Fracture of an infinitely large elastic plate containing a curved crack

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Fracture of an infinitely large elastic plate containing a curved crack

by

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CHAPTER 1. INTRODUCTION

Overview

Fracture mechanics, a relatively new branch of applied mechanics, is being applied to assess the safety of engineering structures containing crack-like defects. Basically, this field of science seeks to quantify the relationship between material strength and crack size [11, 31, 51]. The analysis of crack problems in plane elasticity has intrigued applied mathematicians for over sixty years. The problem of a single crack in an infinite sheet was first solved by Inglis in 1913 [24] with the use of elliptic coordinates. Seven years later, the first quantitative investigation for the relationship between material strength and crack size was presented by Griffith in 1920 [21]. He formulated the fracture theory based on a simple energy balance. Since then, many mathematical approaches with wide ranges of sophistication have been developed and applied to a variety of crack configurations and loading conditions. One of the most notable work in the early stage of fracture mechanics is Irwin's contribution to extend the Griffith's approach to metals by including the energy dissipated by local plastic flow [25]. In the mean time, Orowan [44] independently proposed a similar modification to the Griffith theory. During the same period, Mott [41] studied a rapidly propagating crack by applying Griffith's theory.

Thereafter, the concept of energy release rate, an important progress in solving engineering problems, was proposed by Irwin in 1956 [26]. Shortly afterwards, Irwin [27] employed the approach by Westergaard [72] to show that the stresses and displacements in the vicinity of the crack tip can be described by a single constant, which was later known as the stress intensity factor $K$. During the same period of time, Williams [73] derived the essentially
identical solutions, showing that the stresses near the crack tip vary with $r^{-1/2}$ ($r$ is the radial distance from the crack tip), and singularity is present at the tip when $r$ approaches zero. The stress intensity factor $K$ provides a one parameter characterization of the stress field ahead of a sharp crack and the energy available to propagate the crack. Failure will occur as the stress intensity factor reaches its critical value, called the fracture toughness. Afterwards, the developed theories were rapidly applied in the engineering industry. For example, Wells [70] successfully applied fracture mechanics to show that fuselage failure in Comet jet aircraft resulted from fatigue cracks, caused by weak reinforcement. Another example is that Winne and Wundt [74] studied the failure of large rotors by applying the approach of energy release rate and successfully predict the bursting behavior of large disks. Recently, it has been estimated [30] that the applications of fracture mechanics saved over £50 million in the construction of the Forties Field oil production platforms in the North Sea. Also, it was reported [65] that £16 million was saved by proving a cracked rotor could safely run.

From 1960 to 1961, researchers such as Irwin [28], Barenblatt [5], and Wells [71] developed models to account for the plastic zone ahead of the crack tip. It was proposed by Wells [71] that as significant plasticity precedes failure, the displacements of the crack edges can be alternatively used as a fracture criteria. Under the idealized condition that the plastic deformation flow was modelled by nonlinear elastic deformation, Rice [49] developed another parameter used to characterize the nonlinear elastic behavior of materials. It was showed that the nonlinear energy release rate can be expressed as a line integral, which is independent of the path of integral. Previously, Esheby [17] developed several integrals, similar to what Rice obtained. However, he did not apply them to the crack problems.

Later, this line integral, called the $J$ integral, was related to the stress fields near the crack tip by Hutchinson [23] and Rice and Rosengren [50]. It was shown that the $J$ integral can be viewed as an energy release rate. Reviewing Rice's article, Begley and Landes [6] characterized the fracture toughness of steels with the $J$ integral. Ten years later, they
successfully set up the standard procedures to experimentally test the $J$ integral of metals [7].

Thereafter, in order to apply the concepts of fracture mechanics to design, Shih and Hutchinson [55] developed the theory to analyze the fracture design based upon the $J$ integral. Later, a handbook [35], based upon the methodology, is published by the Electric Power Research Institute.

Recently, this field of fracture mechanics has almost reached its matured stage. The related solution techniques span a wide range of topics such as analytic function theory, integral transforms, boundary collocation, finite difference, finite elements, boundary elements, asymptotic method, etc. These diverse techniques have been applied in a variety of crack configurations and loading conditions. Muskhelishvili's contribution to the development of the complex variable scheme has undoubtedly been a remarkable achievement in solving plane crack problems. As a matter of fact, the complex representation of the general solution to the problems of plane elasticity was originally introduced by Kolosov [34]. Although several authors have obtained several similar complex representations of the solutions before Kolosov, Kolosov was the first one who systematically applied them. For the application of complex representations to crack problems, Muskhelishvili [42] was the first one who efficiently made use of conformal mapping and Cauchy's formulae to solve the problems. One of Muskhelisvili's most valuable contributions was his introducing the concept of "linear relationship". With the concept, mathematical singularities due to geometry or loading can usually be anticipated.

**Background of Linear Elastic Fracture Mechanics**

Since the foundation of the work in this dissertation is laid upon linear elastic fracture mechanics (LEFM), the basic principals of the theory are reviewed here as an introduction to the methodology employed to investigate the crack problems later. To understand the basic principals of this theory, one has to retrieve back to Griffith's theory [21], based upon an energy...
balance argument. For details of this theory, one can refer to the book by Anderson [2]. Alternatively, the other approach to LEFM is based on the singular nature of the stress distribution near a crack tip, which will be briefly described in the following.

It was shown by Irwin [27] that the stress field in the region dominated by the singularity of stress can be regarded as the sum of three invariant stress patterns (Figure 1.1) as follows:

\[ \begin{align*}
\text{Mode I - Opening mode: Crack surface displacements are normal to the crack plane.} \\
\text{Mode II - Forward shear mode: The in-plane shear takes place, and the crack surfaces tend to slide over each other in the plane of the plate.} \\
\text{Mode III - Anti-plane shear mode - The out-of-plane shear takes place, and the crack surfaces tend to slide over each other in the normal direction of the plate. Following the conventional notation for all stress components illustrated in Figure 1.2, the singular stress fields and} \\
\end{align*} \]
associated displacements in the vicinity of the crack tip can be written as follows:

For mode I, the stress field near the crack tip can be described by

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \begin{bmatrix}
\cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \\
\cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \\
\cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}
\end{bmatrix}
\]

(1.1)

\[
\sigma_{zz} = \begin{cases} 
\nu(\sigma_{xx} + \sigma_{yy}), & \text{plane strain} \\
0, & \text{plane stress}
\end{cases}
\]

where \(r\) and \(\theta\) are shown in Figure 1.2, and \(\nu\) is the Poisson's ratio.

Figure 1.2: Stress and displacement components in the vicinity of a crack tip
The displacement components $u$, $v$, and $w$ at a certain point near the crack tip is given by

$$
\begin{bmatrix}
  u \\
  v
\end{bmatrix} = \frac{K_I}{8G\sqrt{2\pi r}} \begin{bmatrix}
  (2K - 1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \\
  (2K + 1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}
\end{bmatrix}
$$

(1.2)

$$
\begin{align*}
  w &= \begin{cases} 
    0 & , \text{plane strain} \\
    -\frac{v}{E} \int (\sigma_{xx} + \sigma_{yy}) \, dz & , \text{plane stress}
  \end{cases}
\end{align*}
$$

where $G$ and $E$ are the shear modulus and modulus of elasticity respectively, and $K$ is an elasticity constant, defined by

$$
K = \begin{cases} 
  (3 - 4v) & , \text{plane strain} \\
  \frac{3 - v}{1 - v} & , \text{plane stress}
\end{cases}
$$

(1.3)

For mode II, the stress components at an arbitrary point near the crack tip are written as

$$
\begin{bmatrix}
  \sigma_{xx} \\
  \sigma_{yy} \\
  \sigma_{xy}
\end{bmatrix} = \frac{K_{II}}{\sqrt{2\pi r}} \begin{bmatrix}
  \sin\frac{\theta}{2} \left( 2 + \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \right) \\
  \cos\frac{\theta}{2} \left( 1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right) \\
  -\sin\frac{\theta}{2} \left( 2 + \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \right)
\end{bmatrix}
$$

(1.4)

$$
\begin{align*}
  \sigma_{zz} &= \begin{cases} 
    v(\sigma_{xx} + \sigma_{yy}) & , \text{plane strain} \\
    0 & , \text{plane stress}
  \end{cases}
\end{align*}
$$
and the near field displacements are described by

\[
\begin{bmatrix}
    u \\
    v \\
    w
\end{bmatrix} = \frac{K_{II}}{8G} \sqrt{\frac{2r}{\pi}} \begin{bmatrix}
    (2\kappa + 3)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \\
    - (2\kappa - 3)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}
\end{bmatrix}
\]

\[w = \begin{cases} 
    0, \text{ plane strain} \\
    -\frac{v}{E} \int (\sigma_{xx} + \sigma_{yy})dz, \text{ plane stress}
\end{cases} \tag{1.5}\]

For mode III, the near field stress and displacement components are written as

\[
\begin{bmatrix}
    \sigma_{xx} \\
    \sigma_{yz}
\end{bmatrix} = \frac{K_{III}}{\sqrt{2\pi r}} \begin{bmatrix}
    -\sin\frac{\theta}{2} \\
    \cos\frac{\theta}{2}
\end{bmatrix}, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0
\]

\[u = v = 0, \quad w = \frac{K_{III}}{G} \sqrt{\frac{2r}{\pi}} \sin\theta \tag{1.6}\]

The parameters \(K_h, K_{II}\) and \(K_{III}\) are called the stress intensity factors (abbreviated to \(K\)-factors in this dissertation). The \(K\)-factor can be interpreted as the singularity intensity of the inverse square root stress at the crack tip, and it follows that the \(K\)-factor is independent of \(r\) and \(\theta\). Obviously, from equations (1.1) \(\sim\) (1.6), it can be seen that the \(K\)-factors must be a function of applied loads and crack size from dimensional considerations. The \(K\)-factors for a single crack with different shapes can be expressed in a general form,

\[K_I = GF_1 \cdot \sigma \sqrt{\pi a}, \quad K_{II} = GF_2 \cdot \sigma \sqrt{\pi a}, \quad K_{III} = GF_3 \cdot \sigma \sqrt{\pi a} \tag{1.7}\]
where $a$ is the half distance between the crack tips, and $GF_1$, $GF_2$, and $GF_3$ are geometric functions corresponding to the applied load $\sigma$ for the mode $I$, $II$ and $III$, respectively.

More details of the theory of LEFM can be seen in the reference [2].

**Present Work**

So far, most of the work done for the through-thickness crack problem is treating the crack as a perfectly straight cut. The only analytical solution to the extension problem of a plate containing a circular arc crack (abbreviated to CAC in this dissertation), was given by Muskhelishvili [42]. The CAC problem was solved by use of an elegant analytic continuation concept he introduced, which leads to the formulation in terms of a Hilbert problem. Later, the same technique was employed by Perlman and Sih [45] to investigate the bending problem of a bi-material plate containing a CAC.

The main object of this dissertation is to investigate the fracture of an elastic plate containing a circular arc crack or an elliptical crack using the complex variable method, a different way of approach from the scheme employed by Muskhelishvili, as well as the conformal mapping technique by Kolosov [34] and Muskhelishvili [42]. A new conformal mapping is devised to transform the contour surface of a circular arc crack or an elliptical crack into a unit circle. With the assistance of the proposed mapping, Cauchy integrals are taken for each term in the corresponding boundary equation. Consequently, the complex stress functions for a general loading condition can be obtained. Thus, by using one of the complex stress functions, the $K$-factors corresponding to a specific loading condition can be also obtained.

Additionally, the interaction between both tips of the crack tips for specific cases is also studied, which exhibits different characteristics from those of a pair of collinear cracks.
The main feature of the obtained solution is that without any limiting process, it can be continuously reduced to the particular case of the straight crack problem simply by setting a mapping variable equal zero. This continuous transition gives accurate determination of the effects that the curvature has on the stress intensity factors for a shallow arc crack.

The dissertation is organized as follows:

Chapter 2 describes all basic equations and the theory of the complex variable scheme, which will be employed to investigate the crack problems later. In this chapter, the concepts leading to an important boundary equation are briefly described. In addition, basic principals of the conformal mapping technique, assistance in solving the boundary equation, is also introduced.

Chapter 3 gives the solution to the fracture problem of a plate weaken by a circular arc crack with all-round tension. In this chapter, a curved crack modelled by a circular arc is subdivided into two categories - a shallow arc crack and a deep arc crack. For a shallow arc crack, the appropriateness of the circular arc model is studied. For a deep arc crack, the interaction between the crack tips is investigated.

Chapter 4 gives the solution to the fracture problem of a CAC with some general loading conditions at infinity. Also, as comparison to the particular case investigated in Chapter 3, the interaction is also studied for the special case when uniform shearing stress acts upon the whole crack edge. Additionally, the approximate solution of the in-plane bending of a large beam containing a CAC is also given.

Chapter 5 gives the solution to the CAC problem for concentrated loads applied at an arbitrary point on the crack edge. This solution is formulated in an independent chapter because the obtained functions can be taken as the Green's function to formulate many other problems with sophisticated loading conditions. In the chapter, to approach the concentrated loading problem, the problem is solved that only part of the crack edge is subjected to uniform normal and shearing stress.
Chapter 6 gives the solution to the CAC problem for general transverse flexural loads applied at infinity. By changing a parameter value, the obtained solution can be reduced to consider the rigid inclusion problem. The formulations for the solution are given to consider a wide range of general flexural loads, including bending, twisting, and out-of-plane shearing.

Chapter 7 gives the $K$-factors of an elliptical crack. The original mapping for the CAC is modified by stretching out the $y$ axis to transform the contour surface of an elliptical crack to a unit circle. Similarly, the particular case when uniform traction act upon the whole crack edge are studied. An example of semi-elliptical crack is given at the end.

Chapter 8 gives the general discussions about the advantage of using the present approach to solve the CAC problem and some observations for the obtained results. Besides, the possible future work is also suggested at the end of this chapter.

Appendix A provides the proof that the mapping function implemented in Chapter 3, 4, 5 and 6 transforms the contour surface of a CAC to a unit circle.

Appendix B provides all relevant functions needed for determination of stress and displacement components for the CAC problem in Chapter 4.

Appendix C provides all relevant functions needed for determination of stress and displacement components for the CAC problem in Chapter 5.

Appendix D provides all relevant functions needed for determination of stress and displacement components for the CAC problem in Chapter 6.

Appendix E provides the proof that the mapping function implemented in Chapter 7 transforms the contour surface of an elliptical crack to a unit circle.

**General Notation**

All of the notations which will be used throughout this dissertation are listed as follows:

\[ \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \] = stress components in the $x$-$y$ plane
\( \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy} \) = strain components in the \( x-y \) plane

\( u, v, w \) = displacement components in the \( x, y, z \) direction

\( G \) = shear modulus

\( \nu \) = Poisson's ratio

\( E \) = modulus of elasticity

\( \kappa = (3-4\nu) \) for plain strain, \( (3-\nu)/(1+\nu) \) for plain stress

\( N = -(3+\nu)/(1-\nu) \)

\( h \) = plate thickness

\( D = Eh^3/[12(1-\nu^2)] \) (flexural rigidity of a plate)

\( z = x + iy \) (coordinate system for a curved crack)

\( z_0 = x_0 + iy_0 \) (coordinate system for applied loads)

\( \omega(\zeta) \) = mapping function

\( a \) = one half of the distance between the crack tips

\( b \) = geometric variable for an ellipse

\( c \) = mapping variable

\( 2\alpha \) = subtending angle of a circular arc crack

\( \alpha_0 \) = leading angle of \( z_0 \) with respect to \( z \)

\( r_0 \) = radius of a circular crack

\( \theta, \theta_0 \) = reference angle of an arbitrary point on the unit circle, measured from the reference line in counterclockwise direction

\( \eta \) = deviation angle

\( R = a/2\sqrt{1-c^2} \)

\( V = b/2\sqrt{1-c^2} \)
$\Re = \frac{b}{a}$ (aspect ratio of an ellipse)

$\Gamma = \text{unit circle}$

$s = \text{boundary surface of a contour}$

$s^* = \text{finite region in a closed contour}$

$s^\infty = \text{finite region outside a closed contour}$

$S_\zeta = \text{unbounded domain in the } \zeta\text{-plane}$

$S_z = \text{unbounded domain in the } z\text{-plane}$

$\xi = \text{boundary value of } \zeta \text{ on the unit circle}$

$z_s = \text{coordinate of the boundary surface in the } z\text{-plane}$

$N_n, N_t = \text{normal and tangential traction applied on the boundary surface}$

$X_n, Y_n = \text{components of the applied traction in the } x, y \text{ direction}$

$X, Y = \text{concentrated forces in the } x, y \text{ direction}$

$\phi, \psi = \text{complex stress functions}$

$P, Q = \text{normal and tangential traction}$

$P', Q' = \text{concentrated normal and tangential force}$

$T_{x_0}, T_{y_0} = \text{tension in the } x_0, y_0 \text{ direction}$

$S_0 = \text{constant shearing stress}$

$M_0 = \text{constant bending moment}$

$I = \text{moment of inertia of a beam}$

$m(s), q(s) = \text{prescribed out-of-plane bending moment and shearing stress along the crack edge}$

$f(\xi) = \text{loading integration function for a boundary surface}$

$\mathcal{F}(\zeta) \text{ or } \mathcal{F}_1(\zeta) = \text{Cauchy integral of } f(\xi)$

$\mathcal{F}_2(\zeta) = \text{Cauchy integral of } \overline{f(\xi)}$

$\mathcal{F}_3 = \text{Cauchy integral of } f(\xi) \text{ for concentrated loads}$
\[ \mathcal{F}_2 \text{ Cauchy integral of } \mathcal{F}(\xi) \text{ for concentrated loads} \]

\[ g(\xi) = 2G(\nu + i \nu) \]

\[ K_I, K_{II}, K_{III} \text{ stress intensity factors of fracture mode } I, II, III \]

\[ GF_1, GF_2, GF_3 \text{ geometric functions for fracture mode } I, II, III \]

\[ NK_1, NK_2 \text{ normalized stress intensity factors for fracture mode } I, II \]
CHAPTER 2. BASIC EQUATIONS AND THEORY REVIEW

Basically, this chapter is to review the principal of the plane theory of elasticity and the concepts of using complex stress functions to solve the governing equations corresponding to the theory. Along with this complex variable scheme, the method of conformal mapping becomes a powerful tool in dealing with the boundary value problem. The method of conformal mapping, the foundation of the present work in the thesis, is also reviewed in this chapter.

In this chapter, all notations are to be used throughout the thesis, except where noted.

Basic Equations

Elasticity equation

As is well known, the plane theory of elasticity is generally catalogued into two cases of equilibrium of elastic bodies which are of considerable interest in practice; one is the case of plane stress and the other is the case of plane strain. For a general three dimensional case, there will be nine stress components (Figure 2.1) relative to the Cartesian coordinate tabulated as follows:

\[
\vec{T} = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix}
\] (2.1)

where \( \vec{T} \) symbolically represents the stress tensor, and \( \sigma_{xy} \), for example, denotes the stress
component on the x-plane in the direction of \( y \). As a result of summation of moments about the axis \((x, y, z)\), it can be easily shown that

\[
\sigma_{yz} = \sigma_{zy}, \quad \sigma_{zx} = \sigma_{xz}, \quad \sigma_{xy} = \sigma_{yx}
\]  

Therefore, the stress tensor \( \mathbf{T} \) turns out to be symmetrical and the number of stress components is reduced to six. Although arbitrary values can be assigned to the six stress components at a point, the variation of these stresses from one point to another is restricted by the status of equilibrium. By summation of the total force in all direction, it can be easily proved that the generalized equilibrium equation can be expressed as

\[
\frac{\partial \sigma_{ij}}{\partial x_i} + g_j = 0 \tag{2.3}
\]

where the convention of tensor for the index \( i \) and \( j \) is used to denote coordinate variables \( x \), \( y \), and \( z \), and \( g \) is the body force, which is neglected in this dissertation.
All the strain components can be linearly related to the total stresses by the following six equations known as the elasticity equations.

\[
E \xi_{xx} = \sigma_{xx} - v(\sigma_{yy} + \sigma_{zz}), \quad \xi_{yy} = \sigma_{yy} / 2G \\
E \xi_{yy} = \sigma_{yy} - v(\sigma_{xx} + \sigma_{zz}), \quad \xi_{zz} = \sigma_{zz} / 2G \\
E \xi_{zz} = \sigma_{zz} - v(\sigma_{xx} + \sigma_{yy}), \quad \xi_{xy} = \sigma_{xy} / 2G
\]

(2.4)

where \( E \) is the modulus of elasticity, \( v \) is the Poisson's ratio, and \( G \) is the shear modulus, defined by \( G = E/(1+v) \).

Compatibility of strain

Though the six strain components on a particular element can be arbitrary, the strains in adjoining elements should be restricted in some manner so that all values are compatible with a continuous distribution of the three displacement components \( u, v, w \) in the \( x, y \) and \( z \) direction, respectively. In the small displacement theory for the \( x-y \) plane while quadratic terms are discarded, the strain components can be expressed as

\[
\xi_{xx} = \frac{\partial u}{\partial x}, \quad \xi_{yy} = \frac{\partial v}{\partial y}, \quad \xi_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\]

(2.5)

These strain components can be differentiated to give

\[
\frac{\partial^2 \xi_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}, \quad \frac{\partial^2 \xi_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}, \quad 2 \frac{\partial^2 \xi_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}
\]

(2.6)

In equations (2.6), the displacement \( u \) and \( v \) can be eliminated to achieve the equation of strain compatibility, written as
By permutating the x-y-z coordinate variables, one can obtain two more similar equations. Similarly, it can be shown that three further equations of a different kind still need to be satisfied. However, equations of this second set are identically satisfied in cases of plain stress and plain strain. Otherwise, all compatibility equations must be taken into account.

Plane stress

In a large group of practical problems of engineering significance, certain approximations may be applied to simplify the case of general three dimensional problems. For the case when the thickness \( h \) of the flat plate is very small as compared with the dimensional size of the plate, the lateral stresses throughout the plate thickness (along the direction of z-axis) are small enough to be neglected. For the plane stress problem, it may be assumed that throughout the plate. Furthermore, it can be also assumed that other stresses \( \sigma_{xx}, \sigma_{yy}, \) and \( \sigma_{xy} \) are independent of the variable \( z \). With these approximation, this idealized state of plane stress actually simplifies the elasticity equation (2.4) into

\[
\frac{\partial^2 \xi_{xx}}{\partial y^2} + \frac{\partial^2 \xi_{yy}}{\partial x^2} - 2 \frac{\partial^2 \xi_{xy}}{\partial x \partial y} = 0
\]  

(2.7)

After the above expressions (2.9) are substituted into the compatibility equation (2.7), one can
readily obtain the differential equation restricting the stress components, written as

\[
\frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{xx}}{\partial y^2} - 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} - \nu \frac{\partial}{\partial x} \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right) - \nu \frac{\partial}{\partial y} \left( \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} \right) = 0
\]  

(2.10)

Inserting the equilibrium equation (2.3) allows the equation (2.10) be reduced to

\[
\frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{xx}}{\partial y^2} - 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0
\]  

(2.11)

Along with the equilibrium equation, the above equation (2.11) restricts admissible stress distribution in case of the plane stress.

**Plane strain**

Alternatively, if the deformation of an element in a body is assumed to happen only in the x-y plane, one may obtain the state of plane strain, which has the following condition

\[
\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0
\]  

(2.12)

As the condition of zero strain along the z-axis is inserted into the elasticity equation (2.4), the longitudinal stress \( \sigma_{zz} \) can be rewritten as

\[
\sigma_{zz} = \nu (\sigma_{xx} + \sigma_{yy})
\]  

(2.13)

When this expression of stress along the z-axis is substituted back into the remaining three elasticity equations, it can lead to modified expressions for strain, written as

\[
E' \varepsilon'_{xx} = \sigma_{xx} - \nu' \sigma_{yy} \quad , \quad E' \varepsilon'_{yy} = \sigma_{yy} - \nu' \sigma_{xx} \quad , \quad E' \varepsilon'_{xy} = (1 + \nu') \sigma_{xy}
\]  

(2.14)
where \( E' = E / (1 - \nu^2) \) and \( \nu' = \nu / (1 - \nu) \). As the modified expressions in (2.14) are inserted back into the strain compatibility equation, all of these new elastic constants are gone and the final expressions appear to be identical with the equations (2.3) and (2.11). Therefore, for both of the plane stress and the plane strain, the same boundary stresses will lead to identical distribution of internal stresses. However, the distribution of strain and displacement are not the same for the both cases.

**Complex Variable Method**

Either for the case of plane stress or for the case of plane strain, it is feasible to express the stress components as a function of \( x \) and \( y \) in such a way that the equilibrium equation (2.3) and the compatibility equation (2.11) are satisfied for all internal points. Since all three stress components \( \sigma_{xx}, \sigma_{yy}, \) and \( \sigma_{xy} \) need to be identified with derivatives of some scalar variables, three scalars \( A, B, \) and \( C \) may be introduced to give general expressions for these three stress components written as

\[
\sigma_{xx} = \frac{\partial^2 A}{\partial x^2} - 2 \frac{\partial^2 B}{\partial x \partial y} + \frac{\partial^2 C}{\partial y^2}
\]

\[
\sigma_{xy} = \frac{\partial^2 A}{\partial y^2} + 2 \frac{\partial^2 B}{\partial x \partial y} + \frac{\partial^2 C}{\partial x^2}
\]

\[
\sigma_{xy} = \frac{\partial^2 A}{\partial x \partial y} - \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right) B - \frac{\partial^2 C}{\partial y^2}
\]

(2.15)

Particularly, if the scalars \( A \) and \( B \) are chosen to be zero, the reduced expressions can be inserted into the stress compatibility equation (2.11), which gives rise to the following biharmonic equation,
In equation (2.16), the function \( C \) is well known as the Airy stress function. On solving the biharmonic equation (2.16) from its boundary condition, one may determine all the stress components through (2.15). However, with \( C \) set equal zero, the stress components can be substituted into the differential equations (2.3) and (2.11) to give the following equations restricting \( A \) and \( B \),

\[
\frac{\partial^4 C}{\partial x^4} + 2 \frac{\partial^4 C}{\partial x^2 \partial y^2} + \frac{\partial^4 C}{\partial y^4} = 0
\]

(2.16)

Since equation (2.17) happens to satisfy the Cauchy-Riemann condition for a function, denoted by \( \mathcal{D} \), having real and imaginary parts \( \nabla^2 A \) and \( \nabla^2 B \) respectively, it can be solved by introducing two analytic functions such that

\[
A + iB = z \phi + \chi
\]

(2.18)

which can be verified by differentiating equation (2.18) to obtain

\[
\mathcal{D} = 4 \phi'(z) + 2 \chi''(z)
\]

(2.19)

which is analytic and therefore satisfies equation (2.17). Obviously, it is advantageous to implement this formulation to solve the problems of plane elasticity since arbitrary analytic functions of \( z \) (equal to \( x + iy \)) can be chosen as long as they can satisfy boundary conditions, whereas admissible real functions \( C \) are restricted by the biharmonic equation.

Substituting the expression (2.18) into equations (2.15) with \( C \) particularly set equal zero and making some operations for complex variables, one may readily prove that all the
stress components can be written as

\[
\sigma_{xx} = 2 \text{Re} \{ \phi' \} - \text{Re} \{ z \phi'' + \psi' \} \\
\sigma_{yy} = 2 \text{Re} \{ \phi' \} + \text{Re} \{ z \phi'' + \psi' \} \\
\sigma_{xy} = \text{Im} \{ z \phi'' + \psi' \}
\]

(2.20)

where \( \text{Re} \) and \( \text{Im} \) represent real and imaginary parts respectively, and a new function \( \psi \) is introduced and defined by \( \psi(z) = \chi'(z) \). On substituting these expressions (2.20) for the stress components into the elasticity equations (2.9) and using the definition of strain (2.5), it follows that the displacement components can be expressed as

\[
u + i v = \frac{(K(j) - z(j)' - i\psi)}{2G}
\]

(2.21)

where for the case of plane stress, \( \kappa \) is written as \((3-v)/(1+v)\). For the case of plane strain, elastic constants should be modified by replacing \( E \) by \( E' \) and \( v \) by \( v' \). As long as these complex stress functions \( \phi \) and \( \psi \) of an arbitrary point in a domain is found for specified boundary conditions, all of the stress and displacement components can be directly obtained through equations (2.20) and (2.21).

**Boundary equation**

In solving the problem with an annulus boundary, it is feasible to choose certain terms of a stress function in a series form to fit them in specified boundary conditions. However, a way of calculating the stress function from boundary stresses is needed for an irregular boundary, which does not consist of terms of regular series. In favor of dealing with an irregular boundary, transformation of the expressions for stresses (2.20) to those in the polar coordinate can be made using.
Thus, bringing in the stress functions for all stress components, one can obtain individual stress component in the polar coordinate, written as

\[ \sigma_{rr} = 2 \text{Re} \{ \phi' \} - \text{Re} \{ e^{2i\theta} (z \phi'' + \psi') \} \]

\[ \sigma_{\theta\theta} = 2 \text{Re} \{ \phi' \} + \text{Re} \{ e^{2i\theta} (z \phi'' + \psi') \} \]

\[ \sigma_{r\theta} = \text{Im} \{ e^{2i\theta} (z \phi'' + \psi') \} \]  

Suppose tractions \( N_n \) and \( N_t \) are applied on a segment of the boundary with its outward normal inclined at an angle \( \eta \) from the x-axis (Figure 2.2). In Figure 2.2, the line, having an angle \( \eta \) from the x-axis, is particularly drawn to pass through the origin of the coordinate, but this is not necessary for a general case. Since these applied tractions are identical with the stresses \( \sigma_{rr} \) and \( \sigma_{r\theta} \) on the contour surface, the traction can be expressed as

![Figure 2.2: Traction on a boundary](image-url)
\[ N_n + iN_t = \phi' + \psi' - e^{-2i\eta}(z\overline{\phi''} + \overline{\psi'}) \]  
(2.24)

Let the path of the boundary be denoted by \( z \) and a segment along the contour surface has a normal which makes an angle \( \eta \) with the x-axis. Since the derivative of an analytic function must be uniquely valued regardless of the direction in which the point is approached, the derivative of an analytic function \( \Omega \) can be evaluated by

\[
\frac{d\Omega}{dz_s} = \frac{\partial x}{\partial z_s} \frac{\partial \Omega}{\partial x} + \frac{\partial y}{\partial z_s} \frac{\partial \Omega}{\partial y} 
\]  
(2.25)

The incremental length \( dz_s \) is given by

\[
dz_s = dx + idy = ds e^{i(\eta + \pi/2)} 
\]  
(2.26)

By use of equation (2.26), the analytic function \( \Omega \) is differentiated and eventually written as

\[
\frac{d\Omega}{dz_s} = ie^{-i\eta} \left( \frac{\partial \Omega}{\partial x} \sin \eta - \frac{\partial \Omega}{\partial y} \cos \eta \right) 
\]  
(2.27)

If this analytic function \( \Omega \), defined by \( \Omega = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \), is differentiated with respect to the incremental length \( dz_s \), it immediately follows that \( d\Omega/dz_s \) is expressed as

\[
\frac{d\Omega}{dz_s} = \phi' + \overline{\phi'} - e^{-2i\eta}(z\overline{\phi''} + \overline{\psi'}) 
\]  
(2.28)

As referring back to equation (2.24), one may see that this analytic function \( \Omega \) is nothing but the integration of the applied traction along the contour surface. Consequently, it gives rise to the following important equation,
\[ \phi(z_s) + \overline{z_s} \phi(z_s) + \overline{\psi(z_s)} = \int_s (N_n + i N_t) \, dz_s \] (2.29)

The integral on the right hand side of equation (2.29), denoted by \( \beta \), is called the loading integration function throughout the rest of this dissertation. Equation (2.29) shows that as long as boundary stresses are specified along the contour surface \( s \), the combination of stress functions \( \Omega \) at an arbitrary point of \( s \) can be directly obtained through integration of the stresses along the boundary. If the specified tractions are written in terms of \( x \) and \( y \) components \( (X_n \) and \( Y_n \) ) instead of normal and tangential stresses, it is more convenient to rewrite the right hand side of equation (2.29) as

\[ \int_s (N_n + i N_t) \, dz_s = i \int_s (X_n + i Y_n) \, ds \] (2.30)

Equation (2.30) can be easily proved by

\[ i \,(X_n + i Y_n) \, ds = [i(N_n \cos \eta - N_t \sin \eta) - (N_n \sin \eta + N_t \cos \eta)] \, ds \]

\[ = ds \, e^{i \eta} \, (iN_n - N_t) \]

\[ = -i \, dz_s \, (iN_n - N_t) \]

\[ = (N_n + i N_t) \, dz_s \] (2.31)

Mathematically speaking, the integration of stresses should yield a constant. However, due to the fact that the addition of any constant will not affect the stress and displacement fields given by equations (2.20) and (2.21), this constant can be omitted.

The boundary equation (2.29) is constructed to solve the first fundamental problem where external stresses are specified along the boundary. The second fundamental problem type is that only displacements of the points along the boundary are given. Completely from
the specified displacements on the boundary, the state of stresses is determined and equation (2.21) can serve as the boundary equation for the second fundamental problem.

From the boundary equations (2.21) and (2.29), it is seen that only particular values of the stress functions on the boundary can be obtained. Further arguments and a special way needed to reconstruct the functions for all points in the domain are discussed next.

Cauchy Integral

Basic properties

Considering the special case when continuously distributed loads on the crack surface are assumed, one may express the stress function in terms of distance along the boundary. Since this stress function must be analytic inside its domain, the function can be deduced for all interior points simply by inspecting its boundary value. However, if the distributed loads are not continuous, the Cauchy integral approach needs to be implemented for relating the general expression of the stress function to its boundary value.

Suppose an domain comprising of a closed loop $s$, as shown in Figure 2.3 (a) and the boundary value of an analytic function $g$ is known. The Cauchy integral theorem states that the general expression for the function $g$ in this domain can be written as

$$ g(z) = \frac{1}{2 \pi i} \int_s \frac{g(z_s) dz_s}{z_s - z} \quad (2.32) $$

in which $g(z_s)$ is the boundary value of the function $g(z)$ and the integral on the right hand side of equation (2.32) is called "Cauchy Integral". The condition for equation (2.32) to be valid is that there must be no pole (singular point) occurring inside the domain of this closed contour. Suppose this condition holds, and then the analytic function $g(z_s)$ can be expressed in the
As the equation (2.33) is inserted into equation (2.32), the terms with positive power can be further written as

\[ \frac{A_n z_s^n}{z_s - z} = A_n \left( z_s^{n_1} + z_s^{n_2} z + \ldots + z_s^{n_l} + \frac{z^n}{z_s - z} \right) \]  

(2.34)

The integration of all terms around the closed contour turn out to be zero except the last term, which gives \( 2\pi i A \). Therefore, as the function \( g(z) \) appears to be analytic within the closed contour, integration of its Cauchy Integral results in the same function with its boundary value \( z \), replaced by the variable \( z \) within its domain.

If the domain is extended to infinity where the function \( g \) is assumed to vanish, the
Cauchy integral of a closed contour inside the domain is

\[ g(z) = -\frac{1}{2\pi i} \int_{C} \frac{g(z_s) dz_s}{z_s - z} \]  
(2.35)

To prove equation (2.35), a closed contour \( z_o \) is drawn to enclose the boundary surface \( s \) as shown in figure 2.3 (b). These two contour surfaces can be so connected that they form a single contour which makes a counterclockwise circuit. When equation (2.32) is applied to this composite contour, it follows that the function \( g(z) \)

\[ g(z) = \frac{1}{2\pi i} \left( \int_{C} \frac{g(z_o) dz_o}{z_o - z} - \int_{s} \frac{g(z_s) dz_s}{z_s - z} \right) \]  
(2.36)

Now, the outer boundary is allowed to expand to infinity. If the function \( g(z) \) vanishes at infinity, the function \( g(z_o) \) should approach zero as the boundary \( z_o \) is at a very large distance from the origin. Eventually, for evaluating the Cauchy Integral of an infinite sheet containing a hole of boundary \( s \), the theorem gives equation (2.35).

**Elementary formulae**

In preceding section, the condition of no singular point occurring in the domain is assumed and additionally the function \( g \) is let vanish at infinity. Herein, a couple of more general formulae will be given to facilitate calculations in many case. The proof of these formulae can be referred to the book by Muskhelishvili [42] and will not be reviewed here.

Let \( s^* \) denote the finite part of the plane bounded by a contour surface and \( s' \) denote the infinite plane outside this contour. The positive direction of integration along the boundary is so defined that \( s^* \) lies on the left. Now it follows that:

(1) Let the function \( g(z) \) be holomorphic in \( s^* \). Then, one can obtain
\[
\frac{1}{2\pi i} \int_{s} \frac{g(z_s)dz_s}{z_s - z} = g(z) \quad , \quad z \in s^+
\]

(2.37)

\[
\frac{1}{2\pi i} \int_{s} \frac{g(z_s)dz_s}{z_s - z} = 0 \quad , \quad z \in s^-
\]

(2.38)

(2) Let the function \(g(z)\) be holomorphic in \(s^\prime\) including the point at infinity. Then, one obtains

\[
\frac{1}{2\pi i} \int_{s} \frac{g(z_s)dz_s}{z_s - z} = -g(z) + g(\infty) \quad , \quad z \in s^-
\]

(2.39)

(3) Let the function \(g(z)\) be holomorphic in \(s^\prime\) except the points \(a_1, a_2, \ldots, a_n\) where it has poles with the principal parts \(G_1(z), G_2(z), \ldots, G_n(z)\). Then, one can obtain

\[
\frac{1}{2\pi i} \int_{s} \frac{g(z_s)dz_s}{z_s - z} = g(z) - G_1(z) - G_2(z) - \ldots - G_n(z) \quad , \quad z \in s^+
\]

(2.40)

(4) Let the function \(g(z)\) be holomorphic in \(s^\prime\) except the points \(a_1, a_2, \ldots, a_n\) and also the point \(\infty\) where it has poles with the principal parts \(G_1(z), G_2(z), \ldots, G_n(z), G_{\infty}(z)\). Then, one obtains

\[
\frac{1}{2\pi i} \int_{s} \frac{g(z_s)dz_s}{z_s - z} = -g(z) + G_1(z) + G_2(z) + \ldots + G_n(z) + G_{\infty}(z) \quad , \quad z \in s^-
\]

(2.41)
Transformation of Boundary Equation

For the purpose of evaluating Cauchy integrals, it is convenient to transform a non-circular boundary contour into a circle. Through a transformation equation \( z = \omega(\zeta) \), all the points in the \( z \) domain \( S_z \) containing the contour \( s \) are mapped onto those exterior to a unit circle in the \( \zeta \) plane (Figure 2.4). Through the transformation equation \( \omega(\zeta) \), the stress function \( \phi(z) \) can be expressed in terms of a new variable \( \zeta \) in the mapped plane. Such a transformation is said to be conformal if the mapping \( \omega(\zeta) \) is analytic and also \( \omega'(\zeta) \neq 0 \) for \( \zeta \in S_\zeta \). As long as the transformation is single-valued, the value of the function \( \phi(z) \) will be numerically identical with the one of \( \phi(\zeta) \) at the corresponding point in the \( \zeta \)-plane. For convenience, the algebraic

\[
S_z \quad S_\zeta
\]

\[
z - \text{plane} \quad \zeta - \text{plane}
\]

Figure 2.4: Transformation of a contour to a unit circle
expression for the function \( \phi \) is kept the same for both planes irrespective of the variable chosen at a point.

Particularly, the boundary value of \( \phi \) at a point \( z = z_0 \) on the contour surface \( s \) is mapped to its corresponding point \( \zeta = \xi \) on the unit circle. Singularities (poles) in \( \omega(\xi) \) could happen to describe corner properties on the boundary \( s \). Through the transformation function \( \omega(\zeta) \), the boundary equation (2.29) for the first fundamental problem can be transformed into the following equation in the auxiliary plane \( \zeta \),

\[
\phi(\xi) + \frac{\omega(\xi)}{\omega'(\xi)} \phi'(\xi) + \psi(\xi) = f(\xi)
\]

(2.41)

where the function \( f(\xi) \) is the loading integration function, formulated in the \( \zeta \)-plane. Recall the statement that the complex stress function at a certain point in the \( z \)-plane has the same value at its corresponding point in the \( \zeta \)-plane and so are the stress components. From the formulations for calculating the stress components (2.20), the stress components can be further rewritten as

\[
\sigma_{xx} + \sigma_{yy} = 4 \text{Re} \left\{ \phi'(\zeta) / \omega'(\zeta) \right\}
\]

\[
\sigma_{yy} - \sigma_{xx} + 2 i \sigma_{xy} = 2 \omega(\zeta) \left( \phi''(\zeta) \omega'(\zeta) - \phi'(\zeta) \omega''(\zeta) \right) / (\omega'(\zeta))^3
\]

(2.42)

\[
+ 2 \psi'(\zeta) / \omega'(\zeta)
\]

In favor of dealing with circular boundary, the obtained stresses from (2.42) can be transformed to express the stress components in the polar coordinate system by

\[
\sigma_{rr} + \sigma_{\theta\theta} = \sigma_{xx} + \sigma_{yy}
\]

\[
\sigma_{\theta\theta} - \sigma_{rr} + 2 i \sigma_{r\theta} = (\sigma_{yy} - \sigma_{xx} + 2 i \sigma_{xy}) e^{2i\theta}
\]

(2.43)
where $\theta$ is the angle measured from the $x$ axis in counterclockwise direction. All the stresses referred to $(x, y)$ axis can be obtained from (2.42). Therefore, bringing in the expressions for all stress components and separating the real and imaginary parts of (2.43), one will obtain all individual stress components in the polar coordinate, written as

\[
\sigma_{rr} = 2Re\{\phi'(\zeta)/\omega'(\zeta)\}
\]

\[-Re\{e^{2i\theta}[\omega'(\zeta)(\phi''(\zeta)\omega'(\zeta) - \phi'(\zeta)\omega''(\zeta))/(\omega'(\zeta))^3 + \psi'(\zeta)/\omega'(\zeta)]\}\]

\[
\sigma_{\theta\theta} = 2Re\{\phi'(\zeta)/\omega'(\zeta)\}
\]

\[+Re\{e^{2i\theta}[\omega'(\zeta)(\phi''(\zeta)\omega'(\zeta) - \phi'(\zeta)\omega''(\zeta))/(\omega'(\zeta))^3 + \psi'(\zeta)/\omega'(\zeta)]\}\]

\[
\sigma_{r\theta} = Im\{e^{2i\theta}[\omega'(\zeta)(\phi''(\zeta)\omega'(\zeta) - \phi'(\zeta)\omega''(\zeta))/(\omega'(\zeta))^3 + \psi'(\zeta)/\omega'(\zeta)]\}\]

Especially for the subsequent studies for a circular arc crack, the above expressions for the stresses components in the polar coordinate system are very useful.

Most of the equations reviewed in this chapter can be found in the reference book by Dugdale and Ruiz [16]. For more details about the theory, the reader can refer to the book by Muskhelishvili [42] and other reference [38].
CHAPTER 3. FRACTURE OF A PLATE WEAKEN BY A CIRCULAR ARC CRACK UNDER BIAXIAL TENSION

In the preceding chapter, the boundary equation for general fundamental problems has been well reviewed. In this chapter, a new conformal mapping is devised to transform the unbounded plane outside a CAC onto the unbounded auxiliary plane containing a unit circle. The mapping function is applied to solve the corresponding boundary equation. By this proposed mapping, direct integrations of stresses along the contour surface can be performed by evaluating Cauchy integrals around the unit circle. Therefore, general complex stress functions can be evaluated from the boundary equation (2.41). Based upon the determined stress function, the stress intensity factor can be evaluated using the approach by Sih, etc. [59]. In the following chapters, the same technique will be implemented to investigate other cases.

In this chapter, the special case when biaxial tension is applied at infinity is investigated.

Superposition of Problems

Uniform stresses applied at infinity

Consider a general case when uniform stresses with components $T_x$ and $T_y$ in the $x$ and $y$ directions and a shear stress $S_{xy}$ are applied at infinity. The stress fields can be described by

$$\phi(z) = (T_x + T_y)z / 4 \quad , \quad \psi(z) = (T_y - T_x + 2iS_{xy})z / 2$$

(3.1)
which can be verified by applying equation (2.20).

When an infinite sheet is subjected to the same amount of tension along the \( x \) and \( y \) directions at infinity (Figure 3.1 (a)), the normal stress in any direction is constant and the tangential shear stress is equal to zero. Under this situation, the tension \( T_x \) and \( T_y \) in (3.1) is now equal to a constant \( T_\infty \), and \( S_\infty \) equals 0. Therefore, the expressions in (3.1) are reduced to

\[
\phi(z) = \frac{T_\infty}{2}, \quad \psi(z) = 0
\]  

(3.2)

which lead to the conditions of \( \sigma_r = \sigma_{\theta \theta} = T_\infty \) and \( \sigma_\phi = 0 \) in the polar coordinate on applying equations in (2.23).

**Superposition**

It is observed that the operations for the stress functions in the boundary equation (2.29) are all linear. For the extension problem with biaxial tension applied at infinity, one may write the boundary equation for the traction-free condition on the crack surface as follows,

\[
\phi_\infty(z) + z \frac{\phi'(z)}{z} + \psi(z) = 0
\]

(3.3)

In equation (3.3), the stress functions \( \phi \) and \( \psi \) can be expressed as

\[
\phi(z) = \phi_0(z) + \phi_\infty(z), \quad \psi(z) = \psi_0(z) + \psi_\infty(z)
\]

(3.4)

where \( \phi_0, \psi_0 \) are functions holomorphic outside the closed contour including the point at infinity and \( \phi_\infty, \psi_\infty \) are the stress functions describing uniform stresses at infinity. For the special case of biaxial tension, \( \phi_\infty \) and \( \psi_\infty \) are given by (3.2). Physically, the expressions in (3.4) can be perceived as the superposition of two problems to get traction-free condition on the crack surface. As shown in Figure 3.1 (b), one case is that an infinite sheet is subjected to
Figure 3.1: (a) Infinite sheet containing a crack under biaxial tension at infinity
(b) Superposition of two problems
biaxial tension at infinity, and the other is that the corresponding stresses to the loads at infinity are applied on the crack surface. Therefore, the stress intensity factor of the stated problem is no more than the one with crack opening stresses applied on the crack surface. When calculating the stresses and displacements in the whole domain, one needs to superpose the both problems.

**Conformal Mapping**

**Mapping Function**

In calculation of stress functions, it is expedient for the purpose of evaluating Cauchy integrals to transform the boundary contour into a unit circle. It is well known that an ellipse can be transformed to a unit circle through the mapping function,

\[
\omega(\zeta) = \frac{a}{2} \left( \zeta + m \zeta^{-1} \right)
\]

(3.5)

By this mapping, the ellipse can be further degenerated into a slit with crack length \(2a\) if \(m\) is set equal to unity. As a matter of fact, the above mapping should be considered as a particular one and can be further modified into a more general form, which transforms a warped ellipse to a unit circle (Figure 3.2). This modified mapping takes the following form,

\[
\omega(\zeta) = \frac{a}{2} \left[ \lambda (\zeta + ic) + \lambda (\zeta + i c)^{-1} \right]
\]

(3.6)

in which \(\lambda\) and \(c\) are some constants used to determine the warped shape.

If the parameter constant \(\lambda\) is set equal to \((1-c^2)^{-1/2}\), the warped shape will be closed and become a contour line. Supposed \(\zeta = \xi = e^{i\theta}\) (where \(\xi\) is denoted as the boundary value of \(\zeta\)
Figure 3.2: Transformation of a closed contour into a unit circle

on the unit circle throughout this paper, and $\theta$ is the angle of the point on the unit circle measured from $\zeta_x$-axis) is substituted into equation (3.6), it can be readily proved that this contour line is nothing but an arc of a circle with radius $r_0 = a\lambda/(2c)$ and chord length $2a$ (see Appendix A for the proof). After this contour line is relocated, the transformation equation (3.6) can be rearranged and becomes a useful mapping function,

$$ z = \omega(\zeta) = R \left( \frac{\zeta^2 + 1}{\zeta + ic} \right) $$

(3.7)

where $R = a(1-c^2)^{-1/2}/2$. Particularly, if the constant $c$ is chosen $0$, the above mapping equation is identical with equation (3.5) with $m = 1$, which is the case for the straight crack problem. This mapping function, considered as extension of the particular form for the straight crack problem, actually transforms the contour surface of a circular arc crack with radius $r_0$ and chord length $2a$ into a unit circle. Also, the domain region outside the circular arc crack (Figure 3.3...
(a) is mapped onto the region outside the unit circle $\Gamma$ (Figure 3.3 (b)).

As shown in Figure 3.3, through this transformation equation (3.7), the crack tips $z_{(1)}$ and $z_{(2)}$, having the coordinates

$$z_{(1,2)} = (-i\pm a - i a \sqrt{1-c^2})$$  \hspace{1cm} (3.8)

are actually mapped to the points $\xi_{(1)}$ and $\xi_{(2)}$, given by

$$\xi_{(1,2)} = (-i\pm \sqrt{1-c^2} - i c)$$ \hspace{1cm} (3.9)

Although the mapping function is well constructed, the variable $c$ needs to be related to the geometry of the crack. To interpret the meaning of the variable $c$, one may draw a triangle connecting the points of crack tips and the origin. Eventually, it can be readily proved that this mapping variable $c$ can be replaced by

$$c = \sin \eta$$ \hspace{1cm} (3.10)

where the angle $\eta$, shown in Figure 3.3, is called "deviation angle" throughout the dissertation. On introducing this nondimensional angle, one may find this is a convenient way to define a circular crack. At one position of the crack surface, there should be two points coincide with each other at the same coordinate. As shown in Figure 3.3, these two arbitrarily chosen points $z_{(U)}$ and $z_{(L)}$ are mapped to their corresponding points in the auxiliary plane, which have the following expressions,

$$\xi_{(U)} = (-c \sin \theta_0 + \sqrt{1-c^2 \cos^2 \theta_0}) e^{-i\theta_0-i\epsilon}$$

$$\xi_{(L)} = (c \sin \theta_0 + \sqrt{1-c^2 \cos^2 \theta_0}) e^{i\theta_0-i\epsilon}$$ \hspace{1cm} (3.11)

in which $\theta_0$ is the angle measured from the reference line, introduced to facilitate the
Figure 3.3: Transformation of (a) a circular arc crack to (b) a unit circle
transformation, and is called "reference angle" by the author. The coordinates of these two points in the $z$-plane can be obtained directly by substituting equations (3.11) into the transformation equation (3.7) and are expressed as

$$z_{(U)} = 2R \cos \theta_0 \left( -i e \cos \theta_0 + \sqrt{1 - e^2 \cos^2 \theta_0} \right)$$

(3.12)

As is obvious from equation (3.12), the corresponding points on the upper and lower crack surface with an angle $\theta_0$ will possess the same coordinate whether this angle is chosen positive or negative.

The solutions obtained in the $\zeta$-plane need to be transformed back to the physical $z$-plane. For this purpose, the transformation equation (3.7) is inverted to express the variable $\zeta$ in terms of $z$ by

$$\zeta = \frac{z \pm \sqrt{z^2 - 4R (R - i e z)}}{2R}$$

(3.13)

In equation (3.8), it is seen that the term $\sqrt{z^2 - 4R (R - i e z)}$ is multiple-valued. This multiple-value problem can be resolved by rewriting the term as

$$\sqrt{z^2 - 4R (R - i e z)} = \sqrt{z - z_{(1)}} \sqrt{z - z_{(2)}}$$

$$= \sqrt{r_{(1)} r_{(2)}} e^{i(\theta_{(1)} + \theta_{(2)})/2}$$

(3.14)

where the radial distance $r_{(1)}$, $r_{(2)}$ and the angles $\theta_{(1)}$, $\theta_{(2)}$ are shown in Figure 3.4. By introducing a branch cut $\theta_1$, $\theta_2 \in [0, 2\pi]$, the sign problem can be automatically resolved. Therefore, the sign "±" in equation (3.13) can be actually replaced by the positive sign "+". In spite of the resolution for the multiple-value problem as shown in equation (3.14), one may still insert the $z$ value into the multiple-value term in a straightforward manner. However, to do so,
circular arc crack

Figure 3.4: Polar coordinate representations for the term $\sqrt{z - z_{(1)}} \sqrt{z - z_{(2)}}$
one has to divide the whole domain into several subdomains as shown in Figure 3.5 and use appropriate sign for each subdomain. As a matter of fact, one of the signs $-\pm$ is supposed to be used to map a point outside the contour surface onto its corresponding point outside the unit circle in the $\zeta$-plane. However, the form with the other sign $+\pm$ will map the same coordinate in the $z$-plane onto the bounded region inside the unit circle, which is not the case of interest and should be discarded. Which form is valid can be readily discerned by examining the mapped coordinate in the $\zeta$-plane and is shown, as an example, in Figure 3.5. In the figure, domain 1 is for the expression with "$+$" sign and domain 2 is for the one with "$-$" sign. For convenience, only positive sign is used throughout the dissertation, provided that the multiple-value problem is resolved by equation (3.14).

**Derivation for Complex Stress Functions**

In Chapter 2, the boundary equation mapped to the auxiliary plane for a general fundamental problem has been reviewed. In this section, the devised mapping function described previously is applied to the boundary equation to determine the complex stress functions. For convenience, the subscript 0, representing holomorphic characteristic in the unbounded domain, is omitted in this section.

Since the analytic function $\phi(\xi)$ represents the boundary value of $\phi(\zeta)$, which is holomorphic within the unbounded domain outside the crack and vanishes at infinity, its Cauchy integral can be written as $-\phi(\zeta)$ by equation (2.35). By use of Cauchy's formulae, the boundary condition equation (2.29) can be further expressed as

$$-\phi(\zeta) + \frac{1}{2\pi i} \int_Y \frac{\omega(\xi)}{\omega'(\xi)} \frac{\phi'(\xi^{-1})}{\xi - \zeta} d\xi = \mathcal{F}(\zeta)$$

(3.15)

where, for brevity, the Cauchy integral of $\mathcal{F}(\xi)$ is denoted as $\mathcal{F}(\zeta)$.
circular arc crack

domain 1: "+" for the inversion formula

domain 2: "-" for the inversion formula

Figure 3.5: Domain regions for appropriate inverse mapping
As stated by the principal of superposition, the $K$-factor at the crack tips of a circular arc crack subjected to biaxial tension at infinity is actually equal to the one with the same amount of normal traction applied on the crack surface. Let the uniform outward normal stress $-P$ (compression) act upon the whole edge of the circular arc crack. Then the quantity $f$, according to its definition, can be written as

$$f(\xi) = -P \cdot R \left( \frac{\xi^2 + 1}{\xi + ic} \right)$$

The expression for $f(\xi)$ in equation (3.16) can be decomposed into

$$f(\xi) = -P \cdot R \left( \xi - ic + \frac{1 - c^2}{\xi + ic} \right)$$

where the first two terms on the right hand side are holomorphic inside the unit circle $\Gamma$, and therefore their Cauchy integrals are actually zero. Consequently, one may apply Cauchy's formulae and obtain

$$\mathcal{F}(\zeta) = \frac{P \cdot R \left( 1 - c^2 \right)}{\xi + ic}$$

Now, the attention is given to the second term on the left hand side of equation (3.15). To evaluate this term, the mapping function $\omega(\xi)$ described by equation (3.3) is substituted in it. After lengthy algebraic operations are performed, the term $\omega(\xi)/\omega'(\xi)$ can be written as

$$\frac{\omega(\xi)}{\omega'(\xi)} = \frac{(1 - c^2)^2}{(\xi + ic)} \cdot \frac{\xi(2c^2 - 1)(c^2 - 1) - ic(1 - c^2) + (\xi - ic)(1 - ic\xi)^2}{\xi^2 + 2ic\xi - 1}$$

It is observed that the second term on the right hand side of equation (3.19), denoted by $H(\xi)$.
herein, is the boundary value of the function, holomorphic inside $\Gamma$ without a pole. Since the function $\phi'(\zeta^{-1})$, the reflection of $\phi'(\zeta)$, is holomorphic inside $\Gamma$, it follows that $H(\xi)\cdot \phi'(\zeta^{-1})$ is the boundary value of the function, also holomorphic inside $\Gamma$ without a pole, and then its Cauchy integral will turn out to be zero. Actually, the first term on the right hand side of equation (3.19) appears to be associated with the principal part. Thus, the Cauchy integral of this term $\omega(\xi)\phi'(\xi^{-1})/\omega'(\xi)$ can be rewritten as

$$
\frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\xi) \phi'(\xi^{-1})}{\omega'(\xi) (\xi - \xi)} d\xi = \frac{(1 - c^2)^2}{2\pi i} \int_{\Gamma} \frac{\phi'(\xi^{-1})}{(\xi + ic)(\xi - \xi)} d\xi \quad (3.20)
$$

Since the integrand in equation (3.20) contains the unknown function $\phi'$, it may give us the first impression that the Cauchy integral still remains unknown. However, in perceiving that a pole at $-ic$ exists inside the unit circle $\Gamma$, one may expand the function $\phi'(\xi)$ in Taylor's series about the point $1/ic$ as follows,

$$
\phi'(\xi) = \phi'(1/ic) + \phi''(1/ic) \cdot (\xi - 1/ic) + ... \\
+ \phi^{n-1}(1/ic) \cdot (\xi - 1/ic)^n_{n=\infty} \quad (3.21)
$$

As taking conjugate operation for each term in equation (3.21), one may obtain

$$
\overline{\phi'(\xi^{-1})} = \overline{\phi'(-1/ic)} + \overline{\phi''(-1/ic) \cdot (\xi^{-1} + 1/ic)} + ... \\
+ \overline{\phi^{n-1}(-1/ic) \cdot (\xi^{-1} + 1/ic)^n_{n=\infty}} \quad (3.22)
$$

On inserting equation (3.22) into the integrant of equation (3.20) and letting $\xi$ approach $-ic$, one may immediately obtain
\[
\frac{\phi'(\xi^{-1})}{\xi + ic} \bigg|_{\xi = -ic} = \frac{\phi'(-1/ic)}{\xi + ic} + \frac{\phi''(-1/ic)}{\xi \cdot ic}
\]  
(3.23)

Apparently, the first term on the right hand side of equation (3.23) appears to be the principal part. Therefore, by applying Cauchy's formulae, one may readily obtain

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\phi'(\xi^{-1})}{(\xi + ic)(\xi - \zeta)} \, d\xi = -\frac{\phi'(-1/ic)}{\zeta + ic}
\]  
(3.24)

Hence, the boundary equation (3.15) can be rewritten as the following equation,

\[
-\phi(\zeta) - \frac{(1 - c^2)^2 \phi'(-1/ic)}{\zeta + ic} = \mathcal{F}(\zeta)
\]  
(3.25)

The above equation cannot be solved for \(\phi(\zeta)\) as long as the second term on the left hand side still remains unknown. To evaluate the term, differentiation with respect to \(\zeta\) is taken for each term in equation (3.25) and let \(\zeta\) equal \(-ic^{-1}\). As a result, the following expression is obtained,

\[
-\phi'(-ic^{-1}) + \frac{(1 - c^2)^2 \phi'(ic^{-1})}{(-ic^{-1} + ic)^2} = \mathcal{F}'(-ic^{-1})
\]  
(3.26)

Taking the conjugate operation and rearranging the above equation will give the constant term, written as

\[
\bar{\phi}'(ic^{-1}) = \frac{(1 - c^2)^2 \phi'(-ic^{-1})}{(-ic^{-1} + ic)^2} - \mathcal{F}'(-ic^{-1})
\]  
(3.27)

As a result of substituting equation (3.27) back into equation (3.26) for determination of the constant \(\phi'(-ic^{-1})\), this constant can be expressed in terms of \(\mathcal{F}'(-ic^{-1})\). In the sequel, the
conjugate of this constant takes the form,

$$\overline{\phi'}(ic^{-1}) = \frac{\overline{\mathcal{F}}'(ic^{-1}) - e^2 \overline{\mathcal{F}}'(-ic^{-1})}{(e^4 - 1)}$$

(3.28)

Consequently, after the above equation is substituted back into the boundary equation (3.25), the expression for the complex stress function \(\phi\) can be written as follows,

$$\phi(\zeta) = -\mathcal{F}(\zeta) - \frac{1 - e^2}{1 + e^2} \cdot \frac{e^2 \mathcal{F}'(-ic^{-1}) - \overline{\mathcal{F}}'(ic^{-1})}{\zeta + ic}$$

(3.29)

Recall that \(\mathcal{F}(\zeta)\) represents the Cauchy integral of the stress integration function \(f\), specified along the crack surface. As long as the traction along the crack surface is specified, one may directly obtain the expression for \(\phi\) from equation (3.29). Thus, this equation (3.29) can be considered as the general expression for the stress function \(\phi\) of the circular arc crack problem.

Now, the attention is turned to the special case when uniform normal stress is applied on the crack surface. For this case, the expression for \(\mathcal{F}(\zeta)\) in equation (3.18) can be inserted into equation (3.29), which gives rise to

$$\phi(\zeta) = \frac{-P \cdot a \sqrt{1 - c^2}}{2(\zeta + ic)(1 + c^2)}$$

(3.30)

Simply for the purpose of verification, it is good to note that if \(c\) is particularly chosen to be zero, the expression for \(\phi(\zeta)\) in (3.30) is identical with the one for the straight cut problem. To calculate the stress and displacement components in the whole domain, one needs to differentiate \(\phi\) with respect to \(\zeta\) and eventually obtains
\[
\frac{d \phi}{dz} = \frac{P (1 - e^2)}{(1 + e^2) (z^2 + 2 i c z - 1)}
\]

\[
\frac{d^2 \phi}{dz^2} = \frac{-4 P (1 - e^2)^{2/3} (z + i c)^3}{a (1 + e^2) (z^2 + 2 i c z - 1)^3}
\]

(3.31)

In addition to the stress function $\phi$, the other stress function $\psi$ is still needed to calculate the internal stresses and displacements. For the purpose of deriving the function $\psi$, one needs to start from the conjugate form of the boundary equation, written as

\[
\bar{\phi}(\xi) + \frac{\omega(\xi)}{\omega'(\xi)} \phi'(\xi) + \psi(\xi) = \bar{f}(\xi)
\]

(3.32)

Due to the fact that the conjugate form of $\phi$ should be holomorphic inside $Y$, it immediately follows that its Cauchy integral will turn out to be zero. Similar to the argument in determining the stress function $\phi$, the function $\psi$ should be also holomorphic inside $Y$ without a pole, and then its Cauchy integral turns out to be $-\psi(\zeta)$. Hence, taking Cauchy integrals for each term of equation (3.32) yields

\[
\psi(\zeta) = -\frac{1}{2\pi i} \int_Y \frac{\bar{f}(\xi^{-1})}{\xi - \zeta} d\xi + \frac{1}{2\pi i} \int_Y \frac{\bar{\omega}(\xi^{-1})}{\omega'(\xi)} \phi'(\xi) d\xi
\]

(3.33)

In arriving at the above equation, the constant term $\psi(\infty)$ has been omitted due to the fact that no contribution of this term is made to all stress components. Further, the Cauchy integral of $\bar{\phi}(\xi^{-1})$ can be dropped since the conjugate of $\phi(\xi)$ is holomorphic inside $Y$, and therefore its Cauchy integral turns out to be zero. In equation (3.33), the Cauchy integral of $\bar{f}(\xi^{-1})$ can be promptly evaluated from Cauchy's formulae, which leads to
\[
\frac{-1}{2\pi i} \int_Y \frac{\tilde{f}(\xi^{-1})}{\xi - \zeta} \, d\xi = \frac{-P \cdot R}{\zeta} \quad (3.34)
\]

Now, the expressions for \(\omega\) and \(\phi\) can be brought into the integrant of the second term on the right hand side of equation (3.33), which becomes

\[
\frac{\omega(\xi^{-1}) \cdot \phi'(\xi)}{\omega'(\xi)} = \frac{P \cdot R (1 - c^2) (1 + \xi^2)}{(1 + c^2) \xi (1 - ic \xi) (\xi^2 + 2ic \xi - 1)} \quad (3.35)
\]

Obviously, there exists one pole at \(1/\text{ic}\) outside the unit circle \(Y\), which is associated with the principal part of the function \(\phi'(\xi) \cdot \omega(\xi^{-1})/\omega'(\xi)\). Accordingly, the function is divided into two terms as follows,

\[
\frac{\omega(\xi^{-1}) \cdot \phi'(\xi)}{\omega'(\xi)} = \frac{P \cdot R (1 - c^2)}{(1 + c^2)} \left( \frac{ic}{1 - ic \xi} + \frac{\xi^2 + 2ic \xi + 1}{\xi (\xi^2 + 2ic \xi - 1)} \right) \quad (3.36)
\]

where the first term on the right hand side is apparently the principal part. By use of Cauchy's formulae, it can be obtained that the Cauchy integral of the term \(\phi'(\xi) \cdot \omega(\xi^{-1})/\omega'(\xi)\) is

\[
\frac{1}{2\pi i} \int_Y \frac{\omega(\xi^{-1}) \cdot \phi'(\xi)}{\omega'(\xi)} \, d\xi = \frac{-P \cdot R (1 - c^2) (\xi^2 + 2ic \xi + 1)}{(1 + c^2) \xi (\xi^2 + 2ic \xi - 1)} \quad (3.37)
\]

As a result of substituting equations (3.34) and (3.37) back into (3.33), the general expression for \(\psi(\zeta)\) is written as

\[
\psi(\zeta) = \frac{-P \cdot a (\zeta^2 + 2ic \zeta - c^2)}{(1 + c^2) \sqrt{1 - c^2 \zeta} (\zeta^2 + 2ic \zeta - 1)} \quad (3.38)
\]

Again, for the purpose of evaluating the stress and displacement components, the process of
differentiation by use of chain rule is needed to obtain

\[
\frac{d\psi(\zeta)}{dz} = \frac{2P(\zeta + ic)^2 [\zeta^4 + 4ic\zeta^3 + (1 - 7c^2)\zeta^2 - 4ic^3\zeta + c^3]}{(1 + c^2)\zeta^2(\zeta^2 + 2ic\zeta - 1)^3}
\] (3.39)

Recall that the problem is solved for the case of uniform normal traction acting upon
the whole crack edge. For the case of biaxial tension applied at infinity, superposition of the
stress functions \(\phi\) and \(\psi\) by equation (3.4) is required. All the stress functions needed to
calculate the internal stress and displacement fields have been determined. Thus, the problem
of biaxial tension at infinity is solved.

**Stress Intensity Factors**

For the plane extension problem, the concepts of stress intensity factors were
originally introduced by Irwin [27] to indicate the strength of singularity at crack tips. Also, it
have been shown by Anderson and Paris [3] that the rate of crack propagation is controlled by
the \(K\)-factor. It was shown by Sih, etc. [59] that the complex variable technique by
Muskhelishvili [42] can be conveniently incorporated into computing \(K\)-factors for various
configurations. In the vicinity of a crack tip \(z - z_{(1)}\), Sih, etc. [59] gave the approach, for plane
extension problems, to evaluate the \(K\)-factors through

\[
K = K_I - iK_{II} = 2\sqrt{2\pi} \lim_{z \to z_{(1)}} \frac{d\phi(z)}{dz}
\] (3.40)

where the subscripts \(I\) and \(II\) denote mode 1 and mode 2 respectively, and the factor \((\pi)^{1/2}\) may
be omitted based upon different convention of definition. In order to evaluate the \(K\)-factors by
making use of equation (3.40), the coordinate system has to be rotated in such a way that the

communicate tip of interest is parallel to the x-axis. For this purpose, an expedient transformation is

so made that the crack tip of interest is not only tangent to the x-axis but also located at the

origin point \( (\zeta(t) = 0) \). This transformation takes the form,

\[
\zeta = \frac{E}{2c^2 - 1 + 2ic\sqrt{1-c^2}} - a \left( 1 + \frac{ic}{\sqrt{1-c^2}} \right)
\]  

(3.41)

where \( E \) is the new transformed coordinate system. The \( K \)-factors for the both crack tips are

the same due to the fact that the applied loads and the crack shape are symmetric about \( y \)-axis.

Particularly, for the left tip at \( \zeta(t) \), substituting equation (3.13) for the expression of \( \zeta \) along

with equation (3.41) into equation (3.40) yields

\[
K_I = \sqrt{\pi a \cdot \frac{\sqrt{1-c^2}}{1+c^2}} \cdot P \quad , \quad K_{II} = \sqrt{\pi a \cdot \frac{c}{1+c^2}} \cdot P
\]  

(3.42)

For the convenience of calculating \( K \)-factors from the crack configuration, equation (3.42)

can be further rewritten by replacing the mapping variable \( c \) with \( \sin \eta \) (from equation (3.10)),

and then one can obtain

\[
K_I = \sqrt{\pi a \cdot \frac{\cos \eta}{1 + \sin^2 \eta}} \cdot P \quad , \quad K_{II} = \sqrt{\pi a \cdot \frac{\sin \eta}{1 + \sin^2 \eta}} \cdot P
\]  

(3.43)

As is well known, the \( K \)-factor of the first mode for the straight cut problem carries the form

of \( K_{I(s)} = (\pi a)^{1/2} P \) where the subscript \((s)\) is adopted to denote the straight crack. Hence, it is

ready to define the geometric functions by \( GF_I = K_I / K_{I(s)} \) and \( GF_{II} = K_{II} / K_{I(s)} \), which are

plotted in Figure 3.6. As is obvious from the figure, the geometric function for mode \( I \) will
Figure 3.6: Geometric functions as a function of
(a) the mapping variable $c$, (b) the angle $\eta$
decrease with increasing $\eta$. However, the other geometric function for mode $II$ will trend in the opposite way.

**Shallow arc crack**

A shallow arc crack is often taken as a perfectly straight one for simplicity in applications. However, this simplification may cause some concerns about the errors. To investigate such simplification, a shallow arc crack can be modelled by an arc of a circle with a large radius. As shown in Figure 3.7, a group of shallow arc cracks, having $\eta$ in the range of $[0, \pi/4]$, are drawn to their real scales. For the present purpose, the variable $c$, restricted in the region $0 \leq c \leq c_0$ ($c_0$ equals $2^{-1/2}$ for a semi-circle), can be expressed as

$$c = \sqrt{\frac{1-\sqrt{1-\rho^2}}{2}} \quad (3.44)$$

where $\rho$ is the nondimensional curvature defined by $\rho = a/r_0$. In the same manner as before, if one is interested in the variation of stress intensity factors due to the change of curvature, equations (3.42) can be expressed in terms of $\rho$ by substituting the expression for $c$ (3.44) into it. Consequently, the $K$-factors can be rewritten as

$$K_I = \sqrt{\pi a} \cdot \frac{\sqrt{2} \cdot \sqrt{1+\sqrt{1-\rho^2}}}{3-\sqrt{1-\rho^2}} \cdot P \quad , \quad K_{II} = \sqrt{\pi a} \cdot \frac{\sqrt{2} \cdot \sqrt{1-\sqrt{1-\rho^2}}}{3-\sqrt{1-\rho^2}} \cdot P \quad (3.45)$$

Once again, for the purpose of verification, it is seen that equation (3.45) is reduced to the one for the straight crack problem as $\rho$ goes to zero ($r_0$ is infinitely large). In fact, for this particular biaxial tension problem, the results obtained from equations (3.45) can be proved to be consistent with those obtained by Sih, etc [59]. It can be seen that the $K$-factors, given by
Figure 3.7: A group of circular arc cracks with the angle $\eta \leq \pi/4$
equations (3.45), can be reduced to consider the straight crack problem without taking any limiting process, therefore equations (3.45) can be considered as more general expressions.

In the same manner, the geometric functions can be also defined in terms of the nondimensional variable $\rho$ by

$$G_F1(\rho) = \frac{\sqrt{2} \cdot \sqrt{1 + \sqrt{1 - \rho^2}}}{3 - \sqrt{1 - \rho^2}}$$

$$G_F2(\rho) = \frac{\sqrt{2} \cdot \sqrt{1 - \sqrt{1 - \rho^2}}}{3 - \sqrt{1 - \rho^2}}$$

which are plotted in Figure 3.8 (a).

Although the $K$-factors can be calculated through equations (3.46) as long as the nondimensional curvature of a shallow arc crack is determined, it is not convenient to express the $K$-factors in this way for the present purpose since determining the curvature from the crack configuration may cause some errors especially for those which are not perfectly circular. Instead, equation (3.43) is a better way to express its $K$-factors to avoid possible errors involved in evaluating the curvature of a shallow arc crack. To estimate the errors caused by modelling the crack as a straight one, one may find it convenient to define the discrepancy by

$$D_{P1}(\eta) = 1 - \frac{\cos \eta}{1 + \sin^2 \eta}$$

$$D_{P2}(\eta) = \frac{\sin \eta}{1 + \sin^2 \eta}$$

which are plotted in Figure 3.8 (b). The angle $\eta$ can be easily measured by geometrical configuration of a curved crack. Then, one may estimate the error percentage from the above equation (3.47) if a shallow arc crack is modelled by a straight cut. These discrepancy functions plotted in Figure 3.8 (b) provides a good and convenient criteria to judge whether the straight crack modelling is appropriate for a shallow arc crack. It is seen from Figure 3.8 (b), the discrepancy of mode $II$, which is greater than the one of mode $I$ for a shallow arc crack,
Figure 3.8: (a) Geometric functions as a function of $\rho$ for shallow arc cracks
(b) Discrepancy functions as a function of $\eta$
increases with the deviation angle in a linear-like relationship while $\eta$ is smaller than 0.2 in radian.

**Deep arc crack**

For a deep arc crack when the deviation angle of the CAC is greater than $\pi/4$ (Figure 3.9), the interaction between the crack tips get more intense as the distance between them become shorter (or the subtending angle $\alpha$ gets larger). The interaction between the crack tips [54] displays different phenomena from those of a pair of collinear cracks. For the simple case of a pair of collinear cracks, one may refer to the work by Barenblatt [5], Erdogan [18], and Sih [56]. In the handbook by Tada in 1973 [66], the formulation of the stress intensity factor for the inner tips is given by

$$\begin{align*}
\begin{bmatrix}
K_I \\
K_{II}
\end{bmatrix} &= \begin{bmatrix} P \\ Q \end{bmatrix} \sqrt{\pi a} \frac{d_0^2 E(k) - a^2}{K(k) a \sqrt{d_0^2 - a^2}} \\
\end{align*}$$

(3.48)

where $k$, $K(k)$, and $E(k)$ are defined by

$$k = \sqrt{1-a^2/d_0^2}, \quad K(k) = \int_0^{\pi/2} \frac{d\Theta}{\sqrt{1-k^2 \sin^2 \Theta}}, \quad E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \Theta} d\Theta,$$

and $a$ and $d_0$ are the half distance between the inner and outer crack tips, respectively. Apparently, the $K$-factors will go to infinity if the distance $a$ approaches zero ($K_I$ and $K_{II}$ diverge with an order of $a^{1/2}$). To investigate whether the same interaction phenomena would happen to the circular arc crack drawn in Figure 3.9, one may find it expedient to express the variable $c$ in terms of $p$ in equation (3.42). As the radius of the crack $r_0$ is fixed and the particular case of $c_0 \leq c < 1$ is considered, $c$ can be replaced by
Figure 3.9: Deep arc cracks
Then, equations in (3.42) become

\[ K_I = \sqrt{\pi a} \cdot \frac{\sqrt{2} \cdot \sqrt{1 - \sqrt{1 - \rho^2}} \cdot P}{3 + \sqrt{1 - \rho^2}}, \quad K_{II} = \sqrt{\pi a} \cdot \frac{\sqrt{2} \cdot \sqrt{1 + \sqrt{1 - \rho^2}} \cdot P}{3 + \sqrt{1 - \rho^2}} \] (3.50)

From equation (3.50), it is interesting to note that as the distance \( a \) approaches zero (\( \rho \to 0 \)), the stress intensity factors do not diverge like those of collinear cracks. In the same manner as before, one may obtain its geometric functions, written as

\[ GF1(\rho) = \frac{\sqrt{2} \cdot \sqrt{1 - \sqrt{1 - \rho^2}}}{3 + \sqrt{1 - \rho^2}}, \quad GF2(\rho) = \frac{\sqrt{2} \cdot \sqrt{1 + \sqrt{1 - \rho^2}}}{3 + \sqrt{1 - \rho^2}} \] (3.51)

which are plotted in Figure 3.10. From this figure, the tendency of how the stress intensity factors converge as \( \rho \) approaches zero can be seen. Contrary to the case of a pair of collinear cracks, as the distance between the crack tips approaches zero (\( a \to 0 \)), the stress intensity factors for both fracture modes I and II vanish.

**Example Case**

The problem of a circular arc crack under biaxial tension at infinity is solved by giving its stress functions \( \phi \) and \( \psi \). Furthermore, the distributions of stresses and displacements can be directly calculated through equations (2.20) and (2.21). In this section, the distributions of
Figure 3.10: Geometric functions as a function of $\rho$ for deep arc cracks
displacements and stresses near the crack edge are numerically calculated and put into graphs to help one visualize the problem. As an example, a sample plate with the properties $v=0.33$ and $E=10 \times 10^6 \text{ psi}$ is assumed to be subjected to unit tension in both $x$ and $y$ directions. To enlarge the scale, a magnification factor of $10^6$ is used.

By use of Westergaard stress functions [72], it can be readily proved that the edge of a straight crack will form an ellipse after biaxial tensions are applied at infinity. Mathematically, if the crack shape is evolved into a circular arc, its crack opening displacements will no longer force the crack edge to form a regular shape modelled by a simple equation. First of all, to calculate the stress and displacement components, one needs to resolve the multiple-value problem (3.13) stated previously. For the purpose of easy programming, the problem is resolved by redefining the variable $\zeta$ by

$$\zeta = \zeta_1 \cdot \delta_1 + \zeta_2 \cdot \delta_2$$

(3.52)

where $\zeta_1$ and $\zeta_2$ are the conversion formulae for domain 1 (using "+" sign) and domain 2 (using "-" sign) respectively, and $\delta_1$ and $\delta_2$ are defined by

$$\delta_1 = \begin{cases} 0, & |\zeta_1| < 1 \\ 1, & |\zeta_1| \geq 1 \end{cases} \quad \delta_2 = \begin{cases} 0, & |\zeta_2| < 1 \\ 1, & |\zeta_2| > 1 \end{cases}$$

(3.53)

After the multiple-value problem is resolved, all mesh points of the unbounded domain in the $\varepsilon$-plane can be easily computed for their displacements and stresses. As an example, a particular case with $c = 0.3$ ($\eta = 1.266$) is investigated. The displacements for all mesh points are shown in Figure 3.11, and the distributions of stresses $\sigma_{xx}$, $\sigma_{yy}$ as well as their contour lines are plotted in Figure 3.12, 3.13 and 3.14, respectively.
Figure 3.11: Deformation of a plate containing a CAC under biaxial tension
Figure 3.12: Distribution of $\sigma_{xx}$ in the vicinity of a CAC with biaxial tension at infinity
Figure 3.13: Distribution of $\sigma_{xy}$ in the vicinity of a CAC with biaxial tension at infinity
Figure 3.14: Distribution of $\sigma_{yy}$ in the vicinity of a CAC with biaxial tension at infinity
CHAPTER 4. SOME IN-PLANE FRACTURE PROBLEMS REGARDING THE TOPICS OF A CIRCULAR ARC CRACK

In Chapter 3, the particular loading condition of biaxial tension applied at infinity substantially simplifies the complexity of the curved crack problem since, by use of the superposition principal, only uniform normal traction applied on the whole crack surface needs to be considered. In this chapter, the general loading condition is considered such that different axial loads combined with shear are applied at infinity (Figure 4.1).

In contrast with the case of uniform normal traction on the crack surface, the interaction behavior is also studied for the case when uniform shearing acts upon the whole crack edge. Additionally, the problem of bending of a strip containing a CAC is approximately solved using the same boundary equation.

General In-Plane Fracture of a Plate Containing a CAC

General extensional loads

As show in Figure 4.1, along with shearing $S_{xy}$, axial tension $T_{x_0}$ and $T_{y_0}$ are applied along axes $x_0$ and $y_0$, respectively. This circular arc crack is assumed to have such general configuration that the tangent of the top point of the crack surface subtends an angle $\alpha_0$ with the axis $x_0$.

Recalling the principal of superposition described in Chapter 3, one needs to formulate this problem in such a way that only corresponding normal and shearing stresses are
Figure 4.1: Plate containing a CAC subjected to extensional loads at infinity
applied along the whole crack edge to satisfy the traction-free condition. First of all, in correspondence with the loading condition at infinity, the stress components at the points situated at the crack surface position have to be determined. Making use of the transformation of stress from the \( z_0 \)-coordinate to the \( \varepsilon \)-coordinate, one can formulate the corresponding stresses in the \( \varepsilon \)-coordinate in terms of the applied stresses as follows,

\[
\begin{align*}
T_x &= \frac{T_{x_0} + T_{y_0}}{2} + \frac{T_{x_0} - T_{y_0}}{2} \cos 2\alpha_0 - S_{xy_0} \sin 2\alpha_0 \\
T_y &= \frac{T_{x_0} + T_{y_0}}{2} - \frac{T_{x_0} - T_{y_0}}{2} \cos 2\alpha_0 + S_{xy_0} \sin 2\alpha_0 \\
S_{xy} &= -\frac{T_{x_0} - T_{y_0}}{2} \sin 2\alpha_0 + S_{xy_0} \cos 2\alpha_0
\end{align*}
\]

(4.1)

where \( T_x, T_y \), and \( S_{xy} \) are the equivalent tensile and shearing stresses in the \( \varepsilon \)-coordinate. From equation (3.1), the complex stress functions corresponding to the loading condition at infinity can be expressed as

\[
\phi = \frac{T_x + T_y}{4} \cdot z, \quad \psi = \frac{T_y - T_x + 2iS_{xy}}{2} \cdot z
\]

(4.2)

Therefore, inserting equations (4.2) back into the expressions (2.23) for the radial and tangential stress in the polar coordinate system yields

\[
\sigma_{rr} = \frac{T_x + T_y}{2} + \frac{T_x - T_y}{2} \cos (2\varpi) + S_{xy} \cdot \sin (2\varpi)
\]

(4.3)

\[
\sigma_{r\theta} = -\frac{T_x - T_y}{2} \sin (2\varpi) + S_{xy} \cdot \cos (2\varpi)
\]

where the angle \( \varpi \) is measured from the reference axis \( x' \) passing through the center of the
circle containing the circular arc crack (shown in Figure 4.1). Further, the expressions for the radial and tangential stress in (4.3) can be rewritten as

\[-\sigma_r + i\sigma_{r\theta} = \frac{T_y - T_x}{2} e^{-2i\varphi} + iS_{xy} e^{2i\varphi} - \frac{T_y + T_x}{2}\]  (4.4)

Since the goal is to integrate these stress components with respect to the coordinate of the crack surface \(z_s\), those terms associated with the angle \(\varphi\) in equation (4.4) have to be converted to \(z_s\) through

\[\varphi = -i \ln \left( \frac{cz_s + i}{R} \right)\]  (4.5)

which is obvious by recalling the radius of a circular arc crack is \(R/c\). Therefore, by substituting the expression for \(\varphi\) (4.5) into equation (4.4), the stress components are given by

\[-\sigma_r + i\sigma_{r\theta} = \frac{(T_y - T_x)}{2} \left( \frac{cz_s + i}{R} \right)^{-2} + iS_{xy} \left( \frac{cz_s + i}{R} \right)^{2} - \frac{(T_y + T_x)}{2}\]  (4.6)

Once the expressions for radial and tangential stresses are obtained, the principal of superposition described in Chapter 3 can be implemented to evaluate corresponding complex stress functions.

**Complex stress functions**

By the principal of superposition, the corresponding crack opening stresses described by equation (4.6) are applied along the crack edge. As before, general formulation of the
corresponding boundary equation to the applied traction on the crack surface \( s \) is still the same, given by equation (3.13). However, the Cauchy integral of the stress integration term \( \mathcal{F}(\zeta) \) needs to be modified. Directly through indefinite integration, the term \( f(\xi) \) corresponding to the crack opening stresses, given by equation (4.6), is eventually expressed as

\[
 f(\xi) = \frac{(T_x - T_y) R^2}{2(c^2 \omega(\xi) + icR)} + \frac{i S_{xy} R}{3c} \left( \frac{c \omega(\xi) + i}{R} \right)^3 \\
 - \frac{(T_x + T_y) \omega(\xi)}{2} 
\]

(4.7)

Basically, the Cauchy integral of \( f(\xi) \), given by equation (4.7), involves integration of three distinct functions of \( \xi \). Before Cauchy's formulae are applied, arrangement to detach its principal part (if there is any) from each term needs to be made. Inserting the expression for \( \omega(\xi) \) into equation (4.7), one may obtain \( f(\xi) \), given by

\[
 f(\xi) = \frac{(T_x - T_y)(\xi + ic) R}{2c\xi(c\xi + i)} + \frac{i S_{xy} R}{3c} \left[ \frac{\xi(c\xi + i)}{\xi + ic} \right]^3 \\
 - \frac{(T_x + T_y)(\xi^2 + 1) R}{2(\xi + ic)}
\]

(4.8)

To apply Cauchy's formulae, one needs to rewrite the functions \( (\xi + ic)/\xi(c\xi + i) \) and \( (\xi^2 + 1)/(\xi + ic) \) in equation (4.8) as

\[
 \frac{\xi + ic}{\xi(c\xi + i)} = c + \left\{ \frac{1 - c^2}{c\xi + i} \right\}, \quad \frac{\xi^2 + 1}{\xi + ic} = \{\xi - ic\} + \frac{1 - c^2}{\xi + ic}
\]

(4.9)

and also separate the function \( \xi^3(c\xi + i)^3/(\xi + ic)^3 \) into
\[
\frac{\xi^3(e^{\xi+i})}{(\xi+i\epsilon)^3} = \{c^3\xi^3 + 3ic^2(1-c^2)\xi^2 - 3c(1-c^2)(1-2c^2)\xi - i(1-c^2)(10c^4 - 8c^2 + 1)\}
- c(1-c^2)(15c^4 - 15c^2 + 3)\xi^2 + (24i\epsilon^5 - 21i\epsilon^3 + 3i\epsilon)\xi - (10c^6 - 8c^4 + c^2)
\]
\[
(\xi+i\epsilon)^3
\] (4.10)

In equations (4.9) and (4.10), all terms in the curly bracket are functions, holomorphic inside \( \gamma \). By applying Cauchy's formulae, the Cauchy integrals of all these holomorphic terms will turn out to be zero. As a result, the Cauchy integral of the function \( f(\xi) \) can be written as

\[
\mathcal{F}(\zeta) = \frac{(T_y - T_x)R}{2\zeta} + \frac{(T_y + T_x)(1-c^2)R}{2(\zeta + ic)}
\]

\[
+ \frac{iS_{xy}R(1-c^2)}{3(\zeta + ic)^3} \left[(15c^4 - 15c^2 + 3)\xi^2 + (24i\epsilon^5 - 21i\epsilon^3 + 3i\epsilon)\xi - (10c^6 - 8c^4 + c^2)\right]
\] (4.11)

Since the expression for \( \phi \) includes the differentiated form of \( \mathcal{F} \) (see equation (3.29)), equation (4.11) is differentiated with respect to \( \zeta \) and the term \( \mathcal{F} \) can be obtained and is written as

\[
\mathcal{F}'(\zeta) = \frac{(T_x - T_y)R}{2\zeta^2} - \frac{(T_y + T_x)(1-c^2)R}{2(\zeta + ic)^2}
\]

\[
- iS_{xy}R(1-c^2) \frac{(5c^4 - 5c^2 + 1)\xi^2 + (6i\epsilon^5 - 4i\epsilon^3)c + c^4 - 2c^6}{(\zeta + ic)^4}
\] (4.12)

Furthermore, due to the involvement of \( d\phi/d\xi \) in computing internal stresses and displacements, consecutive differentiation of \( \mathcal{F} \) of second order is performed to give
\[ \mathcal{F}''(\zeta) = \frac{(T_y - T_x)R}{\zeta^3} + \frac{(T_y + T_x)(1-c^2)R}{(\zeta + ic)^3} \]
\[ + 2 i S_{xy} R (1-c^2) \left( 5c^4 - 5c^2 + 1 \right) \zeta^2 + (4ic^5 - ic^3 - ic)\zeta - c^3 \]
\[ \frac{(\zeta + ic)^5}{(\zeta + ic)^5} \] (4.13)

Consequently, by inserting these modified functions (4.11) and (4.12) for the general in-plane loading condition into the original formulation of \( \phi \) in equation (3.29), one can obtain the complex stress function \( \phi(\zeta) \) for this problem. There is no need to explicitly write down the whole lengthy formulation of \( \phi(\zeta) \) because as long as the expressions for \( \mathcal{F} \) and \( \mathcal{F}' \) in equations (4.11) and (4.12) are built into a computer subprogram, the complex value of \( \phi \) can be directly evaluated through equation (3.29) in a straightforward manner.

Again, for the purpose of computing internal stresses and displacements, consecutive differentiations of \( \phi \) up to second order are needed. Eventually, one can readily obtain

\[
\frac{d\phi}{dz} = \frac{1}{R(\zeta^2 + 2ic\zeta - 1)} \left\{ -(\zeta + ic)^2 \mathcal{F}'(\zeta) + \frac{1-c^2}{1+c^2} \cdot [c^2 \mathcal{F}'(-ic^{-1} - \mathcal{F}'(ic^{-1}))] \right\}
\]
\[
\frac{d^2\phi}{dz^2} = \frac{-(\zeta + ic)^3}{R^2(\zeta^2 + 2ic\zeta - 1)^3} \left\{ 2\mathcal{F}'(\zeta) + (\zeta + ic)\mathcal{F}''(\zeta) + 2(1-c^2)(1+c^2)^{-1} \cdot [c^2 \mathcal{F}'(-ic^{-1} - \mathcal{F}'(ic^{-1}))] \right\}
\] (4.14)

Now, the attention is turned to the other complex stress function \( \psi(\zeta) \) to complete the solution to this problem. Recall the conjugate form of the boundary equation (3.32) which is used to evaluate the stress function \( \psi(\zeta) \) by taking Cauchy integral of each term in the equation. It is imperative to apply Cauchy's formulae again to evaluate the Cauchy integrals of the functions \( f(\xi^{-1}) \) and \( \omega(\xi^{-1}) \cdot \phi'(\xi)/\omega'(\xi) \), which are given by
\[ \tilde{f}(\xi^{-1}) = \frac{(T_x - T_y) R \xi(1 - ic\xi)}{2c(c - i\xi)} - \frac{(T_x + T_y) R(\xi^2 + 1)}{2\xi(1 - ic\xi)} \]

\[- \frac{iS_{xy} R (c - i\xi)^3}{3c\xi^3(1 - ic\xi)^3} \]

\[ \frac{\omega(\xi^{-1})\phi'(\xi)}{\omega'(\xi)} = \frac{(T_y - T_x)}{2} \cdot \frac{R(\xi^2 + 1)(\xi + ic)^2}{\xi^3(1 - ic\xi)(\xi^2 + 2ic\xi - 1)} \]

\[ + \frac{(T_y + T_x)(1 - c^2) R + 2\mathcal{F}_0}{2} \cdot \frac{1}{\xi(1 - ic\xi)(\xi^2 + 2ic\xi - 1)} \]

\[ + \frac{iS_{xy} R(\xi^2 + 1)[(5c^4 - 5c^2 + 1)\xi^2 + (6ic^5 - 4ic^3)\xi + (c^4 - 2c^6)]}{(1 - c^2)^{-1}\xi(1 - ic\xi)(\xi^2 + 2ic\xi - 1)(\xi + ic)^2} \]

where \( \mathcal{F}_0 \) denotes the constant term \([e^2 \mathcal{F}'(-ic^{-1}) - \mathcal{F}'(ic^{-1})](1 - c^2)/(1 + c^2) \). In the same manner as before, all principal parts have to be disassociated from those function in (4.15). As is obvious from equations (4.15), the points at zero and \(-ic\) are those associated with their principal parts. Therefore, the goal is to detach all the terms with the denominator of \( \xi \) or \( \xi + ic \), which can be achieved by basic algebraic operations. For this purpose, all functions of \( \xi \) in the expression of \( \tilde{f}(\xi^{-1}) \) are rewritten as

\[ \frac{\xi(1 - ic\xi)}{c - i\xi} = \left\{ \frac{c\xi + i(1 - c^2)}{c - i\xi} \right\} - \frac{ic(1 - c^2)}{c - i\xi}, \quad \frac{\xi^2 + 1}{\xi(1 - ic\xi)} = \left\{ \frac{\xi + ic}{1 - ic\xi} \right\} + \frac{1}{\xi} \]

\[ \frac{(c - i\xi)^3}{\xi^3(1 - ic\xi)^3} = \frac{-3c(1 - c^2)(1 - 2c^2)\xi^2 - 3ic^2(1 - c^2)\xi + c^3}{\xi^3} \]

\[ + \left\{ \frac{(1 - c^2)[3c^4(1 - 2c^2)\xi^2 + 3ic^3(3 - 5c^2)\xi + (10c^4 - 8c^2 + 1)]}{(i + c\xi)^3} \right\} \]
and all functions of $\xi$ involved in the expression of $\overline{\omega}(\xi^{-1}) \cdot \phi'(\xi) / \omega'(\xi)$ are rewritten as

$$\frac{(\xi^2 + 1)(\xi + ic)^2}{\xi^3(1 - ic \xi)(\xi^2 + 2ic \xi - 1)} = \left\{ \frac{-c(1 - c^2)^2}{i + c \xi} \right\}$$

$$+ \frac{(1 - c^2)^2 \xi^4 + ic(2c^4 - 5c^2 + 4)\xi^3 + (c^4 - 3c^2 + 1)\xi^2 + ic(2 - c^2)\xi + c^2}{\xi^3(\xi^2 + 2ic \xi - 1)}$$

$$= \left\{ \frac{ic(1 - c^2)(1 - 2c^2)}{1 - ic \xi} \right\}$$

where all terms in the curly bracket are holomorphic inside $\Upsilon$. Sequentially, as Cauchy integrations are performed for the boundary equation with its conjugate form, the complex stress function $\psi$ is given by

$$\psi(\zeta) = \frac{(T_x - T_y)R(1 - c^2)}{2(\zeta + ic)} - \frac{[(T_x + T_y)(1 - c^2)R + 2\mathcal{F}_0](\zeta^2 + 2ic \zeta + 1)}{2\zeta(\zeta^2 + 2ic \zeta - 1)}$$

$$+ \frac{iS_y R[3(1-c^2)(1-2c^2)\zeta^2 - 3ic(1-c^2)\zeta + c^2]}{3\zeta^3} - \frac{(T_x + T_y)R}{2\zeta}$$

$$+ \frac{(T_x - T_y)R}{2\zeta^3(\zeta^2 + 2ic \zeta - 1)} \left[ (1 - c^2)^2 \xi^4 + ic(2c^4 - 5c^2 + 4)\xi^3 + (c^4 - 3c^2 + 1)\xi^2 \right.$$

$$+ ic(2 - c^2)\xi - c^2]$$

$$- iS_x R(1 - c^2)\zeta^{-1}(\zeta + ic)^{-2}(\zeta^2 + 2ic \zeta - 1)^{-1} \left[ (1 - c^2)(1 - 2c^2)\zeta^4 + ic(8c^4 - 9c^2 + 2)\zeta^3 + (1 - 2c^2)(5c^4 - 5c^2 + 1)\zeta^2 + ic(4c^4 - 10c^2 + 5)\zeta + c^4(1 - 2c^2) \right]$$

(4.18)
As usual, the differentiated form of \( \frac{d\psi}{dz} \), needed for computing internal stresses and displacements, can be directly obtained through the process of differentiation using the chain rule. Because this expression for \( \frac{d\psi}{dz} \) is too lengthy to write down here and is not needed for evaluating the stress intensity factors, it is listed in Appendix B for reference.

It is noteworthy that for the special case of biaxial tension \((T_{x0} = T_{y0} = P\) and \(S_{xy0} = 0\)), representations for the complex stress functions \( \phi \) and \( \psi \) are actually reduced to those given by equations (3.30) and (3.38).

**Stress intensity factors**

Since the loadings are not geometrically symmetric about the crack configuration, stress intensity factors have to be evaluated separately for each crack tip of interest. In the same manner as before, the stress intensity factors can be directly evaluated through the differentiated complex stress function \( \frac{d\psi}{dz} \). For crack tip 1 (lying on the left hand side of \( y \)-axis), the transformation equation (3.43) is used again to relocate the crack tip such that it is tangent to the new transformed \( \tilde{y} \)-axis at the origin. Now, if the interest is turned to the other crack tip (tip 2), equation (3.41) should be modified into

\[
    z = \frac{\tilde{y}}{1 - 2e^2 + 2ie \sqrt{1-e^2}} + a \left( 1 - \frac{ic}{\sqrt{1-e^2}} \right) \tag{4.19}
\]

Through a lengthy process of coordinate transformation and taking limit, it can be proved that

\[
    \lim_{\varepsilon \to 0} \sqrt{\frac{\tilde{y}}{\zeta^2 + 2ic\zeta - 1}} = \begin{cases} 
    \frac{\sqrt{a ( -ic + \sqrt{1-e^2} )}}{2 \sqrt{2 (1-e^2) }} & (z = z_{(1)}) \\
    \frac{\sqrt{a ( ic + \sqrt{1-e^2} )}}{2 \sqrt{2 (1-e^2) }} & (z = z_{(2)})
\end{cases} \tag{4.20}
\]
Therefore, the limiting values expressed in equations (4.20) can be inserted into the expression for $d\phi/d\varepsilon$ in (4.14). In the sequel, the stress intensity factors can be expressed as

$$K_{(1/2)} = \sqrt{\pi a} \frac{(-\pm) i c + \sqrt{1 - c^2}}{R(1 - c^2)} \left[ (c^2 - 1) \mathcal{F}'(\xi_{(1/2)}) + \mathcal{F}_0 \right]$$  \hspace{1cm} (4.21)$$

where the back slash sign "\" means "or" and will be used throughout the rest of the dissertation. Again, for the special case when biaxial tension are applied ($T_x = T_y = P$ and $S_{xy} = 0$), it can be verified that the expression (4.21) is actually identical with the one given by equation (3.42).

### Bending of a Large Beam Containing a CAC

**Stress distributions**

In this section, consideration is given to the case that a beam-like strip with a circular arc crack appearing inside is subjected to bending at both ends (Figure 4.2). This case under consideration is very practical from the engineering point of view since this could happen in the truss of a structure. Such a problem with a straight crack oriented longitudinally was solved by Muskhelishvili [42]. Using different methods, Lokshin [39] found the solution to the particular case when the straight crack is perpendicular to the axis of the beam. A number of similar problems for circular, elliptic and some other types of holes were solved and investigated in details by Naiman [43]. Herein, the mapping function described in Chapter 3 is employed to investigate the case using the approximation method established by Muskhelishvili [42].

As shown in Figure 4.2, the circular arc crack is oriented in the direction of the $x$-axis...
Figure 4.2: Bending of a strip containing a CAC

along which a bending moment $M_0$ is applied. If a strip is cut from the body and bounded by the straight lines $y = \pm d$, the traction-free condition needs to be satisfied on the edges while purely normal stress $M_0 y / I$ (where $I$ is the moment of inertia for the cross section) will act on the transverse section of the strip. On the one hand, from beam theory [68], this linearly distributed normal stress is given by

$$\sigma_{xx} = -\frac{3 M_0 y}{2 h d^3}$$

where $h$ is the thickness of the beam. On the other hand, the traction-free condition on both edges will enforce $\sigma_{yy} = \sigma_{xy} = 0$ on the surface $y = \pm d$. 
Complex stress functions

To apply the complex variable scheme to this problem, the complex stress functions corresponding to the stress distributions are found to be

\[
\phi_M = \frac{i M_0 z^2}{8 I}, \quad \psi_M = -\frac{i M_0 z^2}{8 I}
\]  

(4.23)

where the subscript \( M \) is used to denote the contribution due to the bending moment \( M_0 \). The expressions in (4.23) can be verified by directly inserting them into the formulation for computing stresses (2.22). First of all, the beam is assumed to be very large as compared with the size of the curved crack \( d >> a \) so that the far field state of stress (at large distance from the crack) is given by (4.22). Moreover, the traction-free condition is also assumed on the crack edge. Since the dimensions of the curved crack is small as compared with the size of the beam, the problem will be solved as if the large beam is an unbounded plate. Under these circumstances, the complex stress functions \( \phi \) and \( \psi \) have to satisfy

\[
\phi(z) = \phi_0(z) + \phi_M(z) \quad , \quad \psi(z) = \psi_0(z) + \psi_M(z)
\]  

(4.24)

where \( \phi_0 \) and \( \psi_0 \) are functions, holomorphic outside the crack including the point at infinity. Apparently, introducing the variable \( \zeta \) in the auxiliary plane, one can obtain

\[
\phi(\zeta) = \phi_0(\zeta) + \frac{i M_0 \zeta^2}{8 I}, \quad \psi(\zeta) = \psi_0 - \frac{i M_0 \zeta^2}{8 I}
\]  

(4.25)

Due to the traction-free condition on the crack surface, substituting equations (4.25) into the boundary equation (2.29), one has to take the term \( f \) as zero. As a result, it is seen that the holomorphic function \( \phi_0 \) and \( \psi_0 \) (outside \( \Gamma \)) have to satisfy the boundary equation (2.29) in such a way that \( f(\zeta) \) is replaced by \( f_0(\zeta) \), given by
\[
f_0(\xi) = -\phi_M(\xi) - \frac{\omega(\xi)}{\omega'(\xi)} \cdot \frac{\phi'_M(\xi)}{\phi'_M(\xi)} - \psi_M(\xi)
\]  

(4.26)

For convenience, the subscript 0 is dropped for the expressions \( \phi_0, \psi_0 \) and \( f_0 \) from now on. However, one must remember to superpose the stress functions \( \phi_M, \psi_M \) to obtain the complete solution to this bending problem. Recall the general formulation of the function \( \phi(\zeta) \), given by equation (3.29). The goal is to evaluate the term \( \mathcal{F}(\zeta) \) (the Cauchy integral of \( f(\xi) \)) and insert the obtained expression into the formulation (3.29). Apparently, bringing all the expressions for \( \phi_M(\xi), \psi_M(\xi), \) and \( \omega(\xi) \) in equation (4.26), one can come up with

\[
f(\xi) = \frac{-3 i M_0 R^2}{16 h d^3} \left( \frac{\xi^2 + 1}{\xi + i c} \right)^2 + \frac{3 i M_0 R^2}{8 h d^3} \frac{(\xi^2 + 1)^2}{\xi (\xi + i c)(1 - i c \xi)}
\]

(4.27)

\[\frac{3 i M_0 R^2}{16 h d^3} \frac{(\xi^2 + 1)^2}{\xi^2 (1 - i c \xi)^2}\]

Obviously, it is seen from equation (4.27) that the points at zero and \(-ic\) are the poles occurring inside \( \Gamma \). Therefore, applying Cauchy's formulae to evaluate the Cauchy integral of \( f(\xi) \), one has to disassociate the principal parts from all functions, holomorphic inside \( \Gamma \) except those poles. Through basic algebraic operations for all functions of \( \xi \), one can readily obtain

\[
\left( \frac{\xi^2 + 1}{\xi + i c} \right)^2 = \left( \frac{\xi^2 - 2 i c \xi + 2 - 3 c^2}{(\xi + i c)^2} \right) + \frac{(1 - c^2)(1 + 3 c^2 - 4 i c \xi)}{(\xi + i c)^2}
\]

\[
\frac{(\xi^2 + 1)^2}{\xi (\xi + i c)(1 - i c \xi)} = \left\{ \frac{i \xi}{c} + \frac{1 + c^2}{c^2} - \frac{1 - c^2}{c^2 (1 - i c \xi)} \right\} + \frac{1}{i c \xi} - \frac{1 - c^2}{i c (\xi + i c)}
\]

(4.28)

\[
\frac{(\xi^2 + 1)^2}{\xi^2 (1 - i c \xi)^2} = \left\{ \frac{2 i c \xi (1 - c^4) - (1 - c^2)(1 + 3 c^2)}{c^2 (i + c \xi)^2} \right\} + \frac{1 + 2 i c \xi}{\xi^2}
\]
where the Cauchy integrals of all terms in the curly bracket will be dropped. Sequentially, applying Cauchy's formulae for all terms in equation (4.27) and making some arrangements yields

\[
\mathcal{F}(\zeta) = \frac{3IM_0R^2}{16hd^3} \left\{ \frac{(1-c^2)(1+3c^2-4ic\zeta) + ic(4\zeta^2+1)-\zeta(1+2c^2)}{(\zeta+ic)^2} \right\} \tag{4.29}
\]

It can be readily verified that for the special case where a straight cut is oriented in the direction of the beam, the expression for \(\mathcal{F}(\zeta)\) in (4.29) is actually reduced to zero, which is consistent with the work by Muskhelishvili [42]. As a matter of fact, a longitudinal straight cut does not influence the state of stress, however a curved crack does. As before, consecutive differentiations of \(\mathcal{F}(\zeta)\) up to second order are needed to calculate stress and displacement components inside the beam. Eventually, the formulation for \(\mathcal{F}'\) is given by

\[
\mathcal{F}'(\zeta) = \frac{3IM_0R^2}{8hd^3} \left\{ \frac{(1-c^2)(2ic\zeta-c^2-1)}{(\zeta+ic)^3} \right\} - \left\{ \frac{2ic\zeta^3-(1+2c^2)\zeta^2+ic(1-c^2)\zeta-c^2}{\zeta^3(\zeta+ic)^2} \right\} \tag{4.30}
\]

Equation (4.30) can be further differentiated once to give \(\mathcal{F}''(\zeta)\), listed in Appendix B. Hence, the corresponding stress function \(\phi\) and its differentiated forms \(d\phi/dz\) and \(d^2\phi/dz^2\) can be immediately obtained through equations (3.29) and (4.14). Since these functions can be obtained in a straightforward manner simply by inserting these modified function \(\mathcal{F}'\) and \(\mathcal{F}''\) into the associated expressions, their explicit formulations do not need to be listed here.

To complete the solution to this in-plane bending problem, one has to evaluate the other stress function \(\psi(\zeta)\) and its derivative \(d\psi(\zeta)/dz\) as well. On applying Cauchy's formulae to evaluate the Cauchy integrals of the terms in the conjugate boundary equation (3.32) to obtain
the function $\psi(\zeta)$, one needs to decompose those functions containing a pole occurring inside the unit circle $\Upsilon$. Taking the conjugate of equation (4.27), one will obtain the whole expression for $\overline{f}(\xi^{-1})$, written as

$$\overline{f}(\xi^{-1}) = \frac{3iM_0R^2(\xi^2 + 1)^2}{16h d^3 \xi^2 (1 - ic\xi)^2} - \frac{3iM_0R^2(\xi^2 + 1)^2}{hd^3 \xi(1 - ic\xi)(\xi + ic)}$$

(4.31)

$$+ \frac{3iM_0R^2(\xi^2 + 1)^2}{16hd^3 (\xi + ic)^2}$$

By inserting the expressions for $\phi'$, and the mapping function $\omega$ as well as its differentiated form $\omega'$ into the function $\overline{\omega}(\xi^{-1}) \cdot \phi'(\xi)/\omega'(\xi)$, one obtains

$$\frac{\overline{\omega}(\xi^{-1}) \phi'(\xi)}{\omega'(\xi)} = \frac{-3iM_0R^2}{8hd^3} \left\{ \frac{(1-c^2)(2ic\xi - 1 - c^2)(\xi^2 + 1)}{\xi(1 - ic\xi)(\xi + ic)(\xi^2 + 2ic\xi - 1)} \right.$$  

$$- \frac{(\xi^2 + 1)(2ic\xi^3 - (1 + 2c^2)\xi^2 + ic(1 - c^2)\xi - c^2)}{\xi^4(1 - ic\xi)(\xi^2 + 2ic\xi - 1)} \right\}$$

(4.32)

$$+ \frac{\mathcal{F}_0(\xi^2 + 1)}{\xi(1 - ic\xi)(\xi^2 + 2ic\xi - 1)}$$

In equation (4.32), the constant term $\mathcal{F}_0$, although defined in the same way as before ($\mathcal{F}_0 = [c^2 \mathcal{F}'(-ic^{-1}) - \overline{\mathcal{F}'}(ic^{-1})](1 - c^2)/(1 + c^2)$), has to adopt its new function $\mathcal{F}'(\zeta)$, given by equation (4.30). It is observed that two poles at the points $0$ and $-ic$ occurs inside the unit circle $\Upsilon$. The function $(\xi^2 + 1)/\xi(1 - ic\xi)(\xi^2 + 2ic\xi - 1)$ has been decomposed for their principal parts in equation (4.17). In a similar way, the functions $(\xi^2 + 1)^2/\xi(\xi + ic)$, $(\xi^2 + 1)^2/(1 - ic\xi)(\xi + ic)$, and $(\xi^2 + 1)^2/\xi(1 - ic\xi)(\xi + ic)$ can be rewritten as
\[
\frac{(\xi^2 + 1)^2}{\xi^2 (1 - i c \xi)^2} = \left\{ \frac{(1 - c^2)(1 + 3 c^2 - 2 i c \xi (1 - c^4))}{c^2(1 - i c \xi)^2} - \frac{1}{c^2} \right\} + \frac{1 + 2 i c \xi}{\xi^2}
\]

\[
\frac{(\xi^2 + 1)^2}{\xi (1 - i c \xi)(\xi + i c)} = \left\{ \frac{1 + c^2}{c^2} + \frac{i \xi}{c} + \frac{i(1 - c^2)}{c^2(i - c \xi)} \right\} + \frac{(1 - i c \xi)}{\xi(\xi + i c)}
\]  

\[(4.33)\]

\[
\frac{(\xi^2 + 1)^2}{(\xi + i c)^2} = \left\{ \xi^2 - 2 i c \xi + 2 - 3 c^2 \right\} + \frac{(1 - c^2)(1 + 3 c^2 - 4 i c \xi)}{(\xi + i c)^2}
\]

and \((\xi^2 + 1)(2 i c \xi - 1 - c^2)/\xi(1 - i c \xi)(\xi + i c)(\xi^2 + 2 i c \xi - 1)\) is rewritten as

\[
\frac{(\xi^2 + 1)(2 i c \xi - 1 - c^2)}{\xi(1 - i c \xi)(\xi + i c)(\xi^2 + 2 i c \xi - 1)} = \frac{-i c^2(1 - c^2)(1 - 2 c^2)}{i + c \xi}
\]

\[+ \xi^{-4}(\xi^2 + 2 i c \xi - 1)^{-1}[i c(1 - c^2)(1 - 2 c^2)\xi^5 - (1 + 5 c^2 - 8 c^4 + 4 c^6)\xi^4
\]

\[+ i c(2 - c^2)(1 - 2 c^2)\xi^3 - (1 + 4 c^2 - 2 c^4)\xi^2 + i c(1 - 2 c^2)\xi - c^2]
\]

\[(4.34)\]

In equations (4.33) and (4.34), it is seen that all terms in the curly bracket are functions, holomorphic inside \(\Gamma\), and therefore their Cauchy integral will turn out to be zero. Sequentially, following exactly the same procedure as before to acquire \(\psi(\zeta)\), one can readily obtain

\[
\psi(\zeta) = \frac{3 i M_0 R^2}{8 h d^3} \left\{ \frac{(1 - c^2)(1 + 3 c^2 - 4 i c \zeta)}{2(\zeta + i c)^2} - \frac{(1 - i c \zeta)}{\zeta(\zeta + i c)} + \frac{1 + 2 i c \xi}{2 \xi^2}
\]

\[+ \frac{(1 - c^2)[i c \zeta^3 - (1 + 3 c^2)\xi^2 + i c(1 - 2 c^2)\zeta - (1 + c^2)]}{\zeta(\zeta + i c)(\zeta^2 + 2 i c \zeta - 1)}\]

\[\frac{G(\zeta)}{\xi^4(\zeta^2 + 2 i c \xi - 1)}
\]

\[= \frac{\mathcal{F}_0(\zeta^2 + 2 i c \xi + 1)}{\zeta(\zeta^2 + 2 i c \xi - 1)}
\]

\[(4.35)\]

where \(G(\zeta)\) is defined by
Additionally, a lengthy expression of $\frac{d\psi}{dz}$ can be obtained directly from the process of differentiation using the chain rule and is listed in Appendix B for reference. Thus, this bending problem is completely solved by approximation.

If the crack size is fairly small as compared with the dimensions of the beam, experiments with models have shown that the approximate solution to the straight crack problem remains sufficiently exact from the practical point of view. For a beam with a circular hole, the approximate solution remains good enough, provided the hole is not larger than $3/5$ of the width of the beam [69]. When the hole is square, Savin [52] found the approximate solution remains exact as long as the hole is not larger than $1/3$ of the width of the beam.

**Stress intensity factors**

Since the bending moment in this problem is symmetrical about the geometry of the beam and the crack, the stress intensity factors at the both crack tips are identical. In evaluating the stress intensity factors through the approach by Sih, etc. [59], one has to relocate the crack tip of interest and take limiting process. Fortunately, perceiving the fact that the only term needed to be evaluated for its limiting value is $\sqrt{2i}/(\zeta^2 + 2ic\zeta + 1)$, one may avoid repeating the complicated process of taking limiting values for each problem with different loading condition. The stress intensity factors, expressed in exactly the same form as equation (4.21), can be directly evaluated by inserting the new function $\mathcal{F}(\zeta)$, defined by equation (4.30), into equation (4.21) and by substituting appropriate constants in equation (4.30). Since these substitutions can be done in a straightforward manner, the final explicit form of the stress intensity factor is not listed here.
A CAC-Containing Plate Subjected to Uniform Shear on the Whole Crack Edge

In Chapter 3, the special case is considered that only uniform normal traction acts upon the crack edge. It is intriguing to see that as the distance between the crack tips becomes very small, the stress intensity factors for the fracture mode $I$ will actually converge to its limiting value - zero. That means the interaction between the crack tips will counteract each other and eventually nullify the stress intensity factors of both tips. This convergence phenomenon piques our interest to study if this kind of convergence would happen to the same situation but with only uniform shear applied on the edge instead (Figure 4.3).

For a shallow arc crack, it is also interesting to investigate how its geometric function would trend. In this section, both of the cases for a shallow and deep arc crack will be investigated.

Figure 4.3: Uniform shear applied on the surface of a CAC
Complex stress functions

Basically, the nature of the case for uniform shear applied on the whole crack edge is the same as the one studied in Chapter 3. For the case when uniform normal traction is applied, only real part of the general expression of applied traction is considered. For the present case, the attention is turned to its imaginary part only. If the applied traction consists of only uniform shear $Q$, the quantity $f(\zeta)$, according to its definition, can be written as

$$f(\zeta) = iQ \cdot R \frac{\xi^2 + 1}{\xi + ic}$$  \hspace{1cm} (4.37)

By making use of the same procedure for Cauchy integration as before, the Cauchy integral of $f(\zeta)$ is now expressed as

$$\mathcal{F}(\zeta) = \frac{-iQ \cdot R \left(1 - c^2\right)}{\zeta + ic}$$  \hspace{1cm} (4.38)

The formulation for the stress function $\phi(\zeta)$ (3.30) still remains the same but with the updated function $\mathcal{F}(\zeta)$, given by equation (4.38). After Cauchy integrals are evaluated for all terms in the corresponding boundary equation and its conjugate form, the final expressions for the complex stress functions $\phi(\zeta)$, $d\phi(\zeta)/dz$, and $\psi(\zeta)$ are given by

$$\phi(\zeta) = \frac{iRQ}{\zeta + ic}, \quad \frac{d\phi(\zeta)}{dz} = \frac{-iQ}{\zeta^2 + 2ic\zeta - 1}$$  \hspace{1cm} (4.39)

$$\psi(\zeta) = \frac{2iRQ}{\zeta(\zeta^2 + 2ic\zeta - 1)}$$

The rest of all other functions of the differentiated form involved in calculating stress and displacement components can be found in Appendix B.
Stress intensity factors

Exactly the same procedure implemented in Chapter 3 is utilized again to evaluate the stress intensity factors for this case. Due to the property of symmetry, the stress intensity factors at the both crack tips are identical. Eventually, the stress intensity factors are found to be

\[ K_I = \sqrt{\pi a} \frac{Q}{1 - c^2}, \quad K_{II} = \sqrt{\pi a} \frac{1}{\sqrt{1 - c^2}} \]  

(4.40)

As is well known, the \( K_{II} \) for a straight crack equals \( \sqrt{\pi a} Q \), denoted by \( K_{I(n)} \) herein.

Therefore, in a similar manner as before, the corresponding geometric functions can be defined by

\[ GF_1(c) = \frac{c}{1 - c^2}, \quad GF_2(c) = \frac{1}{\sqrt{1 - c^2}} \]  

(4.41)

which are plotted in Figure 4.4 (a). Similarly, if the mapping variable \( c \) is converted to the nondimensional angle \( \eta \) by \( c = \sin \eta \), the geometric functions can be rewritten as

\[ GF_1(\eta) = \frac{\sin \eta}{\cos^2 \eta}, \quad GF_2(\eta) = \frac{1}{\cos \eta} \]  

(4.42)

which are plotted in Figure 4.4 (b). From Figure 4.4, the tendency of how the geometric functions vary with the value of \( c \) and the angle \( \eta \) can be seen. From this figure, it is interesting to see that the stress intensity factors for both fracture mode \( I \) and \( II \) are increasing with increasing \( c \) or \( \eta \). The point where two line intersect with each other is for the case of a semi-circular arc crack. In contrast to the case when normal traction is applied, as the angle \( \eta \) continues increasing, the stress intensity factors will drastically increase and finally diverge at the point \( c = 1 \) (\( \eta = \pi/2 \)).

For the case of a shallow arc crack \((0 \leq c < c_0)\), the geometric functions can be expressed
Figure 4.4: Geometric functions for uniform shear as a function of
(a) the mapping variable \( c \), (b) the angle \( \eta \)
in terms of the nondimensional curvature \( \rho \) (from equation (3.44)) as

\[
GF1(\rho) = \frac{\sqrt{2} \cdot \sqrt{1 - \sqrt{1 - \rho^2}}}{1 + \sqrt{1 - \rho^2}}, \quad GF2(\rho) = \frac{\sqrt{2}}{\sqrt{1 + \sqrt{1 - \rho^2}}} \quad (4.43)
\]

In equations (4.43), the geometric functions are plotted as a function of \( \rho \) in Figure 4.5 (a). The discrepancy functions for this case (plotted in Figure 4.5 (b)) are given by

\[
DQ1(\eta) = \frac{\sin \eta}{\cos^2 \eta}, \quad DQ2(\eta) = \sec \eta - 1 \quad (4.44)
\]

Instead, if a deep arc crack is considered \((c_0 < c < 1)\), the conversion equation (3.49) should be used to write the normalized stress intensity factors in terms of the nondimensional curvature \( \rho \). Consequently, the geometric functions for a deep arc crack can be expressed as

\[
GF1(\rho) = \frac{\sqrt{2} \cdot \sqrt{1 + \sqrt{1 - \rho^2}}}{1 - \sqrt{1 - \rho^2}}, \quad GF2(\rho) = \frac{\sqrt{2}}{\sqrt{1 - \sqrt{1 - \rho^2}}} \quad (4.45)
\]

which are plotted in Figure 4.6. Obviously, when the value of \( \rho \) approaches zero, the stress intensity factors will be infinitely large. It is intriguing to investigate how fast the stress intensity factors diverge with decreasing \( a \). For this purpose, limiting process is taken and eventually, it is found that the stress intensity factor of mode I will diverge with an order of \((a)^{-\frac{3}{2}}\), while the stress intensity factor of mode II will diverge with an order of \((a)^{-1/2}\).

**Example case**

In this section, the same example case investigated in Chapter 3 is studied again to visualize the problem with shearing applied on the crack surface. Suppose the sample plate has
Figure 4.5: (a) Geometric functions of a shallow arc crack subjected to shear vs. $\rho$

(b) Discrepancy functions of a CAC subjected to shear as a function of $\eta$
Figure 4.6: Geometric functions of (a) mode I and (b) mode II as a function of $\rho$ for a deep arc crack subjected to uniform shear
the properties $v=0.33$, $E=10 \times 10^6 \text{ psi}$, and the CAC has a configuration of $c = 0.3$ and is subjected to unit shear on the crack edge. As before, a magnification factor of $10^6$ is used to enlarge the scale. The displacements of all mesh points near the circular arc crack can be visualized from Figure 4.7. For a better view, the deformation of the crack surface is separately put into a graph in Figure 4.8. From Figure 4.8, it can be clearly seen that a certain region near the upper crack surface and lower crack surface will overlap. Physically, this is not allowed to happen, and instead the overlapping regions will pop out of the plane. Similarly, all of the stress components are calculated using the functions given in Appendix B. The distributions of $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{xy}$ along with their contours are plotted in Figure 4.9, Figure 4.10, and Figure 4.11, respectively.
Figure 4.7: Displacements of mesh points in the vicinity of a CAC subjected to shear
Figure 4.8: Deformed crack surface subjected to shear
Figure 4.9: Distribution of $\sigma_{xx}$ in the vicinity of a CAC subjected to shear
Figure 4.10: Distribution of $\sigma_{xy}$ in the vicinity of a CAC subjected to shear
Figure 4.11: Distribution of $\sigma_{yy}$ in the vicinity of a CAC subjected to shear
CHAPTER 5. FRACTURE OF A CAC-CONTAINING PLATE WITH PARTIALLY DISTRIBUTED OR CONCENTRATED LOADS

In Chapter 3 and Chapter 4, some particular loading conditions are considered for a CAC-containing plate. Except the one for uniform shear on the crack surface, those cases previously considered are common in most practical situations. However, for a practical case in engineering industry, possible loading conditions could be much more complicated than what was modelled by simple mathematical equations, previously given in Chapter 3 and 4.

In this chapter, the special case is considered where concentrated loads act at an arbitrary point on the crack edge. Because the obtained solution can be taken as the Green's function to formulate many other problems with sophisticated loading conditions, the solution to this problem is very useful from a practical point of view. To investigate this case, a uniform load is first considered, which acts only upon part of the crack edge. Then, starting from the solution to this partial loading problem, one can derive the stress functions of the problem when concentrated loads are applied on the crack surface.

Uniform Loads Partially Distributed on the Surface of a CAC

Complex stress functions

As before, the same mapping \( \omega(\zeta) \), given by equation (3.7), is utilized to transform the unbounded domain outside a circular arc crack (including its surface) onto the unbounded region outside the unit circle (including the circle) in the auxiliary plane. Now, consideration
is given to the particular loading condition that only part of the crack edge is subjected to uniform normal and tangential traction, shown in Figure 5.1. As shown in the figure, the part of crack surface $z_{(L)} - z_{(U)}$ is subjected to uniform normal traction $P$ and tangential traction $Q$. By making use of the devised mapping, the arbitrary points on the upper and lower surface $z_{(U)}$ and $z_{(L)}$ are actually mapped to the points $\xi_{(U)}$ and $\xi_{(L)}$ (see Figure 3.1 (b)), respectively. The corresponding points $\xi_{(U)}$ and $\xi_{(L)}$ given by equation (3.11), have an angle $\theta_c$ measured from the reference line in clockwise and counterclockwise direction, respectively. From the mapping principal, the arc $\xi_{(U)} - \xi_{(L)}$ also carries the same amount of traction corresponding to those in the $z$-plane.

Let the uniform traction $F$, composed of the outward normal stress $-P$ (compression) and the tangential stress $Q$ (clockwise direction), act upon part of the crack edge $z_{(L)} - z_{(U)}$.

Figure 5.1: Uniform traction applied on part of the crack edge in the $z$-plane
and vanish at infinity as shown in Figure 5.1 Accordingly, the traction \( F \) can be expressed by

\[
F = -P + iQ
\]  

(5.1)

For this case, the quantity \( f \), given by equation (2.30), has to be written separately for the loaded and unloaded region as

\[
\Phi(z) = F \cdot R \left( \frac{\xi^2 + 1}{\xi + ic} \right)
\]  

(5.2)

For \( z \in \text{arc } z_{(L)} - z_{(2)} - z_{(U)} \),

\[
= F \cdot z_{(U)}
\]

It is noted that no matter what the direction of integration is, the form of \( \Phi(z) \) is kept the same.

Since Cauchy's formulae apply only to the case when the function in the integrand is continuously distributed along the closed path, they do not work for this case with loads on part of the unit circle. Instead of using Cauchy's formulae, the Cauchy integral of \( \Phi(z) \), denoted as \( \mathcal{F}_1(\zeta) \) herein, can be evaluated directly through integration according to its basic definition.

This Cauchy integration for \( \Phi(z) \) is performed around the unit circle \( \Gamma \) and is found to be

\[
\mathcal{F}_1(\zeta) = \frac{F \cdot R}{2\pi i} \left[ \frac{\xi^2 - 1}{\xi + ic} \cdot \ln \frac{\xi_{(U)} + ic}{\xi_{(L) + ic}} + \left( \frac{\xi^2 + 1}{\xi + ic} - \frac{z_0}{R} \right) \ln \frac{\xi_{(U)} - \zeta}{\xi_{(L)} - \zeta} \right]
\]

(5.3)

where \( z_0 \) is the point at which traction discontinues, given by

\[
z_0 = z_{(U \setminus L)} = R \left( \frac{\xi_{(U \setminus L)}^2 + 1}{\xi_{(U \setminus L) + ic}} \right)
\]

(5.4)

In equation (5.4), \( \xi_{(U)} \) or \( \xi_{(L)} \) can be directly obtained from equation (3.11). Therefore, for the purpose of evaluating the term \( d\Phi/dz \), the differentiation of \( \mathcal{F}_1(\zeta) \) with respect to \( \zeta \) is executed
and takes the following form,

\[ \mathcal{F}_1'(\zeta) = \frac{F \cdot R}{2 \pi i} \left[ \frac{1-c^2}{(\zeta + ic)^2} \ln \frac{\xi(U) + ic}{\xi(U) + ic} + \frac{\xi(U) - \xi(L)}{(\xi(U) - \zeta)(\xi(L) - \zeta)} \left( \frac{\zeta^2 + 1}{\zeta + ic} R \right) + \frac{\zeta^2 + 2ic \zeta - 1}{(\zeta + ic)^2} \ln \frac{\xi(U) - \zeta}{\xi(L) - \zeta} \right] \]

(5.5)

where \(z_0\) is supposed to be given as the loading condition. It follows that the solution of the boundary equation (3.29) with the function \(\mathcal{F}(\zeta)\) replaced by \(\mathcal{F}_1(\zeta)\) is given by

\[ \phi(\zeta) = -\mathcal{F}_1(\zeta) - \frac{1-c^2}{1+c^2} \cdot \frac{c^2 \mathcal{F}_1'(-ic^{-1}) - \mathcal{F}_1'(ic^{-1})}{\zeta + ic} \]

(5.6)

For the purpose of evaluating the stress intensity factor later, \(d\phi(\zeta)/dz\) is obtained through the process of differentiation using the chain rule and is found to be

\[ \frac{d\phi(\zeta)}{dz} = -\mathcal{F}_1'(\zeta) \cdot (\zeta + ic)^2 + \frac{1-c^2}{1+c^2} \cdot \frac{c^2 \mathcal{F}_1'(-ic^{-1}) - \mathcal{F}_1'(ic^{-1})}{R(\zeta^2 + 2ic\zeta - 1)} \]

(5.7)

where the function \(\mathcal{F}_1'(\zeta)\) is given by equation (5.5).

Thus, by substituting the expression for \(\phi'(\zeta)\) into the boundary equation in its conjugate form (3.32), the complex stress function \(\psi(\zeta)\) can be immediately obtained from Cauchy integration. In evaluating the Cauchy integral on the left hand side of the conjugate boundary equation, perceiving the fact that \(\mathcal{F}_1'(\zeta)\) is a holomorphic function outside the unit circle, one needs to disassociate the principal part of \(\overline{\omega(\xi^{-1}) \mathcal{F}_1'(\xi) / \omega'(\xi)}\). Eventually, by making use of Cauchy's formulae, the complex stress function \(\psi(\zeta)\) can be expressed as
\[ \psi(\zeta) = -\mathcal{F}_2(\zeta) + \left( \frac{\zeta^2 + 1}{\zeta(1-ic\zeta)(\zeta^2+2ic\zeta-1)} \right) \left( 1 + \frac{1}{c(1-ic\zeta)} \right) \]

\[ \frac{(1-c^2)^2}{ic(1-ic\zeta)} \left( \mathcal{F}_1^{'}(1/ic) - (\mathcal{F}_1^{'}(1/ic))(\zeta^2+2ic\zeta+1) \right) \]

\[ (1+c^2)\zeta(\zeta^2+2ic\zeta-1) \]

where the function \( \mathcal{F}_2(\zeta) \) is the Cauchy integral of the conjugate of the stress integration function \( \mathcal{f}(\xi) \). After the conjugate operation is performed for the stress integration function, its Cauchy integral can be directly obtained through integration in a fashion as before. As a result, the function \( \mathcal{F}_2(\zeta) \) is expressed as

\[ \mathcal{F}_2(\zeta) = \frac{F\cdot R}{2\pi i} \left[ \frac{c^2-1}{c(c\zeta+i)} \ln \frac{1-ic\xi(\omega)}{1-ic\xi(\ell)} - \ln \frac{\xi(\omega)}{\xi(\ell)} + \left( \frac{\zeta^2+1}{\zeta(1+ic\zeta)} - \frac{z_0}{R} \right) \ln \frac{\xi(\omega)-\zeta}{\xi(\ell)-\zeta} \right] \]  

(5.9)

The first impression of equation (5.9) is that there exists a pole at the point \( \zeta = 1/ic \). As a matter of fact, the first and the third term in the bracket on the right hand side of equation (5.9) together have removed the principal part having the denominator \( (1-ic\zeta) \), and thus, indeed, \( \psi(\zeta) \) is holomorphic throughout the whole plane outside the unit circle \( \Gamma \).

It is noteworthy that if \( \xi(\omega) \) is equal to \( \xi(\ell) \) (by substituting \( \theta_0 = \pi \) into equation (3.11)), equation (5.3) and (5.9) will be reduced to the results of the special case when the whole crack surface is fully loaded. The complex stress functions, given by equation (5.6) and (5.8), along with the expressions for \( \mathcal{F}(\zeta) \) (5.3) and \( \mathcal{F}_2(\zeta) \) (5.9) constitute the solution to the partial loading problem. Consecutive differentiations using the chain rule \( d/dz = d/d\zeta \cdot d\zeta/dz \) to evaluate \( d^2\phi(\zeta)/dz^2 \) and \( d\psi(\zeta)/dz \) are performed and listed in Appendix C for reference.
Stress intensity factors

For this partial loading problem, the partially distributed loads are obviously not symmetric about the configuration of the crack. Thus, different coordinate transformation equations are needed to relocate each of the crack tips of interest. Equations (3.41) and (4.18) are employed for this purpose. Going through the same process of evaluating the stress intensity factors as before, one can obtain the $K$-factors for the crack tip 1 and 2, given by

$$K_1 = K_{I(1)} - iK_{II(1)}$$
$$= \sqrt{\pi a} \cdot \frac{\xi_{(2)}}{R} \left[ -\mathcal{F}_{1}'(\xi_{(1)}) + \frac{c^2 \mathcal{F}_{1}'(i c^{-1}) - \mathcal{F}_{1}'(i c^{-1})}{1+c^2} \right]$$
$$K_2 = K_{I(2)} - iK_{II(2)}$$
$$= \sqrt{\pi a} \cdot \frac{-\xi_{(1)}}{R} \left[ -\mathcal{F}_{1}'(\xi_{(2)}) + \frac{c^2 \mathcal{F}_{1}'(i c^{-1}) - \mathcal{F}_{1}'(i c^{-1})}{1+c^2} \right]$$

where, again, $\xi_{(1)}$ and $\xi_{(2)}$ are coordinates of the crack tip 1 and 2 in the $\zeta$-plane, and the function $\mathcal{F}_{1}'$ is given by equation (5.5).

Example case

As long as the loading condition and the configuration of the circular arc crack are specified ($z_0, c$ and $\alpha$ are determined), the stress intensity factors for the both tips can be directly calculated through equation (4.20) in a straightforward manner. To illustrate this partial loading problem, the example case is studied that only right half of the crack surface is subjected to uniform traction. For the special case ($\theta_0 = \pi/2$), $\xi_{(1)}$ and $\xi_{(2)}$ are equal to $-i$ and $+i$ respectively, and the whole formulations will be significantly simplified. The constant terms $\mathcal{F}_{1}'(\xi_{(1)}), \mathcal{F}_{1}'(\xi_{(2)}),$ and $\mathcal{F}_{1}'(i c^{-1})$ in equation (5.10) can be directly evaluated through equation (5.5) and found to be
\[ F_1' (\xi_1) = \frac{F \cdot R}{2\pi} \left[ \pi - \frac{2}{\sqrt{1-c^2}} - i \ln \frac{1+c}{1-c} \right] \]

\[ F_1' (\xi_2) = \frac{F \cdot R}{2\pi} \left[ \pi + \frac{2}{\sqrt{1-c^2}} - i \ln \frac{1+c}{1-c} \right] \tag{5.11} \]

\[ F_1' (-ic^{-1}) = \frac{F \cdot R}{2\pi} \left[ \frac{-\pi c^2}{1-c^2} + i \left( \frac{2c}{1-c^2} - \ln \frac{1+c}{1-c} \right) \right] \]

After these constants are substituted back into equation (5.10), the final expression for the stress intensity factors of this half-loading case take the following forms,

\[
K_{I_{(12)}} = \sqrt{\pi a \cdot P} \left( \frac{1-c^2}{2} \frac{(-\text{\textdagger})}{\text{\textdagger}} \frac{1}{\pi (1-c^2)} - \frac{c^2\sqrt{1-c^2}}{2 (1+c^2)} \right)
\]

\[
+ \sqrt{\pi a \cdot Q} \left[ \frac{(-\text{\textdagger})}{\text{\textdagger}} \frac{c}{2 (1-c^2)} + \frac{c \sqrt{1-c^2}}{\pi (1+c^2)} - \frac{\sqrt{1-c^2}}{\pi (1+c^2)} \ln \frac{1+c}{1-c} + \frac{c}{\pi \sqrt{1-c^2}} \right]
\]

\[
K_{II_{(12)}} = \sqrt{\pi a \cdot P} \left( \frac{(+\text{\textdagger})}{\text{\textdagger}} \frac{c}{2 (1+c^2)} \right) \tag{5.12}
\]

\[
+ \sqrt{\pi a \cdot Q} \left[ \frac{1}{2 \sqrt{1-c^2}} \frac{(-\text{\textdagger})}{\text{\textdagger}} \frac{1}{\pi (1+c^2)} \frac{c}{\pi (1+c^2)} \ln \frac{1+c}{1-c} \right]
\]

As the special case when the circular arc crack is reduced to a straight one (\(c = 0\)) is concerned, the stress intensity factors (5.12) are reduced to

\[
K_{I_{(12)}} = \sqrt{\pi a \cdot P} \left( \frac{1}{2} \frac{(-\text{\textdagger})}{\text{\textdagger}} \frac{1}{\pi} \right) \tag{5.13}
\]

\[
K_{II_{(12)}} = \sqrt{\pi a \cdot Q} \left( \frac{1}{2} \frac{(-\text{\textdagger})}{\text{\textdagger}} \frac{1}{\pi} \right)
\]
which are in agreement with the results obtained by Paris and Sih [47]. As before, the corresponding geometric functions for the example case can be defined in the same manner. To avoid involving excessive notation, one can separate this case into two situations - pure tension ($Q = 0$) and pure shear ($P = 0$). For the situation of pure tension, the geometric functions for the crack tip I are defined as

$$\begin{align*}
GF1(c) &= \frac{1 - c^2}{2} - \frac{1}{\pi (1 - c^2)} - \frac{c^2 \sqrt{1 - c^2}}{2 (1 + c^2)} \\
GF2(c) &= \frac{c}{2 (1 + c^2)}
\end{align*}$$

(5.14)

which are plotted in Figure 5.2 (a). Also, the mapping variable $c$ can be replaced by the nondimensional angle $\eta$ using equation (3.10), and then equation (5.14) is rewritten as

$$\begin{align*}
GF1(\eta) &= \frac{\cos^2 \eta}{2} - \frac{1}{\pi (\cos^2 \eta)} - \frac{\sin^2 \eta \cos \eta}{2 (1 + \sin^2 \eta)} \\
GF2(\eta) &= \frac{\sin \eta}{2 (1 + \sin^2 \eta)}
\end{align*}$$

(5.15)

which are plotted in Figure 5.2 (b). Since our interest is to investigate the rate of change of the $K$-factors due to the geometric configuration, their absolute values are taken for plotting Figure 5.2. From this figure, it is intriguing to see that in contrast with the fully loaded case investigated in Chapter 3, the stress intensity factor of mode I at crack tip I will approach infinity as the distance between the crack tips approaches zero. However, the stress intensity factor of fracture mode II at crack tip I will converge to some value instead. Another interesting phenomenon observed is that when the value of $c$ equals 0.3783 (or $\eta$ equals 0.388), the stress intensity factor of mode I at crack tip I becomes zero.
Figure 5.2: Geometric functions of a half-loaded CAC with pure tension as a function of (a) the mapping variable $c$, (b) the angle $\eta$
Similarly, for the situation of pure shear, the geometric functions for the crack tip 1 are defined as follows,

\[ GF1(c) = \left( \frac{-c}{2(1-c^2)} + \frac{c\sqrt{1-c^2}}{\pi(1+c^2)} - \frac{\sqrt{1-c^2}}{\pi(1+c^2)} \ln \frac{1+c}{1-c} + \frac{c}{\pi\sqrt{1-c^2}} \right) \]  

\[ GF2(c) = \left( \frac{1}{2\sqrt{1-c^2}} - \frac{1}{\pi(1+c^2)} - \frac{c}{\pi(1+c^2)} \ln \frac{1+c}{1-c} \right) \]  

(5.16)

which are plotted in Figure 5.3 (a). If the mapping variable \( c \) is replaced with \( \sin \eta \), equations (5.16) are rewritten as

\[ GF1(\eta) = \frac{-\sin \eta}{2 \cos^2 \eta} + \frac{\sin \eta \cos \eta}{\pi(1+\sin^2 \pi)} - \frac{\cos \eta}{\pi(1+\sin^2 \eta)} \ln \left( \frac{1+\sin \eta}{1-\sin \eta} \right) + \frac{\tan \eta}{\pi} \]  

\[ GF2(\eta) = \frac{1}{2 \cos \eta} - \frac{1}{\pi(1+\sin^2 \eta)} - \frac{\sin \eta}{\pi(1+\sin^2 \eta)} \ln \left( \frac{1+\sin \eta}{1-\sin \eta} \right) \]  

(5.17)

which are plotted in Figure 5.3 (b). It is observed that for this half-loading problem with pure shear, the stress intensity factors of both fracture modes for the crack tip 1 will diverge to infinity as the distance between the crack tips approaches zero.

By the same way, the geometric function for the crack tip 2 can be also defined. As described in Chapter 3, the formulations for the stress intensity factors can be also written in terms of the nondimensional curvature \( \rho \). However, they have to be dealt with separately for the cases of a shallow arc crack and a deep arc crack in the same manner as what was done in Chapter 3. Since the procedures are of the same philosophy, the formulations to express the stress intensity factors in terms of \( \rho \) for the present case can be easily obtained and will not be given here.
Figure 5.3: Geometric functions of a half-loaded CAC with pure shear as a function of (a) the mapping variable $c$, (b) the angle $\eta$. 
Now, the same example case (a sample plate containing a circular arc crack with \( c = 0.3 \)) investigated in Chapter 3 and Chapter 4 can be studied again for the present loading condition. Also, a magnification factor of \( 10^6 \) is used to help us visualize the displacements of all mesh points. The displacement grids are shown in Figure 5.4. The stress distributions of \( \sigma_{xx} \), \( \sigma_{yy} \), and \( \sigma_{yx} \) as well as their contours are plotted in Figure 5.5, 5.6 and 5.7, respectively.

**Concentrated Loads Applied on the Surface of a CAC**

A special object of interest in this chapter is to formulate the solution to the case when concentrated loads, consisted of an outward normal force \( P' \) and a clockwise tangential force \( Q' \), act at some point \( \xi_s \) on the surface of a circular arc crack (Figure 5.8). Let this point \( \xi_s \) at which loads are applied have an angle \( \Theta \) in the \( \zeta \)-plane, measured from the reference line. From equation (3.11), the point \( \xi_s \) can be written as

\[
\xi_s = \left( c \sin \Theta + \sqrt{1 - c^2 \cos^2 \Theta} \right) e^{i \Theta} - ic
\]

(5.18)

Therefore, by substituting the expression for \( \xi_s \) (5.18) into the mapping function (3.7), the corresponding point of \( \xi_s \) in the \( z \)-plane, denoted by \( z_s \), is expressed as

\[
z_s = 2 R \cos \Theta \left( \sqrt{1 - c^2 \cos^2 \Theta - ic \cos \Theta} \right)
\]

(5.19)

First of all, to formulate concentrated loads on the crack surface, one has to assign uniform tractions applied on a differential increment of the crack surface. For this purpose, the above equation is differentiated with respect to \( \Theta \), and the following equation is obtained,

\[
dz_s = \frac{2R \sin \Theta \left( c \cos \Theta + i \sqrt{1 - c^2 \cos^2 \Theta} \right)^2}{\sqrt{1 - c^2 \cos^2 \Theta}} d\Theta
\]

(5.20)
Figure 5.4: Deformation of a CAC-containing plate half-loaded with tension
Figure 5.5: Distribution of $\sigma_{xx}$ in the vicinity of a CAC half-loaded with tension
Figure 5.6: Distribution of $\sigma_{xy}$ in the vicinity of a CAC half-loaded with tension
Figure 5.7: Distribution of $\sigma_{yy}$ in the vicinity of a CAC half-loaded with tension
Let the uniform traction \( F \), given by equation (5.1), act upon an increment of infinitesimal length \( dz \). Obviously, the concentrated force, denoted by \( F' \), can be written as

\[
F' = P' + iQ' = F \cdot dz
\]

\[
= F \cdot \frac{2R \sin \Theta (c \cos \Theta + i \sqrt{1 - c^2 \cos^2 \Theta})}{\sqrt{1 - c^2 \cos^2 \Theta}} \cdot d\Theta
\]

Thus, an infinitesimal increment of angle \( \Theta \) can be directly obtained by

\[
d\Theta = \frac{F}{F} \cdot \frac{\sqrt{1 - c^2 \cos^2 \Theta}}{2R \sin \Theta (c \cos \Theta + i \sqrt{1 - c^2 \cos^2 \Theta})^2} \]

(5.22)

Basically, the general complex stress functions of this problem are exactly of the same forms as those for the partial loading problem, given by equations (5.6) and (5.8). However, some modifications for the terms \( \mathcal{F}_i(\zeta) \), \( \mathcal{F}_i'(\zeta) \), and \( \mathcal{F}_2(\zeta) \) are necessary. Of course, as intending
to calculate the stress or the displacement components in the domain, one also needs to further
modify the expressions for $F_1''(\zeta)$ and $F_2''(\zeta)$. To obtain these functions for concentrated loads,
denoted by $\mathcal{S}_1(\zeta)$, $\mathcal{S}_1'(\zeta)$, $\mathcal{S}_1''(\zeta)$, $\mathcal{S}_2(\zeta)$, and $\mathcal{S}_2'(\zeta)$ herein, one needs to differentiate the original
functions with respect to $\Theta$ and multiply it by $\partial \Theta$ (see Dugdale and Ruiz [16]). Therefore,
through a lengthy process of differentiation, $\mathcal{S}_1(\zeta)$ and $\mathcal{S}_2(\zeta)$ are found to be

$$
\mathcal{S}_1(\zeta) = \frac{F}{2 \pi i \Theta} \left\{ \frac{c^2-1}{(\zeta+i c)^2} \dot{\theta}_2 + \left( \frac{\zeta^2+1}{\zeta+i c} \right) \dot{\theta}_3 - \ln \frac{\zeta''(\Theta) - \zeta}{\zeta''(\Theta) - \zeta} \right\}
$$

$$
\mathcal{S}_2(\zeta) = \frac{F}{2 \pi i \Theta} \left\{ \frac{c^2-1}{c(\zeta+i c)} \dot{\theta}_4 - \frac{\zeta^2+1}{\zeta} \dot{\theta}_5 + \left( \frac{\zeta^2+1}{\zeta(1+i c)} \right) \dot{\theta}_6 - \ln \frac{\zeta''(\Theta) - \zeta}{\zeta''(\Theta) - \zeta} \right\}
$$

(5.23)

where $\xi''(U_l)$ is given by equations (3.11) with $\Theta_0$ replaced by $\Theta$, and $\dot{\theta}_{1-6}$ are defined by

$$
\dot{\theta}_1 = \frac{\xi''(U) + i c \xi''(U)}{(\xi''(U) + i c)^2} \cdot \xi''(U), \quad \dot{\theta}_2 = \left( \frac{\xi''(U)}{\xi''(L)} + i c \right)
$$

$$
\dot{\theta}_3 = \left( \frac{\xi''(U)}{\xi''(L)} - \frac{\xi''(U)}{\xi''(L)} \right), \quad \dot{\theta}_4 = i c \left( \frac{\xi''(L)}{1+i c \xi''(U)} - \frac{\xi''(U)}{1-i c \xi''(U)} \right)
$$

(5.24)

where $\xi''(U)$ and $\xi''(L)$ are

$$
\xi''(U) = \left( c + i e^{-i \Theta - c^2 \cos \Theta} \right) e^{2i \Theta}, \quad \xi''(L) = \left( c + i e^{i \Theta - c^2 \cos \Theta} \right) e^{-2i \Theta}
$$

(5.25)

By the same way, $\mathcal{S}_1'(\zeta)$, $\mathcal{S}_1''(\zeta)$, and $\mathcal{S}_2'(\zeta)$ can be also obtained and the final simplified
expressions for these functions are listed in Appendix C for reference. Thus, the expressions for the stress functions in equations (5.6) and (5.8) accompanied with the associated functions \( S_1(\zeta), S_1'(\zeta), S_2''(\zeta), S_3(\zeta), \) and \( S_2'(\zeta) \) construct the general solution to the problem of concentrated loading. Numerical calculations for the stress distributions around the crack surface have been carried out, and it has been proved that all of the boundary conditions on the crack surface are satisfied.

Equation (5.10) can still be used to evaluate the stress intensity factors of this problem, however the function \( F_1(\zeta) \) needs to be replaced by \( S_1'(\zeta) \), given in Appendix C. As this curved crack is degenerated into a straight slit \( (c = 0) \), it can be easily proved that the reduced solution is actually identical with the one obtained by Sih, etc. [59].

Formulations for general loading conditions

Once the solution to the concentrated loading problem has been established, one may construct the solution to other loading problems by replacing \( F \) with \( P(\Theta) \, d\zeta \), where \( P(\Theta) \) is the crack opening stresses around the circumference of the crack. Therefore, the Cauchy integrals of the stress integration functions can be written as

\[
\mathcal{F}_1(\zeta) = \frac{R}{2\pi i} \int_0^\pi P(\Theta) \left\{ \frac{e^{2-1}}{(\zeta+ic)} \frac{\partial}{\partial \zeta} \left( \frac{\zeta^{2+1} S}{\zeta+ic} \frac{z_s}{R} \right) \left[ \ln \frac{\sigma(\zeta)}{\sigma(\zeta)} - 1 \right] \right\} d\Theta
\]

\[
\mathcal{F}_2(\zeta) = \frac{R}{2\pi i} \int_0^\pi P(\Theta) \left\{ \frac{e^{2-1}}{c(c\zeta+i)} \frac{\partial}{\partial \zeta} \left( \frac{\zeta^{2+1} S}{\zeta(1-ic\zeta)} \frac{z_s}{R} \right) \left[ \ln \frac{\sigma(\zeta)}{\sigma(\zeta)} - 1 \right] \right\} d\Theta
\]

Without any difficulty, the integrations in equations (5.26) can be performed using Gauss integration or other numerical schemes. Also, other functions associated with the stress integration terms \( \mathcal{F}_1'(\zeta), \mathcal{F}_1''(\zeta), \) and \( \mathcal{F}_2'(\zeta) \) can be obtained by the same way.
For example, formulating the integration terms in (5.26) for a couple of concentrated forces \((X\text{ and } Y)\) and a moment \(M_c\) applied at the point \(z_c\) (shown in Figure 5.9), one needs to calculate the corresponding radial and tangential stress at the points where the surface of a circular arc crack is situated. By displacing the coordinate \(z\) to \(z'\), the arbitrary point \(z'_S\) (in the coordinate \(z'\)) can be written as

\[z'_S = r_0 e^{i\hat{\alpha}} - (m + in)\]  

(5.27)

where, from the geometry, the angle \(\hat{\alpha}\) can be obtained by

\[\hat{\alpha} = \tan^{-1} \frac{1 - 2c^2\cos^2\Theta}{2c\cos\Theta \sqrt{1 - c^2\cos^2\Theta}}\]  

(5.28)

The angle \(\theta'\), which can be also found from the geometry shown in Figure 5.9, is expressed in terms of the angle \(\hat{\alpha}\) as
\( \theta' = -i \ln \left[ \frac{r_0 e^{i \hat{\alpha}} - (m + i n)}{\sqrt{(m - r_0 \cos \hat{\alpha})^2 + (n - r_0 \sin \hat{\alpha})^2}} \right] \)  

(5.29)

From the results obtained by Muskhelishvili [42], the corresponding complex stress functions for the concentrated loads formulated in the \( z' \)-coordinate are given by

\[
\begin{align*}
\frac{d\Phi}{dz'} &= \frac{-X - iY}{2 \pi z'_s (1 + \kappa)} \\
\frac{d\Psi}{dz'} &= \frac{\kappa (X - iY)}{2 \pi z'_s (1 + \kappa)} - \frac{i M_c}{2 \pi (z'_s)^2}
\end{align*}
\]

(5.30)

As a result, by substituting the expressions for the complex stress functions into equations (2.23), the stress components \( \sigma_{rr}, \sigma_{r\theta}, \) and \( \sigma_{\theta\theta} \) in the polar coordinate of \( z' \) can be obtained in a straightforward manner. Now, these stress components must be transformed to those expressed in the \( z_a \)-coordinate so that the superposition principal can be applied for those points on the crack surface. This can be done by

\[
\begin{align*}
\sigma_{rr} &= \frac{\sigma_{rr}' + \sigma_{\theta\theta}'}{2} + \frac{\sigma_{rr}' - \sigma_{\theta\theta}'}{2} \cos (2 \hat{\alpha} - 2 \theta') + \sigma_{\theta\theta}' \sin (2 \hat{\alpha} - 2 \theta') \\
\sigma_{r\theta} &= -\frac{\sigma_{rr}' - \sigma_{\theta\theta}'}{2} \sin (2 \hat{\alpha} - 2 \theta') + \sigma_{\theta\theta}' \cos (2 \hat{\alpha} - 2 \theta')
\end{align*}
\]

(5.31)

where \( \theta' \), shown in Figure 5.5, is given by equation (5.29). To satisfy the traction-free condition on the surface of the circular arc crack, the traction \( \mathcal{T}(\Theta) \), defined by \( \mathcal{T}(\Theta) = -\sigma_n + i \sigma_{\theta} \), is applied on the crack surface. Therefore, through numerical integration, the Cauchy integrals \( \mathcal{F}_1(\zeta) \) and \( \mathcal{F}_2(\zeta) \) along with their derivatives can be evaluated. Hence, the numerical values of the complex stress functions \( \phi \) and \( \psi \) as well as their derivatives corresponding to the applied tractions on the crack surface can be obtained. Eventually, by superposing the values of the stress functions, evaluated through equations (5.30), final
numerical values of \( \phi \) and \( \psi \) as well as their derivatives for the present example can be obtained.

Particularly, if \( m \) and \( n \) are chosen equal to zero (i.e. the concentrated loads are applied at the center of the circle containing the arc), the formulations for \( \sigma_{rr} \) and \( \sigma_{r\theta} \) in (5.31) are reduced to

\[
\sigma_{rr} = \sigma_{rr}' = \frac{(3 + \kappa)(X \cos \hat{\alpha} + Y \sin \hat{\alpha})}{2 \pi r_0 (1 + \kappa)}
\]

\[
\sigma_{r\theta} = \sigma_{r\theta}' = \frac{(\kappa - 1)(X \sin \hat{\alpha} - Y \cos \hat{\alpha})}{2 \pi r_0 (1 + \kappa)} - \frac{M_c}{2 \pi r_0^2}
\]

which can be referred to the reference [16]. If the expression for the angle \( \hat{\alpha} \) (5.28) is written in terms of the angle \( \Theta \), the stress components in the polar coordinate can be expressed as

\[
\sigma_{rr} = -\frac{3 + \kappa}{2 \pi r_0 (1 + \kappa)} \left[ 2X \cos \Theta \sqrt{1 - c^2 \cos^2 \Theta} + Y(1 - 2c^2 \cos^2 \Theta) \right]
\]

\[
\sigma_{r\theta} = \frac{\kappa - 1}{2 \pi r_0 (1 + \kappa)} \left[ X(1 - 2c^2 \cos^2 \Theta) - 2Y \cos \Theta \sqrt{1 - c^2 \cos^2 \Theta} \right] - \frac{M_c}{2 \pi r_0^2}
\]

which will substantially simplify the formulations.

Thus, the expressions for complex stress functions \( \phi \) and \( \psi \) in equations (5.6) and (5.8) along with equations (5.26) construct the solution to the problem with general loading conditions.
CHAPTER 6. FLEXURAL FRACTURE OF A CAC-CONTAINING PLATE

Chapter 3 to Chapter 5 were devoted to solving the problem of in-plane fracture (with loading on the $x$-$y$ plane) of a plate containing a circular arc crack. In this chapter, the problem of flexural fracture (with transverse out-of-plane loading) is studied. Basically, the in-plane fracture problem investigated before is governed by the biharmonic equation [75]. Similarly, problems with flexural bending, twisting, and shearing are also governed by the biharmonic equation. Modifications of formulations for the transverse loading were made by Leknitskii [36] to derive the corresponding boundary equation.

In this chapter, the mapping function devised to transform a circular arc crack to a unit circle in preceding chapters is implemented to investigate the flexural fracture problem using the corresponding boundary equation derived by Leknitskii.

**General Flexural Loads**

In this section, the same mapping function (3.7) is implemented to investigate the case when transverse loads (bending moment, twisting moment, and shearing stress) act on the crack surface and vanish at infinity. General loading condition is formulated into the expressions for the corresponding complex stress functions, and particular loading such as partial loading and concentrated loading considered in the previous work can be also taken into account by modifying the stress integration function.

As an isotropic thin plane free from lateral loads originally lies in the complex plane $z$, 

its deflection $w$ normal to the $z$-plane satisfies the biharmonic equation $\nabla^4 w = 0$. Based on Muskhelishvili's approach by complex variables, the solution of the biharmonic equation can be expressed in terms of two analytic functions $\Phi$ and $\Psi$ in which the underline is used to differentiate the flexure problem with the extension problem previously described. Accordingly, the deflection $w$ can be expressed as

$$w = Re\{z\Phi(z) + \chi(z)\} \quad (6.1)$$

where $\chi(z) = \int \psi(z) dz$.

The bending moments $M_x$ and $M_y$, twisting moment $H_{xy}$, and shearing stresses $Q_x$ and $Q_y$ at any point in the plane are expressed as

$$M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right), \quad M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + v \frac{\partial^2 w}{\partial x^2} \right)$$

$$Q_x = D \frac{\partial}{\partial x} \nabla^2 w, \quad Q_y = D \frac{\partial}{\partial y} \nabla^2 w \quad (6.2)$$

$$H_{xy} = -D(1-v) \frac{\partial^2 w}{\partial x \partial y}$$

Leknitskii [36], by substituting the expression for the deflection (6.1) into equations (6.2), obtained the following results,

$$M_x + M_y = -4D(1+v)Re\{\Phi'(z)\}$$

$$M_y - M_x + 2iH_{xy} = 2D(1-v)[z\Phi''(z) + \Phi'(z)] \quad (6.3)$$

$$Q_x - iQ_y = -4D\Phi''(z)$$

where $D$, as usual, is the flexural rigidity of the plate, defined by $D = E h^3/12(1-v^2)$, and $h$ is the
thickness of the plate. For convenience, the underlines of the complex stress functions, adopted
to denote the transverse flexure problem, will be dropped from now on. Part of the scheme
given by Goland [19] and Yu [75] is followed to solve the problem of transverse flexure shown
in Figure 6.1. For some particular cases of interest as classified by Yu [75] according to the
type of load transmitted through the plate, the problems are given by

(a) Plain bending - A constant bending moment \( M_{\alpha 0} = M_0 \) is applied along its principal axis
\( x_0 \), making an arbitrary angle \( \alpha_0 \) with the \( x \) axis, while the other bending moment \( M_{\beta 0} \) is equal
to zero.

(b) Cylindrical bending - Also, a constant bending moment \( M_{\alpha 0} = M_0 \) takes place along the
direction of \( x_\alpha \)-axis. Additionally, another bending moment \( M_{\beta 0} = \nu M_0 \) is applied to prevent
the anti-elastic curvature from developing.

(c) All-round bending - Constant moments \( M_{\alpha 0} = M_{\beta 0} = M_0 \) are applied along the axis \( x \) and
y, respectively. Under this condition, the distribution of internal moments is all-round, which means the internal moment at an arbitrary point throughout the whole plane is constant and independent of the direction.

(d) Uniform twisting - A constant twisting moment of $H_\alpha = H_0$ is transmitted throughout the plane. For this case, the load condition can be replaced by $M_\alpha = -M_\alpha = M_0$ with $\alpha_0 = 45^\circ$ and the same effect can be produced.

(e) Uniform shearing - constant shear $Q_{x0}$ is transmitted along the direction $x_0$ and throughout the whole plane. For this case, $Q_{x0}$ is equal to a constant $Q_0$ and $Q_{x0}$ is zero.

In a similar manner as was done for the problem of bending of a CAC-containing plate in Chapter 4, one has to formulate the complex stress functions corresponding to the loading conditions from case (a) through case (d) in order to solve the related boundary equation, given later.

Bending and twisting

Since the first four problems are of the same characteristics, they will be solved together and the last one separately. First of all, consider the functions,

$$\phi(z) = A \cdot z + \phi_0(z), \quad \psi(z) = (B + iC)z + \psi_0(z) \quad (6.4)$$

where the constants $A$, $B$, and $C$ are real, and $\phi_0$ and $\psi_0$ are functions, holomorphic in the unbounded domain outside the crack surface and therefore include only negative powers of $z$. Those functions written in terms of the constants $A$, $B$, and $C$ are actually added to satisfy the boundary conditions at infinity. Because the loads are specified along the $z_0$-coordinate system, which makes an arbitrary leading angle $\alpha_0$ with the $z$-coordinate, one has to relate these unknown constants with the applied loads in the $z$-coordinate. As the expressions for these functions (6.4) are substituted into the first two equations of (6.3), one will come up with three
simultaneous equations as follows,

\[ M_x + M_y = -4D(1 + v)A \]
\[ M_y - M_x = 2D(1 - v)B \]
\[ H_{xy} = D(1 - v)C \]

where the moments \( M_x, M_y, \) and \( H_{xy} \) are the equivalent bending and twisting moments in the \( z \)-coordinate corresponding to the applied moments \( M_{x0}, M_{y0}, \) and \( H_{xy0} \) respectively. Equations (6.5) can be solved for these unknown constants \( A, B, \) and \( C, \) which are expressed as

\[ A = \frac{-(M_x + M_y)}{4D(1 + v)}, \quad B = \frac{-(M_x - M_y)}{2D(1 - v)}, \quad C = \frac{H_{xy}}{D(1 - v)} \]  

(6.6)

From equations (6.6), these constants have now been specified according to the corresponding loading condition at infinity in the \( z \)-coordinate. However, the transformation to equate these constants in terms of the moments specified in the \( z_0 \)-coordinate is needed. Noting that the \( z \)-coordinate make an angle \( \alpha_0 \) in clockwise direction with the \( z_0 \)-coordinate, one can relate the equivalent moments in the \( z \)-coordinate to the applied moments through

\[ M_x = \frac{M_{x0} + M_{y0}}{2} + \frac{M_{x0} - M_{y0}}{2} \cos 2\alpha_0 - H_{xy0} \sin 2\alpha_0 \]
\[ M_y = \frac{M_{x0} + M_{y0}}{2} - \frac{M_{x0} - M_{y0}}{2} \cos 2\alpha_0 + H_{xy0} \sin 2\alpha_0 \]
\[ H_{xy} = -\frac{M_{x0} - M_{y0}}{2} \sin 2\alpha_0 + H_{xy0} \cos 2\alpha_0 \]  

(6.7)

Thus, equations (6.7) can be substituted into equation (6.6) to give the constants in terms of the
applied moments. Particularly, the constants can be determined for each of the stated bending and twisting problems from (a) to (d) with specific bending and twisting moment inserted into equation (6.7).

Shear

By observing the last equation of (6.3), the stress functions contributed by the shear have to be in a form of second order to satisfy the boundary condition at infinity. Thus, for the present case, the corresponding complex stress functions can be written as

$$\phi(z) = A_1 z^2 + \phi_0(z), \quad \psi(z) = A_2 z^2 + \psi_0(z) \quad (6.8)$$

where $A_1$ and $A_2$ are complex constants, and the functions $\phi_0$ and $\psi_0$, as before, are holomorphic in the unbounded domain outside the crack surface including the points at infinity. If the expressions for the complex stress functions (6.8) are substituted into equations (6.3), one obtains

$$M_x + M_y = -8D(1 + v) Re \{A_1 z\}$$

$$M_y - M_x + 2i H_{xy} = 4D(1 - v)(A_1 \bar{z} + A_2 z) \quad (6.9)$$

$$Q_x - iQ_y = -8DA_1$$

The equations for internal moments and shears (6.9) hold for large value of $z$. From the state of equilibrium in the direction of $x_0$ and $y_0$, the shear $Q_{x0}$ and $Q_{y0}$ can be written in terms of the shear components $Q_x$ and $Q_y$ by

$$Q_{x0} = Q_x \cos \alpha_0 + Q_y \sin \alpha_0 \quad , \quad Q_{y0} = -Q_x \sin \alpha_0 + Q_y \cos \alpha_0 \quad (6.10)$$

For the stated problem (e), the shear stresses to be transmitted are $Q_{x0} = Q_0$ and $Q_{y0} = 0$. Solving equations (6.10) for $Q_x$ and $Q_y$, one can obtain
Thus, the moments needed to hold the state of equilibrium are

\[ M_x = Q_0 x \cos \alpha_0 + v Q_0 y \sin \alpha_0 \]
\[ M_y = v Q_0 x \cos \alpha_0 + Q_0 y \sin \alpha_0 \]  

(6.12)

These expressions for internal moments to hold the equilibrium situation can then be substituted into (6.9) for determining the constants \( A_1 \) and \( A_2 \). These constants are found to be

\[ A_1 = \frac{\bar{Q}_0}{8D} e^{-i\alpha_0} \tag{6.13} \]

For a more general case when \( Q_{x0} \) and \( Q_{y0} \) are applied at the same time, one can either implement the principal of superposition or modify equations (6.11) by

\[ Q_x = Q_{x0} \cos \alpha_0 + Q_{y0} \sin \alpha_0 \]
\[ Q_y = Q_{x0} \sin \alpha_0 + Q_{y0} \cos \alpha_0 \]  

(6.14)

Thus, equations (6.12) will become

\[ M_x = Q_{x0} (x \cos \alpha_0 + vy \sin \alpha_0) + Q_{y0} (-x \sin \alpha_0 + vy \cos \alpha_0) \]
\[ M_y = Q_{x0} (vx \cos \alpha_0 + y \sin \alpha_0) + Q_{y0} (-vx \sin \alpha_0 + y \cos \alpha_0) \]  

(6.15)

\[ H_{xy} = 0 \]

Sequentially, if equations (6.14) and (6.15) are substituted into equations (6.9), one can obtain the constants \( A_1 \) and \( A_2 \), written as

\[ A_1 = \left\{ \frac{(Q_{x0} - iQ_{y0})e^{-i\alpha_0}}{-8D} \right\} \], \[ A_2 = \left\{ \frac{(Q_{x0} + iQ_{y0})e^{i\alpha_0}}{-8D} \right\} \]  

(6.16)
When the special condition of uniform shear ($Q_{ox} = Q_{ox} = 0$) is considered, these constants will be reduced to those given by equations (6.13).

**General Solution of the Boundary Equation**

For the hole problem that the boundary moment and shear resultant are prescribed, Leknitskii [36] derived the following boundary equation (which can be also referred to the work by Savin [53] and Yu [75]),

\[
N \phi(z) + \bar{\phi}'(\bar{z}) + \psi(z) = \frac{1}{D} \left( \frac{1}{1-v} \right) \int_0^s \left[ m(s) + i \int_0^s q(s) \, ds \right] (dx + idy) \quad (6.17)
\]

where $m(s)$ and $q(s)$ are the prescribed moment and shear resultant along the boundary $s$ and $N$ is defined by $-(3+v)/(1-v)$. For a free boundary along the crack surface (with $m(s) = q(s) = 0$), the whole term on the right-hand side of equation (6.17) turns out to be zero. If one considers the rigid inclusion problem [36], it can be shown that $N$ is equal to 1. By introducing a mapping function $z = \omega(\zeta)$, the term $\phi' (d\phi/dz)$ in equation (6.17) can be rewritten as $\phi'(\zeta)/\omega'(\zeta)$. For the hole problem with general transverse loads applied on the boundary, the boundary equation (6.17) is transformed into

\[
N \phi(\xi) + \frac{\omega(\xi)}{\omega'(\xi^{-1})} \phi'(\xi^{-1}) + \psi(\xi^{-1}) = f(\xi) \quad (6.18)
\]

where the constant $N$ is equal to $-(3+v)/(1-v)$, and the function $f(\zeta)$ now represents the transformed function of the whole term on the right-hand side of equation (6.17) in the \( \zeta \)-plane.

Following the same procedure as before, one can eventually obtain the solution of the boundary equation (6.14), written as
\[ \phi(\zeta) = \frac{(1-c^2)(c^2 \mathcal{F}_1'(i/c) - N \mathcal{F}_1'(i/c))}{N(N^2-c^4)(\zeta+i/c)} - \frac{\mathcal{F}_1(\zeta)}{N} \] (6.19)

where, as before, the function \( \mathcal{F}_1(\zeta) \) denotes the Cauchy integral of the function \( f(\xi) \). To complete the solution to this transverse flexure problem, one must also determine the other stress function \( \psi(\zeta) \). For this purpose, the boundary equation (6.18) is rewritten in its conjugate form as

\[ N\bar{\phi}(\xi) + \frac{\omega(\xi)}{\omega'(\xi)} \phi'(\xi) + \psi(\xi) = \bar{f}(\xi) \] (6.20)

As Cauchy integrals are taken for each term of equation (6.20), the first term on the left hand side will be sifted out, and as a result, the function \( \psi(\zeta) \) is found to be

\[ \psi(\zeta) = -\mathcal{F}_2(\zeta) + \frac{1}{2\pi i} \int_{\gamma} \frac{\omega(\xi) \phi'(\xi)}{\omega'(\xi)} \frac{d\xi}{(\xi-\zeta)} \] (6.21)

where \( \mathcal{F}_2(\zeta) \) is, as before, the Cauchy integral of \( \overline{f(\xi)} \). Reviewing the form of \( \psi(\zeta) \) obtained in Chapter 3, one can note that they are exactly of the same form. However, insertion of the new \( \phi(\zeta) \) in (6.19) is necessary to evaluate \( \psi(\zeta) \), given by equation (6.21).

**Complex Stress Functions**

**General bending and twisting**

The problems with general bending and twisting at infinity (described from case (a) to case (d)), having the common characteristics that the stress functions associated with the loads
possess only the first order of $z$, are assorted as one catalog. Therefore, all the problems can be solved altogether, and different constants are assigned to the final solution for different problems. Referring back to the general expressions for the complex stress functions with $z$ replaced by the transformation function $\omega(\zeta)$, one can write down the stress functions as follows,

$$
\phi(\zeta) = AR \left( \frac{\zeta^2 + 1}{\zeta + ic} \right) + \phi_0(\zeta)
$$

$$
\psi(\zeta) = (B + iC)R \left( \frac{\zeta^2 + 1}{\zeta + ic} \right) + \psi_0(\zeta)
$$

(6.22)

In a similar manner as before, as these expressions for complex stress functions (6.22) are inserted into the boundary equation (6.18) with $f(\xi)$ taken as zero, it can be seen that the holomorphic functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$ will satisfy the boundary equation in such a way that the function $f(\xi)$ is replaced by a function $f_0(\xi)$, given by

$$
f_0(\xi) = R \left[ -A(N+1) \frac{\xi^2 + 1}{\xi + ic} - (B - iC) \frac{\xi^2 + 1}{\xi (1 - ic\xi)} \right]
$$

(6.23)

For convenience, the subscript 0 of the function $f_0(\xi)$ is dropped from now on. As before, when applying Cauchy's formulae to evaluate the Cauchy integral of $f(\xi)$, one needs to disassociate the pole(s) occurring inside the unit circle $\Gamma$ from all functions. For this purpose, $f(\xi)$ is rewritten as

$$
f(\xi) = -AR(N+1) \left( \{\xi - ic\} + \frac{1 - c^2}{\xi + ic} \right) - (B - iC)R \left( \frac{1}{\xi} + \left( \frac{\xi + ic}{1 - ic\xi} \right) \right)
$$

(6.24)

where, as before, the curly bracket is used to denote the functions inside are holomorphic inside
Immediately, it follows that the Cauchy integral of \( f(\zeta) \), denoted by \( \mathcal{F}_1(\zeta) \), turns out to be

\[
\mathcal{F}_1(\zeta) = AR(N+1) \frac{1-c^2}{\xi + ic} + \frac{(B-iC)R}{\zeta}
\]  

(6.25)

Also, \( \mathcal{F}_1(\zeta) \) can be differentiated with respect to \( \zeta \) to obtain \( \mathcal{F}_1'(\zeta) \), written as

\[
\mathcal{F}_1'(\zeta) = -AR(N+1) \frac{1-c^2}{(\xi + ic)^2} - \frac{(B-iC)R}{\zeta^2}
\]  

(6.26)

After the expression for \( \mathcal{F}_1(\zeta) \) (6.25) is substituted into equation (6.19) with appropriate insertion of the constant \( 1/ic \), the final expression for the complex stress function \( \phi_0(\zeta) \) (with its subscript 0 recovered) is written as

\[
\phi_0(\zeta) = \frac{A_0}{\zeta + ic} - \frac{B_0}{\zeta}
\]  

(6.27)

where the constants \( A_0 \) and \( B_0 \) represent

\[
A_0 = A \cdot \frac{R(N+1)(N-c^2)(c^4 + Ne^2 - N)}{N(N^2 - c^4)} + B \cdot \frac{Rc^2 (1-c^2)(N-c^2)}{N(N^2 - c^4)}
\]

\[
+ C \cdot \frac{iRc^2 (1-c^2)(N+c^2)}{N(N^2 - c^4)}
\]  

(6.28)

\[
B_0 = \frac{(B-iC)R}{N}
\]

To complete the solution, the next step is to evaluate the function \( \psi(\zeta) \) from the conjugate boundary equation (6.20). To do this, the expression for \( \phi_0(\zeta) \) in (6.27) is differentiated with respect to \( \zeta \), and then the obtained form of \( \phi_0'(\zeta) \) is substituted into equation (6.20). To evaluate the Cauchy integrals in equation (6.17), one has to rewrite the associated functions as
\[ f(\xi) = -A \cdot R(N+1) \left( \frac{1}{\xi} + \frac{1}{1 - iC \xi} \right) - (B + iC) R \left( \frac{\xi - iC + 1}{\xi + iC} \right) \]

\[ \frac{\omega(\xi)}{\omega'(\xi)} = -A_0 \left( \frac{iC}{1 - iC \xi} + \frac{\xi^2 + 2iC \xi + 1}{\xi^2 + 2iC \xi - 1} \right) + B_0 \left( \frac{iC (1 - C^2)^2}{1 - iC \xi} \right) \]

\[ + \frac{(1 - C^2)^2 \xi^4 + iC (4 - 5C^2 + 2C^4) \xi^3 + [(1 - C^2)^2 - C^2] \xi^2 + iC (2 - C^2) \xi - C^2}{\xi^3 (\xi^2 + 2iC \xi - 1)} \] (6.29)

where all the terms in the curly bracket are holomorphic inside \( \Gamma \), and therefore the Cauchy integrals of all these terms turn out to be zero. Applying Cauchy's formulae to evaluate the Cauchy integral of all the terms in the conjugate boundary equation, one will obtain the function \( \psi_0(\zeta) \) (with its subscript 0 recovered), written as

\[ \psi_0(\zeta) = -AR(N+1) - \frac{(B + iC) R(1 - C^2)}{\zeta + iC} + A_0 \left( \frac{\zeta^2 + 2iC \zeta + 1}{\zeta^2 + 2iC \zeta - 1} \right) \]

\[ - B_0 \frac{(1 - C^2)^2 \zeta^4 + iC (4 - 5C^2 + 2C^4) \zeta^3 + [(1 - C^2)^2 - C^2] \zeta^2 + iC (2 - C^2) \zeta - C^2}{\zeta^3 (\zeta^2 + 2iC \zeta - 1)} \] (6.30)

As before, to calculate the stress and displacement components in the whole domain, it is necessary to differentiate equation (6.30) using the chain rule to obtain \( d\psi_0(\zeta)/d\zeta \). All of the equations needed to compute the stress and displacement components for this general bending and twisting problem are listed in Appendix D for reference.

**General shearing**

Basically, the philosophy of treating the problem with shear is more or less like the one with bending and twisting. The only obvious difference is that the stress functions due to the
applied shearing stress take the form of second order. Although this modification is slight, the whole formulations will become much more complicated. First, consider equation (6.8) with the variable $z$ replaced by the mapping function $\omega(\zeta)$ so that the complex stress functions are written as

$$
\phi(\zeta) = A_1 \cdot R^2 \left( \frac{\zeta^2 + 1}{\zeta + ic} \right)^2 + \phi_0(\zeta)
$$

(6.31)

$$
\psi(\zeta) = A_2 \cdot R^2 \left( \frac{\zeta^2 + 1}{\zeta + ic} \right)^2 + \psi_0(\zeta)
$$

where the functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$ are, as before, holomorphic outside $\Gamma$, and $A_1$ and $A_2$ are complex constants. In the same manner as before, equation (6.31) is substituted into the boundary equation (6.18) with the loading integration function taken as zero. As a result, the boundary equation will be so satisfied by the functions stress functions $\phi_0$ and $\psi_0$ that the function $f(\xi)$ is replaced by a function,

$$
f(\xi) = -N A_1 \cdot R^2 \left( \frac{\xi^2 + 1}{\xi + ic} \right)^2 - \frac{2 A_1 \cdot R^2 (\xi^2 + 1)^2}{\xi(\xi + ic)(1 - i\xi)} - A_2 \cdot R^2 \frac{(\xi^2 + 1)^2}{\xi^2 (1 - i\xi)^2}
$$

(6.32)

Again, equation (6.32) needs to be disassembled for the poles (at the points zero and $-ic$) occurring inside $\Gamma$. The function $f(\xi)$ can be rewritten as

$$
f(\xi) = \left\{ \frac{g_1(\xi)}{(1 - i\xi)^2} \right\} + \frac{g_2(\xi)}{\xi^2 (\xi + ic)^2}
$$

(6.33)

where the function in the curly bracket is holomorphic inside $\Gamma$, and the functions $g_1(\xi)$ and $g_2(\xi)$ represent
\[ g_1(\xi) = NA_1 c^2 R^2 \xi^4 + 2 i c R^2 \left( A_1 + NA_1 - NA_1 c^2 \right) \xi^3 \]
\[ + R^2 (-2A_1 - A_2 + 2c^2 A_1 - NA_1 + 6NA_1 c^2 - 3NA_1 c^4) \xi^2 \]
\[ + 2icR^2 (3A_1 - c^2 A_2 + 3NA_1 - 3NA_1 c^2) \xi \]
\[ + R^2 (3NA_1 c^2 - 2NA_1 + 3c^2 A_2 - 2A_2 - 4A_1) \]  
(6.34)

\[ g_2(\xi) = 2icR^2 (A_1 - A_2 + 2NA_1 - 2NA_1 c^2) \xi^3 \]
\[ - R^2 (2A_1 + A_2 + 2c^2 A_1 - 4c^2 A_2 + NA_1 + 2c^2 NA_1 - 3c^4 NA_1) \xi^2 \]
\[ - 2icR^2 (A_1 + A_2 - c^2 A_2) \xi + R^2 c^2 A_2 \]

It immediately follows from Cauchy's formulae that the Cauchy integral of \( f(\xi) \) is written as

\[ \mathcal{F}_1(\zeta) = \frac{-g_2(\zeta)}{\zeta^2 (\zeta + ic)^2} \]  
(6.35)

As before, to determine the explicit form of the complex stress function \( \phi_0(\zeta) \), equation (6.35) needs to be differentiated once to give

\[ \mathcal{F}_1'(\zeta) = \frac{g_3(\zeta)}{\zeta^3 (\zeta + ic)^3} \]  
(6.36)

where \( g_3(\zeta) \) represents

\[ g_3(\zeta) = -2icR^2 (-A_1 + A_2 - 2NA_1 + 2NA_1 c^2) \zeta^4 \]
\[ + 2R^2 (-2A_1 - A_2 - c^2 A_1 - 3c^2 A_2 + NA_1 + e^4 NA_1) \zeta^3 \]
\[ + 6icR^2 (-A_1 + A_2 + c^2 A_2) \zeta^2 + 2c^2 R^2 (A_1 + 3A_2 - c^2 A_2) \zeta \]
\[ + 2ic^3 R^2 A_2 \]  
(6.37)
For brevity, the constant term 

\[(1-c^2)(c^2 \mathcal{F}'(1/ic) - N \mathcal{F}'(i/c)) / N(N^2-c^4)\]

in the expression for \(\phi(\zeta)\) in (6.19) is denoted by \(\mathcal{F}_c\). After the value \(1/ic\) is inserted into the function \(\mathcal{F}_1(\zeta)\), the constant \(\mathcal{F}_c\) is found to be

\[
\mathcal{F}_c = \frac{2ic^3 R^2 T \{ (c^2-N^2) A_1 + (1-c^2) [N(A_1+2A_2)+2c^2A_2] \}}{N(N^2-c^4)} \tag{6.38}
\]

Thus, the final explicit formulation of \(\phi_0(\zeta)\), obtained by equation (6.19) as well as equation (6.35), is written as

\[
\phi_0(\zeta) = -\mathcal{F}_c + \frac{g_2(\zeta)}{\zeta+ic} \tag{6.39}
\]

Once the shearing condition is prescribed at infinity, the constants \(A_1\) and \(A_2\) can be determined from equation (6.9), and therefore the stress function \(\phi_0\) can be obtained. For the particular problem of uniform shear, the condition of equation (6.11) can be inserted into the related equations, and the final form of the stress function can be obtained in a straightforward manner.

All the related equations to calculate the stress and displacement components are listed in Appendix D for reference.

To complete the solution to the general shearing problem, the same procedure as before is followed to find the other function \(\psi_0(\zeta)\). First, equation (6.39) has to be differentiated with respect to \(\zeta\) once to give

\[
\phi_0'(\zeta) = \frac{\mathcal{F}_c}{(\zeta+ic)^2} - \frac{g_3(\zeta)}{N\zeta^3(\zeta+ic)^3} \tag{6.40}
\]

As before, the expression for \(\phi_0'\) in (6.40) with \(\zeta\) replaced by the boundary value \(\xi\) and all other functions to be evaluated for their Cauchy integrals are disassociated for the poles occurring inside \(\Gamma\). For this purpose, the function \(\mathcal{F}(\xi)\) is rewritten as
\[
f(\xi) = \left\{ R^2(-4A_1 - 2A_2 + 3A_2c^2 - 2NA_1 + 3Nc^2A_1) + \frac{\xi^4g_2(\xi)}{(1-ic\xi)^2} \right\} + \frac{g_4(\xi)}{\xi^2(\xi + ic)^2} \quad (6.41)
\]

where the functions inside the curly bracket are holomorphic inside \( Y \), and the function \( g_4(\xi) \) represents
\[
g_4(\xi) = 2icR^2(A_1 + 2A_2 - 2c^2A_2 - N\overline{A_1})\xi^3
\]
\[-R^2(2A_1 + 2c^2A_1 + 2c^2A_2 - 3c^4A_2 + N\overline{A_1} - 4Nc^2\overline{A_1})\xi^2 \quad (6.42)
\]
\[-2icR^2(A_1 + NA_1 - Nc^2\overline{A_1})\xi + Nc^2R^2\overline{A_1}
\]

Similarly, the function \( \frac{\overline{\omega}(\xi)\phi_0'(\xi)}{\omega'(\xi)} \) is rewritten as
\[
\frac{\overline{\omega}(\xi)\phi_0'(\xi)}{\omega'(\xi)} = \frac{g_5(\xi)}{N\xi^4(\xi + ic)(\xi^2 + 2ic\xi - 1)}
\]
\[+ \left\{ \frac{2(1-c^2)(-\overline{A_1}c^2R^2 - 2\overline{A_2}c^2R^2 + 2\overline{A_2}c^4R^2 + A_1Nc^2R^2) + icN\mathcal{F}_c}{N(1-ic\xi)} \right\} \quad (6.43)
\]

where the function in the curly bracket is holomorphic inside \( Y \), and the function \( g_5(\xi) \) represents
\[
g_5(\xi) = -2ic(1-c^2)R^2[\overline{A_1} - 2\overline{A_2}(1-c^2) + NA_1]\xi^6
\]+2R^2[NA_1(1-c^2)(1+3c^2) + A_1(2-4c^2+3c^4)+\overline{A_2}(1-10c^2+14c^4-6c^6)]\xi^5
\]-2icR^2[2\overline{A_1}(-2+2c^2-c^4) + \overline{A_2}(-5+14c^2-12c^4+4c^6)
\]+NA_1(1-c^2)(1-2c^2)]\xi^4
\]+2R^2[NA_1(1-c^4)+A_1(1-c^2)(2-c^2)+\overline{A_2}(1-9c^2+8c^4-2c^6)]\xi^3
\]+2ic(-3+c^2)R^2[\overline{A_1} - \overline{A_2}(1-2c^2)]\xi^2 + 2c^2R^2[\overline{A_1} - \overline{A_2}(3-2c^2)]\xi - 2ic^3R^2\overline{A_2} \quad (6.44)
\]
As a result of applying Cauchy's formulae to evaluate the Cauchy integrals of (6.20), the final expression for the complex stress function $\psi_0(\zeta)$ is written as

$$\psi_0(\zeta) = \frac{g_4(\xi)}{\xi^2(\xi + i\xi)^2} - \frac{g_5(\xi)}{N\xi^4(\xi + i\xi)(\xi^2 + 2i\xi - 1)} \quad (6.45)$$

Also, this equation (6.45) can be differentiated using the chain rule to give $d\psi_0/d\zeta$, listed in Appendix D.

Both of the explicit expression for the complex stress function $\phi_0(\zeta)$ and $\psi_0(\zeta)$ have been given in terms of the constants $A_1$ and $A_2$, which are determined from the specified shearing condition. The final expressions for the complex stress functions $\phi(\zeta)$ and $\psi(\zeta)$ can then be obtained from equation (6.31). Thus, the problem with general shear is completely solved.

**Stress Intensity Factors**

**General bending and twisting**

For this problem with general bending and twisting, the complex stress function $\phi_0(\zeta)$ is given by equation (6.27). Since the complex stress functions due to the loading at infinity without the disturbance of the crack essentially make no contribution to the stress intensity factors, the same approach implemented before can be applied to determine the $K$-factors through $d\phi_0/d\zeta$. To do so, equation (6.27) can be differentiated using the chain rule to give

$$\frac{d\phi(\zeta)}{d\zeta} = \left(\frac{-A_0}{R} + \frac{B_0(\zeta + i\xi)^2}{R\xi^2}\right)\frac{1}{\xi^2 + 2i\xi - 1} \quad (6.46)$$

Similarly, to evaluate the stress intensity factors for the transverse flexure problem, Sih, etc.
[59] gave the approach as follows,

\[ K = K_f - i K_{III} = -\frac{12\sqrt{2} \pi D (3+v)}{h^2} \lim_{z \to z_1} \left( \frac{d \phi(z)}{dz} \right) \quad (6.47) \]

Thus, it immediately follows from the substitution of equation (6.46) along with equation (4.19) into equation (6.47) that the stress intensity factors for this general bending and twisting case can be expressed as

\[ K_{(1/2)} = -\frac{6 \sqrt{\pi a D (3+v)} \left[ (-1^+ \cdot) i c + \sqrt{1-c^2} \right]}{R \ h^2 (1-c^2)} \left\{ -A_0 + \frac{B_0 (1-c^2)}{-i c (+\cdot) \sqrt{1-c^2}} \right\} \quad (6.48) \]

where the constants \( A_0 \) and \( B_0 \) are given by equation (6.28).

**General shear**

For determining the \( K \)-factors for the general shearing case, differentiation of the complex stress function \( \phi_\phi(\zeta) \) using the chain rule is performed and yields

\[ \frac{d \phi(\zeta)}{dz} = \left[ \frac{\mathcal{F}_c}{R} - \frac{g_3(\zeta)}{R N \zeta^3 (\zeta + ic)} \right] \frac{1}{\zeta^2 + 2 i c \zeta - 1} \quad (6.49) \]

Thus, by applying equations (6.47) and (4.19), the stress intensity factors can be written as

\[ K_{(1/2)} = \frac{6 \sqrt{\pi a D (3+v) \cdot ((-\cdot) \zeta)} (2/1)}{R h^2 (1-c^2)} \left[ \mathcal{F}_c - \frac{g_3(\zeta)}{N \xi^3 (1/2) \cdot ((+\cdot) \sqrt{1-c^2})} \right] \quad (6.50) \]

where the constant \( \mathcal{F}_c \) is given by equation (6.38), and \( \xi_{(1)} \) (or \( \xi_{(2)} \)) is the corresponding...
coordinate of the crack tip 1 (or crack tip 2) in the \( \zeta \)-plane, given by equation (3.9).

The equations given for these bending, twisting, and shearing problems have taken general loading conditions into account. However, due to the generality of the obtained solutions, these formulations involve too many parameters for one to get insight into the problems. Like the extension problem investigated in Chapter 3, if only bending moment or shearing stress is uniformly applied on the whole crack edge, all pertinent equations will be substantially simplified. By use of the superposition principal stated in Chapter 3, the stress intensity factors for the all-round bending problem of case (c) are actually identical with those for the case when uniform bending moment is prescribed on the whole crack edge. Although impractical, if uniform shearing stress is applied, the whole formulations can be also considerably simplified. In the subsequent section, these two typical examples will be studied as illustrations of the transverse flexure problems.

**Example Cases**

In this section, the typical loading condition is considered that uniform bending moment (or shearing stress) is applied on the whole crack surface. Recall the formulation of the stress function \( \phi_0 \) for general flexural loads, given by equation (6.19). To apply equation (6.47) to evaluate the stress intensity factors, equation (6.19) can be differentiated using the chain rule to give \( d\phi(\zeta)/dz \), expressed as

\[
\frac{d\phi(\zeta)}{dz} = \frac{(1-c^2)^2(c^2 \mathcal{F}_1/(1/\iota c) - N \mathcal{F}_1/(i\iota c))}{NR(N^2-c^4)(\zeta^2+2i\iota c\zeta-1)} \cdot \frac{\mathcal{F}_1/(\zeta+i\iota c)^2}{NR(\zeta^2+2i\iota c\zeta-1)} \quad (6.51)
\]
As a result of substitution of equations (6.51) and (4.19) into equation (6.47), the stress intensity factors can be obtained by

\[
K_{(1/2)} = \frac{6D(3+v)\gamma((\xi_{(2)})\xi_{(1)})\sqrt{\pi a} c^2}{Rh^2} \left[ \frac{(1-c^2)(\xi_{(2)}-\xi_{(1)})}{N^2-c^4} \right]
\]

(6.52)

where the function \( \mathcal{F} \), as before, represents the differentiated form of the Cauchy integral of the function \( f(\xi) \), and \( \xi_{(1)} \) (or \( \xi_{(2)} \)) is the corresponding coordinate of the crack tip 1 (or 2) in the \( \zeta \)-plane, given by equation (3.9). If the distribution of the flexural loads along the crack surface is specified, the Cauchy integral of the function \( f(\xi) \) can be evaluated either by Cauchy's formulae or through direct integration. Eventually, the stress intensity factors can be obtained in a straightforward manner after everything is inserted into equation (6.52).

**Uniform bending moment applied on the whole crack edge**

Considering the particular case when uniform bending moment acts upon the whole crack surface, one can either start either from the formulation for the special case of all round bending or from the formulation for general loading condition. If one chooses to start from the all round bending case, then the loading integration function given by (6.23) with appropriate constants substituted in the equation should be used. If formulating general loading on the crack surface is the preferred way to start, equation (6.17) should be used instead. No matter which way one chooses to start from, it is good to note that identical result will be reached. As a matter of fact, this identity verifies the principal of superposition, described in Chapter 3.

For this problem with constant bending moment \( M_0 \) applied on the crack surface, starting from either way, one may obtain the loading integration function \( f(\xi) \), written as
Thus, the Cauchy integral of \( f(\xi) \), obtained by applying Cauchy's formulae, is expressed as

\[
\mathcal{F}_1(\zeta) = \frac{M_0 R (1-c^2)}{D (1-v)(\zeta + i c)}
\]

(6.54)

Once \( \mathcal{F}_1(\zeta) \) is differentiated with respect to \( \zeta \) to get \( \mathcal{F}'_1(\zeta) \), all related constants can be directly substituted into equation (6.52). Sequentially, the stress intensity factors turn out to be

\[
K_I = \frac{6 M_0 \sqrt{\pi} a}{h^2} \cdot \frac{\sqrt{1-c^2}}{N+c^2}, \quad K_{III} = \frac{6 M_0 \sqrt{\pi} a}{h^2} \cdot \frac{N c}{N+c^2}
\]

(6.55)

It is interesting to note that when this curved crack is degenerated into a straight cut \( (c = 0) \), one may obtain \( K_{I(c)} = 6 M_0 \sqrt{\pi} a / h^2 \), \( K_{III(c)} = 0 \), which are exactly identical with the results obtained by Sih [59]. It was observed by Sih that the factor \( 6M_0 / h^2 \) corresponds to the extensional stress \( P \) in the uniform extension problem. Furthermore, upon observing the above result, one may define the geometric function in the same manner as before. These geometric functions are written as

\[
GF1(\eta) = \frac{N \sqrt{1-c^2}}{N+c^2}, \quad GF3(\eta) = \frac{N c}{N+c^2}
\]

(6.56)

As before, the geometric functions can be also rewritten in terms of the angle \( \eta \) as follows,

\[
GF1(\eta) = \frac{(3+v) \cos \eta}{(3+v)-(1-v)\sin^2 \eta}, \quad GF3(\eta) = \frac{(3+v) \sin \eta}{(3+v)-(1-v)\sin^2 \eta}
\]

(6.57)
For a stainless steel plate \( (v = 0.28) \), the geometric functions are graphed as a function of \( c \) and \( \eta \) in Figure 6.2 (a) and (b), respectively. Also, for a aluminum \( (v = 0.33) \) and lead plate \( (v = 0.43) \), these factors are plotted in Figure 6.3 and Figure 6.4, respectively. In the figures (Figure 6.2 ~ 6.4), it can be seen that the geometric functions for the both fracture modes I and III will be identical when the curved crack is a semi-circular crack \( (c = c_0 = 2^{1/2}) \). By comparing these graph for the effect of the Poisson's ratio, it is observed that as the Poisson's ratio is increased, the geometric functions will be reduced. In other words, for a ductile material (with higher Poisson's ratio), the percent change of \( K_I \) is more severe than the one for a brittle material (with lower Poisson's ratio) while the percent change of \( K_{III} \) goes in the opposite way. Moreover, it can be seen that the effect of the Poisson's ratio on \( K_I \) is more obvious in a certain range of \( c \) from 0.6 to 0.95. However, the effect of the Poisson's ratio on \( K_{III} \) is negligible as long as \( c \) is below the critical value \( c_0 \). Basically, these figures look similar to those for the extension problem. Another interesting phenomenon observed is that the geometric function for mode III has an almost proportional relation with the deviation angle \( \eta \) up to the point \( \eta = 1 \). Without resorting to complicated formulae for calculation, this simple relation gives us a convenient estimate of \( K_{III} \) directly from the geometry of the curved crack, provided the angle \( \eta \) is smaller than unity.

**Uniform shearing stress applied on the whole crack edge**

In fact, this particular case that only uniform shearing stress is applied along the whole crack surface is not so practical as the one with uniform bending moment since no corresponding applied shear at infinity can produce this case by the superposition principal. Nonetheless, reviewing the important work of Chapter 5 for concentrated loading, one may note that the obtained solution to the problem may be further modified to consider concentrated
Figure 6.2: Geometric functions for a steel plate with all-round bending as a function of (a) the mapping variable \( c \), (b) the angle \( \eta \)
Figure 6.3: Geometric functions for an aluminum plate with all-round bending as a function of (a) the mapping variable $c$, (b) the angle $\eta$. 
Figure 6.4: Geometric functions for a lead plate with all-round bending as a function of (a) the mapping variable $c$, (b) the angle $\eta$. 
shearing forces applied at an arbitrary point on the crack surface. Moreover, the same modification can be made for the case with concentrated bending moment, and therefore the obtained solutions for the both cases with concentrated flexural loads can be implemented as the Green's function to formulate many other transverse flexure problems with sophisticated loading conditions.

For this constant shear problem, one has to resort to the boundary equation (6.17). Let the constant shearing stress, denoted by \( q_0 \) herein, act upon the whole crack surface. From the definition of the loading integration function, it immediately follows that the function \( f(\zeta) \) can be written as

\[
\frac{i q_0 R^2 (\zeta^4 + 2 \zeta^2 + 1)}{2 D (1 - \nu) (\zeta + i\epsilon)^2} \quad (6.58)
\]

As before, the function \( f(\zeta) \) needs to be disassociated for the pole \(-i\epsilon\) occurring inside the unit circle \( \mathcal{Y} \). This can be done by rewriting the function \( f(\zeta) \) as

\[
f(\zeta) = \left\{ \zeta^2 - 2 i\epsilon \zeta + 2 - 3 \epsilon^2 \right\} + \frac{(1 - \epsilon^2)(1 + 3 \epsilon^2 - 4 i\epsilon \zeta)}{(\zeta + i\epsilon)^2} \quad (6.59)
\]

where the functions in the curly bracket are holomorphic inside \( \mathcal{Y} \) without a pole. Therefore, directly by applying Cauchy's formulae, the Cauchy integral of the function \( f(\zeta) \) can be written as

\[
\mathcal{F}_1(\zeta) = \frac{i q_0 R^2 (1 - \epsilon^2)(4 i\epsilon \zeta - 1 - 3 \epsilon^2)}{2 D (1 - \nu) (\zeta + i\epsilon)^2} \quad (6.60)
\]

For determining the stress function \( \phi(\zeta) \) and the \( K \)-factor, differentiation of equation (6.60) with respect to \( \zeta \) is performed and yields
Finally, one can substitute $1/i\zeta$ and the value of $\xi_{(r)}$ into equation (6.52), and then the final expression for the stress intensity factor is written as

$$K = \frac{-6\sqrt{\pi a q_0 N R(i c - \sqrt{1 - c^2})}}{h^2} \left[ \frac{c}{N + c^2} + i\sqrt{1 - c^2} - 2c \right]$$

(6.62)

It is seen that the major difference between the present case for shear and the previous one for bending is that the stress intensity factor for this example has an order of $a^{3/2}$. If the variable $N$ is replaced by $(3 + v)/(v - 1)$, the stress intensity factor can be rewritten as

$$K = K_r - iK_{III}$$

$$= \frac{3q_0(3 + v)\sqrt{\pi a^3}}{h^2(1 - v)} \cdot \frac{ic - \sqrt{1 - c^2}}{\sqrt{1 - c^2}} \left[ \frac{c^3(v - 1)}{3 + v + c^2(v - 1)} - 2c + i\sqrt{1 - c^2} \right]$$

(6.63)

Specifically, let $c$ equal zero (for the problem with straight crack), and then the stress intensity factor for this case is reduced to

$$K_{(s)} = 0 , \quad K_{III(s)} = \frac{3q_0(3 + v)\sqrt{\pi a^3}}{h^2(1 - v)}$$

(6.64)

where the subscript $s$, as before, is used to represent the straight crack. In the same manner as before, the stress intensity factor given by equation (6.63) can be normalized by $K_{III(s)}$ to give the geometric function. Therefore, the geometric function is written as

$$GR(c) = \frac{ic - \sqrt{1 - c^2}}{\sqrt{1 - c^2}} \left[ \frac{c^3(v - 1)}{3 + v + c^2(v - 1)} - 2c + i\sqrt{1 - c^2} \right]$$

(6.65)
Also, the geometric function can be rewritten in terms of the angle \( \eta \) as

\[
GR(\eta) = \frac{e^{-i\eta}}{\cos \eta} \left[ \frac{(v-1)\sin^3 \eta}{3 + v + (v-1)\sin^2 \eta} - 2\sin \eta + i\cos \eta \right]
\]  

(6.66)

As an example, the geometric function is plotted for a stainless steel plate \((v = 0.28)\) as a function of \( c \) and \( \eta \) in Figure 6.5 (a) and (b), respectively. From this figure, it is seen that the geometric function for mode I will converge to some value as the both crack tips get close enough with each other, however it will diverge for mode III.
Figure 6.5: Geometric functions for a stainless steel plate with flexural shear on the crack as a function of (a) the mapping variable $c$, (b) the angle $\eta$. 
CHAPTER 7. FRACTURE OF A PLATE CONTAINING AN ELLIPTICAL CUT

From Chapter 3 to Chapter 6, the fracture of a plate (with in-plane extensional loads and transverse flexural loads) weaken by a circular arc crack is investigated. From a practical point of view, the mapping function, devised to transform the contour surface of a circular arc crack to a unit circle, is found useful especially for the case when a shallowly curved crack is modelled by a short arc of a circle with large radius. Undoubtedly, besides the classical model of the straight cut, this modelling of a circular arc crack provides another useful crack model for engineering analysis.

As a deeply curved crack is concerned, which has a larger height than its half-base length, the circular arc model previously described is no longer appropriate. Instead, the model with an elliptical shape may be a better substitute as far as the geometry of the crack is concerned. Under this situation, even though the involved computation will be much more complicated, it will be better to adopt the elliptical crack model. In this chapter, to provide a better model for such a curved crack, the original mapping is modified to transform the contour surface of an elliptical cut (shown in Figure 7.1) to a unit circle. As shown in the figure, the major axes of the elliptical cuts (drawn to their real scales) are along the y-axis, and $\beta$ is the aspect ratio, which is defined by the ratio of the major axis length to the minor axis length. The same scheme employed previously for the model of a circular arc crack can be implemented again, but much more complicated computation will be involved for the present case.

As an example, the $K$-factors of a semi-elliptical crack is numerically calculated as a function of its aspect ratio and compared with the results for a straight crack with total crack
Figure 7.1: A group of circular arc cracks and elliptical cracks

length of the magnitude of the ellipse's semi-major axis.

Mapping Function

Recall the mapping function (3.7), devised to transform the contour surface of a circular arc crack to a unit circle. For the present purpose, the circular arc model may be so modified by stretching out its y-axis that the circular arc is evolved into an elliptical shape. By this means, the original mapping function is modified to the form as follows,
\[ \omega(\zeta) = R \cdot \text{Re} \left\{ \frac{\zeta^2 + 1}{\zeta + i c} \right\} + i V \cdot \text{Im} \left\{ \frac{\zeta^2 + 1}{\zeta + i c} \right\} \]  

(7.1)

where the constants \( R \) and \( V \) are defined by \( R = a \left(1 - c^2\right)^{1/2} / 2 \), \( V = b \left(1 - c^2\right)^{1/2} / 2 \), and \( a, b, c \) are variables to determine the crack configuration. By use of conjugate operation, equation (7.1) can be further expressed as

\[ \omega(\zeta) = \frac{R + V}{2} \cdot \frac{\zeta^2 + 1}{\zeta + i c} + \frac{R - V}{2} \cdot \frac{\bar{\zeta}^2 + 1}{\bar{\zeta} - i c} \]  

(7.2)

Although this transformation, by which the unbounded domain containing an elliptical crack in the \( z \)-plane is mapped onto the unbounded one containing the unit circle in the \( \zeta \)-plane, is not only invertible but also single-valued, its derivative \( dz/d\zeta \) is not well defined everywhere except those points on the crack surface. However, the transformation is well defined on the crack surface where \( \zeta \) can be replaced by \( 1/\bar{\zeta} \) so that as evaluating its stress intensity factor, one may approach the crack tip of interest along the crack surface. Therefore, the mapping function for those points on the crack surface can be rewritten as

\[ \omega(\bar{\zeta}) = \frac{R + V}{2} \cdot \frac{\bar{\zeta}^2 + 1}{\bar{\zeta} + i c} + \frac{R - V}{2} \cdot \frac{\bar{\zeta}^2 + 1}{\bar{\zeta}(1 - i c \bar{\zeta})} \]  

(7.3)

In fact, this mapping function (7.3) transforms the surface of an elliptical crack with the semi-major axis length \( R b / (a c) \), the semi-minor axis length \( R / c \), and a base width (the distance between crack tips) \( 2a \) into a unit circle in the \( \zeta \)-plane. For the present purpose, \( b \) is always kept greater than \( a \) so that the major axis in \( y \) direction is larger than the minor axis in \( x \) direction to resemble a deeply curved crack. The proof that the locus of the unit circle will cast a segment of an ellipse is provided in Appendix E.
Considering the function \( \omega(\zeta) \), given by equation (7.3) with the boundary value \( \xi \) replaced by \( \zeta \), one can see that there is obviously a pole \( \zeta = 1/\imath \) occurring in the unbounded domain. Although the mapping is single-valued, this pole is not permitted in the unbounded domain outside the unit circle \( \Upsilon \). Any calculation for those points in the vicinity of the pole will become unstable and therefore yield erroneous results. Thus, those points in the vicinity of the pole are called malfunctioning points by the author. For the present purpose of this chapter to investigate only the stress intensity factors of the crack tips, the proposed mapping function can indeed work well since the approaching path to the crack tips is taken along the crack surface. Another evidence, verifying the validity of this mapping function, is that according to the characteristic of the Hilbert problem (refer to the reference [42]), the boundary value on the unit circle \( \Upsilon \) is unique (i.e. \( \omega(\xi) = \overline{\omega(\xi)} \)) so that the condition \( \phi^* = \phi \) on the crack surface can be met regardless of the approach direction (from the unbounded "+" or the closed "-" region).

An important observation for the practical implementation of this mapping is that, as the value \( V \) is set smaller than the one of \( R \) so that the major axis of the ellipse resides on the \( x \)-axis, the whole results will be invalidated. Principally, this is due to the fact that the malfunctioning points will be so close to the crack tips that the singularity of the crack tips will be largely increased. Under several numerical experiments, if \( V \) is chosen a larger value than \( R \), this mapping is assured to work very well. Since the goal is to determine the stress intensity factors by approaching the crack tips along the crack surface, all involved functions will be expressed in terms of the boundary value later.

Similarly, the inverse transformation can be written as

\[
\zeta = \frac{\pm \sqrt{\xi^2 - 4R(R - ic\xi)}}{2R} \quad (7.4)
\]
where $\Gamma$ is defined by $\Gamma = \frac{z(1 + R/V)}{2} + \frac{\bar{z}(1 - R/V)}{2}$, and the sign "±" is also defined in the same manner as before for the circular arc model. Similarly, the crack tips are mapped to $\xi_{(1)}$ and $\xi_{(2)}$ in the $\zeta$-plane, given by equation (3.9).

**Solution of the Boundary Equation**

For the points on the boundary, the term $\frac{\omega(\xi)}{\omega'(\xi^{-1})}$ in the boundary equation can be expressed as

$$\frac{\omega(\xi)}{\omega'(\xi^{-1})} = \frac{(\xi + i c)(\xi^2 + 1)(1 - i c \xi)[(R + S)(1 - i c \xi)\xi + (R - S)(\xi + i c)]}{c^2(R - S)\xi^2 + 2i c \xi - 1}(\xi - P_1)(\xi - P_2)(\xi - P_3)(\xi - P_4)$$

(7.5)

where $P_1 \sim P_4$ are four poles, given by

$$P_{12} = \frac{1}{2c} \left[ -i \left( \begin{array}{c} -1 \pm \sqrt{\frac{a+b}{a-b}} \ \ + \ -1 \pm \sqrt{\frac{a-b}{a+b}} \end{array} \right) \right]$$

(7.6)

The pair of poles $P_1$ and $P_2$ appear inside the unit circle $\Upsilon$, and the other pair of poles $P_3$ and $P_4$ fall outside of $\Upsilon$. It can be proved that if the value of $b$ approaches the value of $a$, $P_3$ and $P_4$ will approach infinity and the limits of $P_1$ and $P_2$ are equal to $-ic$, which is exactly the case for the circular arc crack. As before, to evaluate the Cauchy integral of the term associated with $\omega(\xi)/\omega'(\xi^{-1})$, equation (7.5) is further rewritten as the form,

$$\frac{\omega(\xi)}{\omega'(\xi^{-1})} = \frac{C_0}{\xi} + \frac{C_1}{\xi - P_1} + \frac{C_2}{\xi - P_2} + \mathcal{H}(\xi)$$

(7.7)
where $C_0$, $C_1$, and $C_2$ are constants to be determined, and $\mathcal{H}(\xi)$ is some holomorphic function inside the unit circle. Due to the fact that the Cauchy integral of the function $\mathcal{H}(\xi); \frac{\phi'(\xi^{-1})}{\xi}$, also holomorphic inside $\Upsilon$, will be dropped, the function $\mathcal{H}(\xi)$ does not need to be determined. The constants $C_1$ (or $C_2$) can be determined by multiplying $(\xi^{-P_1})$ (or $(\xi^{-P_2})$) on both sides of equation (7.7) and then substituting $\xi = P_1$ (or $\xi = P_2$) into it. Eventually, after a process of basic algebraic operations, one will obtain these constants, written as

$$C_1 = \frac{(P_1 + ic)(P_1^2 + 1)(1 - icP_1)[(a + b)(1 - icP_1)P_1 + (a - b)(P_1 + ic)]}{c^2(a - b)P_1(P_1^2 + 2icP_1 - 1)(P_1 - P_2)(P_1 - P_3)(P_1 - P_4)}$$

$$C_2 = \frac{(P_2 + ic)(P_2^2 + 1)(1 - icP_2)[(a + b)(1 - icP_2)P_2 + (a - b)(P_2 + ic)]}{c^2(a - b)P_2(P_2^2 + 2icP_2 - 1)(P_2 - P_1)(P_2 - P_3)(P_2 - P_4)}$$

(7.8)

As a matter of course, $\phi(\xi)$ is a function, holomorphic outside the unit circle, and therefore it should bear the form,

$$\phi(\xi) = \frac{d_1}{\xi} + \frac{d_2}{\xi^2} + \ldots + \frac{d_n}{\xi^n} \quad , \quad |\xi| > 1$$

(7.9)

where $d_1 \sim d_n$ are constants. Therefore, differentiation of $\phi(\xi)$ with respect to $\xi$ will yield

$$\phi'(\xi) = -\frac{d_1}{\xi^2} - 2\frac{d_2}{\xi^3} - \ldots - n\frac{d_n}{\xi^{n+1}} \quad , \quad |\xi| > 1$$

(7.10)

Sequentially, taking conjugate for each term in equation (7.10), one will obtain

$$\overline{\phi'(\xi)} = -\overline{d_1} \xi^2 - 2\overline{d_2} \xi^3 - \ldots - n\overline{d_n} \xi^{n+1} \quad , \quad |\xi| < 1$$

(7.11)
From equation (7.11), it follows that the function \( C_0 \phi'(\xi^{-1})/\xi \) should be holomorphic inside the unit circle, and then the Cauchy integral of this term will be dropped out. Hence, there is no need to determine the constant \( C_0 \). At last, by applying Cauchy's formulae, the Cauchy integral of the term \( \omega(\xi) \cdot \phi'(\xi^{-1})/\omega'(\xi^{-1}) \) can be written as

\[
\frac{1}{2 \pi i} \int \frac{\omega(\xi) \cdot \phi'(\xi^{-1})}{\omega'(\xi^{-1})} d\xi = -\frac{C_1 \phi'(P_1^{-1})}{\xi - P_1} - \frac{C_2 \phi'(P_2^{-1})}{\xi - P_2}
\]  

(7.12)

Thus, the boundary equation for the present problem can be rewritten as

\[
-\phi(\xi) - \frac{C_1 \phi'(P_1^{-1})}{\xi - P_1} - \frac{C_2 \phi'(P_2^{-1})}{\xi - P_2} = \mathcal{F}_1(\xi)
\]  

(7.13)

Since the function \( \phi' \) is unknown, equation (7.13) still remains unsolved unless the unknown terms \( \phi'(P_1^{-1}) \) and \( \phi'(P_2^{-1}) \) are determined. Unlike the boundary equation obtained for the problem of a circular arc crack, there are two unknown constants \( \phi'(P_1^{-1}) \) and \( \phi'(P_2^{-1}) \) to be determined in equation (7.13). In a similar manner as what was done for the problem of a circular arc crack, equation (7.13) is differentiated with respect to \( \xi \), and then let \( \xi = P_1^{-1} \) and \( \xi = P_2^{-1} \) be substituted into the differentiated equation, respectively. Eventually, one will arrive at two simultaneous equations,

\[
-\phi'(P_1^{-1}) + \frac{C_1 \phi'(P_1^{-1})}{(P_1^{-1} - P_1)^2} + \frac{C_2 \phi'(P_2^{-1})}{(P_1^{-1} - P_2)^2} = \mathcal{F}'_1(P_1^{-1})
\]  

(7.14)

\[
-\phi'(P_2^{-1}) + \frac{C_1 \phi'(P_1^{-1})}{(P_2^{-1} - P_1)^2} + \frac{C_2 \phi'(P_2^{-1})}{(P_2^{-1} - P_2)^2} = \mathcal{F}'_1(P_2^{-1})
\]
Further, the above simultaneous equations are written in their conjugate forms,

\[
\phi'(\overline{P}_1^{-1}) = \frac{C_1 \cdot \phi'(\overline{P}_1^{-1})}{(\overline{P}_1^{-1} - \overline{P}_1)^2} + \frac{C_2 \cdot \phi'(\overline{P}_2^{-1})}{(\overline{P}_2^{-1} - \overline{P}_1)^2} - \mathcal{F}'(\overline{P}_1^{-1})
\]

(7.15)

\[
\phi'(\overline{P}_2^{-1}) = \frac{C_1 \cdot \phi'(\overline{P}_1^{-1})}{(\overline{P}_2^{-1} - \overline{P}_1)^2} + \frac{C_2 \cdot \phi'(\overline{P}_2^{-1})}{(\overline{P}_2^{-1} - \overline{P}_2)^2} - \mathcal{F}'(\overline{P}_2^{-1})
\]

Therefore, the expressions for these two constants \(\phi'(\overline{P}_1^{-1})\) and \(\phi'(\overline{P}_2^{-1})\), given by equations (7.15), can be substituted back into equations (7.14). After rearrangement is made to factor these two unknown constants out of the equations, these two simultaneous equations can be rewritten in the forms,

\[
CM_{11} \phi'(1/\overline{P}_1) + CM_{12} \phi'(1/\overline{P}_2) = BM_1
\]

\[
CM_{21} \phi'(1/\overline{P}_1) + CM_{22} \phi'(1/\overline{P}_2) = BM_2
\]

(7.16)

where the coefficients \(CM_{11}, CM_{12}, CM_{21},\) and \(CM_{22}\) are given by

\[
CM_{11} = \left[ -1 + \frac{C_1 C_1}{(1/\overline{P}_1 - \overline{P}_1)^2 (1/\overline{P}_1 - \overline{P}_1)^2} + \frac{C_2 C_1}{(1/\overline{P}_1 - \overline{P}_2)^2 (1/\overline{P}_2 - \overline{P}_1)^2} \right]
\]

(7.17)

\[
CM_{12} = \left[ \frac{C_1 C_2}{(1/\overline{P}_1 - \overline{P}_1)^2 (1/\overline{P}_1 - \overline{P}_2)^2} + \frac{C_2 C_2}{(1/\overline{P}_1 - \overline{P}_2)^2 (1/\overline{P}_2 - \overline{P}_2)^2} \right]
\]

\[
CM_{21} = \left[ \frac{C_1 C_1}{(1/\overline{P}_2 - \overline{P}_1)^2 (1/\overline{P}_1 - \overline{P}_1)^2} + \frac{C_2 C_1}{(1/\overline{P}_2 - \overline{P}_2)^2 (1/\overline{P}_2 - \overline{P}_1)^2} \right]
\]

\[
CM_{22} = \left[ -1 + \frac{C_1 C_2}{(1/\overline{P}_2 - \overline{P}_1)^2 (1/\overline{P}_1 - \overline{P}_2)^2} + \frac{C_2 C_2}{(1/\overline{P}_2 - \overline{P}_2)^2 (1/\overline{P}_2 - \overline{P}_2)^2} \right]
\]
and the constants $BM_1$ and $BM_2$ represent

$$BM_1 = \mathcal{F}'_1(1/\mathcal{P}_1) + \mathcal{F}'_1(1/\mathcal{P}_2) \frac{C_1}{(1/\mathcal{P}_1 - \mathcal{P}_1)^2} + \mathcal{F}'_1(1/\mathcal{P}_2) \frac{C_2}{(1/\mathcal{P}_1 - \mathcal{P}_2)^2}$$

$$BM_2 = \mathcal{F}'_1(1/\mathcal{P}_2) + \mathcal{F}'_1(1/\mathcal{P}_1) \frac{C_1}{(1/\mathcal{P}_2 - \mathcal{P}_1)^2} + \mathcal{F}'_1(1/\mathcal{P}_2) \frac{C_2}{(1/\mathcal{P}_2 - \mathcal{P}_2)^2}$$

Taking conjugate operation for each term in equations (7.16) and following the conventional notation of tensor (i.e., repeated index are referred to as the summation), one may rewrite these two simultaneous equations as

$$D_{11} \overline{\phi}'(\mathcal{P}_1^{-1}) + D_{12} \overline{\phi}'(\mathcal{P}_2^{-1}) = \mathcal{F}'_1(\mathcal{P}_1^{-1}) + C_{11} \cdot \mathcal{F}'_1(\mathcal{P}_1^{-1}) + C_{12} \cdot \mathcal{F}'_1(\mathcal{P}_2^{-1})$$

$$D_{21} \overline{\phi}'(\mathcal{P}_1^{-1}) + D_{22} \overline{\phi}'(\mathcal{P}_2^{-1}) = \mathcal{F}'_1(\mathcal{P}_2^{-1}) + C_{21} \cdot \mathcal{F}'_1(\mathcal{P}_1^{-1}) + C_{22} \cdot \mathcal{F}'_1(\mathcal{P}_2^{-1})$$

where the coefficients $D_{ij}$ and $C_{ij}$ are defined by $D_{ij} = C_{ik} \cdot C_{kj} - \delta_{ij}$; $C_{ij} = C_{j} (\mathcal{P}_i^{-1} - \mathcal{P}_j)^{-2}$, ($i, j = 1, 2$), and $\delta_{ij}$ is the Kronecker Delta, defined as usual. For brevity, equations (7.19) can be further abbreviated to a tensor form,

$$D_{ij} \overline{\phi}'(\mathcal{P}_i^{-1}) = \mathcal{F}'_1(\mathcal{P}_i^{-1}) + C_{ij} \cdot \mathcal{F}'_1(\mathcal{P}_j^{-1})$$

Equation (7.19) can be solved for these two unknown constants $\overline{\phi}'(\mathcal{P}_i^{-1})$ and $\overline{\phi}'(\mathcal{P}_j^{-1})$.

Consequently, these two constants can be obtained by

$$\overline{\phi}'(\mathcal{P}_i^{-1}) = D_{ij}^{-1} \left[ \mathcal{F}'_1(\mathcal{P}_j^{-1}) + C_{ji} \cdot \mathcal{F}'_1(\mathcal{P}_i^{-1}) \right]$$

Hence, the complex stress function $\phi(\xi)$ for the points on the crack surface can be directly determined by substituting the obtained constants (7.21) into equation (7.13).
As a matter of fact, the obtained expression for the complex stress function $\phi$ is only valid for those points which are away from the malfunctioning points. For the present case under study, this complex stress function will stay exact for those points in the vicinity of the crack surface. To apply the obtained stress function to calculate the stress and displacement components, one has to note that the points under investigation have to be far away from the malfunctioning points. In the same manner as before, the other function $\psi(\xi)$ can be also obtained from evaluating the Cauchy integrals of all terms in the boundary equation with its conjugate form. However, due to the complexity in expressing all pertinent functions, the derivation for the other function $\psi(\xi)$ is formidable. Even though the complex stress function $\psi(\xi)$ can be obtained, the calculation for internal stresses and displacements has certain limitation by those malfunctioning points. Fortunately, in evaluating the stress intensity factors, the information about the complex stress function $\psi(\xi)$ is not needed anyway so that this function will not be given here. The present object of interest is to determine the stress intensity factor at the crack tips of the elliptical crack so that the effort is focused on determining the stress function $\phi(\xi)$ and its derivative $d\phi(\xi)/dz$.

**Stress Intensity Factors**

As far as the $K$-factors are concerned, the obtained expression for the function $\phi(\xi)$ along the boundary surface needs to be differentiated with respect to $\xi$ to give

$$\phi'(\xi) = \sum_{i=1}^{2} \frac{C_i \cdot \phi'(\mathcal{P}_i^{-1})}{(\xi - \mathcal{P}_i)^2} - \mathcal{F}_1'(\xi)$$

(7.22)

In order to evaluate the stress intensity factors at the crack tips through the approach by Sih, etc. [59], it is necessary to transform the original coordinate system in the same manner as before.
First of all, one needs to get the expression for \( \frac{d\phi}{dz} \) by

\[
\frac{d\phi}{dz} = \phi'(\xi) \frac{(R+S)(\xi^2 + 2ic\xi - 1) + (R-S)(\xi^2 + 2ic\xi - 1)}{2(\xi + ic)^2 \xi^2 (1 - ic\xi)^2}^{-1}
\]

(7.23)

Thus, from equation (3.40), the stress intensity factor at the left crack tip \( z = z_{(1)} \) is found to be

\[
K = 2\sqrt{2\pi} \lim_{z \to 0} \sqrt{z} \phi'(\xi) \frac{(R+S)(\xi^2 + 2ic\xi - 1) + (R-S)(\xi^2 + 2ic\xi - 1)}{2(\xi + ic)^2 \xi^2 (1 - ic\xi)^2}^{-1}
\]

(7.24)

\[
= 4\sqrt{2\pi} \frac{\phi'(\xi_{(1)})\xi_{(1)}^2 + ic(1 - ic\xi_{(1)})^2}{(R+S)\xi_{(1)}^2 + (R-S)(\xi_{(1)} + ic)^2} \lim_{z \to 0} \frac{\sqrt{z}}{\xi_{(1)}^2 + 2ic\xi_{(1)} - 1}
\]

where \( z \) is the new transformed coordinate system, relocating the elliptical crack in such a way that the left tip is tangent to the origin point of this coordinate system. In the above equation, when taking the limit, one will need to evaluate the term \( \sqrt{z}/(\xi_{(1)}^2 + 2ic\xi_{(1)} - 1) \big|_{z \to 0} \), which belongs to the type of zero divided by zero. To evaluate this limiting value, one has to rewrite \( \xi_{(1)} \) in terms of \( z \) by equation (7.4) and then transform \( z \) to the new coordinate \( z \) by

\[
z = \frac{z\sqrt{a^2(2c^2 - 1)^2 + 4b^2c^2(1 - c^2)}}{a(2c^2 - 1) + 2ibc\sqrt{1 - c^2}} - a - \frac{icb}{\sqrt{1 - c^2}}
\]

(7.25)

Eventually, one can obtain the stress intensity at the left crack tip, written as

\[
K_{(1)} = 2\sqrt{\pi} \frac{\phi'(\xi_{(1)})\sqrt{a^2(1 - 2c^2)^2 + 4b^2c^2(1 - c^2)}}{- (R+S)\xi_{(1)} - (R-S)(1 - c^2)\xi_{(1)}^{-1}(1 - ic\xi_{(1)})^2}
\]

(7.26)
For the right crack tip \( z = \tau_{(2)} \), the transformation equation (7.25) needs to be so modified that the right crack tip is tangent to the x-axis at the origin of the new transformed coordinate. This can be done by modifying the transformation equation (7.25) into

\[
Z = \sqrt{a^2(2c^2 - 1)^2 + 4b^2c^2(1 - c^2)} \sqrt{a} + \frac{i cb}{\sqrt{1-c^2}}
\]

(7.27)

Then, the same procedure of evaluating its limiting value can be followed. Sequentially, the stress intensity factor at the right crack tip turns out to be

\[
K_{(2)} = 2\sqrt{\pi} \frac{\phi'(\xi_{(2)})}{(R + S)\xi_{(2)} + (R - S)(1 - c^2)\xi_{(2)}^{-1}(1 - i c \xi_{(2)})^{-2}}
\]

(7.28)

To evaluate the stress intensity factors at the both tips from equations (7.26) and (7.28), one needs to first calculate the constants \( \phi'(P_1^{-1}) \) and \( \phi'(P_2^{-1}) \) by equation (7.21) and then evaluate \( \phi'(\xi_{(1)}) \) and \( \phi'(\xi_{(2)}) \) from equation (7.22). An example case will be given next to illustrate the approach to evaluate the stress intensity factors for this elliptical crack problem.

Example Case

Particularly, if only uniform traction \( F, \) defined by \( F = -P + i Q, \) are applied on the whole crack edge, one can come up with this loading integration function \( f(\xi), \) given by

\[
f(\xi) = \frac{F(R + V)}{2} \left( \frac{\xi^2 + 1}{\xi + i c} \right) + \frac{F(R - V)}{2} \left( \frac{\xi^2 + 1}{\xi - i c \xi^2} \right)
\]

(7.29)
The Cauchy integral of the first term on the right hand side of equation (7.29) has been determined in Chapter 3 with different constant in its front. Similarly, the pole(s) occurring inside the unit circle can be also disassociated from the second term on the right hand side of equation (7.29). As a result of applying Cauchy’s formulae, the final expression for $\mathcal{F}_1(\xi)$ is written as

$$\mathcal{F}_1(\xi) = \frac{F(R + S)(c^2 - 1)}{2(\xi + ic)} - \frac{F(R - S)}{2\xi}$$

To determine $\phi'(\xi)$, equation (7.30) is differentiated with respect to $\xi$ to give

$$\mathcal{F}_1'(\xi) = \frac{F(R + S)(1 - c^2)}{2(\xi + ic)^2} + \frac{F(R - S)}{2\xi^2}$$

Thus, for this all-round tension problem, the function $\phi'(\xi)$ is given by

$$\phi'(\xi) = \frac{F(R + S)(c^2 - 1)}{2(\xi + ic)^2} - \frac{F(R - S)}{2\xi^2} + \frac{C_i \Phi'(\bar{\mathcal{P}_i}^{-1})}{(\xi - \mathcal{P}_i)^2}$$

As an example of the very direct solution obtained in this chapter, the particular case is considered when an infinite plane contains a curved crack modelled by an elliptical cut with a remote stress field consisting of all-round tensile stress $P$. For this particular case, from the principal of superposition, the stress intensity factors at the both tips are identical while only uniform normal stress $P$ is applied on the crack surface. Thus, the shear stress $Q$ is set equal to zero in the following study.

The formulations for the stress intensity factors of an elliptical crack involve much more calculations than those for a circular arc crack. Looking into the expression for the stress intensity factors, one may find that the applied stress $P$ can be factored out if a new function $\mathcal{G}(\xi)$ is defined by
where the aspect ratio $\mathcal{R}$ is defined by $b/a$. Consequently, the stress intensity factor can be rewritten as

$$K = \sqrt{\pi a \cdot P \cdot GF(\mathcal{R}, c)}$$  \hspace{1cm} (7.34)$$

where $GF(\mathcal{R}, c)$ is the geometric function, given by

$$GF(\mathcal{R}, c) = \frac{4 \sqrt{(1 - 2 c^2)^2 + 4 \mathcal{R} c^2 (1 - c^2)}}{(1 + \mathcal{R}) \xi_{(1)} + (1 - \mathcal{R}) (1 - c^2) \xi_{(1)}^{-1} (1 - i c \xi_{(1)})^{-2}}$$

$$\cdot \left\{ \frac{C_i D_j^{-1} [E(P_j^{-1}) + C_{ij} \Xi(P_i^{-1})]}{(\xi_{(1)} - P_j)^2 - \frac{1 - \mathcal{R}}{\xi_{(1)}} - 1 - \mathcal{R}} \right\} \hspace{1cm} (7.35)$$

The factor $\sqrt{a}$ is associated with the crack size, and the variables $\mathcal{R}$ and $c$ basically governs the shape of the elliptical crack. Generally speaking, by suitable choice of $a$, $\mathcal{R}$, and $c$, one can obtain an elliptical crack of any configuration which is symmetric about $y$-axis. For the present object of interest, $\mathcal{R} > 1$ is assumed. Also, as an example of illustration, let $c$ be equal to its critical value ($c = c_0 = 1/\sqrt{2}$), which is the case for a semi-elliptical crack. To gain insight into how the stress intensity factor varies with the aspect ratio, the geometric function $GF(\mathcal{R}, c_0)$ for the both fracture modes $I$ and $II$ are numerically computed and plotted in Figure 7.2.

As the aspect ratio is large enough, the stress intensity factor for mode $I$ will be close to the value for a straight crack with the total length of $R b/(c a)$, which is the magnitude of the semi-major axis, and the stress intensity factor of mode $II$ will approach zero. Therefore, to
Figure 7.2: Geometric functions of a semi-elliptical crack as a function of the aspect ratio
perceive this tendency, one may normalize the stress intensity factors for both modes by the stress intensity factor of a straight crack with the crack length $R b/(c a)$. Instead of taking the stress intensity factor of a straight crack as the factor of normalization, these newly defined normalized stress intensity factors, denoted by $NK1$ and $NK2$, are given by

$$
NK1 = \frac{K_I}{\sqrt{\pi D_0/2}} \quad , \quad NK2 = \frac{K_{II}}{\sqrt{\pi D_0/2}}
$$

where $D_0$ is the length of the semi-major axis $R b/(c a)$. These normalized stress intensity factors are plotted in Figure 7.3. From this figure, the trend of how a deeply curved elliptical crack (with large aspect ratio) resembles the characteristics of a straight crack can be seen.
Figure 7.3: Normalized $K$-factor ($NK$) of a semi-elliptical crack as a function of the aspect ratio.
CHAPTER 8. DISCUSSION AND FUTURE WORK

Conclusion and General Discussion

The proposed mapping function, transforming the unbounded domain outside the crack surface of a CAC onto the unbounded region outside the unit circle, is indeed a powerful tool in dealing with the CAC problem. In contrast with the other way of approach by treating the case as the Hilbert problem, the results obtained by the present approach can be reduced to consider the particular case of a straight crack simply by setting the mapping variable $c$ equal to zero. Instead, if the problem is solved by the approach of Hilbert problem, the only way to reduce the results to those for a straight crack is to let the subtending angle $2\alpha$ of the CAC be very small and its radius $r_0$ very large so that the limiting value of $2r_0\alpha$ can be taken as the total length of straight crack. However, the formulations for both of the two problems of different crack shapes still need to be expressed in two different forms.

The wonders of this mapping is to integrate the problem of a CAC with the one of a straight crack. Consequently, as one intends to build this scheme into computer codes for calculating the stress and displacement components, unique formulations can be implemented for these two different problems. The only thing needed to consider the straight crack problem is to change the value of variable $c$. Moreover, the use of the variable $\eta$ in the final expressions for the stress intensity factors is tremendously expedient from the practical point of view. Especially for a fairly shallow arc crack, even almost flat, it is difficult to accurately estimate its radius and subtending angle if one wishes to model it as a circular arc crack. With no doubt, measure of the angle $\eta$ along with the distance between the crack tips
will significantly reduce the errors caused from estimating the curvature.

From the study in Chapter 3, it is seen that under the condition of all-round tension, the interaction between the crack tips will counteract the stress intensity factor of each other if the distance between the crack tips is assumed to be very small. On contrary, if only uniform shearing stress is applied on the crack surface instead (studied in Chapter 4), the interaction will go in the opposite way so that eventually, the stress intensity factors will diverge to infinity as the distance between the crack tips approaches zero. Therefore, the conclusion may be drawn that when the crack tips of a CAC are very close to each other, the stress intensity factors will be dominated by the applied shear.

From a practical point of view, a shallowly curved crack is more likely to happen than a deeply curved crack for ductile materials. For an almost flat crack, the relevant analysis for such a crack is traditionally made by treating it as a straight one. As studied in Chapter 3 for the problem of extensional fracture, the maximum error percentage by such modelling is almost linear with the angle $\eta$, provided $\eta$ is smaller than 0.2. This criterion provides us a simple rule in judging if the straight crack model is appropriate to model a shallowly curved crack. Similarly, for the transverse flexure problem investigated in Chapter 5, the analysis for such modelling can be also made to provide a good criterion.

The problems solved in the dissertation belong to the first fundamental boundary value problems that only traction is prescribed along the boundary. For the boundary value problems of the second type (displacements are prescribed along the boundary instead), the same procedure can be followed to solve the corresponding boundary equation for the complex stress function $\phi(\zeta)$. In the same manner, the other function $\psi(\zeta)$ can be also determined.

For the work in Chapter 7 studying the problem of an elliptical crack, there is limitation to the geometry of the elliptical crack. The elliptical crack has to be symmetric about the $y$-axis. Since the way of approach to solve this problem is starting from the mapping for a circular arc crack, there is no way to modify the solution to consider an unsymmetrical elliptical crack. The
three parameters used to specify the elliptical crack are \( a \) (half distance between crack tips), \( c \) (mapping variable for a CAC), and \( \Re \) (aspect ratio of the ellipse). In the author's opinion, it is possible to solve this problem in the other way of treating it as the Hilbert problem using the elliptical coordinate system. By this approach, four parameters are needed to specify the general configuration of an unsymmetrical elliptical crack. If this scheme works, the whole elliptical crack problem can be completely solved.

**More General Conditions**

All work done in this dissertation considers the fracture of an infinitely large elastic plate weaken by a circular arc crack. Principally, there are three assumptions made here, which significantly simplify the problem. First, the plate is assumed to be large enough so that the disturbance of the stress field due to the presence of a crack will vanish at infinity. However, for a practical case in engineering industry, the plate subjected to loads is always finite in size so that one has to take the boundary effect into account. This case for a finite geometry is so common that many numerical schemes are emerging as powerful tools to deal with such boundary problems, such as the finite element method and the boundary element method. For examples, the most commonly seen boundary problems solved before are - (1) a rectangular plate weaken by a straight crack subjected to biaxial tension (Figure 8.1), (2) an infinite long strip containing a straight crack subjected to uniaxial tension (Figure 8.2) and (3) a semi-infinite plane with a straight crack on the edge subjected to uniaxial tension at infinity (Figure 8.3). Kobayashi [32] used the method of collocation combined with Westergaard's stress function to investigate case (1) and case (3). The numerical results obtained by Kobayashi are claimed to agree with experimental results within 5-7%. Isida [29] investigated case (2) using Laurent series expansions of complex variables. Also, Sneddon [63] solved the boundary value problem like case (2) by a systematic use of the theory of integral transforms.
Figure 8.1: A rectangular plate weaken by a circular arc crack subjected to biaxial tension

Figure 8.2: An infinite long strip containing a straight crack subjected to uniaxial tension
Another extremely effective technique which combines modified versions of conformal mapping and boundary collocation arguments was first presented by Bowie and Neal [10] to analyze an internal crack in a finite geometry like case (1). In the modified mapping-collocation scheme, five conditions corresponding to summation of the moment, summation of forces in $x$ and $y$ direction, and normal and tangential stress are imposed at the collocation points to improve the accuracy.

**Future Work**

**Boundary effect**

Now, if the straight crack is replaced by the one with circular arc shape, the whole problems will become much more complicated. Many of the techniques appearing to theoretically solve the boundary value problem of a straight crack were using the characteristic of symmetry about the axis along the crack surface, which will substantially simplify the
problem. However, no longer does this characteristic hold for the boundary value problem of a circular arc crack. For example, the scheme proposed by Bowie [9] expands the boundary conditions in series form and then impose an extra condition of least square to minimize the errors. This extra condition provides a linear system of simultaneous equations for the determination of the unknown coefficients in the expanded series. However, if a circular arc crack is considered instead, these unknown coefficients will contain conjugate parts. As a result, the argument of least square will lead to difficulty in evaluating the differentiation of a complex constant with respect of a real constant. Instead, author tried to solve the problem of edge crack using the conventional mapping collocation scheme. The collocation made for many points on the crack edge will also provide a linear system of equations for determination of these complex coefficients. Eventually, the author comes up with the result that its stress intensity factor is about 1.121-1.122 (π α)^1/2, which is quite consistent with the result 1.1215 (π α)^1/2 obtained by Hartranft and Sih [22], Koiter [33], Sneddon and Das [63], and Stallybrass [64]. However, as the same scheme is implemented to consider the same problem but with a circular arc crack on the edge instead, the whole numerical results will become totally unstable. For the future work, the modified mapping collocation scheme can be implemented to investigate the unsolved edge crack problem. Since extra conditions are imposed to reduce the errors, the author believes this scheme should work for this case. Also, the modified mapping collocation scheme is expected to work for the problem of a finite geometry containing an internal circular arc crack.

Recently, the boundary element method is emerging as a powerful numerical technique to deal with boundary value problems. As is well known, the primary advantage of the boundary element method is that discretization only on the boundary is required. For the application of the boundary element method to the crack problems, one can find various references [1, 4, 8, 12-15, 20, 46, 61, 67]. To overcome the singularity problem in overlapping crack surfaces, several techniques have been developed such as the displacement discontinuity
method [13], the specialized Green’s function method [14], and multidomain method [8]. However, due to the singularity nature of the crack tip, there exists some accuracy and convergence difficulty in evaluating the stress intensity factors at the crack tip.

**Anisotropy**

Another assumption made in the work of this dissertation is the plate is made of an isotropic elastic material. Crystals of many materials are known to possess directional properties in mechanical stiffness. One of the typical examples is composite materials, which are widely used in the engineering industry nowadays. If the elastic properties at a point in the body differ in the rectangular coordinate directions $x$, $y$, and $z$, the body is said to be rectilinearly anisotropic. For two-dimensional problems, Leknitskii [37] established formulations in terms of the complex functions $\phi$ and $\psi$. Sih, Paris and Irwin [60] were the first ones to have found the stress field around the crack tip in a generally anisotropic elastic body (Figure 8.4). Sih and Liebowitz [58] reduced the problem to be one of the Riemann-Hilbert type. Therefore, in the future, the whole work done in this dissertation can be modified to take rectilinear anisotropy into account. Scanning through the techniques developed for this purpose, the author suggests that the work of the modified mapping-collocation technique by Bowie and Neal [10] is a good candidate to numerically solve the problem.

**General crack shape**

The conformal mapping scheme implemented in this dissertation is truly an expedient way to solve the crack problem. However, it is extremely difficult to find the appropriate mapping function which can transform a crack shape of one’s desire into a unit circle. Although a circular arc crack or an elliptical crack could be applied as a model for some crack with an unknown shape, it will be more practical if one can solve the problem for a plate
preferred directions of materials

Figure 8.4: Crack in a generally anisotropic body

containing a crack with a general shape. It is difficult but not impossible to find the appropriate mapping function. As a matter of course, the appropriate mapping will not take a nice mathematic form, something looks like the function in this dissertation. However, it is possible to formulate the mapping function in a numerical way to meet the mapping requirements. Once the mapping function is established, similar procedures to what was done in the dissertation can be followed to solve the problem.
APPENDIX A. PROOF OF THE MAPPING FOR A CAC

First, consider the mapping function,

$$ z = \omega(\xi) = R \left( \xi + i c \frac{1 - c^2}{\xi + i c} \right) $$ (A.1)

For the geometry of the unit circle illustrated in Figure 3.3 (b), an arbitrary point on the unit circle can be expressed in the form,

$$ \xi = (c \sin \theta_0 + \sqrt{1 - c^2 \cos^2 \theta_0}) e^{i \theta_0} $$ (A.2)

where the angle $\theta_0$ is measured from the reference line in counterclockwise direction. Let the expression for $\nu$ in (A.2) be substituted into (A.1). Consequently, it can be readily obtained that the corresponding $x$ and $y$ coordinates of this point $\nu$ bear the forms,

$$ x = Re\{\omega(\xi)\} = 2R \cos \theta_0 \sqrt{1 - c^2 \cos^2 \theta_0} $$ (A.3)

$$ y = Im\{\omega(\xi)\} = 2R \sin \theta_0 $$

Then, the process of eliminating the parameter $\theta_0$ is going as follows:

$$ x^2 = 4R^2 \cos^2 \theta_0 (1 - c^2 \cos^2 \theta_0) $$

$$ = 4R^2 (1 - \sin^2 \theta_0) [1 - c^2 (1 - \sin^2 \theta_0)] $$ (A.4)

$$ = (2R - y/c)[2R (1 - c^2) + cy] $$

$$ = 4R^2 (1 - c^2) + [2R c - 2R (1 - c^2)/c] y - y^2 $$

After rearrangement is made for the parameters $x$ and $y$, the locus of the unit circle in the $z$-
plane can be expressed as

\[
\left[ y - R \left( 2c - \frac{1}{c} \right) \right]^2 + x^2 = \left( \frac{R}{c} \right)^2 \quad (A.5)
\]

Obviously, due to the fact \( y \) is always positive (from equation (A.3)), the locus is actually an arc of a circle (Figure A.1) with a radius \( R/c \) and its center is located at \( (0, R(2c-1/c)) \).

![Figure A.1: Locus of a circular arc crack](image)

Now, if the transformation \( z = z' + 2icR \) is introduced so that the point at the top of this arc is tangent to the origin of the new coordinate \( z' \), the mapping function (A.1) will become

\[
z' = \omega(\xi) = R \left( \frac{\xi + ic + \frac{1 - c^2}{\xi + ic}}{\xi + ic} \right) - 2icR \quad (A.6)
\]

\[
= R \left( \frac{\xi^2 + 1}{\xi + ic} \right)
\]

Thus, if the prime is dropped for the new coordinate system, the mapping function used in Chapter 3 is obtained.
APPENDIX B. STRESS FUNCTIONS FOR CHAPTER 4

Complex Stress Functions for general extensional loadings in Chapter 4:

\[
\phi(\zeta) = -\mathcal{F}(\zeta) - \frac{\mathcal{F}_0}{\zeta + ic}
\]

\[
\frac{d\phi}{dz} = \frac{1}{R(\zeta^2 + 2ic\zeta - 1)} \{ -(\zeta + ic)^2 \mathcal{F}'(\zeta) + \mathcal{F}_0 \}
\]

\[
\frac{d^2\phi}{dz^2} = \frac{-(\zeta + ic)^3}{R^2(\zeta^2 + 2ic\zeta - 1)^3} \{ 2 \mathcal{F}'(\zeta) + (\zeta + ic) \mathcal{F}''(\zeta) + 2 \mathcal{F}_0 \}
\]

\[
\psi(\zeta) = \frac{(T_x - T_y)R(1 - c^2)}{2(\zeta + ic)} - \frac{(T_x + T_y)R}{2\zeta}
\]

\[
+ \frac{iS_{xy}R[3(1-c^2)(1-2c^2)\zeta^2 - 3ic(1-c^2)\zeta + c^2]}{3\zeta^3}
\]

\[
- \frac{[(T_x + T_y)(1-c^2)R + 2F_0](\zeta^2 + 2ic\zeta + 1)}{2\zeta(\zeta^2 + 2ic\zeta - 1)}
\]

\[
+ \frac{(T_x - T_y)R}{2\zeta^3(\zeta^2 + 2ic\zeta - 1)} \left[ (1-c^2)^2\zeta^4 + ic(2c^4 - 5c^2 + 4)\zeta^3 
\right]
\]

\[
+ (c^4 - 3c^2 + 1)\zeta^2 + ic(2 - c^2)\zeta - c^2 
\]

\[
- \frac{iS_{xy}R(1-c^2)}{\zeta(\zeta + ic)^2(\zeta^2 + 2ic\zeta - 1)} \left[ (1-c^2)(1-2c^2)\zeta^4 + ic(8c^4 - 9c^2 + 2)\zeta^3 
\right]
\]

\[
+ (1-2c^2)(5c^4 - 5c^2 + 1)\zeta^2 - ic^3(4c^4 - 10c^2 + 5)\zeta + c^4(1-2c^2) \right] 
\]
\[
\frac{d\psi}{dz} = \frac{(T_y - T_x)(1 - c^2)}{2(\zeta^2 + 2ic\zeta - 1)} + \frac{(T_y + T_x)(\zeta + ic)^2}{2\zeta^2(\zeta^2 + 2ic\zeta - 1)} \\
- iS_{xy}(\zeta + ic)^2[(1 - c^2)(1 - 2c^2)\zeta^2 - 2ic(1 - c^2)\zeta + c^2] \\
\zeta^4(\zeta^2 + 2ic\zeta - 1) \\
- [(T_y + T_x)(1 - c^2)R + 2\mathcal{F}_0] \frac{(\zeta + ic)^2[\zeta^4 + 4ic\zeta^3 + 4(1 - c^2)\zeta^2 + 4ic\zeta - 1]}{2R\zeta^2(\zeta^2 + 2ic\zeta - 1)^3} \\
+ \frac{(T_x - T_y)(\zeta + ic)^2}{2\zeta^4(\zeta^2 + 2ic\zeta - 1)^3}[-(1 - c^2)^2\zeta^6 - 2ic(2c^4 - 5c^2 + 4)\zeta^5 + (4c^6 - 14c^4 + 19c^2 - 4)\zeta^4 \\
- 4ic(1 - c^2)(3 - c^2)\zeta^3 + (1 + 14c^2 - 5c^4)\zeta^2 + 2ic(2 + 3c^2)\zeta - 3c^2] \\
\text{(B.3)} \\
- \frac{iS_{xy}(1 - c^2)}{\zeta^2(\zeta + ic)(\zeta^2 + 2ic\zeta - 1)^3}[-(1 - 2c^2)(1 - c^2)\zeta^7 - ic(3 - 15c^2 + 14c^4)\zeta^6 + \\
(38c^6 - 53c^4 + 24c^2 - 4)\zeta^5 + ic(-8 + 60c^2 - 103c^4 + 50c^6)\zeta^4 \\
+ (1 - 3c^2 - 48c^4 + 96c^6 - 32c^8)\zeta^3 - ic(1 + 3c^2 + 16c^4 - 44c^6 + 8c^8)\zeta^2 \\
+ c^4(3 + 4c^2)(1 - 2c^2)\zeta + ic(1 - 2c^2)] \\
\mathcal{F}_0 = \left[ e^2 \mathcal{F}'(-ic^{-1}) - \overline{\mathcal{F}'(ic^{-1})} \right] (1 - c^2)/(1 + c^2) \\
\text{(B.4)} \\
\mathcal{F}(\zeta) = \frac{(T_y - T_x)R}{2\zeta} + \frac{(T_y + T_x)(1 - c^2)R}{2(\zeta + ic)} \\
+ \frac{iS_{xy}R(1 - c^2)}{3(\zeta + ic)^3} \left[ (15c^4 - 15c^2 + 3)\zeta^2 + (24ic^5 - 21ic^3 + 3ic)\zeta \\
- (10c^6 - 8c^4 + c^2) \right] \\
\text{(B.5)}
\[ \mathcal{F}'(\zeta) = \frac{(T_x - T_y) R}{2 \zeta^2} - \frac{(T_y + T_x)(1 - c^2) R}{2(\zeta + ic)^2} \]

\[ - \frac{i S_{xy} R(1 - c^2)}{(\zeta + ic)^4} [(5c^{4} - 5c^{2} + 1)\zeta^{2} + (6i c^{5} - 4c^{3})\zeta + (c^{4} - 2c^{6})] \]  

\[ \mathcal{F}''(\zeta) = \frac{(T_y - T_x) R}{\zeta^{3}} + \frac{(T_y + T_x)(1 - c^2) R}{(\zeta + ic)^{3}} \]

\[ + 2i S_{xy} R(1 - c^2) \frac{(5c^{4} - 5c^{2} + 1)\zeta^{2} + (4i c^{5} - ic^{3} - ic)c^{6}}{(\zeta + ic)^{5}} \]
APPENDIX C. STRESS FUNCTIONS FOR CHAPTER 5

For uniform loads on part of the crack edge:

\[ \phi(\zeta) = -\mathcal{F}_1(\zeta) - \frac{1-c^2}{1+c^2} \cdot \frac{e^2 \mathcal{F}_1'(\zeta) - \mathcal{F}_1'(i\zeta^{-1})}{\zeta + ic} \]  \hspace{1cm} (C.1)

\[ \frac{d\phi(\zeta)}{dz} = \frac{-\mathcal{F}_1'(\zeta) \cdot (\zeta + ic)^2}{R (\zeta^2 + 2ic\zeta - 1)} + \frac{1-c^2}{1+c^2} \cdot \frac{e^2 \mathcal{F}_1'(\zeta) - \mathcal{F}_1'(i\zeta^{-1})}{R (\zeta^2 + 2ic\zeta - 1)} \]  \hspace{1cm} (C.2)

\[ \frac{d^2\phi(\zeta)}{dz^2} = \frac{(\zeta + ic)^3}{(\zeta^2 + 2ic\zeta - 1)^3} \left\{ \frac{2(1-c^2)\mathcal{F}_1'(\zeta) - (\zeta + ic)(\zeta^2 + 2ic\zeta - 1)\mathcal{F}_1''(\zeta)}{R^2} ight. \\
\left. - \frac{2(1-c^2)[e^2 \mathcal{F}_1'(\zeta) - \mathcal{F}_1'(i\zeta^{-1})]}{R^2(1+c^2)} \right\} \]  \hspace{1cm} (C.3)

\[ \psi(\zeta) = -\mathcal{F}_2(\zeta) - \frac{(1-c^2)^2 \mathcal{F}_1'(\zeta)}{ic(1-ic\zeta)} + \frac{(\zeta^2 + 1)(\zeta + ic)^2 \mathcal{F}_1'(\zeta)}{\zeta(1-ic\zeta)(\zeta^2 + 2ic\zeta - 1)} \]  \
\[ - \frac{(1-c^2)(\zeta^2 + 2ic\zeta + 1)[e^2 \mathcal{F}_1'(\zeta) - \mathcal{F}_1'(i\zeta^{-1})]}{(1+c^2)\zeta(\zeta^2 + 2ic\zeta - 1)} \]  \hspace{1cm} (C.4)

\[ \frac{d\psi(\zeta)}{dz} = \frac{(\zeta + ic)^2 \psi'(\zeta)}{R (\zeta^2 + 2ic\zeta - 1)} \]  \hspace{1cm} (C.5)
where the function $\psi'$ has the following form,

$$
\psi' = -\mathcal{F}_2''(\zeta) + \frac{(1-c^2)^2}{(1-ic\zeta)^2}
$$

$$
+ \frac{(c^4+4ic\zeta^3+(4-4c^2)\zeta^2+4ic\zeta-1)(1-c^2)[c^2\mathcal{F}_1'(-ic^-)-\mathcal{F}_1'(ic^-)]}{\zeta^2(\zeta^2+2ic\zeta-1)^2(1+c^2)}
$$

$$
+ \zeta^{-2}(1-ic\zeta)^{-2}(\zeta^2+2ic\zeta-1)^{-2}[\zeta^6-(2ic^3-8ic)\zeta^5+(2c^4-13c^2-4)\zeta^4
$$

$$
-(12ic^3+8ic)\zeta^3+(6c^4+10c^2-1)\zeta^2+6ic^3\zeta-c^2] \cdot \mathcal{F}_1'(\zeta)
$$

$$
+ \zeta^{-1}(1-ic\zeta)^{-2}(\zeta^2+2ic\zeta-1)^{-2}[-ic\zeta^2+(1+4c^2)\zeta^6+(5ic^3+4ic)\zeta^5
$$

$$
-(3c^2+2c^4)\zeta^4+(3ic+2ic^3)\zeta^3-(1+6c^2+2c^4)\zeta^2-(2ic+3ic^3)\zeta+c^2] \cdot \mathcal{F}_1''(\zeta)
$$

and the functions $\mathcal{F}_1(\zeta)$, $\mathcal{F}'_1(\zeta)$, $\mathcal{F}''_1(\zeta)$, $\mathcal{F}_2(\zeta)$, and $\mathcal{F}_3(\zeta)$ take the following forms:

$$
\mathcal{F}_1(\zeta) = \frac{F\cdot R}{2\pi i} \left[ \frac{c^2-1}{\zeta+ic} \ln \frac{\xi_{(U)}+ic}{\xi_{(L)}+ic} + \left( \frac{\zeta^2+1}{\zeta+ic} - \frac{z_0}{R} \right) \ln \frac{\xi_{(U)}-\zeta}{\xi_{(L)}-\zeta} \right]
$$

$$
\mathcal{F}_1'(\zeta) = \frac{F\cdot R}{2\pi i} \left[ \frac{1-c^2}{(\zeta+ic)^2} \ln \frac{\xi_{(U)}+ic}{\xi_{(L)}+ic} + \frac{\xi_{(U)}-\xi_{(L)}}{(\xi_{(U)}-\zeta)(\xi_{(L)}-\zeta)} \left( \frac{\zeta^2+1}{\zeta+ic} - \frac{z_0}{R} \right) \ln \frac{\xi_{(U)}-\zeta}{\xi_{(L)}-\zeta} \right]
$$

$$
\mathcal{F}_1''(\zeta) = \frac{F\cdot R}{2\pi i} \left[ \frac{(\xi_{(U)}-\xi_{(L)})(\xi_{(U)}+\xi_{(L)}-2\zeta)}{(\xi_{(U)}-\zeta)^2(\xi_{(L)}-\zeta)^2} \left( \frac{\zeta^2+1}{\zeta+ic} - \frac{z_0}{R} \right) \ln \frac{\xi_{(U)}+ic}{\xi_{(L)}+ic} \right]
$$

$$
+ \frac{2(c^2-1)}{(\zeta+ic)^3} \ln \frac{\xi_{(U)}+ic}{\xi_{(L)}+ic} \left[ \frac{2(\xi_{(U)}-\xi_{(L)})(\zeta^2+2ic\zeta-1)}{(\xi_{(U)}-\zeta)(\xi_{(L)}-\zeta)(\zeta+ic)^2} \right]
$$
For concentrated loads on the crack surface (the same expressions for the stress functions as those for the case of partially distributed loads but with $\mathcal{F}$ replaced by $\mathcal{S}$):

\[
\mathcal{S}_1(\zeta) = \frac{F}{2\pi i \theta_1} \left\{ \frac{e^2 - 1}{(\zeta + ic)^2} \theta_2 + \left( \frac{\zeta^2 + 1}{\zeta + ic} - \frac{z_0}{R} \right) \theta_3 - \ln \frac{\xi(\zeta)}{\xi(\xi - \zeta)} \right\}
\]

\[
\mathcal{S}_1'(\zeta) = \frac{F}{2\pi i \theta_1} \left\{ \frac{1 - e^2}{(\zeta + ic)^2} \theta_2 + \left( \frac{\zeta^2 + 1}{\zeta + ic} - \frac{z_0}{R} \right) \theta_3 - \frac{\xi(\zeta) - \xi(\xi - \zeta)}{(\xi(\xi - \zeta))(\xi(\xi - \zeta))} \theta_1 + \frac{\zeta^2 + 2ic\zeta - 1}{(\zeta + ic)^2} \right\}
\]

\[
\mathcal{S}_1''(\zeta) = \frac{F}{2\pi i \theta_1} \left\{ \frac{2(e^2 - 1)}{(\zeta + ic)^3} \theta_2 + \left( \frac{\zeta^2 + 1}{\zeta + ic} - \frac{z_0}{R} \right) \theta_3 - \frac{(\xi(\zeta) - \xi(\xi - \zeta))(\xi(\xi) + \xi(\xi - \zeta) - 2\zeta)}{(\xi(\xi - \zeta))(\xi(\xi - \zeta))^2} \theta_1 + \frac{2(\zeta^2 + 2ic\zeta - 1)}{(\zeta + ic)^2} + \frac{2(1 - e^2)}{(\zeta + ic)^3} + \frac{\zeta^2 + 2ic\zeta - 1}{(\zeta + ic)^2} \theta_3 \right\}
\]
\[
S_1'(\zeta) = \frac{F}{2\pi i \theta_1} \left\{ \frac{1 - c^2}{(\zeta + ic)^2} \hat{\theta}_2 + \left( \frac{\zeta^2 + 1}{\zeta + ic} \right) \hat{\theta}_7 - \frac{\xi(U) - \xi(L)}{(\xi(U) - \zeta)(\xi(L) - \zeta)} \hat{\theta}_1 \right\} \\
+ \frac{\zeta^2 + 2 ic \zeta - 1}{(\zeta + ic)^2} \hat{\theta}_3
\]

\[
S_2'(\zeta) = \frac{F}{2\pi i \theta_1} \left\{ \frac{1 - c^2}{(c \zeta + i)^2} \hat{\theta}_4 + \left( \frac{\zeta^2 + 1}{\zeta(1 - ic \zeta)} \right) \hat{\theta}_7 + \frac{c^2 + 2 ic \zeta - 1}{\zeta^2(1 - ic \zeta)^2} \hat{\theta}_3 \right\} \\
- \frac{\xi(U) - \xi(L)}{(\xi(U) - \zeta)(\xi(L) - \zeta)} \hat{\theta}_6
\]

where \( \hat{\theta}_1 \sim \hat{\theta}_8 \) are given by

\[
\hat{\theta}_1 = \frac{\xi(U) - \xi(W)}{(\xi(U) + ic)^2}, \quad \hat{\theta}_2 = \frac{\xi(U) - \xi(U)}{\xi(U) + ic} \\
\hat{\theta}_3 = \frac{\xi(U) - \xi(U)}{\xi(U) - \zeta}, \quad \hat{\theta}_4 = ic \left( \frac{\xi(U) - \xi(U)}{1 - ic \xi(U)} \right) \\
\hat{\theta}_5 = \frac{\xi(U) - \xi(U)}{\xi(U)}, \quad \hat{\theta}_6 = \frac{\xi(U) - \xi(U)}{\xi(U)^2(1 - ic \xi(U))^2} \\
\hat{\theta}_7 = \frac{\xi(U) - \xi(U)}{(\xi(U) - \zeta)(\xi(U) - \zeta)} \left( \frac{\xi(U) - \zeta}{\xi(U)^2 - \zeta^2} \right) \frac{\xi(U) - \xi(U)}{(\xi(U) - \zeta)(\xi(U) - \zeta)^2} \frac{\xi(U) - \xi(U)}{(\xi(U) - \zeta)(\xi(U) - \zeta)^2} \\
\hat{\theta}_8 = \frac{2\xi(U) - \xi(U) - \xi(U)}{(\xi(U) - \zeta)^2(\xi(U) - \zeta)^2} - 2(\xi(U) - \xi(U)) \frac{\xi(U) - \xi(U)}{\xi(U) - \zeta} \frac{\xi(U) - \xi(U)}{(\xi(U) - \zeta)^3}
\]

(C.13) (C.16) (C.17)
\[ \xi_{(U)}' \text{ and } \xi_{(L)}' \text{ are} \]

\[ \xi_{(U)}' = \left( c + i \frac{e^{-i\theta} - c^2 \cos \Theta}{\sqrt{1 - c^2 \cos^2 \Theta}} \right) e^{2i\theta}, \quad \xi_{(L)}' = -\left( c + i \frac{e^{i\theta} - c^2 \cos \Theta}{\sqrt{1 - c^2 \cos^2 \Theta}} \right) e^{-2i\theta} \] (C.18)
APPENDIX D. STRESS FUNCTIONS FOR CHAPTER 6

For the problem with general bending and twisting moments at infinity:

\[ \phi(\zeta) = A R \left( \frac{\zeta^2 + 1}{\zeta + i c} \right) + \phi_0(\zeta) \]  
(D.1)

\[ \phi_0(\zeta) = \frac{A_0}{\zeta + i c} - \frac{B_0}{\zeta} \]  
(D.2)

\[ \frac{d\phi(\zeta)}{dz} = A - \frac{A_0}{R(\zeta^2 + 2ic\zeta - 1)} + \frac{B_0(\zeta + ic)^2}{R\zeta^2(\zeta^2 + 2ic\zeta - 1)} \]  
(D.3)

\[ \frac{d^2\phi(\zeta)}{dz^2} = \frac{2A_0(\zeta + ic)}{R^2(\zeta^2 + 2ic\zeta - 1)^3} - \frac{2B_0(\zeta + ic)^3(\zeta^3 + 3ic\zeta^2 - 3ic^2\zeta - ic)}{R^2\zeta^3(\zeta^2 + 2ic\zeta - 1)^3} \]  
(D.4)

\[ \psi(\zeta) = (B + iC)R \left( \frac{\zeta^2 + 1}{\zeta + i c} \right) + \psi_0(\zeta) \]  
(D.5)

\[ \psi_0(\zeta) = \frac{AR(N + 1)}{\zeta} - \frac{(B + IC)R(1 - c^2)}{\zeta + ic} + \frac{A_0(\zeta^2 + 2ic\zeta + 1)}{\zeta(\zeta^2 + 2ic\zeta - 1)} \]  
(D.6)

\[ -B_0(1 - c^2)^2 \zeta^4 + ic(4 - 5c^2 + 2c^4)\zeta^3 + [(1 - c^2)^2 - c^2]\zeta^2 + ic(2 - c^2)\zeta - c^2 \]  
\[ \frac{\zeta^3(\zeta^2 + 2ic\zeta - 1)}{\zeta^3} \]

\[ \frac{d\psi(\zeta)}{dz} = (B + iC)R \frac{A(N + 1)(\zeta + ic)^2}{\zeta^2(\zeta^2 + 2ic\zeta - 1)} + \frac{(B + IC)(1 - c^2)}{\zeta^2 + 2ic\zeta - 1} + B_0 \frac{(\zeta + ic)^2g_0(\zeta)}{R\zeta^4(\zeta^2 + 2ic\zeta - 1)^3} \]  
(D.7)

\[ \frac{A_6(\zeta^4 + 4ic\zeta^3 + 4(1 - c^2)\zeta^2 + 4ic\zeta - 1)(\zeta + ic)^2}{R\zeta^2(\zeta^2 + 2ic\zeta - 1)^3} \]
where the function $g_0(\zeta)$ represents

$$g_0(\zeta) = (1-c^2)^2\zeta^6 + 2ic(4-5c^2+2c^4)\zeta^5 + (4-19c^2+14c^4-4c^6)\zeta^4$$

$$+ 4ic(1-c^2)(3-c^2)\zeta^3 - (1+14c^2-5c^4)\zeta^2 - 2ic(2+3c^2)\zeta + 3c^2$$

and the constants $A, B, C, A_0$, and $B_0$ are

$$A = \frac{-(M_x + M_y)}{4D(1 + \nu)} \quad , \quad B = \frac{-(M_x - M_y)}{2D(1 - \nu)} \quad , \quad C = \frac{H_{xy}}{D(1 - \nu)}$$

$$M_x = \frac{M_{x_0} + M_{y_0}}{2} + \frac{M_{x_0} - M_{y_0}}{2} \cos 2\alpha_0 - H_{xy_0} \sin 2\alpha_0$$

$$M_y = \frac{M_{x_0} + M_{y_0}}{2} - \frac{M_{x_0} - M_{y_0}}{2} \cos 2\alpha_0 + H_{xy_0} \sin 2\alpha_0$$

$$H_{xy} = -\frac{M_{x_0} - M_{y_0}}{2} \sin 2\alpha_0 + H_{xy_0} \cos 2\alpha_0$$

$$A_0 = A \cdot \frac{R(N+1)(N-c^2)(c^4 + Nc^2 - N)}{N(N^2 - c^4)} + B \cdot \frac{Rc^2(1-c^2)(N-c^2)}{N(N^2 - c^4)} + C \cdot \frac{iRc^2(1-c^2)(N+c^2)}{N(N^2 - c^4)}$$

$$B_0 = \frac{(B-iC)R}{N}$$

For the problem with general shearing stresses at infinity:

$$\phi(\zeta) = A_1 \cdot R^2 \left( \frac{\zeta^2 + 1}{\zeta + ic} \right)^2 + \phi_0(\zeta)$$
\[ \phi_0(\zeta) = \frac{-\mathcal{F}_c}{\zeta + ic} + \frac{g_2(\zeta)}{N\zeta^2(\zeta + ic)^2} \]  
(D.13)

\[ \frac{d\phi(\zeta)}{dz} = 2A_1R \frac{\zeta^2 + 1}{\zeta + ic} + \frac{\mathcal{F}_c}{R(\zeta^2 + 2ic\zeta - 1)} - \frac{g_3(\zeta)}{NR\zeta^3(\zeta + ic)(\zeta^2 + 2ic\zeta - 1)} \]  
(D.14)

\[ \frac{d^2\phi(\zeta)}{dz^2} = 2A_1 - \frac{2\mathcal{F}_c(\zeta + ic)^3}{R^2(\zeta^2 + 2ic\zeta - 1)^3} + \frac{g_3(\zeta)}{N\zeta^4(\zeta^2 + 2ic\zeta - 1)^3} \]  
(D.15)

\[ \psi(\zeta) = A_2 \cdot R^2 \left( \frac{\zeta^2 + 1}{\zeta + ic} \right)^2 + \psi_0(\zeta) \]  
(D.16)

\[ \psi_0(\zeta) = \frac{g_4(\xi)}{\xi^2(\xi + ic)^2} - \frac{g_5(\xi)}{N\xi^4(\xi + ic)(\xi^2 + 2ic\xi - 1)} \]  
(D.17)

\[ \frac{d\psi(\zeta)}{dz} = 2A_2R \frac{\zeta^2 + 1}{\zeta + ic} + \frac{g_7(\zeta)}{\zeta^3(\zeta + ic)(\zeta^2 - 2ic\zeta - 1)} + \frac{g_8(\zeta)}{\zeta^5(\zeta^2 + 2ic\zeta - 1)^3} \]  
(D.18)

where the functions \( g_2(\zeta) \sim g_4(\zeta) \) are defined by

\[ g_2(\zeta) = 2icR^2(\overline{A_1} - \overline{A_2} + 2NA_1 - 2NA_1c^2)\zeta^3 \]

\[ -R^2(2\overline{A_1} + \overline{A_2} + 2c^2\overline{A_1} - 4c^2\overline{A_2} + NA_1 + 2c^2NA_1 - 3c^4NA_1)\zeta^2 \]  
(D.19)

\[ -2icR^2(\overline{A_1} + \overline{A_2} - c^2\overline{A_2})\zeta + R^2c^2\overline{A_2} \]
\[ g_3(\zeta) = -2icR^2(-\overline{A_1} + \overline{A_2} - 2NA_1 + 2NA_1c^2)\zeta^4 \]
\[ + 2R^2(-2\overline{A_1} - \overline{A_2} - c^2\overline{A_1} + 3c^2\overline{A_2} - NA_1 + c^4NA_1)\zeta^3 \]  
\[ + 6icR^2(-\overline{A_1} + \overline{A_2} + c^2\overline{A_2})\zeta^2 + 2c^2R^2(\overline{A_1} + 3\overline{A_2} - c^2\overline{A_2})\zeta \]
\[ + 2ic^3R^2\overline{A_2} \quad (D.20) \]

\[ g_4(\zeta) = 2icR^2(A_1 + 2A_2 - 2c^2A_2 - NA_1)\zeta^3 \]
\[ - R^2(2A_1 + A_2 + 2c^2A_1 + 2c^2A_2 - 3c^4A_2 + NA_1 - 4c^2A_1)\zeta^2 \quad (D.21) \]
\[ - 2icR^2(A_1 + NA_1 - Nc^2A_1)\zeta + Nc^2R^2\overline{A_1} \]

\[ g_5(\zeta) = -2ic(1 - c^2)R^2[-\overline{A_1} - 2\overline{A_2}(1 - c^2) + NA_1]\zeta^6 \]
\[ + 2R^2[NA_1(1 - c^2)(1 + 3c^2) + \overline{A_1}(2 - 4c^2 + 3c^4) \]
\[ + \overline{A_2}(1 - 10c^2 + 14c^4 - 6c^6)]\zeta^5 \]
\[ - 2icR^2[2\overline{A_1}(-2 + 2c^2 - c^4) + \overline{A_2}(-5 + 14c^2 - 12c^4 + 4c^6) \quad (D.22) \]
\[ + NA_1(1 - c^2)(1 - 2c^2)]\zeta^4 \]
\[ + 2R^2[NA_1(1 - c^4) + \overline{A_1}(1 - c^3)(2 - c^2) + \overline{A_2}(1 - 9c^2 + 8c^4 - 2c^6)]\zeta^3 \]
\[ + 2ic(-3 + c^2)R^2[-\overline{A_1} - \overline{A_2}(1 - 2c^2)]\zeta^2 + 2c^2R^2[-\overline{A_1} - \overline{A_2}(3 - 2c^2)]\zeta \]
\[ - 2ic^3R^2\overline{A_2} \]
\[ g_6(\zeta) = 4ic[A_1 - A_2 + 2N_1(1 - c^2)]\zeta^2 - 6[2A_1(1 + c^2) + A_2(1 - c^2)(1 + 3c^2)]\zeta^6 \]

\[ -12ic[A_1(4 + c^2) + A_2(3 - 5c^2) + NA_1(1 - c^4)]\zeta^5 \]

\[ + 2[A_1(2 + 36c^2 + 2c^4) + A_2(1 + 42c^2 - 38c^4) + NA_1(1 + c^4 - 2c^6)]\zeta^4 \]

\[ + 12ic[A_1(1 + 4c^2) + A_2(1 + 8c^2 - 4c^4)]\zeta^3 \]

\[-6c^2[2A_1(1 + c^2) + A_2(4 + 9c^2 - 2c^4)]\zeta^2 \]

\[-4ic^3[A_1 + A_2(5 + 3c^2)]\zeta^4 + 6A_2c^4 \]

\[ g_7(\zeta) = 2icR[-A_1 - 2A_2(1 - c^2) + NA_1]\zeta^4 \]

\[ + 2R[A_1(2 + c^2) + A_2(1 - c^4) + NA_1(1 - 3c^2)]\zeta^3 \]

\[-6icR[-A_1(1 + N) + NA_1c^2]\zeta^2 - 2ic^2[A_1(1 + N) + NA_1(2 - c^2)]\zeta \]

\[-2ic^3NRA_1 \]
\[ g_8(\zeta) = -2i e^R(1 - e^2)[-A_1 - 2A_2(1 - e^2) + NA_1] \zeta^9 \]
\[ + 4R^2[A_1(2 - 4e^2 + 3e^4) + A_2(1 - 10e^2 + 14e^4 - 6e^6) + NA_1(1 - e^2)(1 + 3e^2)] \zeta^8 \]
\[ - 2ie^R[A_1(-19 + 23e^2 - 13e^4 + 3N - 9Ne^2 + 6Ne^4) \]
\[ + A_2(-20 + 72e^2 - 72e^4 + 26e^6) + NA_1(2 + 7e^3)(e^2 - 1)] \zeta^7 \]
\[ + 4R[A_1(-19e^2 + 15e^4 - 6e^6 + 3N - 9Ne^2 + 6Ne^4) \]
\[ + A_2(2 - 35e^2 + 62e^4 - 42e^6 + 12e^8) + NA_1(2 + 3e^3)(e^2 - 1)] \zeta^6 \]
\[ - 2ie^R[A_1(-31 + 40e^2 - 18e^4 + 4e^6 - N + Ne^2 + 4Ne^4 - 4Ne^6) \]
\[ - 2A_2(2 - e^2)(5 - 28e^2 + 14e^4 - 4e^6) - 2NA_1(1 - e^2)(5 + 6e^2)] \zeta^5 \]
\[ + 4R[A_1(-2 - 22e^2 + 11e^4 - 2e^6) + A_2(-1 - 20e^2 + 58e^4 - 26e^6 + 4e^8) \]
\[ - NA_1(1 - e^4)(1 + 2e^2)] \zeta^4 + 2ie^R[A_1(-11 - 27e^2 + 5e^4) \]
\[ + 2A_2(-5 - 20e^2 + 29e^4 - 5e^6) - NA_1(1 - e^4)] \zeta^3 \]
\[ + 4e^2R[A_1(5 + 3e^2) + A_2(9 + 10e^2 - 6e^4)] \zeta^2 \]
\[ + 2ie^3R[3A_1 + 2A_2(7 + 2e^2)] \zeta - 8e^4R A_2 \]

and the constants \( A_1 \) and \( A_2 \) are

\[ A_1 = \left( \frac{Q_{x_0} - iQ_{y_0} e^{-i\eta_0}}{-8D} \right), \quad A_2 = \left( \frac{Q_{x_0} + iQ_{y_0} e^{i\eta_0}}{-8D} \right) \] 

(D.26)
APPENDIX E. PROOF OF THE MAPPING FOR AN ELLIPTICAL CRACK

First, consider the mapping function

\[ z = \omega(\zeta) = \frac{R + V}{2} \left( \zeta + ic + \frac{1 - c^2}{\zeta + ic} \right) + \frac{R - V}{2} \left( \zeta^{-1} - ic + \frac{1 - c^2}{\zeta^{-1} - ic} \right) \]  \hspace{1cm} (E.1)

For the points on the crack surface, the mapping function can be written as

\[ z_x = \omega(\zeta) = \frac{R + V}{2} \left( \zeta + ic + \frac{1 - c^2}{\zeta + ic} \right) + \frac{R - V}{2} \left( \zeta^{-1} - ic + \frac{1 - c^2}{\zeta^{-1} - ic} \right) \]  \hspace{1cm} (E.2)

The coordinate of an arbitrary point on the unit circle (in the \( \zeta \)-plane), which has an angle \( \theta_0 \) measured from the reference line, is given by equation (A.2). Thus, by substituting the expression for the coordinate of this arbitrary point (A.2) into equation (E.2) and making some algebraic operations, one can obtain

\[ z_s = 2 R \cos \theta_0 \sqrt{1 - c^2 \cos^2 \theta_0} + 2 i c V \sin^2 \theta_0 \]  \hspace{1cm} (E.3)

so that the \( x \) and \( y \) coordinate can be written as

\[ x_s = 2 R \cos \theta_0 \sqrt{1 - c^2 \cos^2 \theta_0} \hspace{1cm} , \hspace{1cm} y_s = 2 c V \sin^2 \theta_0 \]  \hspace{1cm} (E.4)

Since \( \theta_0 \) is the reference angle at an arbitrary point on the unit circle, equation (E.4) is the implicit function of the locus of the unit circle in the \( z \)-plane. Thus, the variable \( \theta_0 \) can be eliminated by rewriting the parameter \( x_s \) as
Equation (E.5) can be further rewritten as

\[
\frac{x_s^2}{R^2} + \frac{y_s - 2 V (c - 1/(2c))}{V^2} = c^2 \tag{E.6}
\]

Obviously, the expression (E.6) is the parametric equation for a locus of an ellipse with its origin located at \((0, 2V(c-1/2c))\). Also, it is proved that the length of its semi-major axis is \(V/c\), and its semi-minor axis has the length of \(R/c\). Additionally, from equation (E.4), it is seen that because \(y_s\) is always positive, the locus of this ellipse has only the part with positive \(y_s\) value (Figure E.1). Further, rearrangement can be made for equation (E.2) to give

\[
z_s = \frac{R + V}{2} \left( 2ic + \frac{\xi^2 + 1}{\xi + ic} \right) + \frac{R - V}{2} \left( -2ic + \frac{\xi^2 + 1}{\xi(1 - ic\xi)} \right) \tag{E.7}
\]

\[
= \frac{R + V}{2} \left( \frac{\xi^2 + 1}{\xi + ic} \right) + \frac{R - V}{2} \left( \frac{\xi^2 + 1}{\xi(1 - ic\xi)} \right) + 2icV
\]

If the \(z\)-coordinate is relocated to \(z'\) (by \(z = z' + 2icV\)), equation (E.7) will become

\[
z_s' = \frac{R + V}{2} \left( \frac{\xi^2 + 1}{\xi + ic} \right) + \frac{R - V}{2} \left( \frac{\xi^2 + 1}{\xi(1 - ic\xi)} \right) \tag{E.8}
\]
Now, the prime sign can be dropped in equation (E.8), and thus it is proved that the mapping function employed in Chapter 7 transforms the contour surface of an elliptical crack into a unit circle.
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