Relation algebras and vertex conditions in graph theory

Jerzy Wojdyło
Iowa State University

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Relation algebras and vertex conditions in graph theory

by

Jerzy Wojdyło

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics
Major Professor: Jonathan D. H. Smith

Iowa State University
Ames, Iowa
1998

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This is to certify that the Doctoral dissertation of

Jerzy Wojdylo

has met the dissertation requirements of Iowa State University

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Committee Member

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Committee Member

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Committee Member

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Major Professor

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For the Major Program

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For the Graduate College
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CHAPTER 1  INTRODUCTION

The study of association schemes has been described as "the character-theoretical study of combinatorial objects" or as "group theory without groups" [BI84, preface]. Sometimes association schemes appear in the literature under the name of "cellular rings". An association scheme \((Q, \Gamma)\), comprising a certain kind of partition \(\Gamma\) of the direct square of a finite, nonempty set \(Q\), corresponds to a binary relation algebra. A superscheme \((Q, \Gamma^*)\) comprises a certain family of partitions of the direct powers \(Q^{n+2}\) for each natural number \(n\). Superschemes correspond to Krasner (relation) algebras. J.D.H. Smith [Smi94] showed that an association scheme \((Q, \Gamma)\) can be extended to a superscheme \((Q, \Gamma^*)\) if and only if it is a permutation group scheme, i.e. there exists a transitive, multiplicity-free permutation group \(G\) acting on \(Q\) such that each partition \(\Gamma^n\) of the superscheme is the set of orbits of \(G\) acting componentwise on \(Q^{n+2}\).

Strongly regular graphs are very well known objects in graph theory. They are finite graphs which are regular (i.e. every vertex has the same number of edges coming out of it) and such that for every two distinct vertices, the number of common neighbors depends only on whether these two vertices are adjacent or not. Note that in the case of a strongly regular graph \(G\) we are considering 3-vertex subgraphs of the graph \(G\) which have distinguished pairs of vertices. This can be generalized to \(t\)-vertex subgraphs with distinguished pairs of vertices. With some conditions described later in Definitions 2.2.4 and 2.2.5, if for a graph \(G\) and every \(k\), \(2 \leq k \leq t\), the number of \(k\)-vertex subgraphs with a distinguished pair of vertices depends only on whether these two vertices are adjacent or not, we say that the graph \(G\) is \(t\)-regular (or satisfies the \(t\)-vertex condition). This
dissertation describes connections between the first \( t \) levels of a superscheme and graphs which satisfy the \((t + 3)\)-vertex condition.

The paper "Relation Algebras and \( t \)-vertex Condition Graphs" (Chapter 2) answers the natural question: what prevents an association scheme from being built up any further to a superscheme beyond the partition \( \Gamma' \)? A scheme can be associated with a graph. This scheme is an association scheme if and only if the graph is strongly regular. A height \( t \) presuperscheme consists of the bottom \( t \) levels of a superscheme. A construction of a presuperscheme associated with a graph is given. The main theorem shows that if a presuperscheme associated with a graph is of height \( t \), then the graph satisfies the \((t + 3)\)-vertex condition. Shrikhande's graph provides an example of an association scheme that cannot be extended beyond the bottom level.

The generalization of this result is discussed in the second paper "Presuperschemes and Colored Directed Graphs" (Chapter 3). A similar construction of a presuperscheme associated with a colored, directed graph is given. The scheme associated with a colored, directed graph is an association scheme if and only if the graph is strongly regular. Again, the main theorem shows that if a presuperscheme associated with a colored, directed graph is of height \( t \), then the graph satisfies the \((t + 3)\)-vertex condition. Since the graph is colored and directed this result refers to all association schemes, unlike in the previous chapter, where association schemes consisting of three classes only were considered. This paper also discusses another description of the 2-vertex condition found in the literature.

The paper "On a 4-regular Graph with an Inextensible Association Scheme" (Chapter 4) studies the converse of the main theorem from the previous chapter. This converse does not hold. We give an example of a colored, directed graph which satisfies the 4-vertex condition, but whose presuperscheme cannot be extended beyond the bottom (0-th) level. In particular, the example exhibits an association scheme that cannot be extended to a presuperscheme of positive height.
**Dissertation organization**

This dissertation includes three papers entitled “Relation Algebras and t-vertex Condition Graphs”, “Presuperschemes and Colored Directed Graphs”, and “On a 4-regular Graph with an Inextensible Association Scheme” as Chapters 2, 3 and 4 respectively. Chapter 5 summarizes the general conclusions and contains some open problems for future research.

Appendix A contains Pascal program *graph15* and three outputs from the program, files *ADJ_MAT.TXT*, *ARCS_COLOR1* and *ARCS_COLOR2*. The program creates the adjacency matrix *A* of the graph *G* from Chapter 4. First it follows the construction of the adjacency matrix *C* of the cyclotomic scheme *GF(q)* for *q* = 7 and *d* = 3 [SS]. Then it constructs the adjacency matrix *A* of the graph *G* following [FKM94, Theorem 2.6.6]. The matrix *A* is saved to the text file *ADJ_MAT.TXT*. Next the program creates the class *C01* of the scheme associated with the graph *G*, which consists of pairs of vertices *(x,y)* connected by an arc of color 1, and saves it to the text file *ARCS_COLOR1*. Finally, the program creates the class *C02* of the scheme associated with the graph *G*, which consists of pairs of vertices *(x,y)* connected by an arc of color 2, and saves it to the text file *ARCS_COLOR2*.

Appendix B contains Pascal program *three_vert* and the text file *THREE_VERT.TXT*. The program verifies the 3-vertex condition for the graph *G* from Chapter 4 by computing the numbers of locally ordered 3-vertex subgraphs of all types with respect to all arcs of color 1 and 2 and with respect to all vertices (arcs of color 0). The results are written to the file *THREE_VERT.TXT* and are summarized in Figures 4.1 and 4.2 in Chapter 4.

Appendix C contains Pascal program *four_vert* and the text file *FOUR_VERT.TXT*. The program verifies the 4-vertex condition for the graph *G* from Chapter 4 by computing the numbers of locally ordered 4-vertex subgraphs of all types with respect to
all arcs of color 1 and 2 and with respect to all vertices (arcs of color 0). The results are written to the file FOUR_VERT_TXT and are summarized in Figures 4.3 and 4.4 in Chapter 4.

Appendix D contains Pascal program pentagon and the text file PENTAGON_TXT. It shows that the graph $G$ from Chapter 4 does not satisfy the 5-vertex condition, by computing the numbers of clockwise oriented pentagons with inscribed counterclockwise oriented pentagrams with respect to all arcs of color 1. The results are written to the file PENTAGON_TXT and are summarized in Figure 4.5 in Chapter 4.

Appendix E contains Pascal program first_level and three files CLASSC1_K_TXT, CONST_C1KK_01_TXT and SMALL_CLASS_TXT. Program first_level creates a class $C^1_k$ of the first level of a presuperscheme associated with the graph $G$ from Chapter 4. It consists of triples of vertices, $(v_1, v_2, v_3)$, of all locally ordered 3-vertex subgraphs of the graph $G$ whose type is given by the $3 \times 3$ matrix on page 42. All triples are written to the file CLASSC1_K_TXT. Then first_level computes the constant $c(1, k, k; 0, 1)$ and writes the values of this constant to the file CONST_C1KK_01_TXT. Finally first_level checks for which triples $(v_1, v_2, v_3)$ from the class $C^1_k$ the constant $c(1, j, k; 0, 1) = 0$ (here $C^1_j = C^1_k$) and writes them to the file SMALL_CLASS_TXT.

References


CHAPTER 2  RELATION ALGEBRAS AND $t$-VERTEX CONDITION GRAPHS

A paper submitted to European Journal of Combinatorics

Jerzy Wojdyło

Abstract

The scheme associated with a graph is an association scheme iff the graph is strongly regular. Consider the problem of extending such an association scheme to a superscheme. The obstacles can be expressed in terms of $t$-vertex conditions. If a graph does not satisfy the $t$-vertex condition, a presuperscheme associated with it cannot be erected beyond the $(t-3)$rd level. We give an example of an association scheme which is not extendible to a superscheme: it cannot be extended beyond the bottom level of a presuperscheme.

2.1 Introduction

An association scheme $(Q, \Gamma)$, comprising a certain kind of partition $\Gamma$ of the direct square of a finite, nonempty set $Q$, corresponds to a binary relation algebra. A superscheme $(Q, \Gamma^*)$ comprises a certain family of partitions of the direct powers $Q^{n+2}$ for each natural number $n$. Superschemes correspond to Krasner (relation) algebras. J.D.H. Smith [Smi94] showed that an association scheme $(Q, \Gamma)$ can be extended to a superscheme $(Q, \Gamma^*)$ iff it is a permutation group scheme, i.e. there exists a transitive,
multiplicity-free permutation group $G$ acting on $Q$ such that each partition $\Gamma^n$ of the superscheme is the set of orbits of $G$ acting componentwise on $Q^{n+2}$.

A natural question arises: what prevents an association scheme from being built up any further to a superscheme beyond the partition $\Gamma^t$? A scheme can be associated with a graph. This scheme is an association scheme if and only if the graph is strongly regular. A height $t$ presuperscheme consists of the bottom $t$ levels of a superscheme. The current paper contains a construction of a presuperscheme associated with a graph. The main theorem shows that if a presuperscheme associated with a graph is of height $t$, then the graph satisfies the $(t + 3)$-vertex condition. Shrikhande’s graph provides an example of a scheme that cannot be extended beyond the bottom level.

2.2 Basic definitions

**Definition 2.2.1** [BI84] Let $Q$ be a finite, non-empty set. An association scheme $(Q, \Gamma)$ on $Q$ is a partition $\Gamma = \{C_1, \ldots, C_s\}$ of the direct square $Q^2$ such that four conditions are satisfied:

(A1) $C_1 = \{(x, x) | x \in Q\}$;

(A2) $\forall C_i \in \Gamma, \{(y, x) | (x, y) \in C_i\} \in \Gamma$;

(A3) $\forall C_i \in \Gamma, \forall C_j \in \Gamma, \forall C_k \in \Gamma, \exists c(i, j, k) \in \mathbb{N}. \forall (x, y) \in C_k$,

$|\{(z \in Q | (x, z) \in C_i, (z, y) \in C_j\}| = c(i, j, k)$;

(A4) $\forall 1 \leq i, j, k \leq s, c(i, j, k) = c(j, i, k)$.

**Definition 2.2.2** [Smi94] Let $Q$ be a finite non-empty set. A superscheme $(Q, \Gamma^*)$ on $Q$ is a family of partitions $\Gamma^n = \{C^n_1, \ldots, C^n_s\}$ of the direct powers $Q^{n+2}$, for each natural number $n$, such that:

(S1) $C^n_1 = \{(x, x) | x \in Q\}$. 
(S2) \( \forall m, n \in \mathbb{N}, \forall f : \{1, \ldots, m + 2\} \rightarrow \{1, \ldots, n + 2\}, \forall C^*_j \in \Gamma^n, \)
\( f^*(C^*_j) = \{(x_1, \ldots, x_{m+2}) \mid \exists (y_1, \ldots, y_{n+2}) \in C^*_j, \forall 1 \leq i \leq m + 2, x_i = y_{f(i)}\} \)
is an element of \( \Gamma^m; \)

(S3) \( \forall (m, n) \in \mathbb{N}^2, \forall C^*_i \in \Gamma^m, \forall C^*_j \in \Gamma^n, \forall C^*_{k+n} \in \Gamma^{m+n}, \)
\( \exists c(i, j, k; m, n) \in \mathbb{N}. \forall (x_0, \ldots, x_m, y_0, \ldots, y_n) \in C^*_{k+n}. \)
\( \{|z \in Q \mid (x_0, \ldots, x_m, z) \in C^*_i, (z, y_0, \ldots, y_n) \in C^*_j\}| = c(i, j, k; m, n); \)

(S4) \( \forall 1 \leq i, j, k \leq s_0, c(i, j, k; 0, 0) = c(j, i, k; 0, 0). \)

We call \( \Gamma^t \), the partition of \( Q^{t+2} \), the \textit{t-th level of the superscheme}. A class \( C^*_i \in \Gamma^t \) is denoted as \textit{diagonal} if \( C^*_i = f^*(C^*_j) \) for some \( C^*_j \in \Gamma^k \) and some \( f : \{1, \ldots, t + 2\} \rightarrow \{1, \ldots, k + 2\} \), where \( k < t \), i.e. if elements of \( C^*_i \) have repeated entries.

Later on we will need modified versions of (S2) and (S3). Define

\[ \mathbb{N}_t = \{n \in \mathbb{N} \mid n \leq t\} \quad \text{and} \quad \mathbb{N}_t^2 = \{(m, n) \in \mathbb{N}^2 \mid m + n \leq t\}. \]

Then consider the following conditions:

(\( S2_t \)) \( \forall m, n \in \mathbb{N}_t, \forall f : \{1, \ldots, m + 2\} \rightarrow \{1, \ldots, n + 2\}, \forall C^*_j \in \Gamma^n, \)
\( f^*(C^*_j) = \{(x_1, \ldots, x_{m+2}) \mid \exists (y_1, \ldots, y_{n+2}) \in C^*_j, \forall 1 \leq i \leq m + 2, x_i = y_{f(i)}\} \)
is an element of \( \Gamma^m; \)

(\( S3_t \)) \( \forall (m, n) \in \mathbb{N}_t^2, \forall C^*_i \in \Gamma^m, \forall C^*_j \in \Gamma^n, \forall C^*_{k+n} \in \Gamma^{m+n}, \)
\( \exists c(i, j, k; m, n) \in \mathbb{N}. \forall (x_0, \ldots, x_m, y_0, \ldots, y_n) \in C^*_{k+n}. \)
\( \{|z \in Q \mid (x_0, \ldots, x_m, z) \in C^*_i, (z, y_0, \ldots, y_n) \in C^*_j\}| = c(i, j, k; m, n). \)

A \textit{height \( t \) presuperscheme} \( (Q, \Gamma^t) \) on \( Q \) is a family of partitions \( \Gamma^n = \{C^*_1, \ldots, C^*_t\} \)
of the direct powers \( Q^{n+2} \), for each number \( n \in \mathbb{N}_t \), such that the conditions (S1), (S2t), (S3t) and (S4) are satisfied.

It is easy to see that the family \( \Gamma^n \) of partitions of \( Q^{n+2} \) for each natural number \( n \) is a superscheme if and only if it is a height \( t \) presuperscheme for each natural number \( t \).
**Definition 2.2.3** [BCN89, CDH80] By $G = (V, E)$ we understand an undirected finite graph without loops or multiple edges.

- $V$ is the set of vertices of $G$ and sometimes will be denoted by $V(G)$.
- $E$ is the set of edges of $G$ and sometimes will be denoted by $E(G)$.

Additionally, $N = N(G) = E(\overline{G})$ denotes the set of nonedges of $G$, i.e. the set of edges of the complementary graph $\overline{G}$.

$H$ is a subgraph of $G$, $H \subseteq G$, if $H$ is an induced subgraph of $G$, i.e. $H = (V(H), E(H))$ where $V(H) \subseteq V(G)$ and $\forall x, y \in V(H) , \{x, y\} \in E(H) \Leftrightarrow \{x, y\} \in E(G)$.

A graph $G$ is regular if it is connected and every vertex has the same valency.

**Definition 2.2.4** [BIK89, Iva89] Two subgraphs $H$ and $H'$ of $G$ are of the same type with respect to $\{x, y\} \subseteq V(G)$ if and only if:

1. $\{x, y\} \subseteq V(H) \cap V(H')$.

2. There exists an isomorphism of $H$ onto $H'$ fixing $x$ and $y$.

If $x = y$ we say that $H$ and $H'$ are of the same type with respect to the vertex $x$.

If $\{x, y\} \in E(G)$ then we say that $H$ and $H'$ are of the same type with respect to the edge $\{x, y\}$.

If $\{x, y\} \in N(G)$ then we say that $H$ and $H'$ are of the same type with respect to the nonedge $\{x, y\}$.

**Definition 2.2.5** [BIK89, Iva89] We say that a graph $G$ satisfies the $t$-vertex condition with respect to edges (or that $G$ is $t$-edge regular) if for every $i$, $2 \leq i \leq t$, the number of $i$-vertex subgraphs of every fixed type with respect to an edge $\{x, y\}$ is the same for all edges (i.e. does not depend on the choice of the edge $\{x, y\}$).

We say that a graph $G$ satisfies the $t$-vertex condition with respect to nonedges (or that $G$ is $t$-nonedge regular) if for every $i$, $2 \leq i \leq t$, the number of $i$-vertex subgraphs...
of every fixed type with respect to a nonedge \( \{x, y\} \) is the same for all nonedges (i.e. does not depend on the choice of the nonedge \( \{x, y\} \)).

We say that a graph \( G \) satisfies the \( t \)-vertex condition with respect to vertices (or that \( G \) is \( t \)-vertex regular) if for every \( i \), \( 2 \leq i \leq t \), the number of \( i \)-vertex subgraphs of every fixed type with respect to a vertex \( x \) is the same for all vertices (i.e. does not depend on the choice of the vertex \( x \)).

**Definition 2.2.6** [BIK89, Iva89] A graph \( G \) which satisfies the \( t \)-vertex condition with respect to edges, nonedges and vertices is said to satisfy the \( t \)-vertex condition (or to be \( t \)-regular). Obviously a \( t \)-regular graph is \( k \)-regular for every \( 2 \leq k \leq t \).

**Example 2.2.7** [BIK89, Iva89] A graph \( G \) is regular iff it satisfies the 2-vertex condition. □

### 2.3 Strongly regular graphs

**Definition 2.3.1** [BCN89, CDH80] A graph \( G \) is strongly regular (or an SRG) with the parameters \( (n, k, \lambda, \mu) \) if \( G \) is regular, each pair of adjacent vertices has \( \lambda \) common neighbors and each pair of nonadjacent vertices has \( \mu \) common neighbors. The parameters \( n \) and \( k \) respectively denote the number of vertices of \( G \) and the valency of each vertex.

**Proposition 2.3.2** [BC55, BIK89, Iva89] A graph \( G \) is strongly regular iff it is 3-regular.

**Proof.** Let \( G \) be an SRG with the parameters \( (n, k, \lambda, \mu) \).

It is clear that every 3-regular graph is strongly regular.

To show that an SRG \( G \) is 3-regular, we have to express the numbers of 3-vertex subgraphs of \( G \) of the same type with respect to edges, nonedges and vertices in terms of the parameters \( n, k, \lambda \) and \( \mu \). This is done in Figures 2.1, 2.2, 2.3.
\{x, y\} \in E \hspace{1cm} \text{Type} \hspace{1cm} \text{Number of subgraphs}

\[
\begin{array}{ccc}
\text{x} & \rightarrow & \text{z} \\
\downarrow & & \\
\text{y} & & \\
\end{array}
\hspace{1cm} \lambda = c(2, 2, 2)
\]

\[
\begin{array}{ccc}
\text{x} & \rightarrow & \text{z} \\
\downarrow & & \\
\text{y} & & \\
\end{array}
\hspace{1cm} k - 1 - \lambda = c(3, 2, 2)
\]

\[
\begin{array}{ccc}
\text{x} & \rightarrow & \text{z} \\
\downarrow & & \\
\text{y} & & \\
\end{array}
\hspace{1cm} k - 1 - \lambda = c(2, 3, 2)
\]

\[
\begin{array}{ccc}
\text{x} & \rightarrow & \text{z} \\
\downarrow & & \\
\text{y} & & \\
\end{array}
\hspace{1cm} n - 2k + \lambda = c(3, 3, 2)
\]

Figure 2.1 Types of 3-vertex subgraphs with respect to the edge \{x, y\}.

\{x, y\} \in N \hspace{1cm} \text{Type} \hspace{1cm} \text{Number of subgraphs}

\[
\begin{array}{ccc}
\text{x} & \rightarrow & \text{z} \\
\downarrow & & \\
\text{y} & & \\
\end{array}
\hspace{1cm} \mu = c(2, 2, 3)
\]

\[
\begin{array}{ccc}
\text{x} & \rightarrow & \text{z} \\
\downarrow & & \\
\text{y} & & \\
\end{array}
\hspace{1cm} k - \mu = c(3, 2, 3)
\]

\[
\begin{array}{ccc}
\text{x} & \rightarrow & \text{z} \\
\downarrow & & \\
\text{y} & & \\
\end{array}
\hspace{1cm} k - \mu = c(2, 3, 3)
\]

\[
\begin{array}{ccc}
\text{x} & \rightarrow & \text{z} \\
\downarrow & & \\
\text{y} & & \\
\end{array}
\hspace{1cm} n - 2 - 2k + \mu = c(3, 3, 3)
\]

Figure 2.2 Types of 3-vertex subgraphs with respect to the nonedge \{x, y\}.
$x \in V$

<table>
<thead>
<tr>
<th>Type</th>
<th>Number of subgraphs</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Graph 1" /></td>
<td>( \frac{1}{2}k\lambda )</td>
</tr>
<tr>
<td><img src="image2" alt="Graph 2" /></td>
<td>( \frac{1}{2}k(k - 1 - \lambda) )</td>
</tr>
<tr>
<td><img src="image3" alt="Graph 3" /></td>
<td>( \frac{1}{2}k(k - 1 - \lambda) = \frac{1}{2}(n - 1 - k)\mu )</td>
</tr>
<tr>
<td><img src="image4" alt="Graph 4" /></td>
<td>( \frac{1}{2}k(n - 2k + \lambda) = \frac{1}{2}(n - 1 - k)(k - \mu) )</td>
</tr>
<tr>
<td><img src="image5" alt="Graph 5" /></td>
<td>( \frac{1}{2}(n - 1 - k)(k - \mu) )</td>
</tr>
<tr>
<td><img src="image6" alt="Graph 6" /></td>
<td>( \frac{1}{2}(n - 1 - k)(n - 2 - 2k + \mu) )</td>
</tr>
</tbody>
</table>

Figure 2.3 Types of 3-vertex subgraphs with respect to the vertex $x$. 
Thus an SRG is a 3-regular graph. □

**Definition 2.3.3 [CDH80]** Let \( G = (V, E) \) be a graph. The *scheme associated with* \( G \) is \((V, \Gamma)\), where \( \Gamma = \{ \hat{V}, \hat{E}, \hat{N} \} \), for \( \hat{V} = \{(x, x) | x \in V\} = C_1 \), \( \hat{E} = \{(x,y)|\{x,y\} \in E(G)\} = C_2 \) and \( \hat{N} = \{(x,y)|\{x,y\} \in N(G)\} = C_3 \).

**Proposition 2.3.4 [CDH80]** The scheme associated with a graph is an association scheme iff the graph is strongly regular. □

**Proof.** (\( \Rightarrow \)) Let \( x, y \in V \). Then from Definition 2.2.1, (A3), we have:

\[
\exists k = c(2,2,1) \in \mathbb{N}. \forall (x,x) \in \hat{V}, |\{z \in V|(x,z) \in \hat{E}, (z,x) \in \hat{E}\}| = k, \text{ so the graph is regular and } \\
\exists \lambda = c(2,2,2) \in \mathbb{N}. \forall (x,y) \in \hat{E}, |\{z \in V|(x,z) \in \hat{E}, (z,y) \in \hat{E}\}| = \lambda, \text{ so each pair of adjacent vertices have } \lambda \text{ common neighbors and } \\
\exists \mu = c(2,2,3) \in \mathbb{N}. \forall (x,y) \in \hat{N}, |\{z \in V|(x,z) \in \hat{E}, (z,y) \in \hat{E}\}| = \mu, \text{ so each pair of nonadjacent vertices have } \mu \text{ common neighbors.}
\]

Thus the conditions of strong regularity are satisfied.

(\( \Leftarrow \)) Now we need to show that the scheme associated with an SRG satisfies conditions (A1), (A2), (A3), (A4) of Definition 2.2.1. Let \( G = (V, E) \) be an SRG. Let \((V, \Gamma), \Gamma = \{ \hat{V}, \hat{E}, \hat{N} \}\), be the associated scheme, where \( C_1 = \hat{V}, C_2 = \hat{E}, C_3 = \hat{N} \).

Then conditions (A1) and (A2) are obviously satisfied.

(A3) goes by direct check:

If \( C_k = C_1 = \hat{V} \) then

\[
c(i,j,1) = \begin{cases} 1 & i = j = 1 \\ |C_i| & i = j \neq 1 \\ 0 & i \neq j \end{cases}
\]

If \( C_k = C_2 = \hat{E} \) and \( C_i = C_1 = \hat{V} \) then

\[
c(1,j,2) = \begin{cases} 1 & j = 2 \\ 0 & j \neq 2 \end{cases}
\]
If \( C_k = C_2 = \hat{E} \) and \( C_j = C_1 = \hat{V} \) then
\[
c(i, 1, 2) = \begin{cases} 
1 & i = 2 \\
0 & i \neq 2 
\end{cases}.
\]

If \( C_k = C_3 = \hat{N} \) and \( C_i = C_1 = \hat{V} \) then
\[
c(1, j, 3) = \begin{cases} 
1 & j = 3 \\
0 & j \neq 3 
\end{cases}.
\]

If \( C_k = C_3 = \hat{N} \) and \( C_j = C_1 = \hat{V} \) then
\[
c(i, 1, 3) = \begin{cases} 
1 & j = 3 \\
0 & j \neq 3 
\end{cases}.
\]

If all \( C_i, C_j, C_k \) are different from \( C_1 = \hat{V} \) then the list of \( c(i, j, k) \) is given in the proof of Proposition 2.3.2. Thus (A3) is satisfied. Moreover (A4) is satisfied now, too. Hence the scheme associated with an SRG is an association scheme. \( \square \)

### 2.4 The main theorem

The question arises: can Proposition 2.3.4 be generalized? The answer is given by the connection between height \( t \) presuperschemes and \((t + 3)\)-regular graphs.

Let \( G = (V, E) \) be a graph.

**Definition 2.4.1** Let \( G = (V, E) \) be a graph with totally ordered vertex set \( V = \{v_1 < v_2 < \ldots < v_n\} \). Then \( G \) with such ordered vertices is called a *locally ordered graph*.

Let \( G \) be a locally ordered graph. The **skeleton** \( S(G) \) of the locally ordered graph \( G \) is the list of edges of \( G \) between its ordered vertices, i.e.

\[
S(G) = \{\{i, j\} \subseteq \{1, \ldots, n\} | \{v_i, v_j\} \in E(G); \{v_1 < v_2 < \ldots < v_n\} = V(G)\}.
\]

The skeletons \( S(G) \) and \( S(H) \) of two locally ordered graphs \( G = (\{v_1 < v_2 < \ldots < v_n\}, E(G)) \) and \( H = (\{w_1 < w_2 < \ldots < w_m\}, E(H)) \) are *isomorphic* if \( m = n \) and \( \forall i, j \in \mathbb{N}_n, \{v_i, v_j\} \in E(G) \iff \{w_i, w_j\} \in E(H) \).
Definition 2.4.2 *G-compatible partition* $\Gamma^t$ of $V^{t+2}$.

For $t = 0$, the $G$-compatible partition of $V^2$ is the scheme associated with the graph, $\Gamma^0 = \{ \hat{V}, \hat{E}, \hat{N} \}$.

For $t > 0$ a $G$-compatible partition of $V^{t+2}$ is defined in the following way.

The diagonal classes of $\Gamma^t$ are of the form:

$$ f^k(C_{j-1}^t) = \{(x_1, \ldots, x_{t+2}) | \exists (y_1, \ldots, y_{t+1}) \in C_{j-1}^t. \forall 1 \leq i \leq t + 2, x_i = y_{f(i)} \} $$

for some function $f : \{1, \ldots, t + 2\} \rightarrow \{1, \ldots, t + 1\}$ and some $C_{j-1}^t \in \Gamma^{t-1}$.

A nondiagonal class $C_k^t \in \Gamma^t$ consists of elements which have the same skeleton. (Note that we can treat elements of nondiagonal classes of $\Gamma^t$ as locally ordered $(t + 2)$-vertex subgraphs of the graph $G$.) Thus there is a bijective correspondence between nondiagonal classes $C_k^t$ and isomorphism types of skeletons.

Definition 2.4.3 A set of $G$-compatible partitions which satisfies the conditions of a presuperscheme from Definition 2.2.2 is called a *presuperscheme associated with a graph* $G$.

Theorem 2.4.4 If a presuperscheme associated with a graph $G$ is of height $t$, then the graph $G$ satisfies the $(t + 3)$-vertex condition.

2.5 Proof of the main theorem

The proof of Theorem 2.4.4 involves an inductive construction of the levels of a presuperscheme associated with the graph. This procedure is illustrated by a flow-diagram (Figure 2.4).

Case $t = 0$. If the $G$-compatible partition $\Gamma^0$ is an association scheme $(V, \Gamma^0)$, then by Propositions 2.3.4 and 2.3.2 the graph $G$ satisfies the 3-vertex condition.
Figure 2.4 Flow-diagram of the construction of a presuperscheme associated with the graph $G$.
Case $t > 0$. Suppose there exists a height $t$ presuperscheme $\{\Gamma^0, \ldots, \Gamma^{t-1}, \Gamma^t\}$ associated with the graph $G$. Then partitions $\{\Gamma^0, \ldots, \Gamma^{t-1}\}$ form a height $t-1$ presuperscheme and by induction, the graph $G$ is $(t+2)$-regular.

To prove the $(t+3)$-vertex condition, it suffices to show that the number of $(t+3)$-vertex subgraphs of $G$ of the same type with respect to an edge $\{x, y\} \in E(G)$ (or nonedge $\{x, y\} \in N(G)$ or vertex $x \in V$) does not depend on the choice of $\{x, y\}$ (or $x$).

Let $F$ be a $(t+3)$-vertex subgraph of $G$ of some type $T$, containing $\{x, y\}$. Fix an ordering of vertices of $F$. WLOG, the ordered vertices of $F$ are: $\{x < z < y < a_1 < \ldots < a_t\}$.

Let $I$ be the locally ordered 2-vertex subgraph of $F$ with vertices $\{x < z\}$.

Let $J$ be the locally ordered $(t+2)$-vertex subgraph of $F$ with vertices $\{z < y < a_1 < \ldots < a_t\}$.

Let $K$ be the locally ordered $(t+2)$-vertex subgraph of $F$ with vertices $\{x < y < a_1 < \ldots < a_t\}$.

Let $S(I)$, $S(J)$, $S(K)$ be the skeletons of the graphs $I$, $J$, $K$ respectively.

Let $C_i^0$ be the class of the partition $\Gamma^0$ corresponding to the skeleton $S(I)$.

Let $C_{j_i}^i, \ldots, C_{j_u}^i$ be all the classes of the partition $\Gamma^i$ corresponding to the skeleton $S(J)$.

Let $C_{k_i}^i, \ldots, C_{k_v}^i$ be all the classes of the partition $\Gamma^i$ corresponding to the skeleton $S(K)$.

Then the number of $(t+3)$-vertex subgraphs of type $T$ with respect to $\{x, y\}$ is equal to:

\[
\sum_{q=1}^u \sum_{p=1}^u \frac{|C_{k_q}^i|c(i, j_p, k_q; 0, t)}{nk|\text{Stab}_{x,y}(\text{Aut}(F))|} \quad \text{for } \{x, y\} \in E(G) \text{ and }
\]

\[
\sum_{q=1}^u \sum_{p=1}^u \frac{|C_{k_q}^i|c(i, j_p, k_q; 0, t)}{n(n - 1 - k)|\text{Stab}_{x,y}(\text{Aut}(F))|} \quad \text{for } \{x, y\} \in N(G),
\]
where \( \text{Stab}_{x,y}(\text{Aut}(F)) \subseteq \text{Aut}(F) \) is the stabilizer of \( x \) and \( y \) in the automorphism group of the graph \( F \).

Note that in each case the numbers do not depend on the choice of \( \{x,y\} \) but only on the type \( T \) (on the skeleton \( S(F) \)).

Thus the \((t+3)\)-vertex condition with respect to edges and nonedges is satisfied.

To show the \((t+3)\)-vertex condition with respect to vertices we proceed similarly.

The number of \((t+3)\)-vertex subgraphs of type \( T \) with respect to the vertex \( x \) is equal to:

\[
(2.5.3) \quad \frac{\sum_{q=1}^{u} \sum_{p=1}^{u} |C_{k_q}^i c(i,j_p,k_q;0,t)|}{n|\text{Stab}_x(\text{Aut}(F))|} \quad \text{for } x \in V(G).
\]

So graph \( G \) satisfies the \((t+3)\)-vertex condition. \( \square \)

**Remark 2.5.1** The question whether the reverse conclusion in Theorem 2.4.4 holds will be discussed in a forthcoming paper.

M. H. Klin conjectures in [FKM94] that there exists a sufficiently large \( t_0 \) (\( t_0 \geq 6 \)) such that any strongly regular graph satisfying the \( t_0 \)-vertex condition is of rank 3. A similar question can be posed for presuperschemes: does there exist \( t_0 \) such that any height \( t_0 \) presuperscheme can be extended to a superscheme?

### 2.6 Example

We give an example of a well known graph whose presuperscheme cannot be extended beyond the bottom level.

**Example 2.6.1** [BCN89, Shr59] Consider Shrikhande’s Graph \( G \) (Figure 2.5), where identically labeled pairs of vertices are identified (i.e. the graph is drawn on a torus). The graph \( G \) is an SRG with parameters \((n,k,\lambda,\mu) = (16,6,2,2)\), so by Proposition 2.3.2 it
is 3-regular. However $G$ is not 4-regular. Consider the 4-vertex subgraph $F$ (Figure 2.6) with respect to the nonedge $\{x, y\}$. If the nonedge $\{x, y\}$ is $\{f, h\}$ then the number of such subgraphs is 0, while if the nonedge $\{x, y\}$ is $\{f, o\}$, then the number of such subgraphs is 1. Thus the presuperscheme associated with $G$ has only the bottom level $\Gamma^0 = \{\hat{V}, \hat{E}, \hat{N}\}$.

References


CHAPTER 3  PRESUPERSCHEMES AND COLORED DIRECTED GRAPHS

A paper submitted to Journal of Combinatorial Theory, Series B

Jerzy Wojdylo

Abstract

The scheme associated with a graph is an association scheme iff the graph is strongly regular. Consider the problem of extending such an association scheme to a superscheme in the case of a colored, directed graph. The obstacles can be expressed in terms of $t$-vertex conditions. If a graph does not satisfy the $t$-vertex condition, a presuperscheme associated with it cannot be erected beyond the $(t - 3)$rd level.

3.1 Introduction

An association scheme $(Q, \Gamma)$, comprising a certain kind of partition $\Gamma$ of the direct square of a finite, nonempty set $Q$, corresponds to a binary relation algebra. A superscheme $(Q, \Gamma^*)$ comprises a certain family of partitions of the direct powers $Q^{n+2}$ for each natural number $n$. Superschemes correspond to Krasner (relation) algebras. J.D.H. Smith [Smi94] showed that an association scheme $(Q, \Gamma)$ can be extended to a superscheme $(Q, \Gamma^*)$ iff it is a permutation group scheme, i.e. there exists a transitive,
multiplicity-free permutation group $G$ acting on $Q$ such that each partition $\Gamma^n$ of the superscheme is the set of orbits of $G$ acting componentwise on $Q^{n+2}$.

A natural question arises: what can prevent an association scheme from being built up any further to a superscheme beyond the partition $\Gamma^t$? A scheme can be associated with a graph. This scheme is an association scheme if and only if the graph is strongly regular. A height $t$ presuperscheme consists of the bottom $t$ levels of a superscheme. The current paper contains a construction of a presuperscheme associated with a colored, directed graph. If the scheme associated with a colored directed graph is an association scheme, the graph is strongly regular. The main theorem shows that if a presuperscheme associated with a colored, directed graph is of height $t$, then the graph satisfies the $(t+3)$-vertex condition.

### 3.2 Basic definitions

**Definition 3.2.1** [BI84] Let $Q$ be a finite, non-empty set. A (non-commutative) association scheme $(Q, \Gamma)$ on $Q$ is a partition $\Gamma = \{C_0, \ldots, C_s\}$ of the direct square $Q^2$ such that three conditions are satisfied:

(A1) $C_0 = \{(x, x) | x \in Q\};$

(A2) $\forall C_i \in \Gamma, \{(y, x) | (x, y) \in C_i\} \in \Gamma;$

(A3) $\forall C_i \in \Gamma, \forall C_j \in \Gamma, \forall C_k \in \Gamma, \exists c(i, j, k) \in \mathbb{N}. \forall (x, y) \in C_k,$

$$\{|z \in Q | (x, z) \in C_i, (z, y) \in C_j\} = c(i, j, k).$$

**Definition 3.2.2** [Smi94] Let $Q$ be a finite non-empty set. A (non-commutative) superscheme $(Q, \Gamma^*)$ on $Q$ is a family of partitions $\Gamma^n = \{C_0^n, \ldots, C_s^n\}$ of the direct power $Q^{n+2}$, for each natural number $n$, such that:

(S1) $C_0^0 = \{(x, x) | x \in Q\};$

(S2) $\forall m, n \in \mathbb{N}, \forall f : \{1, \ldots, m + 2\} \rightarrow \{1, \ldots, n + 2\}, \forall C_j^n \in \Gamma^n,$
\[ f^*(C^n_j) = \{(x_1, \ldots, x_{m+2}) \mid \exists (y_1, \ldots, y_{n+2}) \in C^n_j. \forall 1 \leq i \leq m+2, x_i = y_{f(i)}\} \]
is an element of \( \Gamma^m; \)

(S3) \( \forall (m, n) \in \mathbb{N}^2, \forall C^n_i \subset \Gamma^m, \forall C^n_j \subset \Gamma^n, \forall C^{m+n}_k \subset \Gamma^{m+n}, \)
\[ \exists c(i, j, k; m, n) \subset \mathbb{N}. \forall (x_0, \ldots, x_m, y_0, \ldots, y_n) \in C^{m+n}_k, \]
\[ |\{z \in Q \mid (x_0, \ldots, x_m, z) \in C^n_i, (z, y_0, \ldots, y_n) \in C^n_j\}| = c(i, j, k; m, n). \]

Later on we will need modified versions of (S2) and (S3). Define
\[ N_t = \{n \in \mathbb{N} | n \leq t\} \text{ and } N^2_t = \{(m, n) \in \mathbb{N}^2 | m + n \leq t\}. \]

Then consider the following conditions:

(S2t) \( \forall m, n \in N_t, \forall f : \{1, \ldots, m + 2\} \longrightarrow \{1, \ldots, n + 2\}, \forall C^n_j \subset \Gamma^n, \)
\[ f^*(C^n_j) = \{(x_1, \ldots, x_{m+2}) \mid \exists (y_1, \ldots, y_{n+2}) \in C^n_j. \forall 1 \leq i \leq m+2, x_i = y_{f(i)}\} \]
is an element of \( \Gamma^m; \)

(S3t) \( \forall (m, n) \in N^2_t, \forall C^n_i \subset \Gamma^m, \forall C^n_j \subset \Gamma^n, \forall C^{m+n}_k \subset \Gamma^{m+n}, \)
\[ \exists c(i, j, k; m, n) \subset \mathbb{N}. \forall (x_0, \ldots, x_m, y_0, \ldots, y_n) \in C^{m+n}_k, \]
\[ |\{z \in Q \mid (x_0, \ldots, x_m, z) \in C^n_i, (z, y_0, \ldots, y_n) \in C^n_j\}| = c(i, j, k; m, n). \]

A height t (non-commutative) presuperscheme \((Q, \Gamma^*_t)\) on \(Q\) is a family of partitions \( \Gamma^n = \{C^n_0, \ldots, C^n_t\} \) of the direct power \( Q^{n+2} \), for each number \( n \in N_t \), such that the conditions (S1), (S2t) and (S3t) are satisfied.

Remark 3.2.3 Additional commutativity axioms were used in [Smi94] and [Woj]:

(A4) \( \forall 1 \leq i, j, k \leq s, c(i, j, k) = c(j, i, k), \)

(S4) \( \forall 1 \leq i, j, k \leq s_0, c(i, j, k; 0, 0) = c(j, i, k; 0, 0). \)

However, they are not used in proofs of theorems in [Smi94], therefore results from this paper can be translated to the non-commutative case.
Definition 3.2.4 [FKM94] Let $G$ be a colored, directed graph with vertex set $V$ and mapping $E : V^2 \rightarrow \{0, 1, \ldots, d\}$ which assigns to each ordered pair $(x, y)$ of vertices the value $E(x, y)$, called the color of the arc from $x$ to $y$.

Let $K \subseteq V$. Then $K$ induces a colored, directed subgraph $G(K)$ of $G$ with vertex set $K$ and mapping $E_K = E|_K : K^2 \rightarrow \{0, 1, \ldots, d\}$.

Definition 3.2.5 [FKM94] Let $K_1, K_2 \subseteq V$ and $x$ and $y$ be two (not necessarily distinct) vertices of $V$ which belong both to $K_1$ and $K_2$. We say that subgraphs $G(K_1)$ and $G(K_2)$ are of the same type with respect to the pair $(x, y)$ if there is an isomorphism from the subgraph $G(K_1)$ to $G(K_2)$ which maps $x$ to $x$ and $y$ to $y$.

Definition 3.2.6 [FKM94] We say that a colored graph $G$ satisfies the $t$-vertex condition on the arcs of color $i$, $0 \leq i \leq d$, if for every $k$, $2 \leq k \leq t$, the number of $k$-vertex subgraphs of each fixed type, with respect to an ordered pair of vertices $(x, y)$ joined by an arc of color $i$, is the same for all arcs of color $i$.

We call a colored graph $G$, which satisfies the $t$-vertex condition on the arcs of all its colors, a graph with the $t$-vertex condition (or a $t$-regular graph or a graph with depth $t$). Obviously a $t$-regular graph is $k$-regular for every $2 \leq k \leq t$.

Definition 3.2.7 A colored, directed graph $G = (V, E)$ is regular if three conditions are satisfied:

$$(R1) \ \forall u, v \in V, \ E(u, v) = 0 \Leftrightarrow u = v,$$

$$(R2) \ \forall 0 \leq i \leq d, \ \exists! 0 \leq j \leq d. \ \forall u, v \in V, \ E(u, v) = i \Leftrightarrow E(v, u) = j,$$

$$(R3) \ \forall 0 \leq i \leq d, \ \exists c_i \in \mathbb{N}. \ \forall u \in V, \ |\{v \in V \mid E(u, v) = i\}| = c_i.$$

Clearly, the number of arcs of a given color coming into a vertex is the same for all vertices.
Example 3.2.8 [BIK89, FKM94, Iva89] If a colored, directed graph $G$ is regular, then it satisfies the 2-vertex condition.

Conversely, if a colored, directed graph satisfies the 2-vertex condition and the condition $(R1)$ of Definition 3.2.7, then the graph is regular. □

Remark 3.2.9 A different description of a colored, directed graph with the 2-vertex condition can be found in the literature:

$\Gamma$ is a graph with the 2-vertex condition if and only if all nonisolated vertices in each of its one colored subgraphs have the same valence.

However the following example shows that this description is incorrect.

Example 3.2.10 Consider the colored, directed graph on four vertices, with 2-colored arcs: $G = (V, E)$, where $V = \{a, b, c, d\}$ and $E(a, a) = E(b, b) = E(c, c) = E(d, d) = E(a, b) = E(b, c) = E(c, d) = E(d, a) = 0$ and $E(b, a) = E(a, d) = E(d, c) = E(c, b) = E(a, c) = E(c, a) = E(b, d) = E(d, b) = 1$. The graph $G$ does not satisfy the 2-vertex condition on the arcs of color 1. Indeed, consider the type $T$ of the 2-vertex subgraph $K = (K, E_K)$, where $K = \{x, y\}$ and $E_K(x, x) = E_K(y, y) = E_K(y, x) = 0$ and $E_K(x, y) = 1$. If we choose the pair $(x, y)$ to be $(b, a)$ then the number of 2-vertex subgraphs of the type $T$ with respect to the pair $(x, y)$ joined by an arc of color 1 is 1. If we choose the pair $(x, y)$ to be $(b, d)$ then the number of 2-vertex subgraphs of the type $T$ with respect to the pair $(x, y)$ joined by an arc of color 1 is 0.

3.3 The 3-vertex condition

Remark 3.3.1 [BCN89, CDH80] An undirected graph $G$ is strongly regular (or an SRG) with the parameters $(n, k, \lambda, \mu)$ if $G$ is regular, and each pair of adjacent vertices has $\lambda$ common neighbors and each pair of nonadjacent vertices has $\mu$ common neighbors.
The parameters $n$ and $k$ respectively denote the number of vertices of $G$ and the valency of each vertex.

**Proposition 3.3.2** [BC55, BIK89, FKM94, Iva89, Woj] An undirected graph $G$ is strongly regular iff it is 3-regular. □

**Definition 3.3.3** We say that a colored, directed graph $G = (V, E)$ is strongly regular (or an SRG) if it satisfies the 3-vertex condition.

**Definition 3.3.4** [CDH80, Woj] Let $G = (V, E)$ be a colored, directed graph. The scheme associated with $G$ is $(V, \Gamma)$, where $\Gamma = \{C_0, C_1, \ldots, C_d\}$, where $C_i = E^{-1}(i)$ for $0 \leq i \leq d$.

**Proposition 3.3.5** [FKM94] If the scheme associated with a colored, directed graph is an association scheme, then the graph is strongly regular. Conversely, if a colored, directed graph is strongly regular and satisfies the condition $(R1)$ of Definition 3.2.7, then the scheme associated with the graph is an association scheme.

**Proof.** Let $(V, \Gamma)$, with $\Gamma = \{C_0, C_1, \ldots, C_d\}$, be the scheme associated with the colored, directed graph $G = (V, E)$. Since $(V, \Gamma)$ is an association scheme, the condition $(A1)$ of Definition 3.2.1 implies the condition $(R1)$ of Definition 3.2.7. The condition $(A2)$ of Definition 3.2.1 implies the condition $(R2)$ of Definition 3.2.7. Finally the condition $(A3)$ of Definition 3.2.1 with $k = 0$ implies the condition $(R3)$ of Definition 3.2.7. Therefore the graph is regular, and hence satisfies the 2-vertex condition. The condition $(A3)$ with $1 \leq k \leq d$, together with $(R2)$, implies that the number of 3-vertex subgraphs of each fixed type with respect to an ordered pair of vertices $(x, y)$ joined by an arc of color $k$, is the same for all arcs of color $k$. Therefore the graph satisfies the 3-vertex condition.
Suppose the graph $G = (V, E)$ is strongly regular. Then it satisfies the 3-vertex condition, and therefore the 2-vertex condition. It also satisfies the condition $(R1)$ of Definition 3.2.7. Let $(V, \Gamma)$, with $\Gamma = \{C_0, C_1, \ldots, C_d\}$, be the scheme associated with the colored, directed graph $G = (V, E)$. Since the graph $G$ satisfies the condition $(R1)$ of Definition 3.2.7 and the 2-vertex condition, it is regular (see Example 3.2.8). Therefore the conditions $(R1)$, $(R2)$ and $(R3)$ of Definition 3.2.7 are satisfied. But the condition $(R1)$ means that the scheme $(V, \Gamma)$ satisfies the condition $(A1)$ of Definition 3.2.1. The condition $(R2)$ means that the scheme $(V, \Gamma)$ satisfies the condition $(A2)$ of Definition 3.2.1. The condition $(R3)$ means that the scheme $(V, \Gamma)$ satisfies the condition $(A3)$, with $k = 0$, of Definition 3.2.1. And finally, since the graph $G$ satisfies the 3-vertex condition, for every color $k$, $1 \leq k \leq d$, the number of 3-vertex subgraphs of each fixed type, with respect to an ordered pair of vertices $(x, y)$ joined by an arc of color $k$, is the same for all arcs of color $k$. But this means that the condition $(A3)$ (with $1 \leq k \leq d$) of Definition 3.2.1 is satisfied, and therefore the scheme $(V, \Gamma)$ is an association scheme. \(\square\)

3.4 The main theorem

The question arises: can Proposition 3.3.5 be generalized? The answer is given by a connection between height $t$ presuperschemes and $(t + 3)$-regular graphs.

Let $G = (V, E)$ be a colored, directed graph.

**Definition 3.4.1** Let $G = (V, E)$ be a graph with totally ordered vertex set $V = \{v_1 < v_2 < \ldots < v_n\}$. Then $G$ with such ordered vertices is called a *locally ordered graph*.

Let $G$ be a locally ordered graph. The *skeleton* $S(G)$ of the locally ordered graph $G$ is the incidence matrix of the graph $G$ with $(i, j)$-th entry equal to $E(v_i, v_j)$.

**Definition 3.4.2** $G$-compatible partition $\Gamma^t$ of $V^{t+2}$. 
For $t = 0$, the $G$-compatible partition $\Gamma^0$ of $V$ is the scheme associated with the graph, $\Gamma^0 = \{C_0, C_1, \ldots, C_d\}$.

For $t > 0$ the $G$-compatible partition $\Gamma^t$ of $V$ is defined in the following way.

The diagonal classes of $\Gamma^t$ are of the form:

$$f^*(C_j^{t-1}) = \{(x_1, \ldots, x_{t+2}) \mid \exists (y_1, \ldots, y_{t+1}) \in C_j^{t-1}. \forall 1 \leq i \leq t + 2. x_i = y_{f(i)}\}$$

for some function $f : \{1, \ldots, t + 2\} \rightarrow \{1, \ldots, t + 1\}$ and some $C_j^{t-1} \in \Gamma^{t-1}$.

A nondiagonal class $C_k^t \in \Gamma^t$ consists of all elements which have the same skeleton. (Note that we can treat elements of nondiagonal classes of $\Gamma^t$ as locally ordered $(t + 2)$-vertex subgraphs of the graph $G$.) Thus there is a bijective correspondence between nondiagonal classes $C_k^t$ and isomorphism types of skeletons.

**Definition 3.4.3** A set of $G$-compatible partitions which satisfies the conditions of a presuperscheme from Definition 3.2.2 is called a *presuperscheme associated with the graph* $G$.

**Theorem 3.4.4** If a presuperscheme associated with a colored, directed graph $G$ has $t$ levels, then the graph $G$ satisfies the $(t + 3)$-vertex condition.

### 3.5 Proof of the main theorem

The proof of Theorem 3.4.4 involves an inductive construction of the levels of a presuperscheme associated with the graph. This procedure is illustrated by a flow-diagram (Figure 3.1).

**Case** $t = 0$. If the $G$-compatible partition $\Gamma^0$ is an association scheme $(V, \Gamma^0)$, then by Proposition 3.3.5 the graph $G$ satisfies the 3-vertex condition.

**Case** $t > 0$. Suppose there exists a height $t$ presuperscheme $\{\Gamma^0, \ldots, \Gamma^{t-1}, \Gamma^t\}$ associated with the graph $G$. Then the partitions $\{\Gamma^0, \ldots, \Gamma^{t-1}\}$ form a height $t - 1$ presuperscheme, and by induction, the graph $G$ is $(t + 2)$-regular.
Figure 3.1  Flow-diagram of the construction of a presuperscheme associated with the graph $G$
To prove the \((t + 3)\)-vertex condition, it suffices to show that the number of \((t + 3)\)-vertex subgraphs of \(G\) of the same type with respect to an ordered pair of vertices \((x, y)\) joined by an arc of color \(l\), \(0 \leq l \leq d\), does not depend on the choice of the arc of color \(l\).

Let \(F\) be a \((t + 3)\)-vertex subgraph of \(G\) of some type \(T\), containing \(\{x, y\}\). Fix an ordering of the vertices of \(F\). WLOG, the ordered vertices of \(F\) are: \(\{x < z < y < a_1 < \ldots < a_t\}\).

Let \(I\) be the locally ordered 2-vertex subgraph of \(F\) with vertices \(\{x < z\}\).

Let \(J\) be the locally ordered \((t + 2)\)-vertex subgraph of \(F\) with vertices \(\{z < y < a_1 < \ldots < a_t\}\).

Let \(K\) be the locally ordered \((t + 2)\)-vertex subgraph of \(F\) with vertices \(\{x < y < a_1 < \ldots < a_t\}\).

Let \(S(I), S(J), S(K)\) be the skeletons of the graphs \(I, J, K\) respectively.

Let \(C^0_i\) be the class of the partition \(\Gamma^0\) corresponding to the skeleton \(S(I)\).

Let \(C^t_j\) be the class of the partition \(\Gamma^t\) corresponding to the skeleton \(S(J)\).

Let \(C^t_k\) be the classes of the partition \(\Gamma^t\) corresponding to the skeleton \(S(K)\).

Then the number of \((t + 3)\)-vertex subgraphs of type \(T\) with respect to \((x, y)\) joined by an arc of color \(l\) is equal to:

\[
(3.5.1) \quad \frac{|C^t_k|c(i, j, k; 0, t)}{nc_l|Stab_{x,y}(Aut(F))|} \quad \text{for } E(x, y) = l, 1 \leq l \leq d.
\]

where \(Stab_{x,y}(Aut(F)) \subseteq Aut(F)\) is the stabilizer of \(x\) and \(y\) in the automorphism group of the graph \(F\), and \(c_l\) denotes the number of arcs of color \(l\) going out from the vertex \(x\).

Note that in each case, the numbers do not depend on the choice of \((x, y)\), but only on the type \(T\) (on the skeleton \(S(F)\)).

The number of \((t + 3)\)-vertex subgraphs of type \(T\) with respect to \((x, x)\) is equal to:
\[(3.5.2) \quad \frac{|C_k^t|c(i, j, k; 0, t)}{n|\text{Stab}_x(\text{Aut}(F))|} \quad \text{for } l = 0.\]

So the graph \( G \) satisfies the \((t + 3)\)-vertex condition. \( \Box \)

**Remark 3.5.1** The question whether the reverse conclusion in Theorem 3.4.4 holds will be discussed in a forthcoming paper.

**References**


CHAPTER 4 ON A 4-REGULAR GRAPH WITH AN INEXTENSIBLE ASSOCIATION SCHEME

A paper submitted to Graphs and Combintorics

Jerzy Wojdylo

Abstract

The scheme associated with a graph is an association scheme iff the graph is strongly regular. Consider the problem of extending such an association scheme to a superscheme in the case of a colored, directed graph. If a presuperscheme associated with a graph is of height \( t \), then the graph satisfies the \((t + 3)\)-vertex condition. On the other hand, the current paper provides an example of a strongly regular graph, satisfying the 4-vertex condition, whose association scheme cannot be extended to a presuperscheme of height 1.

4.1 Introduction

An association scheme \((Q, \Gamma)\) comprises a certain kind of partition \(\Gamma\) of the direct square of a finite, nonempty set \(Q\). A superscheme \((Q, \Gamma^*)\) comprises a certain family of partitions of the direct powers \(Q^{n+2}\) for each natural number \(n\). Superschemes correspond to Krasner (relation) algebras. J.D.H. Smith [Smi94] showed that an association scheme \((Q, \Gamma)\) can be extended to a superscheme \((Q, \Gamma^*)\) iff it is a permutation
group scheme, i.e. there exists a transitive, multiplicity-free permutation group $G$ acting on $Q$ such that each partition $\Gamma^n$ of the superscheme is the set of orbits of $G$ acting componentwise on $Q^{n+2}$.

A natural question arises: what can prevent an association scheme from being built up any further to a superscheme beyond the partition $\Gamma'$? A scheme can be associated with a graph. This scheme is an association scheme if and only if the (colored, directed) graph is strongly regular. A height $t$ presuperscheme consists of the bottom $t$ levels of a superscheme. The Main Theorem in [Wojb, Woja] shows that if a presuperscheme associated with a (colored, directed) graph is of height $t$, then the graph satisfies the $(t + 3)$-vertex condition. The current paper shows that the reverse conclusion does not hold. It contains an example of a colored, directed graph which satisfies the 4-vertex condition, but whose presuperscheme cannot be extended beyond the bottom (0-th) level. In particular, the example exhibits an association scheme that cannot be extended to a presuperscheme of positive height.

### 4.2 Basic definitions

#### Definition 4.2.1 [Bl84]
Let $Q$ be a finite, non-empty set. A (non-commutative) association scheme $(Q, \Gamma)$ on $Q$ is a partition $\Gamma = \{C_0, C_1, \ldots, C_s\}$ of the direct square $Q^2$ such that three conditions are satisfied:

1. $(A1)$ $C_0 = \{(x, x) | x \in Q\}$;
2. $(A2)$ $\forall C_i \in \Gamma, \{(y, y) | (x, y) \in C_i\} \in \Gamma$;
3. $(A3)$ $\forall C_i \in \Gamma, \forall C_j \in \Gamma, \forall C_k \in \Gamma, \exists c(i, j, k) \in \mathbb{N}, \forall (x, y) \in C_k,
\{|z \in Q | (x, z) \in C_i, (z, y) \in C_j\| = c(i, j, k)$.

#### Definition 4.2.2 [Smi94]
Let $Q$ be a finite non-empty set. A (non-commutative) superscheme $(Q, \Gamma^n)$ on $Q$ is a family of partitions $\Gamma^n = \{C_0^n, \ldots, C_i^n\}$ of the direct power
$Q^{n+2}$, for each natural number $n$, such that:

(S1) \[ C^0 = \{(x, x) | x \in Q\}; \]

(S2) \[ \forall m, n \in \mathbb{N}, \forall f : \{1, \ldots, m + 2\} \rightarrow \{1, \ldots, n + 2\}, \forall C^m_j \in \Gamma^n, \]

\[ f^*(C^n_j) = \{(x_1, \ldots, x_{m+2}) | \exists (y_1, \ldots, y_{n+2}) \in C^n_j, \forall 1 \leq i \leq m + 2, x_i = y_{f(i)}\} \]

is an element of $\Gamma^m$;

(S3) \[ \forall (m, n) \in \mathbb{N}^2, \forall C^m_i \in \Gamma^m, \forall C^n_j \in \Gamma^n, \forall C^{m+n}_k \in \Gamma^{m+n}, \]

\[ \exists c(i, j, k; m, n) \in \mathbb{N}, \forall (x_0, \ldots, x_m, y_0, \ldots, y_n) \in C^{m+n}_k, \]

\[ |\{z \in Q | (x_0, \ldots, x_m, z) \in C^m_i, (z, y_0, \ldots, y_n) \in C^n_j\}| = c(i, j, k; m, n). \]

Later on we will need modified versions of (S2) and (S3). Define

\[ N_t = \{n \in \mathbb{N} | n \leq t\} \text{ and } N^2_t = \{(m, n) \in \mathbb{N}^2 | m + n \leq t\}. \]

Then consider the following conditions:

(S2) \[ \forall m, n \in \mathbb{N}_t, \forall f : \{1, \ldots, m + 2\} \rightarrow \{1, \ldots, n + 2\}, \forall C^m_i \in \Gamma^n, \]

\[ f^*(C^n_j) = \{(x_1, \ldots, x_{m+2}) | \exists (y_1, \ldots, y_{n+2}) \in C^n_j, \forall 1 \leq i \leq m + 2, x_i = y_{f(i)}\} \]

is an element of $\Gamma^m$;

(S3) \[ \forall (m, n) \in \mathbb{N}_t^2, \forall C^m_i \in \Gamma^m, \forall C^n_j \in \Gamma^n, \forall C^{m+n}_k \in \Gamma^{m+n}, \]

\[ \exists c(i, j, k; m, n) \in \mathbb{N}, \forall (x_0, \ldots, x_m, y_0, \ldots, y_n) \in C^{m+n}_k, \]

\[ |\{z \in Q | (x_0, \ldots, x_m, z) \in C^m_i, (z, y_0, \ldots, y_n) \in C^n_j\}| = c(i, j, k; m, n). \]

A height $t$ (non-commutative) presuperscheme $(Q, \Gamma^n_t)$ on $Q$ is a family of partitions $\Gamma^n = \{C^n_0, \ldots, C^n_n\}$ of the direct power $Q^{n+2}$, for each number $n \in \mathbb{N}_t$, such that the conditions (S1), (S2) and (S3) are satisfied.

**Definition 4.2.3** [FKM94] Let $G$ be a colored, directed graph with vertex set $V$ and mapping $E : V^2 \rightarrow \{0, 1, \ldots, d\}$ which assigns to each ordered pair $(x, y)$ of vertices the value $E(x, y)$, called the color of the arc from $x$ to $y$. 
Let $K \subseteq V$. Then $K$ induces a colored, directed subgraph $G(K)$ of $G$ with vertex set $K$ and mapping $E_K = E|_K : K^2 \rightarrow \{0, 1, \ldots, d\}$.

**Definition 4.2.4** [FKM94] Let $K_1, K_2 \subseteq V$ and $x$ and $y$ be two (not necessarily distinct) vertices of $V$ which belong both to $K_1$ and $K_2$. We say that subgraphs $G(K_1)$ and $G(K_2)$ are of the same type with respect to the pair $(x, y)$ if there is an isomorphism from the subgraph $G(K_1)$ to $G(K_2)$ which maps $x$ to $x$ and $y$ to $y$.

**Definition 4.2.5** [FKM94] We say that a colored graph $G$ satisfies the $t$-vertex condition on the arcs of color $i$, $0 \leq i \leq d$, if for every $k$, $2 \leq k \leq t$, the number of $k$-vertex subgraphs of each fixed type, with respect to an ordered pair of vertices $(x, y)$ joined by an arc of color $i$, is the same for all arcs of color $i$.

We call a colored graph $G$, which satisfies the $t$-vertex condition on the arcs of all its colors, a graph with the $t$-vertex condition. It is also known as a $t$-regular graph or a graph with depth $t$.

**Remark 4.2.6** Note, that if a graph $G$ satisfies the $t$-vertex condition, then for every $k < t$ the graph $G$ also satisfies the $k$-vertex condition.

**Definition 4.2.7** A colored, directed graph $G = (V, E)$ is regular if three conditions are satisfied:

1. $(R1)$ $\forall u, v \in V, E(u, v) = 0 \iff u = v$,
2. $(R2)$ $\forall 0 \leq i \leq d$, $\exists 0 \leq j \leq d$. $\forall u, v \in V, E(u, v) = i \iff E(v, u) = j$,
3. $(R3)$ $\forall 0 \leq i \leq d$, $\exists c_i \in \mathbb{N}$. $\forall u \in V$, $|\{v \in V \mid E(u, v) = i\}| = c_i$.

Clearly, the number of arcs of a given color coming into a vertex is the same for all vertices.

**Example 4.2.8** [BIK89, FKM94, Iva89] If a colored, directed graph $G$ is regular, then it satisfies the 2-vertex condition.
Conversely, if a colored, directed graph satisfies the 2-vertex condition and the condition 
(R1) of Definition 4.2.7, then the graph is regular. □

4.3 Association schemes and the 3-vertex condition

Remark 4.3.1 [BCN89, CDH80] An undirected graph G is strongly regular (or an SRG) with the parameters 
(n, k, λ, μ) if G is regular, and each pair of adjacent vertices 
has λ common neighbors and each pair of nonadjacent vertices has μ common neighbors. 
The parameters n and k respectively denote the number of vertices of G and the valence 
of each vertex.

Proposition 4.3.2 [BC55, BIK89, FKM94, Iva89, Wojb] An undirected graph G is 
strongly regular iff it is 3-regular. □

Definition 4.3.3 [Woja] We say that a colored, directed graph G = (V, E) is strongly 
regular (or an SRG) if it satisfies the 3-vertex condition.

Definition 4.3.4 [Woja] Let G = (V, E) be a colored, directed graph. The scheme 
associated with G is (V, Γ), where Γ = {C₀, C₁, ..., Cₙ} with Cᵢ = E⁻¹(i) for 0 ≤ i ≤ d.

Proposition 4.3.5 [FKM94, Woja] If the scheme associated with a colored, directed 
graph is an association scheme, then the graph is strongly regular. 
Conversely, if a colored, directed graph is strongly regular and satisfies the condition 
(R1) of Definition 4.2.7, then the scheme associated with the graph is an association 
scheme.

4.4 Presuperschemes and t-vertex condition

The question arises: can Proposition 4.3.5 be generalized? The answer is given by a 
connection between height t presuperschemes and (t + 3)-regular graphs.
Let $G = (V, E)$ be a colored, directed graph.

**Definition 4.4.1** [Wojb, Woja] Let $G = (V, E)$ be a graph with totally ordered vertex set $V = \{v_1 < v_2 < \ldots < v_n\}$. Then $G$ with such ordered vertices is called a *locally ordered graph*.

Let $G$ be a locally ordered graph. The *skeleton $S(G)$ of the locally ordered graph G* is the incidence matrix of the graph $G$ with $(i, j)$-th entry equal to $E(v_i, v_j)$.

**Definition 4.4.2** [Wojb, Woja] $G$-compatible partition $\Gamma^t$ of $V^{t+2}$.

For $t = 0$, the $G$-compatible partition $\Gamma^0$ of $V$ is the scheme associated with the graph, $\Gamma^0 = \{C_0, C_1, \ldots, C_d\}$.

For $t > 0$ the $G$-compatible partition $\Gamma^t$ of $V$ is defined in the following way.

The diagonal classes of $\Gamma^t$ are of the form:

$$f^*(C_{j-1}^t) = \{(x_1, \ldots, x_{t+2}) \mid \exists (y_1, \ldots, y_{t+1}) \in C_{j-1}^t \forall 1 \leq i \leq t+2, x_i = y_{f(i)}\}$$

for some function $f : \{1, \ldots, t+2\} \rightarrow \{1, \ldots, t+1\}$ and some $C_{j-1}^t \in \Gamma^{t-1}$.

A nondiagonal class $C_k^t \in \Gamma^t$ consists of elements which have the same skeleton. (Note that we can treat elements of nondiagonal classes of $\Gamma^t$ as locally ordered $(t+2)$-vertex subgraphs of the graph $G$.) Thus there is a bijective correspondence between nondiagonal classes $C_k^t$ and isomorphism types of skeletons.

**Definition 4.4.3** [Wojb, Woja] A set of $G$-compatible partitions which satisfies the conditions of a presuperscheme from Definition 4.2.2 is called a *presuperscheme associated with the graph G*.

**Theorem 4.4.4** [Wojb, Woja] If a presuperscheme associated with a graph $G$ has $t$ levels, then the graph $G$ satisfies the $(t+3)$-vertex condition.
4.5 The inextensible association scheme

The question arises: does the reverse conclusion in Theorem 4.4.4 hold? The negative answer is provided by the following example.

Consider a 2-colored directed graph $G$ on 15 vertices given by the following adjacency matrix:

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 2 \\
2 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 2 \\
2 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 \\
2 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 2 \\
2 & 2 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 \\
2 & 1 & 1 & 2 & 1 & 2 & 2 & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 \\
1 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 2 \\
1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 2 \\
1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 2 \\
\end{bmatrix}
\]

Remark 4.5.1 By Definition 4.2.7 and Example 4.2.8 the graph $G$ satisfies the 2-vertex condition. □

Remark 4.5.2 [FKM94, SS] The graph $G$ satisfies the 3-vertex condition.

The list of types of 3-vertex subgraphs with respect to an arc of color 1 is given in Figure 4.1. A arrow from $x$ to $y$ means $E(x, y) = 1$ and $E(y, x) = 2$. This is also the list of types of 3-vertex subgraphs with respect to an arc of color 2, if each of the 3-vertex
<table>
<thead>
<tr>
<th>Type</th>
<th>Number of subgraphs</th>
<th>Type</th>
<th>Number of subgraphs</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="x,y" alt="Diagram 1" /></td>
<td>3</td>
<td><img src="x,y" alt="Diagram 1" /></td>
<td>4</td>
</tr>
<tr>
<td><img src="x,y" alt="Diagram 2" /></td>
<td>3</td>
<td><img src="x,y" alt="Diagram 2" /></td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 4.1 Types of 3-vertex subgraphs with respect to an arc $(x, y)$.

<table>
<thead>
<tr>
<th>Type</th>
<th>Number of subgraphs</th>
<th>Type</th>
<th>Number of subgraphs</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="x,x" alt="Diagram 3" /></td>
<td>21</td>
<td><img src="x,x" alt="Diagram 3" /></td>
<td>28</td>
</tr>
<tr>
<td><img src="x,x" alt="Diagram 4" /></td>
<td>21</td>
<td><img src="x,x" alt="Diagram 4" /></td>
<td>21</td>
</tr>
</tbody>
</table>

Figure 4.2 Types of 3-vertex subgraphs with respect to an arc $(x, x)$.

subgraphs is flipped upside-down. The list of types of 3-vertex subgraphs with respect to an arc of color 0 is given in Figure 4.2. □

**Remark 4.5.3** [FKM94, SS] The graph $G$ satisfies the 4-vertex condition. The list of types of 4-vertex subgraphs with respect to an arc of color 1 is given in Figure 4.3. This is also the list of types of 4-vertex subgraphs with respect to an arc of color 2, if each of the 4-vertex subgraphs is flipped upside-down. The list of types of 4-vertex subgraphs with respect to an arc of color 0 is given in Figure 4.4. □
<table>
<thead>
<tr>
<th>Type</th>
<th>Number of subgraphs</th>
<th>Type</th>
<th>Number of subgraphs</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Subgraph 1" /></td>
<td>3</td>
<td><img src="image2" alt="Subgraph 2" /></td>
<td>6</td>
</tr>
<tr>
<td><img src="image3" alt="Subgraph 3" /></td>
<td>6</td>
<td><img src="image4" alt="Subgraph 4" /></td>
<td>6</td>
</tr>
<tr>
<td><img src="image5" alt="Subgraph 5" /></td>
<td>3</td>
<td><img src="image6" alt="Subgraph 6" /></td>
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<td>6</td>
<td><img src="image10" alt="Subgraph 10" /></td>
<td>6</td>
</tr>
<tr>
<td><img src="image11" alt="Subgraph 11" /></td>
<td>6</td>
<td><img src="image12" alt="Subgraph 12" /></td>
<td>6</td>
</tr>
</tbody>
</table>

Figure 4.3 Types of 4-vertex subgraphs with respect to an arc \((x, y)\).
Figure 4.4 Types of 4-vertex subgraphs with respect to an arc \((x, x)\).
Remark 4.5.4 The graph $G$ does not satisfy the 5-vertex condition.

A critical type of a 5-vertex subgraph is in Figure 4.5. $\square$

Proposition 4.5.5 The graph $G$ is strongly regular and the scheme associated with the graph $G$ is an association scheme. However, its presuperscheme cannot be extended beyond the 0-th level.

Proof. By Remark 4.5.2 and Definition 4.3.3 the graph $G$ is strongly regular. By Proposition 4.3.5 the scheme associated with the graph $G$

$$\Gamma^0 = \Gamma = \{C_0, C_1, C_2\}, \text{ where } C_k = \{(i, j) | E(i, j) = k\} \text{ for } k = 0, 1, 2$$

is an association scheme.

A presuperscheme associated with the graph $G$ cannot be extended beyond the bottom level. For, suppose that $\Gamma^1$ is a $G$-compatible partition of $V^3$ which is the first level of a presuperscheme associated with the graph $G$. Consider a locally ordered 3-vertex subgraph $G(K)$, where $K = \{2 < 12 < 4\}$. The skeleton, $S$, of this subgraph is

<table>
<thead>
<tr>
<th>Arc</th>
<th>Number of subgraphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x, y) = (3, 10)$</td>
<td>3</td>
</tr>
<tr>
<td>$(x, y) = (3, 7)$</td>
<td>4</td>
</tr>
<tr>
<td>$(x, y) = (3, 9)$</td>
<td>5</td>
</tr>
</tbody>
</table>

Figure 4.5 A critical type of a 5-vertex subgraph with respect to an arc $(x, y)$. 
given by the matrix:

\[
\begin{bmatrix}
0 & 1 & 1 \\
2 & 0 & 1 \\
2 & 2 & 0
\end{bmatrix}.
\]

Let \( C_k^1 \) be the class of the partition \( \Gamma^1 \) containing \((2, 12, 4)\). Note that all elements of this class have the same skeleton \( S \). Let \( C_j^1 \) be any class of the partition \( \Gamma^1 \) corresponding to the skeleton \( S \). Recall that the class \( C_0^0 \) consists of pairs \((x, y)\) of vertices of the graph \( G \) such that \( E(x, y) = 1 \).

Now let us compute the constant \( c(1, j, k; 0, 1) \) as in Definition 4.2.2. Choose \((x_0, y_0, y_1) = (2, 12, 4) \in C_k^1\). By a direct check we can see that:

\[
c(1, j, k; 0, 1) = |\{ z \in V | (2, z) \in C_0^0, (z, 12, 4) \in C_j^1 \}| = 0.
\]

Since the value of \( c(i, j, k; 0, 1) \) does not depend on the choice of \((x_0, y_0, y_1) \in C_k^1\), the class \( C_k^1 \) consists of triplets \((x_0, y_0, y_1)\) such that for any class \( C_j^1 \in \Gamma^1 \) corresponding to the skeleton \( S \),

\[
|\{ z \in V | (x_0, z) \in C_0^0, (z, y_0, y_1) \in C_j^1 \}| = 0.
\]

By a computer check we find that there are 21 such triplets: \((2, 12, 4), (2, 14, 3), (2, 15, 6), (3, 9, 7), (3, 13, 5), (3, 15, 4), (4, 9, 5), (4, 10, 8), (4, 14, 6), (5, 10, 6), (5, 11, 2), (5, 15, 7), (6, 9, 8), (6, 11, 7), (6, 12, 3), (7, 10, 2), (7, 12, 8), (7, 13, 4), (8, 11, 3), (8, 13, 2), (8, 14, 5)\).

Thus the class \( C_k^1 \) has only 21 elements.

Now, consider the function \( f : \{1, 2\} \rightarrow \{1, 2, 3\}, f(1) = 1, f(2) = 2 \) as in the condition (S2\(_r\)) of Definition 4.2.2. Then \( f^*(C_k^1) = C_0^0 \) because for any \((x_0, y_0, y_1) \in C_k^1\), \( E(x_0, y_0) = 1 \). But this is a contradiction, since the class \( C_0^0 \) has 105 elements and the class \( C_k^1 \) has only 21 elements. Therefore the 1-st level of a presuperscheme associated with the graph \( G \) does not exist. □
References


CHAPTER 5 CONCLUSION

Not every association scheme can be extended to a superscheme. The obstacles that prevent any extension beyond level $t$ can be expressed in terms of graph theory. If a graph is strongly regular, the scheme associated with the graph is an association scheme. However, if the graph does not satisfy the $(t + 3)$-vertex condition, the (association) scheme associated with the graph cannot be extended beyond level $t$.

On the other hand, in the most general case of colored, directed graphs, the $(t + 3)$-vertex condition is not sufficient for an association scheme associated with a graph to be extendible to a presuperscheme of height $t$.

There are not many examples of non-rank 3 graphs which satisfy the $t$-vertex condition for $t > 4$. A. V. Ivanov writes [Iva89]:

Necessary and sufficient conditions which ensure that a graph with the $t$-vertex condition ($t > 4$) is the graph with the $(t + 1)$-vertex condition as well seem to be more complicated.

One of the possible ways to construct these graphs is to find association schemes which can be extended to presuperschemes of height 1, 2, 3, etc. However this is not an easy problem either, and the current approach seems to confirm Ivanov's remark.

Open problems

There are interesting open problems that appeared during the research on presuperschemes and $t$-regular graphs. Here are some of them.
Does the \( t \)-vertex condition on a simple graph imply the existence of a presuperscheme of height \( t - 3 \)? Does the \( t \)-vertex condition on a colored, undirected graph imply the existence of a presuperscheme of height \( t - 3 \)? How important is it for an association scheme to be symmetric in order to be expandable to a presuperscheme of height \( t - 3 \) and eventually to a superscheme?

M. H. Klin conjectures [FKM94] that there exists a sufficiently large \( t_0 \) (\( t_0 \geq 6 \)) such that any strongly regular graph satisfying the \( t_0 \)-vertex condition is of rank 3. What happens in the case where the graph is colored or directed?

Does this conjecture transfer to association schemes? Does there exist a sufficiently large \( l_0 \), independent from the cardinality of the set \( Q \), such that if any association scheme \((\Gamma, Q)\) can be extended a presuperscheme of height \( l_0 \), then it can be extended to a superscheme?

Are there any connections between presuperschemes and hypergraphs similar to those between (association) schemes and graphs?

Perhaps there can be found an infinite family of graphs such that for every \( t \) there exists a member of this family which satisfies the \( t \)-vertex condition but does not satisfy the \((t + 1)\)-vertex condition. The same may be true for association schemes. There may exist an infinite family of association schemes such that for every \( l \) there exists an association scheme that can be extended to a presuperscheme of height \( l \), but not of height \( l + 1 \).

References

[Iva89] A. V. Ivanov. Non rank 3 strongly regular graphs with the 5-vertex condition. 
APPENDIX A  COMPUTER PROGRAM - GRAPH

Program graph15

PROGRAM graph15(ADJ_MAT_TXT, ARCS_COLOR1, ARCS_COLOR2);

TYPE
cyclomatrix = ARRAY[1..7,1..7] OF integer;
adjmatrix = ARRAY[1..15,1..15] OF integer;

VAR
  a : adjmatrix;
  c : cyclomatrix;
  ADJ_MAT_TXT, ARCS_COLOR1, ARCS_COLOR2 : text;
  i, j : integer;

BEGIN
  FOR i:=1 TO 7 DO FOR j:=1 TO 7 DO
  BEGIN
    IF ((j-i) mod 7 = 1) OR ((j-i) mod 7 = 2) OR ((j-i) mod 7 = 4) THEN c[i,j]:=1;
    IF ((j-i) mod 7 = 3) OR ((j-i) mod 7 = 5) OR ((j-i) mod 7 = 6) THEN c[i,j]:=2;
    IF i=j THEN c[i,j]:=0;
  END;

  a[1,1]:=0;
  FOR j:=2 TO 8 DO a[1,j]:=1;
  FOR j:=9 TO 15 DO a[1,j]:=2;
  FOR i:=2 TO 8 DO a[i,1]:=2;
  FOR i:=9 TO 15 DO a[i,1]:=1;

END;
FOR i:=2 TO 8 DO FOR j:= 2 TO 8 DO {top left}  
a[i,j]:=c[i-1, j-1]; { quarter}

FOR i:=2 TO 8 DO FOR j:= 9 TO 15 DO {top }  
IF j-7=i THEN a[i,j]:=1 { right }  
ELSE a[i,j]:= c[j-8,i-1]; { quarter}

FOR i:=9 TO 15 DO FOR j:= 2 TO 8 DO {bottom }  
IF i-7=j THEN a[i,j]:=2 { left }  
ELSE a[i,j]:= c[j-1,i-8]; { quarter}

FOR i:=9 TO 15 DO FOR j:= 9 TO 15 DO {bottom }  
a[i,j]:=c[j-8,i-8]; { right }

rewrite(ADJ_MAT_TXT);
writeln(ADJ_MAT_TXT,'Adjacency matrix of the graph from Chapter 4:');
writeln(ADJ_MAT_TXT);
FOR i:=1 TO 15 DO { Saving the}  
BEGIN { matrix A }
    FOR j:=1 TO 15 DO write(ADJ_MAT_TXT,a[i,j]:2); { to the }  
        writeln(ADJ_MAT_TXT); { text file }
END; {ADJ_MAT_TXT}
writeln(ADJ_MAT_TXT);

rewrite(ARCS_COLOR1); { Pairs of }  
writeln(ARCS_COLOR1,'Arcs of color 1 (class C-0_1):'); { vertices }  
writeln(ARCS_COLOR1); { (x,y) }  
FOR i:=1 TO 15 DO {connected }  
BEGIN { by an arc }
    FOR j:=1 TO 15 DO {of color 1}
        IF a[i,j]=1 THEN write(ARCS_COLOR1,('(',i:2,',',j:2,')') ');  
            writeln(ARCS_COLOR1); { saved to the}  
        END; { text file }
writeln(ARCS_COLOR1); {ARCS_COLOR1 }
rewrite(ARCS_COLOR2); { Pairs of }
writeln(ARCS_COLOR2,'Arcs of color 2 (class C<0_2):'); { vertices }
writeln(ARCS_COLOR2); { (x,y) }
FOR i:=1 TO 15 DO
BEGIN
  BEGIN {connected }
    FOR j:=1 TO 15 DO
    IF a[i,j]=2 THEN write(ARCS_COLOR2,'(',i:2',
writeln(ARCS_COLOR2); {saved to the}
writeln(ARCS_COLOR2); {text file }
writeln(ARCS_COLOR2); {ARCS_COLOR2 }
END.

Output from the program graph15

File ADJ_MAT_TXT:

Adjacency matrix of the graph from Chapter 4:

0 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2
2 0 1 1 2 1 2 1 2 1 2 1 1 1 1 1
2 2 0 1 1 2 1 2 1 2 1 2 1 2 1 2
2 2 2 0 1 1 2 1 1 2 2 2 1 2 1 2
2 1 2 2 0 1 1 2 2 1 1 1 2 2 2 1
2 2 1 2 2 0 1 1 1 2 1 1 1 1 1 2
2 1 2 1 2 0 1 1 2 1 1 2 1 2 1 1
2 1 1 2 1 2 0 2 2 1 2 1 1 1 1 2
1 2 2 2 1 2 1 0 2 2 1 2 1 1 1 1
1 1 2 2 2 1 2 1 1 0 2 1 2 1 1 1
1 1 2 2 2 1 2 1 1 0 2 2 1 2 1 2
1 2 1 1 2 2 1 1 0 2 1 0 2 1 1 1
1 2 1 2 1 1 2 2 1 2 1 1 1 0 2 2
1 2 2 1 2 1 1 2 2 1 2 1 2 1 1 0

File ARCS_COLOR1:

Arcs of color 1 (class C<0_1):

( 1, 2) ( 1, 3) ( 1, 4) ( 1, 5) ( 1, 6) ( 1, 7) ( 1, 8)
( 2, 3) ( 2, 4) ( 2, 6) ( 2, 9) ( 2,12) ( 2,14) ( 2,15)
( 3, 4) ( 3, 5) ( 3, 7) ( 3, 9) ( 3,10) ( 3,13) ( 3,15)
( 4, 5) ( 4, 6) ( 4, 8) ( 4, 9) ( 4,10) ( 4,11) ( 4,14)
( 5, 2) ( 5, 6) ( 5, 7) ( 5,10) ( 5,11) ( 5,12) ( 5,15)
File ARCS_COLOR2:

Arcs of color 2 (class $C^{-0.2}$):

( 1, 9) ( 1,10) ( 1,11) ( 1,12) ( 1,13) ( 1,14) ( 1,15)
( 2, 1) ( 2, 5) ( 2, 7) ( 2, 8) ( 2,10) ( 2,11) ( 2,13)
( 3, 1) ( 3, 2) ( 3, 6) ( 3, 8) ( 3,11) ( 3,12) ( 3,14)
( 4, 1) ( 4, 2) ( 4, 3) ( 4, 7) ( 4,12) ( 4,13) ( 4,15)
( 5, 1) ( 5, 3) ( 5, 4) ( 5, 8) ( 5, 9) ( 5,13) ( 5,14)
( 6, 1) ( 6, 2) ( 6, 4) ( 6, 5) ( 6,10) ( 6,14) ( 6,15)
( 7, 1) ( 7, 3) ( 7, 5) ( 7, 6) ( 7, 9) ( 7,11) ( 7,15)
( 8, 1) ( 8, 4) ( 8, 6) ( 8, 7) ( 8, 9) ( 8,10) ( 8,12)
( 9, 2) ( 9, 3) ( 9, 4) ( 9, 6) ( 9,10) ( 9,11) ( 9,13)
(10, 3) (10, 4) (10, 5) (10, 7) (10,11) (10,12) (10,14)
(11, 4) (11, 5) (11, 6) (11, 8) (11,12) (11,13) (11,15)
(12, 2) (12, 5) (12, 6) (12, 7) (12, 9) (12,13) (12,14)
(13, 3) (13, 6) (13, 7) (13, 8) (13,10) (13,14) (13,15)
(14, 2) (14, 4) (14, 7) (14, 8) (14, 9) (14,11) (14,15)
(15, 2) (15, 3) (15, 5) (15, 8) (15, 9) (15,10) (15,12)
APPENDIX B  COMPUTER PROGRAM – 3-VERTEX CONDITION

Program 3-vert

PROGRAM three_vert(ADJ_MAT_TXT, ARCS_COLOR1, ARCS_COLOR2, THREE_VERT_TXT);

TYPE
  pair = ARRAY[1..2] OF integer;
  adjmatrix = ARRAY [1..15,1..15] OF integer;
  tabl = ARRAY[1..105] OF integer;
  class = ARRAY [1..105] OF pair;

VAR
  ADJ_MAT_TXT, ARCS_COLOR1, ARCS_COLOR2, THREE_VERT_TXT : text;
  p : pair;
  a : adjmatrix;
  kt : class;
  i, j, k, color, xv, yv, xy, rozmiar : integer;
  c : char;

PROCEDURE konstarc;

VAR
  pc, pi, pj, pk, pv, koniec : integer;
  cconst : tabl;

BEGIN
  pc:=0;
  FOR pv:=1 TO 15 DO
    IF (a[kt[1,i],pv]=xv)
      AND (a[kt[1,2], pv]=yv)
    THEN pc:=pc+1;
  cconst[i]:=pc;
  koniec:=1;

END;}
FOR pk:=2 TO rozmiar DO
BEGIN
  pc:=0;
  FOR pv:=1 TO 15 DO
    IF (a[kt[pk,1],pv]=xv)
      AND (a[kt[pk,2], pv]=yv)
      THEN pc:=pc+1;
  pj:=1;
  FOR pi:=1 TO koniec DO
    IF cconst[pi]=pc THEN pj:=pj*0;
    IF pj=1 THEN
      BEGIN
        koniec:=koniec+1;
        cconst[koniec]:=pc;
      END;
END;

writeln(THREE_VERT_TXT,'Number of triangles of type');
writeln(THREE_VERT_TXT,' 0', color:2, xv:2);
writeln(THREE_VERT_TXT,3-color:2, ' 0',yv:2);
write(THREE.VERT_TXT,3-xv:2, 3-yv:2, '0',' is : ' ) ;
FOR pi:=1 TO koniec DO
  write(THREE.VERT.TXT, cconst[pi]:3);
writeln(THREE_VERT.TXT,'.');
writeln(THREE_VERT_TXT);
END;

PROCEDURE konstvert;
VAR
  pc, pi, pj, pu, pv, pw, koniec : integer;
  cconst : tabl;
BEGIN
  pc:=0;
  pu:=1;
  FOR pv:=1 TO 15 DO FOR pw:=1 TO 15 DO
    IF (a[pu,pv]=xv)
      AND (a[pu, pw]=yv)
      AND (a[pv,pw]=xy)
      THEN pc:=pc+1;
  cconst[1]:=pc;
  koniec:=1;
END;
FOR pu:=2 TO 15 DO BEGIN
   pc:=0;
   FOR pv:=1 TO 15 DO FOR pw:=1 TO 15 DO
      IF (a[pu,pv]=xv) AND (a[pu,pw]=yv) AND (a[pv,pw]=xy)
         THEN pc:=pc+1;
   pj:=1;
   FOR pi:=1 TO koniec DO
      IF cconst[pi]=pc THEN pj:=pj*0;
   IF pj=1 THEN
      BEGIN
         koniec:=koniec+1;
         cconst[koniec]:=pc;
      END;
END;
BEGIN
   writeln(THREE_VERT_TXT,'Number of triangles of type');
   writeln(THREE_VERT_TXT,' 0', xv:2, yv:2);
   writeln(THREE_VERT_TXT,3-xv:2, ' 0',xy:2);
   writeln(THREE_VERT_TXT,3-yv:2, 3-xy:2, ' 0',' is:');
   FOR pi:=1 TO koniec DO
      write(THREE_VERT_TXT, cconst[pi]:3);
   writeln(THREE_VERT_TXT,'.');
   writeln(THREE_VERT.TXT);
END;

BEGIN
   reset(ADJ.MAT.TXT);
   readln(ADJ.MAT.TXT);
   readln(ADJ.MAT.TXT);
   FOR i:=1 TO 15 DO
      BEGIN
         FOR j:=1 TO 15 DO read(ADJ.MAT.TXT,a[i,j]);
         readln(ADJ.MAT.TXT);
      END;
   rozmiar:=0;
   reset(ARCS_COLOR1);
   readln(ARCS_COLOR1);
   readln(ARCS_COLOR1);
   FOR i:=0 TO 14 DO
      { Then it counts }
      { the number, pc, }
      { of 3-vertex }
      { subgraphs }
      { of the same type }
      { with respect }
      { to the remaining }
      { vertices of the }
      { graph and, }
      { if pc is }
      { different from }
      { previous ones, }
      { stores it }
      { in the vector }
      { cconst. }
      { Finally }
      { konstvert }
      { writes }
      { the number }
      { of }
      { 3-vertex }
      { subgraphs }
      { of the }
      { fixed type }
      { to the file }
      { THREE_VERT_TXT. }
      { Beginning of }
      { the program. }
      { This part reads }
      { the matrix }
      { from the file }
      { ADJ_MAT.TXT }
      { and stores it }
      { as A=a[i,j]. }
   { This part }
   { reads pairs }
   { of vertices }
   { (x,y) }
BEGIN
FOR j:=1 TO 7 DO
BEGIN
    read(ARCS_COLOR1,c);
    read(ARCS_COLOR1,kt[7*i+j,1]);
    read(ARCS_COLOR1,c);
    read(ARCS_COLOR1,kt[7*i+j,2]);
    read(ARCS_COLOR1,c);
    read(ARCS_COLOR1,c);
    rozmiar:=rozmiar+1;
END;
readln(ARCS_COLOR1);
END;
color:=1;
BEGIN
FOR j:=1 TO 7 DO
BEGIN
    read(ARCS_COLOR2,c);
    read(ARCS_COLOR2,kt[7*i+j,1]);
    read(ARCS_COLOR2,c);
    read(ARCS_COLOR2,kt[7*i+j,2]);
END;
readln(ARCS_COLOR2);
END;
rewrite(THREE_VERT_TXT);
write(THREE_VERT_TXT,'With respect to an arc of color ','color:1');
writeln(THREE_VERT_TXT,', i.e. E(x,y)=',color:1,', x<y<z.');
writeln(THREE_VERT_TXT);
FOR xv:=1 TO 2 DO
    FOR yv:=1 TO 2 DO
        konstarc;
writeln(THREE_VERT_TXT, 'Checked ',rozmiar:3,' arcs of color ','color:1,'.');
writeln(THREE_VERT_TXT);
rozmiar:=0;
reset(ARCS_COLOR2);
readln(ARCS_COLOR2);
readln(ARCS_COLOR2);
FOR i:=0 TO 14 DO
BEGIN
    FOR j:=1 TO 7 DO
    BEGIN
        read(ARCS_COLOR2,c);
        read(ARCS_COLOR2,kt[7*i+j,1]);
        read(ARCS_COLOR2,c);
        read(ARCS_COLOR2,kt[7*i+j,2]);
    END;
END;

read(ARCS_COLOR2,c);  \{u=1..105, v=1,2\}
read(ARCS_COLOR2,c);  \{ The number of \}
read(ARCS_COLOR2,c);  \{arcs of color 2\}
rozmiar:=rozmiar+1;  \{ is "rozmiar" \}
END;
readln(ARCS_COLOR2);  \{ The color of \}
END;  \{ arcs (x,y) \}
color:=2;  \{ is "color". \}

writeln(THREE_VERT_TXT);
writeln(THREE_VERT_TXT);
write(THREE_VERT_TXT,'With respect to an arc of color ',color:1);
writeln(THREE_VERT_TXT,'
', i.e. E(x,y)=',color:1, ', x<y<c.');</
writeln(THREE_VERT_TXT);

FOR xv:=1 TO 2 DO
  FOR yv:=1 TO 2 DO
    konstarc;
writeln(THREE_VERT_TXT, 'Checked ',rozmiar:3,' arcs of color ',color:1,'.');
writeln(THREE_VERT_TXT);

writeln(THREE_VERT_TXT);
write(THREE_VERT_TXT,'With respect to an arc of color 0 (vertex ');
writeln(THREE_VERT_TXT,', i.e. E(x,x)=0');
writeln(THREE_VERT_TXT);
FOR xv:=1 TO 2 DO
  FOR yv:=1 TO 2 DO
    FOR xy:=1 TO 2 DO konstvert;
END.

Output from program 3-vert: file THREE_VERT.TXT

With respect to an arc of color 1, i.e. E(x,y)=1, x<y<c.

Number of triangles of type
0 1 1
2 0 1
2 2 0 is: 3.
Number of triangles of type
0 1 1
2 0 2
2 1 0 is: 3.

Number of triangles of type
0 1 2
2 0 1
1 2 0 is: 4.

Number of triangles of type
0 1 2
2 0 2
1 1 0 is: 3.

Checked 105 arcs of color 1.

With respect to an arc of color 2, i.e. $E(x,y)=2$, $x<y<c$.

Number of triangles of type
0 2 1
1 0 1
2 2 0 is: 3.

Number of triangles of type
0 2 1
1 0 2
2 1 0 is: 4.

Number of triangles of type
0 2 2
1 0 1
1 2 0 is: 3.

Number of triangles of type
0 2 2
1 0 2
1 1 0 is: 3.

Checked 105 arcs of color 2.
With respect to an arc of color 0 (vertex), i.e. $E(x,x)=0$

Number of triangles of type
0 1 1
2 0 1
2 2 0 is: 21.

Number of triangles of type
0 1 1
2 0 2
2 1 0 is: 21.

Number of triangles of type
0 1 2
2 0 1
1 2 0 is: 28.

Number of triangles of type
0 1 2
2 0 2
1 1 0 is: 21.

Number of triangles of type
0 2 1
1 0 1
2 2 0 is: 21.

Number of triangles of type
0 2 1
1 0 2
2 1 0 is: 28.

Number of triangles of type
0 2 2
1 0 1
1 2 0 is: 21.

Number of triangles of type
0 2 2
1 0 2
1 1 0 is: 21.
APPENDIX C  COMPUTER PROGRAM – 4-VERTEX CONDITION

Program 4-vert

PROGRAM four_vert (ADJ_MAT_TXT, ARCS_COLOR1, ARCS_COLOR2, FOUR_VERT_TXT);

TYPE
  pair      = ARRAY[1..2] OF integer;
  adjmatrix = ARRAY [1..15,1..15] OF integer;
  tabl     = ARRAY[1..105] OF integer;
  class     = ARRAY [1..105] OF pair;

VAR
  ADJ_MAT_TXT, ARCS_COLOR1, ARCS_COLOR2, FOUR_VERT_TXT : text;
  p                        : pair;
  a                        : adjmatrix;
  kt                       : class;
  i, j, k, color, rozmiar  : integer;
  tlv, t1w, t2v, t2w, uv, uw, uz, vw, vz, wz : integer;
  c                        : char;

PROCEDURE konstarc;

VAR
  pc, pi, pj, pk, pl, pm, pn, pv, pw, koniec : integer;
  cconst                                   : tabl;

BEGIN
  pc:=0;
  FOR pv:=1 TO 15 DO FOR pw:=1 TO 15 DO
    IF (a[kt[1,1],pv]=tlv)
      AND (a[kt[1,2],pv]=t1v)
      AND (a[kt[1,1],pw]=t1w)
      AND (a[kt[1,2],pw]=t2w)
      AND (a[kt[2,1],pv]=t2v)
      AND (a[kt[2,2],pv]=t2v)
      AND (a[kt[2,1],pw]=t2w)
      AND (a[kt[2,2],pw]=t2w)
      THEN cconst[pc]:=c;
      pc:=pc+1;
  END;
END konstarc;

PROCEDURE konstarc;

VAR
  pc, pi, pj, pk, pl, pm, pn, pv, pw, koniec : integer;
  cconst                                   : tabl;

BEGIN
  pc:=0;
  FOR pv:=1 TO 15 DO FOR pw:=1 TO 15 DO
    IF (a[kt[1,1],pv]=tlv)
      AND (a[kt[1,2],pv]=t1v)
      AND (a[kt[1,1],pw]=t1w)
      AND (a[kt[1,2],pw]=t2w)
      THEN cconst[pc]:=c;
      pc:=pc+1;
  END;
END konstarc;
AND (a[pv,pw]=vw)
THEN pc:=pc+1;
cconst[1]:=pc;
koniec:=1;
FOR pk:=2 TO rozmiar DO
BEGIN
pc:=0;
FOR pv:=1 TO 15 DO FOR pw:=1 TO 15 DO
IF (a[kt[1,1],pv]=tlv)
AND (a[kt[1,2],pv]=t2v)
AND (a[kt[1,1],pw]=t1w)
AND (a[kt[1,2],pw]=t2w)
AND (a[pv,pw]=vw)
THEN pc:=pc+1;
pj:=1;
FOR pi:=1 TO koniec DO
IF cconst[pi]=pc THEN pj:=pj*0;
IF pj=1 THEN
BEGIN
koniec:=koniec+1;
cconst[koniec]:=pc;
END;
END;
writeln(FOUR_VERT.TXT,'Number of 4-vertex subgraphs of type');
writeln(FOUR_VERT.TXT,' 0', color:2, tlv:2, t1w:2);
writeln(FOUR_VERT.TXT,3-color:2, ' 0', t2v:2, t2w:2);
writeln(FOUR_VERT.TXT,3-tlv:2, 3-t2v:2, ' 0',vw:2);
write(FOUR_VERT.TXT,3-t1w:2. 3-t2w:2,3-vw:2,' 0 is:');
FOR pi:=1 TO koniec DO write(FOUR_VERT.TXT, cconst[pi]:3);
writeln(FOUR_VERT.TXT,'.');
writeln(FOUR_VERT.TXT);
END;

PROCEDURE konstvert;
VAR
pc, pi, pj, pu, pv, pw, pz, koniec : integer;
cconst : tabl;
BEGIN
pc:=0;
pu:=1;
FOR pv:=1 TO 15 DO

FOR pw:=1 TO 15 DO
  FOR pz:=1 TO 15 DO
    THEN pc:=pc+1;
  cconst[1]:=pc;
konec:=1;
FOR pu:=2 TO 15 DO
BEGIN
  pc:=0;
  FOR pv:=1 TO 15 DO
    FOR pw:=1 TO 15 DO
      FOR pz:=1 TO 15 DO
        THEN pc:=pc+1;
      pj:=1;
      FOR pi:=1 TO konec DO
        IF cconst[pi]=pc THEN pj:=pj*0;
      IF pj=1 THEN
        BEGIN
          konec:=konec+1;
          cconst[konec]:=pc;
        END;
    END;
  END;
END;
writeln(FOUR_VERT_TXT);writeln(FOUR_VERT_TXT,'Number of 4-vertex subgraphs of type'); { the }
writeln(FOUR_VERT_TXT,' 0', uv:2, uw:2, uz:2); { number of }
writeln(FOUR_VERT_TXT,3-uv:2, ' 0', vw:2, vz:2); { 4-vertex }
writeln(FOUR_VERT_TXT,3-uw:2, 3-vw:2, ' 0', wz:2); { subgraphs }
write(FOUR_VERT_TXT,3-uz:2, 3-vz:2, 3-wz:2,' 0 is:'); { of the }
FOR pi:=1 TO konec DO write(FOUR_VERT_TXT, cconst[pi]:3); { fixed type }
writeln(FOUR_VERT_TXT,'.'); { to the file }
END;

{ subgraphs }
{ of the type: }
{ t1t2 t1v t1w }
{ t2v t2w }
{ vw }
{ with respect }
{ to the first }
{ arc and }
{ stores it }
{ in the vector }
{ cconst. }
{ Then it counts }
{ the number }
{ pc }
{ of 4-vertex }
{ subgraphs }
{ of the same }
{ type with }
{ respect to the }
{ remaining arcs }
{ and, if pc is }
{ different from }
{ previous ones, }
{ stores it }
{ in the vector }
{ cconst. }
{ Finally }
{ konstvert }
{ writes }
{ the }
{ number of }
{ 4-vertex }
{ subgraphs }
{ of the }
{ fixed type }
{ to the file }
{ FOUR_VERT_TXT. }
BEGIN

reset(ADJ_MAT_TXT);
readln(ADJ_MAT_TXT);
readln(ADJ_MAT_TXT);
FOR i:=1 TO 15 DO
  BEGIN
    FOR j:=1 TO 15 DO read(ADJ_MAT_TXT,a[i,j]);
    readln(ADJ_MAT_TXT);
  END;

rozmiar:=0;
reset(ARCS_COLOR1);
readln(ARCS_COLOR1);
readln(ARCS_COLOR1);
FOR i:=0 TO 14 DO
  BEGIN
    FOR j:=1 TO 7 DO
      BEGIN
        read(ARCS_COLOR1,c);
        read(ARCS_COLOR1,kt[7*i+j,1]);
        read(ARCS_COLOR1,c);
        read(ARCS_COLOR1,kt[7*i+j,2]);
        read(ARCS_COLOR1,c);
        read(ARCS_COLOR1,c);
        rozmiar:=rozmiar+1;
      END;
    readln(ARCS_COLOR1);
  END;
color:=1;

rewrite(FOUR_VERT_TXT);
write(FOUR_VERT_TXT,'Numbers of 4-vertex subgraphs of types ');
writeln(FOUR_VERT_TXT,'given by 4x4 matrices with respect to ');
write(FOUR_VERT_TXT,'arcs of color ',color:1,', ');
writeln(FOUR_VERT_TXT,'i.e. E(a,b)=',color:1,', a<b<c<d. ');
writeln(FOUR_VERT.TXT);
```
i:=0;
FOR tlv:=l TO 2 DO
  FOR t2v:=l TO 2 DO
    FOR tlw:=l TO 2 DO
      FOR t2w:=l TO 2 DO
        FOR vw:=l TO 2 DO
          BEGIN
            konstarc;
i:=i+1;
            IF (i=8) THEN FOR j:=l TO 2 DO
              writeln(FOUR_VERT_TXT);
            IF (i=24) THEN FOR j:=l TO 4 DO
              writeln(FOUR_VERT_TXT);
          END;
        END;
      END;
    END;
  END;
END;
writeln(FOUR_VERT_TXT, 'Checked ',rozmiar:3,' arcs of color ',color:1,'.');
writeln(FOUR_VERT_TXT);
writeln(FOUR_VERT_TXT);
writeln(FOUR_VERT_TXT);
rozmiar:=0;
reset(ARCS_COLOR2);
writeln(FOUR_VERT_TXT);
readln(ARCS_COLOR2);
writeln(FOUR_VERT_TXT);
writeln(FOUR_VERT_TXT);
FOR i:=0 TO 14 DO
  BEGIN
    FOR j:=1 TO 7 DO
      BEGIN
        read(ARCS_COLOR2,c);
        read(ARCS_COLOR2,kt[7*i+j,1]);
        read(ARCS_COLOR2,c);
        read(ARCS_COLOR2,kt[7*i+j,2]);
        read(ARCS_COLOR2,c);
        read(ARCS_COLOR2,c);
        read(ARCS_COLOR2,c);
        rozmiar:=rozmiar+1;
      END;
    readln(ARCS_COLOR2);
  END;
writeln(FOUR_VERT_TXT, 'Numbers of 4-vertex subgraphs');
writeln(FOUR_VERT_TXT,' of types given by 4x4 matrices with respect to');
color:=2;
writeln(FOUR_VERT_TXT,' arcs of color ',color:1,'. ');```
writeln(FOUR_VERT_TXT, 'i.e. E(a,b)=', color:1,' , a<b<c<d.');
writeln(FOUR_VERT_TXT);

i:=0;
FOR t1v:=1 TO 2 DO
  FOR t2v:=1 TO 2 DO
    FOR t1w:=1 TO 2 DO
      FOR t2w:=1 TO 2 DO
        FOR vw:=1 TO 2 DO
          BEGIN
            konstarc;
i:=i+1;
            IF (i=8) THEN FOR j:=1 TO 4 DO
             writeln(FOUR_VERT_TXT);
            IF (i=24) THEN FOR j:=1 TO 4 DO
             writeln(FOUR_VERT_TXT);
          END;
FOR tlv:=1 TO 2 DO
  FOR t2v:=1 TO 2 DO
    FOR t1w:=1 TO 2 DO
      FOR t2w:=1 TO 2 DO
        FOR vw:=1 TO 2 DO
          BEGIN
            konstarc;
i:=i+1;
            IF (i=8) THEN FOR j:=1 TO 4 DO
             writeln(FOUR_VERT_TXT);
            IF (i=24) THEN FOR j:=1 TO 4 DO
             writeln(FOUR_VERT_TXT);
          END;
BEGIN
  writeln(FOUR_VERT_TXT, 'Checked ',rozmiar:3,' arcs of color ',color:1,'.');
writeln(FOUR_VERT_TXT);
writeln(FOUR_VERT_TXT);
writeln(FOUR_VERT_TXT);
write(FOUR_VERT_TXT,'Numbers of 4-vertex subgraphs of types given by ');
writeln(FOUR_VERT_TXT,'4x4 matrices with respect to');
writeln(FOUR_VERT_TXT,'arcs of color 0, i.e. E(a,b)=0, a<b<c<d.');
writeln(FOUR_VERT_TXT);

i:=0;
FOR uv:=1 TO 2 DO
  FOR uw:=1 TO 2 DO
    FOR uz:=1 TO 2 DO
      FOR vw:=1 TO 2 DO
        FOR vz:=1 TO 2 DO
          FOR wz:=1 TO 2 DO
            BEGIN
              konstarc;
i:=i+1;
              IF (i MOD 8 = 0) AND (i <> 64) THEN FOR j:=1 TO 4 DO
               writeln(FOUR_VERT_TXT);
              END;
END.
Output from program 4-vert: file FOUR_VERT.TXT

Numbers of 4-vertex subgraphs of types given by 4x4 matrices with respect to arcs of color 1, i.e. $E(a,b)=1$ $a<b<c<d$.

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<tr>
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0 1 2 2
2 0 1 2
1 2 0 2
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Checked 105 arcs of color 1.
Numbers of 4-vertex subgraphs of types given by 4x4 matrices with respect to arcs of color 2, i.e. $E(a,b)=2$, $a<b<c<d$.

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Number of 4-vertex subgraphs of type 0 2 2 1 1 0 1 1 1 2 0 2 2 1 1 0 is: 6.

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Number of 4-vertex subgraphs of type 0 2 2 2 1 0 1 1 1 2 0 2 1 1 1 0 is: 3.
Numbers of 4-vertex subgraphs of types given by $4 \times 4$ matrices with respect to arcs of color 0, i.e. $E(a,b)=0$, $a<b<c<d$.

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APPENDIX D  COMPUTER PROGRAM – PENTAGON

Program pentagon

PROGRAM pentagon (ADJ_MAT_TXT, ARCS_COLOR1, PENTAGON_TXT);

TYPE
   pair   = ARRAY[1..2] OF integer;
   triple = ARRAY[1..3] OF integer;
   matrix = ARRAY [1..15,1..15] OF integer;
   tabl   = ARRAY[1..105] OF integer;
   class  = ARRAY [1..105] OF pair;
   triclass = ARRAY [1..1000] OF triple;

VAR
   p          : pair;
   a          : matrix;
   kt, c3,c4,c5 : class;
   tc3,tc4,tc5, temp : triclass;
   i,j,k,l3,l4,l5,t13,t14,t15,color,rozmiar : integer;
   t1v,t1w,tiz,t2v,t2w,t2z,uv,uw,uz,vt,vw,vz,wt,zt : integer;
   ADJ_MAT_TXT, ARCS_COLOR1, PENTAGON_TXT : text;
   c          : char;

PROCEDURE konst;

VAR
   pc,pi,pj,pk,pl,pm,pn,pv,pw,pz,koniec : integer;
   cconst : tabl;

BEGIN
   l3:=1;        { Procedure }
   l4:=1;        { konst }
   l5:=1;        { counts }
   t13:=0;       { the number, pc, }
   t14:=0;       { of 5-vertex }
t15:=0;
FOR pk:=1 TO rozmiar DO
BEGIN
pc:=0;
pl:=0;
FOR pv:=1 TO 15 DO FOR pw:=1 TO 15 DO FOR pz:=1 TO 15 DO
IF (a[k][pk,1],pv]=t1v) AND (a[k][pk,2], pv]=t2v)
AND (a[k][pk,1],pw]=t1w) AND (a[k][pk,2], pw]=t2w)
AND (a[k][pk,1],pz]=t1z) AND (a[k][pk,2], pz]=t2z)
AND (a[pv,pw]=vw) AND (a[pv,pz]=vz) AND (a[pw,pz]=wz)
THEN BEGIN
pc:=pc+1;
pl:=pl+1;
temp[pl,1]:=pv;
temp[pl,2]:=pw;
temp[pl,3]:=pz;
END;
END;

IF pc=3 THEN BEGIN
BEGIN
c3[13,1]:=kt[pk,1];
c3[13,2]:=kt[pk,2];
FOR pm:=1 TO 3 DO FOR pn:=1 TO 3 DO
tc3[3*t13+pm,pn]:=temp[pm,pn];
tl3:=tl3+1;
l3:=l3+1;
END;
END;

IF pc=4 THEN BEGIN
BEGIN
c4[14,1]:=kt[pk,1];
c4[14,2]:=kt[pk,2];
FOR pm:=1 TO 4 DO FOR pn:=1 TO 3 DO
tc4[4*t14+pm,pn]:=temp[pm,pn];
tl4:=tl4+1;
l4:=l4+1;
END;
IF \( pc=5 \) THEN
BEGIN
\[
c_{5}[15,1]:=k_t[p_k,1];
\]
\[
c_{5}[15,2]:=k_t[p_k,2];
\]
FOR \( pm:=1 \) TO 5 DO
FOR \( pn:=1 \) TO 3 DO
\[
t_{c5}[5*t_{15}+pm,pn]:=t_{\text{temp}}[pm,pn];
\]
t_{15}:=t_{15}+1;
15:=15+1;
END;
END;

\{ This part writes the type of the 5-vertex subgraph \}
\{ to the file PENTAGON_TXT. \}
write(PENTAGON_TXT,'Numbers of 5-vertex subgraphs of the type');
writeln(PENTAGON_TXT,' given by the 5x5 matrix');
writeln(PENTAGON_TXT,' 0', color:2, tlv:2, tlw:2, tlz:2);
writeln(PENTAGON_TXT,3-color:2, ' 0',t2v:2, t2w:2, t2z:2);
writeln(PENTAGON_TXT,3-tlv:2, 3-t2v:2, ' 0',vw:2, vz:2);
writeln(PENTAGON_TXT,3-tlw:2, 3-t2w:2,3-vw:2,' 0', wz:2);
writeln(PENTAGON_TXT,3-tlz:2, 3-t2z:2,3-vz:2, 3-wz:2,' 0 ');
writeln(PENTAGON_TXT,'with respect to an arc of color 1 are 3, 4 and 5.');
writeln(PENTAGON_TXT);
writeln(PENTAGON_TXT);

\{ This part lists in the file PENTAGON_TXT \}
\{ all arcs of color 1 which have 3 pentagons. \}
\{ For every arc, it lists the remaining triples of vertices \}
\{ of each of the 3 pentagons. \}
write(PENTAGON_TXT,' Arc ');
writeln(PENTAGON_TXT,' Remaining triples of vertices of pentagons');
writeln(PENTAGON_TXT);
FOR \( pi:=1 \) TO 13-1 DO
BEGIN
write(PENTAGON_TXT, '(',c_{3}[pi,1]:2,',',c_{3}[pi,2]:2,')': '');
FOR \( pm:=1 \) TO 3 DO
BEGIN
write(PENTAGON_TXT,'(',t_{c3}[3*(pi-1)+pm,1]:2,',');
write(PENTAGON_TXT,t_{c3}[3*(pi-1)+pm,2]:2,',');
write(PENTAGON_TXT,t_{c3}[3*(pi-1)+pm,3]:2,')');
END;
END;
writeln(PENTAGON_TXT);
END;
writeln(PENTAGON_TXT);
write(PENTAGON_TXT,'Number of arcs of color 1 in this subclass is');
writeln(PENTAGON_TXT,13-1:3,'.'调节);
writeln(PENTAGON_TXT);
writeln(PENTAGON_TXT);
{ This part lists in the file PENTAGON.TXT }  
{ all arcs of color 1 which have 4 pentagons. }  
{ For every arc, it lists the remaining triples of vertices }  
{ of each of the 4 pentagons. }
write(PENTAGON_TXT,' Arc ');
writeln(PENTAGON_TXT,' Remaining triples of vertices of pentagons');
writeln(PENTAGON.TXT);
FOR pi:=1 TO 14-1 DO 
BEGIN
  write(PENTAGON_TXT, '(',c4[pi,1]:2,',',c4[pi,2]:2,') : ');
  FOR pm:=1 TO 4 DO 
  BEGIN
    write(PENTAGON_TXT, '(',tc4[4*(pi-1)+pm,1]:2,',') ;
    write(PENTAGON_TXT,tc4[4*(pi-1)+pm,2]:2,',');
    write(PENTAGON_TXT,tc4[3*(pi-1)+pm,3]:2,') , ');
  END;
  writeln(PENTAGON_TXT);
END;
writeln(PENTAGON_TXT);
write(PENTAGON_TXT,'Number of axes of color 1 in this subclass is');
writeln(PENTAGON_TXT,14-1:3,'.'调节);
writeln(PENTAGON_TXT);
writeln(PENTAGON_TXT);
{ This part lists in the file PENTAGON.TXT }  
{ all arcs of color 1 which have 5 pentagons. }  
{ For every arc, it lists the remaining triples of vertices }  
{ of each of the 5 pentagons. }
write(PENTAGON_TXT,' Arc ');
writeln(PENTAGON_TXT,' Remaining triples of vertices of pentagons');
writeln(PENTAGON_TXT);
FOR pi:=1 TO 15-1 DO 
BEGIN
  write(PENTAGON_TXT, '(c5[pi,1]:2,',',c5[pi,2]:2): '); 
  FOR pm:=1 TO 5 DO 
  BEGIN
    write(PENTAGON_TXT,'(tc5[5*(pi-1)+pm,1]:2,') ;
    write(PENTAGON_TXT,tc5[5*(pi-1)+pm,2]:2,');
    write(PENTAGON_TXT,tc5[5*(pi-1)+pm,3]:2'), ');
  END;
writeln(PENTAGON_TXT);
END;
writeln(PENTAGON_TXT);
writeln(PENTAGON_TXT);
write(PENTAGON_TXT,'Number of arcs of color 1 in this subclass is');
writeln(PENTAGON_TXT,14-1:3,'.');
writeln(PENTAGON_TXT);
writeln(PENTAGON_TXT);
IF rozmiar = 13+14+15-3 THEN 
  writeln(PENTAGON_TXT,'All ',rozmiar:3,' arcs of color 1 were checked.');
END;

BEGIN 
  reset(ADJ_MAT_TXT);
  readln(ADJ_MAT_TXT);
  readln(ADJ_MAT_TXT);
  FOR i:=1 TO 15 DO 
  BEGIN
    FOR j:=1 TO 15 DO read(ADJ_MAT_TXT,a[i,j]);
    readln(ADJ_MAT_TXT);
  END;
  rozmiar:=0;
  reset(ARCS_COLOR1);
  readln(ARCS_COLOR1);
  readln(ARCS_COLOR1);
  FOR i:=0 TO 14 DO 
  BEGIN
    FOR j:=1 TO 7 DO 
    BEGIN
      read(ARCS_COLOR1,c);
      read(ARCS_COLOR1,kt[7*i+j,1]);
      read(ARCS_COLOR1,c); 
    END;
  END;
read(ARCS_COLOR1,kt[7*i+j,2]);   \{as KT=kt[u,v]. \}
read(ARCS_COLOR1,c);           \{u=1..105, v=1,2\}
read(ARCS_COLOR1,c);   \{The number of\}
read(ARCS_COLOR1,c);       \{arcs of color 1\}
rozmiar:=rozmiar+1;       \{is "rozmiar"\}
END;
readln(ARCS_COLOR1);         \{arcs (x,y)\}
END;
color:=1;                  \{is "color".\}

rewrite(PENTAGON_TXT);        \{This part defines the type\}
\{of the 5-vertex subgraph.\}
\{The vertices are: t1, t2, v, w, z.\}
\{t1v:=2 means that the\}
\{arc from t1 to v is of color 2,\}
\{etc.\}
konst;

END.

Output from program pentagon: file PENTAGON_TXT

Numbers of 5-vertex subgraphs of the type given by the 5x5 matrix
\[
\begin{bmatrix}
0 & 1 & 2 & 1 & 2 \\
2 & 0 & 1 & 2 & 1 \\
1 & 2 & 0 & 1 & 2 \\
2 & 1 & 2 & 0 & 1 \\
1 & 2 & 1 & 2 & 0
\end{bmatrix}
\]
with respect to an arc of color 1 are 3, 4 and 5.

Arc Remaining triples of vertices of pentagons

(1, 2): (12, 8, 14) (14, 5, 15) (15, 7, 12)
(1, 3): (9, 8, 13) (13, 2, 15) (15, 6, 9)
(1, 4): (9, 7, 10) (10, 2, 14) (14, 3, 9)
(1, 5): (10, 8, 11) (11, 3, 15) (15, 4, 10)
(1, 6): (9, 5, 11) (11, 2, 12) (12, 4, 9)
(1, 7): (10, 6, 12) (12, 3, 13) (13, 5, 10)
(1, 8): (11, 7, 13) (13, 4, 14) (14, 6, 11)
(2, 9): (5, 6, 8) (7, 4, 5) (8, 3, 7)
(3,10): (2, 4, 8) (6, 7, 2) (8, 5, 6)
Number of arcs of color 1 in this subclass is 42.

Arc Remaining triples of vertices of pentagons

( 2, 3):    ( 5, 6,13), ( 7,14, 5), (10, 6, 7), (13,12,10),
( 2, 4):    ( 5,12,10), ( 8, 3, 8), (10,15,10), (11, 3,11),
( 2, 6):    ( 7, 4,11), ( 8,15, 5), (11,14,11), (13, 4, 7),
( 3, 4):    ( 6, 7, 7), ( 8,15,13), (11, 7, 8), (14,13,14),
( 3, 5):    ( 2,14, 5), ( 5, 6, 9), (13, 7, 2),
(15, 1):    ( 5,12,10), ( 8, 3, 8), (10,15,10), (11, 3,11),
(15, 13):   ( 2, 4,10), ( 3,13, 2), (10, 6, 3),
(15, 15):   ( 2, 4,10), ( 3,13, 2), (10, 6, 3),
Number of axes of color 1 in this subclass is 21.

Arc Remaining triples of vertices of pentagons

(2, 12): (1, 6, 11), (8, 14, 1), (8, 14, 10), (10, 6, 11), (11, 9, 8),
(2, 14): (1, 4, 10), (5, 15, 1), (5, 15, 13), (10, 9, 5), (13, 4, 10),
(2, 15): (1, 3, 13), (7, 12, 1), (7, 12, 11), (11, 3, 13), (13, 9, 7),
(3, 9): (1, 4, 14), (8, 13, 1), (8, 13, 12), (12, 4, 14), (14, 10, 8),
(3, 13): (1, 7, 12), (2, 15, 1), (2, 15, 11), (11, 7, 12), (12, 10, 2),
(3, 15): (1, 5, 11), (6, 9, 1), (6, 9, 14), (11, 10, 6), (14, 5, 11),
(4, 9): (1, 6, 12), (7, 10, 1), (7, 10, 15), (12, 11, 7), (15, 6, 12),
(4, 10): (1, 5, 15), (2, 14, 1), (2, 14, 13), (13, 5, 15), (15, 11, 2),
(4, 14): (1, 8, 13), (3, 9, 1), (3, 9, 12), (12, 8, 13), (13, 11, 3),
(5, 10): (1, 7, 13), (8, 11, 1), (8, 11, 9), (9, 7, 13), (13, 12, 8),
(5, 11): (1, 6, 9), (3, 15, 1), (3, 15, 14), (9, 12, 3), (14, 6, 9),
(5, 15): (1, 2, 14), (4, 10, 1), (4, 10, 13), (13, 2, 14), (14, 12, 4),
(6, 9): (1, 3, 15), (5, 11, 1), (5, 11, 14), (14, 3, 15), (15, 13, 5),
(6, 11): (1, 8, 14), (2, 12, 1), (2, 12, 10), (10, 8, 14), (14, 13, 2),
(6, 12): (1, 7, 10), (4, 9, 1), (4, 9, 15), (10, 13, 4), (15, 7, 10),
(7, 10): (1, 4, 9), (6, 12, 1), (6, 12, 15), (9, 14, 6), (15, 4, 9),
(7, 12): (1, 2, 15), (3, 13, 1), (3, 13, 11), (11, 2, 15), (15, 14, 3),
(7, 13): (1, 8, 11), (5, 10, 1), (5, 10, 9), (9, 8, 11), (11, 14, 5),
(8, 11): (1, 5, 10), (7, 13, 1), (7, 13, 9), (9, 5, 10), (10, 15, 7),
(8, 13): (1, 3, 9), (4, 14, 1), (4, 14, 12), (9, 15, 4), (12, 3, 9),
(8, 14): (1, 2, 12), (6, 11, 1), (6, 11, 10), (10, 2, 12), (12, 15, 6),
(9, 5): (2, 14, 10), (6, 8, 2), (10, 8, 11), (11, 1, 6), (11, 14, 6),
Number of arcs of color 1 in this subclass is 21.

All 105 arcs of color 1 were checked.
APPENDIX E  COMPUTER PROGRAM – FIRST_LEVEL

Program first_level

PROGRAM first_level(ADJ_MAT_TXT, ARCS_COLOR1, CLASSC1_K_TXT,
                  CONST_C1KK_01_TXT, SMALL_CLASS_TXT);

TYPE triple = ARRAY[1..3] OF integer;
  matrix = ARRAY[1..15,1..15] OF integer;
  tabl  = ARRAY [1..15] OF integer;
  klasa = ARRAY[1..315] OF triple;

VAR
  kt, jt, small : klasa;
  a : matrix;
  x, y, z, i, j, rozmiarj, rozmiark, smallsize : integer;
ADJ_MAT_TXT, ARCS_COLOR1, CLASSC1_K_TXT : text;
CONST_C1KK_01_TXT, SMALL_CLASS_TXT : text;
c : char;

PROCEDURE konstcijk;

VAR
  pc, pi, pj, pk, pl, pv, koniec : integer;
  cconst : tabl;

BEGIN
  pc:=0;
  pk:=1;
  FOR pv:=1 TO 15 DO
    FOR pj:=1 TO rozmiarj DO
      IF (a[kt[pk,i],pv]=1)
        { Procedure       }
        { konstcijk       }
        { counts the     }
        { intersection    }
        { constants       }

        { Procedure       }
        { konstcijk       }
        { counts the     }
        { intersection    }
        { constants       }

    END FOR;
  END FOR;
END;

PROCEDURE konstcijk;
\begin{verbatim}
AND (pv=jt[pj,1])
AND (kt[1,2]=jt[pj,2])
AND (kt[1,3]=jt[pj,3])
THEN pc:=pc+1;
cconst[1]:=pc;
koniec:=1;
FOR pk:=2 TO rozmiark DO
BEGIN
  pc:=0;
  FOR pv:=1 TO 15 DO
    FOR pj:=1 TO rozmiarj DO
      IF (a[kt[pk,1],pv]=1)
         AND (pv=jt[pj,1])
         AND (kt[pk,2]=jt[pj,2])
         AND (kt[pk,3]=jt[pj,3])
      THEN pc:=pc+1;
    pl:=1;
  FOR pi:=1 TO koniec DO
    BEGIN
      IF cconst[pi]=pc THEN pl:=pj+1;
    END;
  IF pl=1 THEN
  BEGIN
    koniec:=koniec+1;
cconst[koniec]:=pc;
  END;
END;

write(CONST_C1KK_01_TXT,'There are ',koniec:2);
write(CONST_C1KK_01_TXT,' constants c(i,j,k;0,1) : ');
FOR pi:=1 TO koniec DO
BEGIN
  write(CONST_C1KK_01_TXT, cconst[pi]:2);
  IF pi<koniec
     THEN write(CONST_C1KK_01_TXT,' ','
     ELSE writeln(CONST_C1KK_01_TXT,'.' );
  END;
writeln(CONST_C1KK_01_TXT); 
\end{verbatim}
PROCEDURE konstsmall;

VAR
    pc, pi, pj, pk, pv, koniec : integer;
    cconst : tabl;
BEGIN
    pc:=0;
    smallsize:=0;
    FOR pk:=1 TO rozmiarj DO
        BEGIN
            pc:=0;
            FOR pv:=1 TO 15 DO
                BEGIN
                    FOR pj :=1 TO rozmiark DO
                        IF (a[kt[pk,1],pv]=1)
                            AND (pv=jt[pj,1])
                            AND (kt[pk,2]=jt[pj,2])
                            AND (kt[pk,3]=jt[pj,3])
                            THEN pc:=pc+1;
                    IF pc=0 THEN
                        BEGIN
                            smallsize:=smallsize+1;
                            FOR pi:=1 TO 3 DO small[smallsize,pi]:=kt[pk,pi];
                        END;
                END;
        END;
    END;
BEGIN
    reset(ADJ_MAT_TXT);
    readln(ADJ_MAT_TXT);
    readln(ADJ_MAT_TXT);
    FOR i:=1 TO 15 DO
        BEGIN
            FOR j:=1 TO 15 DO read(ADJ_MAT_TXT,a[i,j]);
            readln(ADJ_MAT_TXT);
        END;
rozmiarj:=0;
rozmiark:=0;
FOR x:=1 TO 15 DO
    FOR y:=1 TO 15 DO
FOR z:=1 TO 15 DO
  IF (a[x,y]=1) AND (a[x,z]=1) AND (a[y,z]=1) THEN
    BEGIN
      rozmiarj:=rozmiarj+1;
      jt[rozmiarj,1]:=x;
      jt[rozmiarj,2]:=y;
      jt[rozmiarj,3]:=z;
      rozmiark:=rozmiark+1;
      kt[rozmiark,1]:=x;
      kt[rozmiark,2]:=y;
      kt[rozmiark,3]:=z;
    END;
  END;
rewrite(CLASS1_K_TXT);
FOR i:=1 TO rozmiark DO
  BEGIN
    write(CLASS1_K_TXT,'(',kt[i,1]:2,',')
    writeln(CLASS1_K_TXT,kt[i,2]:2,',')
    writeln(CLASS1_K_TXT,kt[i,3]:2,')');
  END;
writeln(CLASS1_K_TXT,' size:')
writeln(CLASS1_K_TXT,rozmiark:7)
rewrite(CONST_C1KK_01_TXT);
konstcijk;
rewrite(SMALL_CLASS_TXT);
konstsmall;
write(SMALL_CLASS_TXT,'Triples from the class C\-1\,k '); {This}
writeln(SMALL_CLASS_TXT,'corresponding to c(i,j,k;0,1) = 0:'); {part}
writeln(SMALL_CLASS_TXT);
FOR z:=1 TO smallsize DO
  BEGIN
    write(SMALL_CLASS_TXT,'(',small[z,1]:2,',')
    write(SMALL_CLASS_TXT,small[z,2]:2,',')
    write(SMALL_CLASS_TXT,small[z,3]:2,')');
    END;
    IF z MOD 7 = 0 THEN writeln(SMALL_CLASS_TXT); {to the file}
writeln(SMALL_CLASS_TXT);
**Output from program first_level.pa**

File CLASSC1_K_TXT:

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**Class size:** 315
There are 3 constants $c(i,j,k;0,1)$: 1, 2, 0.

Triples from the class $C^{-1}_k$ corresponding to $c(i,j,k;0,1) = 0$:

( 2,12, 4) ( 2,14, 3) ( 2,15, 6) ( 3, 9, 7) ( 3,13, 5) ( 3,15, 4) ( 4, 9, 5)
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( 6,12, 3) ( 7,10, 2) ( 7,12, 8) ( 7,13, 4) ( 8,11, 3) ( 8,13, 2) ( 8,14, 5)