Boundary element solutions to wave scattering by surface anomalies on a fluid-solid interface

Shivanand Shenoy
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Boundary element solutions to wave scattering by surface anomalies on a fluid-solid interface

Shendy, Shivanand, Ph.D.

Iowa State University, 1994

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<td>$\rho$</td>
<td>density of the fluid medium</td>
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<td>$p$</td>
<td>pressure in the fluid medium</td>
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CHAPTER 1. INTRODUCTION

Introduction

The research described in this dissertation concerns the reflection, transmission, and scattering of ultrasonic waves incident on a fluid-solid interface. The waves are time harmonic and usually generated within the fluid. All analyses are performed in the Fourier transformed domain. In the absence of surface irregularities or ‘dimples’, longitudinal and compressional waves are transmitted in the solid and acoustic waves get reflected back into the fluid. Irregularities in the solid surface cause a deviation in the above model to produce ‘scattered’ waves in the two media. For completeness, the case of a finite elastic scatterer in an infinite fluid domain is also presented. This is the starting point in the development of the semi-infinite model analysis.

This chapter contains (i) a brief history of ultrasonic wave analysis, (ii) the fluid-solid interaction problem, (iii) the integral equation formulation, and (iv) the numerical technique used to solve the integral equations – the boundary element method.

Ultrasonic Wave Scattering

Lord Rayleigh, a pioneer in the mathematical analysis of acoustics, first studied the scattering of sound [1]. Due to the complexity of the solution, he restricted himself to ‘small’ (compared to the wavelength) scatterers. Later, Morse extended the analysis to rigid, immovable circular cylinders and large spheres [2]. Morse also tabulated the phase-angles for the partial scattered waves to simplify the Bessel function dependence. Morse, Lowan,
Feshbach and Lax later extended this to include the transmitted compressional waves. Faran carried the analysis further to include the presence of transmitted shear waves [3].

Ying and Truell were the first to examine the scattering of an acoustic wave propagating in a fluid [4]. In this paper, they studied the scattering effects of a plane longitudinal wave by three different obstacles – an isotropically elastic sphere, a spherical cavity, and a rigid sphere. Hickling [5] studied the scattering by solid elastic spheres in water. Doolittle and Überall extended this to elastic cylinders [6].

Recently, the NDE (Non Destructive Evaluation) community has shown considerable interest in ultrasonic analysis. Typically, ultrasonic waves are passed into materials using a fluid coupling medium. Anomalies in the material disturb (scatter) the wave field. The measured scattered wave field is analyzed to determine the nature of the anomaly. Typical information obtained includes material property, size, shape, location, and orientation of the anomaly.

**Fluid-Solid Interaction**

Junger's paper [7] contains a historical study of elastic structures under acoustic excitation. The text by Junger and Feit [8] is a classic reference on this subject as well. The problems of time-harmonic acoustic wave scattering from submerged elastic structures, and transmission of elastic waves into the solid, are of interest to many fields including structural acoustics, seismology and geophysics.

The interaction of an acoustic wave in the fluid with the submerged elastic body is a coupled problem. This requires a simultaneous solution of the vibrational response of the
elastic medium and the acoustic field. When a solid body is immersed in a fluid with reasonable impedance, some of its vibrational characteristics, such as the natural frequency, change.

Early research attempted to predict this interaction using analytical methods. This method was only feasible when the geometries were simple. The techniques included, among others, separating the differential equations with the aid of special co-ordinate systems. The need to model complex geometries and the advent of high-speed computing facilities led to the development of special numerical schemes including the finite difference, finite element and more recently, the boundary element method.

The basis of early numerical approaches was the variational formulation. Gladwell et al. [9, 10] used this method to solve radiation problems involving air-plate and air-membrane systems. Ohayon and Valid solved the variational formulation using a coupled finite element technique [11]. Stephen used a finite difference approach to the problem [12]. Many others used a finite element model of the solid with an integral equation model of the acoustic domain using different coupling methods [13, 14, 15, 16]. In most of these schemes, the domains are discretized and the resulting coupled system of equations, in matrix form, are solved using iterative procedures. The acoustic and elastic equations are solved separately using appropriate interface conditions. All of these methods have a major drawback – they require the discretization of an infinite domain. They also require the discretization of the interior of the solid.
Boundary Element Analysis

The boundary element approach addresses this infinite domain problem in the most elegant fashion. The first step is to derive a surface integral representation of the boundary value problem. There are three methods commonly used to obtain the integral equation. They are – Betti’s reciprocal work theorem, the weighted residual method and the method of superposition from linear operator theory. The integrand in these equations is singular at a point on the integration surface. This singular integral equation is the BIE – boundary integral equation. The next step is to solve the BIE numerically by discretizing the boundary into surface elements. This numerical scheme is the BEM – boundary element method.

The BIE approach has been successfully used to solve a variety of continuum mechanics problems. Jaswon was the first to use this method to solve potential problems [17]. Rizzo [18] and Cruse [19] extended this to elastostatics. Later developments include the works of Shaw et al. [20], Seybert et al. [21] in radiation and scattering of acoustic waves, Rizzo et al. [22] in radiation and scattering of elastic waves, and Rizzo et al. [23] in the elastic half-space scattering problem. Goswami et al. [24, 25] combined the acoustic scattering BEM analysis with the elastodynamic analysis to solve the scattering problem of an elastic body submerged in a fluid.

Kane [26] has tabulated the relative merits and demerits of the BEM, FEM and finite difference methods in a new text on boundary element analysis. The advantages of the BEM
over the FEM and finite difference schemes are:

- The dimensions of the problem are reduced
- Infinite domains are handled elegantly
- Direct calculation of boundary data
- Gradients do not depend on volume mesh

Some of the disadvantages are:

- Requires a fundamental solution and an integral relation
- Matrices are usually non-symmetric
- Not well suited for non-homogeneous problems
- Poor results when surface to volume ratio is large

Various studies have shown an increase in speed (by a factor of 4 to 10) when the BEM is used in place of the FEM. Of course, the major reason to use the BEM is its ability to solve a variety of problems that cannot be easily solved using finite elements. It is hoped that this dissertation will illustrate this point.

**BEM and the Fluid-Solid Half-Space Problem**

The focus of this dissertation is the problem of ultrasonic scattering from surface breaking anomalies. In this problem, ultrasonic waves are generated in the fluid coupling medium using an acoustic transducer. The fluid and solid media are assumed to be infinite in both the horizontal and vertical planes in comparison to the wave length of the generated wave. Alternately, the surface anomalies which produce the scattered waves are assumed to
be small compared with the radius of the discretized half-plane.

Unlike the applications described so far, this problem results in BIEs' with non-closed surfaces. This requires a truncated model of the infinite half-space surface (or the use of specialized elements). Goswami et al. [28] dealt with this problem using specialized waves which had a limited bandwidth. These waves allowed the model to be truncated outside of the wave bandwidth. A new technique is developed in this dissertation. This method puts no limitations on the type of incident wave, and the truncated surface size is determined by the fact that scattering effects are only felt in a small neighborhood of the anomaly. This phenomena has previously been studied and verified (in the absence of the fluid medium) by various people including Sanchéz-Sesma [29] and Gonsalves et al. [30].

Structure of the Dissertation

The dissertation is explicit and may be read as a self-contained document. Chapter 2 deals with the scattering problem of an elastic body immersed in a fluid. The BIE is derived and a new solution technique – MRBEM (multi-region BEM) is presented. In Chapter 3, this MRBEM is extended to the ‘fluid-solid half-space with fluid-filled surface dimples’ scattering problem In Chapter 4, the previous analysis is repeated replacing the fluid-filled dimples with air. Chapter 5 contains a comparative study of the dimple problem in three situations – solid-vacuum-vacuum, solid-fluid-fluid and solid-fluid-air, with the incident wave, in all three cases, coming from the solid. The appendices contains the radiation condition, the various fundamental solutions, and the integral representation and identities used.
CHAPTER 2. FULL-SPACE SCATTERING

Introduction

The formulation, and numerical solution, to the ultrasonic wave scattering problem due to a finite elastic body submerged in an infinite fluid medium, are described in this chapter. An aluminum sphere submerged in water is used to illustrate the developed techniques. The method, as presented, may be used for any homogeneous elastic body in an inviscid fluid. It is also independent of the shape of the elastic body. The incident field in the fluid is assumed to be generated by a plane wave. These waves are usually generated, using acoustic transducers, within the fluid. Other types of waves may be used as well.

Goswami et al. [24, 25] modeled and solved this problem using two methods – a direct, and an iterative boundary element method. The usual technique, as described in the above paper, is used to derive the fluid and solid integral equations. However, a multi-region boundary element scheme is employed to numerically solve these equations. This generic scheme provides the basis for the solution techniques used in the next few chapters. The derivations are presented here for completeness.

The fluid-solid scattering problem is a two region problem. The solid region is an elastic body with a finite surface boundary. The governing elastodynamic equations are in a vector form with six variables – three displacements, and three tractions. The fluid region is infinite and bounded by two surfaces – the solid surface, and an imaginary sphere, of infinite radius, which encloses the solid (Figure 2.1). The core of the method involves the coupling of these integral equations using interface conditions.
Figure 2.1: Scattering model for the fluid-solid interaction problem
Goswami's [24, 25] scheme involved solving four equations. Three of these equations described the elastodynamic field in the solid. The fourth equation described the acoustic field in the fluid. Upon discretization, this system of linear equations gave rise to conditioning problems due to the nature of the two mathematically dissimilar fields. In the multi-region approach, the fluid integral equation (scalar) is mathematically converted to a vector form. The fluid region is then treated (for numerical purposes) as the second solid region. Since the two regions share a common interface, three common equations, involving interface displacements, are obtained and solved. The displacements obtained are then used to calculate the tractions on the interface. The pressure on the fluid surface is obtained using these tractions and the interface conditions. The normal gradient of the pressure, which is similarly related to the surface displacements, is obtained using the interface conditions. This method of solving the integral equations does not suffer from ill-conditioning.

The Model

The interaction model used to describe the problem is shown in Figure 2.2. The fluid region $V_f$ (acoustic medium) encompasses the finite solid region $V_s$ (elastodynamic region). The solid is bounded by the closed surface $B$. The acoustic field in the fluid is characterized by the total pressure $p$ and the total pressure gradient $q = \frac{dp}{dn}$ in the normal direction. As is common in scattering problems, the total fields $p$ and $q$, defined to be the sum of the incident and scattered fields, are given by

$$ p = p^I + p^S $$

(2.1)
\[ q = q^I + q^S \] (2.2)

The scattered field \( q^S \) is required to satisfy the Sommerfeld radiation conditions at infinity (Appendix B). A unit normal \( \mathbf{n}_f \) is defined in the outward direction to the fluid. The incident beam \( q^I \) of frequency \( \omega \) is generated within the fluid region \( V_f \).

The elastodynamic field in the solid region \( V_s \), characterized by the elastic displacement \( \mathbf{u} \) and traction \( \mathbf{t} \), is given by the vector equations

\[
\mathbf{u} = \mathbf{u}^I + \mathbf{u}^S \quad (2.3)
\]
\[
\mathbf{t} = \mathbf{t}^I + \mathbf{t}^S \quad (2.4)
\]

The elastodynamic response of the displacement \( \mathbf{u}(x,t) \) in \( V_s \), in the absence of body forces, is governed by the Cauchy-Navier equations

\[
(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}(x,t)) + \mu \nabla^2 \mathbf{u}(x,t) = \rho_s \frac{\partial^2 \mathbf{u}(x,t)}{\partial t^2} \quad (2.5)
\]

where \( \rho_s \) is the density of the solid and \( \lambda \) and \( \mu \) are the Lamé constants.

Similarly the propagation of the pressure wave \( p(x,t) \) in the fluid is governed by the Helmholtz wave equation

\[
\nabla^2 p(x,t) = \frac{1}{c_f^2} \frac{\partial^2 p(x,t)}{\partial t^2} \quad (2.6)
\]

where \( c_f \) is the wave speed in the fluid. The current study being limited to time harmonic analysis, harmonic variation \( e^{-i\omega t} \) is assumed for all field variables. The quantities \( p, p^I, \) and \( p^S \) satisfy the Helmholtz wave equation 2.6 and \( \mathbf{u} \) satisfies the Cauchy-Navier
elastodynamic equation 2.5.

Under the time-harmonic assumption, equations 2.5 and 2.6 reduce to the elliptic differential equations

\[
\left(\frac{k_l^2}{k_f^2} - 1\right)\nabla(\nabla \cdot \mathbf{u}(x)) + \nabla^2 \mathbf{u}(x) + k_f^2 \mathbf{u}(x) = 0 \quad x \in V_s
\]  

(2.7)

\[
(\nabla^2 + k_f^2)p(x) = 0 \quad x \in V_f
\]  

(2.8)

where \(k_l\) and \(k_s\) are the longitudinal and shear wave numbers in the solid, \(k_f\) is the acoustic wave number in the fluid, \(c_l\) and \(c_s\) are the longitudinal and shear wave speeds in the solid, and

\[
k_f = \frac{\omega}{c_f}, \quad k_l = \frac{\omega}{c_l}, \quad k_s = \frac{\omega}{c_s}
\]  

(2.9)

\[
c_l = \sqrt{\frac{\lambda + 2\mu}{\rho_s}}, \quad c_s = \sqrt{\frac{\mu}{\rho_s}}
\]  

(2.10)

At the fluid-solid interface \(S\), the acoustic pressure \(p\) and the solid traction \(t\) are required to satisfy the interface condition

\[
p(x)n_f(x) = -t(x) \quad x \in S
\]  

(2.11)

Similarly, the pressure gradient \(q\) and the displacement \(u\) are required to satisfy the interface condition

\[
q(x) = \rho_f \omega^2 \mathbf{u}(x) \cdot n_f(x) \quad x \in S
\]  

(2.12)

where \(\rho_f\) is the fluid density.
Figure 2.2: BIE model of the fluid-solid interaction problem
Boundary Integral Equations

The differential equations 2.7 and 2.8 and boundary conditions 2.11 and 2.12 are transformed to integral equations involving the field quantities \((p, q, u, \text{ and } t)\) and known solutions. These known solutions are referred to as the fundamental solutions or free space Green's functions. The boundary element technique depends on the existence of the fundamental solution for a given problem.

For the fluid region which exists on the exterior of the surface \(S\), Green's second theorem is used to convert equation 2.8 to an integral equation. For a point \(x_f\) in the fluid, the representation integral over boundary \(S\) for the pressure \(p\) at the point \(x_f\) is given by (Appendix C),

\[
p(x_f) = \int_S \left[ F(r)q(y) - G(r)p(y) \right] ds(y) + p'(x_f) \quad x_f \in V_f
\]

(2.13)

Similarly, the representation integral over the boundary \(S\) for the displacement \(u\) at a point \(x_s\) in the solid is obtained using Betti's reciprocal relation [31] and is given by (Appendix C),

\[
u(x_s) = \int_S \left[ U(r)t(y) - T(r)u(y) \right] ds(y) \quad x_s \in V_s
\]

(2.14)

In equations 2.13 and 2.14, the field point \(y\) is a point on the boundary, the source point \(x\) is either in the fluid or the solid (depending on the subscript), and \(r(x,y)\) is the distance vector between points \(x\) and \(y\) as shown in Figure 2.2. The kernel functions \(F, G, U\) and \(T\) are the full-space fundamental solutions representing the responses to point disturbances in the fluid and the solid.

The second step in the boundary element method is the derivation of the boundary
integral equations from the representation integrals. This is achieved by moving the point $x$ to the boundary. In this limit process the point $x$ is brought to the boundary or integration path. As a result, when the integration point and $x$ coincide, the kernels are singular. The kernel functions $F$ and $U$ are $O(1/r)$ whereas $G$ and $T$ exhibit $O(1/r^2)$ behavior. The $O(1/r)$ singularities are termed weak and can be numerically integrated using a polar coordinate transformation which effectively removes the singularity (Rizzo, Shippy, and Rezayat [32]). The $O(1/r^2)$ or strong singularities can also be removed through regularization and use of certain identities of the fundamental solution (Liu and Rudolphi [33]). This process is outlined in the following paragraphs.

To distinguish between the static and dynamic problems associated with the two regions, an appropriate superscript ($S =$ static, $D =$ dynamic) is used. The fundamental kernels to Laplace's equation are used to regularize the dynamic kernels of the Helmholtz equation. Similarly, the elastostatic kernels are used to regularize the elastodynamic kernels. In the limit as $r \to 0$, the dynamic kernels asymptotically converge to their static counterpart (Appendix A); hence their use.

The acoustic boundary integral equation is obtained, from the representation integral 2.13 in the following manner, using the fundamental solutions presented in Appendix A. At the point $x_f$ in the fluid, the total pressure $p_0$ is given by

$$p_0 = p(x_f)$$

(2.15)

The representation integral using the normal $n_f$ is

$$p_0 = \int_S [F^D q - G^D p] ds + p_f$$

(2.16)
where

\[ p = p(y), \quad q = q(y), \quad p^I = p^I(x), \quad y \in S \] (2.17)

The static Green's function \( G^S \) is used to modify this as follows

\[ p_0 + \int_S \left[ G^D - G^S \right] ds + \int_S G^S (p - p_0) ds + p_0 \int_S G^S ds = \int_S F^D ds + p^I \] (2.18)

Using the identity in Appendix C, when \( x_f \) is outside the region enclosed by \( S \),

\[ \int_S G^S ds = 0 \] (2.19)

The boundary \( S \) may be divided into two parts - \( S_r \) and \( S_s \), where \( S_s \) is the surface where the integrands become singular and

\[ S = S_r \cup S_s \] (2.20)

The surface \( S_s \) is usually the boundary element which contains the collocation point.

As the point \( x_f \) is moved to the boundary \( S_s \), equation 2.18 is rewritten as the acoustic BIE

\[ p_0 + \int_{S_r} \left[ G^D - G^S \right] ds + \int_{S_r} G^S (p - p_0) ds + \int_{S_r} G^D ds - \int_{S_s} G^S p_0 ds = \int_S F^D ds + p^I \] (2.21)

where the integrals \( \int_{S_r} \left[ G^D - G^S \right] ds \), \( \int_{S_r} G^S (p - p_0) ds \), and \( \int_S F^D ds \) are weakly singular and can be numerically evaluated using the polar coordinate transformation method developed by Rizzo, Shippy and Rezayat [32].

A similar process is used to obtain the elastodynamic BIE. At an interior point \( x_s \) in the solid, the displacement \( u \) may be written as

\[ u_0 = u(x_s) \] (2.22)
and the representation integral with respect to the normal $\mathbf{n}$, is

$$u_0 = \int_S [U^D t - T^D u] ds \tag{2.23}$$

where

$$u = u(y), \quad t = t(y), \quad y \in S \tag{2.24}$$

Using the elastostatic free space Green's traction tensor $T^S$ (Appendix A), equation 2.23 can be rewritten as

$$u_0 = \int_S U^D t ds - \int_S [T^D - T^S] u ds - \int_S T^S (u - u_0) ds - u_0 \int_T T^S ds \tag{2.25}$$

Using the identity (Appendix C),

$$\int_S T^S ds = -I \tag{2.26}$$

where $I$ is the identity tensor, equation 2.25 can be written as

$$\int_S [T^D - T^S] u ds - \int_S T^S (u - u_0) ds = \int_S U^D t ds \tag{2.27}$$

The integrals are now weakly singular and do pose no computational problems. Using the same technique as in the acoustic case, the boundary $S$ is divided into $S_s$ and $S_r$ where $S_s$ corresponds to the singular element containing the collocation node. In the limit when $x \to y \in S$, equation 2.27 can be rewritten as the elastodynamic BIE

$$\int_{S_s} T^D u ds - u_0 \int_{S_s} T^S ds + \int_{S_s} [T^D - T^S] u ds - \int_{S_s} T^S (u - u_0) ds = \int_S U^D t ds \tag{2.28}$$
Multi-Region Boundary Element Method

Analytical (closed form) solutions to the BIEs are extremely rare and usually exist when the geometry and boundary conditions are simple [31]. Therefore, BIE are usually solved using the BEM. The solution technique involves: (a) the discretization of the boundary surface $S$ into curvilinear triangular or quadrilateral elements; (b) approximation of the field variables on the surface using isoparametric quadratic shape functions; and (c) use of suitable Gaussian quadrature formulas for numerical integration.

The boundary $S$ (which is also the fluid-solid interface) is first discretized into a set of surface elements. Quadratic isoparametric shape functions are used to approximate the field variables and the geometry. The pressure, pressure gradient, displacement, and traction may thus be piecewise interpolated between the element nodes. The BIEs are now converted, to a discretized form at each node of $S$, using a Gaussian quadrature scheme. The set of linear equations obtained are solved for the field quantities at the boundary nodes. After the boundary information is obtained, farfield and interior field values are calculated using the representation integrals (Equations 2.21 and 2.28).

A modular program was developed to solve the fluid and solid BIE using a multi-region idea [46]. This approach simplifies code modification to solve a variety of problems (scalar, vector, scattering, radiation) with various boundary conditions, element types, and integration orders.

Figure 2.3 illustrates typical boundary elements used in three dimensional problems. These are the linear triangular (three node), quadratic triangular (six node), linear quadrilateral
(four node), and quadratic quadrilateral (eight node) elements. The quadratic (patch) elements are essential for accuracy. Higher order elements may also be used to increase accuracy, usually at the expense of increased computation time. The usual convention in node numbering is counter-clockwise for the normal \( n \) to be outgoing.

For an element with \( n \) nodes, the surface geometry of the element is approximated using a quadratic surface passing through the nodes. The global cartesian coordinate \( x \) of a point on the element is related to the local element coordinate system \( X_{ij} \) by

\[
x_i = \sum_{j=1}^{n} H_j(\xi, \eta) X_{ij} \quad i = 1, 2, 3 \quad (2.29)
\]

where \( H_j \) is the shape function in the local \((\xi, \eta)\) system as shown in Figure 2.4.

The shape functions for the three node triangular and six node triangular elements are given by

\[
\begin{align*}
H_1 &= \xi \\
H_2 &= \eta \\
H_3 &= 1 - \xi - \eta 
\end{align*} \quad (2.30)
\]

and

\[
\begin{align*}
H_1 &= \xi(2\xi - 1) \\
H_2 &= \eta(2\eta - 1) \\
H_3 &= (1 - \xi - \eta)(1 - 2(\xi + \eta)) \\
H_4 &= 4\xi \eta \\
H_5 &= 4\eta(1 - \xi - \eta) \\
H_6 &= 4\xi(1 - \xi - \eta) 
\end{align*} \quad (2.31)
\]

In each case, \( \xi \) and \( \eta \) are the two independent coordinates. The shape functions for the four node linear quadrilateral element are given by

\[
H_j = \frac{1}{4} (1 + \xi_0)(1 + \eta_0) \quad j = 1, 2, 3, 4 \quad (2.32)
\]
Figure 2.3: Typical boundary elements and their node numbering
Figure 2.4: Typical boundary elements and their mapping
and the eight node quadratic quadrilateral shape functions are

\[
H_j = \begin{cases} 
\frac{1}{4} (1 + \xi_0)(1 + \eta_0)(\xi_0 + \eta_0 - 1) & j = 1, 2, 3, 4 \\
\frac{1}{2} (1 + \xi^2)(1 - \eta_0) & j = 5, 7 \\
\frac{1}{2} (1 + \xi_0)(1 - \eta^2) & j = 6, 8 
\end{cases}
\]  
(2.33)

where \(\xi_0 = \xi \xi_j\) and \(\eta_0 = \eta \eta_j\) and \((\xi_j, \eta_j)\) are the coordinates of node \(j\).

A variable \(f\) over an element is represented by

\[
f(\xi, \eta) = \sum_{j=1}^{n} H_j(\xi, \eta)f_j 
\]
(2.34)

where \(n\) is the number of nodes on the element, \(f_j\) are the nodal values of \(f\) on the element and \(H_j\) are the shape functions.

The shape functions used to approximate the field variables may be different from those used to approximate the geometry. This is especially true when certain characteristics are expected in the field variables in which case higher order approximations may be necessary.

The transformation of the integrals into linear boundary element equations is performed as shown for a typical integral

\[
II = \int_S f(y)K(x, y)ds(y) 
\]
(2.35)

where \(f(y)\) is the field variable and \(K(x, y)\) is the corresponding kernel function. If the boundary \(S\) is divided into \(M\) elements with \(n\) nodes on each element and element area \(d_n\), the integral \(II\)
can be represented as

\[ II = \sum_{i=1}^{M} \sum_{j=1}^{n} f_{ij} \left[ \int_{dS} H_j(\xi, \eta)K(x, y(\xi))J(\xi, \eta)ds(\xi, \eta) \right] \]

\[ = \sum_{i=1}^{M} \sum_{j=1}^{n} f_{ij} \int_{dS} L(r(x, \xi, \eta))ds(\xi, \eta) \tag{2.36} \]

where \( f_{ij} \) is the value of \( f \) at the \( j \)-th node of the \( i \)-th element, and \( J(\xi, \eta) \) is the Jacobian of the coordinate transformation such that

\[ ds = J(\xi, \eta)d\xi d\eta \tag{2.37} \]

The integrals are now evaluated using Gaussian quadrature, as follows

\[ \int_{dS} L ds = \int_{dS} L d\xi d\eta \]

\[ = \int_{-1}^{1} \int_{-1}^{1} L(\xi, \eta)d\xi d\eta \]

\[ = \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} W_p W_q L(\xi_p, \xi_q) \tag{2.38} \]

where \( W_p \) and \( W_q \) are the weight factors, \( n_1 \) and \( n_2 \) are the number of Gauss points, and \( \xi_p \) and \( \xi_q \) are the abscissas of the Gauss points. The values of the weight factors and the abscissas can be found in the text by Stroud and Secrest [34] and most books on the finite and the boundary element method.

In the case of weakly singular integrals (when the integrand has a \( O(1/r) \) singularity), the integration, as shown above, is slightly modified. A polar coordinate system is used. This converts the area \( dS \) into the form \( \rho d\rho d\theta \). This extra \( \rho \), which is \( O(\rho) \) for sufficiently small \( \rho \), removes the singularity.
The discretization of the boundary integral equations (surface S being discretized into \( N = M n \) nodes where \( M = \) number of elements and \( n = \) nodes/element) produces a finite system of linear algebraic equations with complex coefficients. This matrix is usually non-symmetric and fully populated unlike a similar matrix which arises in finite element analysis. In the discrete matrix form, the acoustic BIE 2.21 may be written as

\[
[A]{p} = [B]{q} + {p'}
\]  
\[\text{(2.39)}\]

and the elastodynamic BIE 2.28 as

\[
[C]{u} = [D]{t}
\]  
\[\text{(2.40)}\]

where \( A \) and \( B \) are complex \( N \times N \) matrices and \( C \) and \( D \) are complex \( 3N \times 3N \) matrices.

Equation 2.39 corresponds to Region 1 (acoustic), which is scalar, has one equation per node. Equation 2.40 corresponding to Region 2 (elastodynamic), is a vector equation with three scalar equations per node. These equations are coupled by the interface conditions. The interface conditions are now used to transform the scalar acoustic matrix to a vector form. This vector form is coupled with the elastodynamic matrix using the interface condition on the traction. The resulting system is then solved for the nodal displacements. This process, henceforth called the MRBEM, is described below.

The first step in the solution is the conversion of the scalar matrix (\( N \times N \)) to a vector form (\( 3N \times 3N \)). The outward normal \( n \) (actually \( n_r \)) at each node on the fluid side of the
interface may be written in a matrix form \( N \) as

\[
N = \begin{bmatrix}
  n_1 & n_2 & n_3 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
  0 & 0 & 0 & n_1^2 & n_2^2 & n_3^2 & 0 & \ldots & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
  0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \ldots & n_1^N & n_2^N & n_3^N
\end{bmatrix}
\]  

(2.41)

where the superscript indicates the node number and the subscript represents the coordinate axis. This matrix is \( N \times 3N \) in size. The transpose of this matrix is denoted by \( N^T \). The interface condition may now be written as

\[
\{ t \} = -N^T \{ p \}
\]  

(2.42)

where \( \{ t \} \) is the column vector \((3N \times 1)\) of nodal tractions and \( \{ p \} \) is the column vector \((N \times 1)\) of nodal pressures. It should be noted that the \( \{ t \} \) can be considered as fluid tractions.

Similarly, the conditions on \( \{ q \} \) are

\[
\{ q \} = (\rho_f \omega^2)N\{ u \}
\]  

(2.43)

where \( \rho_f \) is the fluid density, \( \omega \) is the input wave frequency, and \( \{ u \} \) represents the fluid displacements.

The acoustic BIE eqn. 2.39 may be written as

\[
\{ p \} = A^{-1}B\{ q \} + A^{-1}\{ p^I \}
\]  

(2.44)

Using eqn. 2.43, and pre-multiplying by \( N^T \), eqn. 2.44 may be written as

\[
N^T\{ p \} = (\rho_f \omega^2)N^TA^{-1}BN\{ u \} + N^TA^{-1}\{ p^I \}
\]  

(2.45)
Using eqn. 2.42, the acoustic BIE now becomes the desired elastodynamic type BIE

\[-\{t\}^F = \left(\rho_f \omega^2\right) N^T A^{-1} B N \{u\} + N^T A^{-1} \{p^l\}\]  

(2.46)

where the superscript F indicates that the region is a fluid.

The elastodynamic BIE 2.40 is written as

\[\{t\}^S = D^{-1} C \{u\}\]  

(2.47)

where the superscript S indicates that the region is a solid.

Using the interface condition,

\[\{t\}^F + \{t\}^S = \{0\}\]  

(2.48)

equations 2.46 and 2.47 may be combined to obtain the matrix equation

\[\left[\left(-\rho_f \omega^2\right) N^T A^{-1} B N + D^{-1} C\right] \{u\} = N^T A^{-1} \{p^l\}\]  

(2.49)

This equation may now be solved using LINPACK or any other suitable package for the total displacement \{u\}. Substitution of the resultant vector \{u\} in equation 2.47 will give the tractions. Using the interface conditions in equations 2.42 and 2.43, the pressure and pressure gradient in the fluid may also be calculated. With the boundary information thus calculated, field information in the interiors of the solid and the fluid may be calculated using the corresponding representation integral.

**Numerical Results**

Various problems were solved to verify the accuracy and efficiency of the algorithms and programs developed. The results obtained were compared with those obtained by Goswami [24,25] for spherical solids (aluminum, brass) in fluid media (water, glycerine).
These plane waves were of the form
\[ p^I(x, y, z) = p_0 e^{-ik_fz} \]  
(2.50)

where \( p_0 \) is the incident pressure amplitude of the time harmonic wave traveling in the negative \( z \) direction. The scattered field \( p^s \) was normalized with respect to the fluid wave number \( k_f \) and the farfield radius \( R \) such that
\[ P = \frac{p^s}{p^I} kR \]  
(2.51)

The quantity \( P \) was plotted in the polar graphs in Figures 2.7-2.14 with \( 0^\circ \) being the forward scattering angle and \( 180^\circ \) being the back scattering angle.

Two specific fluid-solid combinations were used – a brass sphere of unit radius immersed in glycerine, and an aluminum sphere of unit radius in water. Wave numbers, ranging from \( k_f = 1 \) to \( k_f = 7 \), were used to illustrate the versatility of the code. Two different meshes (Table 2.1, Figures 2.5-2.6) were used to illustrate the solution dependence on discretization. The results presented in Goswami [25] were used to compare the accuracy of the solutions. The FORTRAN codes were run on a DEC Alpha machine.

Table 2.1: BEM meshes used in the fullspace problem

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Nodes</th>
<th>Elements</th>
<th>CPU time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse</td>
<td>74</td>
<td>24</td>
<td>16</td>
</tr>
<tr>
<td>Fine</td>
<td>164</td>
<td>54</td>
<td>122</td>
</tr>
</tbody>
</table>
Figure 2.5: Discretization of a sphere using the coarse mesh

Figure 2.6: Discretization of a sphere using the fine mesh
As seen in Figure 2.7, both meshes worked well at capturing the scattered farfield in the backscatter direction. However, in the forward direction, the peak amplitude fell short when the coarse mesh was used.

Doubling the frequency of the wave significantly magnified the errors (Figure 2.8) generated using the coarse mesh. The peak back scattering amplitude was incorrect by almost a factor of 50%.

As seen in Figure 2.9, when the frequency was increased by a factor of 3, the forward scattering amplitudes increased considerably when compared with the back scatter. The errors were more apparent along the z-axis.

Figure 2.10 shows the farfield scattered pressure amplitudes for a large value of $k_f$. The fine mesh worked remarkably well while the coarse mesh gave completely incorrect results. This is primarily due to the fact that the wavelength of the incident wave was much smaller than the characteristic length of an element.

Figures 2.11-2.13 show the scattered farfield obtained for an aluminum sphere immersed in water using a range of frequencies. As before in the case of the brass sphere in glycerine, the fine mesh was required to accurately capture the behavior of the front and back scatter amplitudes.

It should be noted that in all cases, the results are presented without comparison with other published work. The results, however, have been compared and verified with those presented by Goswami [25].
Figure 2.7: Farfield scattered pressure (brass sphere in glycerine, $k_f = 1$)
Figure 2.8: Farfield scattered pressure (brass sphere in glycerine, $k_f = 2$)
Figure 2.9: Farfield scattered pressure (brass sphere in glycerine, $k_f = 3$)
Figure 2.10: Farfield scattered pressure (brass sphere in glycerine, $k_f = 7$)
Figure 2.11: Farfield scattered pressure (aluminum sphere in water, $k_f = 3$)
Figure 2.12: Farfield scattered pressure (aluminum sphere in water, $k_f = 5$)
Figure 2.13: Farfield scattered pressure (aluminum sphere in water, \( k_f = 7 \))
Conclusions

The coarse mesh gave poor results when used with high frequencies but worked very well at low frequencies. This was expected because the ratio of the wave length to the characteristic length of an element decreased with increasing frequency. In general, good results may be obtained when this ratio is close to 1. The density of the mesh affects the solution to the problem. Coarse meshes often produce accuracy errors. Similarly, elements with large aspect ratios result in ill-conditioning. The technique of creating a good mesh is a research subject in its own right and is usually developed with experience. A simple method used to determine an appropriate mesh is to check the convergence of the BEM results using meshes of increasing density.

Since the integrands involve terms of type $e^{ikr}$, the frequency $\omega$ plays a crucial role in the choice of an integration scheme. In general, higher order Gaussian schemes are required for accuracy when using high frequencies [32].

In general, the multi-region approach was efficient and also as accurate as the regular approach used by Goswami [26]. The advantages of the method are better illustrated in the half-space problems of the next few chapters. The only disadvantage was the higher storage requirements by the boundary element matrices.
CHAPTER 3. HALF-SPACE SCATTERING – FLUID-FILLED DIMPLE

Introduction

This chapter presents the specialization of the boundary element method for modeling the reflection, transmission, and scattering of an ultrasonic wave, through an infinite fluid-solid interface, in the presence of surface indentations. The method developed here may be used for any type of wave – plane, Rayleigh, etc. It can also be easily extended (using hypersingular elements) to deal with surface breaking cracks. The work is motivated by the need of the NDE community to detect surface breaking flaws using immersion testing.

Previous work in this area has been restricted to either special surfaces (flat) or special waves (Gaussian). These include: Luco and Apsel for a layered half-space [36], Rizzo et al. [23] for an elastic half-space, and Neubauer for a bounded beam [37]. Goswami [25] first solved the fluid-solid curved surface in the presence of a Gaussian beam using the boundary element method. This method had no restrictions on the shape of the surface. It did, however, require the use of a specialized wave with band-limiting properties, namely the Gaussian wave. The method developed in this chapter has no such restrictions.

The Scattering Model

The model presented here (Figure 3.1) is a modification of the full space model developed in Chapter 2. While the full-space model posed the problem using a scattering formulation, the half-space model will be posed as a radiation problem. This approach is similar to that used by Gonsalves et al. [30] to solve the elastodynamic half-space problem.
Figure 3.1: Fluid-solid half-space with a fluid filled dimple – 2-D schematic
The advantages of this method are:

- Any type of wave can be used
- Only a small neighborhood of the flaw needs to be modeled
- Easily programmed using the MRBEM

The interaction model used to describe the problem is shown in Figure 3.1. The fluid-solid interface is assumed, without loss of generality, to be the surface $z = 0$. The fluid (acoustic medium) fills the semi-infinite region $V_f (z > 0)$. The solid (elastodynamic medium) covers the semi-infinite region $V_s (z \leq 0)$. The fluid also fills the dimple (flaw) on the solid surface. The acoustic field in the fluid is characterized by the total pressure $p$ and the total pressure gradient $q$ ($dp/dn$) in the normal direction. The total fields $p$ and $q$, defined to be the sum of the incident, reflected and scattered fields, are given by

$$p = p^I + p^R + p^s$$  \hspace{1cm} (3.1)\]
$$q = q^I + q^R + q^s$$  \hspace{1cm} (3.2)\]

where the scattered fields $p^s$ and $q^s$ are equal to zero in the absence of the scatterer (dimple). These scattered fields are required to satisfy the Sommerfeld radiation conditions at infinity (Appendix B). A unit normal $n_f$ is defined in the outward direction to the fluid. The incident beam $p^I$ of frequency $\omega$ is generated within the fluid region $V_f$.

The elastodynamic field in the solid region $V_s$, characterized by the elastic displacement $u$ and traction $t$, is given by the vector equations

$$u = u^t + u^s$$  \hspace{1cm} (3.3)\]
\[ t = t^t + t^s \]  (3.4)

The time-harmonic elastodynamic response of the displacement \( u(x) \) in \( V_t \), in the absence of body forces, is governed by the Cauchy-Navier equation 2.7. The propagation of the time-harmonic pressure wave \( p(x) \) in the fluid is governed by the Helmholtz wave equation 2.8.

At the fluid-solid interface \( S \), the acoustic pressure \( p \) and the solid traction \( t \) are required to satisfy the interface conditions
\[
\begin{align*}
  p(x)n_f(x) &= -t(x) \quad x \in S \\
  q(x) &= \rho_f \omega^2 u(x) \cdot n_f(x) \quad x \in S
\end{align*}
\]  (3.5)  (3.6)

where \( \rho_f \) is the fluid density.

**The Incident Wave Model**

The modeling of the incident wave is crucial to the solution of the fluid-solid interface problem when posed as a radiation problem. It is therefore presented, in this section, in detail. To illustrate the technique, a plane compressional wave propagating in the \( xz \) plane is used. The technique is general and can be easily adapted to other types of waves. The starting point is the analysis of plane waves in various references including Ewing *et al.* [27], Miklowitz [38], Aki and Richards [39], and Burridge [40].

The contributions of the scatterer (dimple) are the quantities \( p^i, q^i, u^i, \) and \( t^i \). Hence, for the purposes of this work, the incident wave information includes the corresponding incident, reflected, and transmitted quantities. This model is shown in Figure 3.2.
Figure 3.2: Fluid-Solid half-space incident plane wave model
When a plane compressional incident wave, generated within the fluid, given by

$$\Phi_1 = A_1 e^{ik_f(x \sin(\theta) - z \cos(\theta))}$$  \hspace{1cm} (3.7)

hits the fluid-solid half-space interface, the reflected acoustic wave is given by

$$\Phi_2 = A_2 e^{ik_f(x \sin(\theta) + z \cos(\theta))}$$  \hspace{1cm} (3.8)

Two waves, a longitudinal wave $\Phi_3$ and a shear wave $\Psi_3$, given by

$$\Phi_3 = A_3 e^{ik_l(x \sin(\alpha) - z \cos(\alpha))}$$  \hspace{1cm} (3.9)

$$\Psi_3 = B_3 e^{ik_r(x \sin(\beta) - z \cos(\beta))}$$  \hspace{1cm} (3.10)

are transmitted into the solid.

The reflection and transmission angles are related by Snell’s law, given by

$$\frac{\sin(\theta)}{c_f} = \frac{\sin(\alpha)}{c_l} = \frac{\sin(\beta)}{c_t}$$  \hspace{1cm} (3.11)

where $c_f$ is the wave speed in the fluid, and $c_l$ and $c_t$ are the longitudinal and shear wave speed in the solid respectively.

The total incident pressure $P$ in the fluid is given by

$$P = \rho_f \omega^2 (\Phi_1 + \Phi_2)$$  \hspace{1cm} (3.12)

and the corresponding displacements in the solid are

$$U_x = \frac{\partial \Phi_3}{\partial x} - \frac{\partial \Psi_3}{\partial z}$$

$$U_y = 0$$  \hspace{1cm} (3.13)

$$U_z = \frac{\partial \Phi_3}{\partial z} + \frac{\partial \Psi_3}{\partial x}$$

The tractions are calculated from the displacements using Hooke’s law.
Using Snell’s law, the reflection and transmission angles are calculated. Using the interface conditions, given in equations 3.5 and 3.6, and the fact that the fluid cannot support shear stresses, the amplitude ratios $A_2/A_1$, $A_3/A_1$, and $B_3/A_1$ may be calculated. Thus for an incident plane wave, with amplitude $A_1$ and angle of incidence $\theta$, the reflected and transmitted amplitudes and angles are easily calculated. With this information, the incident, reflected, and transmitted pressures, pressure gradients, displacements, and tractions can be calculated at all points on the boundary ($S_H \cup S_D$). These quantities will be used as the driving force for generating the solution in the following section.

**Boundary Element Formulation**

Even though the two regions are half-spaces, the problem associated with each region is modeled using the respective full-space fundamental solution. As shown in Figure 3.1, two infinite boundaries $S^+_\infty$ and $S^-\infty$ are introduced to adapt the problem to the boundary element method. The scattered field quantities $p^s$ and $q^s$ are required to satisfy the Sommerfeld radiation condition (Appendix B) at $S^\infty$. Similarly the quantities $u^s$ and $t^s$ are required to satisfy the Sommerfeld radiation condition at $S^\infty$. This requires

\[
\int_{S^\infty} \left\{ Fq^s - Gp^s \right\} ds = 0 \quad (3.14)
\]

\[
\int_{S^\infty} \left\{ Ut^s - Tu^s \right\} ds = 0 \quad (3.15)
\]

The fluid domain $V_f$ is bounded by $S^+_\infty$, $S_H$, and $S_D$, and the solid domain $V_s$ is bounded by $S^-\infty$, $S_H$, and $S_D$. The boundary $S_H$ is further split into two parts, $S_HU$ and $S_HM$. 

where $S_{HU}$ is the unmodeled part of the interface and $S_{HM}$ is the modeled part as shown in Figure 3.3. Since specialized infinite elements are not used, only a small portion of the infinite $xy$ plane, represented by $S_{HM}$, is modeled. The integrals corresponding to $S_{HU}$ are ignored since the scattered effects are only felt in a small neighborhood of the scatterer $S_D$. This is explained in more detail later in this section.

![Figure 3.3: Model of the interface](image)

The starting point in the derivation of the acoustic BIE is the representation integral,

$$p^s(x) + \int_S G(r) p^s(y) \, ds = \int_S F(r) q^s(y) \, ds \quad x \in V_f, \, y \in S \quad (3.16)$$

where $S = S_D + S_{HM} + S_{HU}$ is the boundary. This integral can be written as

$$p^s_0 + \int_{S_D+S_{HM}} G p^s \, ds = \int_{S_D+S_{HM}} F q^s \, ds + \int_{S_{HU}} \left( F q^s - G p^s \right) \, ds \quad (3.17)$$

where

$$\int_{S^ vap}{\left( F q^s - G p^s \right) \, ds = 0} \quad (3.18)$$

due to the radiation condition (eqn. 3.14) and

$$\int_{S_{HU}} \left( F q^s - G p^s \right) \, ds = 0 \quad (3.19)$$

because the scattered field quantities $p'$ and $q'$ are zero on the unmodeled part. Hence the
acoustic representation integral for the half-space problem with a fluid-filled dimple (using the full-space fundamental solution) is

\[ p_0^s + \int_{S_{D+SHM}} G^s p \, ds = \int_{S_{D+SHM}} F q \, ds \]  \hspace{1cm} (3.20)

The elastodynamic representation integral for the solid is similarly derived starting from

\[ u^s(x) + \int_S T(r) u^s(y) \, ds = \int_S U(r) t^s(y) \, ds \quad x \in V_f, \, y \in S \] \hspace{1cm} (3.21)

where \( S = S_D + S_{HM} + S_{HU} + S_{\infty} \), to obtain

\[ u_0^s + \int_{S_{D+SHM}} T u^s \, ds = \int_{S_{D+SHM}} U t^s \, ds \] \hspace{1cm} (3.22)

The superscript \( s \) is dropped in the rest of the discussion and all field variables are assumed to be scattered data unless otherwise mentioned. It should also be noted that the normal used in equation 3.20 is \( n_f \), and in equation 3.22 is \( n_s \).

Using equation 3.20, the acoustic BIE is now obtained by taking the field point \( x \in V_f \) to the boundary \( S = S_D + S_{HM} \). The boundary \( S \), just described, is now divided into two parts – a singular part \( S_0 \) corresponding to the element containing the collocation point, and a regular part \( S_1 \). The static Green’s function \( G^s \) is used to regularize equation 3.20 as follows

\[ p_0 + \int_{S_1} G^D p \, ds + \int_{S_0} (G^D - G^S) p \, ds + \int_{S_0} G^S (p - p_0) \, ds + \int_{S_0} G^S p_0 \, ds = \int_{S} F^D q \, ds \] \hspace{1cm} (3.23)
Simplifying, equation 3.23 is written as

\[
p_0 \left[ 1 + \int_{S_0} G^S \, ds \right] + \int_{S_i} G^D \, ds + \int_{S_0} (G^D - G^S) \, ds + \int_{S_0} G^S (p - p_0) \, ds = \int_S F^D \, q \, ds \tag{3.24}
\]

Using the integral identities in Appendix C, integral \( \int G^S \, ds \) can be expanded as

\[
\int_{S_0} G^S \, ds = \int_{S_D + S_{HU} + S_{HM}} G^S \, ds - \int_{S_i} G^S \, ds = -\frac{1}{2} - \int_{S_i} G^S \, ds
\tag{3.25}
\]

Thus, equation 3.24 becomes the acoustic BIE

\[
p^I + p_0 \left[ \frac{1}{2} - \int_{S_i} G^S \, ds \right] + \int_{S_i} G^D \, ds + \int_{S_0} (G^D - G^S) \, ds + \int_{S_0} G^S (p - p_0) \, ds = p^I + \int_S F^D \, q \, ds \tag{3.26}
\]

The elastodynamic BIE is obtained as

\[
u^I + u_0 \left[ \frac{1}{2} I - \int_{S_i} T^S \, ds \right] + \int_{S_i} T^D \, ds + \int_{S_0} (T^D - T^S) \, ds + \int_{S_0} T^S (u - u_0) \, ds = u^I + \int_S U^D \, t \, ds \tag{3.27}
\]

where \( I \) is the identity matrix, using the identity (Appendix C)

\[
\int_{S_D + S_{HU} + S_{HM}} T^S \, ds = -\frac{1}{2} I
\tag{3.28}
\]

**Multi-Region Boundary Element Method**

The boundary integral equations obtained here differ in two aspects with their counterparts in the previous chapter. The current formulation uses the scattered field
quantities whereas the full-space problem was modeled using the total field. The \textit{diagonal} terms have an \textit{add-on} factor of 0.5 instead of 1. Equations 3.26 and 3.27 are discretized using the process described in Chapter 2 to obtain the matrix forms

\begin{align*}
[A] [p^s] &= [B] [q^s] \quad (3.29) \\
[C] [u^s] &= [D] [t^s] \quad (3.30)
\end{align*}

These systems of equations cannot be solved in their current form because the interface conditions are only satisfied by the total field. Moreover, since all the variables are unknown, there is no \textit{driving force} to generate a solution. Hence, they are modified to replace the scattered field variables with their total field counterparts.

The acoustic equation 3.29 is written as

\begin{equation}
[p^s] = A^{-1} B [q^s] \quad (3.31)
\end{equation}

The total incident fields can be written as

\begin{align*}
p^i &= p^{\text{incident}} + p^{\text{reflected}} \\
q^i &= q^{\text{incident}} + q^{\text{reflected}} \\
u^i &= u^{\text{transmitted}} \\
t^i &= t^{\text{transmitted}}
\end{align*} \quad (3.32)

Adding $p^i$ to both sides of equation 3.31,

\begin{equation}
[p^i] = A^{-1} B ([q^i] - [q^i]) + \{p^i\} \quad (3.33)
\end{equation}

where the superscript $t$ indicates the total field. Using the matrices $N$ and $N^T$ from Chapter 2,
equation 3.3 is multiplied by $N^T$ to get

$$N^T\{p^f\} = N^T A^{-1} B\{q^f\} + N^T\{p^l\}$$ (3.34)

Using the interface conditions, equation 3.34 is written in the traction/displacement form as,

$$\{t^f\}^F = (-\rho_f \omega^2)N^T A^{-1} B N\{u^f\} + N^T A^{-1} B\{q^f\} - N^T\{p^l\}$$ (3.35)

where the superscript $F$ indicates a fluid region.

Similarly, equation 3.30 is written as

$$\{t^s\} = D^{-1} C\{u^s\}$$ (3.36)

Adding $t^f$ to both sides, equation 3.36

$$\{t^f\}^S = D^{-1} C\{u^f\} - D^{-1} C\{u^s\} + \{t^f\}$$ (3.37)

Using the interface condition,

$$\{t^f\}^F + \{t^f\}^S = \{0\}$$ (3.38)

the combined system equation is

$$\left[D^{-1} C - \rho_f \omega^2 N^T A^{-1} B N\right]\{u^f\} = -N^T A^{-1} B\{q^f\} + N^T\{p^l\} + D^{-1} C\{u^s\} - \{t^f\}$$ (3.39)

The right hand side of equation 3.39 is the vector containing the incident and reflected wave field quantities at all the nodes on $S_D$ and $S_{HM}$. Thus, any type of wave may be used, provided this analytical information is available. A typical boundary element model for the fluid-filled dimple on the half-space interface is shown in Figure 3.4.
• Parameters
  - Frequency = 894,000 Hz
  - Radius of dimple = 1
  - Fluid = Water
    • $K = 6.0$
  - Solid = Aluminum
    • $K_l = 1.4010$
    • $K_t = 2.8656$
  - Nodes/Elements
    • 241 nodes
    • 80 elements
      - 8 tria., 24 quad. on hemisphere
      - 48 quad. on flat surface

Figure 3.4: Typical BEM model of a fluid-filled dimple on the fluid-solid interface
Numerical Results

To illustrate the techniques developed, a boundary element mesh was constructed as shown in Figure 3.4. An aluminum object, immersed in water, contains the water filled dimple. A circular area, four times the dimple radius, was discretized. This area adequately captured the scattering information while satisfying the band-limiting condition on the scattered field. The material properties used are tabulated in Table 3.1.

Graphs of the total and scattered boundary pressures, as well as the total and scattered back-scattered pressures are presented in Figures 3.5-3.8. In all cases, a frequency of 894 kHz ($k_f = 6$) was used and the resulting pressures normalized using the incident pressure amplitude. The back-scattered pressure amplitudes were obtained at various distances from the center of the half space surface.

Table 3.1: Fluid-Solid interaction material properties

<table>
<thead>
<tr>
<th>Properties</th>
<th>Units</th>
<th>Al</th>
<th>Br</th>
<th>Lucite</th>
<th>Tungs./C</th>
<th>Solder</th>
<th>Water</th>
<th>Glycerine</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_f$</td>
<td>cm/sec</td>
<td>638000</td>
<td>430000</td>
<td>272000</td>
<td>666000</td>
<td>301000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_l$</td>
<td>cm/sec</td>
<td>312000</td>
<td>215000</td>
<td>134000</td>
<td>398000</td>
<td>145000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_t$</td>
<td>cm/sec</td>
<td>13.50</td>
<td>20.27</td>
<td>33.22</td>
<td>12.93</td>
<td>29.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta_{\text{crit}}$ (water)</td>
<td>degrees</td>
<td>17.42</td>
<td>26.37</td>
<td>44.60</td>
<td>16.67</td>
<td>39.38</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 3.5 shows the amplitudes of the scattered pressure calculated on the fluid-solid interface. Since the scattered field is due only to the presence of the dimple, it decays when measured far from the scatterer. The peak pressure is obtained at the bottom of the dimple when the incident wave is normal to the x-axis in the xz-plane. The location of the point where the maximum pressure is obtained shifts towards the right of the z-axis since the wave is incident at angles measured counterclockwise from the z-axis. The scattered pressure was normalized using the total pressure on the flat surface in the absence of the dimple. Figure 3.6 shows the total (incident+reflected+scattered) pressure distribution on the water-aluminum interface obtained for various angles of incidence of a plane wave in the xz-plane. The overall behavior was identical to that of the scattered field in Figure 3.5.

Figure 3.5: Boundary pressure in the xy plane of a fluid-filled dimple
Figure 3.6: Total normalized boundary pressure on a fluid-filled dimple
Figure 3.7 contains the scattered pressure measured in the fluid along the axis of incidence (backscatter). The scattered pressure was computed at radial distances (measured from the origin of the coordinate system) ranging from half the dimple radius to four times the radius. As expected, the plot shows a decaying trend. These values were computed using the representation integral for the acoustic region. The scattered field did not vary much with the angle of incidence. The data was normalized using the total incident field on the flat part of the interface. The total pressure field, plotted in Figure 3.8, was obtained from the scattered field in Figure 3.7, by adding the incident and reflected fields.
Figure 3.8: Total back-scattered pressure in the fluid due to the dimple
Conclusions

Since exact solutions are non-existent in problems of this nature, it is very difficult to verify the accuracy of the results. To compensate for this inadequacy, various other checks were performed on the program, including reciprocity and convergence checks. The reciprocity check is described in Chapter 5. To check for convergence, various meshes of increasing density and discretized area were used. The results in Figures 3.5-3.8 are a consequence of this optimization. Increasing the mesh density did not significantly improve the results. Also, discretizing a surface area of radius greater than 4 times the dimple radius did not change the results much. The major change was a slight reduction in the peak amplitudes. The overall shape of the curve did not change.

Since the program used to solve the fluid-filled problem is a modified version of the full-space program developed in Chapter 2, the accurate results obtained in that chapter provided further confidence in the workings of the software. The solutions, obtained and tested in Chapter 5 for an identical problem in the absence of the fluid, also affirmed the accuracy. Naturally, the best check would be to compare with experimental or exact solutions. It is hoped that further work in this area will validate the results obtained here.
CHAPTER 4. HALF-SPACE SCATTERING – AIR-FILLED DIMPLE

Introduction

This chapter presents the specialization of the boundary element method for modeling the reflection, transmission, and scattering of an ultrasonic wave, through an infinite fluid-solid interface, in the presence of hollow surface indentations. These defects usually occur when air bubbles are trapped on the fluid-solid interface during immersion testing in acoustic microscopy. The method developed here is an extension of the process developed in Chapter 3 that was used to solve a similar problem where the defect was filled with the coupling fluid. This solution technique may be used for any type of wave – plane, Rayleigh, etc. It can also be easily extended (using hyper-singular elements) to deal with surface breaking cracks.

The Scattering Model

The model presented here (Figure 4.1) is a modification of the fluid-filled dimple model developed in Chapter 3. This case, similar to the fluid-filled problem, is posed as a radiation problem.

The advantages of this method are:

- Any type of wave can be used
- Only a small neighborhood of the flaw needs to be modeled
- Easily programmed using the MRBEM
- System matrix is compact and limited to nodes on the fluid-solid interface
Figure 4.1: Fluid-solid half-space with a air bubble – 2-D schematic
The interaction model used to describe the problem is shown in Figure 4.1. The fluid-solid interface is assumed, without loss of generality, to be the surface $z = 0$. The fluid (acoustic medium) fills the semi-infinite region $V_f \ (z > 0)$. The solid (elastodynamic medium) covers the semi-infinite region $V_s \ (z \leq 0)$. The dimple (flaw) on and within the solid surface shares a pressure-free surface (flat part) with the fluid and a traction-free surface (hemispherical part) with the solid. The half-space interface (excluding the dimple) constitutes the fluid-solid interface where the usual interface conditions are satisfied. The acoustic field in the fluid is characterized by the total pressure $p$ and the total pressure gradient $q \ (dp/dn)$ in the normal direction. The total fields $p$ and $q$, defined to be the sum of the incident, reflected and scattered fields, are given by

$$p = p^I + p^R + p^S$$  \hspace{1cm} (4.1)  \\
$$q = q^I + q^R + q^S$$  \hspace{1cm} (4.2)

where the scattered fields $p'$ and $q'$ are equal to zero in the absence of the scatterer (dimple). These scattered fields are required to satisfy the Sommerfeld radiation conditions at infinity (Appendix B). A unit normal $\mathbf{n}_f$ is defined in the outward direction to the fluid. The incident beam $p^I$ of frequency $\omega$ is generated within the fluid region $V_f$.

The elastodynamic field in the solid region $V_s$, characterized by the elastic displacement $u$ and traction $t$, is given by the vector equations

$$u = u^I + u^S$$  \hspace{1cm} (4.3)  \\
$$t = t^I + t^S$$  \hspace{1cm} (4.4)

The time-harmonic elastodynamic response of the displacement $u(x)$ in $V_s$, in the
absence of body forces, is governed by the Cauchy-Navier equation 2.7. The propagation of
the time-harmonic pressure wave \( p(x) \) in the fluid is governed by the Helmholtz wave
equation 2.8.

At the fluid-solid interface \( S \), the acoustic pressure \( p \) and the solid traction \( t \) are
required to satisfy the interface conditions

\[
\begin{align*}
  p(x)n_f(x) &= -t(x) \quad x \in S \\
  q(x) &= \rho_f \omega^2 u(x) \cdot n_f(x) \quad x \in S
\end{align*}
\]

where \( \rho_f \) is the fluid density.

The incident wave model is identical to that presented in Chapter 3. The contributions
of the scatterer (air bubble) are the quantities \( p', q', u', \) and \( t' \). Hence, for the purposes of this
work, the incident wave information includes the corresponding incident, reflected, and
transmitted quantities (Figure 3.2).

**Boundary Element Formulation**

The derivation of the representation integral and the boundary integral equation is
identical to that presented in the corresponding section of Chapter 3. Even though the two
regions are half-spaces, the problem associated with each region is modeled using the
respective full-space fundamental solution. The scattered field quantities \( p^s \) and \( q^s \) are
required to satisfy the Sommerfeld radiation condition (Appendix B) at \( S_\infty^+ \). Similarly the
quantities \( u^s \) and \( t^s \) are required to satisfy the Sommerfeld radiation condition at \( S_\infty^- \). This
requires the identities,
\[ \int_{S_\infty^+} \{ F_q^S - G_p^S \} ds = 0 \quad (4.7) \]
\[ \int_{S_\infty^-} \{ U_t^S - T u^S \} ds = 0 \quad (4.8) \]

The fluid domain \( V_f \) is bounded by \( S_\infty^+ \), \( S_H \), and \( S_{DF} \), and the solid domain \( V_s \) is bounded by \( S_\infty^- \), \( S_H \), and \( S_{DS} \). The boundary \( S_H \) is further split into two parts, \( S_{HU} \) and \( S_{HM} \), where \( S_{HU} \) is the unmodeled part of the interface and \( S_{HM} \) is the modeled part as shown in Figure 4.2. The integrals, corresponding to \( S_{HU} \), are ignored since the scattered effects are only felt in a small neighborhood of the scatterer \( S_{DF} + S_{DS} \).

\[ p^t = 0 \]
\[ t^t = 0 \]

Figure 4.2: Model of the interface containing the air-filled bubble

The boundary integral equations are derived in the same manner as shown in Chapter 3. The acoustic representation integral for the half-space problem with a fluid-filled dimple (using the full-space fundamental solution) is

\[ p_0^S + \int_{S_{DF} + S_{HM}} G_p^S ds = \int_{S_{DF} + S_{HM}} F_q^S ds \quad (4.9) \]
The elastodynamic representation integral for the solid is similarly derived as

\[ u_s^0 + \int_{S_{DS} + S_{HM}} T u^s \, ds = \int_{S_{DS} + S_{HM}} U t^s \, ds \]  

(4.10)

The superscript \( s \) is dropped in the rest of the discussion and all field variables are assumed to be scattered data unless otherwise mentioned. It should also be noted that the normal used in equation 4.9 is \( n_f \) and in equation 4.10 is \( n_r \).

Using equation 4.9, the acoustic BIE is now obtained by taking the field point \( x \in V_f \) to the boundary \( S = S_{DF} + S_{HM} \). The boundary \( S \), just described, is now divided into two parts – a singular part \( S_0 \) corresponding to the element containing the collocation point, and a regular part \( S_1 \). The static Green's function \( G^S \) is used to regularize equation 4.9 to obtain the acoustic BIE

\[ P^I + p_0 \left[ \frac{1}{2} \int_{S_1} G^S \, ds \right] + \int_{S_1} G^D \, ds + \int_{S_0} \left( G^D - G^S \right) ds + \int_{S_0} G^S \left( p - p_0 \right) ds = P^I + \int_{S} F^D \, q ds \]  

(4.11)

Similarly, the elastodynamic BIE is obtained as

\[ u^I + u_0 \left[ \frac{1}{2} I - \int_{S_1} T^S \, ds \right] + \int_{S_1} T^D \, ds + \int_{S_0} \left( T^D - T^S \right) \, ds + \int_{S_0} T^S \left( u - u_0 \right) \, ds = u^I + \int_{S} U^D \, t ds \]  

(4.12)

**Multi-Region Boundary Element Method**

The boundary integral equations obtained here differ in two aspects with their counterparts in Chapter 3. The fluid-filled dimple problem required the solution of a matrix corresponding to all nodes and elements in the discretized model. The air-bubble problem only requires the solution of a smaller system of equations – corresponding to those elements and
nodes on the fluid-solid interface. Unlike the fluid-filled case, the scattered field quantities also satisfy the usual conditions on the fluid-solid interface. This occurs due to the fact that the interface elements lie on the surface $S_H$, where the incident and reflected fields satisfy the interface conditions by design. Consequently, the solution obtained will consist of the scattered field. The fluid-filled case led to a system of equations with total field variables. Equations 4.11 and 4.12 are discretized using the process described in Chapter 2 to obtain the matrix forms

$$[A][p^s] = [B][q^s] \quad (4.13)$$

$$[C][u^s] = [D][t^s] \quad (4.14)$$

These systems of equations are solved after some matrix manipulations described below. This process takes advantage of the pressure-free and traction-free boundary conditions to reduce the size of the system matrix.

The total incident fields can be written at all points as

$$p^I = p^{\text{incident}} + p^{\text{reflected}}$$
$$q^I = q^{\text{incident}} + q^{\text{reflected}} \quad (4.15)$$
$$u^I = u^{\text{transmitted}}$$
$$t^I = t^{\text{transmitted}}$$

Additionally, on the pressure-free surface $S_{DF}$,

$$p^{\text{total}} = p^{\text{incident}} + p^{\text{reflected}} + p^{\text{scattered}} = 0 \quad (4.16)$$
Hence, the scattered pressure is given by the relation

\[ p^{\text{scattered}} = -(p^{\text{incident}} + p^{\text{reflected}}) \]  

(4.17)

Similarly, on the traction-free surface \( S_{DS} \),

\[ t^{\text{scattered}} = -(t^{\text{transmitted}}) \]  

(4.18)

After dropping the superscript \( s \) on the field quantities (assumed scattered), eqn. 4.13 can be written in an expanded form as

\[
\begin{bmatrix}
A_{DD} & A_{DH} \\
A_{HD} & A_{HH}
\end{bmatrix}
\begin{bmatrix}
p_D \\
p_H
\end{bmatrix} =
\begin{bmatrix}
B_{DD} & B_{DH} \\
B_{HD} & B_{HH}
\end{bmatrix}
\begin{bmatrix}
q_D \\
q_H
\end{bmatrix}
\]  

(4.19)

where \( A_{DD}, A_{DH}, B_{DH}, \) etc., are sub-matrices. (Note: The subscript DH indicates that the collocation point is on \( S_{DF} \) and the integration point is on \( S_H \)). Similarly, the vector \( p_D \) represents the pressure at the nodes of surface \( S_{DF} \), etc. Equation 4.19 is modified, by moving the columns of the matrices, as follows

\[
\begin{bmatrix}
(-\rho \omega^2) B_{DD} & A_{DH} \\
(-\rho \omega^2) B_{HD} & A_{HH}
\end{bmatrix}
\begin{bmatrix}
\hat{q}_D \\
\hat{q}_H
\end{bmatrix} =
\begin{bmatrix}
0 & B_{DH} \\
0 & B_{HH}
\end{bmatrix}
\begin{bmatrix}
\hat{q}_D \\
\hat{q}_H
\end{bmatrix} -
\begin{bmatrix}
A_{DD} p_D \\
A_{HD} p_D
\end{bmatrix}
\]  

(4.20)

where

\[ \hat{q}_i = \frac{q_i}{\rho \omega^2} \quad i = D, H \]  

(4.21)

Using eqns. 4.15 and 4.17,

\[
\begin{bmatrix}
(-\rho \omega^2) B_{DD} & A_{DH} \\
(-\rho \omega^2) B_{HD} & A_{HH}
\end{bmatrix}
\begin{bmatrix}
\hat{q}_D \\
\hat{q}_H
\end{bmatrix} =
\begin{bmatrix}
0 & B_{DH} \\
0 & B_{HH}
\end{bmatrix}
\begin{bmatrix}
\hat{q}_D \\
\hat{q}_H
\end{bmatrix} -
\begin{bmatrix}
-A_{DD} p_D \\
-A_{HD} p_D
\end{bmatrix}
\]  

(4.22)
which is now written as

\[
S \begin{bmatrix} \hat{q}_D \\ \hat{p}_H \end{bmatrix} = R \begin{bmatrix} 0 \\ \hat{q}_H \end{bmatrix} - \begin{bmatrix} d_D^T \\ d_H^T \end{bmatrix}
\] (4.23)

Inverting matrix S and multiplying this matrix equation by \( N^T \) (eqn. 2.41), eqn. 4.23 becomes

\[
-N^T \begin{bmatrix} \hat{q}_D \\ \hat{p}_H \end{bmatrix} = -N^T S^{-1} R \begin{bmatrix} 0 \\ \hat{q}_H \end{bmatrix} + N^T S^{-1} \begin{bmatrix} d_D^T \\ d_H^T \end{bmatrix}
\] (4.24)

This equation is further simplified to the form

\[
\begin{bmatrix} -u_D^\text{Fluid} \\ t_H^\text{Fluid} \end{bmatrix} = -N^T S^{-1} R \begin{bmatrix} 0 \\ u_H^\text{Fluid} \end{bmatrix} + N^T S^{-1} \begin{bmatrix} d_D^T \\ d_H^T \end{bmatrix}
\] (4.25)

Hence, the interface fluid tractions are given by the equation

\[
\{t_H\}^\text{Fluid} = -N^T S^{-1} R \{u_H\}^\text{Fluid} + N^T S^{-1} \{d_H^T\}
\] (4.26)

A similar approach is used on the elastodynamic matrix equation 4.14. After dropping the superscript \( s \) on the field quantities, it is expanded as

\[
\begin{bmatrix} C_{DD} & C_{DH} \\ C_{HD} & C_{HH} \end{bmatrix} \begin{bmatrix} u_D \\ u_H \end{bmatrix} = \begin{bmatrix} D_{DD} & D_{DH} \\ D_{HD} & D_{HH} \end{bmatrix} \begin{bmatrix} t_D \\ t_H \end{bmatrix}
\] (4.27)

where the subscript DD corresponds to the contribution of the nodes on \( S_{DS} \). Applying some transformations on the column sub-matrices, equation 4.27 is written as

\[
\begin{bmatrix} 0 & C_{DH} \\ 0 & C_{HH} \end{bmatrix} \begin{bmatrix} u_D \\ u_H \end{bmatrix} = \begin{bmatrix} -C_{DD} & D_{DH} \\ -C_{HD} & D_{HH} \end{bmatrix} \begin{bmatrix} t_D \\ t_H \end{bmatrix} + \begin{bmatrix} D_{DD} t_D \\ D_{HD} t_D \end{bmatrix}
\] (4.28)

Using the identity equation 4.18, equation 4.28 is simplified to the form

\[
\begin{bmatrix} 0 & C_{DH} \\ 0 & C_{HH} \end{bmatrix} \begin{bmatrix} u_D \\ u_H \end{bmatrix} = \begin{bmatrix} -C_{DD} & D_{DH} \\ -C_{HD} & D_{HH} \end{bmatrix} \begin{bmatrix} u_D \\ t_H \end{bmatrix} + \begin{bmatrix} -D_{DD} t_D \\ -D_{HD} t_D \end{bmatrix}
\] (4.29)
and can be represented by

\[
L\begin{bmatrix} 0 \\ \bf{u}_H \end{bmatrix} = M\begin{bmatrix} \bf{u}_D \\ \bf{t}_H \end{bmatrix} + \begin{bmatrix} \bf{b}_D^T \\ \bf{b}_H^T \end{bmatrix}
\]  

(4.30)

Upon inverting \( M \), the equation is further simplified to the desired form

\[
\begin{bmatrix} \bf{u}_D \\ \bf{t}_H \end{bmatrix}^{\text{Solid}} = M^{-1}L\begin{bmatrix} 0 \\ \bf{u}_H \end{bmatrix}^{\text{Solid}} - M^{-1}\begin{bmatrix} \bf{b}_D^T \\ \bf{b}_H^T \end{bmatrix}
\]  

(4.31)

Hence, the interface solid tractions are

\[
\begin{bmatrix} \bf{t}_H \end{bmatrix}^{\text{Solid}} = M^{-1}L\begin{bmatrix} \bf{u}_H \end{bmatrix}^{\text{Solid}} - M^{-1}\begin{bmatrix} \bf{b}_H^T \end{bmatrix}
\]  

(4.32)

It should be noted that equations 4.26 and 4.32 are the reduced set \((3m)\) of matrix equations corresponding to \( m \) nodes on the fluid-solid interface. The field quantities (without superscripts) in these equations correspond to the scattered data. Adding these equations and using the interface condition,

\[
\begin{bmatrix} \bf{t}_H \end{bmatrix}^{\text{Solid}} + \begin{bmatrix} \bf{t}_H \end{bmatrix}^{\text{Fluid}} = \{0\}
\]  

(4.33)

the combined system equation is given by

\[
\begin{bmatrix} M^{-1}L - N^TS^{-1}RN \end{bmatrix}\begin{bmatrix} \bf{u}_H \end{bmatrix} = \begin{bmatrix} M^{-1}\begin{bmatrix} \bf{b}_H^T \end{bmatrix} - N^TS^{-1}\begin{bmatrix} \bf{d}_H^T \end{bmatrix} \end{bmatrix}
\]  

(4.34)

which can now be solved for the interface displacements \( \bf{u}_H \).

Upon calculation of \( \bf{u}_H \), equation 4.32 may be used to compute \( \bf{u}_D \) on the hemispherical part \( S_{DS} \) of the air bubble and the tractions \( \bf{t}_H \) on \( S_H \). Similarly, equation 4.25 may now be used along with the interface conditions to obtain the pressure on \( S_H \) and pressure gradients on the flat part \( S_{DF} \) of the bubble. All quantities thus computed are scattered fields.
Numerical Results

To illustrate the techniques developed in this chapter, numerical experiments were performed by using a hemispherical air bubble on the surface of an aluminum solid immersed in water. The incident wave was generated in the fluid at three different angles of incidence measured counterclockwise from the z-axis. The scattered pressure obtained was added to the incident and reflected pressures to obtain the total pressure on the boundary nodes of the fluid. Unlike the fluid bubble case, all nodes on the fluid boundary lie on the half space. All numerical results obtained and plotted in Figures 4.3-4.7 were normalized using the amplitude of the input wave (incident + reflected). In all cases, a frequency of 894 kHz was used which correspond to a wave number of 6 in the fluid. The material properties are tabulated in Table 3.1.

Figure 4.3 shows the amplitudes of the scattered pressure calculated on the fluid interface (z = 0) in the xz-plane. It should be noted that the scattered field is due only to the presence of the air-bubble. Hence, as expected, it decays when measured far from the scatterer. The peak pressure is obtained on the fluid-bubble interface when the scattered pressure is the complement of the incident and reflected pressure.

Figure 4.4 shows the total (incident+reflected+scattered) pressure distribution on the water interface obtained for various angles of incidence of a plane wave in the xz-plane. Since this pressure was normalized using the total (incident+reflected) pressure in the absence of the scatterer, the values far away from the air-bubble were close to unity.
Figure 4.3: Scattered pressure amplitude on the fluid boundary
Figure 4.4: Total pressure amplitude on the fluid boundary
Figure 4.5 contains the scattered pressure measured in the fluid along the axis of incidence (backscatter). The scattered pressure was computed at radial distances (measured from the origin of the coordinate system) ranging from half the bubble radius to four times the radius. As expected, the plot shows a decaying trend. These values were computed using the representation integral for the acoustic region. Unlike the corresponding case of a fluid-filled bubble, the scattered field did vary with the angle of incidence. The data was normalized using the total incident field on the flat part of the interface.

The total pressure field, plotted in Figure 4.6, was obtained from the scattered field in Figure 4.5, by adding the incident and reflected fields. The results were normalized, as before, using the total incident field on the flat part of the interface.

![Figure 4.5: Scattered pressure amplitude in the back scatter direction in the fluid](image-url)
Figure 4.6: Total pressure amplitude in the back scatter direction in the fluid.
Conclusions

Again, due to the absence of exact solutions in problems of this nature, it is very difficult to verify the accuracy of the results. As before, various other tests were performed on the program, including reciprocity and convergence checks. The reciprocity check is described in Chapter 5. To check for convergence, various meshes of increasing density and discretized area were used. The results in Figures 4.3-4.6 are a consequence of this optimization. Increasing the mesh density did not significantly improve the results. Also, discretizing a surface area of radius greater than 4 times the dimple radius did not change the results much. The major change was a slight reduction in the peak amplitudes. The overall shape of the curve did not change.

Since the program used to solve the air-bubble problem is a modified version of the full-space program developed in Chapter 2, the accurate results obtained in that chapter are the primary source of confidence in the software. The solutions, obtained and tested in Chapter 5 for an identical problem in the absence of the fluid, also affirm the accuracy. Naturally, the best check would be to compare with experimental or exact solutions. It is hoped that further work in this area will validate the results obtained here.

It should be noted that the backscattered pressures obtained in the case of an air-bubble are significantly lower than those obtained when the fluid fills the bubble.
CHAPTER 5. WAVES GENERATED IN THE SOLID – A COMPARITIVE STUDY

Introduction

A lack of availability of analytical and experimental data prevents the verification of the numerical results presented in Chapters 3 and 4. Numerical results obtained by generating a plane wave within the solid are presented for various solid half-space boundary conditions. Specifically, the effects of this wave are analyzed in three different situations – (1) traction-free half-space interface, (2) fluid-solid half-space interface with the fluid-filled dimple, and (3) fluid-solid half-space interface with air trapped in the dimple. The results obtained in the traction-free surface case are compared with other published work [29, 31]. The results obtained in the fluid-filled dimple and air-filled dimple are then compared with the traction-free surface results to validate the working of the computer programs developed. Reciprocity checks are also performed to lend confidence to the numerical results.

The Scattering Model

The models used in this chapter are similar those in Figures 3.1 and 4.1. The major difference is the generation of the incident wave within the solid instead of the fluid. The problems, in all three cases, are posed as radiation problems as shown in Chapters 3 and 4. The half-space surface is assumed to be the surface z = 0 and the solid fills the region z \leq 0 except for the region enclosed by the dimple.

In the presence of the fluid in the region z > 0, the acoustic field is characterized by the total pressure p and the total pressure gradient q in the normal direction \( n_r \). These fields, p and
where the scattered fields are zero in the absence of the scatterer (dimple). The scattered fields satisfy the Sommerfeld radiation conditions at infinity (Appendix B).

The elastodynamic fields in the solid, characterized by the elastic displacement $u$ and traction $t$, are given by the vector equations

$$u = u^{\text{incident}} + u^{\text{reflected}} + u^{\text{scattered}}$$

$$t = t^{\text{incident}} + t^{\text{reflected}} + t^{\text{scattered}}$$

where the scattered fields satisfy the Sommerfeld radiation conditions and are zero in the absence of the dimple.

The time-harmonic elastodynamic response of the displacement is governed by the Cauchy-Navier equation 2.7. The time-harmonic pressure in the fluid is governed by the Helmholtz wave equation 2.8. In the presence of the fluid, the conditions on the fluid-solid interface, given by

$$p(x)n_f(x) = -t(x) \quad x \in S$$

$$q(x) = \rho_f \omega^2 u(x) \cdot n_f(x) \quad x \in S$$

are satisfied by the acoustic and elastodynamic field variables.

The incident, reflected, and transmitted fields (Figure 5.1) are obtained as shown in the section on the incident wave model of Chapter 3. For incidence angles within the critical angle
Figure 5.1: The incident wave model for an L-wave generated within the solid
two reflected waves – a longitudinal wave and a shear wave are generated in the solid, and a pressure wave is transmitted into the fluid. The angles of incidence, reflection and transmission satisfy Snell’s law (equation 3.11). In the presence of a scatterer, such as a surface dimple, scattered waves are also generated. In the absence of the fluid, transmitted waves do not exist.

The Boundary Element Solution

In the absence of the fluid, the half-space surface \((z = 0)\) and the dimple surface satisfy the zero-traction conditions. Hence,

\[
\mathbf{r}^{\text{incident}} + \mathbf{r}^{\text{reflected}} + \mathbf{r}^{\text{scattered}} = \mathbf{0} \quad (5.7)
\]

Equation 5.7 may be written in a more convenient form as

\[
\mathbf{t}^s = -\left(\mathbf{r}^{\text{incident}} + \mathbf{r}^{\text{reflected}}\right) = \mathbf{t}^f \quad (5.8)
\]

The elastodynamic BIE is obtained, as shown in Chapter 4, as

\[
\begin{aligned}
\mathbf{u}_0^s &\left[\frac{1}{2} \mathbf{I} - \int T^S ds\right] + \int T^D u^s ds + \int (T^D - T^S) u^s ds + \int T^S (u^s - u_0^s) ds = \int U^D t^s ds \\
\end{aligned} \quad (5.9)
\]

where the dynamic kernels \(T^D\) and \(U^D\) and the static kernel \(T^S\) are given in Appendix A. The integration regions \(S_0\) corresponds to the boundary element containing the collocation node and \(S_1\) corresponds to the rest of the discretized boundary.

The BIE 5.9 is discretized, as described in Chapter 2, to the matrix form

\[
[C]\{\mathbf{u}^s\} = [D]\{\mathbf{t}^s\} \quad (5.10)
\]
and may be written, using equation 5.8, as

\[ [C] [u^s] = \{b\} \]  \hspace{1cm} (5.11)

where

\[ \{b\} = [D] [-l^1] \]  \hspace{1cm} (5.12)

The matrix equation 5.11 can be solved using Linpack to obtain the scattered displacements. The total fields may then be calculated using equations 5.7 and 5.8.

The analyses for the fluid-filled dimple and air-filled dimple are identical to those presented in Chapters 3 and 4 respectively.

**Numerical Results**

To verify the working of the computer program, numerical results obtained in the traction-free surface case were compared with those presented by Sanchéz-Sesma [29] and Manolis *et al.* [31]. Two test cases were used – a valley (dimple), and a ridge (bump) on the half-space solid surface.

The first case (Figure 5.2) represents a valley on an infinite solid surface \((z = 0)\). The solid fills the region \(z \geq 0\) and has a Poisson’s ratio = 0.25. The incident L-wave \((k_1 = \pi/4)\) is normal to the half-plane and propagating in the negative \(z\) direction. The surface of the dimple and the half-plane was modeled with 193 nodes using 64 eight-node isoparametric elements. The horizontal \((U_x)\) and vertical \((U_z)\) displacement amplitudes on the surface in the \(xz\)-plane were plotted (Figure 5.3).
Figure 5.2: Valley on a traction-free half-space surface

Figure 5.3: Displacement amplitudes due to scattering by a valley
The second case presented is the scattering of a plane wave ($k_0 = \pi/2$) generated within a solid ($\nu = 0.30$) by an axisymmetric ridge as shown in Figure 5.4. This model was discretized using 189 nodes with 60 eight-node isoparametric elements. The total vertical ($U_z$) and horizontal ($U_x$) displacements were plotted in the $xz$-plane as shown in Figure 5.5.

![Scattering model of a ridge on a traction-free half-plane](image1)

**Figure 5.4: Scattering model of a ridge on a traction-free half-plane**

![Displacement amplitudes due to scattering by a ridge](image2)

**Figure 5.5: Displacement amplitudes due to scattering by a ridge**
Figure 5.6 contains a plot of the scattered displacement amplitudes on the surface of the aluminum model immersed in water. These displacements are normalized using the total (incident + reflected) incident displacement amplitudes on the flat part of the interface. The incident wave, traveling in the solid and normal to the half-space, is completely reflected when the surface is traction free. In the presence of the fluid, some of the wave is transmitted through into the fluid. Hence, as expected, the displacements are strongest when there is no transmission. This transmission is maximum in the presence of the fluid-filled bubble and moderate in the presence of the air-bubble.
Figure 5.7 shows the amplitudes of the scattered pressure calculated on the fluid-solid interface. Since the scattered field is due only to the presence of the dimple, it decays when measured far from the scatterer. The peak pressure is obtained around the bottom of the dimple when the incident wave is not normal to the x-axis in the xz-plane. The location of the point where the maximum pressure is obtained shifts towards the right of the z-axis. This is also expected since the wave is incident at angles measured clockwise from the negative z-axis. The angles represent the transmission angles of the pressure wave in the fluid.

Figure 5.8 shows the total (transmitted + scattered) pressure distribution on the water-aluminum interface obtained for various angles of incidence of a plane wave in the xz-plane. The overall behavior was identical to that of the scattered field in Figure 5.7.

Figure 5.7: Scattered pressure amplitudes in the xz-plane for a fluid-filled bubble
Figure 5.8: Total pressure amplitudes in the xz-plane for a fluid-filled bubble
Figure 5.9 contains the scattered pressure measured in the fluid along the axis of transmission (forward scatter). The scattered pressure was computed at radial distances (measured from the origin of the coordinate system) ranging from half the bubble radius to four times the radius. As expected, the plot shows a decaying trend. These values were computed using the representation integral for the acoustic region. The scattered field did not vary much with the angle of incidence. The data was normalized using the total transmitted field on the flat part of the interface.

The total pressure field, plotted in Figure 5.10, was obtained from the scattered field in Figure 5.9, by adding the transmitted field. The results were normalized, as before, using the transmitted field on the flat part of the interface.

![Graph](image-url)

**Figure 5.9: Backscattered pressure (scattered) for a fluid-filled bubble**
Figure 5.10: Backscattered pressure (total) for a fluid-filled bubble
Figure 5.11 shows the amplitudes of the scattered pressure calculated on the fluid-interface ($z = 0$). The scattered field is due only to the presence of the air-bubble. Hence, as expected, it decays when measured far from the scatterer. The angles shown below represent the transmission angles of the pressure wave in the fluid. The peak pressure is of unit magnitude since the fluid-bubble interface is pressure (transmitted + scattered) free. As expected, the results, when the transmission angle is greater than zero, are skewed to the right.

The transmitted surface pressures were added to the scattered pressures in Figure 5.11 and plotted in Figure 5.12. The results were normalized using the transmitted field information in the absence of the air-bubble.

![Figure 5.11: Scattered pressure in the xz-plane for an air bubble](image)
Figure 5.12: Total pressure amplitudes in the xz-plane for an air bubble
As before, the backscattered pressure was computed using the acoustic representation integral. The results obtained were plotted, as shown in Figure 5.13, along the axis of transmission in the fluid.

The transmitted pressure was added to the scattered pressure plotted in Figure 5.13 and normalized using the incident transmitted pressure on the fluid-solid interface. The results, shown in Figure 5.14 show trends similar to those in the case of the fluid-filled dimple (Figure 5.10)

Figure 5.13: Backscattered pressure (scattered) for an air-filled bubble
Figure 5.14: Backscattered pressure (total) for an air bubble
Acoustic and Elastodynamic Reciprocity

Reciprocity checks, as defined by Neerhoff [47], were performed on the software developed for the full-space, solid half-space, and fluid-solid half-space problems. The checks provide a validation of all the results obtained.

The following integral was used to compute the reciprocity of the acoustic region:

$$R = \int_{S} (p^1 q^2 - p^2 q^1) ds$$  \hspace{1cm} (5.13)

In the above equation, $p^1$ and $p^2$ (and their normal gradients $q^1$ and $q^2$), represent the boundary pressure on $S$, obtained using two different angles of incidence (example, $0^\circ$ and $10^\circ$). In the case of the full-space problem of Chapter 2, the surface $S$ was the surface of the solid scatterer. In the case of the half-space problems with the dimple and the air-bubble, $S$ represents the surface $S^+_\infty + S_D + S_H$, where $S_D$ is the dimple surface and $S_H$ is the half-space interface. The normalized quantities

$$Q_f = \frac{\int_{S} (p^1 q^2 - p^2 q^1) ds}{\int_{S} (p^1 q^1) ds}$$ \hspace{1cm} (5.14)

$$Q_s = \frac{\left| \int_{S} (u^1 t^2 - u^2 t^1) ds \right|}{\left| \int_{S} (u^1 t^1) ds \right|}$$ \hspace{1cm} (5.15)

were used to obtain the percentage error. The quantity $Q_f$, in all cases, was within 5%. The
displacements $u^1$ and $u^2$ represent the displacements on surface $S$ for incident plane waves at different angles. The quantity $Q$, was less than 0.05 in all cases.

**Conclusions**

The surface displacements shown in Figure 5.6 and the reciprocity checks in the previous section provide sufficient confidence in the results presented in this dissertation. The results obtained for related problems where exact solutions exist further solidify the validity of the techniques developed. The decaying nature of the scattered field validates the assumptions required in the modeling of the problems. The non-dependence of the techniques on the shape of the scatterer and the convergence of results with increasing mesh density and surface discretization illustrate the versatility and the accuracy of the multi-region method in the solution of a wide variety of problems.

Future work should incorporate the presence of cracks and crack-like anomalies on the surface using hyper-singular elements. The method of solution is independent of the nature of the defect. Other extensions would be the use of specialized waves for specific problems. The only change required would be the generation of the incident wave information.
REFERENCES


APPENDIX A: FUNDAMENTAL SOLUTIONS

Acoustics

Helmholtz’s Equation

The propagation of time harmonic acoustic waves is modelled by the wave equation

\[(\nabla^2 + k^2)p = 0\]  \hspace{1cm} (A.1)

where \(k = \omega/c\) is the wave number, \(\omega\) is the frequency, \(c\) is the wave speed, and \(p\) is the pressure. The fundamental solution associated with this problem is the solution that satisfies

\[(\nabla^2 + k^2)F^D(x, y) = -\delta(r)\]  \hspace{1cm} (A.2)

where \(F^D(x, y)\) is the pressure at point \(x\) due to a point source of dilatation at point \(y\), and \(r\) is the radius vector from \(y\) to \(x\). The fundamental solution (derived in Morse et al. [36]) is given by the expression

\[F^D(r) = \frac{e^{ikr}}{4\pi} \]  \hspace{1cm} (A.3)

and its normal derivative \(G^D\) is

\[
\frac{\partial F^D}{\partial n} = G^D(r) = (-1 + ikr) \left[ \frac{e^{ikr}}{4\pi r^2} \right] (r \cdot n) \]  \hspace{1cm} (A.4)

Laplace’s Equation

When \(\omega = 0\), equation A.1 becomes Laplace’s equation given by

\[\nabla^2 p = 0\]  \hspace{1cm} (A.5)
The corresponding fundamental solution $F^S(x,y)$ is the solution of

$$\nabla^2 F^S(x,y) = -\delta(r)$$  \hspace{1cm} (A.6)

where $F^S$ is given by

$$F^S(r) = \frac{1}{4\pi r}$$  \hspace{1cm} (A.7)

and its normal derivative is

$$\frac{\partial F^S}{\partial n} = G^S(r) = -\frac{1}{4\pi r^2}(r \cdot n)$$  \hspace{1cm} (A.8)

It should be noted that in the limit as $r \to 0$, both the dynamic fundamental solution and its normal derivative exhibit the same behavior. This particular aspect of these mathematical quantities is the reason why the static kernels are used to regularize their dynamic counterparts.

**Elasticity**

**Elastostatic Green’s Tensor**

In the absence of body forces, the equilibrium equation of linear elasticity is

$$\sigma_{ij,j}(x) = 0$$  \hspace{1cm} (A.9)

where $\sigma_{ij}$ represents the cartesian components of the stress vector and are related to the displacements $u_k$ by Hooke’s law

$$\sigma_{ij} = C_{ijkl}u_{k,l}$$  \hspace{1cm} (A.10)

with the tractions $t_j$ given by

$$t_j = \sigma_{ij}n_j$$  \hspace{1cm} (A.11)
In the case of linear, isotropic materials, two material constants, \(\lambda\) and \(\mu\), are adequate to describe the material. This reduces the coefficient matrix to the form

\[
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)
\]  

Equation A.9 can now be written as

\[
(\lambda + \mu)(u_{k,k})_i + \mu \delta_{ik} u_{k,ll} = 0
\]

and the corresponding equation for the fundamental solution is

\[
(\lambda + \mu)(U^S_{km,k})_i + \mu \delta_{ik} U^S_{km,ll} = -\delta_{im} \delta(r)
\]

The quantity \(U^S_{km}(x,y)\) represents the \(k\)-th component of displacement at \(x\) due to a concentrated unit force acting in the \(m\)-th direction at \(y\). The solution to equation A.14 [32] is

\[
U^S_{ij}(r) = \frac{1}{16\pi\mu(1-\nu)} \left( \frac{1}{r} \right) \left[ (3-4\nu)\delta_{ij} + r_i r_j \right]
\]

and the associated traction (using eqns. A.10 and A.11) is

\[
T^S_{ij}(r) = \frac{2\nu-1}{8\pi(1-\nu)} \left( \frac{1}{r^2} \right) \left[ \frac{\partial}{\partial n_i} \left( \delta_{ij} + \frac{3r_i r_j}{1-2\nu} \right) + r_i n_j - r_j n_i \right]
\]

where the comma indicates partial derivatives with respect to co-ordinates at \(y\), and the normal derivatives are taken at \(y\).

**Elastodynamic Green’s Tensor**

The time-harmonic elastodynamic equation of motion is similarly given by

\[
(\lambda + \mu)U^D_{ij,ik}(x,y) + \mu U^D_{kj,ll}(x,y) + \rho \omega^2 U^D_{ij}(x,y) = -\delta_{ij} \delta(x-y)
\]

The quantity \(U^D_{ij}(x,y)\) represents the displacement component at \(x\) in the \(i\)-th direction due to
a unit time-harmonic point force applied at $y$ in the $j$-th direction. The solution to this equation is presented in various references \[31,32\] and is given by

$$U_{ij}^D (r) = \frac{1}{4\pi\rho} \frac{1}{r^3} \left\{ \delta_{ij} \left[ (k_t r)^2 e_t + D \right] + C r_i r_j \right\}$$ \hspace{1cm} (A.18)

where $\lambda$ and $\mu$ are the Lamé constants, $\rho$ is the material density, and

$$c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$c_t = \sqrt{\frac{\mu}{\rho}}$$

$r = |x - y|$ 

$$k_\alpha = \frac{\omega}{c_\alpha}, \quad \alpha = t, l$$ \hspace{1cm} (A.19)

$$e_\alpha = e^{ik_\alpha r}$$

$$D = \Gamma_t e_t - \Gamma_l e_l; \quad \Gamma_\alpha = -1 + ik_\alpha r$$

$$C = \Omega_t e_t - \Omega_l e_l; \quad \Omega_\alpha = 3 - 3ik_\alpha r - k_\alpha^2 r^2$$

and the corresponding traction is

$$T_{ij}^D (r) = \frac{1}{4\pi\rho} \frac{1}{r^4} \left\{ \lambda \beta_1 + \mu \beta_2 + 2\mu \beta_3 \right\}$$ \hspace{1cm} (A.20)

where

$$\beta_1 = e_t (k_t r)^2 \Gamma_t r_j n_i$$

$$\beta_2 = e_t (k_t r)^2 \Gamma_t \left( \delta_{ij} \frac{\partial r}{\partial n} + r_j n_j \right)$$

$$\beta_3 = \left[ C \left( \delta_{ij} \frac{\partial r}{\partial n} + r_j n_j + r_j n_i \right) + F r_i r_j \frac{\partial r}{\partial n} \right]$$ \hspace{1cm} (A.21)

$$F = H_t e_t - H_l e_t$$

$$H_\alpha = 15 - 15ik_\alpha r - 6k_\alpha^2 r^2 + ik_\alpha^3 r^3$$
APPENDIX B: THE SOMMERFELD RADIATION CONDITIONS

The Sommerfeld radiation conditions provide a criterion for the uniqueness of the solution to the wave equation (acoustic or elastodynamic) for scattering in an unbounded domain. The conditions require that the scattered field always be outgoing.

In the case of the acoustic field $p$, the condition [25,46] states that

$$\lim_{r \to \infty} r \left( \frac{\partial p}{\partial r} - ikp \right) = 0$$  \hspace{1cm} (B.1)

where $k$ is the wave number, and $r$ is the distance to any field point.

The radiation conditions imply that for a unique solution to the Helmholtz equation in the unbounded domain $S_\infty$, where the boundary data of the problem are prescribed over a finite domain $S$, the solution at a point far away from $S$ should behave quantitatively as the Green's function $F(r)$. Hence, the solution $p$ must behave as

$$p(r) = O\left(\frac{1}{r}\right) \hspace{1cm} r \to \infty$$  \hspace{1cm} (B.2)

and for the direction of propagation to be outgoing,

$$\frac{\partial p}{\partial r} - ikp = o\left(\frac{1}{r}\right) \hspace{1cm} r \to \infty$$  \hspace{1cm} (B.3)
APPENDIX C: INTEGRAL REPRESENTATIONS AND IDENTITIES

Acoustic Integral Equation

The starting point of all boundary element analysis is the integral representation. The derivation of the acoustic integral is presented here as an illustration of the technique commonly used. Similar procedures are used to derive the elastodynamic equation. The acoustic wave equation is given by

\[ (\nabla^2 + k^2) p = 0 \] (C.1)

The fundamental solution \( F \) satisfies the equation

\[ (\nabla^2 + k^2) F = -\delta(x - y) \] (C.2)

From Green's second identity, if \( S \) is the boundary and \( V \) is the domain,

\[ \int_V [p \nabla^2 F - F \nabla^2 p] dV = -\int_S [p F - F q] ds \] (C.3)

where

\[ G = \frac{\partial F}{\partial n} \quad q = \frac{\partial p}{\partial n} \] (C.4)

From C.1 and C.2

\[ p(x) = \int_S [p G - F q] ds \] (C.5)

In the presence of a finite scatterer (Figure C.1), the scattered and incident fields satisfy

\[ (\nabla^2 + k^2) p^S(x) = 0 \quad x \in V_e \] (C.6)
Figure C.1: Full space scattering model
and
\[
\left( \nabla^2 + k^2 \right) p^I(x) = 0 \quad x \in V_e + V_i \quad (C.7)
\]

Using Green's second identity, the integral representation of the scattered field in the external domain may be written as

\[
C p^S(x_0) = \int_{S_\infty + S} \left( p^S(y)G(r) - q^S(y)F(r) \right) ds \quad y \in S \quad (C.8)
\]

where \( C = 0 \) when \( x_0 \) lies in \( V_e \) and 1 when \( x_0 \) lies in \( V_i \). Similarly, the representation integral for \( p^I \) in the interior domain is

\[
C p^I(x_0) = \int_{S} \left( p^I(y)G(r) - q^I(y)F(r) \right) ds \quad y \in S \quad (C.9)
\]

where \( C = 0 \) when \( x_0 \) lies in \( V_i \) and 1 when \( x_0 \) lies in \( V_e \). Since \( p^S \) satisfies the Sommerfeld radiation conditions, the integral in C.8 corresponding to \( S_\infty \) vanishes. Hence,

\[
p^S(x) = \int_{S} \left( p^S(y)G(r) - q^S(y)F(r) \right) ds \quad x \in V_e \quad (C.10)
\]

Similarly, in equation C.9, for \( x \) in \( V_e \)

\[
0 = \int_{S} \left( p^I(y)G(r) - q^I(y)F(r) \right) ds \quad x \in V_e \quad (C.11)
\]

The above equation remains unchanged with the normal \( n \), since \( q = -n \). Combining equations C.10 and C.11, the following relation may be obtained

\[
p^S(x) = \int_{S} \left( p(y)G(r) - q(y)F(r) \right) ds \quad x \in V_e \quad (C.12)
\]
Integral Identities

Consider the finite scatterer $V_j$, enclosed by surface $S$, and surrounded by $V_f$ as shown in Figure C.1. Let $x_e$ and $x_i$ be points in the fluid and solid respectively. Let $F^S$ and $U^S$ be the static fundamental solutions to Laplace’s equation and the elastostatic equation respectively. If $G^S$ and $T^S$ are the corresponding normal gradient and traction fundamental solutions, then the following identities [25] hold true:

$$\int_{S} T^S_q(x_i, y) ds = -I \quad (C.13)$$

$$\int_{S} T^S_n(x_e, y) ds = 0 \quad (C.14)$$

$$\int_{S} T^S_n(x_e, y) ds = -I \quad (C.15)$$

$$\int_{S_{\infty}} G^S_q(x_i, y) ds = -I \quad (C.16)$$

$$\int_{S} G^S_n(x_e, y) ds = 0 \quad (C.17)$$

$$\int_{S_{\infty}} G^S_n(x_e, y) ds = -I \quad (C.18)$$

These identities are obtained by using the static representation integrals and imposing constant displacement and pressure conditions. This results in zero tractions and pressure gradients which in turn simplify the integrals to the identities above.
In the case of the half-space scatterers, consider the model shown in Figure 3.1. Hence

\[
\frac{1}{2} S_{\infty} = S^- = S^+ \tag{C.19}
\]

Thus, from equation C.15,

\[
\int_{S^-} T_{n_1}^S \, ds = -\frac{1}{2} I \tag{C.20}
\]

Using the domain \(V_f\) as an exterior region, equation C.15 may be written as

\[
\int_{S^- + S_H + S_D} T_{n_1}^S \, ds = -I \tag{C.21}
\]

Hence, from C.20 and C.21,

\[
\int_{S_H + S_D} T_{n_1}^S \, ds = -\frac{1}{2} I \tag{C.22}
\]

Similarly, using \(V_e\) as an exterior region,

\[
\int_{S_H + S_D} G_{nr}^S \, ds = -\frac{1}{2} \tag{C.23}
\]