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Oscillatory solutions in conservation laws related to nonexistence of weak self-similar Riemann solutions

George Robert Peters
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Oscillatory solutions in conservation laws related to nonexistence of weak self-similar Riemann solutions

by

George Robert Peters

A dissertation submitted to the graduate faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Major Professors: Sunčica Čanić and Dragan Mirković

Iowa State University

Ames, Iowa

1997

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For the Graduate College
To Linda
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ABSTRACT

We study Riemann solutions of inviscid systems of conservation laws obtained as a viscous limit of an associated parabolic system. This limit depends on the positive definite viscosity matrix. Specifically, we consider Riemann problems with shock initial data, i.e., the initial data for which the right and left states correspond to a Lax admissible shock. We are particularly interested in what happens with a Riemann solution if this shock does not admit a viscous profile due to the presence of a Hopf bifurcation and limit cycles in the dynamical system associated to the viscous entropy criterion.

We focus our study on two classes of models: the shallow water equations and a three-phase flow model arising in petroleum engineering. For these models with Riemann data in the strictly hyperbolic region, it is proved that there exists no weak self-similar Riemann solution. Instead, numerical simulations provide solutions exhibiting continuously generated oscillations. We prove that the limit of these oscillatory solutions, as the viscosity goes to zero, satisfies the system of conservation laws in a measure-valued sense. We conjecture that in the three-phase flow model this solutions corresponds to interspersing of different phases.
1 INTRODUCTION

A Riemann problem is an initial value problem for a system of conservation laws with initial data that consists of two constant states connected with a jump discontinuity. These problems are very important in more general Cauchy (initial-value) problems. This is due to the important work done by James Glimm where he constructed solutions to general initial-value problems using Riemann problem solutions. See Ref. [14].

Solutions of Cauchy problems for conservation laws typically exhibit discontinuities. More specifically they contain shocks, which are defined as jump discontinuities traveling at a specified speed. To allow for nondifferentiability, the conservation law is rewritten in the "weak form" or integral form. Standard weak self-similar solutions of Riemann problems consist of constant states, jump discontinuities (shock waves), and continuously changing parts (rarefaction waves). The change to weak form takes care of the problem of discontinuities, but introduces multiple solutions. To distinguish between shocks we consider physical and those we do not we introduce admissibility or entropy criteria. The following two entropy criteria are best known. The first is an entropy criterion introduced by Peter Lax, based on stability, and called the Lax admissibility condition. See Refs. [18, 23]. The condition that we use in this thesis is the viscosity admissibility condition, introduced by I.M. Gelfand. See Ref. [13]. This condition reintroduces some of the physics that was ignored when dissipation was neglected in the derivation of the conservation law. We consider the parabolic problem obtained by adding a dissipative term specified by the positive definite matrix $D(U)$.

$$U_t + F(U)_x = \epsilon(D(U)U_x)_x.$$
A shock is considered admissible if it is the limit of traveling wave solutions of the parabolic problem as $\varepsilon$ approaches zero. The matrix $D(U)$ has typically been chosen to be the identity matrix. This certainly simplifies the analysis, but it is often not a realistic choice and can have a large effect on solutions that we consider admissible.

In addition to the classical shock and rarefactions waves, there are transitional waves. The admissibility of transitional waves depends sensitively on the form of $D$. See, for example, Refs. [17, 15].

The central question of this work concerns the behavior of Riemann solutions in cases where shock waves that appear in the solution do not admit a viscous profile for a physical diffusion $D$. To answer this question we consider Riemann problems with shock initial data, i.e., the data for which our initial left and right states correspond to the left and right states in a shock wave which satisfies the Lax admissibility condition. We are particularly interested in what happens with a Riemann solution if the shock is not viscous admissible.

In this thesis we show that in the shallow water equations and in a three-phase flow model, there is initial data for which it is not possible to construct a weak self-similar Riemann solution. This nonexistence arises naturally for certain diffusion matrices $D$ in models which do not allow transitional shock waves (e.g., gas dynamics, the shallow water equations, etc.). In models which allow transitional waves (e.g., three phase flow in porous media, reactive gas dynamics, etc.), it is often possible to use these waves to construct an alternative solution. In fact, transitional waves are known to cause nonuniqueness of solutions. See Refs. [2, 3, 6]. However, there are situations in which there is nonexistence, despite the potential for transitional waves. See Ref. [5]. So in one of these instances of nonexistence, what does the solution of the parabolic problem

$$U_t + F(U)_x = \varepsilon DU_{xx} \quad (1.1)$$

look like as $\varepsilon$ goes to zero?
To investigate this, we numerically solved (1.1) for different values of $\epsilon$ using a linearized Crank-Nicolson method. We found that the solutions exhibit persistent and stable oscillations of the state variables that increase in frequency as $\epsilon$ decreases. We proved that a sequence of solutions $\{U^\epsilon, \epsilon \to 0\}$ converges (in the weak-* sense of $L^\infty$) to a measure-valued solution of the hyperbolic system. In all of the cases we study, the left and right states lie in the strictly hyperbolic region, and the diffusion matrix is positive definite.

Hermano Frid and I-Shih Liu have also worked with oscillation and measure-valued solutions in conservation laws. See Refs. [11, 12, 20]. The most relevant of their articles to this thesis is [11]. In contrast to this thesis Frid and Liu in [11] solve a Riemann problem with initial data lying in the elliptic region of state space and not the strictly hyperbolic region. Riemann problems with initial data in the elliptic region are quite different than those with initial data in the strictly hyperbolic part. In addition, they solve the system of conservation laws with a first order method instead of solving the parabolic system. This results in their dissipation coming from the numerics, rather that being intentionally selected.

This thesis is organized as follows. In Chapter 2 we introduce conservation laws, scale-invariant solutions, and shock admissibility criteria. The shallow water equations and a model of three-phase flow in porous media are derived in Chapter 3. Non-existence of standard weak solutions coming from these models is discussed in Chapter 4.

Measure-valued solutions are introduced in Chapter 5. These solutions arise in the limit of the numerical solutions. The numerical method used and the weak-* convergence of the solutions is covered in Chapter 6. This chapter also deals with the expectation values of the measure-valued solutions. In Chapter 7 we show some numerical results obtained for the specific examples introduced in Chapter 4. The final chapter contains the conclusions and is followed by three Appendices. The first discusses the Majda-Pego region. The second shows and discusses some numerical results using the Lax-Friedrichs
and Lax-Wendroff methods. Finally, the third shows in detail certain functions used in the three-phase flow example.
2 CONSERVATION LAWS

We consider a system of conservation laws in one space dimension

\[ U_t + F(U)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \]  

(2.1)

where the state variable \( U \) lies in the state space \( U \subseteq \mathbb{R}^2 \), and the flux \( F : U \to \mathbb{R}^2 \) is continuously differentiable. In this thesis we will restrict our attention to the initial value problem, or Cauchy problem, where \( U \) at time zero is specified for all \( x \),

\[ U(x, 0) = U_0(x), \quad -\infty < x < \infty. \]  

(2.2)

These equations are termed conservation laws because equations modeling conservation of mass, momentum, and energy can be put into this form. The state variable \( U \) is a density function of the property being conserved. The typically nonlinear \( F \) is the flux of \( U \). Due to the nonlinearity of \( F \), these equations have a tendency to form discontinuities in finite time even when starting with smooth initial data. To study the propagation of discontinuities we often consider problems with Riemann initial conditions,

\[ U(x, 0) = \begin{cases} 
U_L, & x < 0 \\
U_R, & x > 0 
\end{cases}. \]  

(2.3)

A system of conservation laws is said to be hyperbolic if the eigenvalues of the Jacobian of \( F \) are real, and titled strictly hyperbolic if those eigenvalues are also distinct. The set of points in the state space where the eigenvalues of \( F \) are real and distinct is called the region of strict hyperbolicity. The set of points where the eigenvalues are complex conjugate is called the elliptic region. All of the Riemann problems, i.e., conservation
law systems with Riemann initial conditions, we solve in this thesis have $U_L$ and $U_R$ in the region of strict hyperbolicity.

**Weak Solutions**

Obviously, when jump discontinuities exist, we lose differentiability and the form of system (2.1) is no longer valid. To remedy this situation we consider a "weak form" of the equations. Multiplying through by a test function and integrating by parts yields the weak form

$$\int_0^\infty \int_{-\infty}^{\infty} U_t \phi_t + F(U)\phi_x dx dt + \int_{-\infty}^{\infty} U_0(x)\phi(x,0) dx = 0, \quad (2.4)$$

where $\phi(x,t) \in C_0^\infty$, the set of all real-valued infinitely differentiable functions of compact support on $\mathbb{R} \times [0,\infty)$.

**DEFINITION 2.1** The function $U(x,t)$ is called a weak solution of the Cauchy problem given by equations (2.1) and (2.2), if $U(x,t)$ satisfies equation (2.4) for all real-valued $\phi(x,t) \in C_0^\infty(\mathbb{R} \times [0,\infty))$.

Weak solutions of Riemann problems are of the form $U(x,t) = \hat{U}(x/t)$ consisting of constant states, jump discontinuities (shock waves), and continuously changing components (rarefaction waves). The reason for seeking scale-invariant solutions is the fact that if $U(x,t)$ is a weak solution of a Riemann problem then $U^\alpha(x,t) = U(\alpha x, \alpha t)$ is also a solution for any $\alpha > 0$.

**Shock Waves**

Let $U$ be a weak solution to the Cauchy problem (2.1), (2.2) that contains a jump discontinuity. Assume that this discontinuity is a curve in $x,t$-space given by $\hat{x}(t)$, and that $U$ is smooth away from $\hat{x}(t)$. Let $D$ be a bounded region containing part of this discontinuity, but not containing any of the line $t = 0$, and let $\phi$ be a test function with
support strictly contained in $D$. We define $D_1$ as the portion of $D$ to the left of the discontinuity and $D_2$ as the rest. See Figure 2.1. Since $U$ is a weak solution,

$$0 = \int \int_D U \phi_t + F(U) \phi_x dx dt$$

$$= \int \int_{D_1} U \phi_t + F(U) \phi_x dx dt + \int \int_{D_2} U \phi_t + F(U) \phi_x dx dt. \quad (2.5)$$

We will now concentrate on the $D_1$ part. By Green's theorem we know that

$$\int \int_{D_1} (U \phi_t + F(U) \phi_x) dx dt = \int_{\partial D_1} F(U) \phi t - U \phi dx. \quad (2.6)$$

It is also true that

$$\int \int_{D_1} (U \phi_t + (F(U) \phi)_x) dx dt =$$

$$\int \int_{D_1} U \phi_t + U \phi_t + F(U) \phi_x + F(U) \phi_x dx dt. \quad (2.7)$$

Since $U$ is smooth over $D_1$, the first term plus the third term of the right hand side of this equation equals zero, which implies that (2.6) and (2.7) yield

$$\int \int_{D_1} U \phi_t + F(U) \phi_x dx dt = \int_{\partial D_1} F(U) \phi t - U \phi dx. \quad (2.8)$$
Now define \( U \) on the left side of \( \dot{x}(t) \) to be \( U_- \), and \( U \) on the right to be \( U_+ \). Also define \( P_1 \) and \( P_2 \) to be the left and right intersection points respectively of \( \dot{x}(t) \) with \( \partial D \). Since \( \phi \)'s support is contained within \( \partial D \),

\[
\int_{\partial D_1} F(U) \phi dt - U \phi dx = \int_{P_1}^{P_2} F(U_-) \phi dt - U_- \phi d\bar{x}.
\]

(2.9)

Similarly, it follows that

\[
\int_{\partial D_2} F(U) \phi dt - U \phi dx = -\int_{P_1}^{P_2} F(U_+) \phi dt - U_+ \phi d\bar{x}.
\]

(2.10)

By equation (2.5), we have

\[
0 = \int_{P_1}^{P_2} (F(U_-) - F(U_+)) \phi dt + (U_+ - U_-) \phi d\bar{x}
\]

\[
= \int_{t_1}^{t_2} \phi \left( (F(U_-) - F(U_+)) + (U_+ - U_-) \frac{d\bar{x}}{dt} \right) dt
\]

(2.11)

from which it follows that

\[
(F(U_-) - F(U_+)) + (U_+ - U_-) \frac{d\bar{x}}{dt} = 0
\]

(2.12)

since \( \phi \) is arbitrary. Therefore a jump discontinuity between two constant states \( U_- \) and \( U_+ \) traveling at speed \( s_0 \), \( (U_-, U_+, s_0) \), satisfies the weak equation if the Rankine-Hugoniot condition,

\[
s(U_+ - U_-) = F(U_+) - F(U_-),
\]

(2.13)

holds.

With a fixed \( U_- \), this is a system of 2 equations in 3 unknowns. The solutions consist of a one-parameter family of curves in state space. For a fixed left state \( U_- \), the set of right states that satisfy equation (2.13) for some \( s \) is called the Hugoniot locus of \( U_- \). If we denote the parameter by \( \zeta \), our system is

\[
s(\zeta)(U(\zeta) - U_-) = F(U(\zeta)) - F(U_-).
\]

(2.14)
Since $U_-$ is in the strictly hyperbolic region, locally there are 2 distinct curves in the Rankine-Hugoniot locus, each tangent at $U_-$ to an eigenvector of $F'(U_-)$. See Ref. [23]. This is suggested by the following calculation. Let $\zeta_0$ be a value of $\zeta$ such that $U(\zeta_0) = U_-$. Taking the derivative of both sides of equation (2.14) with respect to $\zeta$ gives

$$\dot{s}(\zeta)(U(\zeta) - U_-) + s(\zeta)\dot{U}(\zeta) = F'(U(\zeta))\dot{U}(\zeta).$$  \hspace{1cm} (2.15)$$

Evaluating this at $\zeta_0$ then produces

$$F'(U_-)\dot{U}(\zeta_0) = s(\zeta_0)\dot{U}(\zeta_0).$$  \hspace{1cm} (2.16)$$

This indicates that at $\zeta_0$, $\dot{U}$ is an eigenvector of $F'(U_-)$ and $s(\zeta_0)$ is the corresponding eigenvalue. Note that since $F'(U_-)$ has two distinct eigenvalues, it also has two distinct eigenvectors.

**Rarefaction Waves**

Rarefaction waves are smoothly changing scale-invariant solutions to Riemann problems. Consider a solution of the form $U(x, t) = \hat{U}(\xi)$, where $\xi \equiv x/t$. This solution must satisfy the conservation law (2.1), which can be rewritten as

$$\ddot{U}(\xi)(-x/t^2) + F'(\hat{U})\dot{U}'(\xi)(1/t) = 0, \text{ or } (2.17)$$

$$\ddot{U}(\xi)(-\xi) + F'(\hat{U})\dot{U}'(\xi) = 0, \hspace{1cm} (2.18)$$

where $\hat{U}'(\xi)$ denotes the derivative with respect to $\xi$. It then follows that either

1. $\hat{U}''(\xi) = 0$ ($U$ is a constant solution), or

2. $\xi = \lambda_i(\hat{U}(\xi))$ and $\hat{U}'(\xi) = \alpha(\xi)\vec{r}_i(\hat{U}(\xi))$, where $\lambda_i$ and $\vec{r}_i$ are the eigenvalues and eigenvectors of $F'$

is true. In the second case, we have a system of ordinary differential equations specified by

$$\hat{U}'(\xi) = \alpha(\xi)\vec{r}_i(\hat{U}(\xi)).$$  \hspace{1cm} (2.19)$$
If $F$ is smooth, which it is the cases we consider, then to have a unique solution to this differential equation we need to specify an initial condition. To this end let $\dot{U}(\xi_L) = U_L$.

We determine $\alpha$ from the following calculation:

$$
\xi = \lambda_i(\dot{U}(\xi)) \text{ differentiated with respect to } \xi \text{ gives,}
1 = \nabla \lambda_i(\dot{U}(\xi)) \cdot \dot{U}'(\xi),
1 = \nabla \lambda_i(\dot{U}(\xi)) \cdot \alpha(\xi) \mathcal{F}_i(\dot{U}(\xi)),
\alpha(\xi) = 1/\left[\nabla \lambda_i(\dot{U}(\xi)) \cdot \mathcal{F}_i(\dot{U}(\xi))\right]. \tag{2.20}
$$

Clearly for $\alpha$ to be well determined we need the denominator on the right hand side of equation (2.20) to be nonzero.

**Definition 2.2** The system (2.1) is said to be genuinely nonlinear if $\nabla \lambda_i(U) \cdot \mathcal{F}_i(U) \neq 0$ for any $U$ in state space.

A point $U$ in state space is said to be a point of genuine nonlinearity if $\nabla \lambda_i(U) \cdot \mathcal{F}_i(U) \neq 0$.

**Definition 2.3** An $i$-rarefaction wave is a continuous solution of the form

$$
U(x,t) = \begin{cases} 
U_L & x/t \leq \lambda_i(U_L), \\
\dot{U}(x/t) & \lambda_i(U_L) < x/t < \lambda_i(U_R), \\
U_R & x/t \geq \lambda_i(U_R),
\end{cases} \tag{2.21}
$$

where $\dot{U}(x/t)$ satisfies:

$$
\begin{align*}
\dot{U}'(\xi) &= \frac{1}{\nabla \lambda_i(U) \cdot \mathcal{F}_i(\dot{U})} \mathcal{F}_i(\dot{U}), \quad \xi_L < \xi < \xi_R, \\
\dot{U}(\xi_L) &= U_L,
\end{align*} \tag{2.22}
$$

$\xi = \lambda(\dot{U})$ increases monotonically from $\xi_L$ to $\xi_R$, and if genuine nonlinearity holds at all $\ddot{U}$.

The function $\lambda$ is monotonic along the solution due to the genuine nonlinearity. As mentioned before there are also transitional rarefaction waves.

**Definition 2.4** A transitional rarefaction wave is a rarefaction wave that changes from a faster family to a slower family.

See Ref. [17]. This cannot happen in a strictly hyperbolic region since it is necessary that $\lambda_{i+1} = \lambda_i$ at the point where the families change.
Composite Waves

An i-composite wave is an i-shock followed immediately by an i-rarefaction or an i-rarefaction followed by an i-shock. A shock followed by a rarefaction of the same family occurs when the speed of an i-shock ending at a point $U^*$ is equal to the characteristic speed $\lambda_i(U^*)$. A rarefaction followed by a shock of the same family can occur when genuine nonlinearity breaks down. The i-rarefaction wave ends at a point $U^*$ with a shock of speed $\lambda_i(U^*)$.

Admissibility Criteria

Changing to the weak form allows discontinuities in the solution, but also allows for multiple solutions. To distinguish between shocks we consider 'physically reasonable' and those we do not, we apply admissibility conditions.

Viscosity Admissibility

The viscosity admissibility criterion attempts to recover some of the physics behind the problem by considering the parabolic system

$$U_t^* + F(U^*)_x = \epsilon(D(U^*)U^*_x)_x,$$

where $D(U)$ denotes a diffusion matrix or dissipation matrix. In this work we shall consider $D(U) = D$ to be a constant positive definite matrix. To determine whether a shock $(U_-, U_+, s)$ is admissible or not, we look for traveling wave solutions $U^*(\zeta) = U^*(\zeta + s)$ of equation (2.23), such that $\lim_{\zeta \to -\infty} U^*(\zeta) = U_-$, and $\lim_{\zeta \to \infty} U^*(\zeta) = U_+$. We substitute $U^*$ into (2.23) yielding

$$-\frac{s}{\epsilon} \frac{d}{d\zeta} U^* + \frac{1}{\epsilon} \frac{d}{d\zeta} F(U^*) = \epsilon D(U) \frac{1}{\epsilon^2} \frac{d^2}{d\zeta^2} U^*. \quad (2.24)$$

Multiplying through by $\epsilon$ and integrating from $-\infty$ to $\zeta$ gives

$$\frac{d}{d\zeta} U^* = D^{-1}(U)[-s(U^* - U_-) + F(U^*) - F(U_-)]. \quad (2.25)$$
Equation (2.25) is a three-parameter family of dynamical systems with the parameters \( U_- \in \mathbb{R}^2, \ s \in \mathbb{R} \). A critical point or fixed point of a dynamical system \( U' = X(U') \) is a point where \( X(U') = 0 \). So for a fixed \( U_- \), a point \( U_+ \) satisfying the Rankine-Hugoniot condition is a critical point of the dynamical system (2.25). Notice that \( U_- \) is also a critical point. Hyperbolic critical points, defined as critical points where the eigenvalues of \( X' \) have nonzero real parts, are separated into several categories. A hyperbolic fixed point, \( U_- \), is: a node if the eigenvalues are real and of the same sign, a saddle if the eigenvalues are real and of opposite sign, and a spiral if the eigenvalues are complex conjugate. A node or a spiral is an attractor if the real parts of the eigenvalues are negative, and a repellor if the real parts of the eigenvalues are positive. A shock \((U_-, U_+, s)\) is admissible if there exists a connecting orbit from \( U_- \) to \( U_+ \) in the dynamical system (2.25).

**Definition 2.5** A shock \((U_-, U_+, s)\) admits a viscous profile if there exists a connecting orbit in the dynamical system (2.25) traversed in the direction from \( U_- \) to \( U_+ \).

If \((U_-, U_+, s)\) admits a viscous profile and \( U_- \) and \( U_+ \) are hyperbolic fixed points, then an admissible shock wave is one of the four following types:

- a 1-wave, which corresponds to a repellor, \( U_- \), to saddle, \( U_+ \), connecting orbit.
- a 2-wave, which corresponds to a saddle, \( U_- \), to attractor, \( U_+ \), connecting orbit.
- a transitional wave, which corresponds to a saddle to saddle connecting orbit.
- an overcompressive wave, which corresponds to a repellor, \( U_- \), to attractor, \( U_+ \), connecting orbit.

**Lax Admissibility**

Another common criterion used is the Lax admissibility condition. This condition requires that both of the characteristics of one and only one family approach the shock.
Therefore, a Lax 1-shock must have

\[ s < \lambda_1(U_-), \quad \lambda_1(U_+) < s < \lambda_2(U_+), \]

(2.26)

and a Lax 2-shock

\[ s > \lambda_2(U_+), \quad \lambda_1(U_-) < s < \lambda_2(U_-), \]

(2.27)

where \( \lambda_1(U) \) and \( \lambda_2(U) \) are the eigenvalues of \( F'(U) \) numbered so that \( \lambda_1(U) < \lambda_2(U) \).

There is a relationship between the viscosity admissibility condition and the Lax admissibility condition. If we assume that \( D = I \), then the type of fixed points of the dynamical system (2.25) is given by the eigenvalues of \( F'(U_{cr}) - sI \). These eigenvalues are \( \lambda_1(U) - s \) and \( \lambda_2(U) - s \). If the inequalities in (2.26) hold, then \( U_- \) is a repellor and \( U_+ \) is a saddle point. If the inequalities in (2.27) hold, then \( U_- \) is a saddle point and \( U_+ \) is an attractor. However, this does not imply that there is a connecting orbit between the two critical points, and indeed this is often the case. So even with \( D = I \) there are Lax admissible shocks that are not viscous admissible. However, viscous admissibility for \( D = I \) and Lax admissibility are locally equivalent. See Ref. [9].

**Constructing Riemann Solutions**

A wave group is a collection of waves of the same family, shock waves, rarefaction waves, or composite waves, occurring in a Riemann solution in increasing speed. Riemann solutions consist of a series of wave groups of increasing speed separated by constant states and joining \( U_L \) to \( U_R \). For example, a solution may consist of a 1-wave group, followed by a 2-wave group, etc. However, not all wave groups need to be present.

To construct Riemann solutions we use wave curve analysis. The \( i \)-shock curve through \( U_L \) is the set of states \( U \) that can be connected to \( U_L \) by an admissible \( i \)-shock. Locally, with the Lax admissibility criterion, this is half of the \( i \)-branch of the Rankine-Hugoniot curve through \( U_L \). The \( i \)-rarefaction curve through \( U_L \) is the set of states \( U' \).
that can be connected to $U_L$ by an $i$-rarefaction wave. Locally this is another half-curve. These two curves meet and are tangent at $U_L$. Together these two curves form the $i$-wave curve, or the forward $i$-wave curve. The wave curve can be extended globally by allowing composite waves. A backward $i$-wave curve through a point $U_R$ is the set of states $U$ such that $U_R$ lies on the forward $i$-wave curve through $U$.

To solve a Riemann problem, the forward $1$-wave curve through $U_L$ is constructed. If $U_R$ lies on this curve then the solution is just a single $1$-shock, $1$-rarefaction, or a $1$-composite wave. If not, then the backward $2$-wave curve through $U_R$ is constructed. If there exists an intersection, $U_M$, of these two curves, the solution consists of a $1$-wave from $U_L$ to $U_M$ followed by a $2$-wave from $U_M$ to $U_R$. If $U_M = U_L$, that is $U_L$ lies on the backward $2$-wave curve through $U_R$, then the solution is a single $2$-wave from $U_L$ to $U_R$. It is important that we have $1$-waves followed by $2$-waves, since wave speeds must increase. The speeds of the waves must increase, so it is impossible to have a $2$-wave followed by a $1$-wave, except in the rare case of a transitional rarefaction wave.

In addition to $1$ and $2$-shock waves, $1$ and $2$-rarefaction waves, and $1$ and $2$-composite waves, Riemann solutions may contain non-classical waves, such as transitional or under-compressive shock waves, transitional rarefaction waves, and/or overcompressive shock waves. Transitional shocks and transitional rarefactions do not belong to a specific family. Transitional shock waves can occur before, after, or between $1$-waves and $2$-waves. The speeds of the waves still have to be increasing order. Overcompressive waves must always occur at the end of the wave sequence.
3 MODELS

In this thesis we focus our study on two models: the shallow water equations and a three-phase flow model arising in petroleum engineering. We will show that for certain positive definite viscosity matrices and Riemann initial data, there is no Riemann solution in which all shocks satisfy the viscosity entropy criterion. We will then show that in these situations, oscillatory solutions exist which satisfy the system of conservation laws in a measure-valued sense.

In this chapter we present the derivation of the model equations.

Shallow Water Equations

Water flowing in a narrow trough can be modeled by a system of differential equations with one space dimension. The two variables are height \( h(x, t) \) and velocity \( v(x, t) \). The velocity is assumed to be constant along vertical slices of water. The width of the trough and the density of the water, \( \rho \), are assumed to be constant. The width then divides out of the following conservation equations. The mass of the water in the interval \([x_1, x_2]\) at time \( t \) is

\[
\int_{x_1}^{x_2} \rho h(x, t) dx
\]  

(3.1)

The rate of change of that mass is given by

\[
\frac{d}{dt} \int_{x_1}^{x_2} \rho h(x, t) dx = \rho h(x_1, t)v(x_1, t) - \rho h(x_2, t)v(x_2, t).
\]  

(3.2)
Integrating both sides from $t_1$ to $t_2$ gives
\[ \int_{x_1}^{x_2} \rho h(x, t_2)dx - \int_{x_1}^{x_2} \rho h(x, t_1)dx = \int_{t_1}^{t_2} (\rho h(x_1, t)v(x_1, t) \]
\[ \quad - \rho h(x_2, t)v(x_2, t))dt. \]  
(3.3)

Assuming differentiability of $h$ and of $hv$ allows us to write
\[ \int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{d}{dt} \rho h(x, t)dx dt = -\int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{d}{dx} (\rho h(x, t)v(x, t))dx dt. \]  
(3.4)

Since this must be true for all $x_1, x_2, t_1,$ and $t_2,$ we obtain the differential form of the mass continuity equation,
\[ h_t + (hv)_x = 0. \]  
(3.5)

The conservation of momentum equation comes from Newton's Second Law, which states that the rate of change of momentum = force. Considering a control volume $[x_1, x_2],$ the left side of this equation is the rate of change of the internal momentum plus the momentum flux across the boundaries, and the right side is given by pressure $p,$ at the boundaries. Therefore,
\[ \frac{d}{dt} \int_{x_1}^{x_2} \rho h(x, t)v(x, t)dx + \rho h(x_2, t)v^2(x_2, t) - \rho h(x_1, t)v^2(x_1, t) \]
\[ = -p(x_2) + p(x_1, t) \]  
(3.6)

As in the conservation of mass, we obtain the differential form of the momentum equation,
\[ (\rho hv)_t + (\rho hv^2 + p)_x = 0. \]  
(3.7)

The pressure comes from a hydrostatic law stating that at depth $y$ the pressure is $\rho gy.$ with $g$ the gravitational constant. The total pressure at each boundary is found by integrating $y$ from 0 to $h(x, t).$ This gives $p = \frac{1}{2}\rho gh^2.$ Substituting this into (3.7) and dividing by $\rho$ gives
\[ (hv)_t + (hv^2 + \frac{1}{2}gh^2)_x = 0. \]  
(3.8)
Expanding the derivatives and using $h_t = - (h_x v + h v_x)$ from (3.5) gives

$$-v(h_x v + h v_x) + v_t h + (h_x v^2 + 2 h v v_x + g h h_x) = 0 \quad (3.9)$$

Canceling terms and dividing by $h$ we have

$$v_t + (v^2/2 + g h)_x = 0. \quad (3.10)$$

Finally, let $\phi = gh$. Equation (3.5) implies the shallow water equations of the form.

$$\begin{bmatrix} v \\ \phi \end{bmatrix}_t + \begin{bmatrix} v^2/2 + \phi \\ v \phi \end{bmatrix}_x = 0. \quad (3.11)$$

**Three-phase Flow Model**

We are concerned with three-phase Buckley-Leverett flow. See Ref. [1]. The three phases are gas, water, and oil. For simplicity we assume that rock porosity, $\phi$, rock permeability, $\kappa$, and the mass densities of the three phases, $\rho_i$ ($i = 1, 2, 3$), are constant. In addition, we assume that the phase viscosities, $\mu_i$, are constant.

The saturation of a phase, $s_i$, is defined to be the ratio of the volume of that phase to the pore volume. Assuming that all pores are filled with our three phases gives

$$s_1 + s_2 + s_3 = 1. \quad (3.12)$$

Let us define $s_1$ to be the saturation of gas, $s_2$ the saturation of water, and $s_3$ the saturation of oil. We then define the other subscripted variables to correspond accordingly (e.g. $\mu_2$ is the viscosity of water).

Working in one space dimension, we write the conservation of gas, water, and oil as

$$\begin{align*}
(p_1 s_1 \phi)_t + (p_1 v_1)_x &= 0, \\
(p_2 s_2 \phi)_t + (p_2 v_2)_x &= 0, \\
(p_3 s_3 \phi)_t + (p_3 v_3)_x &= 0.
\end{align*} \quad (3.13)$$
where \( v_i \), the \( i \)th velocity, is the volume of the \( i \)th phase moving across a unit of area in a unit of time.

The conservation of momentum comes from Darcy's Law and is given by

\[
v_i = -\lambda_i \frac{\partial p}{\partial x}, \quad (i = 1, 2, 3),
\]

where \( p \) is pressure, \( \lambda_i \) is the phase mobility of the \( i \)th phase, and \( \kappa \) is the rock permeability. This phase mobility is equal to the relative permeability \( k_i \) divided by the corresponding viscosity \( \mu_i \). The choice of relative permeability functions will be discussed later.

Dividing each of the equations in (3.13) by its corresponding density and adding yields

\[
\frac{\partial (v_1 + v_2 + v_3)}{\partial x} = 0.
\]

Therefore, \( v_T = v_1 + v_2 + v_3 \) is a function of time alone. Summing the momentum equations (3.14), solving for \( p_x \), and substituting back into each of those equations yields

\[
v_i = \frac{-\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} v_T,
\]

for \( i = 1, 2, 3 \).

Substitution into the first two equations of (3.13) followed by division by \( \phi \) and corresponding densities produces

\[
(s_1)_t + \left( \frac{-\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{v_T}{\phi} \right)_x = 0,
\]

\[
(s_2)_t + \left( \frac{-\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \frac{v_T}{\phi} \right)_x = 0.
\]

The third saturation is found by (3.12). Assuming that the total velocity is constant and letting \( \dot{t} = t \frac{v_T}{\phi} \) gives our system of conservation laws

\[
(s_1)_{\dot{t}} + \left( \frac{-\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \right)_x = 0,
\]

\[
(s_2)_{\dot{t}} + \left( \frac{-\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \right)_x = 0.
\]
The most common diffusive force used for this system comes from capillary pressure. Capillary pressure is defined as the difference between the pressures in two different phases in the pores. Let the capillary pressures be given by

\[ p_{c1} \equiv p_1 - p_3 \]
\[ p_{c2} \equiv p_2 - p_3 \]

where \( p_i \) is the pressure in the \( i \)th phase. We make the typical assumption that \( p_{c1} \) is an increasing function of \( s_1 \) and \( p_{c2} \) is an increasing function of \( s_2 \). See Ref. [1]. This changes our conservation of momentum equations to

\[
\frac{d}{dx} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} -\lambda_1 \left( \frac{\partial p_3}{\partial x} + \frac{\partial p_{c1}}{\partial x} \right) \\ -\lambda_2 \left( \frac{\partial p_3}{\partial x} + \frac{\partial p_{c2}}{\partial x} \right) \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix} \frac{d}{dx} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} - \begin{bmatrix} \frac{\partial \kappa}{\partial x} \\ \frac{\partial \kappa}{\partial x} \end{bmatrix} \kappa.
\]

Proceeding as before gives the dissipative system

\[
\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} -\lambda_1 \frac{s_1}{\lambda_1 + \lambda_2 + \lambda_3} \\ -\lambda_2 \frac{s_2}{\lambda_1 + \lambda_2 + \lambda_3} \end{bmatrix} = \begin{bmatrix} \kappa \\ \kappa \end{bmatrix} \frac{d}{dx} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix},
\]

where the capillary diffusion matrix is given by

\[ D \equiv \begin{bmatrix} \lambda_1 (\lambda_3 + \lambda_2) \frac{\partial p_{c1}}{\partial x} & -\lambda_1 \lambda_2 \frac{\partial p_{c2}}{\partial x} \\ -\lambda_1 \lambda_2 \frac{\partial p_{c1}}{\partial x} & \lambda_2 (\lambda_3 + \lambda_1) \frac{\partial p_{c2}}{\partial x} \end{bmatrix} \frac{\kappa}{\lambda_1 + \lambda_2 + \lambda_3}. \]

See Ref. [1].

Following Ref. [22] we define our viscosity and our relative permeabilities by assuming that

\[
\mu_1 = \mu_2 = \mu_3 = 1, \text{ and that } \kappa_i = s_i^2 \text{ for } (i = 1, 2, 3).
\]
Replacing $\ell$ by $t$ and $s_3$ by $1 - s_1 - s_2$ in the system (3.18) gives

$$
(s_1)_t + \left( \frac{-s_1^2}{s_1^2 + s_2^2 + (1 - s_1 - s_2)^2} \right)_x = 0,
$$

and

$$
(s_2)_t + \left( \frac{-s_2^2}{s_1^2 + s_2^2 + (1 - s_1 - s_2)^2} \right)_x = 0,
$$

which is the Corey-Pope-Marchesin model. This system has a single point where hyperbolicity fails, called the umbilic point, at $s_1 = s_2 = 1/3$. To avoid the non-generic situation of the single umbilic point, we perturb the model to have, instead, a small elliptic region "centered" around $(1/3, 1/3)$. To produce a model with a nontrivial elliptic region we need only to add some appropriate linear terms to the flux in system (3.24). It is not clear, however, that this makes any physical sense. Instead, we modify the relative permeabilities in such a way to give the same result. We let

$$
\kappa_1 = s_1^2 + a_1(c, s_1, s_2) \quad \text{and} \quad \kappa_2 = s_2^2 + a_2(c, s_1, s_2),
$$

where $c$ is a measure of the size of the elliptic region, and $a_1$ as well as $a_2$ approach zero uniformly as $c \to 0$. The functions $a_1$ and $a_2$ are selected so that the flux of the resulting system is the same as the flux of (3.24) plus an appropriate linear term. The functions $a_1$ and $a_2$ are given explicitly in Appendix C. With $a_1$ and $a_2$ small, we have not altered our model significantly. We selected our modifications to the relative permeabilities so that the system with an elliptic region has the form

$$
U_t + \left( G(U) + M \begin{bmatrix} 0 & 2c \\ -2c & 0 \end{bmatrix} M^{-1} U \right)_x = 0
$$

where $U \equiv [s_1, s_2]^T$, $M$ is an invertible matrix whose purpose will be seen shortly, and $G$ is the flux function from system (3.24).

To simplify the analysis we consider the quadratic flux function obtained by the expansion of the flux from equation (3.26) about the center of the elliptic region. The resulting quadratic model is then put into \textit{elliptic normal form} by using the equivalence transformations listed below.
DEFINITION 3.1 A quadratic model with a flux function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is in the elliptic normal form if the Jacobian of $F$ equals

$$F'(U, a, b, c) = \begin{bmatrix} au + bv & bu + v + c \\ bu + v - c & u \end{bmatrix}, \ a \neq 1 + b^2, \quad (3.27)$$

where $U = [u, v]^T$, and $a, b, \text{and } c > 0$ are constants.

Any quadratic model with a bounded elliptic region can be put into this form by two equivalence transformations. See Ref. [7]. The first is a change of independent variable

$$\dot{t} = t, \quad (3.28)$$
$$\dot{x} = x - st, \quad (3.29)$$

and the second is a linear transformation of $U$

$$\dot{U} = MU + U_0, \quad (3.30)$$

where $U_0$ is a constant vector.

We first expand the flux function of equation (3.26) about $[1/3, 1/3]^T$, and drop all cubic and higher order terms, with

$$M = \begin{bmatrix} -2/3 & 0 \\ 1/3 & -\sqrt{3}/3 \end{bmatrix}, \ U_0 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}, \ \text{and } s = 2. \quad (3.31)$$

Applying the above transformations gives a quadratic system in normal form,

$$\dot{u} + \begin{bmatrix} -\dot{u}^2 + \dot{v}^2 + 2cv \\ 2\dot{u}\dot{v} - 2c\dot{u} \end{bmatrix} = 0. \quad (3.32)$$

Note that this system has an elliptic region centered at $[0, 0]^T$ with a radius of $c$. 
4 NON-EXISTENCE OF SOLUTIONS

Introduction

In this chapter we study specific Riemann problems for the models discussed in the previous chapter. For these examples we show that a scale-invariant weak solution consisting of constant states, rarefaction waves, and viscous admissible shock waves, does not exist. This will be shown explicitly for the shallow water example. The three-phase flow nonexistence proof is the topic a paper by Čanić. See Ref. [5]. Some basic ideas of the proof for the second example will be discussed here. These two examples are the ones for which we find measure-valued solutions in Chapter 7.

Shallow Water Equations

As outlined in Chapter 3, the shallow water equations are
\[
\begin{bmatrix}
v \\
\phi
\end{bmatrix}_t + \begin{bmatrix} v^2/2 + \phi \\ v\phi \end{bmatrix}_x = \begin{bmatrix} 0 \\
0 \end{bmatrix}
\] (4.1)

where \( v \) is the velocity and \( \phi \) is the depth times a constant gravitational acceleration.

Letting

\[
U = \begin{bmatrix} v \\ \phi \end{bmatrix} \quad \text{and} \quad F(U) = \begin{bmatrix} v^2/2 + \phi \\ v\phi \end{bmatrix},
\]

we study the Riemann problem for this system with

\[
U_L = \begin{bmatrix} .00 \\ .12 \end{bmatrix} \quad \text{and} \quad U_R = \begin{bmatrix} -.1840 \\ .0642 \end{bmatrix}.
\] (4.2)
For a viscosity matrix we use the positive definite matrix

\[
D = \begin{bmatrix}
13.0116 & -5.9144 \\
-5.9144 & 13.0116
\end{bmatrix}.
\] (4.3)

The \(U_L, U_R,\) and \(D\) were chosen so that there exists a Lax admissible 2-shock from \(U_L\) to \(U_R,\) but not a self-similar Riemann solution where the shocks satisfy the viscous entropy condition. The left and right states lie in the strictly hyperbolic region of state space. These choices are not degenerate. More specifically, there is a whole class of \(U_Ls, U_Rs,\) and \(Ds\) such that the nonexistence occurs.

The waves that could be used to construct a scale-invariant solution are 1 and 2-rarefaction waves, viscous admissible 1 and 2-shock waves, composite waves, viscous admissible transitional shock waves, transitional rarefaction waves, and overcompressive waves. We will show that there cannot be transitional rarefaction waves, transitional shock waves, overcompressive waves or composite waves, which leaves only "classical" waves.

Transitional rarefaction waves occur when a 2-rarefaction wave is joined on its right by a 1-rarefaction wave. See Ref. [17]. At the point of connection, the eigenvalues of \(F'(U)\) must coincide. Since

\[
F'(U) = \begin{bmatrix}
v & 1 \\
\phi & v
\end{bmatrix}
\]

the eigenvalues of \(F'(U)\) are \(\lambda_1 = v - \sqrt{\phi}\) and \(\lambda_2 = v + \sqrt{\phi}.\) Note that the strictly hyperbolic region is the open half-plane where \(\phi > 0,\) and the eigenvalues coincide only at \(\phi = 0.\) Hence for a transitional rarefaction wave to occur, \(\lambda_2\) must be increasing as it approaches \(\phi = 0.\) However, \(\nabla \lambda_2 \cdot \vec{n} > 0,\) where \(\vec{n} = [0, 1]\) is the normal to the boundary of the elliptic region. So \(\lambda_2\) is decreasing as it approaches the line of coincidence, thus we cannot have a transitional rarefaction.
The following lemma shows that our system cannot have transitional shocks because for any positive definite $D$, we cannot have two saddle points in the dynamical system associated with the admissibility criterion. For this we need the Rankine-Hugoniot conditions, which are

\[ -s(v - v_-) + v^2/2 + \phi - v_-^2/2 - \phi_- = 0, \quad \text{and} \]

\[ -s(\phi - \phi_-) + v\phi - v_-\phi_- = 0. \]  

(4.4)

(4.5)

The second equation gives

\[ s = \frac{v\phi - v_-\phi_-}{\phi - \phi_-} \quad \text{when} \quad \phi \neq \phi_- . \]  

(4.6)

Recall that for viscous admissible shocks we are looking for a traveling wave solution $U((x - st)/\epsilon)$ that limits to a Riemann solution

\[ U(x, t) = \begin{cases} 
U_-, & x < st \\
U_+, & x > st 
\end{cases} . \]

For this we need a $U(\zeta)$, where $\zeta = (x - st)/\epsilon$, that solves the system of ODE's

\[ U' = D^{-1} \left\{-s(U - U_-) + F(U) - F(U_-)\right\} \]  

(4.7)

with $D$ as our viscosity matrix.

**Lemma 4.1** For any $D$ positive definite, this dynamical system (4.7) associated with the shallow water equations, cannot have two saddle points.

**Proof:** Let $X = D^{-1} \{-s(U - U_-) + F(U) - F(U_-)\}$. Since we are interested in saddle points of the dynamical system, we want to see where $\det(X') < 0$. Since $D^{-1}$ is positive definite, it has no influence on the sign of $\det(X')$. The Jacobian of $X$ is given by $X' = D^{-1} \{F''(U) - sI\}$. Let

\[ A \equiv F''(U) - sI = \begin{bmatrix} v - s & 1 \\
\phi & v - s \end{bmatrix} . \]  

(4.8)
so that \( \det(X') = \det(A) = (v - s)^2 - \phi \). We will show that, given \( U_- = [v_-, \phi_-]^T \), there does not exist a \([v, \phi]^T\) satisfying the Rankine-Hugoniot equations, where both of the inequalities, \((v_- - s)^2 - \phi_- < 0\) and \((v - s)^2 - \phi < 0\), are satisfied. That is, there are not two saddle points in the dynamical system \( U_\zeta = X \).

Fix a left state \([v_-, \phi_-]^T\) and a value \( s \). This sets the dynamical system \( U_\zeta = X \). Now consider a second fixed point \([v, \phi]^T\) and assume that both \([v_-, \phi_-]^T\) and \([v, \phi]^T\) are saddle points. This implies that \((v_- - s)^2 - \phi_- < 0\) and \((v - s)^2 - \phi < 0\). First consider the determinant at \([v, \phi]^T\). This point must lie on a Hugoniot curve through \([v_-, \phi_-]^T\). so we substitute for \( s \) using (4.6) into \((v - s)^2 - \phi < 0\) giving

\[
\left( \frac{v - v\phi - v_- \phi_-}{\phi - \phi_-} \right)^2 - \phi < 0,
\]

\[
\Rightarrow \left( \frac{v\phi - v_\phi_- - v\phi + v_\phi_-}{\phi - \phi_-} \right)^2 - \phi < 0,
\]

\[
\Rightarrow (v_-\phi_- - v\phi_-)^2 < \phi(\phi - \phi_-)^2.
\]

\[
\Rightarrow (v - v_-)^2\phi_-^2 < \phi(\phi - \phi_-)^2.
\]

\[
\Rightarrow -\sqrt{\phi_-} \left| \frac{\phi - \phi_-}{\phi_-} \right| + v_- < v < \sqrt{\phi_-} \left| \frac{\phi - \phi_-}{\phi_-} \right| + v_-.
\]

Note that \( \phi \) and \( \phi_- \) are greater than zero since we are restricting ourselves to the strictly hyperbolic region. The inequality for the determinant at \([v_-, \phi_-]^T\) solves for \( v \) similarly.

\[
\left( \frac{v_- - v\phi - v_- \phi_-}{\phi - \phi_-} \right)^2 - \phi_- < 0,
\]

\[
\Rightarrow -\sqrt{\phi_-} \left| \frac{\phi - \phi_-}{\phi_-} \right| + v_- < v < \sqrt{\phi_-} \left| \frac{\phi - \phi_-}{\phi_-} \right| + v_-.
\]

To eliminate \( v \) from these inequalities we use the Rankine-Hugoniot equations. Substituting for \( s \) in equation (4.4) gives

\[
-v\phi - v_- \phi_- (v - v_-) + v^2/2 + \phi - v_\phi_-^2/2 - \phi_- = 0.
\]

Multiplying through by \((\phi - \phi_-)\) then produces

\[
(v_-\phi_- - v\phi)(v - v_-) + (v^2/2 + \phi - v_\phi_-^2/2 - \phi_-)(\phi - \phi_-) = 0.
\]
By multiplying and collecting like terms we get
\[
-1/2(\phi + \phi_-)v^2 + v_-(\phi + \phi_-)v - v_-^2/2(\phi + \phi_-) + (\phi - \phi_-)^2 = 0. \tag{4.13}
\]
\[
\Rightarrow v^2 - 2v_-v + v_-^2 - 2(\phi - \phi_-)^2 = 0. \tag{4.14}
\]

From the quadratic formula it then follows that
\[
v = v_- \pm \frac{\sqrt{2|\phi - \phi_-|}}{\sqrt{\phi + \phi_-}}. \tag{4.15}
\]

We then substitute this into the inequality (4.9), which gives
\[
-\sqrt{\phi} \left| \frac{\phi - \phi_-}{\phi_-} \right| < \pm \frac{\sqrt{2|\phi - \phi_-|}}{\sqrt{\phi + \phi_-}} < \sqrt{\phi} \left| \frac{\phi - \phi_-}{\phi_-} \right|, \quad \text{or} \tag{4.16}
\]
\[
-\frac{\sqrt{\phi}}{\phi_-} < \pm \frac{\sqrt{2}}{\sqrt{\phi + \phi_-}} < \frac{\sqrt{\phi}}{\phi_-}. \tag{4.17}
\]

To have two saddle points the above inequality must be satisfied at the same time as the inequality derived from (4.10)
\[
-\frac{\sqrt{\phi_-}}{\phi} < \pm \frac{\sqrt{2}}{\sqrt{\phi + \phi_-}} < \frac{\sqrt{\phi_-}}{\phi}. \tag{4.18}
\]

These simplify to
\[
\frac{\phi_-}{\sqrt{\phi}} < \frac{\sqrt{\phi + \phi_-}}{\sqrt{2}} \quad \text{and} \quad \frac{\phi}{\sqrt{\phi_-}} < \frac{\sqrt{\phi + \phi_-}}{\sqrt{2}}. \tag{4.19}
\]

It must be that \( \phi \neq \phi_- \), since \([v_-, \phi_-]^T\) is the only point on the Hugoniot curves where \( \phi = \phi_- \). Let \( \phi > \phi_- \). Therefore, there exists a \( d > 0 \) such that \( \phi = \phi_- + d \). We square both sides of the right inequality in (4.19) giving
\[
\frac{\phi_-}{\sqrt{\phi^-}} < \frac{\sqrt{\phi + \phi_-}}{\sqrt{2}} \Rightarrow \frac{\phi_-}{\phi} < \frac{\phi + \phi_-}{2}.
\]
\[
\Rightarrow \frac{\phi_-^2 + 2\phi_-d + d^2}{\phi_-} < \phi_- + \frac{d}{2}.
\]
\[
\Rightarrow \frac{3d + d^2}{\phi_-} < 0.
\]
This is a contradiction since both \( d \) and \( \phi_- \) are positive. Similarly, if \( \phi < \phi_- \) then the other inequality will result in a contradiction. Therefore, there cannot be two saddle points in the dynamical system. Since \( s \) was selected arbitrarily, there cannot be a pair of saddle points in any dynamical system associated to the viscous admissibility of this problem.

In addition, overcompressive waves cannot occur in this model. The proof is nearly identical to that of the nonexistence of transitional shock waves. For an overcompressive wave to exist we have to have a dynamical system with a repellor and an attractor. This is not possible, as we will show below, because we cannot have two fixed points at which the determinant of \( X' \) is positive.

**THEOREM 4.1** Given \( U_- = [v_- \phi_-]^T \), there does not exist a \([v, \phi]^T\) satisfying the Rankine-Hugoniot equations where both \((v-s)^2 - \phi > 0\) and \((v_- - s)^2 - \phi_- > 0\).

**PROOF:** As we said before, the proof follows that of the transitional shock waves. Let us assume that we have two fixed point satisfying the inequalities in the statement of the theorem. These reduce to

\[
v < -\sqrt{\phi} \left| \frac{\phi - \phi_-}{\phi_-} \right| + v_- \quad \text{or} \quad v > \sqrt{\phi} \left| \frac{\phi - \phi_-}{\phi_-} \right| + v_-
\]

and

\[
v < -\sqrt{\phi_-} \left| \frac{\phi - \phi_-}{\phi} \right| + v_- \quad \text{or} \quad v > \sqrt{\phi_-} \left| \frac{\phi - \phi_-}{\phi} \right| + v_-
\]

respectively. As before, we eliminate \( v \) using the Rankine-Hugoniot equations. With simplification this gives,

\[
\frac{\sqrt{\phi + \phi_-}}{\sqrt{2}} < \frac{\phi_-}{\sqrt{\phi}} \quad \text{and} \quad \frac{\sqrt{\phi + \phi_-}}{\sqrt{2}} < \frac{\phi}{\sqrt{\phi_-}}.
\]

It must be that \( \phi \neq \phi_- \), since \([v_-, \phi_-]^T\) is the only point on the Hugoniot curves where \( \phi = \phi_- \). Let \( \phi > \phi_- \). Therefore, there exists a \( d > 0 \) such that \( \phi = \phi_- + d \). We square
both sides of the left inequality in (4.22) giving
\[
\frac{\phi + \phi_-}{2} < \frac{\phi^2}{\phi}.
\]
\[
\phi^3 + \phi \phi_- - 2\phi_-^2 < 0,
\]
\[
\phi_-^2 + 2\phi_- d + d^2 + \phi_-^2 + \phi_- d - 2\phi_- < 0,
\]
\[
3\phi_- d + d^2 < 0.
\]
This is a contradiction since both \(d\) and \(\phi_-\) are positive. Similarly if \(\phi < \phi_-\) then the other inequality will result in a contradiction. Therefore, there cannot be two points in the dynamical system with positive determinants. And since \(s\) was selected arbitrarily there cannot be a pair of such points in any dynamical system associated to the viscous admissibility of this problem.

Our wave curves for this example do not have any composite wave parts. A wave curve through a state \(U_0\) changes from a rarefaction wave to a composite wave at the inflection point. At this point \(\nabla \lambda_i(U) \cdot \vec{r}_i(U) = 0\), where the \(\vec{r}_i\)'s are the eigenvalues of \(F'(U)\). Since \(\vec{r}_1 = [1, -\sqrt{\phi}]^T\), \(\vec{r}_2 = [1, \sqrt{\phi}]^T\), \(\lambda_1 = v - \sqrt{\phi}\), and \(\lambda_2 = v - \sqrt{\phi}\), we have \(\nabla \lambda_i \cdot \vec{r}_i = 3/2\). Therefore, \(\nabla \lambda_i \cdot \vec{r}_i \neq 0\) for all \(U\) and we cannot have any composite waves with a shock on the right.

We also cannot have composite waves with a shock on the left, called composite waves sonic on the left. Composite waves sonic on the left occur only when the \(i\)-shock speed, \(s\), equals \(\lambda_i\) at some point on the \(i\)-shock curve. Consider the shock branch of the 1-wave curve through \(U_L\). Using equations (4.4) and (4.6) we find that the shock branch of the 1-wave curve satisfies
\[
v = \frac{-\sqrt{2}(\phi - \phi_L)}{\sqrt{\phi + \phi_L}} \tag{4.23}
\]
with \(\phi > \phi_L\). We also used the fact that \(v_L = 0\). Again, using \(s\) from equation (4.6) and \(\lambda_1 = v - \sqrt{\phi}\), we see that \(s = \lambda_1\) only when
\[
v = \frac{\sqrt{\phi}(\phi - \phi_L)}{\phi_L} \tag{4.24}
\]
The curves in equations (4.23) and (4.24) do not intersect for \( \phi > \phi_L \), since one is always positive and the other always negative. Therefore the shock part of the 1-wave does not have any left sonic points. It is also true that the shock part of the backward 2-wave curve through \( U_R \) does not have any left sonic points.

The only waves left to construct a standard Riemann solution are the "classical" shock waves and rarefaction waves. To do this, we construct the forward 1-wave curve through \( U_L \) and the backward 2-wave curve through \( U_R \), as depicted in Figure 4.1. A point of intersection of these two curves would give a solution. However, the gap in the backward 2-wave curve, due to viscous inadmissibility, keeps the curves from crossing. The part of the curve near the Hopf bifurcation point, which includes \( U_L \), is inadmissible due to a limit cycle in the dynamical system keeping the saddle point from connecting to the attractor at \( U_R \). The rest of the inadmissible points \( U_* \) fail due to the dynamical system having a repellor at \( U_R \), which makes an admissible 2-shock from \( U_* \) to \( U_R \) impossible.
While Figure 4.1 is only local, we need to be sure that these wave curves do not intersect elsewhere. The 1-shock branch through $U_L$ has the property that
\[
\frac{dv}{d\phi} = -\frac{\sqrt{2}/2\phi + 3\sqrt{2}/2\phi_L}{(\phi + \phi_L)^{3/2}}.
\] (4.25)

This derivative is negative for all $\phi$ in the domain ($\phi > \phi_L$). Similarly the backward 2-shock branch through $U_R$ satisfies
\[
\frac{dv}{d\phi} = \frac{\sqrt{2}/2\phi + 3\sqrt{2}/2\phi_R}{(\phi + \phi_R)^{3/2}}
\] (4.26)

which is always positive. Hence these curves never meet. They also will not intersect the 1 or 2-rarefaction branches given by
\[
v = -2\sqrt{\phi} + 2\sqrt{\phi_L} \text{ and } v = 2\sqrt{\phi} - 2\sqrt{\phi_R} + v_R
\] (4.27)

respectively. Therefore we have proved the following theorem.

**THEOREM 4.2** The Riemann problem defined by equations (4.1) and (4.2) does not have a scale-invariant solution in which all shock waves admit viscous profiles with the viscosity matrix given by (4.3).

### Three-phase Flow Model

The equations for three-phase flow from Chapter 3 are given by
\[
\begin{bmatrix}
u \\
v
\end{bmatrix}_t + \begin{bmatrix}
-u^2 + v^2 + 2cv \\
2uv - 2cu
\end{bmatrix}_x = 0.
\] (4.28)

We consider the Riemann problem for this system where $c = 0.23$,

\[
U_L = \begin{bmatrix}
0.2110 \\
-0.1755
\end{bmatrix} \text{ and } U_R = \begin{bmatrix}
0.2800 \\
-0.1128
\end{bmatrix}.
\] (4.29)

For the viscosity matrix we use the positive definite matrix
\[
D = \begin{bmatrix}
0.0970 & -0.0209 \\
-0.0209 & 0.6750
\end{bmatrix}.
\] (4.30)
The $U_L$, $U_R$, and $D$ were chosen so that there exists a Lax admissible 1-shock from $U_L$ to $U_R$, but not a self-similar Riemann solution where the shocks satisfy the viscosity entropy condition. The left and right states lie in the strictly hyperbolic region of state space. Again, choice of $U_L$, $U_R$, and $D$ is not degenerate. The observed behavior of this example occurs with a class of $U_{LS}$, $U_{RS}$, and $D$s.

In Ref. [5] it is shown that there is no self-similar Riemann solution. The main result of Ref. [5] is summed up in the followin.

**THEOREM 4.3** Consider the Riemann problem defined by (4.28) and (4.29). Then there exists no Riemann solution in which all shock waves admit viscous profiles with the viscosity matrix given by (4.30).

The proof relies on considering all possible solutions involving classical and non-classical waves. In contrast with the shallow water equations, this model allows transitional shock waves, which are known to cause non-uniqueness of Riemann solutions (see Refs. [17, 15]) and therefore introduce an additional difficulty in showing nonexistence of solutions. There can be straight line transitional waves, i.e., two saddle points with a straight line connecting orbit, and curved transitional waves. In [5] S. Čanić deals with all of the possible Riemann solutions and shows that there are none.

The main approach presented in the proof can be summarized as follows. First, it is shown that there cannot be a solution constructed strictly with (viscous admissible) classical waves. As in the shallow water equations, the 1-wave curve through $U_L$ does not intersect the backward 2-wave curve through $U_R$. After this, it is proven that a solution beginning or ending with a transitional wave does not exist. This implies that the only possibility is a 1-wave followed by one or more transitional waves, and ending with a 2-wave. All of the possible solutions starting with a 1-wave must then be considered.

The 1-wave curve through $U_L$ in this model has three shock parts and one rarefaction part. In [5], the possibility of a solution starting with a 1-wave to one of these pieces is considered with each piece handled in turn. In each of these instances, a point is
reached where no more transitional waves can be constructed. The critical argument is that either there are no more connecting orbits in transitional waves, or the speed of the concatenated wave is less than the speed of the proceeding wave. In the end, none of these can produce a solution with any number of transitional waves proceeding the final 2-wave. Hence a Riemann solution with all shocks satisfying the viscous admissibility condition does not exist.
5 MEASURE-VALUED SOLUTIONS

In the remainder of this thesis, we analyze the solution type for the models with shock initial data discussed in Chapters 3 and 4.

The main result of this work is that in the situations described in Chapter 4, when no self-similar Riemann solution exists, there are oscillatory solutions satisfying the systems of conservation laws in a measure-valued sense. To present this result, we first introduce measure-valued solutions.

The concept of measure-valued solutions of conservation laws was first developed by DiPerna in [10], and Tartar in [24]. The following Lemma due to Tartar is what we use to generate our measure-valued functions in Chapter 6. These functions are shown in the same Chapter to be measure-valued solutions of hyperbolic problems in the sense defined below.

**Lemma 5.1** If $U_k(x,t)$ is a sequence in $L^\infty(\mathbb{R} \times [0,\infty), \mathbb{R}^2)$, satisfying $U_k(x,t) \in K$ almost everywhere, where $K$ is a compact subset of $\mathbb{R}^2$, then there exists a family of probability measures $\nu_{x,t}$ and a subsequence $U_k$ such that for all $h \in C(\mathbb{R}^2)$

\[
\lim h(U_k(x,t)) \rightarrow \overline{h}(x,t) \equiv \langle \nu_{x,t}, h \rangle = \int_{K \subset \mathbb{R}^2} h(U) d\nu_{x,t}(U) \quad (5.1)
\]

where $\rightarrow$ denotes convergence in the weak-* topology of $L^\infty$.

This convergence means that for any $g \in L^1(\mathbb{R} \times [0,\infty), \mathbb{R})$, one has

\[
\lim_{k \rightarrow \infty} \iint h(U_k(x,t)) g(x,t) dx dt = \iint \overline{h}(x,t) g(x,t) dx dt. \quad (5.2)
\]
**Definition 5.1** Let $P(\mathbb{R}^2)$ denote the set of all probability measures of $\mathbb{R}^2$. We say that $\nu: \mathbb{R} \times [0, \infty) \to P(\mathbb{R}^2)$ is a Measure-Valued Solution of the system of conservation laws (2.1) if

1. $\nu(x,t) = \nu_{x,t}$ has support in a fixed compact set for a.e. $(x,t)$, and
2. for all test functions $\phi \in C_0^\infty$, it is true that
   \[
   \int_0^\infty \int_{-\infty}^\infty \left\{ \langle \nu_{x,t}, U \rangle \phi_t + \langle \nu_{x,t}, F(U) \rangle \phi_x \right\} dxdt + \int_{-\infty}^\infty U_0(x) \phi(x,0)dx = 0. \tag{5.3}
   \]

We do not calculate the measures explicitly, but approximate the expectation values $\langle \nu_{x,t}, U \rangle$ and $\langle \nu_{x,t}, F(U) \rangle$ using the $U_k$'s that generate the measures.

The sequence $U_t$ is obtained as follows. We define $U_1(x,t,\epsilon)$ as the numerical solution of the parabolic PDE

\[
U_t + F(U)_x = \epsilon DU_{xx},
\]

with Riemann initial data, using a linearized Crank-Nicolson method as described in the next chapter with $\Delta x$ and $\Delta t$ selected for stability. We then let $U_t(x,t,\epsilon_t = \epsilon/\ell)$ be the numerical solution of

\[
U_t + F(U)_x = \frac{\epsilon}{\ell} DU_{xx},
\]

with $\Delta x/\ell$ and $\Delta t/\ell$. 
6 NUMERICAL METHOD AND THEORY

Introduction

We solve the parabolic equation (2.23) with Riemann initial data using a linearized Crank-Nicolson method. In the region of strict hyperbolicity, with a "class" of viscosity matrices which are positive definite and symmetric, we observe that, in the region of nonexistence of a self-similar weak solution, oscillations occur which converge in the limit, as epsilon goes to zero, to a measure-valued solution. These oscillations are not a characteristic of the numerical method used, but of the equations themselves.

When a viscous profile does exist, our numerical solution coincides with the traveling wave solution.

The Numerical Method and Stability

We use a linearized Crank-Nicolson method as used by Beam and Warming. See Ref. [4]. It is a second order implicit finite difference method. Let $\Delta x$ and $\Delta t$ represent the increments in space and time respectively and let the numerical solution at a grid point be denoted by $U^n_j$, where $U^n_j$ approximates the exact solution to (2.23) at $x = j\Delta x$, $t = n\Delta t$. We define the piecewise constant approximate solution by $U(x, t) \equiv U^n_j$ for all $(x, t) \in [x_j - \Delta x/2, x_j + \Delta x/2] \times [t_n, t_n + \Delta t)$. For each time step the solution of the following block-tridiagonal system gives the increments $\delta U^n_j \equiv U^{n+1}_j - U^n_j$:

\[
\frac{1}{\Delta t} \delta U^n_j + \frac{1}{2\Delta x} \{ F(U^n_{j+1}) - F(U^n_{j-1}) + \frac{1}{2} F'(U^n_{j+1}) \delta U^n_{j+1} - \frac{1}{2} F'(U^n_{j-1}) \delta U^n_{j-1} \} = \frac{1}{(\Delta x)^2} \epsilon D \{ U^n_{j+1} - 2U^n_j + U^n_{j-1} + \frac{1}{2} [ \delta U^n_{j+1} - 2\delta U^n_j + \delta U^n_{j-1} ] \}.
\] (6.1)
The linearization assumption, \( F(U_j^{n+1}) \approx F(U_j^n) + F'(U_j^n) \delta U_j^n \), is made to avoid the solution of a nonlinear system at each step. The block-tridiagonal nature of this method is more apparent when rewritten in the form,

\[
\begin{align*}
\left\{ -\frac{1}{4\Delta x} F'(U_{j-1}^n) & - \frac{1}{2(\Delta x)^2} \epsilon D \right\} \delta U_{j-1}^n + \left\{ \frac{1}{\Delta t} I + \frac{1}{(\Delta x)^2} \epsilon D \right\} \delta U_j^n \\
+ \left\{ \frac{1}{4\Delta x} F'(U_{j+1}^n) - \frac{1}{2(\Delta x)^2} \epsilon D \right\} \delta U_{j+1}^n \\
= & -\frac{1}{2\Delta x} [F(U_{j+1}^n) - F(U_{j-1}^n)] \\
+ & \frac{1}{(\Delta x)^2} \epsilon D [U_{j+1}^n - 2U_j^n + U_{j-1}^n].
\end{align*}
\] (6.2)

The following invariance property of this method will prove useful later: if \( \Delta x, \Delta t \), and \( \epsilon \) are replaced by \( \Delta x/k, \Delta t/k, \) and \( \epsilon/k \) respectively, the system of equations does not change. If we start with Riemann initial data, all the \( U_j^0 \)'s with negative \( j \) have one value, and all the \( U_j^0 \)'s with nonnegative \( j \) have another, regardless of what \( \Delta x \) is. This, along with the facts that our system of equations does not change, except in size when scaled by \( k \), and that changes in \( U \) move with a finite propagation speeds, shows that \( U_j^n \) with \( \Delta x, \Delta t \), and \( \epsilon \) is equal to \( U_j^n \) with \( \Delta x/k, \Delta t/k, \) and \( \epsilon/k \). This is also assuming that the initial disturbance has not reached the edges of the physical domain.

This method, including the linearization, has a local truncation error given by \( \tau = \mathcal{O}(\Delta t((\Delta x)^2 + (\Delta t)^2)) \). Therefore, it is consistent. To show that this method is convergent, at least on linear problems, we must prove stability.

The linearized Crank-Nicolson method applied to the linear system

\[
U_t + AU_x = \epsilon DU_{xx}
\] (6.3)

takes the form

\[
M \delta \tilde{U}^n = (2I - 2M) \tilde{U}^n
\] (6.4)
where

\[
M = \begin{bmatrix}
I + \frac{\mu}{2} \epsilon D - \frac{\lambda}{4} A & -\frac{\mu}{2} \epsilon D + \frac{\lambda}{4} A \\
-\frac{\mu}{2} \epsilon D - \frac{\lambda}{4} A & I + \mu \epsilon D \\
& \ddots & \ddots & \ddots \\
& I + \mu \epsilon D & -\frac{\mu}{2} \epsilon D + \frac{\lambda}{4} A \\
& -\frac{\mu}{2} \epsilon D - \frac{\lambda}{4} A & I + \frac{\mu}{2} \epsilon D + \frac{\lambda}{4} A
\end{bmatrix}
\]  

(6.5)

with \( \mu \equiv \Delta t/(\Delta x)^2 \) and \( \lambda \equiv \Delta t/\Delta x \). The vector \( \tilde{U}^n \) is made up of the \( U_j^n \)'s. The equation (6.4) can be rewritten as

\[
M \tilde{U}^{n+1} = (2I - M)\tilde{U}^n, \text{ or } 
\]

\[
\tilde{U}^{n+1} = (2M^{-1} - I)\tilde{U}^n. 
\]

(6.6) (6.7)

Since we are considering the linear problem, the global error at the \( (n+1) \)st step, \( \tilde{e}^{n+1} \), satisfies

\[
\tilde{e}^{n+1} = (2M^{-1} - I)\tilde{e}^n + \tau. 
\]

(6.8)

Therefore, for Lax-Richtmyer \( L^\infty \) stability we need \( \|2M^{-1} - I\|_\infty \leq 1 + \alpha \Delta t \) for some constant \( \alpha \). See Ref. [19]. To show that this is the case consider the following.

Let \( N \) be the matrix such that \( M = I + N \). Considering the rows of \( N \) it is clear that \( \|N\|_\infty \leq 2\epsilon \mu \|D\|_\infty + \frac{\lambda}{2} \|A\|_\infty \). By making \( \Delta t \) sufficiently small we can force this bound to be less than one. The infinity norm of \( N \) must be less than one for the following
Figure 6.1 Shallow Water Equations: The left picture shows the numerical solution, \( u \), of a parabolic problem with \( \epsilon = 2.66 \). The right picture shows the numerical solution, \( u \), of the same parabolic problem with \( \epsilon = 0.133 \).

calculation.

\[
\|2M^{-1} - I\|_{\infty} = \|2(I + N)^{-1} - I\|_{\infty} \\
= \|2(I - N + N^2 - \ldots) - I\|_{\infty} \quad \text{(if } \|N\|_{\infty} < 1) \\
= \|I + 2(-N + N^2 - \ldots)\|_{\infty} \\
\leq 1 + 2\|N\|_{\infty} - I + N - N^2 + \ldots\|_{\infty} \\
\leq 1 + 2\|N\|_{\infty}(\|I\|_{\infty} + \|N\|_{\infty} + \|N^2\|_{\infty} + \ldots) \\
\leq 1 + 2\|N\|_{\infty} \left( \frac{1}{1 - \|N\|_{\infty}} \right) \\
\leq 1 + 2\Delta t\|N/\Delta t\|_{\infty} \left( \frac{1}{1 - \|N\|_{\infty}} \right) \\
= 1 + \alpha \Delta t.
\]

Note that \( N/\Delta t \) is independent of \( \Delta t \). This shows that the method is stable and hence convergent on linear problems.

We have shown that the \( U^n \)'s have a bounded rate of growth in the \( L^\infty \) norm. Therefore the sequence \( U_t \) has a bounded \( L^\infty \) growth rate. Numerically, we observe global \( L^\infty \)-boundedness of \( U_t \) in our nonlinear examples. Figures 6.1 and 6.2 demonstrate the boundedness of two elements in the sequence. The left figures of 6.1 and
6.2 were calculated with $\epsilon = 2.66$, $\Delta x = 9.33$, and $\Delta t = 2.01$. The right figures were calculated with $\epsilon = 0.133$, $\Delta x = 0.4667$, and $\Delta t = 0.1007$. The equations and Riemann initial data are specified in equations (7.1), (7.2), and (7.3) from Chapter 7. Notice that as $\epsilon$ decreases ($\ell$ increases) the amplitude of the waves does not increase, only the frequency of the waves does. This behavior is also seen in the three-phase flow solutions.

The assumption of uniform $L^\infty$-boundedness of the $U_\ell$’s is important in the rest of the thesis.

**A Second Numerical Method**

For extra proof that the oscillations observed with the Crank-Nicolson method are characteristic of the equations and not of the numerics, we implemented a cubic spline Galerkin method. The cubic splines discretize the PDE (2.23) in space giving a system of ODE’s in time. This system is then solved in time with a second order Runge-Kutta routine. More specifically, each point $x_j$ in the space discretization has a cubic spline basis element, $\phi_j(x)$, centered around it. We then look for a solution of the form

$$U^N = \sum_{j=1}^{N} c_j(t) \phi_j(x).$$

(6.9)
Substituting (6.9) for $U$ in equation (2.23), multiplying through by $\phi_i(x)$, integrating over the $x$-domain, and applying integration by parts once to the space derivative terms yields two ordinary differential equations in time. Doing this for each $\phi_i(x)$ yields a large system of ODEs that can be numerically solved for the $\xi_j(t)$s. This is done with a second order Runge-Kutta routine. The resulting method has a truncation error of $O((\Delta x)^4 + (\Delta t)^2)$, which makes it different than the linearized Crank-Nicolson method.

This Galerkin method gave the same oscillations when run with identical initial conditions, suggesting that the oscillations are not numerical aberrations. This is in addition to the fact that, as $\Delta x$ and $\Delta t$ are reduced with a fixed $\epsilon$ and $T$, the oscillations are unchanged when using either method. This is illustrated in Figure 6.3. These graphs show two short runs of the linearized Crank-Nicolson method. Each run used the Riemann initial data from (4.2), the matrix given in (4.3), $T = 320$, and $\epsilon = .25$. The data on the left in Figure 6.3 comes from a run with $\Delta x = .6$ and $\Delta t = .088$. The data on the right in Figure 6.3 comes from a run with $\Delta x = .3$ and $\Delta t = .022$. Despite the different discretizations the results are indistinguishable and contain oscillations. If the oscillations were numerical then they would depend on the choice of $\Delta x$ and $\Delta t$, which they do not.
The Numerical Sequence and Limit

The invariance property of the linear system shown earlier in this chapter encourages us to define a sequence of numerical solutions in the following manner. Let $U_1(x, t, \epsilon)$ be the solution obtained by solving (2.23) with Riemann initial data for some $\epsilon > 0$ using the discretization $\Delta x, \Delta t$. Denote by $U_\ell(x, t, \epsilon)$ the solution of (2.23) with $\epsilon$ replaced by $\epsilon/\ell$, i.e.,

$$U_t + F(U)_x = \epsilon U_{xx},$$

(6.10)

using the discretization $\Delta x/\ell$, $\Delta t/\ell$. Then the following invariance property holds.

**Proposition 6.1** The solution $U_\ell(x, t, \epsilon)$ is equal to $U_1(lx, lt, \epsilon)$.

**Proof:** Given a point $(x, t)$, it is true that $(x, t) \in [(j - \frac{1}{2})\Delta x/\ell, (j + \frac{1}{2})\Delta x/\ell] \times [n\Delta t/\ell, (n + 1)\Delta t/\ell]$ for some integers $j$ and $n$. So $U_\ell(x, t, \epsilon) = U^n_{i,j}$, the numerical solution of (6.10) at the grid point $(j, n)$. Since the point $(lx, lt)$ is in $[(j - \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta x) \times [n\Delta t, (n + 1)\Delta t)$, $U_1(lx, lt, \epsilon) = U^n_{i,j}$. Therefore $U_\ell(x, t, \epsilon) = U_1(lx, lt, \epsilon)$, since $U^n_{i,j} = U^n_{i,j}$.

This fact allows the first approximation to serve as the whole sequence. Therefore, instead of reducing epsilon, we push further out in time.

In both models, numerical evidence shows that the sequence $U_\ell$ of numerical solutions is uniformly bounded in $L^\infty$. Indeed, Figures 6.1 and 6.2 show the functions $u_\ell$ and $v_\ell$ versus $x/t$ as $\epsilon \to 0$ and indicate that the amplitude of the state variables is uniformly bounded, whereas the frequency of the oscillations increases as $\epsilon \to 0$. Therefore, we assume that, in both examples, there exists a constant $C > 0$ such that $\|U_\ell(x, t, \epsilon_\ell)\|_\infty \leq C$ for all $x$, $t$, and $\ell$. Under this assumption we can prove the following proposition.

**Proposition 6.2** If the $U_\ell$s are uniformly bounded in $L^\infty$, then there exists a subsequence $U_k(x, t, \epsilon_k)$ of $U_\ell$ and a measure-valued function $\nu_{x,t}$ such that for all $h \in C(\mathbb{R}^2)$, $h(U_k(x, t)) \to \overline{h}(x, t)$, in the weak-* topology of $L^\infty$. 
PROOF: This is a direct application of Lemma 5.1.

In the following theorem we prove that this function is a measure-valued solution of the system of conservation laws.

**Theorem 6.1** The measure-valued function \( \nu_{x,t} \) generated by the subsequence \( U_k \) is a measure-valued solution to the system of conservation laws.

**Proof:** We need to show that the measure-valued function \( \nu_{x,t} \) is a solution in the sense of Definition 5.1. Recall that \( U^n_{k,j} \) denotes the discrete approximation to the exact solution of the parabolic problem (6.10) (with \( k \) instead of \( \ell \)) at the point \((x_j, t_n) = (j \Delta x_k, n \Delta t_k)\). where \( \Delta x_k = \Delta x/k \), and \( \Delta t_k = \Delta t/k \). We introduce the following numerical flux function

\[
\hat{F}(U^n_k; j) = \frac{1}{2}[F(U^n_{k,j+1}) + \frac{1}{2} F'(U^n_{k,j+1})(U^n_{k,j+1} - U^n_{k,j+1})
+ F(U^n_{k,j}) + \frac{1}{2} F'(U^n_{k,j})(U^n_{k,j} - U^n_{k,j})].
\] (6.11)

In terms of \( \hat{F} \), the linearized Crank-Nicolson method (6.1) can be written as

\[
\frac{1}{\Delta t_k} [U^n_{k,j+1} - U^n_{k,j}] + \frac{1}{\Delta x_k} [\hat{F}(U^n_k; j) - \hat{F}(U^n_k; j - 1)] = \frac{1}{2(\Delta x_k)^2} \epsilon_k D[U^n_{k,j+1} - 2U^n_{k,j} + U^n_{k,j-1} - 2U^n_{k,j+1} + U^n_{k,j-1}].
\] (6.12)

If we multiply through by \( \Delta x_k \Delta t_k \phi(x_j, t_n) \) where \( \phi \in C^\infty_0 \) and sum over all values of \( n \) and \( j \), we get

\[
\Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=\infty}^{\infty} \frac{1}{\Delta t_k} [U^n_{k,j+1} - U^n_{k,j}] \phi(x_j, t_n) + \\
\Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=\infty}^{\infty} \frac{1}{\Delta x_k} [\hat{F}(U^n_k; j) - \hat{F}(U^n_k; j - 1)] \phi(x_j, t_n) = \\
\Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=\infty}^{\infty} \frac{1}{2(\Delta x_k)^2} \epsilon_k D[U^n_{k,j+1} - 2U^n_{k,j} + U^n_{k,j-1} - 2U^n_{k,j+1} + U^n_{k,j-1}] \\
+ U^n_{k,j+1} - 2U^n_{k,j} + U^n_{k,j-1}] \phi(x_j, t_n).
\] (6.13)
Summation by parts can be employed to move the differences onto the \( \phi \)'s.

\[
\Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \frac{1}{\Delta t_k} [\phi(x_j, t_n) - \phi(x_j, t_{n-1})] U_{k,j}^n + \\
\Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \frac{1}{\Delta x_k} [\phi(x_{j+1}, t_n) - \phi(x_j, t_n)] \tilde{F}(U_{k,j}^n; j) = \\
\Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \left[ \varepsilon_k D \frac{1}{2(\Delta x_k)^2} (\phi(x_{j+1}, t_n) - \phi(x_j, t_n)) - 2\phi(x_j, t_n) \\
+ \phi(x_{j-1}, t_n)) U_{k,j}^n + \varepsilon_k D \frac{1}{2(\Delta x_k)^2} (\phi(x_{j+1}, t_n) - \phi(x_j, t_n)) \\
- \phi(x_{j-1}, t_n)) U_{k,j}^{n+1}] + \Delta x_k \sum_{j=-\infty}^{\infty} \phi(x_j, 0) U_{k,j}^0.
\]

(6.14)

Due to the compact support of \( \phi \), all the boundary terms are zero, except the one at \( t = 0 \). Note that the sums are actually finite due to the compact support of \( \phi \). Let \([0, M] \times [-N, N]\) contain the support of \( \phi \), and let \( G \) bound the derivatives of \( \phi \) up to third order, i.e., \( G = \max_{x,t} \{|D^\alpha \phi|, |\alpha| \leq 3\} \). Recall that \( U_{k,j}^n \) is a discrete value of the piecewise continuous function \( U_k \), where \( U_k(x, t) = U_{k,j}^n \) for all \((x, t) \in [x_j - \Delta x_k/2, x_j + \Delta x_k/2) \times [t_n, t_n + \Delta t_k)\). Taking the limit as \( k \to \infty \), i.e., \((\Delta x_k, \Delta t_k, \varepsilon_k) \to (0, 0, 0)\), of equation (6.14), we consider each pair of sums individually. Starting with the first term.

we use a Taylor expansion on \( \phi \) and introduce \( \overline{U} \equiv \phi(x,t) \) to obtain

\[
\lim_{k \to \infty} \Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \frac{1}{\Delta t_k} [\phi(x_j, t_n) - \phi(x_j, t_{n-1})] U_{k,j}^n = \\
\lim_{k \to \infty} \Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \left[ \frac{\Delta t_k}{2} \phi_t(x_j, t_n) + \phi_t(x_{j+1}, t_n) - \phi_t(x_{j-1}, t_n) \right] \\
\cdot [\overline{U}(x_j, t_n) + (U_{k,j}^n - \overline{U}(x_j, t_n))],
\]

(6.15)

where \( \zeta_n \) is a point between \( t_{n-1} \) and \( t_n \). Multiplying the two sums in the square brackets gives four terms. The first satisfies

\[
\lim_{k \to \infty} \Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \phi_t(x_j, t_n) \overline{U}(x_j, t_n) = \\
\int_0^M \int_{-N}^N \overline{U}(x,t) \phi_t(x,t) dx dt.
\]

(6.16)
The second converges to zero by the argument,

$$\lim_{k \to \infty} |\Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \phi_t(x_j, t_n)(U_{k,j}^n - \overline{U}(x_j, t_n))| \leq$$

$$\lim_{k \to \infty} \Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} |(U_{k,j}^n - \overline{U}(x_j, t_n))| \leq$$

$$\lim_{k \to \infty} \Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} [(U_{k,j}^n - \overline{U}_k(x_j, t_n))]$$

$$+ |R_{\overline{U}_k}(x_j, t_n)|,$$  \hspace{1cm} (6.17)

where \(\overline{U}_k\) is the piecewise constant function defined by \(\overline{U}_k(x, t) = \overline{U}(x_j, t_n)\) for all \((x, t) \in [x_j - \Delta x_k/2, x_j + \Delta x_k/2] \times [t_n, t_n + \Delta t_k]\) and \(\overline{U}_k(x, t) + R_{\overline{U}_k}(x, t) = \overline{U}(x, t)\). This allows us to replace the first sum with an integral. Also note that since \(\overline{U}\) is integrable.

\(R_{\overline{U}_k}\) goes to zero uniformly as \(k\) goes to infinity. We then get

$$\lim_{k \to \infty} G \int_0^L \int_{-N}^N |U_k - \overline{U}_k(x, t)| dx dt$$

$$+ \lim_{k \to \infty} \Delta x_k \Delta t_k G \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} |R_{\overline{U}_k}(x_j, t_n)|$$

$$= \lim_{k \to \infty} G \int_0^L \int_{-N}^N |U_k - \overline{U}_k(x, t)| dx dt$$

$$= \lim_{k \to \infty} G \int_0^L \int_{-N}^N |U_k - \overline{U}(x, t) + R_{\overline{U}_k}(x, t)| dx dt$$

$$\leq \lim_{k \to \infty} G \int_0^L \int_{-N}^N |U_k - \overline{U}(x, t)| + |R_{\overline{U}_k}(x, t)| dx dt$$

$$= 0.$$

The remainder goes to zero because \(\overline{U} \in L^1_{loc}\), which is also where \(\overline{h}\) for all continuous \(h\) reside. The difference term goes to zero because \(\overline{U}\) is the weak-* limit of the \(U_k\)'s. The other two terms in (6.15) go to zero due to the second \(\Delta t_k\) term and the integrability of \(\overline{U}\).
The limit of the part of (6.14) involving the numerical flux is of the form,

\[ \lim_{k \to \infty} \Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \frac{1}{\Delta x_k} [\phi(x_{j+1}, t_n) - \phi(x_j, t_n)] \tilde{F}(U^n_{k,j}) = \]

\[ \lim_{k \to \infty} \Delta x_k \Delta t_k \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} [\phi_x(x_j, t_n) + \frac{\Delta x_k}{2} \phi_{xx}(\xi_j, t_n)] \]

\[ \cdot \{ \tilde{F}(x_j, t_n) + \frac{1}{2} [ (F(U^n_{k,j+1}) - \tilde{F}(x_j, t_n)) + \frac{1}{2} F'(U^n_{k,j+1})(U^{n+1}_{k,j+1} - U^n_{k,j+1}) \]

\[ + (F(U^n_{k,j}) - \tilde{F}(x_j, t_n)) + \frac{1}{2} F'(U^n_{k,j})(U^{n+1}_{k,j} - U^n_{k,j}) ] \} \]. \tag{6.18} \]

All of these terms except \( \phi_x(x_j, t_n) \tilde{F}(x_j, t_n) \) will go to zero for the following reasons.

Clearly all the terms involving the extra \( \Delta x_k \) have a zero limit. The difference \( \phi_x(x_j, t_n) \cdot (F(U^n_{k,j}) - \tilde{F}(x_j, t_n)) \) behaves in the same way that \( \phi_t(x_j, t_n) \cdot (U^n_{k,j} - \tilde{U}(x_j, t_n)) \) did, since \( \tilde{F} \) is the weak-* limit of \( F(U_k) \). The \( U_k \)'s are uniformly bounded so the norm of the Jacobian \( F' \) is as well. The difference \( U^{n+1}_{k,j} - U^n_{k,j} \) can be rewritten as \( (U^{n+1}_{k,j} - \tilde{U}(x_j, t_n)) - (U^n_{k,j} - \tilde{U}(x_j, t_n)) \). The discrepancy in indices is eliminated by renumbering before the limit is taken and seeing that the leftover terms go to zero as well. Thus the limit in (6.18) is

\[ \int_0^M \int_{-N}^N \tilde{F}(x,t) \phi_x(x,t) \, dx \, dt. \tag{6.19} \]

The limit of terms of (6.14) involving \( \epsilon_k \) would give an integral of \( \tilde{U}(x,t) \phi_{xx}(x,t) \) if the \( \epsilon_k \) term was not there. With the \( \epsilon_k \) they converge to zero. The boundary term of (6.14) is a simple integral.

Therefore, in the limit as \( k \to \infty \), (6.14) becomes

\[ \int_0^\infty \int_{-\infty}^\infty \{ \tilde{U}(x,t) \phi_t(x,t) + \tilde{F}(x,t) \phi_x \} \, dx \, dt \]

\[ + \int_{-\infty}^\infty \tilde{U}_0(x) \phi(x,0) \, dx = 0, \tag{6.20} \]

for all \( \phi \in C_0^\infty. \)
Expectation Values

We do not compute the measures \( \nu_{x,t} \) directly since we need only to be able to calculate the expectation values \( \overline{h} \) for continuous functions \( h \). To this end we use the invariance property of our numerical method to prove the following theorem.

**Theorem 6.2** For each \( h \in C(\mathbb{R}^n) \), the expectation values \( \langle \nu_{x,t}, h(U) \rangle \), where \( \nu_{x,t} \) is the limit of the subsequence \( U_k(x,t,\epsilon_k) \), satisfy

\[
\overline{h} \equiv \langle \nu_{x,t}, h(U) \rangle = \lim_{T \to \infty} \frac{2}{T^2} \int_0^T h(U_1(\frac{x}{t}, \tau, \tau)) \, d\tau,
\]

whenever the limit converges uniformly for almost every \( x/t \in \mathbb{R} \).

**Proof:** This proof follows that of Frid & Liu in Ref. [11]. We showed before that \( U_k(x,t,\epsilon/k) = U_1(kx,kt,\epsilon) \). With the assumption of \( L^\infty \) stability, Theorem 6.1 shows that the subsequence \( U_k \) generates a measure-valued solution \( \nu_{x,t} \) to our system of conservation laws. Consider a point \((x,t) \in \mathbb{R} \times (0,\infty)\), and let

\[
r_0 = \sqrt{x^2 + t^2}, \quad \theta_0 = \tan^{-1}(t/x),
\]

where the range of \( \tan^{-1} \) is \((0,\pi)\). We also denote by \( \Delta(r, \theta) \) the infinitesimal sector \(|r - r_0| < \Delta r, |\theta - \theta_0| < \Delta \theta, \Delta r > 0, \text{ and } \Delta \theta > 0\). The area of \( \Delta(r, \theta) \) is \( m(\Delta(r, \theta)) = 4r_0 \Delta r \Delta \theta \).

Given a continuous function \( h \in C(\mathbb{R}^m) \), for a.e. \((x,t) \in \mathbb{R} \times (0,\infty)\),

\[
\langle \nu_{x,t}, h(U) \rangle = \lim_{\Delta r + \Delta \theta \to 0} \frac{1}{m(\Delta(r, \theta))} \int_{\Delta(r, \theta)} \langle \nu_{r, \theta}, h(U) \rangle \, dy \, d\tau.
\]

We are interested in \( \overline{h} \) only in the distributional sense so we are working with the right hand side of equation (6.22). If \( h \in C(\mathbb{R}^n) \) is such that the limit in (6.21) converges
uniformly for a.e. $x/t \in \mathbb{R}$, we have

\[
\frac{1}{m(\Delta (r, \theta))} \int \int_{\Delta (r, \theta)} (\nu_{y, \tau}, h(U)) \, dy \, dr
\]  
(6.23)

\[
= \frac{1}{m(\Delta (r, \theta))} \lim_{k \to \infty} \int \int_{\Delta (r, \theta)} h(U_k(y, \tau, \epsilon/k)) \, dy \, dr
\]  
(6.24)

\[
= \frac{1}{m(\Delta (r, \theta))} \lim_{k \to \infty} \int \int_{\Delta (r, \theta)} h(U_1(ky, k\tau, \epsilon)) \, dy \, dr
\]  
(6.25)

\[
= \frac{1}{m(\Delta (r, \theta))} \lim_{k \to \infty} \int \int_{\Delta (r, \theta)} h(U_1(kr, \theta, \epsilon)) \, r \, dr \, d\theta
\]  
(6.26)

\[
= \frac{1}{m(\Delta (r, \theta))} \lim_{k \to \infty} \frac{1}{k^2} \int \int_{\Delta (r, \theta)} h(U_1(r, \theta, \epsilon)) \, r \, dr \, d\theta
\]  
(6.27)

\[
= \lim_{k \to \infty} \frac{1}{4k^2 r_0 \Delta \theta \Delta r} \int_{\theta_0 + \Delta \theta}^{\theta_0 - \Delta \theta} \left[ \int_{k(r_0 - \Delta r)}^{k(r_0 + \Delta r)} h(U_1(r, \theta, \epsilon)) \, r \, dr \right] d\theta
\]  
(6.28)

\[
= \frac{1}{2\Delta \theta} \int_{\theta_0 - \Delta \theta}^{\theta_0 + \Delta \theta} \lim_{k \to \infty} \frac{1}{2k^2 r_0 \Delta r} \int_{k(r_0 - \Delta r)}^{k(r_0 + \Delta r)} h(U_1(r, \theta, \epsilon)) \, r \, dr \, d\theta
\]  
(6.29)

\[
= \frac{1}{2\Delta \theta} \int_{\theta_0 - \Delta \theta}^{\theta_0 + \Delta \theta} \frac{1}{2r_0 \Delta r} \left\{ \frac{1}{2} (r_0 + \Delta r)^2 \lim_{k \to \infty} \frac{2}{k^2 (r_0 + \Delta r)^2} \int_{0}^{k(r_0 + \Delta r)} h(U_1(r, \theta, \epsilon)) \, r \, dr \right\} d\theta
\]  
(6.30)

By the convergence assumption we have that (6.30) equals

\[
\frac{1}{2\Delta \theta} \int_{\theta_0 - \Delta \theta}^{\theta_0 + \Delta \theta} \left\{ \lim_{R \to \infty} \frac{2}{R^2} \int_{0}^{R} h(U_1(r, \theta, \epsilon)) \, r \, dr \right\} d\theta
\]  
(6.31)

which due to the uniformity of the convergence assumption approaches

\[
\lim_{R \to \infty} \frac{2}{R^2} \int_{0}^{R} h(U_1(r, \theta_0, \epsilon)) \, r \, dr
\]  
(6.32)

a.e. as $\Delta \theta \to 0$.

The convergence also implies that (6.24) does not change if the subsequence is replaced with the whole sequence. This means that the whole sequence $U_\epsilon$ converges to the measure-valued solution $\nu_{x,t}$. 

In the numerical simulations the limit (6.21) is approximated by the sum,

\[ \overline{h}(x/t) \approx \frac{2}{n(n+1)} \sum_{k=1}^{n} kh(U_1(k\Delta t, j_k \Delta x)), \quad j_k = \left[ k\Delta t \frac{x}{t} \Delta x \right], \quad (6.33) \]

where \( \Delta \tilde{t} \) is an increment of \( t \), usually larger than \( \Delta t \), such that \( T = n \Delta \tilde{t} \). In the examples studied in this work, this sum appears to converge uniformly. This allows us to make the necessary assumptions for the previous theorem. We believe that the sum converges by observing the partial sums. Figures 6.4, 6.5, and 6.6 show graphs of three partial sums made from a numerical solution of the shallow water equations. Figure 6.4 shows \( \overline{u} \) versus \( x/t \) using 1/100th of the full partial sum. Figure 6.5 shows \( \overline{u} \) versus \( x/t \) using 1/10th of the full partial sum. Finally, Figure 6.6 shows \( \overline{u} \) versus \( x/t \) using the full partial sum. The run that found \( U_1 \) to generate these pictures used the Riemann initial data from (4.2), the matrix given in (4.3), \( \epsilon = .133, \Delta x = .4667, \Delta t = .1007, \tilde{t} = 6.7 \), and \( T = 32000 \). Notice the convergence.
Figure 6.5 Shallow Water Equations: 1/10th of the full partial sum.

Figure 6.6 Shallow Water Equations: The full partial sum.
7 NUMERICAL RESULTS

In this chapter, we present a detailed description of the numerical examples studied in this thesis. For both the shallow water equations and the three-phase flow model, we obtain oscillatory solutions of the parabolic PDE

\[ U_t + F(U)_x = \epsilon D U_{xx} \]

that increase in frequency as \( \epsilon \to 0 \) and stay bounded in amplitude. We calculate expectation values using Theorem 6.2 and show that they satisfy the system of conservation laws in a measure-valued sense. In both examples, the initial data corresponds to a single Lax admissible shock whereas the expectation values of the solutions consist of more than one shock.

The Shallow Water Equations

As described in Chapter 3 we study the parabolic problem for the shallow water equations

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix}
+ \begin{bmatrix}
  u^2/2 + v \\
  u v
\end{bmatrix}_t = \epsilon D \begin{bmatrix}
  u \\
  v
\end{bmatrix}_{xx},
\]

with \( D \) given by

\[
D = \begin{bmatrix}
  13.0116 & -5.9144 \\
  -5.9144 & 13.0116
\end{bmatrix},
\]
Figure 7.1 Shallow Water Equations: The left picture shows the connecting orbit in the dynamical system with $D = I$. The right picture shows the dynamical system when $D$ is given by (7.2). The lack of a connecting orbit is due to a limit cycle surrounding the attractor $U_R$.

and Riemann data

$$U_L = \begin{bmatrix} .00 \\ .12 \end{bmatrix} \quad \text{and} \quad U_R = \begin{bmatrix} -.1840 \\ .0642 \end{bmatrix}. \quad (7.3)$$

This is shock initial data corresponding to a Lax 2-shock that does not have a viscous profile. The corresponding phase space portrait is shown in Figure 7.1 on the right. When $D = I$ there exists a connecting orbit in the dynamical system (2.25) replacing $U_-$ with $U_L$. The corresponding phase space portrait is shown in Figure 7.1 on the left. These Figures were generated with the Riemann problem solver [16].

We solve the parabolic problem (7.1) with the initial data given by (7.3) with the linearized Crank-Nicolson method. For $D = I$, $\Delta x = 31.11$, $\Delta t = 21.02$, and $\epsilon = 20$, we obtain the expected shock traveling at a constant speed. With $D$ given by (7.2), $\Delta x = .4667$, $\Delta t = .1007$, and $\epsilon = .133$ we see persistent oscillations. The function $U$ is approximated over the $x$-interval $[-2000, 12000]$ out to time $T = 32000$. Figure 7.2 shows $u$ versus $x/t$ for these two cases. For the second case, as time increases, the oscillations increase in frequency over the same $x/t$ intervals. Let $f$ and $g$ be the first and second
components of the flux function respectively. Compare the oscillatory solution $u$ to its expectation values $\bar{u}$ as shown in Figure 7.3. The expectation values were calculated with equation (6.33) using $\Delta t = 6.7$ out to time $T = 32000$, updating the sum during the run at each $\Delta t$ increment. The plots of $\bar{u}$ and $\bar{g}$ are similar. From Theorem 6.1 and Theorem 6.2 we know that these expectation values represent a measure-valued solution of the hyperbolic problem. Indeed, we verify that this is the case with our numerical simulations. Namely, we show numerically that the expectation values $\bar{u}$, $\bar{v}$, $\bar{f}$, and $\bar{g}$ satisfy the hyperbolic system of conservation laws in a measure-valued sense. First, we observe that the expectation value of the solution, $\bar{U}$, consists of three shocks. Each of the shocks travel with a certain speed $s$ which has to be such that the following form of the Rankine-Hugoniot condition holds:

$$-s(U_+ - U_-) = F_+ - F_-.$$  \hfill (7.4)

Notice that because $F$ is nonlinear, $\bar{F}(U) \neq F(\bar{U})$. Equation (7.4) follows from the definition of a measure-valued solution, see Definition 5.1. Table 7.1 shows the values of
Figure 7.3 Shallow Water Equations: These figures show the expectation values of \( u \) and \( f \) obtained for the numerical simulation shown in Figure 7.2 on the right.

\[ s = \frac{\bar{f}_+ - \bar{f}_-}{\bar{u}_+ - \bar{u}_-} \]

compared to the numerically observed \( s \).

We remark that the middle transition shown in Figure 7.3 is indeed a shock. We conclude this by observing that the further out in time the run is taken, i.e., the more terms used in the approximation of the expectation values, the steeper the transition becomes.

<table>
<thead>
<tr>
<th>( \bar{u}_- )</th>
<th>( \bar{u}_+ )</th>
<th>( f_- )</th>
<th>( f_+ )</th>
<th>( s )</th>
<th>( (f_+ - f_-)/(\bar{u}<em>+ - \bar{u}</em>-) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>-0.132</td>
<td>0.1200</td>
<td>0.0965</td>
<td>0.173</td>
<td>0.177</td>
</tr>
<tr>
<td>-0.132</td>
<td>-0.150</td>
<td>0.0965</td>
<td>0.0923</td>
<td>0.236</td>
<td>0.233</td>
</tr>
<tr>
<td>-0.150</td>
<td>-0.184</td>
<td>0.0923</td>
<td>0.0811</td>
<td>0.330</td>
<td>0.329</td>
</tr>
</tbody>
</table>

Notice again that a Riemann problem with non-profilable shock initial data, produced a measure-valued solution whose expectation values consist of the three shock waves. Note also the location of the middle states, \( \overline{U}_{M1} \) and \( \overline{U}_{M2} \) relative to the oscillations in Figure 7.4. They lie far from the “center” of the oscillations.
Three-phase Flow Model

Adding the dissipation $D$, given by equation (4.30), to the system (3.32) gives

$$
\begin{bmatrix}
u \\
v
\end{bmatrix} + \begin{bmatrix}
-u^2 + v^2 + 2cv \\
2uv - 2cu
\end{bmatrix} + \epsilon D \begin{bmatrix}
u \\
v
\end{bmatrix} = e_D \begin{bmatrix}
u \\
v
\end{bmatrix}.
$$

(7.5)

where

$$
D = \begin{bmatrix}
0.0970 & -0.0209 \\
-0.0209 & 0.6750
\end{bmatrix}
$$

(7.6)

and we let $c = 0.23$.

We consider the Riemann initial value problem with Riemann data given by

$$
U_L = \begin{bmatrix}
.2110 \\
-.1755
\end{bmatrix}
\quad \text{and} \quad
U_R = \begin{bmatrix}
.2800 \\
-.1128
\end{bmatrix}.
$$

(7.7)

As described in Chapter 4, this is again shock initial data which corresponds to a Lax admissible 1-shock solution to the hyperbolic problem. However, this shock does not have a viscous profile. There is a limit cycle surrounding the repellor $U_L$ in the dynamical system (2.25). The corresponding phase space portrait is shown in Figure 7.5.
Figure 7.5 Three-Phase Flow: Limit cycle surrounding the repellor $U_L$.

the shallow water equations, if we let $D = I$, then there is a connecting orbit in state space and the numerical solution consists of the initial shock traveling at a constant speed. However, with $D$ given by (7.6), $\Delta x = .187$, $\Delta t = .192$, and $\epsilon = .215$ we once again have persistent oscillations. The function $U$ is approximated over the $x$-interval $[-13000, 1000]$ out to time $T = 64000$. Figure 7.6 shows $u$ vs. $x/t$ for this case. Again, as time increases, the oscillations increase in frequency over the same $x/t$ interval while maintaining a fixed amplitude. Compare this to the expectation values $\bar{u}$ and $\bar{f}$ in Figure 7.7, where again $f$ and $g$ are the first and second components of the flux function respectively. These expectation values were calculated using equation (6.33) with $\Delta \bar{t} = 10.0$ out to time $T = 64000$. The plots of $\bar{v}$ and $\bar{g}$ are similar. Just as in the shallow water equations, Theorem 6.1 and Theorem 6.2 prove that these shocks are weak solutions of the hyperbolic problem. Again, we have verified this numerically by checking that the expectation values $\bar{U}$ and $\bar{F}$ satisfy the hyperbolic system in a measure-valued sense. That is, we verified that the Rankine-Hugoniot jump conditions (7.4) are satisfied by the two shocks in $\bar{U}$ and $\bar{F}$.

It is difficult to tell from looking that the first transition is, or is converging to, a shock. That it is indeed converging to a jump discontinuity becomes more clear when
Figure 7.6 Three-Phase Flow: Oscillations in $u$ when $D$ is given by (7.6). The frequency of oscillations increases, whereas the amplitude stays uniformly bounded as $\epsilon \to 0$.

Figure 7.7 Three-Phase Flow: The figures show the expectation values of $u$ and $f$. 
Figure 7.8 Three-Phase Flow: The solid line is just a magnified view of the first shock in $\bar{u}$ in Figure 7.7. The dotted line is an approximation of $\bar{u}$ using fewer terms in the approximating partial sum (6.33). Notice that the transition from the left state to the middle state is more abrupt and that the overshoot is gone.

one compares the previous plot of the approximation to $\bar{u}$ to one that uses fewer terms. Figure 7.8 shows a magnified view of two successive approximations to $\bar{u}$. We can see that the run pushed further out in time has considerably shortened the $x/t$ interval where the transition occurs.
8 CONCLUSIONS

We have seen that there are cases of nonexistence of Riemann solutions in the strictly hyperbolic region of two different models. The numerical solution of the associated parabolic problem showed persistent oscillations in the state variables. We showed that in the limit, as the viscosity approached zero, the solutions converged in the weak-* topology of $L^\infty$ to a measure-valued solution of the hyperbolic problem. The expectation values of these measure-valued solutions were approximated and shown to indeed be weak solutions. All of this shows the importance of the choice of the dissipation matrix $D$. Since with some choices of $D$, but not all, we get oscillations.

A number of questions remain. Do the oscillations represent some physical behavior? Are they a failure of the model? Exactly when do these oscillations and measure-valued solutions occur? What other models exhibit this behavior?

We conjecture that in the three-phase flow model the oscillatory solution corresponds to interspersing of different phases. This would explain the rapid rise and fall of the saturation variables.

Furthermore, preliminary results indicate that the behavior studied here can be expected in models in which the state space is not the entire Euclidean space, and the viscosity matrix is not everywhere stable in the sense of Majda and Pego. See Refs. [8] and [21], and Appendix A.

Specifically, in both cases presented in this thesis, the space where the equations are hyperbolic does not correspond to the entire Euclidean space. In the shallow water equations, the state space is given by $\phi \geq 0$. In the three-phase flow model the equations
are hyperbolic outside a circle of radius $c$ (0.23) centered at the origin.

In such situations, typical positive definite viscosity matrices that are different from a multiple of the identity, produce a large region of points in state space that are linearly unstable in the sense of Majda and Pego (see Appendix A). We believe that this phenomenon is responsible for the behavior studied in this thesis.

The behavior cannot be ignored as many models in practical use exhibit these properties. One such example is the three-phase flow model studied in this thesis. Another one is the compressible Euler equations for which the state space corresponds to the half-space $\rho > 0$ ($\rho = \text{density}$). The viscosity for this model imposed by the Navier-Stokes equation is not even positive definite, and therefore fails to satisfy the Majda-Pego stability condition.

Further investigation in this direction needs to be done to identify the mechanisms that lead to the phenomena described in this thesis.
APPENDIX A MAJDA-PEGO REGION

Majda and Pego define what they call a \textit{strictly stable} diffusion matrix $D(U_0)$. See Ref. [21]. The basic idea is to consider the parabolic equation (2.23) linearized about the constant state $U_0$,

$$V^\varepsilon_t + F'(U_0)V^\varepsilon_x = \varepsilon D(U_0)V^\varepsilon_{xx}$$

(A.1)

$$V^\varepsilon(x,0) = V_0(x).$$

Their requirement is that $\lim_{\varepsilon \to 0} V^\varepsilon = V^0$ for all initial data $V_0 \in L^2$, where $V^0$ is the solution of the linearized hyperbolic equation. That is, equation (A.1) must be uniformly well posed in $L^2$ as $\varepsilon \to 0$. This turns out to be a little too permissive since $D \equiv 0$ and some non-positive definite matrices satisfy this condition. Therefore, we require $D$ to be positive definite. In a different sense this strict stability condition is too strict, since matrices as nice as the identity will not be strictly stable if the conservation law has an elliptic region. Following the lead of Čanić and Plohr in [8] we define certain points in state space to be Majda-Pego points.

\textbf{DEFINITION A.1} A state $U_0$ is a Majda-Pego point for the positive definite viscosity matrix $D$ provided that $F'(U_0)$ has distinct real eigenvalues and that the Cauchy problem for $V^\varepsilon_t + F'(U_0)V^\varepsilon_x = \varepsilon D(U_0)V^\varepsilon_{xx}$ is uniformly well-posed in $L^2$ as $\varepsilon \to 0$.

This is equivalent in the 2 by 2 case to

1. $\det D(U_0) > 0$, and

2. $R^{-1} D R(U_0)$ has positive diagonal elements.
where \( R \) is a matrix of eigenvectors of \( F'(U) \). See Refs. [21, 8]. The boundary of the set of Majda-Pego points is related to the Bogdanov-Takens locus. Namely, it has been proven in Ref. [8] that a state \( U_0 \) is a Majda-Pego point for the viscosity matrix \( D \) if and only if \( \text{tr}[-s + F'(U_0)] \) and \( \text{tr}\{D^{-1}[-s + F'(U_0)]\} \) have the same (nonzero) sign for each eigenvalue \( s \) of \( F'(U_0) \). In particular, the boundary of the set of Majda-Pego points is contained in the union of the following curves: the boundary of the elliptic region (the coincidence locus) and the set of all states for which the Bogdanov-Takens bifurcation occurs at \( U_L \), where \( U_R \to U_L \). For more details, see Ref. [8].

Many first order numerical methods used to solve the strictly hyperbolic system,

\[
U_t + AU_x = 0,
\]  

(A.2)

solve the modified equation,

\[
U_t + AU_x = D(A, \Delta x, \Delta t)U_{xx},
\]  

(A.3)

to second order. The diffusive term \( D \) from the modified equation is often a polynomial in \( A \). For example the Lax-Friedrichs method and the upwind method have diffusion given by,

\[
D = \frac{\Delta x^2}{2\Delta t} \left( I - \left( \frac{\Delta t}{\Delta x} A \right)^2 \right) \quad \text{and} \quad D = \frac{\Delta x}{2} A \left( I - \frac{\Delta t}{\Delta x} A \right)
\]

respectively. With an appropriate stability condition on \( \Delta t \), the above matrices will be positive definite.

Notice that if our \( D \) matrix is a polynomial in \( A \) then the matrix of eigenvectors, \( R \), that diagonalizes \( A \) will also diagonalize \( D \). These diagonal elements of \( R^{-1}DR \) are the eigenvalues of \( D \) and are therefore positive. Thus, the region of strict hyperbolicity is the same as the region of Majda-Pego points if the \( D \) matrix used is a polynomial in \( F' \). However, if one wants to specify their diffusion explicitly, and not rely on the method to select it, then \( D \) will probably not be a polynomial in \( F' \).
Figure A.1 A Three-Phase Flow Model: The elliptic region is the interior of the smaller ellipse. The Majda-Pego region is the area outside the larger ellipse.

When the diffusion matrix is added to the hyperbolic system giving a parabolic system, physical considerations are the determining factor. It is not clear that a physically reasonable diffusion will produce a dominant Majda-Pego region, or that the states outside the Majda-Pego region are non-physical. For example, system (3.26), which comes from three-phase flow in porous media, with \( c = .1 \) and the diffusion matrix \( D \), given by

\[
D = \begin{bmatrix}
.026550 & -.003680 \\
-.003680 & .006095
\end{bmatrix}, \tag{A.4}
\]

has a substantial region of points outside the Majda-Pego region. This \( D \) equals the capillary pressure diffusion matrix (3.22) with \( s_1 = .4 \) and \( s_2 = .25 \). Figure A.1 shows the elliptic region and the Majda-Pego region for this system. The elliptic region is the inside ellipse centered at \((1/3, 1/3)\). The Majda-Pego region is the area outside the outer ellipse. Recall that \( s_1 + s_2 \leq 1 \), so only the lower left half of the square is physically reasonable. This shows that a nontrivial part of the state space consists of points that are not Majda-Pego.
APPENDIX B LAX-FRIENDRICH AND LAX-WENDROFF

In Section 7 we used numerical methods to solve the parabolic problem (2.23) that is the result of adding a diffusive term to the hyperbolic system (2.1). For certain choices of Riemann initial data and diffusion matrix $D$, we see persistent oscillations in the numerical solution. We address in this appendix two logical questions that arise from this. What happens if you use a standard numerical method on the hyperbolic problem? Are the oscillations related to numerical dispersion? We answer these questions by solving the shallow water equations with the Lax-Friedrichs method and the Lax-Wendroff method. The Lax-Friedrichs method has numerical diffusion and the Lax-Wendroff method has numerical dispersion.

The Lax-Friedrichs method is an explicit method given by

$$U_j^{n+1} = \frac{1}{2} (U_{j+1}^n + U_{j-1}^n) - \frac{\Delta t}{2\Delta x} (F(U_{j+1}^n) - F(U_{j-1}^n)). \quad (B.1)$$

This is a first order method. Thus, it has numerical diffusion. We used this method on the shallow water equations (3.11) with the initial data given by (7.3). For $\Delta x = 1.00$ and $\Delta t = 0.95$ we obtain a smoothed shock traveling at a constant speed. A plot of $u$ vs. $x/t$ at time $T = 320$ is shown in Figure B.1. Except for the amount of diffusion this is the same solution as was obtained solving the parabolic problem with $D = I$. It is important to note that unless the physical diffusion is a multiple of the identity this solution may not be physically relevant. More generally, using a method with numerical diffusion on any hyperbolic problem may lead to physically unrealistic solutions.

Since the flux $F$ is nonlinear we use the Richtmyer two-step Lax-Wendroff method
Figure B.1 The shallow water equations solved with the Lax-Friedrichs method.

given by

\[
U_{j+1/2}^{n+1/2} = \frac{1}{2} (U_j^n + U_{j+1}^n) - \frac{\Delta t}{2\Delta x} (F(U_{j+1}^n) - F(U_j^n)) \tag{B.2}
\]

\[
U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} \left( F(U_{j+1/2}^{n+1/2}) - F(U_{j-1/2}^{n+1/2}) \right). \tag{B.3}
\]

Since this is a second order method, it is dispersive. We used this method on the shallow water equations (3.11) with the initial data given by (7.3). For \(\Delta x = 0.70\) and \(\Delta t = 1.80\) out to time \(T = 320\) we obtain a fairly sharp shock solution with a small spike just before the shock, as shown in Figure B.2. If we try to remove this spike by reducing \(\Delta t\) to 0.36 we unintentionally increase rather than reduce the amount of dispersion. Figure B.3 shows this plot of \(u\) vs. \(x/t\). Note that the oscillations are dependent on the choice of \(\Delta x\) and \(\Delta t\), unlike the oscillations observed in Chapter 7. This behavior is indicative of dispersion. Note also that the shock being approximated by the Lax-Wendroff method is again the shock obtained solving the parabolic problem with \(D = I\).
Figure B.2 Solution of the shallow water equations using the Lax-Wendroff method with $\Delta x = 0.70$ and $\Delta t = 1.80$.

Figure B.3 Solution of the shallow water equations using the Lax-Wendroff method with $\Delta x = 0.70$ and $\Delta t = 0.36$. 
APPENDIX C RELATIVE PERMEABILITY ADDITIONS

Here we present the explicit formulae for the rational functions $a_1$ and $a_2$ mentioned in Chapter 3. By the addition of the $a_1$ and $a_2$ terms to the relative permeability we change the flux function from the one in system (3.24) to the one in system (3.26). This requires the solution of two nonlinear equations in two unknowns ($a_1$ and $a_2$). We solved these two equations using Maple V. They are as follows.
\[ a_1 = 2c(12cs_1s_2^2 - 86cs_1s_2^2 + 64s_1^7c - 256cs_1^3s_2^2 - 280cs_1s_2^2 \\
- 256cs_1s_2^4 + 192cs_1s_2^4 - 48cs_1^5 - 168cs_1s_2^2 - 112cs_2^6 \\
+ 112cs_2^5 + 216cs_1s_2^4 - 144s_1^6cs_2 + 232s_1^5cs_2 + 88cs_1^4s_2^3 \\
- 16s_1^7c + 80cs_1^3s_2^4 + 40s_1^6c - 176cs_1^4s_2^3 + 304cs_1^2s_2^5 \\
+ 256cs_1^2s_2^5 + 24cs_2^3 - 68cs_2^4 + 98cs_1^4s_2 - 200cs_1^2s_2 \\
- 24cs_1^2s_2 - 16s_1^3c + 32cs_1^5 + 2cs_1^5 - 12cs_1^5 - 4cs_2^3 \\
- 56s_1^2c + 64s_1^6cs_1^3 + 40s_1^7s_2^3c - 96s_1^6cs_1^3 + 80s_1^6s_2^2c \\
+ 96s_1^5s_2^2c - 31^2s_1 - 153^1/2s_1^3 + 63^1/2s_1^2 + 72cs_1^3s_2^3 \\
+ 12s_2^23^1/2 - 32s_2^33^1/2 - 2s_1^33^1/2 - 143^1/2s_1^5 + 43^1/2s_1^4 \\
+ 203^1/2s_2^4 + 2cs_2s_1 + 24cs_2^2s_1^2 + 16cs_1^4s_2^4 - 763^1/2s_2^3s_1 \\
+ 1523^3^3^1/2 - 403^1/2s_1^5s_2^2 + 2363^1/2s_1^3s_2^3 - 823^1/2s_1^3s_4^2 \\
+ 543^1/2s_1^5s_2 - 2833^1/2s_1^3s_2^3 - 2513^1/2s_1^3s_3^2 - 1653^1/2s_1s_2^4 \\
- 103^1/2s_2^5s_2 + 2003^1/2s_1^2s_2^4 - 223^1/2s_1^2s_2^6 + 943^1/2s_1^5s_2^3 \\
- 633^1/2s_2^5s_2 + 1123^1/2s_1^3s_2 + 183^1/2s_2^3s_1 + 1963^1/2s_2^3s_2^2 \\
- 1093^1/2s_1^4s_2 - 563^1/2s_1^2s_2^5 - 743^1/2s_1^2s_2^6 + 1603^1/2s_1^4s_2^2 \\
+ 20s_2^6s_1^3/2 - 42s_2^23^1/2 + 48s_2^33^1/2 - 4s_2^3s_2^3 \\
/ (-3 + 12s_1 - 32c^2s_1s_2^2 - 18s_2^2 + 12s_2 - 12s_1^2 - 8c_2s_1s_2 \\
- 16c^2s_1s_2^2 + 16c^2s_1s_2^2 + 16c^2s_1^6 - 16c^2s_1^6 - 32c^2s_1^5 + 32c^2s_1^5 - 16c^2s_1^3 \\
+ 4c^2s_1^3 + 16c^2s_1^6 - 32c^2s_1^6 + 32c^2s_1^6 - 16c^2s_1^3 \\
+ 4c^2s_1^3 - 16c^2s_1s_2^3 + 36s_1^2s_2 + 16c^2s_1s_2^3 - 32c^2s_1s_2^3 \\
+ 12s_1^2s_2 - 36s_1s_2 - 12s_2^2s_1 + 36s_2^2s_1 - 12s_2^2s_1 \\
- 12s_2^2s_1 - 16c^2s_1s_2^2 - 3s_2^4 + 12s_3^2 + 12s_2^2 + 32c^2s_1s_2^3).\]
\[ a_2 = -2c(24cs_1s_2^3 - 98cs_1s_2^3 + 16s_2^7c - 256cs_1^5s_2^2 - 232cs_1s_2^5 \\
- 304cs_1^5s_2^4 + 144cs_1s_2^6 - 112cs_1^5s_2^2 - 88cs_1^3s_2^4 - 40cs_2^6 \\
+ 48cs_2^5 + 200cs_1s_2^4 - 192s_1^6cs_2 + 280s_1^4cs_2 + 168cs_1^3s_2^3 \\
- 64s_1^7c + 176cs_1^3s_2^4 + 112s_1^6c - 80cs_1^4s_2^3 + 256cs_1^4s_2^2 \\
+ 256cs_1^4s_2^2 + 12cs_2^4 - 32cs_1^3s_2^2 + 86cs_1^3s_2^2 - 216cs_1s_2^4 \\
- 12cs_1^2s_2 + 68cs_1^4 + 16s_1^8c + 4cs_1^2 - 24cs_1^3 - 16cs_1^4s_2^4 \\
- 2cs_2^2 - 40s_1^7cs_1 - 96s_1^5cs_2^3 + 56s_1^7s_2c - 80s_1^6s_2^2 \\
+ 96s_1^6s_2^2c + 64s_1^5s_2^3c - 231/2s_1 - 3231/2s_1 + 1231/2s_1^2 \\
- 72cs_1^3s_2^3 + 6s_2^31/2 - 15s_2^31/2 - s_2^31/2 - 4231/2s_2^5 \\
- 431/2s_1^7 + 2031/2s_1^5 + 4831/2s_1^4 - 2cs_1s_1 - 24cs_1^2s_1^2 \\
- 6331/2s_2s_1 + 11231/2s_2^3s_1 - 5631/2s_1s_2^3 + 23631/2s_1^3s_2^3 \\
- 7431/2s_1^3s_2^4 + 9431/2s_1^3s_2^4 - 25131/2s_1^3s_2^4 - 28331/2s_1^3s_2^4 \\
- 10931/2s_1s_2^4 - 2231/2s_1s_2^4 - 16031/2s_1s_2^4 - 1031/2s_1s_2^4 \\
+ 5431/2s_1s_2^5 - 7631/2s_1s_2^5 - 15231/2s_1s_2^5 + 1831/2s_2s_1 \\
+ 19631/2s_1s_2^5 - 16531/2s_1s_2^5 - 4031/2s_1s_2^5 - 8231/2s_1s_2^5 \\
+ 20031/2s_1s_2^5 + 4s_2^31/2 - 14s_2^31/2 + 20s_2^31/2) \\
/ (-3 + 12s_1 - 32c^2s_1s_2^2 - 18s_2^2 + 12s_2 - 18s_1^2 \\
- 8c^2s_1s_2 - 16c^2s_1s_2^2 + 16c^2s_1s_2^2 + 16c^2s_1s_2^6 - 32c^2s_2^5 \\
+ 32c^2s_1^4 - 16c^2s_1^4 + 4c^2s_1^2 + 16c^2s_1^4 - 32c^2s_2^5 \\
+ 32c^2s_1^4 - 16c^2s_2^2 + 4c^2s_2^2 - 16c^2s_1s_2^3 + 36s_1s_2^3s_2 \\
+ 32c^2s_1s_2^3 - 32c^2s_1s_2^3 - 12s_1^3s_2 - 36s_1s_2 - 18s_2s_1 \\
+ 36s_2s_1 - 12s_1s_2 - 3s_2^2 + 16c^2s_1s_2^2 - 3s_1^4 \\
+ 12s_1^2 + 12s_2^3 + 32c^2s_1s_2^2).}
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