration will provide the nuclear size enhancement up to \( A^{2/3} \). But, in terms of double scattering picture, if we require two soft field operators to come from the same nucleon, we will get a well-known \( A^{1/3} \) enhancement.

In principle, there are double scattering diagrams without such poles. In our calculation, we evaluate only diagrams that have such poles. This approximation is called the leading pole approximation.

In order to perform the integration of momentum fractions, it is convenient to rewrite the double scattering contribution defined in Eq. (3.11) as

\[
\frac{d\sigma^{(D)}_{qA \to \gamma}}{d^3l} = \frac{1}{2x's} \int \frac{dy_1}{2\pi} \frac{dy_2}{2\pi} \frac{1}{2} \langle p_A | F_{\alpha_1}^{-\gamma}(y_2^-) \bar{\psi}_q(0) \gamma^+ \psi_q(y_1^-) F_{\alpha_2}^{\gamma}(y^-) | p_A \rangle
\]

\[
\times \left( -\frac{1}{2} \sigma^{\alpha\beta} \right) \left[ \frac{1}{2} \frac{\partial^2}{\partial k^\alpha_T \partial k^\beta_T} H(y_1^-, y^-, y_2^-, k_T = 0, p, l) \right].
\]

(3.13)

In Eq. (3.13), the modified partonic part \( H \) is defined as

\[
H(y_1^-, y^-, y_2^-, k_T, p, l) = \int dx dx_k dx_{k'} e^{ixp+y_1^-} e^{ixkp+y^-} e^{-ix_kp_y} \bar{H}(x', p', x, x_k, x_{k'}, k_T, p, l)
\]

(3.14)

where the partonic part \( \bar{H} \) is given by diagrams shown in Figure 3.4. It is clear from Eq. (3.14), all integrals of momentum fractions can now be done explicitly without knowing the details of the multiparton matrix elements.
Consider the diagram shown in Figure 3.4a, the final state photon-gluon two particle phase space can be written as

$$
\Gamma = \frac{1}{8\pi^2} \frac{1}{x's + u} \delta \left( x + x_k + \frac{x't}{x's + u} + \frac{-k_T^2 - 2k_T \cdot l}{x's + u} \right). \quad (3.15)
$$

In deriving Eq. (3.15), we have taken the factor $d^3l/E_l$ out of the phase space due to the definition of invariant cross section (e.g., see Eq. (3.13)). Using Eq. (3.15), the contribution to $\hat{H}$ from the diagram shown in Figure 3.4a can be expressed as

$$
\hat{H}_{1a} = \frac{\alpha_s}{2\pi} C_1 \frac{1}{x's + u} I_{1a}(x, x_k, x_{k'}) \frac{1}{x_k - x_{k'} - k_T^2 - x's + u - i\epsilon} \frac{1}{i\epsilon} \delta \left( x + x_k + \frac{x't}{x's + u} + \frac{-k_T^2 - 2k_T \cdot l}{x's + u} \right), \quad (3.16)
$$

where the subscript $1a$ has following convention: “1” stands for the type-1 subprocess, shown in Figure 3.3a; “a” for the real contribution, corresponding to diagrams in Figure 3.4a. In Eq. (3.16), the factor $C_1$ is an overall color factor for the type-1 subprocess. The function $I_{1a}$ in Eq. (3.16) is given by

$$
I_{1a} = \frac{1}{4} \frac{1}{x's + u} \text{Tr} \left[ \gamma \cdot (x'p' + k_T) \gamma \cdot p \gamma \cdot (x'p' + k_T) R_{1a}^{\beta\gamma} \gamma \cdot p \gamma_{1a}^{\alpha\mu} \right] \times (g_{\alpha\beta})(g_{\mu\nu}), \quad (3.17)
$$

where $R_{1a}^{\beta\gamma}$ and $L_{1a}^{\alpha\mu}$ are the right and left blob, respectively, as shown in Figure 3.4a. These blobs include all possible tree Feynman diagrams with the same external partons. Substituting Eq. (3.16) into Eq. (3.14), we obtain

$$
H_{1a} = \frac{\alpha_s}{2\pi} C_1 \frac{1}{x's + u} \int dx_k e^{ix_k p^+(y - y_2^-)} \frac{1}{x_k - k_T^2 + i\epsilon} \quad (3.18)
$$
\[ x \int dx_{k'} e^{ix_{k'}p^+y_2} \frac{1}{x_k - x_{k'} - \frac{k_T^2}{x_s^2} - ie} \]
\[ \times \int dx e^{ixp^+y_1^-} \delta \left( x + x_k + \frac{x't}{x's + u} + \frac{-k_T^2 - 2k_T \cdot l}{x's + u} \right) \times I_{1a}(x, x_k, x_{k'}). \] (3.18)

After performing \(dx_k\) and \(dx_{k'}\) by contour integration, and \(dx\) by the \(\delta\)-function, we derive

\[ H_{1a} = (2\pi\alpha_s) C_1 \frac{1}{x's + u} e^{ixp^+y_1^-} e^{i(k_T^2/x's)p^+(y^- - y_2^-)} \]
\[ \times \theta(-y_2^-) \theta(y_1^- - y^-) I_{1a}(\bar{x}, x_k, x_{k'}) , \] (3.19)

where the \(\theta\)-functions are the results of contour integration, and the momentum fractions for the function \(I_{1a}\) are defined as

\[ \bar{x} = \frac{1}{x's + u} \left[ x't + \frac{u}{x's} - 2k_T \cdot l \right] ; \] (3.20a)
\[ x_k = \frac{k_T^2}{x's} ; \] (3.20b)
\[ x_{k'} = 0 ; \] (3.20c)
\[ x = -\frac{x't}{x's + u} . \] (3.20d)

Similarly, we derive the contribution from the diagram shown in Figure 3.4b as

\[ H_{1b} = (2\pi\alpha_s) C_1 \frac{1}{x's + u} e^{ixp^+y_1^-} e^{i(k_T^2/x's)p^+(y^- - y_2^-)} \]
\[ \times \theta(-y_2^-) \theta(y_1^- - y^-) I_{1b}(x, x_k, x_{k'}) , \] (3.21)

where \(x, x_k\) and \(x_{k'}\) are also defined in Eq. (3.20). Similar to Eq. (3.17), the partonic part \(I_{1b}\) is given by

\[ I_{1b} = \frac{1}{4} \text{Tr} \left[ \gamma \cdot (x'p') R_{1b}^{\beta\nu} \gamma \cdot p L_{1b}^{\alpha\mu} \right] (-g_{\alpha\beta})(-g_{\mu\nu}) . \] (3.22)
The diagram shown in Figure 3.4c has following contribution

\[ H_{1c} = (2\pi\alpha_s) C_1 \frac{1}{x's + u} e^{i x p^+ y_1^-} e^{i \left( \frac{k_T^2}{x's} \right) p^+ (y^- - y_2^-)} \times \theta(y^- - y_2^-) \theta(-y^-) I_{1b}(x, x_k, x_{k'}) . \quad (3.23) \]

In deriving Eq. (3.23), we used the fact that the partonic part when \( I_{1c} = I_{1b} \) when \( x_k \) and \( x_{k'} \) are evaluated at the same values as listed in Eq. (3.20).

Combining \( H_{1a}, H_{1b} \) and \( H_{1c} \) (given in Eqs. (3.19), (3.21), and (3.23), respectively) together, we can obtain the total contribution to \( H \), defined in Eq. (3.14), from the type-1 diagrams shown in Figure 3.3a,

\[ H_1 = H_{1a} + H_{1b} + H_{1c} = (2\pi\alpha_s) C_1 \frac{1}{x's + u} e^{i \left( \frac{k_T^2}{x's} \right) p^+ (y^- - y_2^-)} \theta(-y_2^-) \theta(y^- - y^-) \times \left[e^{i x p^+ y_1^-} I_{1a}(x, x_k, x_{k'}) - e^{i x p^+ y_1^-} I_{1a}(x, x_k, x_{k'})\right] . \quad (3.24) \]

All momentum fractions in Eq. (3.24) are evaluated at the values defined in Eq. (3.20). In deriving Eq. (3.24), we have dropped a term proportional to

\[ \theta(-y_2^-) \theta(y_1^- - y^-) - \theta(-y^- - y_2^-) \theta(y^- - y_2^-) - \theta(-y^- - y^-) \theta(-y^-) \]

as \( y_1^- \sim 1/(xp^+) \to 0 \). Physically, it means that all \( y \) integrations in such term are localized, and therefore, will not give any large nuclear size enhancement.

By substituting Eq. (3.24) into Eq. (3.13), we can obtain the lowest order double scattering contribution from the type-1 diagrams shown in Fig. 3.3a. One important step in getting the final result is taking the derivative with respect to \( k_T \) as defined in Eq. (3.13). Comparing Eq. (3.24) with Eq. (3.13), and knowing the fact that

\[ \left[ e^{i x p^+ y_1^-} I_{1a}(x, x_k, x_{k'}) - e^{i x p^+ y_1^-} I_{1a}(x, x_k, x_{k'})\right]_{k_T=0} = 0 , \quad (3.25) \]
The derivatives on the exponential \( \exp[i \left( k_T^2/x' s \right) p^+(y^- - y_2^-)] \) do not contribute, and therefore, we can set \( \exp[i \left( k_T^2/x' s \right) p^+(y^- - y_2^-)] = 1 \) in Eq. (3.24).

Define the two-quark-two-gluon matrix element \( T_q(x, A) \) as

\[
T_q(x, A) = \int \frac{dy_1^-}{2\pi} e^{iy_1^-} \int \frac{dy_2^-}{2\pi} \theta(y_1^- - y^-) \theta(-y_2^-) \times \frac{1}{2} (p_A | F_\alpha^+(y_2^-) \bar{\psi}_q(0) \gamma^+ \psi_q(y_1^-) F^+\alpha(y^-) | p_A) . \tag{3.26}
\]

In Eq. (3.26), \( F_{\mu\nu} \) and \( \psi_q \) are the field strength and quark field operator, respectively.

Substituting Eq. (3.24) into Eq. (3.13), and use Eq. (3.26), we obtain

\[
E_1 \frac{d\sigma_1^{(D)}}{d^3 l} = \alpha_{em}(4\pi \alpha_s)^2 e_q^2 C_1 \frac{1}{2x's} \frac{1}{x' s + u} \times \left( - \frac{1}{2} g^{\alpha\beta} \right) \frac{1}{2} \frac{\partial^2}{\partial k_T^\alpha \partial k_T^\beta} \left[ T_q(\tilde{x}, A) I_{1a}(\tilde{x}, x_k, x_{k'}) - T_q(x, A) I_{1b}(x, x_k, x_{k'}) \right] , \tag{3.27}
\]

where \( \sigma_1^{(D)} \) stands for the double scattering contribution from the type-1 subprocess shown in Figure 3.3a. It is important to note that although the interference diagrams shown in Figure 3.4b and Figure 3.4c are important in driving Eq. (3.27), the final result depends only on the real diagram shown in Figure 3.4a. That is, the double scattering picture is preserved. The role of interference diagrams is to take care of the infrared sensitivities of the short-distance hard parts.

### 3.3.2 Final Factorized Form

The derivatives with respect to \( k_T \) are straightforward. It is most convenient to reexpress the derivatives in terms of derivatives with respect to \( \tilde{x} \) or \( x \). After working
out the derivatives, we obtain \[18\]

\[
E_l \frac{d\sigma^1(D)}{d^3l} = \alpha_{em} (4\pi \alpha_s)^2 e_q^2 \frac{1}{2x' s} \frac{1}{x' s + u} H_{q\bar{q}} \\
\times 2 \left\{ \left[ \frac{\partial^2}{\partial x^2} \left( \frac{T_q(x, A)}{x} \right) \right] \left( \frac{l_T^2}{(x' s + u)^2} \right) \\
+ \left[ \frac{\partial}{\partial x} \left( \frac{T_q(x, A)}{x} \right) \right] \left( \frac{-u}{x' s(x' s + u)} \right) \right\},
\] (3.28)

where \( x \) is given in Eq. (3.20d). The partonic hard part \( H_{q\bar{q}} \) is defined as

\[
H_{q\bar{q}} = C_1 x I_{1a}(x, x_k = 0, x_{k'} = 0).
\] (3.29)

Following the same derivation, we obtain contributions from the type-2 shown in Figure 3.3b. It is

\[
E_l \frac{d\sigma^2(D)}{d^3l} = \alpha_{em} (4\pi \alpha_s)^2 e_q^2 \frac{1}{2x' s} \frac{1}{x' s + u} H_g \\
\times 2 \left\{ \left[ \frac{\partial^2}{\partial x^2} \left( \frac{T_g(x, A)}{x} \right) \right] \left( \frac{l_T^2}{(x' s + u)^2} \right) \\
+ \left[ \frac{\partial}{\partial x} \left( \frac{T_g(x, A)}{x} \right) \right] \left( \frac{-u}{x' s(x' s + u)} \right) \right\},
\] (3.30)

In Eq. (3.30), \( T_g(x, A) \) is the four-gluon matrix element and defined as

\[
T_g(x, A) = \int \frac{dy_1^-}{2\pi} e^{ixp^+ y_1^-} \int \frac{dy_2^-}{2\pi} \theta(y_1^- - y_2^-) \theta(-y^-) \\
\times \frac{1}{xp^+} \langle p_A | F^{\sigma+}(y_2^-) F^{+\alpha}(0) F^{+\alpha}(y_1^-) F^{+\sigma}(y^-) | p_A \rangle,
\] (3.31)

with \( F_{\mu\nu} \) as the field strength.

In Eq. (3.30), the partonic hard part \( H_g \) is defined as

\[
H_g = C_2 I_{2a}(x, x_k = 0, x_{k'} = 0),
\] (3.32)
where $C_2$ is the overall color factor for the type-2 diagrams, and $I_{2a}$ is given by the real diagrams shown in Figure 3.5, and defined as

$$I_{2a} = \frac{1}{4} \text{Tr} \left[ \gamma \cdot (x'p' + x_kp + k_T) R_{2a}^{\beta \nu} \gamma \cdot l' L_{2a}^{\alpha \mu} \right] \left(-g_{\alpha \beta}\right) \left(-g_{\mu \nu}\right), \quad (3.33)$$

where $l' = x'p' + (x + x_k)p - l$ is the momentum carried by the quark going to the final state.

Similarly, for the type-3 diagrams, as sketched in Figure 3.3c, we obtain

$$E_l \frac{d\sigma(3)}{d^3l} = \alpha em (4\pi \alpha_s)^2 e_q^2 \frac{1}{2x's} \frac{1}{x's + u} H_q$$

$$\times 2 \left\{ \left[ \frac{\partial^2}{\partial x^2} \left( \frac{T_q(x, A)}{x} \right) \right] \left( \frac{P_T^2}{(x's + u)^2} \right) \right.$$

$$\left. + \left[ \frac{\partial}{\partial x} \left( \frac{T_q(x, A)}{x} \right) \right] \left( \frac{-u}{x's(x's + u)} \right) \right\}. \quad (3.34)$$

The partonic hard part $H_q$ in Eq. (3.34) is defined as

$$H_q = C_3 x I_{3a}(x, x_k = 0, x_k l = 0), \quad (3.35)$$

Figure 3.5: The real "Compton" diagrams corresponding to the four-gluon matrix element.
where $C_3$ is the overall color factor for the type-3 diagrams, and $I_{3a}$ is given by the real diagrams shown in Figure 3.6, and defined as

$$I_{3a} = \frac{1}{4} \text{Tr} \left[ \gamma \cdot p R^{\beta \nu} \gamma \cdot l' L_{3a}^{\alpha \mu} \right] \left( -g_{\alpha \beta} \right) \left( -g_{\mu \nu} \right),$$

where $l'$ is the same as that defined in Eq. (3.33).

Figure 3.6: The real “Compton” diagrams corresponding to the two-quark-two-gluon matrix element.

The twist-four matrix element $T_i(x, A)$ with $i = q, g$ in Eqs. (3.26) and (3.31) are originally introduced in Ref. [18]. They provide information about nuclear matter and its interaction with high energy probes, and are as fundamental as the normal parton distributions.

3.4 Analytic Result

To calculate the nuclear dependence parameter $\alpha(l)$ defined in Eq. (3.7), we need to evaluate both contributions of single scattering and double scattering.
For double scattering, the partonic short-distance hard parts, defined in Eqs. (3.29), (3.32) and (3.35), can be easily evaluated by calculating corresponding Feynman diagrams shown in Figures 3.4, 3.5, and 3.6. They are given by

\[ H_{qq} = \left( \frac{2}{27} \right) \left( \frac{-u}{x's + u} + \frac{x's + u}{-u} \right) \], \hspace{1cm} (3.37a)

\[ H_g = \left( \frac{1}{36} \right) \left( \frac{x's}{x's + u} + \frac{x's + u}{x's} \right) \], \hspace{1cm} (3.37b)

\[ H_q = \left( \frac{1}{16} \right) \left( \frac{x's}{-u} + \frac{-u}{x's} \right) \]. \hspace{1cm} (3.37c)

After convoluted Eqs. (3.28), (3.30), and (3.34) with corresponding parton distributions from the beam, following Eq. (3.4), we obtain the complete analytical expressions for the double scattering contribution in hadron-nucleus collisions:

\[
E_l \frac{d\sigma^{(D)}_{hA \rightarrow \gamma^{(l)}}}{d^2 \vec{l}} = \alpha_{em}(4\pi \alpha_s)^2 \int dx' dx \delta \left( x - \frac{-x't}{x's + u} \right) \left( \frac{1}{x's} \right) \left( \frac{1}{x's + u} \right) \times \sum_q e_q^2 \left[ f_{q/h}(x') \Phi_q(x, x', A) H_{qq} + f_{q/h}(x') \Phi_g(x, x', A) H_g + f_{g/h}(x') \Phi_q(x, x', A) H_q \right],
\] (3.38)

where \( \sum_q \) runs over all quark and antiquark flavors. In Eq. (3.38), the functions \( \Phi_i \) with \( i = q, g \) represent the effective parton flux from the nucleus, are given by

\[
\Phi_i = \left[ \frac{\partial^2}{\partial x^2} \left( \frac{T_i(x, A)}{x} \right) \right] \left( \frac{1}{x's + u} \right)^2 + \left[ \frac{\partial}{\partial x} \left( \frac{T_i(x, A)}{x} \right) \right] \left( \frac{-u}{x's(x's + u)} \right). \hspace{1cm} (3.39)
\]

The \( T_i(x, A) \) with \( i = q, g \) in Eq. (3.39) are the twist-four matrix elements in nuclei, given by Eqs. (3.26) and (3.31).

For single scattering, following Eq. (3.3), the lowest order invariant cross section
for direct photon production is given by Ref. [4]

\[ E \frac{d\sigma^{(S)}_{hN \rightarrow \gamma l}}{d\Omega_l} = \sum_{a,b} \int dx' f_a(x') \int dx f_b / N(x) \delta \left( x - \frac{x't}{x's + u} \right) \times \alpha_{em} \alpha_s \left( \frac{1}{s} \right) \left( \frac{1}{x's + u} \right) |M_{ab \rightarrow \gamma}|^2 , \] (3.40)

where \( \sum_{a,b} \) run over all gluon, quark and antiquark flavors; and the matrix elements for the "Annihilation" and "Compton" subprocesses, sketched in Figure 3.1, are given by

\[ |M_{q\bar{q} \rightarrow \gamma g}|^2 = e_q^2 \left( \frac{4}{9} \right) 2 \left( \frac{\hat{u}}{\hat{t}} + \frac{\hat{u}}{-\hat{u}} \right) ; \] (3.41a)

\[ |M_{qg \rightarrow \gamma q'}|^2 = e_q^2 \left( \frac{1}{6} \right) 2 \left( \frac{-\hat{t}}{\hat{s}} + \frac{\hat{s}}{-\hat{t}} \right) ; \] (3.41b)

where \( e_q \) is the fractional charge carried by a quark of type "q". The invariants \( \hat{s}, \hat{t} \) and \( \hat{u} \) are usual Mandelstam invariants for the parton-parton subprocess. They are related to those at the hadron-nucleon interaction by

\[ \hat{s} = x' x s, \quad \hat{t} = x' t, \quad \hat{u} = x u ; \] (3.42a)

\[ s = (p' + p)^2, \quad t = (p' - l)^2, \quad u = (p - l)^2 . \] (3.42b)

### 3.5 Model for Multi-parton Distribution and Numerical Results

Using the analytical results presented in Eqs. (3.40) and (3.38), we can evaluate the nuclear dependence parameter \( \alpha(l) \) defined in Eq. (3.7), and compare our numerical results with recent data from Fermilab experiment E706 [20].

The nuclear dependence parameter \( \alpha(l) \) defined in Eq. (3.7) depends on contributions from both single scattering and double scattering. All these contributions
depend on the nonperturbative parton distributions or multi-parton correlation functions. In deriving the following numerical results, the Set 1 pion distributions of Ref. [26] are used for pion beams; and the CTEQ3L parton distributions of Ref. [27] are used for free nucleons. The twist-4 multi-parton correlation functions defined in Eq. (3.26) and Eq. (3.31) have not been well-measured. By comparing the definition of these twist-4 correlation functions with the normal twist-2 parton distributions [28], authors of Ref. [18] proposed following approximate expressions for the twist-4 correlation functions,

\[ T_i(x, A) = \lambda^2 A^{1/3} f_{i/A}(x, A) \]  

(3.43)

where \( i = q, \bar{q}, \) and \( g \). The \( f_{i/A} \) are the effective twist-2 parton distributions in nuclei, and the factor \( A^{1/3} \) is proportional to the size of the nucleus. The proportionality constant, \( \lambda^2 \), has a dimension of \([\text{energy}]^2\) due to the dimension difference between twist-4 and twist-2 matrix elements. The value of \( \lambda^2 \) was estimated in Ref. [29] by using the measured nuclear enhancement of the momentum imbalance of two jets in photon-nucleus collisions [30, 31], and was found

\[ \lambda^2 \approx 0.05 \sim 0.1 \text{GeV}^2. \]  

(3.44)

This value is not too far away from the naive expectation from the dimensional analysis, \( \lambda^2 \sim \Lambda_{\text{QCD}}^2 \). In the calculation below, we use \( \lambda^2 = 0.1 \text{ GeV}^2 \). Therefore, the numerical results presented here can be viewed as the upper limit of the theoretical predictions.

The \( A^{1/3} \) dependence of the twist-4 multiparton correlation functions, introduced in Eq. (3.43), is not unique. From the definition of the correlation functions in Eqs. (3.26) and (3.31), the lack of oscillation factors for both \( y^- \) and \( y_2^- \) integrals
can, in principle, give nuclear enhancement proportional to $A^{2/3}$. The $A^{1/3}$ dependence is a result of the assumption that the positions of two field strengths (at $y^-$ and $y^+_2$, respectively) are confined within one nucleon.

In Eq. (3.43), the effective nuclear parton distributions $f_i/A$ should have the same operator definitions of the normal parton distributions with free nucleon states replaced by the nuclear states. For a nucleus with $Z$ protons and atomic number $A$, we define

$$f_i/A(x, A) = A \left( \frac{N}{A} f_i/N(x) + \frac{Z}{A} f_i/P(x) \right) R_i^{\text{EMC}}(x, A),$$

where $f_i/N(x)$ and $f_i/P(x)$ with $i = q, q, g$ are normal parton distributions in a free neutron and proton, respectively; and $N = A - Z$. The factor $R_i^{\text{EMC}}$ takes care of the EMC effect in these effective nuclear parton distributions. We adopted the $R_i^{\text{EMC}}$ from Ref. [32], since they fit data well. However, at fixed target energies, the $x$ values covered by the direct photon experiments are large and out of the nuclear shadowing region. The integration over $dx'$ in Eqs. (3.40) and (3.38) averages out the EMC effect from the large $x$ region. Actually, one can neglect the $R_i^{\text{EMC}}$ in Eq. (3.45).

Using parton distributions and correlation functions introduced above, and the analytical results presented in Eqs. (3.40) and (3.38), we can predict the nuclear dependence parameter $\alpha(l)$, defined in Eq. (3.7), without any further free parameter.

In Figure 3.7, we compare our numerical predictions for the nuclear dependence parameter with the recent experimental data from Fermilab experiment E706 [20]. The $\alpha_{E706}(l)$ presented in Figure 3.7 is slightly different from that defined in Eq. (3.7). E706 measured the direct photon cross sections with the $\pi^-$ beam on two
different targets: Cu($A = 63.55$) and Be($A = 9.01$); and the $\alpha_{E706}(l)$ was extracted according to following definition

$$\frac{\sigma_{Cu}(l)}{\sigma_{Be}(l)} \equiv \left( \frac{A_{Cu}}{A_{Be}} \right)^{\alpha_{E706}(l)}.$$  \hspace{1cm} (3.46)

The beam energy is $p' = 515$ GeV. It is clear that the theoretical calculation presented in this work is consistent with the data.

It is evident from Figure 3.7 that the nuclear dependence parameter $\alpha_{E706}(l)$
is very close to unity, or equivalently, the Cronin effect for direct photon production is very small, and much smaller than that observed in the single particle inclusive cross sections [13]. One clear difference between the direct photon production and the single particle inclusive cross section is that direct photon production has only initial state multiple scattering, while the single particle inclusive has both initial and final state multiple scattering. In addition, the single particle inclusive cross sections depend on the parton-to-hadron fragmentation functions.

As pointed out in Ref. [18], the multiple scattering contribution is most important when the momentum fraction $x$ from the nuclear correlation functions is large because of the derivatives with respective to the $x$, which were introduced in Eq. (3.39). However, for direct photon production, the kinematics could not fix all parton momentum fractions, and still leave one momentum fraction to be integrated. For example, the $dx'$ in Eqs. (3.40) and (3.38) still needs to be integrated. Because of the steeply falling feature of the distributions and correlation functions, the cross sections in the central region ($x_F \sim 0$) for a given value of $l_T$ is dominated by the distributions with momentum fractions $x' \sim x \sim x_T = 2l_T/\sqrt{s}$, which is less than 0.6 even for the largest value of $l_T$ shown in Figure 3.7. Therefore, the double scattering contribution is relatively small because of the insignificant contributions from the derivative terms, and consequently, the $\alpha_{E706}(l)$ is close to one.

However, for the single particle production, the parton-to-hadron fragmentation functions will effectively enhance the contribution of the large $x$ region because of all fragmentation functions vanish when $z$ goes to 1. Kinematically, the direct photon production corresponds to the single particle production at $z = 1$. Therefore, we expect that single particle production has a larger Cronin effect than the direct photon production.
production at the exact same kinematics, even without including the contribution from the final state multiple scattering.

In the case of a pion beam, the quark-antiquark "annihilation" subprocess, sketched in Figure 3.1a, dominates the production of direct photons at fixed target energies, due to the valence antiquarks in the beam. However, if we use a proton beam, the quark-gluon "Compton" subprocess, as sketched in Figure 3.1b, is more important for the production of direct photons. Therefore, the direct photon production with a proton beam is more sensitive to the gluon distributions in hadrons. Figures 3.8, 3.9, 3.10 and 3.11 show our predictions of the nuclear dependence parameter \( \alpha(l) \), defined in Eq. (3.7), for a \( \pi^- \) and proton beam, respectively.

In Figures 3.8 and 3.10, the nuclear dependence parameter \( \alpha(l) \), defined in Eq. (3.7), is plotted as a function of the photon's transverse momentum \( l_T \) at \( x_F = 0 \). In plotting these figures, a 515 GeV beam energy and a Copper target were assumed. The same parton distributions and correlation functions as for the Figure 3.7 were used. Just as in the case of Figure 3.7, the value of \( \alpha(l) \) is very close to unity for both pion and proton beams. Changing a pion beam to a proton beam does not affect the kinematics of the collisions. The effective values of parton momentum fractions from the nuclear target are the same for both cases. Therefore, as explained above, the values of \( \alpha(l) \) are close to unity due to the fact that the effective parton momentum fractions from the nuclear target are not large enough, and the derivatives do not give the double scattering contribution enough enhancement.

In order to enhance the contribution from double scattering, we need to look for events in the negative \( x_F \) region, where the effective values of parton momentum fractions from the target are larger. In Figures 3.9 and 3.11, we plot the nuclear
dependence parameter $\alpha(l)$, defined in Eq. (3.7), as a function of $x_F$ at $l_T = 4.0$, 6.0 and 8.0 GeV, respectively. The same beam, target and beam energy were used. It is clear that when $x_F$ becomes more negative, the values of $\alpha(l)$ increase. This is consistent with the fact that the more negative values of $x_F$, the larger the effective values of parton momentum fractions from the nuclear target, and, consequently, the larger the derivatives, defined in Eq. (3.39).

The nuclear dependence calculated here is known as a power correction (or a
“high-twist” effect) to the normal single scattering. As one expected, the ratio of double scattering over the single scattering is proportional to $1/l_T^2$, which vanishes as $l_T^2$ increases. However, because of the derivatives with respect to $x$ in Eq. (3.39), the ratio effectively has three types of terms: those proportional to $1/l_T^2$, $1/((1 - x) l_T^2)$, and $1/((1 - x)^2 l_T^2)$, respectively. For fixed values of $x$, all three terms vanish as $1/l_T^2$. However, in our case, the values of $x$ and $l_T^2$ are not independent. When effective values of $x$ from the target partons are small, all three types of terms should show the $1/l_T^2$ behavior. This is clearly evident in Figures 3.9 and 3.11: for a fixed value
of $x_F \sim 0$, $\alpha(l_T)$ decreases as $l_T$ increases. However, as $x_F$ decreases, the effective values of $x$ increase much faster for larger values of $l_T$ due to the phase space limit. As a result, the term proportional to $1/((1 - x)^2 l_T^2)$ becomes more important than the $1/l_T^2$ term. Therefore, it is possible that the nuclear dependence is larger for a larger $l_T$, when effective values of $x$ near 1. Such feature is evident in Figures 3.9 and 3.11 when $x_F$ is very negative. Of course, when $\alpha_s/((1 - x)^2 l_T^2)$ is of order of unity, we will have to take into account all higher power terms [33].

Comparing Figures 3.9 and 3.11, one finds that as $x_F$ grows more negative, the
values of $\alpha(l)$ with a proton beam increases much faster than with a pion beam. This is because the “Compton” subprocess dominates the production of direct photons in the case of a proton beam, and the gluon distribution in a proton falls off much faster than the valence quark distributions as the momentum fraction increases. The fast falling gluon distribution produces larger derivative terms, and therefore, larger values of $\alpha(l)$. Future data from Fermilab experiment E706 with a proton beam can test this feature.
4. PHOTONS WITH ISOLATION CUT

4.1 Why Impose Isolation Cut and Problems Raised

Although the cross section for the inclusive yield of high energy photons is well-defined, and can be calculated reliably within the context of QCD perturbation theory, the inclusive cross section may not be measurable at high energy experiments for observational reasons. Owing to backgrounds from, e.g., $\pi^0 \rightarrow \gamma\gamma$, a single high energy photon is observed and the cross section is measured only when the photon is relatively isolated. Isolation procedures differ in their details in different experiments at electron-positron and hadron-hadron collider facilities. Different choices of isolation definitions change only the details of the analysis, not the basic physics.

Owing to isolation, the predicted photon cross section develops explicit functional dependence on the isolation energy parameter $\epsilon_h$ and isolation cone $\delta$. The $\epsilon_h$ is defined as a ratio of the photon energy, $E_\gamma$, and hadronic energy, $E_{\text{hadron}}$, in the isolation cone.

A proper theoretical treatment of the isolated energetic photon yield requires careful consideration of the origins of both infrared and collinear singularities in QCD perturbation theory. In a theoretical calculation, photon isolation limits the final-state phase space accessible to accompanying gluons (g) and quarks (q). This phase space restriction may break the perfect cancellation of soft singularities in each
order of perturbation theory that guarantees reliable predictions in the inclusive case [34]. The breakdown of the cancellation of infrared singularities appears first at next-to-leading order in the fragmentation contributions. The associated physics can be summarized as follows. For the fragmentation contribution, hadronic energy in the isolation cone has two sources: a) energy from parton fragmentation, $E_{\text{frag}}$, and b) energy from non-fragmenting final-state partons, $E_{\text{cone}}_{\text{partons}}$, that enter the cone. When the maximum hadronic energy allowed in the isolation cone is saturated by the fragmentation energy, $E_{\text{max}} = E_{\text{frag}}$, there is no allowance for energy in the cone from other final-state partons. In particular, if there is a gluon in the final state, the phase space for this gluon becomes restricted. By contrast, isolation does not affect the virtual gluon exchange contribution. Therefore, in the isolated case, there is a possibility that the infrared singularity from the virtual contribution may not be cancelled completely by the restricted real contribution.

In a perturbative QCD (pQCD) calculation of the inclusive yield of photons, the quark-photon collinear singularities that arise in each order of perturbation theory, associated with the hadronic component of the photon, are subtracted and absorbed into quark-to-photon and gluon-to-photon fragmentation functions, $D(z, \mu^2)$, in accord with the factorization theorem [3]. The scale $\mu^2$ denotes the fragmentation scale that separates the non-perturbative domain from the region in which perturbation theory should apply; $z$ is the momentum fraction carried by the observed photon from its parent parton. Since fragmentation is a process in which photons are part of quark, antiquark, or gluon "jets", it is evident that photon isolation reduces the contribution from fragmentation terms. For a small value of the energy resolution parameter, $\epsilon_h$, the isolation cut eliminates most of the contribution from parton-to-
photon fragmentation. However, $\epsilon_h$ may never be equal to zero either experimentally or theoretically. Experimentally, the finiteness of $\epsilon_h$ is guaranteed by detector resolution. Theoretically, the perturbative calculation of the cross section for isolated photons is ill defined if $\epsilon_h$ vanishes [23]. Therefore, the isolated photon cross section always includes a nonperturbative fragmentation contribution.

Fragmentation is modeled in perturbation theory as a collinear process, whereas experimental cone sizes are finite ($\delta \neq 0$) and partons are manifested as sprays of hadrons. Correspondingly, there is an inherent conceptual incompatibility between theoretical, collinear fragmentation functions and empirical fragmentation functions, $D_{\text{exp}}(z, \mu^2, \delta)$, that more naturally would be defined with reference to a cone of specified size. In our calculation, we use the usual collinear fragmentation functions, $D_{c\rightarrow \gamma}(z, \mu^2)$. We derived the dependence of the cross section on cone size $\delta$ and energy isolation $\epsilon_h$. The results display a limited region of phase space in which the incompatibility of the collinear and finite cone size assumptions leads to an apparent infrared sensitivity for isolated photon cross section. Although this limited region is not of practical interest for experiments at current energies, it is important in understanding the impact of such infrared sensitivity on computations of prompt photon production in hadron-hadron reactions.

Previous theoretical studies of isolated prompt photon production in $e^+e^- \rightarrow \gamma X$ include those of Refs. [6, 34, 35, 36]. All four groups at LEP have published papers on prompt photon production [37]. A measurement of the photon fragmentation function from an analysis of the two-jet rate in $Z$ decays is reported by the ALEPH collaboration [38]. Practical aspects of confronting theoretical calculations with data from LEP are addressed in Ref. [12].
4.2 Isolated Photons in $e^+e^- \rightarrow \gamma X$

Our calculations of isolated photon yields in $e^+e^- \rightarrow \gamma X$ are carried out through first order in the electromagnetic coupling strength, $\alpha_{em}$, and the quark-to-photon and gluon-to-photon fragmentation contributions through first order in the strong coupling strength $\alpha_s$ [39]. The final results display the full angular dependence of the cross sections, separated into longitudinal $\sin^2 \theta_\gamma$ and transverse components $(1 + \cos^2 \theta_\gamma)$, where $\theta_\gamma$ is the direction of the $\gamma$ with respect to the $e^+e^-$ collision axis.

4.2.1 General Factorized Formula

The definition of isolated photons in $e^+e^- \rightarrow \gamma X$ is demonstrated in Figure 4.1. In this definition, a cone of half-angle $\delta$ is drawn about the direction of the photon's momentum, and the cross section is defined for photons accompanied by less than a specified amount of hadronic energy in the cone, e.g., $E_{h}^{cone} \leq E_{\text{max}} \equiv \epsilon_h E_{\gamma}$. Based on the definition of isolation, it is clear that the isolated photon cross section is part of the inclusive cross section. The isolated cross section can be written as the inclusive cross section minus a subtraction term [23]. As in the calculation of jet cross sections [40], the singularity structure of subtraction term is much easier to deal with than that of the isolated cross section itself, at the order in perturbation theory in which we are working. The derivation of the subtraction term is done in $n = 4 - 2\epsilon$ dimensions in order to display singularities explicitly. The hard-scattering matrix elements for subtraction terms are identical to those for the fully inclusive case discussed in Chapter 2, but the integration over momentum variables of the unobserved final-state partons is restricted by the requirements of photon isolation.
In $e^+e^- \rightarrow \gamma X$, as sketched in Figure 2.1, the general expression of the cross section for an $m$ parton final state, Eq. (2.1), is valid for both inclusive and isolated cross sections. The difference between the inclusive and isolated cross sections resides in the phase space, $dPS^{(m)}$, integration for the final state partons. For isolated cross sections, because of the isolation condition, not all partons can be integrated over all phase space. For example, partons with energy larger than $\epsilon_h E_\gamma$ are excluded from the cone of isolation about the observed photon.

As proposed in Ref. [23], the isolated cross section can be written as the following difference of cross sections:

$$
E_\gamma \frac{d\sigma^{iso}}{d^3 \ell} = E_\gamma \frac{d\sigma^{incl}}{d^3 \ell} - E_\gamma \frac{d\sigma^{sub}}{d^3 \ell}.
$$

In Eq. (4.1), $E_\gamma d\sigma^{incl}/d^3 \ell$ is the cross section for inclusive photons. It is well-defined in QCD perturbation theory. The factorized form of $E_\gamma d\sigma^{incl}/d^3 \ell$ is given by Eq. (2.11) in Section 2.2 of Chapter 2. The short-distance hard-scattering cross
sections \( E \frac{d\sigma^{incl}}{e^+e^-\rightarrow cX/d^3p_c} \) in Eq. (2.11) are derived in Chapter 2 through one-loop level.

From LEP experiments, cross sections for isolated photons \( E \frac{d\sigma^{iso}}{e^+e^-\rightarrow \gamma X/d^3\ell} \) are well-behaved and finite. Therefore, the subtraction term \( E \frac{d\sigma^{sub}}{e^+e^-\rightarrow \gamma X/d^3\ell} \), defined in Eq. (4.1), should be well-behaved as well. Since the available phase space for the isolated photon cross section is smaller than that for inclusive photons, \( E \frac{d\sigma^{sub}}{e^+e^-\rightarrow \gamma X/d^3\ell} \) should be positive and finite. In terms of the definition given in Eq. (4.1), \( E \frac{d\sigma^{sub}}{e^+e^-\rightarrow \gamma X/d^3\ell} \) can be viewed as a "cross section" for a photon "jet" with photon momentum \( \ell \) and hadronic energy in the "jet" cone \( E_{hcone} \) restricted to be larger than \( E_{max} = \epsilon_h E_\gamma \). The virtue of Eq. (4.1) is that the infrared and collinear singularities of the subtraction term \( \sigma^{sub} \) are much easier to deal with, at the order in which we are working, than those of \( \sigma^{iso} \) itself.

We assume as a working hypothesis that the subtraction term \( \sigma^{sub}_{e^+e^-\rightarrow \gamma X} \) can be factored in the same way as the inclusive cross section and expressed as a convolution:

\[
E \frac{d\sigma^{sub}}{e^+e^-\rightarrow \gamma X} = \sum_c \int_1^\infty \frac{dz}{z} E_c \frac{d\sigma^{sub}}{e^+e^-\rightarrow cX} \left( x_c = \frac{x_\gamma}{z} \right) D^{iso}_{c\rightarrow \gamma}(z, \delta). \tag{4.2}
\]

In Eq. (4.2), we include explicit dependence on the cone size \( \delta \) in the fragmentation functions in order to point out that, in principle, the fragmentation functions extracted from the cross section for isolated photons may depend on the size of isolation cone. Since the inclusive cross section \( \sigma^{incl} \) in Eq. (4.1) is well-defined in QCD perturbation theory, it is our task to show to what extent the short-distance subtraction terms \( \sigma^{sub}_{e^+e^-\rightarrow cX} \) are free from infrared and collinear divergences.

The limits of integration over \( z \) in Eq. (4.2) are fixed by kinematics:
$x_c = x_\gamma / z \leq 1$. However, the isolation condition imposes, in addition, a requirement on the total hadronic energy in the isolation cone, $E_{h}^{\text{cone}} \geq E_{\text{max}} = \epsilon_h E_\gamma$.

Generally, $E_{h}^{\text{cone}}$ in the cone has two sources: $E_{\text{frag}} + E_{\text{partons}}$, as is illustrated in Figure 4.2. The fragmentation component, $E_{\text{frag}}$, is the hadronic component of the "jet" from which the energetic photon itself emerges. In the collinear approximation,

$$E_{\text{frag}} = (1 - z)E_c = \left(\frac{1-z}{z}\right)E_\gamma.$$ (4.3)

The second component, $E_{\text{partons}}$, is the contribution from other final state partons that are emitted into the region of phase space defined by the photon isolation cone.

![Figure 4.2: An isolation cone containing a parton $c$ that fragments into a $\gamma$ plus hadronic energy $E_{\text{frag}}$. In addition, the cone includes a gluon that fragments giving hadronic energy $E_{\text{parton}}$.](image-url)
For the subtraction term,

\[ E_{h}^{\text{cone}} = E_{\text{partons}} + E_{\text{frag}} \]

\[ = E_{\text{partons}} + \left( \frac{1 - z}{z} \right) E_{\gamma} \geq E_{\text{max}} \equiv \epsilon_{h} E_{\gamma}. \] (4.4)

If \( z \leq 1/(1 + \epsilon_{h}) \), or, equivalently, \( E_{\text{frag}} \geq \epsilon_{h} E_{\gamma} \), the constraint \( E_{h}^{\text{cone}} \geq \epsilon_{h} E_{\gamma} \) is satisfied for any value of \( E_{\text{partons}} \). Correspondingly, there is no restriction on the phase space for accompanying final state partons. Consequently, \( \hat{\sigma}_{e^{+}e^{-} \rightarrow cX}^{\text{incl}} \) for \( z \leq 1/(1 + \epsilon_{h}) \).

If \( z > 1/(1 + \epsilon_{h}) \), to satisfy \( E_{h}^{\text{cone}} \geq \epsilon_{h} E_{\gamma} \), it is necessary that

\[ E_{\text{partons}} \geq E_{\text{max}} - E_{\text{frag}} \equiv E_{\text{min}}(z) = \left[ (1 + \epsilon_{h}) - \frac{1}{z} \right] E_{\gamma}. \] (4.5)

Notice that \( E_{\text{min}}(z) > 0 \) as long as \( z > 1/(1 + \epsilon_{h}) \), and \( E_{\text{min}}(z) = 0 \) if \( z = 1/(1 + \epsilon_{h}) \).

Taking into account the constraints from the isolation condition, we can rewrite Eq. (4.2) as

\[ E_{\gamma} \frac{d\sigma_{e^{+}e^{-} \rightarrow cX}}{d\ell} \]

\[ = \sum_{c} \left[ x_{\gamma}, \frac{1}{1 + \epsilon_{h}} \right] \frac{dz}{z^{2}} E_{c} \frac{d\hat{\sigma}_{e^{+}e^{-} \rightarrow cX}^{\text{incl}}}{d^{3}p_{c}} \left( x_{c} = \frac{x_{\gamma}}{z} \right) \left( x_{c} \geq E_{\text{partons}} \geq E_{\text{min}} \right) \]

\[ \times D_{c \rightarrow \gamma}^{\text{iso}}(z, \delta) \]

\[ + \sum_{c} \left[ x_{\gamma}, \frac{1}{1 + \epsilon_{h}} \right] \frac{dz}{z^{2}} E_{c} \frac{d\hat{\sigma}_{e^{+}e^{-} \rightarrow cX}^{\text{sub}}}{d^{3}p_{c}} \left( x_{c} = \frac{x_{\gamma}}{z} \right) \]

\[ \times D_{c \rightarrow \gamma}^{\text{iso}}(z, \delta) \] (4.6)

\[ \equiv \sum_{c} \left[ E_{c} \frac{d\hat{\sigma}_{e^{+}e^{-} \rightarrow cX}^{\text{incl}}}{d^{3}p_{c}} \otimes D_{c \rightarrow \gamma}^{\text{iso}}(z, \delta) \right. \]

\[ + E_{c} \frac{d\hat{\sigma}_{e^{+}e^{-} \rightarrow cX}^{\text{sub}}}{d^{3}p_{c}} \otimes D_{c \rightarrow \gamma}^{\text{iso}}(z, \delta) \]
It is important to note in Eq. (4.6) that the short-distance subtraction terms $E_c d\delta_{sub}^{\gamma\rightarrow cX}$ are needed only for $x_c \leq \min[x\gamma(1 + \epsilon_h), 1] < 1$, if $x\gamma < 1/(1 + \epsilon_h)$.

The modified convolution signs "\(\hat{\otimes}\)" and "\(\check{\otimes}\)" in Eq. (4.6) are defined in the same way as the convolution sign "\(\otimes\)" in Eq. (2.11), except for the limits of the $z$-integration. For functions $A(x_c)$ and $B(z)$, we define

\[
A(x_c) \hat{\otimes} B(z) = \int_{x_\gamma}^{1} \frac{dz}{z^2} A(x_c = x\gamma/z) B(z) ; \quad (4.7a)
\]

\[
A(x_c) \check{\otimes} B(z) = \int_{\max\left[x\gamma, 1/(1 + \epsilon_h)\right]}^{1} \frac{dz}{z^2} A(x_c = x\gamma/z) B(z) ; \quad (4.7b)
\]

\[
A(x_c) \otimes B(z) = \int_{x_\gamma}^{\max\left[x\gamma, 1/(1 + \epsilon_h)\right]} \frac{dz}{z^2} A(x_c = x\gamma/z) B(z) . \quad (4.7c)
\]

Notice the identity $\otimes = \hat{\otimes} + \check{\otimes}$.

Using Eq. (2.11) in Section 2.2 for the inclusive cross section, and (4.6) for the subtraction term, from Eq. (4.1), we can derive a simplified expression for the isolated cross section

\[
\frac{d\sigma^{iso}}{E_\gamma \frac{d^3\ell}{d^3p_c}} = \sum_c \int_{x_\gamma}^{\max\left[x\gamma, 1/(1 + \epsilon_h)\right]} \frac{dz}{z^2} \left( E_c \left. \frac{d\sigma^{incl}}{d^3p_c} \right|_{E_{partons} \geq E_{min}} - E_c \left. \frac{d\delta_{sub}^{\gamma\rightarrow cX}}{d^3p_c} \right|_{E_c^\rightarrow \gamma(z)} \right) D_c^\rightarrow \gamma(z) \nonumber
\]

\[
= \sum_c \left( E_c \left. \frac{d\sigma^{incl}}{d^3p_c} \right|_{E_{c_\gamma \rightarrow \gamma(z)}} - E_c \left. \frac{d\delta_{sub}^{\gamma\rightarrow cX}}{d^3p_c} \right|_{E_{c_\gamma \rightarrow \gamma(z)}} \right) \hat{\otimes} D_c^\rightarrow \gamma(z) . \quad (4.8)
\]

In deriving Eq. (4.8), $D_c^{iso\gamma(z, \delta)} = D_c^\rightarrow \gamma(z)$, is assumed for simplicity.
When \( c = \gamma \), \( D_{c \to \gamma}(z) = \delta(1 - z) \) through order \( O(\alpha_{em}) \), and \( E_{frag} = 0 \) (i.e., \( E_{\text{min}} = \epsilon_{h} E_{\gamma} \)). Therefore, we may rewrite Eq. (4.8) more explicitly as

\[
E_{\gamma} \frac{d\sigma^{\text{iso}}}{e^{+}e^{-}\to\gamma X} = E_{\gamma} \frac{d\sigma^{\text{iso}}}{e^{+}e^{-}\to\gamma X} + \sum_{c=\bar{q},q,g} E_{c} \frac{d\sigma^{\text{iso}}}{e^{+}e^{-}\to cX} \odot D_{c \to \gamma}(z). \tag{4.9}
\]

The isolated short-distance hard-scattering cross sections are defined as

\[
E_{\gamma} \frac{d\sigma^{\text{iso}}}{e^{+}e^{-}\to\gamma X} = E_{\gamma} \frac{d\sigma^{\text{iso}}}{e^{+}e^{-}\to\gamma X} \bigg|_{\text{partons inside cone; } E_{\text{partons}} \geq \epsilon_{h} E_{\gamma}}.
\]

\[
E_{c} \frac{d\sigma^{\text{iso}}}{e^{+}e^{-}\to cX} = E_{c} \frac{d\sigma^{\text{iso}}}{e^{+}e^{-}\to cX} \bigg|_{\text{partons inside cone; } E_{\text{partons}} \geq E_{\text{min}}}.
\]

The value of \( E_{\text{min}} \) is specified in Eq. (4.5).

When \( x_{\gamma} \to 1/(1 + \epsilon_{h}) \), the subtraction term \( \delta^{\text{sub}} \) and, consequently, the isolated cross section \( \delta^{\text{iso}} \) develop a logarithmic singularity, \( \ell n |(1 + \epsilon_{h}) - 1/x_{\gamma}| \). This singularity is discussed in detail later.

### 4.2.2 Lowest Order Contribution

In lowest order, \( O(\alpha_{em}^{o} \alpha_{s}^{o}) \), photon production occurs only through the quark or antiquark fragmentation process, as sketched in Figure 2.3. In this case \( c = q, \bar{q} \) in Eq. (4.8); and there is no direct production of photons. Owing to momentum
balance, the quark and anti-quark have equal but opposite momentum in the overall center-of-mass system, and there can be no accompanying parton in the isolation cone around the fragmenting parton. Therefore, at $O(\alpha^0_{em} \alpha_s^0)$, $x_c = 1$ and

$$\left. \frac{d\hat{\sigma}^{(0)}_{sub}}{d^3p_c} \right|_{\text{partons inside cone}} = 0 . \quad (4.11)$$

Substituting Eq. (4.11) into Eq. (4.8), we derive

$$E_l \frac{d\sigma^{(0)}_{iso}}{d^3l} = \int_{\text{max}}^1 dz \sum_{c=q,\bar{q}} E_c \frac{d\hat{\sigma}^{(0)\text{incl}}}{d^3p_c} \left( x_c = \frac{x}{z} \right) D_{c\to\gamma}(z). \quad (4.12)$$

The lowest order hard-scattering cross section $E_c d\hat{\sigma}^{(0)\text{incl}}_{e^+e^-\to cX}/d^3p_c$ in $n$ dimensions is derived in Section 2.2 of Chapter 2. The result is expressed in Eq. (2.18). Using the expression of $E_c d\hat{\sigma}^{(0)\text{incl}}_{e^+e^-\to cX}/d^3p_c$ in Eq. (2.18), and substituting it into Eq. (4.12), we obtain the lowest order isolated cross section [6]

$$E_l \frac{d\sigma^{(0)\text{iso}}}{d^3l} = \sum_q \left[ \frac{2}{s} F_{q\bar{q}}(s) \right] \alpha_{em} N_c (1 + \cos^2 \theta) \frac{1}{x} D_{q\to\gamma}(x, \mu_F), \quad (4.13)$$

for $x > 1/(1 + \epsilon_h)$. At this order, the isolated cross section vanishes if $x < 1/(1 + \epsilon_h)$. The angles $\theta$ and $\theta_c$ are identical since we take all products of the fragmentation to be collinear. The overall factor of 2 in Eq. (4.13) accounts for the $\bar{q}$ contribution. The result given in Eq. (4.13) is consistent with those derived in Refs. [35, 36].
4.2.3 Factorized Form for the Subtraction Term at One-loop Order

As in the inclusive case, there are three distinct contributions to the subtraction terms for $e^+e^- \rightarrow \gamma X$ at one-loop level in perturbation theory, as shown in Eq. (2.20) in Section 2.3 of Chapter 2. To calculate the short-distance partonic cross sections (hard parts), $\hat{\sigma}_{\text{sub}}$ in Eq. (4.10), we first apply the factorized form in Eq. (4.6) to parton states perturbatively order-by-order in the coupling constants, and we then extract the perturbative expressions for $\hat{\sigma}_{\text{sub}}$.

To derive the direct production term corresponding to Eq. (2.20a), $\hat{\sigma}_{\text{sub}}^{(1)}e^+e^-\rightarrow\gamma X$, we apply Eq. (4.6) perturbatively to first order in $\alpha_{em}$. We obtain

$$
\sigma_{e^+e^-\rightarrow\gamma X}^{(1)\text{sub}} \bigg|_{E_q(\text{or } E_{\bar{q}}) \geq E_h E_\gamma} = \hat{\sigma}_{e^+e^-\rightarrow\gamma X}^{(1)\text{sub}}(x_{q\gamma}) \bar{\otimes} D_{\gamma \rightarrow \gamma}(z) + \hat{\sigma}_{e^+e^-\rightarrow qX}^{(0)\text{sub}}(x_{q \gamma}) \bar{\otimes} D_{q \rightarrow \gamma}^{(1)}(z) + \hat{\sigma}_{e^+e^-\rightarrow qX}^{(1)\text{incl}}(x_{q \gamma}) \bar{\otimes} D_{q \rightarrow \gamma}^{(1)}(z) + (q \rightarrow \bar{q}).
$$

(4.14)

Since the zeroth order subtraction term $\hat{\sigma}_{e^+e^-\rightarrow \gamma X}^{(0)\text{sub}}$ vanishes, Eq. (4.11), and the zeroth order photon-photon fragmentation function $D_{\gamma \rightarrow \gamma}^{(0)}(z) = \delta(1 - z)$, the expression for the short-distance hard part becomes

$$
\hat{\sigma}_{e^+e^-\rightarrow\gamma X}^{(1)\text{sub}}(x_{q\gamma}) = \sigma_{e^+e^-\rightarrow\gamma X}^{(1)\text{sub}} \bigg|_{E_q(\text{or } E_{\bar{q}}) \geq E_h E_\gamma} - \hat{\sigma}_{e^+e^-\rightarrow qX}^{(0)\text{incl}}(x_{q \gamma}) \bar{\otimes} D_{q \rightarrow \gamma}^{(1)}(z) - (q \rightarrow \bar{q}).
$$

(4.15)

In this equation, $\hat{\sigma}_{e^+e^-\rightarrow qX}^{(0)\text{incl}}$ has already been derived in Section 2.2 in the inclusive case, and can be obtained from Eq. (2.18), and the modified convolution "\(\bar{\otimes}\)" is defined in Eq. (4.7c). The perturbative fragmentation function $D_{q \rightarrow \gamma}^{(1)}(z)$ in
\( n \)-dimensions is given by Eq. (2.47) in Section 2.4 of Chapter 2, and \( D_{q\to\gamma}^{(1)}(z) = D_{q\to\gamma}^{(1)}(z) \). Although \( \sigma_{e^+e^-\to\gamma X}^{(1)\text{sub}} \) and the perturbative fragmentation functions \( D_{q\to\gamma}^{(1)}(z) \) are both formally divergent as \( \epsilon \to 0 \), these divergences cancel and leave a finite expression for \( \sigma_{e^+e^-\to\gamma X}^{(1)\text{sub}} \), if the conventional QCD factorization theorem holds [3].

For the gluon fragmentation contribution: \( e^+e^- \to g \to \gamma \) (Eq. (2.20c)), we apply Eq. (4.6) to a gluon state perturbatively to first order in \( \alpha_s \). Letting \( E_g \) be the gluon's energy, we obtain

\[
\sigma_{e^+e^-\to g X}^{(1)\text{sub}} \bigg|_{E_g(\text{or } E\overline{q})} \geq \epsilon_{\text{min}} E_g = \sigma_{e^+e^-\to g X}(x_g) \hat{\sigma}_{g\to g}^{(0)}(z) + \sigma_{e^+e^-\to q X}(x_q) \hat{\sigma}_{q\to g}^{(1)}(z) + \sigma_{e^+e^-\to q X}(x_q) \hat{\sigma}_{q\to g}^{(1)}(z) + (q \to \overline{q}) \, ,
\]

where \( x_g = 2E_g/\sqrt{s} \), and \( x_q = 2E_q/\sqrt{s} \). In Eq. (4.16), the modified convolutions are defined in the same way as in Eq. (4.7), except that \( x_{\gamma} \) is replaced by \( x_g \), and \( \epsilon_h \) is replaced by \( \epsilon_{\text{min}} \):

\[
\epsilon_{\text{min}}(z) = \frac{E_{\text{min}}(z)}{E_g} = (1 + \epsilon_h)z - 1 \geq 0 \, ,
\]

where \( z = x_{\gamma}/x_g \). Because \( z \leq 1 \), there are the restrictions

\[
\epsilon_{\text{min}}(z) \leq \epsilon_h \; ; \quad \frac{1}{1 + \epsilon_{\text{min}}} \geq \frac{1}{1 + \epsilon_h} \, .
\]

Consequently, owing to Eq. (4.11), \( \hat{\sigma}_{e^+e^-\to q X}(x_q) = 0 \). Since \( D_{g\to g}^{(0)}(z) = \delta(1-z) \),
Eq. (4.16) can be rewritten as
\[
\hat{\sigma}_{\text{sub}}^{(1)}(e^+e^- \rightarrow gX(xg)) = \sigma_{\text{sub}}^{(1)}(e^+e^- \rightarrow gX) \bigg|_{E_g(\text{or } E_q) \geq \varepsilon_{\text{min}} E_q} - \hat{\sigma}_{\text{incl}}^{(0)}(e^+e^- \rightarrow qX(xq)) \otimes D_q^{(1)}(q \rightarrow q) (q \rightarrow \bar{q}). \tag{4.19}
\]

In Eq. (4.19), the divergent cross section \( \sigma_{\text{sub}}^{(1)}(e^+e^- \rightarrow gX) \) is evaluated from the Feynman graphs shown in Figure 2.5, and the quark-to-gluon collinear divergences are embedded in the first-order fragmentation function \( D_q^{(1)}(q \rightarrow g) \). The function \( D_q^{(1)}(q \rightarrow g) \) is the same as \( D_{q \rightarrow \gamma}^{(1)} \) of Eq. (2.47), except that \( e_q^2(\alpha_{\text{em}}/2\pi) \) is replaced by \( C_F(\alpha_s/2\pi) \).

The color factor \( C_F = 4/3 \).

For the quark fragmentation contribution: \( e^+e^- \rightarrow q \rightarrow \gamma \) (Eq. (2.20b)), we apply Eq. (4.6) to a quark state perturbatively at first order in \( \alpha_s \). The short-distance hard part for quark fragmentation can be obtained in a fashion similar to the derivation of \( \hat{\sigma}_{\text{sub}}^{(1)}(e^+e^- \rightarrow gX) \):
\[
\hat{\sigma}_{\text{sub}}^{(1)}(e^+e^- \rightarrow qX(xq)) = \sigma_{\text{sub}}^{(1)}(e^+e^- \rightarrow qX) \bigg|_{E_g(\text{or } E_q) \geq \varepsilon_{\text{min}} E_q} - \hat{\sigma}_{\text{incl}}^{(0)}(xq) \otimes D_q^{(1)}(q \rightarrow q) (q \rightarrow \bar{q}). \tag{4.20}
\]

The Feynman graphs that contribute to \( \sigma_{\text{sub}}^{(1)}(e^+e^- \rightarrow qX) \) are sketched in Figure 2.6. In this fragmentation process, the quark is effectively “observed” through \( q \rightarrow \gamma \) fragmentation. For the real gluon emission diagrams, sketched in Figure 2.6a, the gluon and antiquark are not observed and their momenta will be integrated over. Since \( \varepsilon_{\text{min}}(z) \), defined in Eq. (4.17), can be equal to zero, the contribution of the real emission diagrams to \( \sigma_{\text{sub}}^{(1)}(e^+e^- \rightarrow qX) \) shows both infrared and collinear singularities. The infrared singularity associated with soft gluon emission should be cancelled by a
contribution from the virtual gluon exchange diagrams shown in Figure 2.6b, if the conventional QCD factorization theorem holds. When the gluon is parallel to the fragmenting quark, the real emission diagrams manifest a collinear singularity that should be cancelled by the negative term in Eq. (4.20).

4.2.4 Parton Level Cross Sections: $\sigma^{(1)}_{sub}$

In order to derive the short-distance hard parts at one-loop level, defined in Eqs. (4.15), (4.19) and (4.20), we must compute the formally divergent partonic cross sections $\sigma^{(1)}_{sub}$ in $n$-dimensions, for $c = \gamma, g, q$ and $\bar{q}$. We must evaluate Feynman diagrams for $e^+e^- \rightarrow cX$ and $e^+e^- \rightarrow$ three particle final states, $(q\bar{q}\gamma)$ or $(q\bar{q}g)$; and $e^+e^-$ to two particle final states $(q\bar{q})$ with one-loop virtual gluon exchange. Similar to the inclusive cross section derived in Section 2.1 of Chapter 2, the general expression for the subtraction term is:

$$d\sigma^{(1)}_{sub} = \sum_{q,m} \left[ \frac{2}{s} F_q PC(s) \right] e^2 C_q \frac{1}{4} (H_1 + H_2) dPS^{(m)}_{sub},$$

where $m = 2$ or 3 corresponding to the number of final state particles, and the constant $C_q$ is an overall color factor. In Eq. (4.21), the functions $H_1$ and $H_2$ are defined by Eqs. (2.4) and (2.5) in Section 2.1. Since the hard scattering matrix elements for subtraction terms are identical to those for the inclusive case, the corresponding $H_1$ and $H_2$ for each subprocess are identical to those in the inclusive case. They are derived in Sections 2.2, 2.4 and 2.5 of Chapter 2. The factor $dPS^{(m)}_{sub}$ in Eq. (4.21) is the multi-particle phase space element. The two- and three-particle final state phase space elements, $dPS^{(2)}$ and $dPS^{(3)}$ are given by Eq. (A.4) and Eq. (A.35) in the Appendix. In $n = 4 - 2\epsilon$ dimensions, the multi-particle phase space expressions for
the subtraction term are formally identical to those for the calculation of the inclusive cross section [7], except that the limits of integration differ here, owing to the isolation condition. These phase space limits are derived below for each individual subprocess.

4.2.4.1 Derivation of $\hat{\sigma}^{(1)}_{\text{sub}}$ for $e^+e^-\rightarrow \gamma X$

From the derivation for the inclusive case in Section 2.4, following Eq. (2.41), we have,

$$\frac{1}{4} \left( H_1 + H_2^{\text{eff}} \right) = e_q^2 (e \mu)^4 \left\{ (1 + \cos^2 \theta_3 - 2 \epsilon) \left[ \frac{1 + (1 - x_3)^2}{x_3^2} \right] \left( \frac{1}{\hat{y}_{13}} + \frac{1}{\hat{y}_{23}} \right) + (1 + \cos^2 \theta_3 - 2 \epsilon) \left[ -2 - \epsilon \left( \frac{1}{\hat{y}_{13}} + \frac{1}{\hat{y}_{23}} \right) \right] + (1 - 3 \cos^2 \theta_3) \left[ \frac{4(1 - x_3)}{x_3^2} \right] \right\}. \quad (4.22)$$

In Eq. (4.22), we introduce the overall coupling factor and neglect terms that do not contribute in the limit $\epsilon \rightarrow 0$. $x_i$ and $y_{ij}$ in Eq. (4.22) are defined as

$$x_i = \frac{2E_i}{\sqrt{s}}, \quad y_{ij} = \frac{2p_i \cdot p_j}{s}, \quad (4.23)$$

with $i, j = 1, 2, 3$. For convenience, we use labels "1", "2", and "3" for the final-state $q, \bar{q}$ and $\gamma$, respectively; The variable $\hat{y}_{j3}$ with $j = 1, 2$ is defined as $\hat{y}_{j3} = y_{j3} / x_3$; $x_3$ is the same as $x_\gamma$, and $\theta_3$ is equal to $\theta_\gamma$ in this case.

The two $\delta$ functions in $dP S^{(3)}$, Eq. (A.35), are used to do the $dy_{12}$ and $dy_{23}$ integrations, and $y_{13}$ is left as the integration variable. Different choices of the integration variable are of course equivalent. In terms of $y_{13}$ and $x_3$, we have identities

$$y_{12} = 1 - x_3, \quad y_{23} = x_3 - y_{13}. \quad (4.24)$$
The limits of integration over $y_{13}$ are derived from the isolation requirement for the subtraction terms. In our parton level approach, hadronic energy in the isolation cone means a parton must be present inside the isolation cone of the observed photon. Second, this hadronic energy must be larger than $\epsilon_h E_\gamma$. We use $\delta$ to denote the half angle of the isolation cone. The statement that the quark (parton “1”) is inside the isolation cone provides the relationship

$$
\hat{y}_{23} \geq \frac{\cos^2(\delta/2)}{1 - x_3 \sin^2(\delta/2)},
$$

whereas the statement that the antiquark (parton “2”) is inside the isolation cone leads to

$$
\hat{y}_{13} \geq \frac{\cos^2(\delta/2)}{1 - x_3 \sin^2(\delta/2)}.
$$

Using Eq. (4.24) and $0 \leq y_{13} \leq x_3$, we derive the condition that either the quark or the antiquark is inside the isolation cone:

$$
0 \leq \hat{y}_{13} \leq \frac{(1 - x_3) \sin^2(\delta/2)}{1 - x_3 \sin^2(\delta/2)}.
$$

These two regions of the $\hat{y}_{13}$ integration do not overlap as long as $x_3 > 1 - \cot^2(\delta/2)$.

The second part of the isolation constraint is the requirement that the parton’s energy inside the isolation cone be larger than $\epsilon_h E_\gamma$. We derive

$$
\max[0, (1 + \epsilon_h - 1/x_3)] \leq \hat{y}_{13} \leq \frac{(1 - x_3) \sin^2(\delta/2)}{1 - x_3 \sin^2(\delta/2)},
$$

$$
1 - \frac{(1 - x_3) \sin^2(\delta/2)}{1 - x_3 \sin^2(\delta/2)} \leq \hat{y}_{13} \leq \min[1, 1 - (1 + \epsilon_h - 1/x_3)].
$$
For simplicity of notation and to facilitate comparison of our results with those of Ref. [36], we define

\[ y_c = \frac{(1 - x_3) \sin^2(\delta/2)}{1 - x_3 \sin^2(\delta/2)} \Rightarrow (1 - x_3) \frac{\delta^2}{4}, \]

\[ y_m = 1 + \epsilon_h - \frac{1}{x_3}, \]

(4.29)

and set \( x_3 = x_\gamma \). The first of the conditions in Eq. (4.28) indicates that \( y_m \) should be less than \( y_c \). Therefore, the subtraction term should vanish if \( y_m \geq y_c \), i.e.,

\[ \delta_{(1)\text{sub}}^{E-\gamma \to \gamma X} = 0 \quad \text{if} \quad x_\gamma \geq x_\gamma^{\max}(\delta, \epsilon_h). \]

(4.30)

Here, \( x_\gamma^{\max}(\delta, \epsilon_h) \) is expressed as

\[ x_\gamma^{\max}(\delta, \epsilon_h) \equiv \left( \frac{1}{1 + \epsilon_h} \right) \frac{2}{1 + \sqrt{1 - 4 \epsilon_h \sin^2(\delta/2)/(1 + \epsilon_h)^2}}. \]

(4.31)

We note that

\[ x_\gamma^{\max}(\delta, \epsilon_h) \geq \frac{1}{1 + \epsilon_h} \]

(4.32)

if \( \delta \neq 0 \). Eq. (4.30) is actually a condition due to energy-momentum conservation.

We consider next, in turn, the regions \( x_\gamma \leq 1/(1 + \epsilon_h) \) and \( x_\gamma > 1/(1 + \epsilon_h) \). If \( x_\gamma \leq 1/(1 + \epsilon_h) \), the integration over \( \hat{y}_{13} \) for \( d\sigma^{(1)\text{sub}} \) has two separate intervals, defined in Eq. (4.28),

\[ \int d\hat{y}_{13} \Rightarrow \int_0^{y_c} d\hat{y}_{13} + \int_{1-y_c}^{1} d\hat{y}_{13}. \]

(4.33)

Substituting Eq. (4.22) into Eq. (4.21), and performing this \( \hat{y}_{13} \) integration, we derive parton level cross section

\[ \frac{d\sigma^{(1)\text{sub}}}{E_\gamma e^+ e^- \to \gamma X} \]
\begin{align*}
\sum_{q} \left[ \frac{2}{s} F_{q}^{PC}(s) \right] \left[ \frac{2}{s} \frac{\alpha_{em} N_c}{\Gamma(1 - \epsilon)} \right] \frac{1}{x_{\gamma}} e_{q} \left( \frac{\alpha_{em}}{2\pi} \right) \times \left\{ (1 + \cos^{2} \theta_{\gamma} - 2\epsilon) \left( \frac{1 + (1 - x_{\gamma})^{2}}{x_{\gamma}} \right) \left( -\frac{1}{\epsilon} \right) 
+ (1 + \cos^{2} \theta_{\gamma}) \left[ \frac{1 + (1 - x_{\gamma})^{2}}{x_{\gamma}} \right] \left( \ln \left( \frac{s}{\mu_{\gamma}^{2}} \right) + \ln(x_{\gamma}^{2}(1 - x_{\gamma})) \right) 
+ \ln(y_{c}) + \ln(x_{\gamma}(1 - 2y_{c})) \right\} 
+ (1 - 3 \cos^{2} \theta_{\gamma}) \left[ 2 \left( \frac{1 - x_{\gamma}}{x_{\gamma}} \right) \right] (2y_{c}) \right) \right. 
\end{align*}

In Eq. (4.34), the usual modified minimal subtraction is

\[\mu_{\gamma}^{2} = \mu^{2} e^{-\gamma E},\]

where \(\gamma E\) is Euler’s constant. The \(1/\epsilon\) poles in Eq. (4.34) arise from the \(1/\gamma_{13}\) and \(1/\gamma_{23}\) terms in Eq. (4.22). Referring to Eq.(4.15), and using Eqs. (2.18) and (2.47) in Chapter 2, we observe that the \(1/\epsilon\) pole in Eq. (4.34) is exactly cancelled. Consequently, for \(x_{\gamma} \leq 1/(1 + \epsilon_{h})\), the finite short-distance hard part is

\begin{align*}
\sum_{q} \left[ \frac{2}{s} F_{q}^{PC}(s) \right] \left[ \frac{2}{s} \frac{\alpha_{em} N_c}{\Gamma(1 - \epsilon)} \right] \frac{1}{x_{\gamma}} e_{q} \left( \frac{\alpha_{em}}{2\pi} \right) \times \left\{ (1 + \cos^{2} \theta_{\gamma}) \left[ \frac{1 + (1 - x_{\gamma})^{2}}{x_{\gamma}} \right] \left( \ln \left( \frac{s}{\mu_{\gamma}^{2}} \right) + \ln(x_{\gamma}^{2}(1 - x_{\gamma})) \right) 
+ \ln(y_{c}) + \ln(x_{\gamma}(1 - 2y_{c})) \left\} \right. 
+ \left. (1 - 3 \cos^{2} \theta_{\gamma}) \left[ 2 \left( \frac{1 - x_{\gamma}}{x_{\gamma}} \right) \right] (2y_{c}) \right. \right. 
\end{align*}

In Eq. (4.36), we neglect terms of \(O(\delta^{2})\).
If \( x_\gamma > 1/(1 + \epsilon_h) \), the negative term, defined through the \( \bar{\otimes} \) convolution in Eq. (4.15), vanishes. In this region of phase space, there should not be any collinear divergences in the partonic cross section, \( E_\gamma d\sigma^{(1)\text{sub}}_{e^+e^-\rightarrow \gamma X}/d^3\ell \), since there is no counter term to cancel them. If the fragmentation process were exactly collinear, the subtraction term \( E_\gamma d\sigma^{(1)\text{sub}}_{e^+e^-\rightarrow \gamma X}/d^3\ell \) would vanish, kinematically, for \( x_\gamma > 1/(1 + \epsilon_h) \). However, a finite isolation cone \( \delta \neq 0 \) allows for a non-vanishing \( E_\gamma d\sigma^{(1)\text{sub}}_{e^+e^-\rightarrow \gamma X}/d^3\ell \) even when \( x_\gamma > 1/(1 + \epsilon_h) \). Equation (4.30) shows that there is a narrow interval, \( 1/(1 + \epsilon_h) < x_\gamma < x_\gamma^\text{max}(\delta, \epsilon_h) \), in which a non-vanishing \( E_\gamma d\sigma^{(1)\text{sub}}_{e^+e^-\rightarrow \gamma X}/d^3\ell \) is allowed kinematically.

In the narrow interval, \( 1/(1 + \epsilon_h) < x_\gamma < x_\gamma^\text{max}(\delta, \epsilon_h) \), the integration over \( \hat{y}_{13} \) for \( d\sigma^{(1)\text{sub}} \) has two separate regions:

\[
\int d\hat{y}_{13} = \int_{y_m}^{y_c} d\hat{y}_{13} + \int_{1-y_c}^{1-y_m} d\hat{y}_{13} \ .
\]

Eq. (4.29) shows that \( y_m > 0 \). Therefore, \( \hat{y}_{13} > 0 \) and \( \hat{y}_{13} < 1 \). Consequently, there is no collinear divergence and the integration over \( \hat{y}_{13} \) can be done completely in \( n = 4 \) dimensions.

For \( 1/(1 + \epsilon_h) < x_\gamma < x_\gamma^\text{max}(\delta, \epsilon_h) \), we derive

\[
E_\gamma \frac{d\sigma^{(1)\text{sub}}_{e^+e^-\rightarrow \gamma X}}{d^3\ell} = 2 \sum_q \left[ \frac{2}{s} F_q^{PC}(s) \right] \left[ \alpha^2 \epsilon \left( \frac{1}{x_\gamma} \right) \right] \left( \frac{\alpha \epsilon m_{Nc}}{2\pi} \right) e_q \left( \frac{\alpha \epsilon m}{2\pi} \right) \times \left\{ (1 + \cos^2 \theta_\gamma) \left[ \left( 1 + \left( \frac{1 - x_\gamma}{x_\gamma} \right)^2 \right) \left( \ln \frac{y_c}{y_m} + \ln \frac{1 - y_m}{1 - y_c} \right) + 4x_\gamma (y_c - y_m) \right] + (1 - 3 \cos^2 \theta_\gamma) \left( 4 \left( \frac{1 - x_\gamma}{x_\gamma} \right) \right)(y_c - y_m) \right\}
\]
\[ 2 \sum_q \left[ \frac{2}{s} F_q^{PC} (s) \right] \left[ \alpha_{em} N_c \frac{1}{x\gamma} \right] e^2_q \left( \frac{\alpha_{em}}{2\pi} \right) \times (1 + \cos^2 \theta_{\gamma}) \left[ \left( \frac{1 + (1 - x\gamma)^2}{x\gamma} \right) \ln \left( \frac{(1 - x\gamma)\delta^2/4}{1 + \epsilon_h - 1/x\gamma} \right) \right] + O(\delta^2). \]

\[ (4.38) \]

\( E_\gamma d\sigma^{(1)sub}_{e^+ + e^- \to \gamma X}/d^3 \ell \) in Eq. (4.38) manifests a logarithmic divergence, which is of the nature of a collinear divergence, as \( x\gamma \to 1/(1 + \epsilon_h) \). This problem arises from the incompatibility of collinear fragmentation, used to define the counter term in Eq. (4.15), and the cone fragmentation used in the definition of the partonic cross section \( E_\gamma d\sigma^{(1)sub}_{e^+ + e^- \to \gamma X}/d^3 \ell \) in Eq. (4.15). More discussion of this issue is given later.

### 4.2.4.2 Derivation of \( \dot{\sigma}^{(1)sub}_{e^+ + e^- \to \gamma X} \)

Feynman diagrams for the finite short-distance hard part for the gluon fragmentation process, order \( \alpha_s, e^+ e^- \to g \to \gamma \), are shown in Figure 2.5. The similarity between the Feynman diagrams for \( e^+ e^- \to \gamma \), shown in Figure 2.4, and those for \( e^+ e^- \to g \to \gamma \), allows us to exploit the results derived in the previous subsection.

We can derive \( \dot{\sigma}^{(1)sub}_{e^+ + e^- \to \gamma X} \) from the expression for \( \dot{\sigma}^{(1)sub}_{e^+ e^- \to \gamma X} \), given in Eqs. (4.36) and (4.38), by making the following four replacements: \( x\gamma \to x_g; N_c \to N_c C_F; e^2 e^2_q \) of the final photon emission vertex by \( g^2 = 4\pi \alpha_s \); and \( \epsilon_h \to \epsilon_{\min}(z) \). The last replacement is absent for the inclusive case. Here \( z \equiv x\gamma/x_g \).

If \( x_g \leq 1/(1 + \epsilon_{\min}) \), which is the same as \( x\gamma \leq 1/(1 + \epsilon_h) \), these replacements in Eq. (4.36) provide the finite short-distance hard part

\[ E_g \frac{d\sigma^{(1)sub}_{e^+ e^- \to g X}}{d^3 p_g} = 2 \sum_q \left[ \frac{2}{s} F_q^{PC} (s) \right] \left[ \alpha_{em} N_c \frac{1}{x_g} \right] C_F \left( \frac{\alpha_s}{2\pi} \right) \]
\begin{align*}
&x(1 + \cos^2 \theta_3) \left\{ \left( \frac{1 + (1 - x_g)^2}{x_g} \right) \left[ \ln \left( \frac{s}{\mu_{\overline{MS}}^2} \right) \\
&+ \ln(x_g^2(1 - x_g)) + \ln \left( \frac{1 - x_g}{4} \right) \right] + x_g \right\} \\
&+ O(\delta^2). \quad (4.39)
\end{align*}

If \( x_g > 1/(1 + \epsilon_{\text{min}}) \), which is the same as \( x_{\gamma} > 1/(1 + \epsilon_h) \), we make the four replacements in Eq. (4.38) and find

\begin{align*}
&\frac{d\hat{\sigma}^{(1)\text{sub}}}{E_g e^+ e^- \rightarrow gX} \\
&= 2 \sum_q \left[ s F_q^P C(s) \right] \left[ \frac{\alpha_s}{4 \pi} \right] C_F \left( \frac{\alpha_s}{2\pi} \right) \\
&\times (1 + \cos^2 \theta_3) \left\{ \left( \frac{1 + (1 - x_g)^2}{x_g} \right) \ln \left( \frac{(1 - x_g)\delta^2/4}{(1 + \epsilon_h)(x_{\gamma}/x_g) - 1/x_g} \right) \\
&+ O(\delta^2). \right\} \quad (4.40)
\end{align*}

From energy-momentum conservation, we derive an equation similar to Eq. (4.30),

\begin{align*}
&\frac{d\hat{\sigma}^{(1)\text{sub}}}{E_g e^+ e^- \rightarrow gX} = 0 \quad \text{if} \quad x_g \geq x_{\text{max}}(z, \delta, \epsilon_h). \quad (4.41)
\end{align*}

The maximum value \( x_{\text{max}} \) is

\begin{align*}
&x_{\text{max}}(z, \delta, \epsilon_h) \equiv \left( \frac{1}{z(1 + \epsilon_h)} \right) \\
&\times \sqrt{2} \sqrt{1 - \frac{4 \sin^2(\delta/2)(z(1 + \epsilon_h) - 1)}{z^2(1 + \epsilon_h)^2}}. \quad (4.42)
\end{align*}

The gluon fragmentation contribution to the \( O(\alpha_s) \) subtraction term in \( \sigma^{(1)\text{sub}}_{e^+ e^- \rightarrow \gamma X} \) is provided by the convolution

\begin{align*}
&\frac{d\sigma^{(1)\text{sub}}}{E_\gamma e^+ e^- \rightarrow gX \rightarrow \gamma X} \\
&= \int d^3 \ell
\end{align*}
If collinear fragmentation is assumed for $g \to \gamma$, $\theta_3 = \theta_\gamma$. After convolution with the gluon-to-photon fragmentation function, the subtraction term $E_g d\hat{\sigma}_{e^+e^- \to gX} / d^3 p_g$ in Eq. (4.40) develops the same logarithmic divergence $ln(1 + \epsilon_h - 1/x_\gamma)$, as $x_\gamma \to 1/(1 + \epsilon_h)$.

4.2.4.3 Derivation of $\hat{\sigma}_{e^+e^- \to qX}^{(1)sub}$

The Feynman diagrams of the short-distance hard part $\hat{\sigma}_{e^+e^- \to qX}^{(1)sub}$ at order $\alpha_s$ for the quark fragmentation process $e^+e^- \to q \to \gamma$, are shown in Figure 2.6. Because $z \geq \max[x_\gamma, 1/(1 + \epsilon_h)]$ in the convolutions of Eq. (4.8), our general analysis in previous subsections shows that the short-distance hard part $\hat{\sigma}_{e^+e^- \to qX}^{(1)sub}$ is needed only for $x_1 = x_c \leq \min[x_\gamma(1 + \epsilon_h), 1]$. We analyze, in turn, the intervals $x_\gamma < 1/(1 + \epsilon_h)$ and $x_\gamma > 1/(1 + \epsilon_h)$. Discussion of the special case $x_\gamma = 1/(1 + \epsilon_h)$ is reserved for Section 4.3.

If $x_\gamma < 1/(1 + \epsilon_h)$, then $x_1 < 1$ for all values of $z$ in the convolution. In this interval, the virtual diagrams of Figure 2.6b, whose contribution is proportional to $\delta(1 - x_1)$, do not contribute to $\hat{\sigma}_{e^+e^- \to qX}^{(1)sub}$. The square of the matrix element for $\hat{\sigma}_{e^+e^- \to qX}^{(1)sub}$ is therefore the same as that in Eq. (2.52) of Section 2.5. From Eq. (2.52) and Eq. (2.54), we have

$$\frac{1}{4} \left( H_1 + H_2^{eff} \right) = (\epsilon \mu \epsilon)^2 (g_\mu \epsilon)^2 \left\{ \left(1 + \cos^2 \theta_1 - 2\epsilon \right) \left[ \left(1 - \epsilon \right) \left( \frac{y_{13}}{y_{23}} + \frac{y_{23}}{y_{13}} \right) \right] \right\}$$
The overall coupling constant \((e\mu^2)^2(g\mu^2)^2\) is restored in Eq. (4.44). Since \(x_1 < 1\) for all values of \(z\), there is no infrared divergence associated with \(1/(1 - x_1)\) terms in Eq. (4.44).

The isolation condition for the partonic cross section \(\sigma^{(1)ub}_{e^+e^-\rightarrow qX}\) requires that either the gluon or the antiquark be in the isolation cone of the quark and have energy larger than \(E_{\text{min}}\) defined in Eq. (4.5). Following an analysis similar to the one that led to Eq. (4.28), we derive the limits of the integration over \(\hat{y}_{13} \equiv y_{13}/x_1\):

\[
\max[0, \bar{y}_m] \leq \hat{y}_{13} \leq \bar{y}_c;
\]

\[
1 - \bar{y}_c \leq \hat{y}_{13} \leq \min[1, 1 - \bar{y}_m].
\]

In Eq. (4.45), \(\bar{y}_c\) and \(\bar{y}_m\) are defined as

\[
\bar{y}_c \equiv \frac{(1 - x_1)\sin^2(\delta/2)}{1 - x_1\sin^2(\delta/2)} \Rightarrow (1 - x_1)\frac{\delta^2}{4};
\]

\[
\bar{y}_m \equiv (1 + \epsilon_h)z - \frac{1}{x_1};
\]
with \( z = x_\gamma / x_1 \). In analogy to Eq. (4.41),

\[
E_1 \frac{d\sigma^{(1)\text{sub}}}{e^+e^-qX} = 0 \quad \text{if} \quad x_1 \geq x_{\text{max}}(z, \delta, \epsilon_h),
\]

where \( x_{\text{max}}(z, \delta, \epsilon_h) \) is defined in Eq. (4.42).

In the region \( x_\gamma < 1/(1 + \epsilon_h), \bar{y}m < 0 \). The integration over \( \bar{y}_{13} \) has two separate intervals:

\[
\int d\bar{y}_{13} \Rightarrow \int_0^{y_c} d\bar{y}_{13} + \int_{1-y_c}^1 d\bar{y}_{13}.
\]

Integrating over \( \bar{y}_{13} \), we obtain

\[
E_1 \frac{d\sigma^{(1)\text{sub}}}{e^+e^-qX} = \left[ \frac{2}{s} F_q PC(s) \right] \left[ \alpha_{em}^2 N_c \left( \frac{4\pi \mu^2}{(s/4) \sin^2 \theta} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \right] \frac{1}{x_1} C_F \left( \frac{\alpha_s}{2\pi} \right)
\times \left\{ (1 + \cos^2 \theta_1 - 2\epsilon) \left( \frac{1 + x_1^2}{1 - x_1} \right) \left( -\frac{1}{\epsilon} \right) + (1 + \cos^2 \theta_1) \left[ \left( \frac{1 + x_1^2}{1 - x_1} \right) \left( \ln \left( \frac{s}{\mu^2} \right) + \ln(x_1^2(1 - x_1)) \right) + \ln(x_1^2(1 - x_1)) \right] \right\}.
\]

Terms of \( O(\delta^2) \) are neglected. The \( 1/\epsilon \) pole in Eq. (4.49) arises from the \( 1/y_{13} \) term in Eq. (4.44), corresponding to the collinear singularity when the gluon is parallel to the fragmenting quark. Using Eq. (2.18) for \( \hat{\sigma}^{(0)\text{incl}} e^+e^-q' \) and Eq. (2.61) for \( D^{(1)}_{q'_\to q}(z) \) in Chapter 2, and using the fact that \( x_1 < 1 \), one can verify that this collinear singularity is cancelled exactly by the subtraction term in Eq. (4.20).

For the finite short-distance subtraction term in the region \( x_\gamma < 1/(1 + \epsilon_h) \), we
obtain

\[
E_1 \frac{d\sigma_{e^+e^-\rightarrow qX}^{(1)sub}}{d^3p_1} = \left[ \frac{2}{s} F_2^{PC}(s) \right] \left[ \alpha_{em}^2 N_c \frac{1}{x_1} \right] C_F \left( \frac{\alpha_s}{2\pi} \right) \times \left( 1 + \cos^2 \theta_1 \right) \left\{ \left( 1 + x_1^2 \right) \left[ \ell n \left( \frac{s}{\mu^2_{\text{MS}}} \right) + \ell n(x_1^2(1-x_1)) + \ell n \left( \frac{\delta_0^2}{4} \right) \right] \right\} + (1-x_1) \right\}. \tag{4.50}
\]

Turning to the case \( x_1 > 1/(1+\epsilon_{\text{min}}(z)) \), which is the same as \( x_\gamma > 1/(1+\epsilon_h) \), we note that \( \tilde{y}_m > 0 \). The integration region over \( \tilde{y}_{13} = y_{13}/x_1 \) now has the form

\[
\int d\tilde{y}_{13} = \int_{\tilde{y}_m}^{\tilde{y}_c} d\tilde{y}_{13} + \int_{1-\tilde{y}_c}^{1-\tilde{y}_m} d\tilde{y}_{13}. \tag{4.51}
\]

Equation (4.47) shows that there is a nonvanishing \( E_1 d\sigma_{e^+e^-\rightarrow qX}^{(1)sub}/d^3p_1 \) for \( x_1 < 1 \). Since \( x_1 < 1 \) and \( \tilde{y}_{13} \geq \tilde{y}_m > 0 \), there is neither an infrared nor a collinear divergence in this region. We perform the integration of Eq. (4.44) over \( \tilde{y}_{13} \) in \( n = 4 \) dimensions, and we obtain

\[
E_1 \frac{d\sigma_{e^+e^-\rightarrow qX}^{(1)sub}}{d^3p_1} = \left[ \frac{2}{s} F_2^{PC}(s) \right] \left[ \alpha_{em}^2 N_c \frac{1}{x_1} \right] C_F \left( \frac{\alpha_s}{2\pi} \right) \left( 1 + \cos^2 \theta_1 \right) \times \left[ \left( 1 + x_1^2 \right) \ell n \left( \frac{(1-x_1)^2/4}{(1+\epsilon_h)(x_\gamma/x_1) - 1/x_1} \right) \right]. \tag{4.52}
\]

Terms of \( O(\epsilon^2) \) are dropped. In deriving Eq. (4.52), we used the fact that the second term in Eq. (4.20) vanishes in this region.

The general form for the \( O(\alpha_s) \) subtraction term to \( \sigma_{e^+e^-\rightarrow qX}^{(1)sub} \) via quark fragmentation is

\[
E_\gamma \frac{d\sigma_{e^+e^-\rightarrow qX\gamma X}^{(1)sub}}{d^3\ell}.
\]
For collinear fragmentation \( q \to \gamma \), \( \theta_3 = \theta_\gamma \).

As in our discussions of the direct and gluon fragmentation contributions, after convolution with the quark-to-photon fragmentation function, the subtraction term in Eq. (4.53) manifests a logarithmic divergence of the form \( \ln |1 + \epsilon_h - 1/x_\gamma| \) when \( x_\gamma \to 1/(1 + \epsilon_h) \).

4.2.5 One-Loop Contributions to Isolated Cross Section

Combining the one-loop subtraction terms derived in previous subsections and the one-loop contributions to the inclusive cross section derived in Chapter 2, we have the complete one-loop contributions to the cross section for isolated photons in \( e^+e^- \) collisions. Complete one-loop contributions to the inclusive cross section, \( E\gamma d\sigma^{incl}_{e^+e^-\to \gamma X}/d^3\ell \), are summarized in Section 2.6.

The analytical results for the isolated cross sections are presented in this subsection for the intervals: \( x_\gamma < 1/(1 + \epsilon_h) \) and \( x_\gamma > 1/(1 + \epsilon_h) \). The case when \( x_\gamma = 1/(1 + \epsilon_h) \) will be discussed separately in the next section.

4.2.5.1 One-Loop Contribution to Isolated Cross Section when \( x_\gamma < 1/(1 + \epsilon_h) \)

Substituting Eqs. (4.36) and (2.65) into Eq. (4.10a), we derive the one-loop direct
production of $e^+e^- \rightarrow \gamma$ for isolated photons:

$$
E_{\gamma} \frac{d\sigma^{(1)iso}}{d^3 \ell} = 2 \sum_q \left[ \frac{2}{s} F_q PC(s) \right] \left[ \frac{2}{\alpha_{em} N_c} \frac{1}{x_\gamma} \right] e_q^2 \left( \frac{\alpha_{em}}{2\pi} \right) \times \left\{ \left(1 + \cos^2 \theta_\gamma \right) \left[ \left( \frac{1 + (1 - x_\gamma)^2}{x_\gamma} \right) \ln \left( \frac{1}{(1 - x_\gamma) \delta^2/4} - x_\gamma \right) \right]
+ (1 - 3 \cos^2 \theta_\gamma) \left[ \frac{2(1 - x_\gamma)}{x_\gamma} \right] \right\},
$$

(4.54)

where we have dropped terms of $O(\delta^2)$. By integrating over $\theta_\gamma$, Eq. (4.54) is consistent with that derived in Ref. [36].

From Eq. (4.9), we obtain the one-loop gluonic fragmentation contribution to $e^+e^- \rightarrow \gamma$ for isolated photons:

$$
E_{\gamma} \frac{d\sigma^{(1)iso}}{d^3 \ell} = \int_{x_{\gamma}}^{1} \frac{d\ell}{z} \left[ \frac{2}{s} F_q PC(s) \right] \left[ \frac{2}{\alpha_{em} N_c} \frac{1}{x_\gamma} \right] \ln \left( \frac{1}{(1 - x_\gamma) \delta^2/4} - x_\gamma \right)
$$

(4.55)

where the short-distance hard part is given by

$$
E_g \frac{d\sigma^{(1)iso}}{d^3 \ell} = 2 \sum_q \left[ \frac{2}{s} F_q PC(s) \right] \left[ \frac{2}{\alpha_{em} N_c} \frac{1}{x_g} \right] C_F \left( \frac{\alpha_s}{2\pi} \right) \times \left\{ \left(1 + \cos^2 \theta_\gamma \right) \left[ \left( \frac{1 + (1 - x_g)^2}{x_g} \right) \ln \left( \frac{1}{(1 - x_g) \delta^2/4} - x_g \right) \right]
+ (1 - 3 \cos^2 \theta_\gamma) \left[ \frac{2(1 - x_g)}{x_g} \right] \right\}.
$$

(4.56)

In deriving the above result, we substituted Eqs. (4.39) and (2.67) into Eq. (4.10b).
Similarly, we derive the one-loop quark fragmentation contribution to $e^+e^- \rightarrow \gamma$

for isolated photons:

\[
\frac{d\sigma^{(1)iso}_{e^+e^-\rightarrow qX\rightarrow \gamma X}}{d^3\ell} = 2 \sum_q \int_{z_{\min}}^{1} \frac{dz}{z} \left[ \int_{\max}^{1} \frac{dz}{z} \left( \frac{x_1}{z} = \frac{x_{\gamma}}{z} \right) \right]

\times \frac{D_{q\rightarrow \gamma}(z, \mu_{\overline{\text{MS}}})}{z},
\]

(4.57)

where the short-distance hard part is given by

\[
\frac{d\sigma^{(1)iso}_{e^+e^-\rightarrow qX}}{d^3p_q} = \left[ \frac{2}{s} F_q PC (s) \right] \left[ \alpha_s^2 N_c \frac{1}{x_1} \right] C_F \left( \frac{\alpha_s}{2\pi} \right)

\times \left\{ 1 + \cos^2 \theta_{\gamma} \left[ \left( \frac{1}{1 - x_1} \right) \frac{1}{1 - x_1} \right] \ln \left( \frac{1}{(1 - x_1)\delta^2/4} \right) - \frac{3}{2} \left( \frac{1}{1 - x_1} \right) \right.

\left. + \frac{1}{2} (x_1 - 3) \right\}.
\]

(4.58)

Eq. (4.58) was derived by substituting Eqs. (4.50) and (2.69) into Eq. (4.10b). Notice that after integrating over $z$, Eq. (4.57) develops a logarithmic divergence $\ln(1/x_{\gamma} - (1 + \epsilon_h))$ as $x_{\gamma} \rightarrow 1/(1 + \epsilon_h)$. This divergence is caused by the $1/(1 - x_1)$ terms in Eq. (4.58). Detail discussion on this divergence will be given later.

By assuming $D_{q\rightarrow \gamma}(z) = D\bar{q}\rightarrow \gamma(z)$, the one-loop antiquark fragmentation contribution to $e^+e^- \rightarrow \gamma$ for isolated photons is the same as that of quark, given in Eq. (4.57), except the $\sum$ is over $\bar{q}$.
4.2.5.2 One-Loop Contribution to Isolated Cross Section

when $x_\gamma > 1/(1 + \epsilon_h)$

As pointed out in previous subsections, the subtraction terms $\hat{\sigma}^{(1)\text{sub}}$ vanish for $x_\gamma > 1/(1 + \epsilon_h)$, if all fragmentation processes were exactly collinear. But, due to a finite cone size, we have a small phase space where the subtraction terms are finite. Since the value of $\epsilon_h$ is very small in most experiments, very limited data are available in this region. However, understanding the isolated cross section in this region is very interesting and important for understanding the isolated cross section in hadron-hadron collisions.

Kinematics requires the subtraction term $E_\gamma d\hat{\sigma}^{(1)\text{sub}}_{e^+e^-\rightarrow\gamma X}/d^3\ell$ to vanish if $x_\gamma > x_\gamma^{\text{max}}(\delta, \epsilon_h)$, which is defined in Eq. (4.31); and therefore,

$$
E_\gamma \frac{d\hat{\sigma}^{(1)\text{incl}}_{e^+e^-\rightarrow\gamma X}}{d^3\ell} = E_\gamma \frac{d\hat{\sigma}^{(1)\text{iso}}_{e^+e^-\rightarrow\gamma X}}{d^3\ell} \quad \text{if } x_\gamma > x_\gamma^{\text{max}}(\delta, \epsilon_h).
$$

(4.59)

However, if $1/(1 + \epsilon_h) < x_\gamma < x_\gamma^{\text{max}}(\delta, \epsilon_h)$, we derive by using Eqs. (4.38) and (2.65),

$$
E_\gamma \frac{d\hat{\sigma}^{(1)\text{iso}}_{e^+e^-\rightarrow\gamma X}}{d^3\ell} = \sum_q \left\{ \frac{2}{s} F_q^{PC}(s) \right\} \left[ \frac{\alpha s N_c}{x_\gamma} \right] e_q \left( \frac{\alpha s}{2\pi} \right)
$$

$$
\times \left\{ \left(1 + \cos^2 \theta_\gamma \right) \left( \frac{1 + (1 - x_\gamma)^2}{x_\gamma} \right) \left[ \ln \left( \frac{s}{\mu^2_{\text{MS}}} \right) + \ln \left( \frac{1 + \epsilon_h - 1/x_\gamma}{(1 - x_\gamma)^2/4} \right) \right] + \left(1 - 3 \cos^2 \theta_\gamma \right) \left[ \frac{2(1 - x_\gamma)}{x_\gamma} \right] \right\}.
$$

(4.60)

By integrating over $\theta_\gamma$, the result here, given in Eq. (4.60), is consistent with that derived in Ref. [36].
For one-loop fragmentation contributions from parton $c(= g, q, \bar{q})$ to $\gamma$, the subtraction terms again vanish if $x_c > x_{c_{\text{max}}}(z, \delta, \epsilon_h)$, which is defined in Eq. (4.42); and thus,

$$ \frac{d\hat{\sigma}^{(1)\text{iso}}}{E_c e^+ e^- \rightarrow cX} = \frac{d\hat{\sigma}^{(1)\text{incl}}}{E_c e^+ e^- \rightarrow cX} \quad \text{if } x_c > x_{c_{\text{max}}}(z, \delta, \epsilon_h). \quad (4.61) $$

In Eq. (4.61), corresponding inclusive hard parts are given in Eqs. (2.67) and (2.69).

However, if $1/(1 + \epsilon_h) < x_c < x_{c_{\text{max}}}(z, \delta, \epsilon_h)$, the subtraction terms are finite and given in Eqs. (4.40) and (4.52). Combining with the inclusive contribution, we obtain the gluon fragmentation contribution as

$$ \frac{d\hat{\sigma}^{(1)\text{iso}}}{E_g e^+ e^- \rightarrow gX} = 2 \sum_q \left[ \frac{2}{s} F_q^{PC} (s) \right] \left[ \alpha^2 s_{C_{\text{em}}} N_c \frac{1}{x_g} \right] C_F \left( \frac{\alpha_s}{2\pi} \right) $$

$$ \times \left\{ (1 + \cos^2 \theta_\gamma) \left[ \frac{1 + (1 - x_g)^2}{x_g} \right] \left[ \ln \left( \frac{s}{\mu^2_{\text{MS}}} \right) + \ln \left( \frac{x_g^2 (1 - x_g)}{(1 - x_g) \delta^2/4} \right) \right] + \ln \left( \frac{(1 + \epsilon_h) (x_\gamma / x_g) - 1 / x_g}{(1 - x_g) \delta^2/4} \right) \right\} \quad (4.62) $$

for $1/(1 + \epsilon_h) < x_g < x_{c_{\text{max}}}(z, \delta, \epsilon_h)$, and the quark fragmentation contribution

$$ \frac{d\hat{\sigma}^{(1)\text{iso}}}{E_q e^+ e^- \rightarrow qX} = \left[ \frac{2}{s} F_q^{PC} (s) \right] \left[ \alpha^2 s_{C_{\text{em}}} N_c \frac{1}{x_1} \right] C_F \left( \frac{\alpha_s}{2\pi} \right) $$

$$ \times \left\{ (1 + \cos^2 \theta_\gamma) \left[ \frac{1 + x_1^2}{1 - x_1} \right] \left[ \ln \left( \frac{s}{\mu^2_{\text{MS}}} \right) + \ln \left( x_1^2 (1 - x_1) \right) \right] + \ln \left( \frac{(1 + \epsilon_h) (x_\gamma / x_1) - 1 / x_1}{(1 - x_1) \delta^2/4} \right) \right\} $$
for $1/(1 + \epsilon_h) < x_1 < x_1^{\text{max}}(z, \delta, \epsilon_h)$, and $x_1^{\text{max}}(z, \delta, \epsilon_h)$ defined in Eq. (4.42).

It is a common feature for Eqs. (4.60), (4.62) and (4.63) that all three contributions develop the logarithmic divergence $\ln(1 + \epsilon_h - 1/x_\gamma)$ as $x_\gamma \to 1/(1 + \epsilon_h)$. This logarithm is a result of the collinear singularities of the Feynman diagrams shown in Figures 2.4, 2.5 and 2.6a. It corresponds to the situation when a non-observing parton inside the isolation cone becomes almost collinear with the observed parton. Since $x_\gamma > 1/(1 + \epsilon_h)$, which is the same as $x_g$ or $x_q > 1/(1 + \epsilon_{\text{min}}(z))$ for all $z$, the counter terms, defined through $\bar{\Phi}$ in Eqs. (4.15), (4.19) and (4.20), vanish. Consequently, the apparent collinear singularities have nothing with which to cancel.

This problem is caused by the incompatibility between collinear fragmentation, which was used to define the counter terms in Eqs. (4.15), (4.19) and (4.20), and the cone fragmentation used to define the partonic cross section $E_\gamma d\hat{\sigma}_e^{(1)\text{sub}} e^+ e^- \to eX / d^3 \ell$ with $c = \gamma, q, \bar{q}, g$ in Eqs. (4.15) (4.19) and (4.20). We may need to revise our concept of photon fragmentation functions in the case of isolated photons.

### 4.3 Breakdown of Conventional Factorization

As $x_\gamma$ approaches to $1/(1 + \epsilon_h)$, a number of expressions given in the previous section for the isolated cross section will develop a logarithmic divergence like $\ln(1 + \epsilon_h) - 1/x_\gamma$. However, the value of $\epsilon_h$ is an arbitrary parameter chosen for an individual experiment. Certainly, a completely consistent theoretical prediction should not be sensitive to an arbitrary experimental parameter, such as $\epsilon_h$. The
reason for us to have such an unstable result as \( x_\gamma \) approaches to \( 1/(1 + \epsilon_h) \) is due to the breakdown of the conventional perturbative factorization theorem for the cross section of isolated photons. The key issues involved in this problem are the isolation condition and finite cone size for fragmentation.

First, we exam the origin of where this logarithmic divergence comes from by using the results, presented in the last section. Let us start with the situation when \( x_\gamma \) approaches \( 1/(1 + \epsilon_h) \) from below. When \( x_\gamma \) is less than \( 1/(1 + \epsilon_h) \), the expression for the direct production, given in Eq. (4.54), is well-behaved as \( x_\gamma \) approaches \( 1/(1 + \epsilon_h) \). Similarly, the gluonic fragmentation contribution, defined by Eqs. (4.55) and (4.56), is also well-behaved as \( x_\gamma \) approaches \( 1/(1 + \epsilon_h) \). Although \( x_g \) can be equal to 1 when \( x_\gamma = 1/(1 + \epsilon_h) \), the \( \ell n(1 - x_g) \) term in Eq. (4.56) gives a finite contribution after the integration over \( z \). Thus, Eqs. (4.54) and (4.55) should also be valid for \( x_\gamma = 1/(1 + \epsilon_h) \).

However, the quark (or antiquark) fragmentation contribution will develop a logarithmic divergence as \( x_\gamma \) approaches \( 1/(1 + \epsilon_h) \). This divergence is caused by the \( 1/(1 - x_1) \) terms in Eq. (4.58), and can be easily understood as follows. Consider the following general integral

\[
I(x_\gamma, \epsilon_h) \equiv \int_{1/(1 + \epsilon_h)}^1 \frac{dz}{z} \frac{1}{1 - x_1} \ell n^m(1 - x_1) F(z, x_1 = x_\gamma/z),
\]

for \( x_\gamma \leq 1/(1 + \epsilon_h) \),

\[
(4.64)
\]

where \( m = 0, 1, ..., \) and \( F(z, x_1 = x_\gamma/z) \) is any smooth function for the region of integration. The integral, \( I(x_\gamma, \epsilon_h) \) can be thought as a simplified version of the one-loop quark fragmentation contribution defined in Eq. (4.57). The \( 1/(1 - x_1) \) and \( \ell n(1 - x_1) \) are typical factors from the short-distance hard part. It is straightforward
to perform the integration in the variable of $x_1$, and we find

$$I(x_\gamma, \epsilon_h) = \int_{x_\gamma}^{x_\gamma/(1+\epsilon_h)} \frac{dx_1}{x_1} \left( \frac{1}{1-x_1} \right) \ell_n^m (1-x_1) F(z = x_\gamma/x_1, x_1)$$

$$\Rightarrow -\ell_n^m+1 \left( 1 - x_\gamma(1 + \epsilon_h) \right)$$

$$\Rightarrow \infty \quad \text{as } x_\gamma \to 1/(1 + \epsilon_h). \quad (4.66)$$

From this example, it is clear that $1/(1-x_1)$ terms in the short-distance hard part will make the isolated cross section very sensitive to the value of $\epsilon_h$, as $x_\gamma$ approaches $1/(1 + \epsilon_h)$. It seems that the perturbatively calculated cross section for isolated photons becomes logarithmic divergent at a different value of $x_\gamma$ if one chooses a different value of $\epsilon_h$. This certainly does not make sense.

Physically, the reason to have such a logarithmic sensitivity is due to the infrared divergence associated with the limit when the final state gluon’s momentum goes to zero, which is the same as $x_1 \to 1$. Normally, such an infrared divergence is cancelled by virtual diagrams. For example, in calculating the inclusive contribution, the infrared singularity associated with $x_1 \to 1$ from the real diagrams, sketched in Figure 2.6a, is cancelled by the infrared contribution from the virtual diagrams, sketched in Figure 2.6b, which are proportional to $\delta(1 - x_1)$. However, as demonstrated below, such perfect cancellation between real and virtual diagrams may be broken by the isolation conditions.

The isolated photons are defined to be the photons having less than $E_{max} = \epsilon_h E_\gamma$ hadronic energy in the isolation cone. For photons produced from parton fragmentation, as sketched in Figure 4.2, we can have a situation in which the $E_{max}$ in the isolation cone is completely provided by the parton fragmentation; and consequently, no other soft gluons can be allowed into the isolation cone. That is, the
phase space allowed for the final-state non-fragmenting partons is smaller. Therefore, it is possible that the infrared divergences associated with the final-state real gluons cannot be completely cancelled by the virtual diagrams due to the mismatch of the phase space. Clearly, such uncanceled infrared contributions should vanish if the cone size goes to zero.

As an example, let us use the one-loop quark fragmentation contribution to the cross section of isolated photons to demonstrate the breakdown of perfect cancellation of infrared divergences. In order to make the following presentation parallel to the statement in the last paragraph, we calculate the isolated cross section from the quark fragmentation directly without going through the subtraction term [34].

In order to examine the situation when \( x_\gamma \) approaches \( 1/(1 + \epsilon_h) \), we need to evaluate both real and virtual diagrams as shown in Figure 2.6. From Eq. (4.44), we have the matrix element square for the real diagrams as

\[
\frac{1}{4} H_{\text{real}} = \frac{1}{4} \left( H_1 + H_2^{\text{eff}} \right)
\]

\[
= \epsilon^2 \langle \epsilon \mu \rangle^2 \left\{ (1 + \cos^2 \theta_1 - 2\epsilon) \left[ \left( \frac{1 + x_1^2}{1 - x_1} \right) \frac{1}{y_{13}} + \frac{y_{13}}{1 - x_1} \right] + (1 + \cos^2 \theta_1 - 2\epsilon) \left[ -\frac{2}{1 - x_1} - \epsilon \left( \frac{1 - x_1}{y_{13}} + \frac{y_{13}}{1 - x_1} + 2 \right) \right] + \left( 1 - 3 \cos^2 \theta_1 \right) \left[ \frac{2}{x_1} \left( 1 - \frac{y_{13}}{x_1} \right) \right] \right\},
\]

which is the same for both inclusive and isolated cross sections. The key difference is the integration limits for the \( \hat{y}_{13} = y_{13}/x_1 \). Integrating \( \hat{y}_{13} \) from 0 to 1, we can easily obtain the real contribution to the inclusive cross section; and all infrared divergences associated with \( x_1 \to 1 \) are cancelled by the contribution from the virtual diagrams [7]. On the other hand, the isolation conditions will require the \( \hat{y}_{13} \)
integration being divided into three regions:

\[ \int d\hat{y}_{13} = \int_{0}^{\min[\bar{y}_c, \bar{y}_m]} d\hat{y}_{13} + \int_{\bar{y}_c}^{1-\bar{y}_c} d\hat{y}_{13} + \int_{\max[(1-\bar{y}_c), (1-\bar{y}_m)]}^{1} d\hat{y}_{13}, \tag{4.68} \]

where \( \bar{y}_c \) and \( \bar{y}_m \) are defined in Eq. (4.46). In the first region, the condition \( 0 \leq \hat{y}_{13} \leq \bar{y}_c \) ensures that a gluon is in the isolation of the fragmenting quark; and condition \( \hat{y}_{13} \leq \min[\bar{y}_c, \bar{y}_m] \) ensures that the total hadronic energy in the isolation cone is less than \( E_{max} = \epsilon_h E_\gamma \). Similarly, the condition \( \max[(1-\bar{y}_c), (1-\bar{y}_m)] \leq \hat{y}_{13} \leq 1 \) for the third region ensures that the antiquark is in the isolation cone and the total hadronic energy in the isolation cone is less than \( E_{max} \). The second interval represents the situation when neither gluon nor antiquark is in the isolation cone.

From Eq. (4.68), it is clear that the isolated cross section is the same as the inclusive cross section if \( \bar{y}_c \leq \bar{y}_m \), which is consistent with Eq. (4.47). When \( \bar{y}_c > \bar{y}_m \), (i.e., \( (1-x_1)\delta^2/4 > z[(1+\epsilon_h) - 1/x_\gamma] \)), Eq. (4.68) told us that the phase space of the final state gluon (and/or antiquark) for the isolated cross section is smaller than that for the inclusive cross section.

If \( x_\gamma \leq 1/(1+\epsilon_h) \), equivalently, \( \bar{y}_m = z[(1+\epsilon_h) - 1/x_\gamma] \leq 0 \), the \( \hat{y}_{13} \) integration defined in Eq. (4.68) is completely given by the second region,

\[ \int d\hat{y}_{13} \Rightarrow \int_{(1-x_1)\delta^2/4}^{1-(1-x_1)\delta^2/4} d\hat{y}_{13}, \tag{4.69} \]

where we expanded the \( \bar{y}_c \) to the order of \( \delta^2 \). We rewrite the limits of \( \hat{y}_{13} \) integration in Eq. (4.69) as

\[
\begin{align*}
\int_{(1-x_1)\delta^2/4}^{1-(1-x_1)\delta^2/4} d\hat{y}_{13} &= \int_{0}^{1} d\hat{y}_{13} - \int_{0}^{(1-x_1)\delta^2/4} d\hat{y}_{13} - \int_{1-(1-x_1)\delta^2/4}^{1} d\hat{y}_{13}.
\end{align*}
\tag{4.70}
\]
Combining the three-particle phase space of Eq. (A.35), and the matrix element square of Eq. (4.67), the first term in the right-hand-side of Eq. (4.70) provides the complete real contribution for the inclusive cross section [7]

\[
\frac{d\sigma^{(R_1)}}{E_1 \frac{d^3 p_1}{d^3 p_1}} = \left[ \frac{2}{s} F^{PC}(s) \right] \left[ \frac{2}{s} \alpha_s^2 N_c \left( \frac{4\pi \mu^2}{(s/4) \sin^2 \theta_1} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon) x_1} \right] \\
\times C_F \left( \frac{\alpha_s}{2\pi} \right) \left[ \left( \frac{4\pi \mu^2}{s} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \right] \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \\
\times \left\{ \left( 1 + \cos^2 \theta_1 - 2\epsilon \right) \left[ \left( \frac{1 + x_1^2}{1 - x_1} + \frac{3}{2} \delta(1 - x_1) \right) \left( \frac{1}{1 - \epsilon} \right) \\
+ \left( \frac{1 + x_1^2}{1 - x_1} \right) \ln \left( x_1^2 \right) \right. \left. + \left( 1 + x_1^2 \right) \left( \frac{\ln(1 - x_1)}{1 - x_1} \right) + \frac{3}{2} \left( \frac{1}{1 - x_1} \right) \right. \\
\left. + \delta(1 - x_1) \left( \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{7}{2} \right) - \frac{1}{2} \left( 3x_1 - 5 \right) \right] \\
\right\} \\
(4.71)
\]

where the superscript \((R_1)\) stands for the contribution of real gluon emission from the phase space specified by the first term in the right-hand-side of Eq. (4.70). In deriving Eq. (4.71), we used Eq. (2.58). The virtual contribution from the diagrams in Figure 2.6b is given by Eq. (2.59) [7]:

\[
\frac{d\sigma^{(V)}}{E_1 \frac{d^3 p_1}{d^3 p_1}} = \left[ \frac{2}{s} F^{PC}(s) \right] \left[ \frac{2}{s} \alpha_s^2 N_c \left( \frac{4\pi \mu^2}{(s/4) \sin^2 \theta_1} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon) x_1} \right] \\
\times C_F \left( \frac{\alpha_s}{2\pi} \right) \left[ \left( \frac{4\pi \mu^2}{s} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \right] \frac{\Gamma(1-\epsilon)^3 \Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}
\]
where the superscript (V) stands for the virtual contribution.

Combining the virtual (V) contribution and a part of the real contribution (R),

we derive

\[
\begin{align*}
\frac{d\sigma}{d^{3}p_1} & \frac{d\sigma}{e^{+}e^{-}\rightarrow qX} \\
& = \left[ \frac{2}{s} \right] FPC(s) \left[ \alpha_{em}^{2} N_{c} \left( \frac{4\pi\mu^{2}}{(s/4)\sin^{2}\theta_{1}} \right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \frac{1}{x_{1}} \right] \\
& \times C_{F} \left( \frac{\alpha_{s}(\mu^{2})}{2\pi} \right) \left\{ (1 + \cos^{2}\theta_{1} - 2\epsilon) \left[ \frac{1 + x_{1}^{2}}{(1 - x_{1})^{+}} + \frac{3}{2} \delta(1 - x_{1}) \right] \left( \frac{1}{1 - \epsilon} \right) \right\} \\
& + \left[ \frac{2}{s} \right] FPC(s) \left[ \alpha_{em}^{2} N_{c} \left( \frac{1}{x_{1}} \right) C_{F} \left( \frac{\alpha_{s}(\mu^{2})}{2\pi} \right) \right] \\
& \times \left\{ (1 + \cos^{2}\theta_{1}) \left[ \frac{1 + x_{1}^{2}}{(1 - x_{1})^{+}} + \frac{3}{2} \delta(1 - x_{1}) \right] \ell\ln \left( \frac{s}{\mu_{MS}^{2}} \right) \\
& + \left( \frac{1 + x_{1}^{2}}{1 - x_{1}} \right) \ell\ln \left( \frac{x_{1}^{2}}{\mu_{MS}^{2}} \right) + (1 + x_{1}^{2}) \left( \frac{\ell\ln(1 - x_{1})}{1 - x_{1}} \right) + \frac{3}{2} \left( \frac{1}{1 - x_{1}} \right) + \\
& + \delta(1 - x_{1}) \left( \frac{2\pi^{2}}{3} - \frac{9}{2} \right) - \frac{1}{2} (3x_{1} - 5) \right] \\
& + \left( 1 - 3\cos^{2}\theta_{1} \right) \right\}. \quad (4.73)
\end{align*}
\]

As expected, from Eq. (4.73), other than the \(1/\epsilon\) term due to the collinear singularity between the fragmenting quark and the real gluon, the contribution from the phase space given by the first term in the right-hand-side of Eq. (4.70) cancels all infrared singularities from the virtual diagrams in Figure 2.6b.

The third term in Eq. (4.70) does not provide any \(1/\epsilon\) singular terms, and gen-
erates some finite terms which vanish as $\delta^2 \to 0$,

$$E_1 \frac{d\sigma^{(R_3)}}{d^3 p_1} \frac{e^+ e^- \to qX}{O(\delta^2)}. \quad (4.74)$$

However, the second term in Eq. (4.70) generates a number of terms with $1/\epsilon$ poles.

By neglecting all terms of $O(\delta^2)$ or higher, we obtain

$$E_1 \frac{d\sigma^{(R_2)}}{d^3 p_1} \frac{e^+ e^- \to qX}{\frac{2}{s} F^{PC}(s) \left( \alpha_{em} C_F \left( \frac{\alpha_s(\mu^2)}{2\pi} \right) \right) \left( 1 + \cos^2 \theta_1 - 2\epsilon \right) \left( \frac{1}{(1 - x_1)^+} + \frac{3}{2} \delta(1 - x_1) \right) \left( \frac{1}{\epsilon} \right)}$$

By adding all contributions, given in Eqs. (4.73), (4.74) and (4.75), together, we have
the next-to-leading order isolated partonic cross section for $e^+e^-\rightarrow q(p_1) + X$,

$$E_1 \frac{d\sigma_{(1)iso}^{(R+V)}}{d^3p_1} = E_1 \frac{d\sigma_{e^+e^-\rightarrow qX}}{d^3p_1} = \left[ \frac{2}{s} F^{PC}(s) \right] \left[ \alpha_{em}^2 N_c \frac{1}{x_1} \right] C_F \left( \frac{\alpha_s(\mu^2)}{2\pi} \right)$$

$$\times \left\{ (1 + \cos^2 \theta_1) \left[ - (1 + x_1^2) \left( \frac{\ell n(1 - x_1)}{1 - x_1} \right) + \delta(1 - x_1) \left( \frac{2\pi^2}{3} - \frac{9}{2} - \ell n^2 \delta^2 \right) \right] - \frac{3}{2} \left( \frac{1}{1 - x_1} \right) + \frac{1}{2}(x_1 - 3) \right\} + (1 - 3\cos^2 \theta_1) \right\}$$

$$\left[ \frac{4}{s} F^{PC}(s) \right] \left[ \alpha_{em}^2 N_c \left( \frac{4\pi\mu^2}{(s/4)\sin^2 \theta_1} \right) \epsilon \frac{1}{\Gamma(1 - \epsilon)} \frac{1}{x_1} \right]$$

$$\times \left\{ (1 + \cos^2 \theta_1 - 2\epsilon) C_F \left( \frac{\alpha_s(\mu^2)}{2\pi} \right) \left[ \left( \frac{4\pi\mu^2}{s} \right) \epsilon \frac{1}{\Gamma(1 - \epsilon)} \right] \right\} \delta(1 - x_1).$$

(4.76)

From Eq. (4.69), $\gamma_{13}$ is always larger than zero for a fixed value of $x_1 \neq 1$; and therefore, $E_1 d\sigma_{(1)iso}^{(1)iso} / d^3p_1$, defined in Eq. (4.76), is independent of $\mu^2_{\text{MS}}$. Furthermore, because there are no collinear subtraction terms for the isolated partonic cross section, the short-distance isolated partonic cross section $\hat{\sigma}^{(1)iso}$ equals to $\sigma^{(1)iso}$ given in Eq. (4.76). However, the uncanceled poles in Eq. (4.76) for $\hat{\sigma}^{(1)iso}$ signal a breakdown of conventional perturbative QCD factorization [34].

In Eq. (4.76), the singularities corresponding to these uncanceled poles are infrared in nature and, as expected, are proportional to $\delta(1 - x_1)$. As explained above,
the uncanceled poles come from the interval specified by the second term in Eq. (4.70), which corresponds to the mismatch of phase space for soft real and virtual gluons due to isolation constraints. Nevertheless, these poles would be irrelevant if $x_1 \neq 1$. However, $x_1 \equiv x_\gamma/z$, and $x_1 = 1$ is kinematically allowed when $x_\gamma = 1/(1 + \epsilon_h)$. We conclude that the conventional factorization theorem for the cross section of isolated photons in $e^+e^-$ annihilation breaks down when $x_\gamma \sim 1/(1 + \epsilon_h)$, where the value of $\epsilon_h$ is chosen in individual experiments.

4.4 Numerical Results and Discussions

Although the conventional factorization formula breaks down when $x_\gamma \to 1/(1 + \epsilon_h)$, isolated prompt photons in $e^+e^-$ are still interesting physical observables for the most part of the phase space, at least at this order in perturbation theory. The analytical expressions, Eqs. (4.54), (4.55) and (4.57), for cross sections with $x_\gamma$ much less than $1/(1 + \epsilon_h)$ are useful for studying the isolated photons at LEP. But, the expressions, Eqs. (4.60), (4.62) and (4.63), for $x_\gamma > 1/(1 + \epsilon_h)$ should not be taken seriously for comparing with data at LEP because of the logarithm, $\ln((1 + \epsilon_h) - 1/x_\gamma)$, which was caused by the incompatibility between the collinear and cone definitions of the parton fragmentation. Although the phase space for these expressions to be applicable is very small, the behavior of these expressions for $x_\gamma > 1/(1 + \epsilon_h)$ is important for studying the isolated photons in hadronic collisions, where this limited phase space will be integrated over through convolution with parton distributions of beam hadrons.

We evaluated the explicit numerical results of the cross sections for isolated photons at LEP with $x_\gamma$ much less than $1/(1 + \epsilon_h)$. In order to compare the isolated
cross sections with our results in the inclusive case, in the following calculations, expressions of $\alpha_{em}$ and $\alpha_s$, and the values of other parameters and constants are chosen to be the same as those used in the inclusive case. They are described in Section 2.6. In Eq. (4.55) and Eq. (4.57), we also choose the same quark-to-photon fragmentation function $D_{q \rightarrow \gamma}(z, \mu_F^2)$ and the gluon-to-photon fragmentation function $D_{g \rightarrow \gamma}(z, \mu_F^2)$ as those used in the inclusive case. They are given in Section 2.6. These fragmentation functions are used for the illustrative purpose. The results for isolated cross sections presented here are normalized by the leading order total hadronic cross section $\sigma_0$. The expression for $\sigma_0$ is also given in Section 2.6.

Figures 4.3 and 4.4 show the comparison of isolated cross section and the inclusive cross section as functions of photon energy $E_\gamma$ at $\sqrt{s} = 91$ GeV and $\theta_\gamma = 90^\circ$, for two choices of isolation energy parameter: $\epsilon_h = 0.15$ and $\epsilon_h = 0.05$. The isolation cone size is chosen to be: $\delta = 20^\circ$. The value $\epsilon_h = 0.15$ is the isolation energy parameter used in Fermilab CDF experiment, while $\epsilon_h = 0.05$ is for the LEP experiment. Figure 4.5, shows the isolated cross section as a function of $E_\gamma$ at $\sqrt{s} = 91$ GeV for the scattering angle $\theta_\gamma = 90^\circ$, with $\epsilon_h = 0.15$ and $\delta = 20^\circ$. Shown in Figure 4.5 are the total cross section and the four separate contributions from lowest-order fragmentation ("0th-Frag"), $O(\alpha_{em})$ direct production ("Direct"), and the $O(\alpha_s)$ quark and gluon fragmentation contributions ("q-Frag" and "g-Frag"). The same results are displayed in Figure 4.6 as a function of scattering angle $\theta_\gamma$ for $E_\gamma = 20$ GeV. In Figures 4.3–4.6, we set renormalization/fragmentation scale $\mu = E_\gamma$. Dependence of the cross sections on $\mu$ is examined in Figure 4.7 at $\theta_\gamma = 90^\circ$ for fixed $E_\gamma$. The Dependence of the cross section on $\epsilon$ is displayed in Figure 4.8.

Figures 4.3–4.5 show clearly that the isolated photon cross section is divergent
as $x \gamma \rightarrow 1/(1 + \epsilon_h)$. In this region, the conventional perturbative QCD factorization breaks down, as discussed in previous sections. Away from the singular region, the isolated cross section is smaller than the inclusive cross section as we expected.

Evident in Figures 4.5 and 4.6 is the dominant contribution of the $O(\alpha_{em})$ direct production when $x \gamma < 1/(1 + \epsilon_h)$. Because of the isolation cut, the lowest order contribution was excluded from the isolated photon cross section when $x \gamma < 1/(1 + \epsilon_h)$. When $x \gamma > 1/(1 + \epsilon_h)$, the lowest order contribution is the dominant
contribution. But this region is very small. Figure 4.5 also shows each divergent term.

When $x_\gamma$ approaches $1/(1 + \epsilon_h)$ from $x_\gamma < 1/(1 + \epsilon_h)$ side, the divergence comes from the first order ($O(\alpha_s)$) quark fragmentation contribution with infrared origin, which is caused by the mismatch of the phase space of the real and virtual diagrams as explained in the last section. When $x_\gamma \rightarrow 1/(1+\epsilon_h)$ from $x_\gamma > 1/(1+\epsilon_h)$ side, quark and gluon fragmentation are both divergent, which is caused by the incompatibility of the collinear fragmentation function definition and the cone fragmentation we used.
Isolated Cross Sections

Figure 4.5: Photon energy dependence of the normalized invariant cross section for the isolated process $e^+e^- \rightarrow \gamma X$ at $\sqrt{s} = 91$ GeV for $\theta_\gamma = 90^\circ$.

for the subtraction term.

Figure 4.7 shows that the isolated cross section has little scale dependence when $x_\gamma < 1/(1 + \epsilon_h)$. This is because the scale dependence of the hard parts for the complete inclusive cross section and the subtraction term cancelled each other.

From Figure 4.8, we can see that when $\epsilon_h$ becomes larger, the isolated cross section increases. This is expected, because the larger $\epsilon_h$, the more hadronic energy is allowed to be in the isolation cone, and consequently, more events contribute to
4.5 Impact for Calculations of Isolated Photons at High Energy Colliders

As shown in previous sections, the conventional perturbative QCD factorization breaks down for the isolated photons in $e^+e^-$ collisions, in the sense that all collinear and infrared singularities cannot be completely absorbed into the nonperturbative
Isolated Cross Sections

| $\frac{1}{\sigma_0} \frac{d \sigma}{d p}$ (1/GeV²) |
|-----------------|-----------------|
| $10^{-6}$ | $10^{-7}$ |
| $10^{-9}$ | $10^{-10}$ |

<table>
<thead>
<tr>
<th>$\mu/E_\gamma$</th>
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- $\delta=20^0$
- $\epsilon_\beta=0.15$
- $E_\gamma=20$ GeV
- $\theta_\gamma=90^0$
- $\sqrt{s}=91$ GeV

Figure 4.7: Renormalization/factorization scale dependence of the normalized invariant cross section for the isolated process $e^+e^- \rightarrow \gamma X$ at $\sqrt{s} = 91$ GeV for $\theta_\gamma = 90^0$ and $E_\gamma = 20$ GeV.

parton-to-photon fragmentation functions. Although, our examples used in previous section are only at one-loop level in perturbative QCD, similar breakdown will take place at higher order contributions as well. In general, such a breakdown of factorization can also be understood in the following intuitive picture.

The intuition might be best achieved through our definition of the isolated cross section, given in Eq. (4.1). We define the isolated cross section as the difference between the inclusive cross section and a subtraction term. It is clear that the in-
exclusive cross section is well-defined in perturbative QCD. Any possible breakdown of the factorization should be due to the subtraction term. As pointed out in Section 4.2, the subtraction term could be viewed as a "cross section" for a photon "jet" with photon’s momentum $\ell$ and hadronic energy $E_{h}^{cone}$ in the "jet" cone larger than $E_{\text{max}} = \epsilon_{h}E_{\gamma}$. Such "jet cross section" can be viewed as an integrated jet cross section, where the word "integrated" means that the cross section for the "jet" events with hadronic energy integrated from a fixed minimum value ($E_{\text{max}} = \epsilon_{h}E_{\gamma}$) to a
maximum value allowed by the kinematics. Based on our experience in calculations for jet cross sections, such an integrated “jet cross section” could be finite in perturbative QCD, if we do not ask for any detail within the jet [40, 41], such as how an individual parton fragments to a photon; however, if we force ourselves to find out the energy of one specific parton within the “jet”, and ask how this parton fragments into the observed photon, most likely, we will have the incomplete cancellation of infrared and collinear singularities. Introduction of the parton-to-photon fragmentation functions for the cross section of isolated photons is similar to asking for the details within the “jet”.

Based on the above limited study, we conclude that cross sections for isolated photons in $e^+e^-$ collisions may not be able to provide precise information on parton-to-photon fragmentation functions, which are defined in conventional perturbative QCD. We need to revise the definition of the fragmentation function for isolated photon events.

This breakdown of factorization has important implications on the computations of isolated photon production at hadron colliders. If we imagine that the process $e^+e^- \rightarrow \gamma X$ plays the role of parton-parton scattering to produce the photon in hadronic collisions, the center-of-mass frame energy $S$ in all our formulas for $e^+e^- \rightarrow \gamma + X$ is replaced by $\hat{s} = x_1 x_2 S$, and $x_\gamma = E_\gamma / \sqrt{S}$ is replaced by $\hat{x}_\gamma = 1E_\gamma / \sqrt{x_1 x_2 S}$. The extra integration over incoming parton’s momentum fractions will force us to cover all possible phase space for $\hat{x}_\gamma$, which is from $x_\gamma = 2E_\gamma / \sqrt{S}$ to 1. That is, we will have to integrate over the region where the conventional perturbative factorization is in question.

However, because the divergence at the parton level is logarithmic, (i.e., di-
vergence at the $e^+e^-$ level is logarithmic), the integration of parton's momentum
fractions will produce a finite result due to the fact that logarithmic divergence is in-
tegrable. But, integrating over a region where isolated partonic cross sections is much
larger than the inclusive cross section (more precisely, $\hat{\sigma}^{(iso)} \sim \infty$) is NOT phys-
tical. Furthermore, the divergent one-loop result will serve as the subtraction term
for the two-loop direct contribution, which is, in turn, not well-defined. We con-
clude that although the perturbative cross sections for isolated photons in hadronic
collisions could be finite, the partonic "hard parts" are infrared sensitive, including
long-distance effect! It then raises a question of how to redefine isolated events, such
that all infrared sensitivities can be systematically absorbed into some well-defined
long-distance matrix elements, or isolated fragmentation functions.
5. CONCLUSIONS

To extract thermal photon signals from relativistic heavy ion collisions, it is necessary to understand the photon fragmentation processes and calculated the extra source of photon production due to multiple scattering. This thesis provided analytical formulas, as well as numerical results for inclusive photon production in $e^+e^- \rightarrow \gamma X$, which can be used for extracting photon fragmentation functions. We also calculated direct photon production in terms of multiple scattering. We predicted nuclear dependence in direct photon production in hadron-nucleus collisions. This work will lead to a better understanding of photon signals in future experiments on relativistic heavy ion collisions at Brookhaven National Laboratory.

The inclusive photon production in hadronic final states in $e^+e^-$ annihilation is an ideal process to extract photon fragmentation functions. We have computed analytically the direct photon contribution through $O(\alpha_{em})$ and the quark-to-photon and gluon-to-photon fragmentation terms through $O(\alpha_s)$. Our results display the full angular dependence of the cross section, separated into transverse $\left(1 + \cos^2 \theta_{\gamma}\right)$ and longitudinal components. The numerical results are presented. The extraction of the photon fragmentation functions from the inclusive process of $e^+e^- \rightarrow \gamma + X$ are also discussed.

In terms of generalized factorization in QCD perturbation theory, and using the
method, developed in Ref. [18], we predicted the nuclear dependence of direct photon production in hadron-nucleus collisions. The theoretical prediction is consistent with the experimental data. We concluded that the observed small Cronin effect in direct photon production is consistent with the much larger Cronin effect observed in single jet and single particle inclusive cross sections.

Furthermore, we derived the analytic expressions for the isolated photon cross section in hadronic final states of $e^+e^-$ annihilations. Using $e^+e^- \rightarrow \gamma + X$ as an example, we showed that the conventional factorization theorem in perturbative quantum chromodynamics breaks down for isolated photon cross sections in a well defined part of phase space. The cross sections for isolated photons in $e^+e^-$ collisions may not be able to provide precise information on parton-to-photon fragmentation functions, which are defined in conventional perturbative QCD. Such a breakdown of factorization has a very important impact on the computation of isolated photon productions in hadronic collisions.
REFERENCES


APPENDIX

TWO AND THREE PARTICLE PHASE SPACE

We need the expressions for two- and three-particle final state phase space $dPS^{(2)}$ and $dPS^{(3)}$ in $n$ dimensions in terms of the variables we used in our calculations. We work out first the specific case of $e^+e^- \rightarrow q\bar{q}$. The four-vector momenta of $q$ and $\bar{q}$ are $p_q$ and $p_{\bar{q}}$.

The two particle phase space element in $n = 4 - 2\epsilon$ dimensions is

$$dPS^{(2)} = \frac{d^{n-1}p_q}{(2\pi)^{n-1}2E_q} \cdot \frac{d^{n-1}p_{\bar{q}}}{(2\pi)^{n-1}2E_{\bar{q}}} \cdot (2\pi)^n \delta(n)(q - p_q - p_{\bar{q}}). \quad (A.1)$$

In the center of mass frame, $p_q = -p_{\bar{q}}$ and $E_q = E_{\bar{q}}$. Eliminating the $d^{n-1}p_{\bar{q}}$ integration, we obtain

$$dPS^{(2)} = \left(\frac{1}{2\pi}\right)^{n-2} \frac{1}{8} E_q^{n-4} dE_q d\theta \sin^{n-3} \theta d\Omega_{n-3}(p_q) \delta \left(E_q - \frac{1}{2}\sqrt{s}\right). \quad (A.2)$$

Since the square of the invariant matrix element, Eq. (2.17) of the text, depends on $\theta$ but not on other angles, we may perform the integration over $d\Omega_{n-3}$:

$$\Omega_{n-3} = 2\pi \pi^{-\epsilon}/\Gamma(1 - \epsilon). \quad (A.3)$$

We derive

$$dPS^{(2)} = \frac{1}{2} \frac{1}{(2\pi)^3} \frac{d^3p_q}{E_q} \left[\left(\frac{4\pi}{(s/4)\sin^2 \theta}\right)^\epsilon \frac{1}{\Gamma(1 - \epsilon)}\right] \frac{2\pi}{s} \frac{\delta(x_q - 1)}{x_q}, \quad (A.4)$$
with \( x_q = 2E_q/\sqrt{s} \).

For the three particle final state \( e^+e^- \rightarrow q\bar{q}\gamma \), we label the four-vector momenta of \( q, \bar{q}, \) and \( \gamma \) as \( p_1, p_2, \) and \( \ell \). The invariant matrix element of interest to us, as defined in Eqs. (2.30) and (2.34), depends explicitly on the inner products \( p_1 \cdot \ell, p_2 \cdot \ell, \) and \( p_1 \cdot p_2 \) as well as on \( p_1 \cdot k, p_2 \cdot k, \) and \( \ell \cdot k \) where \( k \), defined in Eq. (2.7), is the difference \( k = k_{e^+} - k_{e^-} \) of the four-momenta of the initial \( e^+ \) and \( e^- \). However, all these inner products are not independent. Using momentum conservation and the fact that the momenta \( \ell \) and \( k \) are observables, one may show that \( p_1 \cdot \ell \) and \( p_2 \cdot k \) are the only independent invariants. Note that it is completely equivalent to choose \( p_2 \) instead of \( p_1 \).

For general orientation, it is useful to begin in \( n = 4 \) dimensions to establish the angular variables of integration we would use in that case, before generalizing to \( n \) dimensions. In the overall \( e^+e^- \) center of mass frame, we imagine a coordinate system with the \( \gamma \) defining the \( \alpha \) axis, vector \( \vec{k} \) lying in the \( (x,z) \) plane, and vector \( \vec{p}_1 \) generally having non-zero \( x, y \) and \( z \) components, as shown in Figure 2.8.

\[
\vec{k} = |\vec{k}| \left( \sin \theta_\gamma, 0, \cos \theta_\gamma \right); \tag{A.5}
\]

\[
\vec{p}_1 = |\vec{p}_1| \left( \sin \theta_1 \gamma \cos \phi, \sin \theta_1 \gamma \sin \phi, \cos \theta_1 \gamma \right); \tag{A.6}
\]

\[
p_1 \cdot k = -\vec{p}_1 \cdot \vec{k} = -|\vec{p}_1||\vec{k}| \left( \sin \theta_1 \gamma \sin \theta_1 \gamma \cos \phi + \cos \theta_1 \gamma \cos \theta_1 \gamma \right). \tag{A.7}
\]

The four-dimensional example shows that only the components of \( \vec{p}_1 \) in the \( \vec{\ell}, \vec{k} \) plane contribute to \( \vec{p} \cdot \vec{k} \). We use \( \theta_x \) to denote the \( n \)-dimensional generalization of the four-dimensional azimuthal angular variable \( \phi \), and we will express \( dPS^{(3)} \) in \( n \)-dimensions in terms of integrations over \( \theta_1 \gamma \) and \( \theta_x \). Three particle phase space in
Using the $\delta^{(n)}$ function to eliminate the integrations over $p_2$, we obtain
\begin{equation}
  dPS(3) = \frac{d^{n-1}p_1}{(2\pi)^{n-1}2E_1} \frac{d^{n-1}p_2}{(2\pi)^{n-1}2E_2} \frac{d^{n-1}\ell}{(2\pi)^{n-1}2E_\gamma} (2\pi)^n \delta^{(n)}(q - p_1 - p_2 - \ell). \tag{A.8}
\end{equation}

We take angle $\theta_\gamma$ to be the polar angle of the $\gamma$ with respect to the $e^+e^-$ collision axis in the overall center of mass frame. Since the square of the matrix element does not depend on $\phi_\gamma$, we can integrate over $d\Omega_{n-3}(\ell)$ and $d\phi_\gamma$ independently. Using $\Omega_{n-3}$ from Eq. (A.3), and $\int d\phi_\gamma = 2\pi$, we reexpress Eq. (A.10) as
\begin{equation}
  \frac{d^{n-1}\ell}{(2\pi)^{n-1}2E_\gamma} = \frac{1}{2} \frac{d^3\ell}{(2\pi)^3 E_\gamma} \left( \frac{4\pi}{E_\gamma^2 \sin^2 \theta_\gamma} \right)^{\frac{n-4}{2}} \frac{1}{\Gamma(1 - \epsilon)} \frac{d\Omega_{n-3}(\ell)}{d\phi_\gamma}. \tag{A.11}
\end{equation}

We write $d^{n-1}p_1$ in Eq. (A.9) as
\begin{equation}
  d^{n-1}p_1 = E_1^{n-2} dE_1 \ d\Omega_{n-2}(p_1)
  = E_1^{n-2} dE_1 \ d\theta_{1\gamma} \ d\Omega_{n-3}(p_1), \tag{A.12}
\end{equation}

with
\begin{align}
  d\Omega_{n-3}(p_1) &= d\theta_x \ \sin^{n-4} \theta_x \ d\Omega_{n-4}(p_1) \\
  &= d\cos \theta_x \ (1 - \cos^2 \theta_x)^{\frac{n-5}{2}} \ d\Omega_{n-4}(p_1) \tag{A.13}
\end{align}
Since only the components of $p_1$ in the $\vec{\ell}, \vec{k}$ plane contribute to $p_1 \cdot k$, as shown in Eq. (A.7), all angular variables on which the invariant matrix element depends are displayed explicitly in Eqs. (A.12) and (A.13). We may therefore integrate $d\Omega_{n-4}(p_1)$ in Eq. (A.13) to obtain

$$\Omega_{n-4}(p_1) = \frac{2^{n-4} \pi^{n-4}}{2 \Gamma(n - 4)}.$$

In this frame, $E_2$ in Eq. (A.9) can be expressed as

$$E_2^2 = (\vec{p}_2)^2 = (\vec{p}_1 + \vec{\ell})^2 = E_1^2 + E_2^2 + 2E_1E_\gamma \cos \theta_{1\gamma}.$$

Using Eq. (A.15), we can replace the integration over $d \cos \theta_{1\gamma}$ in Eq. (A.12) by an integration over $dE_2$; for fixed $E_1$,

$$E_1E_\gamma d \cos \theta_{1\gamma} = E_2 dE_2.$$

Substituting into Eq. (A.9), we derive

$$dPS^{(3)} = \frac{1}{2} \frac{1}{(2\pi)^3} \frac{d^3\ell}{E_\gamma} \left[ \left( \frac{4\pi}{E_\gamma^2 \sin^2 \theta_{1\gamma}} \right) \frac{1}{\Gamma(1 - \epsilon)} \right]$$

$$\times \frac{1}{4} \frac{1}{(2\pi)^2} \left( \frac{4\pi\epsilon^2}{E_1^2 \sin^2 \theta_{1\gamma}} \right) \Omega_{n-4}(p_1) (1 - \cos^2 \theta_x)^{-\epsilon - 1/2} d \cos \theta_x$$

$$\times \frac{1}{E_\gamma} \delta(\sqrt{s} - E_\gamma - E_1 - E_2) dE_1 dE_2.$$

It is easy to verify that Eq. (A.17) reduces to the familiar form in four-dimensions when $\epsilon \to 0$.

We introduce new dimensionless variables related to the singularity structure of the invariant matrix elements, given in Eqs. (2.30) and (2.34):

$$y_{12} \equiv \frac{2p_1 \cdot p_2}{q^2}$$
We observe that \( y_{1\ell} = 1 - x_2, \ y_{2\ell} = 1 - x_1, \) and \( y_{12} = 1 - x_\gamma. \) In the center of mass frame, \( q = (\sqrt{s}, 0), \) we have

\[
\begin{align*}
x_1 &= \frac{2E_1}{\sqrt{s}}, \quad \text{and} \quad x_2 = \frac{2E_2}{\sqrt{s}}; \quad (A.24) \\
dE_1 \ dE_2 = \frac{s}{4} \ dx_1 \ dx_2 = \frac{s}{4} \ dy_{1\ell} \ dy_{2\ell}; \quad (A.25) \\
\delta(\sqrt{s} - E_q - E_{\ell} - E_{\gamma}) &= \frac{2}{\sqrt{s}} \delta(1 - y_{1\ell} - y_{2\ell} - y_{12}). \quad (A.26)
\end{align*}
\]

After some algebra, one may verify that

\[
\begin{align*}
E_1^2 \ E_2^2 \ \sin^2 \theta_{1\gamma} &= \frac{s^2}{4} \ y_{1\ell} \ y_{2\ell} \ y_{12}. \quad (A.27)
\end{align*}
\]

Substituting Eqs. (A.25)-(A.27) into Eq. (A.17), we derive

\[
\begin{align*}
dPS(3) &= \frac{1}{2} \ \frac{1}{(2\pi)^3} \ \frac{d^3 \ell}{E_\gamma} \ \left[ \left( \frac{4\pi}{(s/4) \ \sin^2 \theta_\gamma} \right)^\epsilon \ \frac{1}{\Gamma(1 - \epsilon)} \right] \ \frac{1}{x_\gamma} \\
&\times \ \frac{1}{4} \ \frac{1}{(2\pi)^2} \ \left( \frac{4\pi^2}{s} \right)^\epsilon \ \Omega_{n-4}(p_1)(1 - \cos^2 \theta_x)^{-\epsilon - 1/2} \ d \cos \theta_x \\
&\times \ \frac{dy_{1\ell} \ dy_{2\ell} \ \delta(1 - y_{1\ell} - y_{2\ell} - y_{12})}{(y_{1\ell} \ y_{2\ell} \ y_{12})^\epsilon}. \quad (A.28)
\end{align*}
\]

For reference, we record that

\[
\int_{-1}^{1} d \cos \theta_x (1 - \cos^2 \theta_x)^{-\epsilon - 1/2} = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} - \epsilon)}{\Gamma(1 - \epsilon)} \quad (A.29)
\]
\begin{align}
\int_{-1}^{1} d \cos \theta_x \cos^2 \theta_x (1 - \cos^2 \theta_x)^{-\epsilon/2} = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2} - \epsilon)}{\Gamma(2 - \epsilon)}. \tag{A.30}
\end{align}

Dividing Eq. (A.30) by Eq. (A.29), we define the average of \( \cos^2 \theta_x \) in \( n \)-dimensions as

\begin{align}
\langle \cos^2 \theta_x \rangle = \frac{1}{2} \frac{1}{1 - \epsilon}. \tag{A.31}
\end{align}

When \( \epsilon \to 0 \), \( \theta_x \) becomes the azimuthal angular \( \phi \), and Eq. (A.31) is consistent with the 4-dimensional result

\begin{align}
\langle \cos^2 \phi \rangle = \frac{1}{2} \int_{0}^{2\pi} d\phi \cos^2 \phi = \frac{1}{2}. \tag{A.32}
\end{align}

Equation (A.28) is written in a form for which the photon with momentum \( \ell \) is the observed particle, with integrations done over the momenta of other final state partons. When considering \( e^+e^- \to q\bar{q}g \) with \( q \) (or \( \bar{q} \)) fragmenting into the observed \( \gamma \), we require instead \( E_q d\sigma / d^3p_q \). It is useful, therefore, to reexpress Eq. (A.28) in a form that manifests the symmetry of phase space among all three final state particles.

Using \( y_{12} = 1 - x\gamma \), we introduce the identity

\begin{align}
1 = dy_{12} \delta \left(x\gamma - (1 - y_{12})\right). \tag{A.33}
\end{align}

Inserting this identity into Eq. (A.28), we obtain a more symmetric form

\begin{align}
\frac{dPS(3)}{s} &= \frac{1}{2} \frac{1}{(2\pi)^3} \frac{E_\gamma}{E_\gamma} \left[ \left( \frac{4\pi}{(s/4)\sin^2 \theta_\gamma} \right)^\epsilon \frac{1}{\Gamma(1 - \epsilon)} \right] \frac{2\pi \delta(x\gamma - (1 - y_{12}))}{s} \frac{d\Omega_{n-3}(p_1)}{d\Omega_{n-3}(p_1)} \times \frac{dy_{1\ell} dy_{2\ell} dy_{12}}{y_{1\ell}^\epsilon y_{2\ell}^\epsilon y_{12}^\epsilon} \delta(1 - y_{1\ell} - y_{2\ell} - y_{12}). \tag{A.34}
\end{align}

We can use Eq. (A.34) to derive an expression for three-particle phase-space suitable for calculating the process of \( e^+e^- \to q\bar{q}g \). We label \( (q, \bar{q}, g) \) as \( (1, 2, 3) \). We
let the gluon replace the photon $\gamma$ in Eq. (A.34), and we use label "i" (=1,2 or 3) to designate the "observed" one. We derive

\[ dPS^{(3)} = \frac{1}{2} \frac{1}{(2\pi)^3} \frac{d^3p_i}{E_i} \left( \frac{4\pi}{s/4} \sin^2 \theta_i \right) \left( \frac{\delta(x_i - (1 - y_{jh}))}{\Gamma(1 - \epsilon)} \right) \frac{1}{s} \left( \frac{2\pi}{\tau} \right)^\epsilon \frac{1}{\Omega_{n-3}(p_j)} \frac{1}{\Omega_{n-3}} \]

\[ \times \frac{\delta(1 - y_{12} - y_{13} - y_{23})}{\gamma_{12}^s \gamma_{13}^s \gamma_{23}^s}, \quad (A.35) \]

The first line of Eq. (A.35) is identical to $dPS^{(2)}$, Eq. (A.4).