Global aspects of control systems: perspectives from control Lyapunov functions

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Global aspects of control systems:
Perspectives from control Lyapunov functions

by

Chung-Ming Ou

A Dissertation Submitted to the
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Major: Applied Mathematics

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For the Major Department
Signature was redacted for privacy.
For the Graduate College

Iowa State University
Ames, Iowa
1996

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>domain of attraction</td>
</tr>
<tr>
<td>$A^s$</td>
<td>strict domain of attraction</td>
</tr>
<tr>
<td>$\mathcal{MS}(C_i, C_j)$</td>
<td>multistability Region of the invariant set $C_i$ and $C_j$</td>
</tr>
<tr>
<td>$\mathcal{CR}$</td>
<td>chain recurrent set</td>
</tr>
<tr>
<td>$D$</td>
<td>lifted control set of the control set $D$</td>
</tr>
<tr>
<td>$\mathcal{E}$</td>
<td>lifted chain control set of the chain control set $E$</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>control Lyapunov function</td>
</tr>
<tr>
<td>$gl(n, \mathbb{R})$</td>
<td>collection of $n \times n$ real matrices</td>
</tr>
<tr>
<td>$\mathcal{L}A$</td>
<td>Lie algebra (of vector fields)</td>
</tr>
<tr>
<td>$\lambda(x, u)$</td>
<td>Lyapunov exponent</td>
</tr>
<tr>
<td>$x \sim y$</td>
<td>$x$ is controllable to $y$</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>state space: a smooth Riemannian manifold</td>
</tr>
<tr>
<td>$\mathcal{MS}$</td>
<td>multistability region</td>
</tr>
<tr>
<td>$\mathcal{O}^+(x)$</td>
<td>the positive orbit (reachable set ) of $x$</td>
</tr>
<tr>
<td>$\mathcal{O}^-(x)$</td>
<td>the negative orbit of $x$</td>
</tr>
<tr>
<td>$\mathcal{O}^+_{\leq T}(x)$</td>
<td>the reachable set of $x$ up to a finite time $T$</td>
</tr>
<tr>
<td>$\mathcal{O}^+_T(x)$</td>
<td>the reachable set of $x$ at time $T$</td>
</tr>
<tr>
<td>$\mathcal{O}^+_{u,T}(x)$</td>
<td>the reachable set of $x$ at time $T$ under the control $u$</td>
</tr>
</tbody>
</table>
\( \mathbb{P}^{n-1} \) \hspace{1cm} \text{the projective space in } \mathbb{R}^n

\( \prec \) \hspace{1cm} \text{the order between chain control sets}

\( \Psi \) \hspace{1cm} \text{continuous dynamical system}

\( R \) \hspace{1cm} \text{reachable map}

\( (U \times \mathcal{M}, \Phi) \) \hspace{1cm} \text{the control flow}

\( \rho \) \hspace{1cm} \text{the control range}

\( \Sigma_{Ly}(D) \) \hspace{1cm} \text{Lyapunov spectrum over the control set } D

\( \varphi(\cdot, x, u) \) \hspace{1cm} \text{the trajectory w.r.t the control } u \text{ with } \varphi(0, x, u) = x

\( \theta \) \hspace{1cm} \text{phase shift}

\( U \) \hspace{1cm} \text{the space of control functions}

\( V \) \hspace{1cm} \text{Lyapunov function for a dynamical system}

\( \omega(x) \) \hspace{1cm} \text{omega limit set of } x

\( \alpha(x) \) \hspace{1cm} \text{alpha limit set of } x
ACKNOWLEDGEMENTS

This thesis is dedicated to my mother, my brother and my sister. Without their support and encouragement, I would not have made it this far in my academic career.

During my research, I received much help from my advisor, Dr. Wolfgang Kliemann. Without his help, I cannot finish this paper. I'd also like to thank Dr. James Murdock, who constantly helps me improve my knowledge of dynamical systems theory for all these years. For several technical parts of this thesis, I wish to thank Dr. W. Kliemann for profitable discussions about the proof of Theorem 2.31. These numerical simulations of the chemical reactor model introduced in chapter 3 are provided by Dr. Gerhard Häckl.

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CHAPTER 1. INTRODUCTION

Lyapunov function techniques have become one of the standard methods to discuss the qualitative behavior of dynamical systems such as

- Limit behavior of trajectories; for example, LaSalle's Invariance principle [20, Section 3.2].

- Stability of equilibrium points; the Lyapunov direct and indirect method [22, Chapter 5, 6].

- Characterization of the Domain of Attraction of stable equilibrium points.

On the other hand, dynamical system theory has been successfully applied to affine control systems with unbounded control range. These methods include linearization, center manifold theorem and Lyapunov function techniques, see [23, 28]. For affine control systems, Lyapunov functions are used for feedback stabilization of (unstable) equilibrium points. Such techniques still depend on Lyapunov functions of (uncontrolled) dynamical systems. For more details, see [23, Chapter 10].

In this paper, we try a different approach: rather than adapt the Lyapunov functions of uncontrolled dynamical system, we will construct control Lyapunov functions of the following affine control systems:

\[ \Sigma_p : \dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t)X_i(x) \tag{1.1} \]
on a smooth Riemannian manifold $\mathcal{M}$, with $\dim \mathcal{M} = n < \infty$ and $d$ the Riemannian metric. Here the $X_i, i = 0, \ldots, m$ are smooth vector fields. $(u_i)_{i=1,\ldots,m} = u \in \mathcal{U}^\rho$; and $\mathcal{U}^\rho$ is the set of admissible control functions, which is defined as follows:

$$\mathcal{U}^\rho := \{ u : \mathbb{R} \to \mathcal{U}^\rho, \text{measurable} \}.$$ 

$\mathcal{U}^\rho = \rho \cdot U$ for $\rho \geq 0$, where $U \subset \mathbb{R}^m$ is compact and connected and $0 \in \text{int}U$. $\rho$ is called the control range of (1.1). For notation convenience, we will omit the superscript $\rho$ from $\mathcal{U}^\rho$, once we fixed the control range $\rho$. We call the following system the uncontrolled dynamical system of (1.1):

$$\dot{x} = X_0(x).$$  \tag{1.2}

Colonius and Kliemann analyze (1.1) via dynamical systems theory, in particular, topological dynamics. Details are the following.

- In [6], a specific continuous dynamical system corresponding to (1.1), called the control flow, has been defined. Topological mixing (transitive) for the control flow and the control sets of (1.1) are related (Proposition 3.5, Theorem 3.9). On the other hand, the chain transitive set of the control flows is somehow related to the chain control sets of (1.1) (Theorem 4.8).
- In [4], the conditions for limit sets of (1.1) contained in a control set are given (Theorem 4.5, Theorem 4.13).
- In [12], the concept of multistability region, which is one major difference between the limit behavior of (1.1) and that of continuous dynamical systems, have been studied. Moreover, a characterization of such region via relative invariant control sets is given (Theorem 2.12).
The above discussions motivate the following questions which are major topics of this paper:

(a) How to define control Lyapunov functions (CLFs) for the control systems (1.1).

(b) Find sufficient conditions for the existence of control Lyapunov functions and construct such functions.

(c) Find applications of such control Lyapunov functions.

For (a), we note that there are two different definitions of Lyapunov functions for control systems, namely:

(1) CLFs are nonincreasing along every control trajectory, for example, see [31],[32].

(2) CLFs are nonincreasing along some control trajectories. [29].

Roughly speaking, we define control Lyapunov functions in this paper based on (1) (with respect to chain control sets. One advantage of such definition is that there is a connection between CLFs of (1.1) and Lyapunov functions for continuous dynamical systems (with respect to chain recurrent set). Conley [15, 6.4] proves the existence of the latter in compact metric spaces. However, chain control sets are less easier to find than another control structure, called the control sets. Therefore, we will explore the relationships between control sets and chain control sets in the chapter 2. We show that generically every chain control sets is exactly the closure of a control set if $\mathcal{M}$ is compact.

In this paper, we provide two different methods to construct CLFs.

(i) (Analogous to) Conley's Construction: This method is based on the Conley's construction of Lyapunov functions (with respect to chain recurrent sets) of
continuous dynamical systems on compact metric spaces [15, 5.1, 6.4]. We modify this construction to control system (1.1) based on the control structure. The state space $\mathcal{M}$ is not necessarily compact.

(ii) Lyapunov Spectrum Method: The idea of this method is the following: we project bilinear control systems on $\mathbb{R}^n$ to the projective space $\mathbb{P}^{n-1}$, called the projected system, then we analyze the behavior of control trajectories of bilinear control systems via the Lyapunov spectrum over (closures of) control sets of projected systems. On the other hand, the control structures on compact space (in this case, $\mathbb{P}^{n-1}$ are less complicated than those on noncompact space. In fact, the control structures of projected systems reflect those of bilinear control systems from the Lyapunov spectrum point of view. Therefore, we can construct CLFs of bilinear control systems. One advantage of this construction is the following: Such a CLF is strict and defined globally.

As for the third goal (c), we discuss several global aspects of control systems (1.1), namely, (1) limit behavior of the control systems; (2) control flows; (3) multi-stability regions. (4) domains of attraction of (chain) control sets. We are interested in using CLFs to analyze the structure of control systems which are composed by the following questions:

(i) Semi-Global Question: the structure of the dynamics within and/or near a (chain) control set. For example, the stability of control sets.

(ii) Global Question: what is the structure of the set of connecting orbits between (chain) control sets? For example, multistability regions and domains of attraction of (chain) control sets.
Moreover, both semi-global and global questions can be discussed via control flows, which is an interplay between control systems and dynamical systems. Another topic is the stability of a (chain) control set, CLFs play a similar role as Lyapunov functions in dynamical systems in such a case.

For the global question, we analyze domains of attraction of (chain) control sets. In particular, Conley's CLFs give a clear picture about domains of attraction of minimal (chain) control sets with respect to a specific order between (chain) control sets. Another global aspect of control systems is the existence of multistability regions. A point is multistable if the system response from this point exhibits different behavior. A multistability region is the collection of multistable points. We characterize these regions via level sets of Conley's CLFs.

However, the global continuity of CLFs is hard to achieve, at least for Conley's construction and the Lyapunov spectrum method. This is understandable, as we try to construct a control independent function with respect to such a large class of locally integrable control functions \( \mathcal{U} \). On the other hand, if we simply consider the constant control with small control range, bifurcation phenomena for the flow behavior will destroy the uniformity of the control trajectories. However, the semicontinuity property of CLFs in these two constructions can be achieved.

The arrangement of this paper is the following. In chapter 2, we introduce basic concepts from dynamical systems theory and control systems which are essential for discussing affine control systems with bounded control ranges. From dynamical system theory, we discuss chain recurrent sets and \( \omega \)-limit sets. From control systems theory, we discuss control structures such as control sets and chain control sets, and relationship among chain control sets, control sets and Morse sets (in particular
\( \omega \)-limit sets and chain recurrent components). We also define Lyapunov functions of continuous dynamical systems with respect to chain recurrent components (connected components of chain recurrent sets). This idea originated from Conley’s paper [14]. Afterwards, we define control Lyapunov functions similar to that of Lyapunov functions. In chapter 3, we construct CLFs via an analog to Conley’s construction of Lyapunov functions. First we discuss the case that (1.1) admits the unique chain control set. Then such a discuss is generalized for (1.1) admits more than one chain control set. In chapter 4, we construct CLFs of bilinear control systems by analyzing the Lyapunov spectrum of projected systems. We analyze the relationship between projected systems and bilinear control system from the points of views of control structures and flow behavior. In chapter 5, several applications of CLFs to the global pictures of control systems, such as control flows, multistability regions and stability of control structure will be discussed. In the chapter of concluding remarks, we discuss several future investigations of CLFs.
CHAPTER 2. PRELIMINARIES

In this chapter, we introduce some basic ideas of dynamical systems and control systems (with bounded control ranges).

Dynamical Systems Theory

The main concept in the dynamical system theory include ω-limit sets and chain recurrence. Research on ω-limit sets leads to the Poincaré-Bendixson theorem, Pexióto theorem etc, see [24]. Chain recurrence, which is closely related to Bowen’s fruitful technique of shadowing, has become increasingly important in the analysis of dynamical systems, especially in the presence of hyperbolicity. Much of this work has been further developed in the recent systematic study of Akin [1].

In this section, we assume that $\mathcal{M}$ is a complete metric space with a metric $d$.

**Definition 2.1**

$\Psi$ is a continuous dynamical system (flow) on $\mathcal{M}$, i.e., a continuous mapping $\Psi : (-\infty, \infty) \times \mathcal{M} \to \mathcal{M}$ with the following properties:

(i) $\Psi_{t+s} = \Psi_{t+s}$, $t, s \in \mathbb{R}$;

(ii) $\Psi_0(x) = x$, $x \in \mathcal{M}$.

Here $\Psi_t$ denotes the mapping from $\mathcal{M} \to \mathcal{M}$ given by $\Psi_t(x) = \Psi(t, x)$. 
Definition 2.2

The \( \omega \)-limit set of a point \( p \in \mathcal{M} \), denoted by \( \omega(p) \), is the set of those points \( q \in \mathcal{M} \) for which there exists a sequence \( t_n \to \infty \) with \( \Psi_{t_n}(p) \to q \).

Similarly, we define the \( \alpha \)-limit set of a point \( p \in \mathcal{M} \), denoted by \( \alpha(p) \), is the set of those points \( q \in \mathcal{M} \) for which there exists a sequence \( t_n \to -\infty \) with \( \Psi_{t_n}(p) \to q \).

Lemma 2.3

Let \( \mathcal{M} \) be a compact. Then for any \( x \in \mathcal{M} \),

(i) \( \omega(x) \neq \emptyset \).

(ii) \( \omega(x) \) is connected.

(iii) \( \omega(x) \) is positively invariant.

Proof. See [24, Pages 15-16]. \( \blacksquare \)

Definition 2.4

A point \( x \) in the continuous dynamical system \( (\mathcal{M}, \Psi) \) is said to be recurrent if \( x \in \omega(x) \). \( (\mathcal{M}, \Psi) \) is called recurrent, if all \( x \in \mathcal{M} \) are recurrent.

Definition 2.5

For \( \epsilon > 0 \) and \( T > 0 \), an \( (\epsilon, T) \)-chain from \( x \in \mathcal{M} \) to \( y \in \mathcal{M} \) consists of a sequence \( x_0, \ldots, x_k \) in \( \mathcal{M} \) and a sequence \( t_0, \ldots, t_{k-1} \) in \( \mathbb{R} \) such that \( x_0 = x, x_k = y, t_j \geq T \) and

\[
d(\Psi(t_j, x_j), x_{j+1}) \leq \epsilon,
\]

for \( j = 0, \ldots, k-1 \). See Figure 2.1.

For \( A \subseteq \mathcal{M} \) define the chain-limit set by \( \Omega(A) = \{ y \in \mathcal{M}, \text{ for all } \epsilon > 0 \text{ and all } T > 0, \text{ there exists } a \in A \text{ such that there is an } (\epsilon, T) \text{-chain from } a \text{ to } y \} \)
The chain recurrent set is defined as

\[ \mathcal{CR} = \{ x \in \mathcal{M}, x \in \Omega(x) \}. \]

The dynamical system \((\mathcal{M}, \Psi)\) is called \textit{chain recurrent} if \(\mathcal{M} = \mathcal{CR}\).

**Theorem 2.6**

\textit{A compact \(\omega\)-limit set is chain recurrent.}

**Proof.** This theorem was originally proved by Conley [14] [15]. \(\blacksquare\)

**Definition 2.7**

A continuous real valued function \(V\) is a \textit{Lyapunov function} for a continuous dynamical system on \(\mathcal{M}\) if \(V\) satisfies the following conditions:

1. \(V\) is nonincreasing along trajectories.
(ii) \( V \) is strictly decreasing along trajectories outside the chain recurrent set.

In particular, \( V \) is called a \textit{strict} Lyapunov function if \( V \) is constant on each chain recurrent component (a connected component of \( CR \)).

In general, a Lyapunov function is not strict. The following example is taken from [14, Remark 4.1F].

Example 2.8

Let \( M = [0,1] \times [0,1] \subset \mathbb{R}^2 \). Consider the continuous dynamical system given by:

\[
\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= -xy(1-x)(1-y)
\end{align*}
\]  

(2.1)

Then equilibrium points are exactly boundary points \( \partial M \) of \( M \) and \( CR = M \) (see Figure 2.2). Define \( V(x,y) = x \). Then \( V \) is strictly decreasing along trajectories in \( int M = (0,1) \times (0,1) \). i.e., \( V \) is a Lyapunov function which is not strict.

Conley proves the following existence theorem of (strict) Lyapunov functions.

Theorem 2.9 Suppose that \((M,d)\) is a compact metric space. Then for any continuous dynamical system \((M,\Psi)\), there exists a strict Lyapunov function \( V \).

\(\) Proof. See [14, Page 39]. \(\)

Definition 2.10

A continuous dynamical system is said to be \textit{gradient-like} if

(i) The chain recurrent set consists of equilibrium points.
Figure 2.2: The flow behavior of (2.1). Equilibrium points are exactly boundary points $\partial M$.

(ii) There exists a Lyapunov function.

**Proposition 2.11**

Assume that $M$ is compact. Then a continuous dynamical system $(M, \Psi)$ is gradient-like if and only if the chain recurrent set consists of equilibrium points.

**Proof.** We note that $(M, \Psi)$ has a Lyapunov function by Theorem 2.9. The result follows. ■

**Control Systems Theory**

In this paper, we are interested in the following affine control system

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t)X_i(x)$$

(2.2)

on a smooth complete Riemannian manifold $M = n < \infty$. $d$ is the Riemannian metric. Here the $X_i$, $i = 0, 1, \ldots m$ is a smooth vector field. $(u_i)_{i=1}^{m} = u \in U^P$; and $U^P$ is the set of admissible control functions, which is defined as follows.
The first important question about the control system (2.2) is reachability. Denote by
\[ \mathcal{O}^{+}(x) = \{ y \in \mathcal{M} : \text{there is } u \in \mathcal{U} \text{ and } t \geq 0 \text{ with } \varphi(t, x, u) = y \} \]
the positive orbit (reachable set) from \( x \in \mathcal{M} \) (we use the notation \( x \sim y \)) and analogously by
\[ \mathcal{O}^{-}(x) = \{ y \in \mathcal{M} : \text{there is } u \in \mathcal{U} \text{ and } t \leq 0 \text{ with } \varphi(t, x, u) = y \} \]
the negative orbit. We are looking for conditions such that, given \( x, y \in \mathcal{M} \), we have \( y \in \mathcal{O}^{+}(x) \) (or \( x \in \mathcal{O}^{-}(y) \)). Similarly, we define the reachable set with finite time \( T > 0 \) as follows.
\[ \mathcal{O}^{+}_{\leq T}(x) = \{ y \in \mathcal{M} : \text{there is } u \in \mathcal{U} \text{ and } 0 \leq t \leq T \text{ with } \varphi(t, x, u) = y \}, \]
\[ \mathcal{O}^{+}_{T}(x) = \{ y \in \mathcal{M} : \text{there is } u \in \mathcal{U} \text{ with } \varphi(T, x, u) = y \}, \]
\[ \mathcal{O}^{-}_{\geq -T}(x) = \{ y \in \mathcal{M} : \text{there is } u \in \mathcal{U} \text{ and } -T \leq t \leq 0 \text{ with } \varphi(t, x, u) = y \}. \]

Now we discuss some control structure of control systems. This approach via control sets is different from the classical analysis from the controllability point of
view, see e.g. [28]. However, the complete controllability of nonlinear affine control systems can be checked from properties of topological dynamics for the control flow, see [6].

As for the typical open-loop controllability of nonlinear control systems, readers may refer to [28]. Colonius and Kliemann also discuss several open-loop controllability results from the point of view of control sets and their domains of attraction.

**Definition 2.12**

A subset $L$ of $\mathcal{M}$ is forward invariant if

for all $x \in L$ and for all $u \in U$, $\varphi(t,x,u) \in L$, for all $t \geq 0$.

**Definition 2.13**

A subset $D$ of $\mathcal{M}$ is a control set of (2.2) if

(i) $D \subseteq \text{cl}(O^+(x))$ for all $x \in D$.

(ii) For all $x \in D$ there is $u \in U$ with $\varphi(t,x,u) \in D$ for all $t \geq 0$.

(iii) $D$ is maximal (with respect to set inclusion) with these properties.

In particular, if for all $x \in D$, for all $u \in U$, $\varphi(t,x,u) \in D$ for all $t \geq 0$, then we call $D$ invariant.

**Remark 2.14**

By the definition, control sets are connected and pairwise disjoint. We will discuss more properties of control sets later.
Definition 2.15

Let $K \subset M$ be nonempty. The domain of attraction of $K$ in the control system (2.2), denoted by $A(K)$, is defined as follows.

$$A(K) := \{x \in M : O^+(x) \cap K \neq \emptyset\}.$$

Proposition 2.16

Assume that $D$ is a control set of the system (2.2) with nonvoid interior. Then $A(D)$ is open.

Proof. Let $x \in A(D)$ and $z \in O^+(x) \cap D$. Then there are $t > 0$ and $u \in U$ with $\varphi(t, z, u) \in \text{int}D$. By continuous dependence on the initial condition, there is a neighborhood of $z$ and hence of $x$, which can be steered into $\text{int}D$. $\blacksquare$

Definition 2.17

The control system (2.2) satisfies the (local) accessibility condition if

(H) $\dim \mathcal{C}\{X_0(x) + \sum u_i X_i(x); (u_i) = u \in U\}$ has rank $n$ for all $x \in M$.

Definition 2.18

The pair $(u, x) \in U \times M$ is an inner pair, if there are $T > 0$ and $S \geq 0$ with $\varphi(T, x, u) \in \text{int} O^{+}_{\leq T+S}(x)$.

Remark 2.19

It follows from the assumption (H) that $O^{+}_{\leq t}(x)$ has nonvoid interior for all $t > 0$.

We introduce another concept for control structure of (2.2), called the chain control set. First we define the controlled $(\epsilon, T)$-chain.
Definition 2.20

Let $x, y \in \mathcal{M}$ and $\epsilon, T > 0$. A controlled $(\epsilon, T)$-chain from $x$ to $y$ is given by $x_0, \ldots, x_n \in \mathcal{M}, u_0, \ldots, u_{n-1} \in \mathcal{U}$, and $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x, x_n = y$, and

$$d(\varphi(t_j, x_j, u_j), x_{j+1}) < \epsilon, \text{ for } j = 0, 1, \ldots, n - 1.$$ 

See Figure 2.3.

Definition 2.21

For the control system (2.2) a set $E \subset \mathcal{M}$ is called a chain control set, if

(i) for all $x \in E$, there is $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$;

(ii) for all $x, y \in E$ and all $\epsilon, T > 0$, there is a controlled $(\epsilon, T)$-chain from $x$ to $y$; and

(iii) $E$ is maximal with these properties.

In particular, if for all $x \in E$, for all $u \in \mathcal{U}$, $\varphi(t, x, u) \in E$, for all $t \in \mathbb{R}$, then we call $E$ invariant.

Proposition 2.22

Chain control sets are closed, connected, and pairwise disjoint.

Proof. See [4, Lemma 4.7]. □

Remark 2.23

Fix a $u \in \mathcal{U}$, (2.2) is reduced to a continuous dynamical system, called the $u$-system. One advantage of chain control sets is the following. Let $x \in \mathcal{M}$ and $u \in \mathcal{U}$. If $\omega(x, u)$, the $\omega$-limit set of $x$ in the $u$-system, is nonempty, then $\omega(x, u) \subset E$, where
$x_0^{x_1^{x_2^{x_3^{x_4^{x_5^{x_6}}}}}}$

Figure 2.3: Controlled $(\epsilon,T)$-chain. Every trajectory is steered by a control $u_i$. The construction is similar to $(\epsilon,T')$-chain.

$E$ is a chain control set. In general, $\omega(x,u)$ is not necessarily contained in a control set. See [4] for details.

Now we have the following questions: (1) What is the relationship between control sets and chain control sets corresponding to different control ranges? The following theorem proves, roughly speaking, a chain control set corresponding to a smaller control range can be obtained by shrinking those control sets corresponding to larger control ranges. (2) From the definition of chain control sets, we know that every control set $D$ of (2.2) is contained in a chain control set $E$ of (2.2). Can we perturb the control range $\rho$ a little bit larger such that $E \subset D$? (3) How about the relationship between control systems and uncontrolled dynamical systems from the control structure point of views? The next theorem answers those questions. We use the notation $(2.2)^\rho$ to represent the control system (2.2) with a fixed control range.
\[
\rho > 0.
\]

**Definition 2.24**

Fix a \( \rho \geq 0 \). \((u,p) \in \mathcal{U} \times \mathcal{M}\) is a \( \rho - \rho' \) inner pair, if

For all \( \rho \leq \rho' \), there exists \( T > 0 \) and \( S > 0 \) such that \( \varphi(T, p, u) \in \text{int} \mathcal{O}^\rho_{\leq T+S}(p) \),

where \( \mathcal{O}^\rho_{\leq T+S}(p) \) is the reachable set of \( p \) up to time \( T + S \) for the control system \((2.2)^\rho\).

In particular, the control system satisfies the \( \rho - \rho' \) inner pair condition, if

\((I_\rho)\) All \((u, p) \in \mathcal{U} \times \mathcal{M}\) are inner pairs.

**Theorem 2.25**

Let \( \mathcal{M} \) be compact. Fix a \( \rho \geq 0 \) and consider the control system \((2.2)^\rho\). Assume that \((2.2)^\rho\) has only finitely many chain control sets \( E_i^\rho \), \( i = 1, \ldots, l \). We have the following results.

(i) For each \( i \in \{1, \ldots, l\} \) there is an increasing sequence (w.r.t. the control range \( \rho' > \rho \)) of chain control sets \( E_i^\rho \) such that \( E_i^\rho = \bigcap_{\rho' > \rho} E_i^\rho \).

(ii) Assume that for all \( \rho' > \rho \) every \((u, x) \in \bigcup_{i=1}^l E_i^\rho \) is a \( \rho - \rho' \) inner pair,

\[
E_i^\rho := \{(u, x) \in \mathcal{U} \times \mathcal{M} : \varphi(t, x, u) \in E_i^\rho, \forall t \in \mathbb{R}\}.
\]

Then for each \( i \in \{1, \ldots, l\} \) there is an increasing sequence (w.r.t. the control range \( \rho' > \rho \)) of control sets \( D_i^\rho \), such that \( E_i^\rho \subset \text{int} D_i^\rho \) and \( E_i^\rho = \bigcap_{\rho' > \rho} D_i^\rho \).

(iii) Assume that the chain recurrent set of the uncontrolled dynamical system \((2.3)\) has finitely many connected components \( M_i, i = 1, \ldots, l \), and that for all \( \rho > 0 \),
all $x \in \bigcup_{i=1}^{l} M_i$, the point $(0, x) \in U \times M$ is a $0 - \rho$ inner pair. Then for each $i \in \{1, \ldots, l\}$, there is a decreasing sequence (w.r.t. the control range $\rho > 0$) of control sets $D_i^\rho$ of $(2.2)^\rho$ such that $M_i \subset \text{int} D_i^\rho$ and $M_i = \bigcap_{\rho > 0} D_i^\rho$.

(iv) Let the assumption of (iii) be satisfied. Then the system $(2.2)^\rho$ is controllable for all $\rho \leq \rho_0$, for some $\rho_0 > 0$ if and only if the uncontrolled dynamical system (2.3) is chain recurrent.

**Proof.**

(i) See [4, Theorem 5.1].

(ii) See [4, Corollary 5.2].

(iii) See [4, Corollary 5.3].

(iv) See [4, Corollary 5.4].

Theorem 3.9 and Theorem 4.8 in [6] suggest to find conditions under which control sets (or their closures) coincide with chain control sets. We note that for each control set $D \subset M$, there exists a unique chain control set $E \subset M$ such that $\text{cl} D \subset E$. But if $D_1, D_2$ are control sets with $\text{cl} D_1 \cap \text{cl} D_2 \neq \emptyset$, then there is one chain control set $E$ with $\text{cl} D_1 \cup \text{cl} D_2 \subset E$, i.e., in general closures of control sets need not be chain control sets. However, we will show that under the $\rho - \rho'$ inner pair conditions, such a situation can be avoided via small perturbation of the control range. We discuss this in the following. Now we assume that $M$ is compact at this
moment. Let \( C(M) \) denote the set of closed subsets of \( M \) with the Hausdorff metric \( d_H \). We know that \((C(M), d_H)\) is a compact metric space. Assume the uncontrolled dynamical system (2.3) has finitely many chain recurrent components \( M_i, i = 1, \ldots, k \). Consider for \( i = 1, \ldots, k \) the following maps:

\[
D_i : [0, \infty] \to C(M), \rho \to \overline{clD_i}^\rho
\]

(2.4)

with \( D_i^0 = M_i \) and \( M_i \subseteq intD_i^\rho \);

and

\[
E_i : [0, \infty] \to C(M), \rho \to E_i^\rho
\]

(2.5)

with \( E_i^0 = M_i \) and \( M_i \subseteq E_i^\rho \).

The following lemma shows that under the \( \rho - \rho' \) inner pair condition, the maps (2.4) and (2.5) are well defined.

Lemma 2.26

Let \( M \) be compact. Consider the control system (2.2) and assume \((I_\rho)\), for all \( \rho \geq 0 \). Then

(i) For all \( i = 1, \ldots, k \) and all \( 0 \leq \rho \leq \infty \) there are unique control sets \( D_i(\rho) \) and chain control sets \( E_i(\rho) \) satisfying the conditions in (2.4) and (2.5), respectively.

(ii) For all \( i = 1, \ldots, k \) and all \( 0 \leq \rho < \rho' \leq \infty \), we have

\[
D_i(\rho) \subseteq E_i(\rho) \subseteq intD_i(\rho').
\]
Proof.

(i) The assertion for the map $E_i(\cdot)$ is obvious. By $(I \rho)$ with $\rho = 0$ we find, by Theorem 2.25 (ii), there exist control sets $D^\rho_i, \rho > 0$ satisfying the condition in (2.4).

(ii) The first inclusion is trivial. Applying (i) and Theorem 2.25 (ii) implies the second inclusion.

\[\square\]

Definition 2.27

Let $X$ be a metric space and $Y$ a topological space. $F : X \to 2^Y$ is a set-valued map, where $2^Y$ is the collection of all subsets of $Y$.

(i) $F$ is called an upper semicontinuous at $x \in X$, if for any neighborhood $\mathcal{V}$ of $F(x)$, there exists $\delta > 0$ such that

$$x' \in B_X(x, \delta) \Rightarrow F(x') \subset \mathcal{V},$$

where $B_X(x, \delta)$ is the $\delta$-ball of $x$ in $X$.

(ii) $F$ is called a lower semicontinuous at $x \in X$, if for any sequence of elements $x_n \in X$ converging to $x$, there exists a sequence of elements $y_n \in F(x_n)$ converging to $y$.

(iii) $F$ is upper (resp. lower) semicontinuous if it is upper (resp. lower) semicontinuous at any point of $X$.

The following theorem characterizes relations between control sets and chain control sets.
Theorem 2.28

Let $\mathcal{M}$ be compact. Assume the control system $(2.2)^{\rho}$ satisfies $(H)$ for all $\rho > 0$ and the $\rho - \rho'$ inner pair condition $(I_\rho)$, for all $\rho \geq 0$. Then for $i = 1, \ldots, k$, the map $D_i(\rho)$ and $E_i(\rho)$, defined in (2.4) and (2.5), respectively, have the following properties:

(i) $D_i(\rho)$ and $E_i(\rho)$ are strictly increasing in $\rho$.

(ii) $D_i(\rho)$ is lower semicontinuous and hence left continuous; $E_i(\rho)$ is upper semicontinuous and hence right continuous.

(iii) If the mapping $D_i(\rho)$ is continuous at $\rho$ then $\text{cl}D_i(\rho) = E_i(\rho)$.

Proof.

(i) This follows from (2.6), and the fact that chain control sets are closed.

(ii) See [11, Theorem 5.2 (ii)]. The proof is for the case that $\mathcal{M}$ are projective spaces and systems are projected systems of bilinear control systems (for definitions, see Chapter 4 in this paper). However, this proof goes through for our general case.

(iii) As $E_i(\rho)$ is closed, $\text{cl}D_i(\rho) \subset E_i(\rho)$ by (2.6). On the other hand, by (2.6) and the fact that $\text{cl}D_i(\rho) = \text{cl}(\text{int}D_i(\rho))$, we have

$$\text{cl}D_i(\rho) \subset E_i(\rho) \subset \text{cl}D_i(\rho'), \text{ for all } \rho' \geq \rho.$$ 

By continuity of $D_i$ at $\rho$, the result follows.
Theorem 2.28 (iii) tells us under the continuity every chain control set of (2.2) is exactly the closure of one control set. We show the generic property of such a result in Theorem 2.31.

Definition 2.29

The control system (2.2) is perfect, if every chain control set is the closure of one control set.

Lemma 2.30

Let $X$ be a compact metric space. Fix a $\delta > 0$. Then $X$ contains at most finitely many disjoint $\delta$-balls.

Proof. Let $\{N_\beta : \beta \in J\}$ be a infinite collection of disjoint $\delta$-balls of $X$. Then

$$X = B_X(X \setminus P, \frac{\delta}{4}) \cup P,$$

where $P = \bigcup_{\beta \in J} N_\beta$. By the compactness of $X$,

$$X = \bigcup_{i=1}^k N_i \bigcup B_X(X \setminus P, \frac{\delta}{4}),$$

for some finite index subset $\{1, 2, \ldots k\}$ of $J$. We observe that for all $x \in X$, either $x \in B_X(X \setminus P, \frac{\delta}{4})$ or $x$ is unique $N_\beta$, for our choice of $\frac{\delta}{4}$. We conclude that

$$\bigcup_{i=1}^k N_i = \bigcup_{\beta \in J} N_\beta,$$

a contradiction to the infinity of $J$. 

Theorem 2.31

Let $\mathcal{M}$ be compact. Consider the control system $(2.2)^p$ and assume $(I_p)$, for all $p \geq 0$. Then $(2.2)^p$ is perfect, except for possibly countably many $\rho > 0$.

Proof. Fix an $i \in \{1, 2, \ldots, k\}$ and let $D_i(\cdot)$ be the map defined by (2.4). For notation convenience, we define $D_{p} := D_i(\rho)$. Assume that $D$ is discontinuous at $\rho_\alpha$, where $\alpha \in J$, $J$ is an uncountable index set. By Theorem 2.25(ii),

$$cl(D_{\rho_\alpha}) \subset int(D_{\rho'})$$

for all $\rho' > \rho_\alpha$.

Since $D$ is not upper semicontinuous at $\rho_\alpha$, there exists an open set $N_{\rho_\alpha}$ such that

$$N_{\rho_\alpha} \subset int(D_{\rho'}) \setminus cl(D_{\rho_\alpha}),$$

for all $\rho' > \rho_\alpha$. This $N_{\rho_\alpha}$ is independent on $\rho' > \rho_\alpha$ due to the monotonicity of $D_{\rho}$ (w.r.t. $\rho$). Without loss of generality, we may assume that $N_{\rho_\alpha}$ is an open ball with radius $q_\rho$. We note that these $N_{\rho_\alpha}$ are disjoint due to the monotonicity of $D_{\rho}$.

We observe that $\{(\frac{1}{k+1}, \frac{1}{k}], (l, l + 1) : k, l = 1, 2, \ldots\}$ is a countable (disjoint) partition of $(0, \infty) \subset \mathbb{R}$. Without loss of generality, we may assume that $\left(\frac{1}{k_0+1}, \frac{1}{k_0}\right]$ contains uncountably many $q_\alpha$, say $\{q_\beta : \beta \in J'\}$. Then the corresponding (disjoint) open balls of $\mathcal{M}$, $N_\beta$, have radii $q_\beta > \frac{1}{k_0+1}$, for all $\beta \in J'$. But according to Lemma 2.30, it is impossible. Therefore, $D_i(\cdot) = D$. has at most countably many discontinuities. Now the result follows from Theorem 2.28 (iii). ■

For the rest of this chapter, $\mathcal{M}$ need not be compact unless further specified.

We may define a (partial) order on the collection of control sets in the following way: Let $D_1, D_2 \subset \mathcal{M}$ be control sets of (2.2). We define
$D_1 \prec D_2$, if there is $x_1 \in D_1$ with $\mathcal{O}^+(x_1) \cap D_2 \neq \emptyset$.

The following lemma (cf. [10, Lemma 3.11]) investigates some topological properties of control sets via the order defined above.

**Lemma 2.32**

*Consider the control system (2.2) and assume (H).*

(i) *Open control sets are minimal, and closed control sets are maximal w.r.t. $\prec$.*

(ii) *Invariant control sets $C \subseteq \mathcal{M}$ are always closed, and hence maximal w.r.t. $\prec$.*

(iii) *If $K \subseteq \mathcal{M}$ is a compact invariant set of (2.2), then there exist (at least) one open and one closed control set in $K$ and they are exactly the minimal and the maximal elements w.r.t. $\prec$."

The question we are interested in here is, how the control sets of (2.2) are related to the $\omega$-limit sets of (2.3). The following theorem gives an answer, and provides an insight into the control structure of (2.2). Now we assume that the uncontrolled dynamical system (2.3) has a **finest Morse decomposition** $\mathcal{M} = \{M_1, \ldots, M_k\}$, i.e. the $M_i$, $i = 1, \ldots, k$ are the connected components of the chain recurrent set of (2.3) [15, 7.2, Chapter 1]. The next theorem shows that the order of control sets of (2.2) is reflected in that of Morse sets of (2.3) defined as follows:

$M_i \prec M_j$ if there are Morse sets $M_i = M_{i,1}, \ldots, M_{i,p} = M_j$ and points $x_1, \ldots, x_p \in \mathcal{M}$ with $\alpha(x_l) \subseteq M_{i,l}$ and $\omega(x_l) \subseteq M_{i,l+1}$ for $l = 1, \ldots, p - 1$.

Here $\alpha(x_l)$ and $\omega(x_l)$ are the $\alpha$- and $\omega$-limit sets of $x_l$ for the uncontrolled dynamical system (2.3).
Theorem 2.33

Let \( M \) be compact. Assume that for all \( \rho > 0 \), all \( x \in \bigcup_{i=1}^{k} M_i \), we have that \((0, x)\) is an inner pair of \((2.2)^\rho\). Let \( M_i = \bigcap_{\rho>0} D_i^\rho \) as in Theorem 2.25(iii), for \( i = 1, \ldots, k \). Then we have the following result.

(i) If \( M_j < M_k \), then \( D_j^\rho < D_k^\rho \) for all \( \rho > 0 \).

(ii) If \( D_j^\rho, D_k^\rho \) are the control sets with \( M_j = \bigcap_{\rho>0} D_j^\rho, M_k = \bigcap_{\rho>0} D_k^\rho \), and if there is an \( \rho_0 > 0 \) such that for all \( 0 < \rho < \rho_0 : D_j^\rho < D_k^\rho \) then \( M_j < M_k \).

(iii) The invariant control sets of \((2.2)^\rho\) correspond for \( \rho > 0 \) small enough to the maximal sets of \((2.3)\), i.e. to the Morse sets which are attractors.

The proof of Theorem 2.33 can be found in [9, Theorem 9].

Before ending this chapter, we define control Lyapunov functions for the control system \((2.2)\) based on chain control sets. Such a definition is similar to Tsinias' one based on the prolongation limit sets, see [31] and [32]. On the other hand, this definition can be regarded as an analog of Lyapunov functions of dynamical systems, from a control flow point of view. We will discuss this in chapter 5.

Definition 2.34

A real-valued function \( F \) is a control Lyapunov function (CLF) of the control system \((2.2)\), if

(i) \( F \) is nonincreasing along trajectories.

(ii) \( F \) is strictly decreasing along trajectories outside chain control sets.
In particular, $\mathcal{F}$ is a strict control Lyapunov function (SCLF) if $\mathcal{F}$ is constant on each chain control set.

**Remark 2.35**

(i) We do not require control Lyapunov functions to be continuous in this definition, the reason is that such a requirement is too strong.

(ii) The reason this definition is based on chain control sets rather than control sets is the following. We will see in chapter 5 that projections of chain recurrent components of control flows are exactly chain control sets. Therefore, there is a connection between Lyapunov functions for continuous dynamical systems and CLFs of control systems.

**Lemma 2.36**

A control Lyapunov function $\mathcal{F}$ of (2.2) is constant on each control set $D$. Moreover, $\mathcal{F}$ is constant on cl$D$ (the closure of $D$) if $\mathcal{F}$ is continuous on a neighborhood of $D$.

**Proof.** Given $x, y \in D$, where $D$ is a control set of the control system (2.2). Assume that $\varphi(0, x, u_1) = x$ and $\varphi(s, x, u_1) = y$ for some $u_1 \in U$, $s > 0$. Then

$$\mathcal{F}(\varphi(0, x, u_1)) \leq \mathcal{F}(\varphi(s, x, u_1))$$

The reverse inequality is similar. Therefore, $\mathcal{F}(x) = \mathcal{F}(y)$. i.e., $\mathcal{F}$ is constant in $D$.

The second results follows directly. ■
CHAPTER 3. CONTROL LYAPUNOV FUNCTIONS OF CONTROL SYSTEMS

After defining control Lyapunov functions at the end of the last chapter, we start to discuss some sufficient conditions for the existence of CLFs. As mentioned earlier, the existence of CLFs looks impossible, at least highly nontrivial. However, inspired by Conley's construction of strict Lyapunov functions of continuous dynamical systems, we can actually construct CLFs for certain control systems.

First, we deal with some direct geometrical constructions of CLFs. It is feasible, as we see in the first section, when control trajectories show some uniformity. We will see that every one-dimensional control system and some two-dimensional control systems will fall into this category.

In the second section related to Conley's construction of CLFs, we will discuss two cases individually, namely, control systems with a unique control set and with more than one control set. Linear control systems on $\mathbb{R}^n$ do admit unique control sets; while the control system (2.2) with $\mathcal{M}$ compact has more than one control set. As an example, we will discuss a chemical reactor model.
Direct Geometrical Constructions

General control systems with bounded control range defined on $\mathbb{R}^1$ is given by

$$\dot{x} = f(x, u), \quad (3.1)$$

where $u \in \mathcal{U} = \{ u : \mathbb{R}^1 \to \rho \cdot U, \text{measurable}\}$, and $U \subset \mathbb{R}^1$ compact, connected with $0 \in \text{int } U$. $\rho > 0$ is the control range.

Assumption 3.1

We assume that $f$ is a continuous function in both components, such that for every $u \in \mathcal{U}$ there are at most finitely many zeros.

As usual, the uncontrolled dynamical system is given by

$$\dot{x} = f(x, 0). \quad (3.2)$$

Definition 3.2

If $x_0 \in \mathbb{R}^1$ is a common equilibrium point of the vector field $f(\cdot, u)$, for all $u \in \mathcal{U}$ then we call $x_0$ a singular point of the control system (3.1). This system is called a singular system. Otherwise, the system is called a regular system.

Example 3.3

Consider the following control system on $\mathbb{R}^1$:

$$\dot{x} = x + u x, \quad u \in [-1, 1]. \quad (3.3)$$

Then $x = 0$ is a unique equilibrium point for the uncontrolled dynamical system $\dot{x} = x$ and also a singular point for this control system.
Remark 3.4

At the singular point $x_0$, $\{x_0\}$ is a one-point invariant control set (singular control set).

The next lemma characterizes chain recurrent sets of one-dimensional continuous dynamical systems.

Lemma 3.5

For every continuous dynamical system on $\mathbb{R}^1$, the chain recurrence set $CR$ is the set of equilibrium points.

Proof. Suppose that there exists $p \in CR$ such that $p$ is not an equilibrium point. We note that there exists a sufficiently small $\epsilon > 0$ such that all $s_1, s_2 \in (p - \epsilon, p + \epsilon)$ with $s_1 < 0 < s_2$ satisfy the following condition:

Either $\varphi(s_1, p) < p < \varphi(s_2, p)$ or $\varphi(s_1, p) < p < \varphi(s_2, p)$.

Without loss of generality, we may consider the first case. It implies that

$$\varphi(t, p) \to z \in \mathbb{R}^1,$$

if there is an equilibrium point $z > p$ and no other equilibrium in $(p, z)$. Otherwise, $\varphi(t, p) \to \infty$ as $t \to \infty$, i.e., $p \not\in CR$, a contradiction. $\blacksquare$

Lemma 3.6 Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be a continuous function with finitely many zeros. Consider the continuous dynamical system $\dot{x} = f(x)$ on $\mathbb{R}^1$. Then it is gradient-like.

Proof. We define $V : \mathbb{R}^1 \to \mathbb{R}$ as follows. First we define $V$ at these equilibrium points $\{x_1, \ldots, x_k\}$ such that

$$\{x_i\} < \{x_j\} \Rightarrow V(x_i) > V(x_j),$$
for $1 \leq i, j \leq k$. Then we define

$$V(x) = L_{[x_i, x_{i+1}]}(x) \text{ for } x \in [x_i, x_{i+1}],$$

where $L_{[x_i, x_{i+1}]}$ is the affine function between $x_i$ and $x_{i+1}$, $i = 1, \ldots, k - 1$. Then $V$ is a strict Lyapunov function. By Lemma 3.5, this system is gradient-like.

For one dimensional control system (3.1), Colonius and Kliemann characterize chain control sets and control sets via zeros of $f$, see [10]. The next lemma describes the flow behavior between chain control sets.

**Lemma 3.7**

Consider the one-dimensional control system (3.1). Then we have the following results.

(i) There are finitely many chain control sets. These chain control sets are either a compact interval or a one-point set.

(ii) For any $x$ outside closure of control sets, $f(x, u) > 0$ for all $u \in U$ if and only if $f(x, 0) > 0$.

(iii) (3.1) is perfect if every chain control set contains a unique control set.

**Proof.**

(i) See [10, Theorem 3.16(i)].

(ii) Let $[a_i, b_i]$ be the closure of a control set of (3.1), $i = 1, 2, \ldots$. Suppose that there exists a $x \in (b_i, a_{i+1})$ such that $f(x, 0) > 0$ and $f(x, \bar{u}) \leq 0$, for some control $\bar{u} \in U$. Then for some $\epsilon > 0$ we have $x \sim y$ by $u = 0$, and $y \sim x$ by $\bar{u}$,
for all $y \in (x, x + \epsilon)$. By (i), we may adjust $\epsilon > 0$ sufficiently small such that $(x, x + \epsilon)$ contains no control sets. This implies that $x$ must belong to a control set, a contradiction.

(iii) We note that every control set $D$ is contained in a chain control set $E$, i.e. $clD \subseteq E$ as $E$ is closed. By (ii), $clD = E$. Otherwise, for $x \in E \setminus clD$, $f(x, u) > 0$ for all $u \in U$ (or $f(x, u) < 0$, for all $u \in U$). This implies that $x$ can not be chain controllable to itself. On the other hand, every chain control set $E'$ contains a unique control set $D'$. Similarly, by (ii), $E' = clD'$. Therefore, the system is perfect.

Theorem 3.8

Consider the one-dimensional control system (3.1). Then (3.1) admits a continuous SCLF.

Proof. Without loss of generality, we may assume that every chain control set is a compact interval. The following construction of SCLF is also applicable if (3.1) admits some one point chain control set. Let $E_i = [a_i, b_i]$ be a chain control set, $i = 1, 2, \ldots, k$. First we define a real-valued function $\mathcal{F}$ on these chain control sets such that $\mathcal{F}$ is constant on each $E_i$ and

$$E_i < E_j \Rightarrow \mathcal{F}(E_i) > \mathcal{F}(E_j), \text{ for } 1 \leq i, j \leq k.$$

Then we define

$$\mathcal{F}(x) = L[b_i, a_{i+1}](x), \ i = 1, \ldots, k - 1, \text{ for } x \in (b_i, a_{i+1}) \text{ outside chain control sets.}$$

$L$ is the affine function.
Figure 3.1: Continuous SCLF of the one-dimensional control system (3.1). There are three chain control sets $E_i = [a_i, b_i]$, $i = 1, 2, 3$. $E_i \prec E_2$, $i = 1, 3$.

For $x \in (-\infty, a_1)$, $F$ can be any strictly increasing (resp. decreasing) linear function as the control trajectory is increasing (resp. decreasing) in $(a_1, \infty)$. Similarly, $F$ can be constructed in $(b_k, \infty)$, see Figure 3.1. $F$ is a continuous SCLF of (3.1).

In general, a strict Lyapunov function of the uncontrolled dynamical system (3.2) is not necessarily a SCLF of (3.1). However, we have the following result.

**Corollary 3.9**

Consider the one-dimensional control system (3.1) Then a strict Lyapunov function $V$ of (3.2) is also a SCLF of (3.1) if and only if every chain control set is an equilibrium point of (3.2).

**Proof.** By Lemma 3.7(ii), $V$ is also a continuous SCLF outside chain control sets. Since $V$ is constant on each chain recurrent component which is an equilibrium point by Lemma 3.5, the result follows.
Example 3.10

Consider the following regular control system defined on $\mathbb{R}^1$,

$$\dot{x} = (x^2 - 1) - u, \quad u \in U = [-0.5, 0.5]. \quad (3.4)$$

The uncontrolled dynamical system $\dot{x} = x^2 - 1$ has two equilibrium points $x_1 = 1$ (source) and $x = -1$ (sink). The chain control sets of (3.4) are $E_1 = [\frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{2}]$ and $E_2 = [-\frac{\sqrt{3}}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]$ and $E_1 < E_2$, see Figure 3.2. A piecewise linear SCLF of (3.4) is shown in Figure 3.3.

To finish this subsection, we consider the linear single-input control system of the following form:

$$\dot{x} = ax + bu \quad (3.5)$$
on $\mathbb{R}^1$; $a, b \neq 0$; $u \in U = \{u : \mathbb{R} \to U$, measurable $\}$, and $U = [-\rho, \rho] \subset \mathbb{R}$, $\rho > 0$.

Without loss of generality, we may assume that $b > 0$.

We note that for every constant function $u \in U$ (or $u \in U$), there is a unique equilibrium point $x = -\frac{b}{a}u$. As usual, the uncontrolled dynamical system is given by $\dot{x} = ax$. We discuss several basic properties of (3.5) as follows. For $u_0 \in U$, the $u_0$-system means the (continuous) dynamical system induced by the control system (2.2) with $u = u_0$.

Theorem 3.11

Consider the control system (3.5). Then we have the following results.

(i) The chain recurrent set $CR$ of the uncontrolled dynamical system is $\{0\}$.

(ii) The system (3.5) has the unique control set $[\frac{b}{a} \rho, -\frac{b}{a} \rho]$ if $a < 0$; $[-\frac{b}{a} \rho, \frac{b}{a} \rho]$ if $a > 0$. $clD$ is also the unique chain control set of (3.5).
Figure 3.2: Two chain control sets of (3.4), $E_1$ and $E_2$. $E_1 \prec E_2$. The boundaries of $E_1$ and $E_2$ are determined by $u = 0.5$ and $u = -0.5$. 
Figure 3.3: SCLF of (3.4). The flow behavior between control sets shows a uniformity for all $u \in U$.

(iii) If $a < 0$ then $\varphi(t, x, u) \to [\frac{b}{a}\rho, -\frac{b}{a}\rho]$ as $t \to \infty$, for all $x \notin [\frac{b}{a}\rho, -\frac{b}{a}\rho]$ and for all $u \in U$.

(iv) Let $a > 0$. If $x \in (-\infty, -\frac{b}{a}\rho)$ then $\varphi(t, x, u) \to -\infty$ as $t \to \infty$. If $x \in (\frac{b}{a}\rho, \infty)$ then $\varphi(t, x, u) \to \infty$ as $t \to \infty$.

(v) The system (3.5) admits a continuous SCLF.

Proof.

(i) This result follows from Lemma 3.5.

(ii) We note that for every constant control $u \in U$ there is a unique equilibrium point $x = -\frac{b}{a}u$ for this $u$-system. For $u = \rho$ and $-\rho$, we get the boundary of the control set $D$, see Figure 3.4. As there is no zero for $ax + bu$ if $x \notin D$ and the boundary of control sets are determined by zeros of $ax + bu$, we conclude that $D$ is the unique control set of (3.5). Since $D$ is the unique control set, $clD = D$ is the unique chain control set of (3.5) by [10, Theorem 3.16(i)].
Figure 3.4: Control structure of the system (3.5). $D$ is a unique control set, which is also a chain control set. There is no other control sets and chain control sets outside $D$. The boundary of $D$ is decided by the constant controls $u = -\rho$ and $u = \rho$.

(iii),(iv) These are direct results from Lemma 3.7 (ii).

(v) By Theorem 3.8.

Conley's Construction

The direct geometrical construction of CLFs has at least one disadvantage: we need to find some uniformity of control trajectories. In general, it is impractical. We have seen this construction is valid for control systems defined on $\mathbb{R}^1$ as the behavior of control trajectories outside chain control sets are very simple. We have also seen that there is a uniformity of control trajectories for planar decoupling control systems.

In this section, we will follow Conley's construction of Lyapunov functions with respect to chain recurrent sets to construct CLFs of affine control systems defined on
However, before we start, we remember some lessons learned from the dynamical systems theory. We know that in $\mathbb{R}^n$ Lyapunov functions do not exist globally for linear (time invariant) systems with a trajectory escaping to infinity. Otherwise, by Lyapunov direct method of stability, the origin will be asymptotically stable with domain of attraction equal to $\mathbb{R}^n$. This gives us some insight about control systems whose trajectory behavior is more complicated than that of dynamical systems. We should expect the existence of a semi-global CLF than a global one. A semi-global CLF is the one defined on a connected subset of $\mathcal{M}$.

In this section, we use the following notations for different reachable sets of $K \subset \mathcal{M}$. Fix $a \in \mathcal{U}$ and $t \in [0, \infty)$.

$\mathcal{O}_{u,t}(K) := \{ y \in \mathcal{M} : y = \varphi(t', x, u), x \in K, \text{ for some } t' \geq t \}.$

$\mathcal{O}_{u,[0,t]}(K) := \{ y \in \mathcal{M} : y = \varphi(t', x, u), x \in K, \text{ for some } 0 \leq t' \leq t \}.$

$\mathcal{O}_{u}^{+}(K) := \{ y \in \mathcal{M} : y = \varphi(t', x, u), x \in K, \text{ for some } t' \geq 0 \}.$

Throughout this section of Conley's construction, we always assume (H) for the control system (2.2).

Control Systems with a Unique Control Set

We first discuss the case that (2.2) has a unique chain control set.

Assumption 3.12

(A1) The control system (2.2) has a unique chain control set $E$, $E \not= \mathcal{M}$. We denote such a system by (2.2)$_{A1}$.
Later on, we will show that linear control systems satisfy the Assumption 3.12. First we define global attractors and reference points of \((2.2)_{A1}\) as follows.

**Definition 3.13**

An invariant chain control set \(E\) is a *global attractor* if \(cl\mathcal{O}^+_u(x) \cap E \neq \emptyset\), for every \(x \in \mathcal{M}\) and every \(u \in \mathcal{U}\).

**Definition 3.14**

\(p \in \mathcal{M} \setminus E\) is a *reference point* if \(\mathcal{O}^+(p) \cap E \neq \emptyset\).

From now on, we fix a reference point \(p\) for the control system \((2.2)_{A1}\).

**Lemma 3.15**

Let \(E\) be a unique chain control set of \((2.2)_{A1}\) and \(p\) be a reference point. Define \(l_p : \mathcal{M} \to [0,1] \subseteq \mathbb{R}\) by

\[
l_p(r) := d(r,E)/[d(r,E) + d(r,p)],
\]

where \(d\) is the (Riemannian) metric of \(\mathcal{M}\). Then \(l_p^{-1}(0) = E\) and \(l_p^{-1}(1) = \{p\}\). \(l_p\) is continuous on \(\mathcal{M}\); \(l_p(\mathcal{M}) = [0,1]\).

**Proof.** These are direct results from the definition of \(l_p\). ■

**Lemma 3.16**

Consider the system \((2.2)_{A1}\). Let \(E\) be the global attractor. Fix \(u \in \mathcal{U}\). Then we have the following results.

(i) For any (open) neighborhood of \(E\), \(W\), there exists some \(i \geq 0\) such that

\[
\mathcal{O}^+_u,\geq_i(clW) \subseteq W.
\]
(ii) Let $W$ be an open neighborhood of $E$. Then for any compact subset $K \subset M$, $K \cap E = \emptyset$, there is some $i \geq 0$ such that
\[ \mathcal{O}_{u, \geq i}^+(K) \subset W. \]

Proof.

(i) Consider the $u$-system. Then the $\omega$-limit set of $W$ is contained in $E$. For the rest of the proof, see [15, 5.1 A].

(ii) For any $x \in K$ and let $t(x) \geq 0$ be the first hitting time of the trajectory $\varphi(\cdot, x, u)$ to $W$, i.e., $\varphi(t(x), x, u) \in W$ and $\varphi(t, x, u) \notin W$, for all $t \leq t(x)$. Then $t(x)$ is continuous on $K$ by classical dependence theorem on initial conditions. By compactness of $K$, $t(x)$ attains maximum at some $x = x_0 \in K$. This $t(x_0)$ is a choice for $i$.

Lemma 3.17

Continuing from Lemma 3.15. Fix $u \in U$. Define $k_u : M \to [0, 1] \subset \mathbb{R}$ by
\[ k_u(r) = \sup \{ l_p(t, r, u) : t \geq 0 \}. \]

Then

(i) $k_u^{-1}(0) \subset E$. In particular, $k_u^{-1}(0) = E$ if $E$ is invariant.

\[ \mathcal{O}_u^-(p) = k_u^{-1}(1). \]

$k_u(M) \subset [0, 1]$. 
(ii) $k_u$ is nonincreasing along control trajectories $\varphi(\cdot, x, u)$, for all $x \in \mathcal{M}$.

(iii) $k_u$ is continuous on $\mathcal{O}_u^{-}(p)$. In particular, $k_u$ is continuous on $\mathcal{M}$ if $E$ is a global attractor.

Proof.

(i) Let $x \in k_u^{-1}(0)$. Then $k_u(x) = 0$. This implies that $l_p(\varphi(t, x, u)) = 0$ for all $t \geq 0$. By Lemma 3.15, $\varphi(t, x, u) \in E$, for all $t \geq 0$. In particular, $x \in E$. If $E$ is invariant, then $l_p(\varphi(t, x, u)) = 0$, for any $x \in E$, and for all $t \geq 0$. This implies that $k_u(E) = 0$, i.e., $E = k_u^{-1}(0)$.

Let $r \in \mathcal{O}_u^{-}(p)$. Then $p = \varphi(T, r, u)$ for some $T \geq 0$. We note that

$$k_u(r) \geq l_p(\varphi(T, r, u)) = l_p(p) = 1,$$

i.e., $k_u(r) = 1$. We conclude that $\mathcal{O}_u^{-}(p) \subset k_u^{-1}(1)$. On the other hand, if $r \in k_u^{-1}(1)$ then there exists a sequence $t_n \geq 0$ such that $l_p(\varphi(t_n, r, u)) \to 1$, as $n \to \infty$. Then $\varphi(t_n, r, u) \to p$, as $n \to \infty$. As $p \not\in E$, $p$ can not be an equilibrium point for the u-system. Hence $\varphi(T, r, u) = p$, for some $T \geq 0$, i.e., $r \in \mathcal{O}_u^{-}(p)$. Hence $k_u^{-1}(1) \subset \mathcal{O}^{-}(p)$.

(ii) The fact that $k_u$ is nonincreasing along control trajectories $\varphi(\cdot, x, u)$ is simply a property of the sup function. Define $k : \mathcal{U} \times \mathcal{M} \to \mathbb{R}$ by $k(u, r) = k_u(r)$.

By Lemma 3.16 and the continuity of $\varphi(t_0, \cdot, u)$ on $\mathcal{M}$, $l_p(\varphi(t_0, \cdot, u))$ is also continuous on $\mathcal{M}$. The crucial part is to show that $k_u$ is continuous.

Let $x_n \to p$. Then $l_p(\varphi(0, x_n, u)) \to l_p(\varphi(0, p, u)) = 1$, by the classical theorem of continuous dependence on initial conditions. We observe that

$$l_p(\varphi(0, x_n, u)) \leq k_u(\varphi(0, x_n, u)) \leq 1,$$
for all \( n \). This implies \( k_u(x_n) \to 1 = k_u(p) \) and \( k_u \) is continuous at \( p \). Similarly, \( k_u \) is continuous on \( \mathcal{O}_u^{-}(p) \).

Assume that \( E \) is a global attractor. We first show that \( k_u \) is continuous on \( E \).

Given \( \epsilon \in [0, 1/2] \), choose any neighborhood \( W \) of \( E \) such that \( l_p(W) < \epsilon \). By Lemma 3.16, there is some \( \hat{t} \geq 0 \) such that \( \mathcal{O}^{+}_{u,\geq \hat{t}}(clW) \subset W \). Then

\[
k_u(\mathcal{O}^{+}_{u,\geq \hat{t}}(clW)) < \epsilon.
\]

For each \( y \in clW \), we define \( t_y : 0 \leq t_y \leq \hat{t} \) as follows. For \( y \in E \), set \( t_y = 0 \). For \( y \in clW \setminus E \), set \( t_y = \hat{t} \) if \( \varphi(\hat{t}, y, u) \not\in E \); set \( t_y \) such that \( \varphi(t, y, u) \in W \) for all \( t \in [t_y, \hat{t}] \). Consider the set \( N \subset W \):

\[
N := \{ z \in \mathcal{O}^{+}_{\geq t_y}(y), \text{ for some } y \in clW \}.
\]

Then for all \( x \in N \), \( \varphi(t, x, u) \in W \), for all \( t \geq 0 \). Hence \( k_u(x) < \epsilon \), for all \( x \in N \).

We also note that \( E \subset N \), as \( E \) is invariant and \( E \subset W \).

Now we show that every point of \( E \) is an interior point in \( N \). As \( E \) is closed, \( E = intE \cup \partial E \). It suffices to consider \( x \in \partial E \). For any \( x \in \partial E \), if \( x \in \mathcal{O}^{+}_{\hat{t}}(W) \) then by the homeomorphism property of \( \varphi(\hat{t}, \cdot, u) \), \( \mathcal{O}^{+}_{\hat{t}}(W) \) is an open set in \( N \). Hence \( x \in intN \). Assume that \( x \not\in \mathcal{O}^{+}_{\hat{t}}(W) \). If \( x = \varphi(t, y) \), for some \( t \geq \hat{t} \), then \( x \in \mathcal{O}^{+}_{\hat{t}}(W) \subset N \), by the homeomorphism property of \( \varphi(t, \cdot, u) \), \( x \in intN \).

Now we only need to consider the case that \( x = \varphi(t, y, u) \), for some \( t < \hat{t} \). Then \( t \geq t_y \), otherwise \( x \not\in W \) by the definition of \( t_y \), i.e. \( x \not\in E \). Again by the homeomorphism property of \( \varphi(t, \cdot, u) \), \( x \in intN \).

Now we conclude that \( E \subset intN \). For any \( x \in E \), \( k_u(B(x)) < \epsilon \), for some open ball \( B(x) \subset N \) of \( x \). Therefore \( k_u \) is continuous on \( E \).
Given \( r \notin E \). Let \( W \) be a neighborhood of \( E \) such that \( \sup l_p(W) < l_p(r) \).
Choose \( K \) a compact neighborhood of \( r \) such that \( K \cap \{p\} = \emptyset \) and

\[
\sup l_p(W) < \inf l_p(K).
\]

Since \( E \) is a global attractor, there is some \( \hat{t} > 0 \) such that \( O^+_{u, \geq \hat{t}}(K) \subset W \) by Lemma 3.16(ii). With this choice of \( \hat{t} \), \( r' \in K \) implies

\[
k_u(r') = \sup l_p(O^+_u(r')) = \sup l_p(O^+_u[0, \hat{t}](r')).
\]

Now \( k_u \) is continuous at \( r \) as \( \sup l_p(O^+_u[0, \hat{t}]) \) depends continuously on \( r' \)

We improve the nonincreasing property of \( k_u \) to the strictly decreasing one in the next lemma.

**Lemma 3.18**

Continuing from Lemma 3.17. Fix a \( u \in \mathcal{U} \). Define \( g_u : \mathcal{M} \to \mathbb{R} \) by

\[
g_u(r) = \int_0^\infty e^{-s} k_u(\varphi(s, r, u))ds.
\]

Then

(i) \( g_u(\mathcal{M}) \subset [0, 1] \). \( g_u^{-1}(0) \subset E \). In particular, \( g_u^{-1}(0) = E \) if \( E \) is invariant.

(ii) \( g_u \) is nonincreasing along trajectories \( \varphi(\cdot, x, u) \). If \( E \) is a global attractor, then \( g_u \) is strictly decreasing along control trajectories outside \( E \).

(iii) \( g_u \) is continuous on \( O^-_u(p) \). Moreover, if \( E \) is a global attractor, then \( g_u \) is continuous \( \mathcal{M} \).
Proof. (i) and (iii) are direct results from Lemma 3.17 and properties of the improper integral. We also note that this improper integral is convergent for all \( r \in M \). Now for \( t_1 > t_2 \geq 0 \) with \( \varphi(t_i, r, u) \not\in E \). \( i = 1, 2 \). Then

\[
g_u(\varphi(t_2, r, u)) - g_u(\varphi(t_1, r, u))
\]

\[
= \int_0^\infty e^{-s}[k_u(\varphi(s, \varphi(t_2, r, u), u)) - k_u(\varphi(s, \varphi(t_1, r, u), u))]ds
\]

For this \( \varphi(t_i, r, u), i = 1, 2 \), there is a \( \delta > 0 \) such that

\[
\varphi(\delta + t_1, r, u) \in E, \text{ and } \varphi(\delta + t_2, r, u) \not\in E.
\]

Hence the last expression is strictly negative as the nonegative integrand is not identically zero in this case.

Finally we have to show that the integrand is either identically zero if this control trajectory \( \varphi(\cdot, r, u) \) never enters \( E \). Let \( \delta := t_1 - t_2 > 0 \) and \( \epsilon := k_u(\varphi(t_1, r, u)) > 0 \). Assume that

\[
P(s) := k_u(\varphi(s + t_2, r, u)) - k_u(\varphi(s + t_1, r, u)) = 0, \text{ for all } s \geq 0.
\]

Now we may choose \( \delta > 0 \) such that \( \delta = \lambda(t_1 - t_2), \) for some positive integer \( \lambda \), and

\[
k_u(\varphi(\delta + t_1, r, u)) \leq \frac{\epsilon}{2}.
\]

Otherwise, \( k_u(\varphi(t_n, r, u)) \geq \frac{\epsilon}{2} > 0 \), for some \( t_n \to \infty \). We observe that

\[
\lambda \delta + t_2 = \lambda(t_1 - t_2) + t_2
\]

\[
= t_1 + (\lambda - 1)t_1 - \lambda t_2 + t_2
\]

\[
= t_1 + (\lambda - 1)t_1 - (\lambda - 1)t_2
\]

\[
= t_1 + (\lambda - 1)\delta.
\]
Hence

\[ k_u(\varphi(\lambda \delta + t_2, r, u)) = k_u(\varphi((\lambda - 1)\delta + t_1, r, u)) \]
\[ = k_u(\varphi((\lambda - 1)\delta + t_2, r, u)) \]
\[ = k_u(\varphi((\lambda - 2)\delta + t_1, r, u)) \]
\[ = \ldots \]
\[ = k_u(\varphi(t_1, r, u)) \]
\[ = \epsilon. \]

However,

\[ k_u(\varphi(\lambda \delta + t_2, r, u)) = k_u(\varphi(\lambda \delta + t_1, r, u)) \leq \frac{\epsilon}{2}. \]

a contradiction. Therefore \( P(s) \) is not identically zero. \( \blacksquare \)

Lemma 3.19

Given \( \gamma > 0 \). Let \( \{ f_\alpha : M \to [-\gamma, \gamma] \subset \mathbb{R} : \alpha \in \mathcal{I} \} \) be a family of (uniformly) bounded, lower semicontinuous function on \( M \). Define

\[ F(x) := \sup_{\alpha \in \mathcal{I}} f_\alpha(x). \]

Then \( F \) is well defined and lower semicontinuous on \( M \).

Proof. \( F \) is well defined as \( f_\alpha(M) \subset [-\gamma, \gamma] \) for all \( \alpha \in \mathcal{I} \). Given \( \epsilon > 0 \). Fix a \( x_0 \in M \). Then there exists an \( \alpha \in \mathcal{I} \) such that \( F(x_0) \leq f_\alpha(x_0) + \frac{\epsilon}{2} \). By the continuity of this \( f_\alpha \), there is a \( \delta > 0 \) such that \( f_\alpha(x) - f_\alpha(x_0) > -\frac{\epsilon}{2} \), for all \( x \in B_M(x_0, \delta) \).

We conclude that

\[ F(x) - F(x_0) \geq F(x) - f_\alpha(x_0) - \frac{\epsilon}{2} \]
\[
\begin{align*}
\geq f(x) - f(x_0) - \frac{\epsilon}{2} \\
> -\frac{\epsilon}{2} - \frac{\epsilon}{2} &= -\epsilon
\end{align*}
\]

for all \( x \in B_M(x_0, \delta) \). Hence \( F \) is lower semicontinuous on \( M \). □

The following theorem is one of the major results in this chapter: the existence of a SCLF for (2.2)_{A1} which is lower semicontinuous on \( M \).

**Theorem 3.20**

Consider the control system (2.2)_{A1} with a unique chain control set \( E \) and let \( p \) be a reference point. Assume that \( E \) is a global attractor, then (2.2)_{A1} admits a lower semicontinuous SCLF on \( M \). \( F^{-1}(0) = E \) and \( F(M) \subset [0,1] \).

**Proof.** We define \( F : M \rightarrow \mathbb{R} \) by

\[
F(x) := \sup \{ gu(x) : u \in U \}.
\]

By Lemma 3.19, \( F \) is well defined and lower semicontinuous on \( M \). Consider a control trajectory \( \varphi(\cdot, r, u) \). Let \( t_2 > t_1 \geq 0 \). Given \( \epsilon > 0 \), then there is a \( u_2 \in U \) such that \( F(\varphi(t_2, r, u)) < gu_2(\varphi(t_2, r, u)) + \epsilon \). Define \( \bar{u} : \mathbb{R} \rightarrow U \subset \mathbb{R}^m \) by

\[
\bar{u}(t) = \begin{cases} 
  u(t) & t \leq t_2 \\
  u_2(t) & t > t_2.
\end{cases}
\]

Then \( \bar{u} \in U \). We first observe that by the definition of \( \bar{u} \) and a property of trajectories, we have

\[
\varphi(s, \varphi(t_2, r, u), u_2) = \varphi(s + t_2, r, \bar{u}) = \varphi(s, \varphi(t_2, r, \bar{u}), \bar{u}),
\]
for any $s \geq 0$. This implies that $g_{u_2}(\varphi(t_2, r, u)) = g_{\bar{u}}(\varphi(t_2, r, \bar{u}))$. Hence

$$
\mathcal{F}(\varphi(t_2, r, u)) < g_{u_2}(\varphi(t_2, r, u)) + \epsilon
= g_{\bar{u}}(\varphi(t_2, r, \bar{u})) + \epsilon
\leq g_{\bar{u}}(\varphi(t_1, r, \bar{u})) + \epsilon
= g_{\bar{u}}(\varphi(t_1, r, u)) + \epsilon
\leq \mathcal{F}(\varphi(t_1, r, u)) + \epsilon.
$$

This implies that $\mathcal{F}(\varphi(t_2, r, u)) \leq \mathcal{F}(\varphi(t_1, r, u))$, for $\epsilon > 0$ is arbitrary. $\mathcal{F}$ is nonincreasing along every control trajectory. In particular, $\mathcal{F}$ is strictly decreasing along the control trajectories outside $E$, as $g_{\bar{u}}(\varphi(t_2, r, \bar{u})) < g_{\bar{u}}(\varphi(t_1, r, \bar{u}))$ in the above third inequality, for any $\bar{u}$ corresponding to each $\epsilon$ and $r \notin E$.

Rest of results follow by Lemma 3.18. ■

Examples: Linear Affine Control Systems

We consider the following linear affine control systems

$$
\dot{x} = Ax + \sum_{i=1}^{m} u_i(t)B_i = Ax + Bu,
$$

where $A \in gl(n, \mathbb{R})$, $B_i$ are $n \times 1$ column vectors, for $i = 1, 2, \ldots m$. $B = [B_1|\ldots|B_m]$. $x \in \mathbb{R}^n$. We assume that system (3.6) has bounded control range $\rho$ which is defined in the beginning if the section 2.2. Unbounded control system $\Sigma_\infty$ means the system (3.6) with unbounded control range. The index of controllability for $\Sigma_\infty$ is the rank of the following matrix:

$$
[B|AB|A^2B|\ldots|A^{m-1}B].
$$
Remark 3.21

(i) Consider the linear control system (3.6). Suppose that the index of controllability for unbounded control system \( \Sigma_\infty \) is \( k \leq n \). Then every control set of (3.6) is contained in the \( k \)-dimensional controllable subspace in \( \mathbb{R}^n \). The controllable subspace of \( \Sigma_\infty \) is generated by \([B|AB|A^2B|\ldots|A^kB] \), \( k \) independent column vectors. See [28, Section 3.3].

(ii) Assume that \( A \) is asymptotically stable. Then (3.6) has the unique control set \( D \) and the unique chain control set \( E \). Moreover \( E = clD \), i.e. this system is perfect. \( E \) and \( D \) contains the unique equilibrium point \( x = 0 \) of the uncontrolled dynamical system \( \dot{x} = Ax \). \( E \) is a global attractor.

Theorem 3.22

Consider the linear control system (3.6). Assume that \( A \) is asymptotically stable. Then this linear control system admits a lower semicontinuous SCLF on \( \mathbb{R}^n \).

Proof. The result follows by Remark 3.21(ii) and Theorem 3.20. ■

Control Systems with More Than One Control Set

In this subsection, we will construct CLFs via Conley's construction on affine control systems (2.2) with more than one control set. For this case, (2.2) have no global attractor. However, we may define the strict domains of attraction of chain control sets, such that invariant chain control sets play a role similar to global attractors in these regions.
For perfect control systems, we may define an order, \( \prec \), between chain control sets, which is similar to that between control sets. Let \( E_1 \) and \( E_2 \) be two chain control sets of (2.2). Then we define

\[
E_1 \prec E_2 \text{ if there is } x_1 \in E_1 \text{ with } \mathcal{O}^+(x_1) \cap E_2 \neq \emptyset.
\]

**Definition 3.23**

Let \( E \) be a chain control set of (2.2). The *strict domain of attraction* of \( E \) is defined by

\[
A^s(E) := \{ y \in \mathcal{M} : \text{cl} \mathcal{O}^+_U(y) \cap E \neq \emptyset, \text{ for all } u \in \mathcal{U} \}.
\]

In this subsection, we assume that the system (2.2) has more than one control set and satisfies the following assumption.

**Assumption 3.24**

(A2) Consider the control system (2.2) which is perfect. Suppose that there are at least one maximal chain control set \( E \) and a minimal chain control set \( E^* \) with respect to \( \prec \) and \( E^* \prec E \). We call these two chain control sets a pair of chain control sets.

We denote this system by \((2.2)_{A2}\).

If \( \mathcal{M} \) is compact then (2.2) admits such a pair of chain control sets by Lemma 2.32 (iii). In general, there are more than one pair of chain control sets. The idea of Conley's construction of CLFs for \((2.2)_{A2}\) is the following. First we construct a semi-global CLF with respect to every pair of chain control sets. See Theorem 3.28 and
Corollary 3.29. Secondly, we somehow 'glue' these semi-global CLFs together and construct a global CLF on the state space $\mathcal{M}$. We will illustrate this by a chemical reactor model at the end of this subsection.

Now we are in a position to prove an existence theorem of CLFs of the system $(2.2)_A^2$. Basically, Conley's construction in this case is an analog of the previous subsection. The idea is the following: $E$ plays the similar role as a global attractor, and $E^*$ as a reference point.

**Lemma 3.25 (Analogous to Lemma 3.15)**

Consider the control system $(2.2)_A^2$. Define $l : \mathcal{M} \rightarrow [0, 1] \subset \mathbb{R}$ by

$$l(r) := \frac{d(r, E)}{d(r, E) + d(r, E^*)}.$$ 

Then $l$ is continuous on $\mathcal{M}$, $l^{-1}(0) = E$, $l^{-1}(1) = E^*$, and $l(\mathcal{M}) = [0, 1]$.

**Lemma 3.26 (Analogous to Lemma 3.17)**

Continuing from Lemma 3.25. Fix a $u \in \mathcal{U}$. Define $k_u : \mathcal{M} \rightarrow \mathbb{R}$ by

$$k_u(r) = \sup\{l(\varphi(t, r, u)) : t \geq 0\}.$$ 

Then

(i) $k_u^{-1}(0) \subset E$. In particular, $k_u^{-1}(0) = E$ if $E$ is invariant. $k_u^{-1}(1) = \mathcal{O}_u^-(E^*)$, $k_u(\mathcal{M}) \subset [0, 1]$.

(ii) $k_u$ is nonincreasing along control trajectories $\varphi(\cdot, x, u)$, for all $x \in \mathcal{M}$.

(iii) $k_u$ is continuous on $\mathcal{O}_u^-(E^*)$. In particular, $k_u$ is continuous on $E^*$ if $\mathcal{M}$ is compact.
Lemma 3.27 (Analogous to Lemma 3.18)

Continuing from Lemma 3.26. Fix $u \in U$. Define $g_u : M \to \mathbb{R}$ by

$$g_u(r) = \int_0^\infty e^{-s} k_u(\varphi(s,r,u)) ds.$$ 

Then

(i) $g_u^{-1}(0) \subset E$. In particular, $g_u^{-1}(0) = E$ if $E$ is invariant.

(ii) $g_u$ is nonincreasing along control trajectories $\varphi(\cdot, x, u)$, for all $x \in M$. Moreover, in $\mathcal{A}^s(E)$, $g_u$ is strictly decreasing along control trajectories outside $E$ if $E$ is invariant.

(iii) $g_u$ is continuous on $\mathcal{O}_u^{-}(E^*)$. In particular, $g_u$ is continuous on $E^*$ if $M$ is compact.

(iv) $g_u$ is continuous on $\mathcal{A}^s(E)$ if $E$ is invariant.

Theorem 3.28

Consider the control system (2.2)$_A$. Then

(i) There is a CLF $\mathcal{F} : M \to [0,1] \subset \mathbb{R}$, such that $\mathcal{F}^{-1}(0) = E$ and $\mathcal{O}^{-}(E^*) \subset \mathcal{F}^{-1}(1)$.

(ii) $\mathcal{F}$ in (i) is a lower semicontinuous SCLF on $\mathcal{A}^s(E)$ if $E$ is invariant. $\mathcal{F}$ is also lower semicontinuous on $\mathcal{O}^{-}(E^*)$. 
Proof. We define $\mathcal{F} : \mathcal{M} \to [0, 1]$ by

$$\mathcal{F}(r) = \sup_{u \in \mathcal{U}} g_u(r).$$

Then $\mathcal{F}$ is well defined and lower semicontinuous by Lemma 3.19; and the proof is similar to that of Theorem 3.20. ■

Corollary 3.29

Consider the system (2.2). Let $\mathcal{M}$ be compact. Then

(i) There are a minimal chain control set $E^*$ and a maximal chain control set $E$ with $E^* \prec E$. $E$ is invariant. $E^* = \mathcal{O}^-(E^*)$.

(ii) There is a CLF $\mathcal{F} : \mathcal{M} \to [0, 1] \subset \mathbb{R}$ such that $\mathcal{F}^{-1}(0) = E$, $E^* \subset \mathcal{F}^{-1}(1)$.

(iii) $\mathcal{F}$ is lower semicontinuous on $E^*$ and $\mathcal{A}^s(E)$.

(iv) $\mathcal{F}$ is a SCLF in $\mathcal{A}^s(E)$. In particular, $\mathcal{F}$ is a SCLF on $\mathcal{M}$ if $E$ and $E^*$ are the only chain control sets of (2.2).

Proof. (i) is by Lemma 2.32(iii) and the fact $\mathcal{O}^{-1}(E^*) = E^*$ if $\mathcal{M}$ is compact. (ii), (iii) and the first part of (iv) follow by Theorem 3.28. As for second part of (iv), we note that $\mathcal{A}^s(E) = \mathcal{M} \setminus \mathcal{A}(E^*) = \mathcal{M} \setminus E^*$. The result follows now by the first part of (iv). ■

Example 3.30

Consider the following one-dimensional control system determined by projecting the two-dimensional bilinear control system

$$\dot{v}(t) = u_1(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v(t) + u_2(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} v(t)$$
onto the projective space $\mathbb{P}^1$, with $U = \{0, 1\} \times [-1, 0]$. We parameterize $\mathbb{P}^1$ via the angle as

$$\mathbb{P}^1 = \{ \theta : -\frac{\pi}{2} < \theta \leq \frac{\pi}{2} \}.$$  

We also define the metric $d$ on $\mathbb{P}^1$ as follows.

$$\begin{align*}
d(\theta_1, \theta_2) &= |\theta_1 - \theta_2|, \quad \theta_i \neq \frac{\pi}{2}, i = 1, 2 \\
d(\theta, \frac{\pi}{2}) &= \frac{\pi}{2} - \theta, \quad \theta \in [0, \frac{\pi}{2}] \\
d(\theta, -\frac{\pi}{2}) &= \theta + \frac{\pi}{2}, \quad \theta \in (-\frac{\pi}{2}, 0).
\end{align*}$$

Now this projected system can be described by the following differential equation (w.r.t. $\theta$)

$$\dot{\theta} = (u_2 - u_1) \cos \theta \sin \theta.$$  

With the transformation $r = \tan \theta$, the above equation becomes

$$\dot{r} = (u_2 - u_1)r.$$  

There are two chain control sets for this system, namely, $E_1 = \{\frac{\pi}{2}\}$ and $E_2 = \{0\}$ with $E_1 \prec E_2$. Now we construct the Conley's SCLF for this one-dimensional control system.

(i) For $\theta \in [0, \frac{\pi}{2}]$, $l(r) = \frac{2\theta}{\pi}$;

$$k_u(r) = \frac{2\theta}{\pi}, \text{ for all } u \in U;$$

$$gu(r) = \frac{2\theta}{\pi} \cdot \frac{1}{(u_1 - u_2 + 1)}, \text{ for all } u \in U;$$

$$\mathcal{F}(r) = \frac{2\theta}{\pi}.$$  

We observe that $1 \leq u_1 - u_2 + 1 \leq 3$. 
(ii) For $\theta \in (-\frac{\pi}{2}, 0)$, $l(r) = \frac{-2\theta}{\pi}$;

$k_u(r) = \frac{-2\theta}{\pi}$, for all $u \in U$;

$g_u(r) = \frac{-2\theta}{\pi} \cdot \frac{1}{u_1 - u_2 + 1}$, for all $u \in U$;

$\mathcal{F}(r) = \frac{-2\theta}{\pi}$.

Also see Figure 3.5.

Example 3.31

Consider the following two-dimensional bilinear control system (also see [13, Example 4.11]).

$$
\dot{v}(t) = u_1(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v(t) + u_2(t) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v(t) + u_3(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} v(t) \quad (3.7)
$$

with $U = [0, 2] \times [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$. The control sets of the projected system of (3.7) on the projective space $\mathbb{P}^1$ in $\mathbb{R}^2$ are given by

$$
D_1 = \pi \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2; v_2 = \alpha v_1, \alpha \in (-\sqrt{2}, -\frac{1}{2}) \right\},
$$
Figure 3.6: Control sets of the projected system of (3.7). A lower semicontinuous strict control Lyapunov function, $\mathcal{F}$ is defined. In particular, $F$ is continuous on $E_1 = \text{cl}D_1$.

$$D_2 = \pi \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}; v_2 = \alpha v_1, \alpha \in \left[\frac{1}{2}, \sqrt{2}\right] \right\},$$

where $\pi$ denotes the natural identification of points in $\mathbb{R}^2$ as subspaces, i.e., as elements in $\mathbb{P}^1$, which is compact. We note that $D_1$ is open and $D_2$ is closed. By Lemma 2.32 (i) and (iii), we have $D_1 \prec D_2$. The chain control sets are $E_1 = \text{cl}D_1$, $E_2 = D_2$ and $E_1 \prec E_2$, see Figure 3.6. This projected system is perfect. By the Corollary 3.29, there is a CLF $\mathcal{F}$ such that $\mathcal{F}(0) = E_2$ and $\mathcal{F}(1) = E_1$. We note that $\mathcal{A}^s(E_2) = \mathbb{P}^1 \setminus E_1$. Therefore we conclude that $\mathcal{F}$ is a SCLF, which is lower semicontinuous on $\mathbb{P}^1 \setminus E_1$, and $\mathcal{F}$ is continuous on $E_1$,.
Figure 3.7: Phase portrait of the chemical reactor model (3.8) with $u = 0$. There are two stable equilibrium points: $x_0$ and $x_2$, and one unstable equilibrium point: $x_1$.

Example 3.32 (Chemical Reactor model: Arrhenius Dynamics)

The model of a well-stirred chemical reactor can be described by the following equation

$$
\begin{pmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
  -x_1 - a(x_1 - x_c) + Bb(1 - x_2)e^{x_1} \\
  -x_2 + b(1 - x_2)e^{x_1}
\end{pmatrix} - u(t) \begin{pmatrix}
  x_1 - x_c \\
  0
\end{pmatrix}. \tag{3.8}
$$

Here $x_1$ is the (dimensionless) temperature, $x_2$ is the product concentration, $x_c > 0$ is the coolant temperature, and $a, b, B$ are positive technical constants. The control input $u$ is the heat transfer coefficient, and the state space is $\mathcal{M} = (0, \infty) \times (0, 1)$. For the numerical results below we have chosen $x_c = 1.0, a = 0.15, b = 0.05, B = 7.0$, and $U = [-0.15, 0.15]$, see Poore [25] for the system behavior with different parameter values.

For the parameter above, the equation (3.8) with constant control $u(t) = u \in U$
has three equilibrium points in $\mathcal{M}$, namely,

$$x_0(u) = \begin{pmatrix} \alpha \\ 0.05e^\alpha/(1 + 0.05e^\alpha) \end{pmatrix},$$
$$x_1(u) = \begin{pmatrix} \beta \\ 0.05e^\beta/(1 + 0.05e^\beta) \end{pmatrix},$$
$$x_2(u) = \begin{pmatrix} \gamma \\ 0.05e^\gamma/(1 + 0.05e^\gamma) \end{pmatrix},$$

where $\alpha < \beta < \gamma$ are the zeros of the transcendental equation

$$-1.15x + 0.15 - u(x - 1) + 0.35e^x/(1 + 0.05e^x) = 0.$$ 

$x_0(u)$ is stable, $x_1(u)$ is hyperbolic, i.e., the linearization of the uncontrolled dynamical system about $x_1$ has a positive and a negative eigenvalue; $x_2(u)$ is stable. For the phase portrait of the uncontrolled dynamical system, see Figure 3.7.

For small control range, Theorem 2.25(iii) yields the existence of exactly three control sets with nonvoid interior $D_i$ with $x_i(0) \in \text{int}D_i$, $i = 0, 1, 2$. $D_0$ and $D_2$ are invariant control sets and $D_1$ is a variant control set, see Figure 3.8. Also there are three chain control sets $E_i$ such that $E_i = D_i$ for $i = 0, 2$ and $E_1 = clD_1$. We note that $E_1 \prec E_0$ and $E_1 \prec E_2$. We consider the compact subset $K = [0, 7] \times [0, 1]$ of $\mathcal{M}$. The domain of attraction of $D_1$ (in $K$) is shown in Figure 3.9. Hence there exist lower semicontinuous SCLFs in $\mathcal{A}^e(E_0)$ (to the left of $\mathcal{A}(D_1)$) and in $\mathcal{A}(E_2)$ (to the right of $\mathcal{A}(D_1)$), say $\mathcal{F}^0$ and $\mathcal{F}^2$, respectively.

We can glue these two SCLFs together to form a global SCLF. We define this global SCLF $\mathcal{F} : K \to [0, 1] \subset \mathbb{R}$ as follows:
Figure 3.8: Three control sets of the chemical reactor model (3.8). $D_0$ and $D_2$ are invariant; $D_1$ is variant.

Figure 3.9: Domain of attraction $A(D_1)$ of the control set $D_1$ of the chemical reactor model (3.8).
$\mathcal{F}(x) := \mathcal{F}^i(x)$, if $x \in \mathcal{A}^s(E_i) \cup \mathcal{A}(D_1)$, $i = 0, 2$.

$\mathcal{F}$ is well defined based on the following two facts:

$$\mathcal{F}^0(\mathcal{A}(D_1)) = \mathcal{F}^2(\mathcal{A}(D_1)) = 1,$$

and

$$K = [0, 7] \times [0, 1] = \mathcal{A}^s(E_0) \cup \mathcal{A}^s(E_2) \cup \mathcal{A}(D_1),$$

which is a disjoint union.
CHAPTER 4. CONTROL LYAPUNOV FUNCTIONS OF BILINEAR
CONTROL SYSTEMS

The local study of ordinary differential equations and smooth dynamical systems via linearization techniques and Lyapunov exponents goes back to Lyapunov's work in 1892. In the time dependent case, Oseledec's multiplicative ergodic theorem shows how to obtain results about Lyapunov exponents, invariant manifolds, exponential stability, and behavior under small perturbations. Likewise, entropy theory, bifurcation theory, strange attractors etc. can be closely related to Lyapunov exponents of dynamical systems.

We can actually define Lyapunov exponents for control systems with a similar fashion. The following question arises: Can we determine the trajectory behavior via (control) Lyapunov exponents? For general control systems, this question is hard to answer. However, Colonius and Kliemann [8, 11] analyze the control structure of the projected systems of bilinear control systems via Lyapunov exponents.

In this chapter, we continue discussing the trajectory behavior of bilinear control systems via Lyapunov spectrum and then find SCLFs. In chapter 3, we find a semi-global, lower semicontinuous SCLF of (2.2) via the Conley's construction, if (2.2) admits a global attractor. The reason that such a SCLF is semi-global is that we need a reference point to construct it. However, in this chapter, we can construct a
global lower semicontinuous SCLF of bilinear control system (4.1) via the Lyapunov spectrum, if (4.1) admits a global attractor.

Control Structure of Bilinear Control Systems on Projective Spaces

We consider the following bilinear control systems

\[ \dot{x} = A_0 x + \sum_{i=1}^{m} u_i A_i x := A(u(t))x \]  

(4.1)
on \mathbb{R}^n. \ A_i \in \mathfrak{gl}(n, \mathbb{R}), \text{ for all } i = 0, 1, \ldots, m. \ U \subset \mathbb{R}^m \text{ is compact and convex with } 0 \in \text{int}U, \text{ the interior of } U. \ As \text{ usual, the solution of } (4.1) \text{ corresponding to } u \in \mathcal{U} \text{ and initial value } x \in \mathbb{R}^n \setminus \{0\} \text{ will be denoted by } \varphi(t, x, u), t \in \mathbb{R}. \text{ We note that for a given control function } u \in \mathcal{U}, (4.1) \text{ is a linear homogeneous differential equation with time varying coefficients.}

Bilinear control systems are useful for analyzing the local behavior of nonlinear control systems while linearization techniques are applied. Therefore, understanding bilinear control systems may broaden our knowledge of nonlinear control systems, at least from local analysis point of views. For control systems with unbounded control ranges, this approach may be referred to [23, chapter 3]. Colonius and Kliemann [13] describe this method for control systems with bounded control ranges.

In this chapter, rather than discussing linearization techniques, we focus on global pictures of bilinear control systems. In particular, the exponential growth behavior of control trajectories of bilinear systems. This can be studied via Lyapunov exponents and Lyapunov spectrum of the associated angular systems on projective spaces \( \mathbb{P}^{n-1} \) (called projected systems), obtained by identifying opposite point on the sphere in \( \mathbb{R}^n \). The projected system of (4.1) (let \( s \) be the angular component) is
described by the following equation:

\[ \dot{s}(t) = h_0(s(t)) + \sum_{i=1}^{m} u_i(t) h_i(s(t)) := h(u(t), s(t)), \quad (4.2) \]

where \( h_j(s) := (A_j - sA_j s \cdot I)s, \) for \( i = 0, 1, \ldots, m. \)

For projected system (4.2), the local accessibility is equivalent to the following Lie algebra rank condition:

\[ \dim \mathcal{L} \mathcal{A}\{h(\cdot, u) : u \in U\}(x) = n - 1, \forall x \in \mathbb{P}^{n-1}. \]

**Remark 4.1**

(H) for (4.1) implies (HP) for (4.2).

Before discussing CLFs, we need to understand more about the control structure of bilinear control systems. In general, it is difficult to find control sets of control systems (2.2) except by numerical simulations. However, one advantage of studying the bilinear control systems (4.1) via the projected systems (4.2) is that we can actually compute control sets (with nonvoid interior) of (4.2). It turns out that these control sets are closely related to generalized eigenspaces of the elements in the following systems semigroup \( \mathcal{S} \) defined by

\[ \mathcal{S} = \{ \exp(t_k B_k) \ldots \exp(t_1 B_1) ; t_j \geq 0, B_j = A(u_j), u_j \in U, j = 1, \ldots k, k \in \mathcal{N} \}, \quad (4.3) \]

where \( \mathcal{N} \) is the collection of positive integers. In fact, this system semigroup \( \mathcal{S} \) is contained in the following system group \( \mathcal{G} \):

\[ \mathcal{G} = \{ \exp(t_k B_k) \ldots \exp(t_1 B_1) ; t_j \in \mathbb{R}, B_j = A(u_j), u_j \in U, j = 1, \ldots k, k \in \mathcal{N} \}. \quad (4.4) \]
$\mathcal{G}$ is a Lie group acting naturally on $\mathbb{R}^n \setminus \{0\}$ and on $\mathbb{P}^{n-1}$. For $t > 0$ denote by $\mathcal{S}_{\leq t}$ the subset of $\mathcal{S}$ with $\sum_{j=1}^{k} t_j \leq t$. We note that by (H), the interior of $\mathcal{S}_{\leq t}$ in the systems Lie group $\mathcal{G}$ is nonvoid, for details, see [7], [11]. The control structure of systems (4.2) is realized by the following theorem.

**Theorem 4.2**

Consider bilinear control system (4.1) and assume (H). Then for the projected control system (4.2) we have the following results.

(i) There are $1 \leq k \leq n$ control sets $D_1, \ldots, D_k$ with nonvoid interior in $\mathbb{P}^{n-1}$, called the main control sets.

(ii) The interior of the main control sets are the connected components of

$$\mathbb{P}V := \{\mathbb{P}E(\lambda) | \lambda \in \text{spec}(g), g \in \text{int}\mathcal{S}\},$$

where $\text{spec}(g)$ is the spectrum of $g$. $\mathbb{P}E(\lambda)$ denoted the projection of the generalized eigenspace of $g$ corresponding to $\lambda$ onto the projective space $\mathbb{P}^{n-1}$.

(iii) The main control sets are linearly ordered by $D_i \prec D_j$ if there exist $x_i \in D_i$, $x_j \in D_j$ and $g \in \mathcal{S}_{\leq t}$ with $gx_i = x_j$.

(iv) For every $t > 0$ and every $g \in \text{int}\mathcal{S}_{\leq t}$ and every $\lambda \in \text{spec}(g)$, there is a main control set $D_i$ such that the generalized eigenspace $E(\lambda)$ satisfies

$$\mathbb{P}(E(\lambda)) \subset \text{int}D_i,$$

the interior of the main control sets consists exactly of these elements $x \in \mathbb{P}^{n-1}$ which are eigenvectors for a (real) eigenvalue of some $g \in \mathcal{S}_t \cap \text{int}\mathcal{S}_{\leq t+1}$ for some $t > 0$. $\mathcal{S}_t$ is the subset of $\mathcal{S}$ with $\sum_{j=1}^{k} t_j = t$. 
(v) For every \( g \in S \) and every \( \lambda \in \text{spec}(g) \) there is some main control set \( D_i \) with
\[
\mathcal{P}E(\lambda) \cap \text{cl}D_i \neq \emptyset; \text{ for every main control set } D_i \text{ and every } g \in S \text{ there is a } \lambda \in \text{spec}(g) \text{ with } \mathcal{P}E(\lambda) \cap \text{cl}D_i \neq \emptyset.
\]

(vi) The control set \( C := D_k \) is closed and invariant and \( \text{cl}C = \cap_{x \in \mathcal{Pcl}O^+} (x) \): the control set \( C^* := D_1 \) is open and \( \text{cl}C^* = \cap_{x \in \mathcal{Pcl}O^-} (x) \); all other main control sets are neither open nor closed.

**Proof.** See [7, Theorem 3.10]. ■

The control structure of the projected system (4.2) with \( n = 2 \) turns out to be fairly simple. In particular, (4.2) is perfect in most cases, see (i) and (iii) in the following theorem.

**Theorem 4.3**

Consider the bilinear control system (4.1) with \( n = 2 \) and assume (H). Then the control structure of (4.2) has the following three possibilities:

(i) The projected system has two main control sets \( D_1 < D_2 \) and two chain control sets \( E_1 = \text{cl}D_1, E_2 = D_2 \) (Figure 4.1).

(ii) The projected system has two main control sets \( D_1 < D_2 \), but one chain control set \( E = \mathcal{P}^1 \).

(iii) The projected system has one main control set \( D = \mathcal{P}^1 \), hence the chain control set is \( E = \mathcal{P}^1 \).

**Proof.** By Theorem 4.2(i), there are at most two main control sets. The results follow by [11, Corollary 4.9]. ■
In this section, we introduce Lyapunov exponents for the bilinear control system (4.1) and the Lyapunov spectrum for the projected systems (4.2). For $n = 2$, the Lyapunov spectrum can be characterized completely by constant controls [19]. For $n > 2$, the Lyapunov spectrum can be estimated via an outer approximation and an inner approximation. The outer approximation uses chain recurrent components of the projected control flow and the associated Morse spectrum, while the inner approximation considers the eigenspaces of periodic perturbations and associated Floquet exponents. Ideas from geometrical nonlinear control theory are used to combined these two approaches, these results can be referred to [11].
**Definition 4.4**

*Lyapunov exponent* of the solution $\varphi(t, x_0, u)$ of (4.1) is defined as

$$
\lambda(x_0, u) = \limsup_{t \to \infty} \frac{1}{t} \log \|\varphi(t, x_0, u)\|
$$

with $\varphi(0, x, u) = x \neq 0$ and $\| \cdot \|$ can be any norm on $\mathbb{R}^n$.

**Remark 4.5**

For constant $A(u)$ of (4.1), Lyapunov exponents are the real parts of the eigenvalues of $A(u)$; for periodic $A(u)$, they are the corresponding Floquet exponents. In any case, $\lambda(x, u) < 0$ if and only if $\varphi(t, x, u)$ converges to zero faster than any exponential $\exp(at)$ with $\lambda(x, u) < a < 0$ (and slower than those with $a < \lambda(x, u)$).

**Definition 4.6**

Let $D$ be a main control set of the projected system (4.2). The *Lyapunov spectrum* of (4.1) over $\text{cl}D$ (the closure of $D$) is defined by

$$
\sum_{Ly}(\text{cl}D) := \{ \lambda(u, p)| (u, p) \in \mathcal{U} \times D, \exists T \geq 0 \implies \hat{\varphi}(t, p, u) \in \text{cl}D, \forall t \geq T \},
$$

where $\hat{\varphi}(\cdot, s_0, u)$ denotes the solution of (4.2) with $\hat{\varphi}(0, s_0, u) = s_0 \in \mathbb{P}^{n-1}$.

The Lyapunov spectrum of the system (4.1) is

$$
\sum_{Ly} = \{ \lambda(u, x); (u, x) \in \mathcal{U} \times \mathbb{R}^n, x \neq 0 \}.
$$

**Remark 4.7**

The Lyapunov exponents of (4.1) define a map

$$
\lambda : \mathcal{U} \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}
$$

in fact, they are defined on the control flow associated with (4.1). The flow point of view allows us to use concepts and techniques from topological dynamics for the
analysis of the Lyapunov spectrum. The details of control flows corresponding to control systems will be introduced in Section 5.1.

**Theorem 4.8** Consider the control system \((4.1)\). Assume \((H)\) for \((4.1)\), for all \(\rho > 0\); and \((I)_\rho\) for \((4.2)\), for all \(\rho \geq 0\). Let \(D^\rho_1, \ldots, D^\rho_{k(\rho)}\) be the main control sets of \((4.2)\). Then for except at most countably many \(\rho\), we have the following results.

(i) Each \(\sum_Ly(\text{cl}D_i^\rho)\) is a closed interval, for all \(i = 1, \ldots k(\rho)\).

(ii) \(\sum_Ly\) of the system \((4.1)\) is a union of closed intervals \(\bigcup_{i=1}^{k(\rho)} \sum_Ly(\text{cl}D_i^\rho)\).

**Proof.**

(i) See [8, Theorem 3].

(ii) The result follows by [8, Theorem 3(2)] and [11, Corollary 5.6].

The following theorem is a complete characterization of the Lyapunov spectrum of \((4.1)\) with \(n = 2\).

**Theorem 4.9**

Consider bilinear control systems \((4.1)\) with \(n = 2\). Assume \((H)\) for \((4.1)\).

(i) If all eigenvalues of \(A(u)\) for all constant control \(u \in U\) are real, then

(a) there are two disjoint control sets \(D_1\) and \(D_2\), \(D_1\) is open, \(D_2\) is closed in \(\mathbb{P}^1\).

(b) \(\sum_Ly = \sum_Ly(\text{cl}D_1) \cup \sum_Ly(D_2)\).
(c) \( \sum_{\lambda_y} (D_2) = [\gamma, \kappa] \) and all \( \lambda \in [\gamma, \kappa] \) are (maximal) real eigenvalues of some \( u \in \mathcal{U} \),

(d) \( \sum_{\lambda_y} (clD_1) = [\gamma^*, \kappa^*] \) and all \( \lambda \in [\gamma^*, \kappa^*] \) are (minimal) real eigenvalues of some \( u \in \mathcal{U} \),

(e) the intervals \( \sum_{\lambda_y} (clD_1) \) and \( \sum_{\lambda_y} (D_2) \) can overlap strictly, i.e., with interior points.

(ii) If all eigenvalues of \( A(u) \) for all constant control \( u \in U \) are complex, then

(a) there is one control set \( D = P^1 \),

(b) \( \sum_{\lambda_y} = \sum_{\lambda_y}(D) \),

(c) \( \sum_{\lambda_y}(D) = [\gamma, \kappa] \) and all \( \lambda \in [\gamma, \kappa] \) are Floquet exponents of some periodic \( \bar{u} \in \mathcal{U} \) with periodic trajectory \( \phi(\cdot, 0, \bar{u}) \) in \( P^1 \).

(iii) If there exist \( u_1, u_2 \in \mathcal{U} \) such that \( A(u_1) \) has real eigenvalue and \( A(u_2) \) has complex eigenvalues, then

(a) there is one control set \( D = P^1 \),

(b) \( \sum_{\lambda_y}(D) = [\gamma, \kappa] \) and all \( \lambda \in [\gamma, \kappa] \) are either real eigenvalues or Floquet exponents of some periodic \( \bar{u} \in \mathcal{U} \)

Proof. Joseph proves this theorem [19, Theorem 4.1] by considering all \( u \in \mathcal{U} \). By [11, Corollary 4.9], it suffices to consider only constant control \( u \in U \) for (i) and (ii).
In this section, we are interested in the bilinear control systems (4.1) satisfying the following assumption (A3). We denote such a system by (4.1)\textsubscript{A3}.

(A3) Consider the bilinear control system (4.1)\textsuperscript{p}. Assume (H) for (4.1)\textsuperscript{p}, for all \( \rho > 0 \). Assume (I\textsubscript{p}) for (4.2)\textsuperscript{p}, for all \( \rho \geq 0 \). Fix a \( \rho > 0 \), let \( D_1^p, \ldots, D_k^p \) be main control sets of (4.2)\textsuperscript{p}. We assume that \( \sum Ly(cI D_i^p) \) is a closed interval, for all \( i = 1, \ldots, k \).

We denote the bilinear control system (4.1) with control range \( \rho > 0 \) satisfying (A3) by (4.1)\textsubscript{A3}.

\textbf{Control Lyapunov Functions of Bilinear Control Systems}

Now we are in a position to analyze flow behavior of (4.1). The idea is to analyze flow behavior in each cone generated by main control sets of (4.2).

The following theorem shows that control sets of (4.1) are those cones generated by particular main control sets.

\textbf{Theorem 4.10}

Consider the bilinear control system (4.1)\textsuperscript{p} \( \text{A3} \). Let \( D^p \) be a main control set. Then we have the following results:

(i) If \( 0 \in \text{int} \sum Ly(clD^p) \) then the system is completely controllable in the cone \( K \) (generated by \( D^p \)),

\[ K := \{ap | p \in \text{int}D^p, \alpha > 0 \}. \]

(ii) If \( 0 \notin \sum Ly(cl(D^p)) \) then the system is not controllable in the cone \( K \) generated by \( D^p \), i.e., for any control set \( C \) of (4.1)\textsubscript{A3}, \( C \cap K = \emptyset \).
Proof. See [8, Theorem 3(2), Theorem 7]. ■

Remark 4.11 The only case not covered by the theorem above is when

$$0 \in \partial \Sigma_L y(clD^0),$$

the boundary of $\Sigma_L y(clD^0)$.

The following lemma describes flow behavior of (4.1) in those cones where the system is not completely controllable.

Lemma 4.12

Consider $(4.1)^\ominus A_3$. Let $D^0$ be a main control set and $K$ be the cone generated by $D^0$. Fix $x \in K$.

(i) Assume that $\max \Sigma_L y(clD^0) < 0$. If $\varphi(\hat{t}, x, u) = ax$ for some $\hat{t} > 0$ then $a < 1$.

(ii) Assume that $\min \Sigma_L y(clD^0) > 0$. If $\varphi(\hat{t}, x, u) = ax$ for some $\hat{t} > 0$ then $a > 1$.

Proof.

(i) Assume that $\varphi(\hat{t}, x, u) = ax$, for some $\hat{t} > 0$, some $a \geq 1$. Define $v : [0, \infty) \to U$ as follows:

$$v(t) := \begin{cases} u(t) & 0 \leq t < \hat{t} \\ u(t - \hat{t}) & t \geq \hat{t}, \end{cases}$$

where $l$ is a nonnegative integer such that $0 \leq (t - \hat{t}) < \hat{t}$. Then $v$ is a periodic function (with period $\hat{t}$) and $v \in \mathcal{U}$. We note that $\varphi(l\hat{t}, x, u) = \alpha^lx$, for every nonnegative integer $l$ as (4.1) is a linear homogeneous system. We compute the
following Lyapunov exponent,

$$\lambda(x, v) = \limsup_{t \to \infty} \frac{1}{t} \log \| \varphi(t, x, v) \| \geq \limsup_{t \to \infty} \frac{1}{t} \log \| \alpha^t x \|$$

$$= \frac{1}{t} \log \alpha > 0,$$

a contradiction.

(ii) The proof is similar to (i).

\[\square\]

The next lemma shows some limit behavior of control trajectories of bilinear systems.

**Lemma 4.13**

Consider the bilinear control system (4.1). Let $\hat{\phi}$ be the solution of the projected system (4.2).

(i) For any $x \in \mathbb{P}^{n-1}$, there is a $u \in \mathcal{U}$ and a $T > 0$ such that

$$\hat{\phi}(t, x, u) \in \text{int}D_k, \text{ for all } t \geq T.$$

(ii) Assume that (4.2) is perfect. Then for any $x \in \mathbb{P}^{n-1}$, for any $u \in \mathcal{U}$, there is a $T \geq 0$ such that

$$\hat{\phi}(t, x, u) \in \text{cl}D_j, \text{ for some } j \text{ and all } t \geq T.$$

(iii) For any $x \in \mathbb{R}^n$, there is a $u \in \mathcal{U}$ and $T > 0$ such that

$$\varphi(t, x, u) \in \text{cl}K, \text{ for all } t \geq T,$$
Where $K$ is the cone generated by $D_k$.

(iv) Assume that (4.2) is perfect. Then for any $x \in \mathbb{R}^n$, and any $u \in \mathcal{U}$, there is a $T \geq 0$ such that

$$\varphi(t, x, u) \in \text{cl}K(D_j), \text{ for some } j \text{ and all } t \geq T.$$  

Where $K(D_j)$ is the cone generated by $D_j$.

**Proof.** (i) We note that $\mathbb{P}^{n-1}$ is compact and $D_K$ is (the unique) invariant control set. The results follows by [4, Lemma 6.1]. (ii) is the direct result from (i).

We note that the $\omega(x, u)$, the $\omega$-limit of $x$ in the projected $u$-system, is nonempty, as $\mathbb{P}^{n-1}$ is compact. Moreover, $\omega(x, u) \subset \text{cl}D_j$ by Remark 2.23, for some chain control set $\text{cl}D_j$ of (4.2). As $\omega$-limit set is positive invariant, the result follows. (iv) is the direct result from (iii). \qed

**Remark 4.14**

The assumption $\max \sum_{\mathcal{L}y}(\text{cl}D_k) < 0$ implies that

$$\max \sum_{\mathcal{L}y}(\text{cl}D_j), \text{ for all } j = 1, 2, \ldots, k - 1.$$  

See [13, Theorem 4.10(iii)]. It guarantees that (4.1) has a global attractor \{0\} by Theorem 4.10(ii). Moreover, by Lemma 4.13(iv) and the property of lim sup, for any $x \in \mathbb{R}^n$ and any $u \in \mathcal{U}$, we have

$$||\varphi(t, x, u)|| \leq e^{ct},$$  

for $t > 0$ sufficiently large. $c := \max \sum_{\mathcal{L}y}(\text{cl}D_k)$. 

The main theorem is this chapter is to construct a SCLF for (4.1) on $\mathbb{R}^n$. This can be regarded as an improvement of the Conley's construction of SCLFs via the Lyapunov spectrum method.

**Theorem 4.15**

Consider the bilinear control system $(4.1)^{A3}$. Assume that

$$\max \sum L_y(\text{cl}D_k^{\rho}(\cdot)) < 0.$$ 

Then $(4.1)^{A3}$ admits a lower semicontinuous SCLF on $\mathbb{R}^n$.

**Proof.** Let $c := \max \sum L_y(\text{cl}D_k^{\rho}(\cdot))$. Fix $a \in \mathcal{U}$ and $b : 0 < b < -c$. Then by Remark 4.14, for any $x \in \mathbb{R}^n$, there exists $t_x > 0$ such that

$$||\varphi(t, x, u)|| < e^{ct}, \text{ for } t > t_x.$$ 

Moreover, we may choose $t \varphi(t, x, u)$ such that (i) $t \varphi(t, x, u)$ is increasing along the control trajectory $\varphi(\cdot, x, u)$ as $t \geq 0$ increases. (ii) $t$ is continuous at any $x \in \mathbb{R}^n$.

Define $g_u : \mathbb{R}^n \rightarrow [0, \infty) \subset \mathbb{R}$ by

$$g_u(x) = \int_0^\infty e^{bt} ||\varphi(t, x, u)|| dt,$$

where $|| \cdot ||$ can be any norm on $\mathbb{R}^n$. Then $g_u$ is well defined on $\mathbb{R}^n$ as

$$g_u(x) \leq \int_0^\infty e^{bt} e^{ct} dt < \frac{1}{b+c} < \infty.$$ 

g_u is continuous at any $x \in \mathbb{R}^n$ as $t$ and $\varphi(t, \cdot, u)$ are continuous at any $x \in \mathbb{R}^n$.

Now, for all $t \geq 0$,

$$g_u(\varphi(t, x, u)) = \int_{t \varphi(t, x, u)}^{\infty} ||e^{b\tau} \varphi(\tau, \varphi(t, x, u), u)|| d\tau.$$
\[= \int_1^\infty e^\beta ||\varphi(t + t, x, u)||dt \]
\[= e^{-\beta t} \int_1^\infty e^\beta ||\varphi(\eta, x, u)||d\eta, \eta = t + \tau \]

We reach our first goal: \( gu \) is strictly decreasing along control trajectories \( \varphi(\cdot, x, u) \), as \( e^{-\beta t} \) is strictly decreasing, and \( \int_1^\infty e^\beta ||\varphi(\eta, x, u)||d\eta \) is non-increasing as \( t \) increases. Our task now is to find a function \( F \) such that \( F \) is strictly decreasing along every control trajectory.

We define \( F \) as follows.

\[ F := \sup_{u \in \mathcal{U}} gu(r). \]

Then \( F \) is well defined on \( \mathbb{R}^n \) as \( 0 \leq F \leq -\frac{1}{b+c} \) for all \( u \in \mathcal{U} \). Consider a control trajectory \( \varphi(\cdot, r, u), r \in \mathbb{R}^n \) and \( u \in \mathcal{U} \). Let \( t_2 > t_1 \geq 0 \). Given \( \epsilon > 0 \), let \( u_2 \in \mathcal{U} \) satisfy

\[ F(\varphi(t_2, x, u)) < gu_2(\varphi(t_2, x, u)) + \epsilon. \]

Define \( \bar{u} : [0, \infty) \rightarrow \mathbb{R}^m \) by

\[ \bar{u}(t) = \begin{cases} 
  u(t) & 0 \leq t \leq t_2 \\
  u_2(t) & t > t_2.
\end{cases} \]

Then we observe that by the definition of \( \bar{u} \) and the property of trajectories, we have

\[ \varphi(s, \varphi(t_2, x, u), u_2) = \varphi(s + t_2, x, \bar{u}) = \varphi(s, \varphi(t_2, x, \bar{u}), \bar{u}), \]

for any \( s \geq 0 \). This implies that \( gu_2(\varphi(t_2, x, u)) = g\bar{u}(\varphi(t_2, x, \bar{u})) \). Hence

\[ F(\varphi(t_2, x, u)) < gu_2(\varphi(t_2, x, u)) + \epsilon \]
\[\begin{align*}
g_\bar{u}(\varphi(t_2, x, \bar{u})) + \epsilon &< g_\bar{u}(\varphi(t_1, x, \bar{u})) + \epsilon \\
&\leq \mathcal{F}(\varphi(t_1, x, \bar{u})) + \epsilon \\
&= \mathcal{F}(\varphi(t_1, x, u)) + \epsilon.
\end{align*}\]

Since \(\epsilon > 0\) is arbitrary, \(\mathcal{F}(\varphi(t_2, x, u)) < \mathcal{F}(\varphi(t_1, x, u))\). □

**Corollary 4.16**

Consider the bilinear control system \((4.2)^\rho_{A3}\) with the unique main control set \(D^\rho\). Assume that \(\max \sum L_y(cD^\rho) < 0\). Then this system admits a SCLF on \(\mathbb{R}^n\).

**Proof.** This is trivial. □

**Example 4.17 (Controlled Linear Oscillator)**

Consider the linear oscillator with perturbation in the restoring force

\[\ddot{y} + 2b\dot{y} + (1 + u(t))y = 0, \quad (4.7)\]

where \(u(t) \in [-1, 1]\). Setting \((x_1, x_2) = (y, \dot{y})\) we can write

\[\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} + u(t) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x, \quad (4.8)\]

\(x \in \mathbb{R}^2\). For \(b^2 < \sqrt{2}\) the projected system on \(P^1\) has a unique main control set \(D = P^1\), compare e.g. [5, section 6]. If \(b > 0.25\), then \(\max \sum L_y(cD) < 0\), also see [5, section 6]. (4.8) admits a SCLF by Corollary 4.16.
CHAPTER 5. GLOBAL ANALYSIS OF CONTROL SYSTEMS: APPLICATIONS OF CONTROL LYAPUNOV FUNCTIONS

In this chapter, we discuss several applications of control Lyapunov functions to global characterizations of control systems with bounded control ranges. Such a global analysis is one major advantage of our CLFs than other definitions of CLFs, e.g., Sontag's [29, 30]. The latter basically analyzes the local behavior of control systems.

Some global aspects of control systems are illustrated by the following arrangement. In section 1, we analyze control systems via control flows, which provide us a way of thinking of control systems as dynamical systems. In section 2, we discuss a particular phenomenon appeared in control systems, that is, multistability regions. Conley's construction of CLFs provide a nontrivial characterization of these regions. In section 3, we discuss the stability of (chain) control sets. This can be regarded as a control version of structural stability in dynamical systems theory.

Control Flows of Control Systems

Dynamical systems theory has developed a large tool-box for the analysis of systems. These concepts and results, together with ideas from control theory, should lead to a better understanding of various control theoretic problems and their solu-
tions. First, we need a way of thinking of control systems as dynamical systems. One is the concept of the control flow and we describe as follows.

The idea is, roughly speaking, to associate an infinite dimensional dynamical systems (called the control flow) to the control system (2.2), which is defined as follows:

\[ \Phi : \mathbb{R} \times U \times M \rightarrow U \times M \]

by

\[ \Phi(t, u, x) = (\theta_t u, \varphi(t, x, u)), \]

where \( \theta_t u(\cdot) = u(t + \cdot) \) is the usual phase shift by \( t \); \( \varphi \) is the solution of the control system (2.2).

The first step in analyzing the control flow is the definition of an appropriate topology on \( U \), the space of control functions. \( U \) will be equipped with the weak*-topology of \( L^\infty(\mathbb{R}, \mathbb{R}^m) = (L^1(\mathbb{R}, \mathbb{R}^m))^* \), because it implies the uniform convergence on compact time intervals of the corresponding trajectories of (2.2), and it seems appropriate for the control flow \( \Phi \) on \( U \times M \).

**Theorem 5.1** Consider the control system (2.2) and its control flow \( \Phi \). We have the following results.

(i) \((U \times M, \Phi)\) is a continuous dynamical system.

(ii) \( U \) is a compact, separable metric space.

(iii) \((U, \theta)\) is a chain recurrent continuous dynamical system.
Proof.

(i) See [6, Lemma 3.4].

(ii) Define a metric on \( \mathcal{U} \) by

\[
d_{\mathcal{U}}(u, v) := \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \frac{1}{1 + \int_{\mathbb{R}} < u(t) - v(t), x_k(t) > \, dt} \right|
\]

where \( \{x_k, k = 1, 2, \ldots, \infty \} \) is a countable, dense subset of \( L^\infty(\mathbb{R, \mathbb{R}^m}) \). The conclusion were shown in [6, Lemma 2.1].

(iii) See [6, Proposition 2.6].

While using this control flow \( \Phi \) as in interplay between control theory and dynamical systems theory, the following problems arise.

(a) How can typical control theoretical concepts (like controllability) be expressed in term of dynamical systems concepts for the control flow, and vice versa? what can be learned from the global and limit structure analysis for control systems.

(b) How is the control flow related to the long term behavior of uncontrolled dynamical system, whose dynamics we want to control?

(c) What is the long term behavior of the control flow, i.e., where do the controlled trajectories end up for \( t \to \infty (\rightarrow -\infty) \). This will answer stabilization, asymptotic controllability and robustness question.

For (a) and (b), the concepts such as chain recurrence and topological mixing play important roles, see [4]. Colonius and Kliemann discuss (c) in [6] and also provides a generic result in [4].
Definition 5.2

Let $D \subseteq \mathcal{M}$ be a control set of (2.2) with nonvoid interior. The lifted control set $\mathcal{D} \subseteq \mathcal{U} \times \mathcal{M}$ of $D$ is defined by

$$\mathcal{D} := \text{cl}\{ (u, x) \in \mathcal{U} \times \mathcal{M}, \varphi(t, x, u) \in \text{int}D, \forall t \in \mathbb{R} \},$$

where the closure is taken with respect to the weak*-topology in $\mathcal{U}$ and the given topology on $\mathcal{M}$.

Remark 5.3

We observe that $\mathcal{D}$ as well as the set $\{ (u, x) \in \mathcal{U} \times \mathcal{M}, \varphi(t, x, u) \in \text{int}D, \forall t \in \mathbb{R} \}$ are invariant under the control flow $\Phi$. If $D$ is bounded in the metric given by the Riemannian structure on $\mathcal{M}$, then $\mathcal{D}$ is compact.

Definition 5.4

Let $(\mathcal{M}, \Psi)$ be a continuous dynamical system. It is called topologically transitive, if there exists some $x \in \mathcal{M}$ such that $\omega(x) = \mathcal{M}$, and topologically mixing, if for any two open sets $V_1, V_2 \subseteq \mathcal{M}$, there exist $T_0 \in \mathbb{R}, T_1 > 0$ such that for all $k \in \mathbb{N},$

$$\Psi(-kT_1 + T_0, V_1) \cap V_2 \neq \emptyset.$$
Theorem 5.5

Consider the control system (2.2) and assume (H). Let \( D \subset U \times M \) with
\[
\text{int}(\Pi_M D) \neq \emptyset.
\]

Then \( D \) is maximally topologically mixing if and only if there exists a control set \( D \) such that \( D \) is the lifted control set of \( D \).

In this case, \( D \) is unique and
\[
\text{int} D = \text{int}(\Pi_M D) \text{ and } \text{cl} D = \Pi_M D.
\]

Proof. See [6, Theorem 3.9]. \( \blacksquare \)

The following theorem gives some semi-global pictures of control systems via control flows.

Theorem 5.6

Let \( D \) be a control set of (2.2) with nonvoid interior. Let \( D \) be the lifted control set of \( D \). Then we have the following results.

(i) The periodic points of \( \Phi \) are dense in \( D \).

(ii) \( \Phi|_D \) is topologically mixing and transitive.

(iii) \( \Phi|_D \) has sensitive dependence on initial conditions.

Proof. See [6, Proposition 3.5]. \( \blacksquare \)
Remark 5.7

\[ \Phi|_\mathcal{D} \text{ is chaotic by Devaney's definition ([16, Definition 8.5])}. \]

Similarly, we may define lifted chain control sets and discuss the behavior of control flows in lifted chain control sets.

Definition 5.8

For a chain control set \( E \) of (2.2). We define the lifted chain control set of \( E \) as

\[ E := \{ (u, x) \in \mathcal{U} \times \mathcal{M}, \varphi(t, x, u) \in E, \forall t \in \mathbb{R} \}. \]

Theorem 5.9

Consider the control system (2.2). We have the following results.

(i) Let \( E \subset \mathcal{M} \) be a chain control set. Then the lifted chain control \( E \subset \mathcal{U} \times \mathcal{M} \) of \( E \) is a maximal invariant connected component of chain recurrent set of the control flow \( (\mathcal{U} \times \mathcal{M}, \Phi) \).

(ii) Let \( E \subset \mathcal{U} \times \mathcal{M} \) be a maximal invariant connected component of chain recurrent set of \( (\mathcal{U} \times \mathcal{M}, \Phi) \). Then \( \Pi \mathcal{M} E \) is a chain control set of (2.2).

Proof. See [4, Theorem 4.9]. \( \square \)

Theorem 5.10

Consider the control system (2.2). Let \( \mathcal{M} \) be compact. Then the control flow \( (\mathcal{U} \times \mathcal{M}, \Phi) \) admits a strict Lyapunov function.
Proof. We note that $\mathcal{U} \times \mathcal{M}$ is also a compact metric space by Theorem 5.1(ii). Then the result follows from Theorem 2.9. □

Remark 5.11

In general, control flows are not gradient-like. We observe that the chain recurrent sets strictly contain equilibrium points.

In general, we cannot construct CLFs of (2.2) via Lyapunov functions of the control flow. However, the following theorem shows the converse is true, namely, we can construct Lyapunov functions of the control flow via CLFs of (2.2).

Theorem 5.12

Consider the control system (2.2). We have the following results:

(i) If $\mathcal{F}$ is a continuous CLF of (2.2) then $V := \mathcal{F} \circ \Pi_{\mathcal{M}}$ is a Lyapunov function of the control flow. In particular, if $\mathcal{F}$ is a continuous SCLF of (2.2) then $V := \mathcal{F} \circ \Pi_{\mathcal{M}}$ is a strict Lyapunov function of the control flow.

(ii) If $\mathcal{F}$ is a lower semicontinuous CLF of (2.2) then $V := \mathcal{F} \circ \Pi_{\mathcal{M}}$ is a lower semicontinuous Lyapunov function of the control flow. In particular, if $\mathcal{F}$ is a lower semicontinuous SCLF of (2.2) then $V := \mathcal{F} \circ \Pi_{\mathcal{M}}$ is a lower semicontinuous strict Lyapunov function of the control flow.

Proof. We simply need to prove second part of (i). Let $\mathcal{E} \subset \mathcal{U} \times \mathcal{M}$ be a chain recurrent component for the control flow $(\mathcal{U} \times \mathcal{M}, \Phi)$. Let $(u, x)$ and $(v, y) \in \mathcal{E}$. Then

$$x = \Pi_{\mathcal{M}}(u, x) \in \Pi_{\mathcal{M}}\mathcal{E} \text{ and } y = \Pi_{\mathcal{M}}(v, y) \in \Pi_{\mathcal{M}}\mathcal{E}.$$
\( \Pi_{\mathcal{M}} \mathcal{E} \) is a subset of a chain control set of the control system (2.2), for \( \Pi_{\mathcal{M}} \mathcal{E} \) is connected and chain controllable. We conclude that

\[
V(u, x) = \mathcal{F} \circ \Pi_{\mathcal{M}}(u, x) = \mathcal{F}(x) = \mathcal{F}(y) = V(v, y),
\]
i.e., \( V \) is constant on each chain recurrent component of the control flow.

Let \( \Phi(\cdot, x, u) \) be a trajectory outside the chain recurrent set of the control flow. Then

\[
V(\Phi(t, x, u)) = \mathcal{F}(\Pi_{\mathcal{M}}(\Phi(t, x, u))) = \mathcal{F}(\varphi(t, x, u)).
\]

We note that \( \varphi(\cdot, x, u) \) is a control trajectory outside chain control sets by Theorem 5.9. This implies \( V \) is strictly decreasing along \( \Phi(\cdot, x, u) \). We conclude that \( V \) is a strict Lyapunov function of the control flow. \( \blacksquare \)

**Multistability Regions**

Recently, the phenomenon of bistability (or better multistability) in control systems has attracted considerable attention: a point is multistable if the system response from this point exhibits different limit behavior. While in (deterministic) dynamical system, the trajectory from an initial point converges to its unique limit set (or possibly to \( \infty \)).

Colonius et al. [12] characterize multistability regions via related invariant control sets. In this section, we characterize multistability regions by Conley’s CLFs. In this section, we still consider the control system (2.2).
Definition 5.13

A point $x_0 \in M$ is multistable, if there exist two invariant control sets $C_1, C_2$ with $x_0 \in \mathcal{A}(C_i)$ for $i = 1, 2$. The collection of such points is called the multistability region, denoted by $\mathcal{M}S$. We also use the following notation

$$\mathcal{M}S(C_1, C_2) = \{x \in M : x \in \mathcal{A}(C_i), i = 1, 2\}$$

to represent the multistability region w.r.t. $C_1$ and $C_2$.

Remark 5.14

The multistability region $\mathcal{M}S$ can be expressed as $\bigcup_{i \neq j} \mathcal{M}S(C_i, C_j)$, where $C_i, C_j$ are (distinct) invariant control sets of (2.2). We note that $\mathcal{M}S(C_i, C_j)$ may be empty.

We now proceed to describe $\mathcal{M}S$ more precisely. In order to avoid certain degeneracies on the boundary $\partial L$ of the compact, forward invariant set $L \subset M$, we require that all limit sets of the control system (2.2) are uniformly bounded away from $\partial L$. The following strong invariance condition (cf. [12, section 2]) turns out to be sufficient:

(SI) $L = \text{cl}(\text{int} L)$, and for all $x \in \mathcal{M}S \cap \text{int} L$ there exists $\epsilon(x) > 0$ such that whenever $\varphi(t, x, u) \in \mathcal{M}S$ for some $t \geq 0$, $u \in U$ then $d(\varphi(t, x, u), \partial L) \geq \epsilon(x)$; there exists $\epsilon_0$ such that for all $x \in \text{cl} \mathcal{M}S$ and $u \in U$ we have that if

$$y = \lim_{k \to \infty} \varphi(t_k, x, u) \in \mathcal{M}S,$$

for some sequence $t_k \to \infty$, then $d(y, \partial L) \geq \epsilon_0$. Here $d$ denotes the (Riemannian) metric on the state space $M$. 
The next lemma characterizes $\mathcal{MS}(C_i, C_j)$ via domains of attraction of control sets.

Lemma 5.15

Consider the control system (2.2) on $L \subset \mathcal{M}$ with two invariant control sets $C_1$ and $C_2$, $L$ is compact forward invariant. Assume (SI). Then we have the following results.

(i) If $x \in \mathcal{MS}(C_1, C_2) \neq \emptyset$ then there is a control set $D \subset \text{clO}^+(x)$.

(ii) Assume that $D \prec C_i$, $i = 1, 2$, and $D$ is maximal with this property, i.e. $D \prec D'$ and $D' \prec C_i$, $i = 1, 2$, for a control set $D'$ implies $D = D'$. Then

$$A(D) = \mathcal{MS}(C_1, C_2)$$

Proof.

(i) See [12, Proposition 2.9].

(ii) See [12, Theorem 2.12, Corollary 2.10]


Theorem 5.16

Consider the control systems (2.2) on $L$ which is perfect and assume (SI). Let $C_1$ and $C_2$ be two distinct invariant control sets in $L$. and $D$ is the only control set in $L$ with $D \prec C_i$, $i = 1, 2$. Then there exist CLFs $\mathcal{F}_i : L \to [0, 1]$ such that

$$\mathcal{MS}(C_1, C_2) = (\mathcal{F}_i)^{-1}(1),$$

and $(\mathcal{F}_i)^{-1}(0) = C_i$, $i = 1, 2$. 

Figure 5.1: Multistability region of the chemical reactor model (3.8). This region is exactly the domain of attraction of $D_1, \mathcal{A}(D_1)$.

**Proof.** First we note that there is no other control set $\hat{D}$ such that $\hat{D} \subset D$. By Lemma 5.15(ii), $\mathcal{A}(D) = \mathcal{MS}(C_1, C_2)$. By Corollary 3.29 applied to $L$, there exist CLFs $\mathcal{F}_i$ such that $\mathcal{F}_i^{-1}(1) = \mathcal{A}(D)$ and $\mathcal{F}_i^{-1}(0) = C_i, i = 1, 2$. ■

**Example 5.17**

We consider again the chemical reactor model, which is perfect, see Example 3.32. We see that $L = [0, 7] \times [0, 1]$ is compact forward invariant. $D_1$ is the only control set in $L$ with $D_1 \subset D_i, i = 0, 2$. The multistability region is given by

$$\mathcal{MS} = \mathcal{MS}(D_0, D_2) = \mathcal{A}(D_1),$$

which is equal to $(\mathcal{F}_i)^{-1}(1), i = 0, 2$, see Figure 5.1.

**Stability of Control Structures**

In this section, we investigate the connection of stability of (chain) control sets and control Lyapunov functions, which is analogous to dynamical systems theory.
One of the major results states that a system of ordinary differential equations is asymptotically stable in a neighborhood of a compact set $K$ if and only if there exists a positive definite function on $K$ which is strictly decreasing on the solutions outside $K$. These are the Lyapunov direct method and the converse Lyapunov theorem.

In this section, we will give a sufficient condition for the stability of a (chain) control set via control Lyapunov functions. Our proof is basically following Tsinias' [32] with some modification. We consider the control system (2.2) in this section as usual.

We define the reachable map of the control systems (2.2). Let $\mathbb{R}_{+}$ be the set of nonnegative numbers.

**Definition 5.18**

Consider the control system (2.2). The reachable map $R: \mathbb{R}_{+} \times \mathcal{M} \to 2^{\mathcal{M}}$ is defined by

$$R(t, x) := \mathcal{O}^+_t(x).$$

Where

$$\mathcal{O}^+_t(x) := \{y \in \mathcal{M} : y = \varphi(t, x, u), \text{ for some } u \in \mathcal{U}\},$$

which is defined in section 2.2. $2^{\mathcal{M}}$ denotes the collection of all subsets of $\mathcal{M}$.

Tsinias discusses properties of reachable maps in the following lemma, whose proof can be seen in [32].
Lemma 5.19

Consider the reachable map $R$ of the control system (2.2).

(i) Let $K$ be a compact subset of $\mathcal{M}$ with $R(t,x) \not\subseteq K$, for some $x \in K$ and $t > 0$.

Then there exists a $t' < t$ such that $R(t',x) \cap \partial K \neq \emptyset$.

(ii) $R$ is transitive, i.e., $R(t_1 + t_2, x) = R(t_1, R(t_2, x))$, for any $t_1, t_2 \geq 0$, $x \in \mathcal{M}$.

Definition 5.20

A nonempty subset $K \subset \mathcal{M}$ is called stable, if for any neighborhood $O$ of $K$, there exists a neighborhood $W$ of $K$, such that $R(t,W) \subset O$, for all $t \geq 0$. Otherwise, $K$ is unstable. Here $R$ is the reachable map.

Theorem 5.21

Consider the system (2.2) with a bounded chain control set $E$. Assume there exists a lower semicontinuous CLF $\mathcal{F}$ on $W$, a neighborhood of $E$, such that

$$\mathcal{F}(x) > \mathcal{F}(E), \text{ for all } x \in W \setminus E.$$  

Then $E$ is stable.

Proof. Let $\epsilon > 0$ with $clB(E, \epsilon) \subset W$ and $m = \min\{\mathcal{F}(x) : x \in \partial B(E, \epsilon)\}$. Then $m > \mathcal{F}(E)$. Let $\mu : \mathcal{F}(E) < \mu < m$, and

$$O := \{x \in clB(E, \epsilon), \mathcal{F}(x) < \mu\}.$$  

We show that $R(t,O) \subset O$, for any $t \geq 0$.

If not, we may assume that $R(t,O) \not\subset O$, for some $t \geq 0$. Then by Lemma 5.19(i), we may assume that there exists a $y \in \partial O$ and $x \in O$ such that $y \in R(t', x)$, for some $0 < t' < t$. Let $y = \varphi(t', x, u)$, for some $u \in \mathcal{U}$. We may consider the following two cases.
(i) If \( x \in O \setminus E \), then \( \mathcal{F}(y) \leq \mathcal{F}(x) < \mu \). On the other hand, \( \mathcal{F}(y) = \mu \), for \( y \in \partial O \), a contradiction.

(ii) If \( x \in E \), then there is a \( z \in \partial E \) such that \( z = \varphi(t'', x, u) \), for some \( t'' \leq t' \), and \( \varphi(\tau, x, u) \in E \), for all \( \tau : 0 \leq \tau \leq t'' \). Now we have

\[
\mathcal{F}(y) = \mathcal{F}(\varphi(t', x, u)) \\
\leq \mathcal{F}(\varphi(t'', x, u)) \\
= \mathcal{F}(z) \\
= \mathcal{F}(E)
\]

This implies that \( \mathcal{F}(y) = \mathcal{F}(E) \), i.e., \( y \in E \). This is also a contradiction.

Now we reach a conclusion that \( R(t, O) \subset O \) and \( E \) is stable. ■

In chapter 3, we construct lower semicontinuous CLFs (or SCLFs) via Conley’s constructions. We combine these results from chapter 3 with Theorem 5.21 in the following corollary.

**Corollary 5.22**

Consider the control system (2.2) and assume (I). Then we have the following results.

(i) For \((2.2)_{A1}\), if \( E \) is a global attractor then \( E \) is stable.

(ii) For \((2.2)_{A2}\), \( E^* \) is unstable. If \( E \) is invariant then \( E \) is stable.

(iii) Assume that \( \mathcal{M} \) is compact. Then maximal chain control sets are stable and minimal ones are unstable.
Proof.

(i) This is a result from Theorem 3.20 (ii) and Theorem 5.21 with $W = \mathcal{A}^g(E)$.

(ii) This is a result from Theorem 3.28(i) (ii) and Theorem 5.21 with $W = \mathcal{A}^g(E)$.

(iii) This is a result from Corollary 3.29(i) (ii) and Theorem 5.21 with $W = \mathcal{A}^g(E)$. 

■
CHAPTER 6. CONCLUDING REMARKS

In this paper, we explore further relationship between two control structures, namely, control sets and chain control sets. We prove that generically each chain control set is exactly the closure of a control set if the state space is compact.

We have shown two systematic ways to construct control Lyapunov functions: Conley's Construction for affine control systems and Lyapunov spectrum method for bilinear control systems. CLFs are lower semicontinuous at least semi-globally for the first method. As for the second method, CLFs are globally lower semicontinuous. Conley's construction is a method adapting dynamical systems theory to control theory. On the other hand, Lyapunov spectrum method analyzes bilinear control systems via lower dimensional control systems (projected systems). There are yet several open problems.

(i) Under what conditions, these CLFs are continuous on the whole state space \( M \).

(ii) Is it possible to find CLFs numerically? We need to choose a proper control \( u' \in \mathcal{U} \) such that \( g_{u'}(r) \) (defined in chapter 3) is very closed to \( \sup_{u \in \mathcal{U}} g_u \). The first idea come to the author is to consider extreme points in the function space \( \mathcal{U} \).

(iii) Is every CLF constant on multistability regions? We prove that Conley's CLFs
are constant on these regions.

Several contributions of this paper include technical constructions of CLFs and global analysis of control systems. We know more about applications of topological dynamics to control systems theory. Further research is promising.
BIBLIOGRAPHY


