Tight Money Policies and Inflation Revisited

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Tight Money Policies and Inflation-Revisited*

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Abstract

This paper reconsiders the link between tight money policies and inflation in the spirit of Sargent and Wallace’s (1981) influential paper “Some Unpleasant Monetarist Arithmetic”. A standard neoclassical model with production, capital, bonds, and return-dominated currency is used to study the long-run effects on inflation of a tightening of monetary policy engineered via an open market sale of bonds. The potential for tight money policies to be inflationary (unpleasant arithmetic) exists even when the real interest rate is below the growth rate of the economy, and such equilibria can be stable. In contrast, when monetary policy is conducted via a fixed inflation-rate rule, the only stable equilibrium is the one that exhibits pleasant arithmetic. The two monetary policy rules therefore produce sharply different predictions about the likely observability of unpleasant arithmetic in real world economies.

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1 Introduction

Standard conventional wisdom suggests that a tightening of monetary policy ought to be accompanied by a reduction in inflation. There is even ample evidence, at least from highly inflationary economies like Germany and Argentina in the post-war period, to support such a notion. A counterexample to this well-accepted line of thinking is provided by Sargent and Wallace in their classic paper “Some Unpleasant Monetarist Arithmetic”. In that paper, they show that in a standard model where monetary policy cannot affect the fixed real interest rate, it is quite possible for a tightening of monetary policy to precipitate an increase in inflation (“unpleasant monetarist arithmetic”). This “spectacular” result is however predicated on the satisfaction of one crucial yet controversial condition: the after-tax real rate of interest has to be higher than the real rate of growth of the economy.

Subsequent to the publication of the Sargent and Wallace paper, a lively debate has ensued in which a number of researchers have questioned the relevance of this condition for real world economies, and have gone on to wonder if unpleasant arithmetic is nothing but a theoretical curiosum. Darby (1984) and more recently, Espinosa and Russell (1998), have pointed out for example that for post-war U.S, the after-tax real rate of return on government debt is close to zero and that the real growth rate of GDP is over 3 percent. The implication is clear: since we live in a low real-interest-rate world, the only arithmetic we are likely to observe is of the “pleasant” variety. This paper is part of a line of research dating back to Miller and Sargent (1984) and more recently, Bhattacharya, Guzman, and Smith (1998) and Espinosa and Russell (1999, 2001) that questions this implication. At a broader level, it seeks to understand the long-run impact of monetary policy on inflation in low real-interest-rate economies using a model with neoclassical production and capital in which money has real effects. Although these other papers have revealed the possibility of obtaining unpleasant arithmetic in low interest economies when the marginal product of capital is variable, ours is the first to focus on this case. In a sense, our paper is a direct descendant of Bhattacharya, Guzman, and Smith (1998) who delivered unpleasant arithmetic in a low real-rate economy but with a fixed marginal product of capital.

To that end, we produce a model that is a hybrid of Diamond (1965) and Wallace (1984). The former component essentially allows us to replace the assumption of a linear production technology in Sargent and Wallace (1981) with the assumption of a neoclassical production technology. As a

1Throughout the paper, we use the term “real interest rate” to signify the real return on one-period government bonds, and not the real marginal product of capital.
result, the real interest rate is no longer fixed. The latter component permits us to study equilibria where money is held even when dominated by competing stores of value. In our setup then, the single final good may be used to purchase government bonds directly or it may be stored to yield capital the following period. The latter activity is assumed to be intermediated, and also subject to a standard currency reserve requirement. Capital dominates money in rate of return. We focus on equilibria where the reserve requirement binds. In such equilibria, bonds dominate money in rate of return, and the marginal product of capital exceeds the real return on bonds. It remains possible for the real rate of return on government bonds to stay below the long run real rate of growth (unity in our case) of the economy.

Following Wallace (1984), we assume that the monetary authority chooses the bonds-money ratio in the initial period and keeps the ratio fixed for all time. Changes in this bond-money ratio may then be thought of as permanent open market operations. As in Wallace (1984), a tightening of monetary policy therefore implies a permanent cut in this ratio. In such a world, and for fairly general specifications of preferences and technology, we provide a complete characterization of the necessary and sufficient conditions needed to obtain unpleasant monetarist arithmetic in steady states. In particular, we show if household saving is mildly sensitive to its return, then multiple equilibria (with the same bonds-money ratio) are possible. At some of these many equilibria, the volume of seigniorage revenue the government collects is increasing in the inflation rate. These are the ones that are the most empirically relevant. At those equilibria, tight money necessarily reduces the capital stock, and may even be inflationary. In particular, if the initial real interest rate exceeds the growth rate of the economy, then the unpleasant arithmetic of Sargent and Wallace necessarily obtains. Recall that our principal focus is on equilibria with real interest rates below unity. We show that even in this case, unpleasant arithmetic remains a strong possibility.

Sharper results are obtained when saving is return-invariant. In this case, there is a unique monetary stationary equilibrium (if any). At this equilibrium an increase in the inflation rate raises the volume of seigniorage revenues. This equilibrium may have an associated real interest rate that is high or low. If the initial real interest rate is greater than unity, unpleasant arithmetic holds. Even at the equilibrium with a real interest rate less than unity, it is still possible for unpleasant (and pleasant) arithmetic to be observed. Specializing to a model economy with Cobb-Douglas technology and logarithmic utility, we go on to show that pleasant (unpleasant) arithmetic

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2 This result has been obtained in slightly different environments by Miller and Sargent (1984), Bhattacharya, Guzman, and Smith (1998), and Espinosa and Russell (1998, 1999).
is a strong possibility for low (high) values of monetary-policy tightness.\(^3\) Two additional points deserve mention here. Unlike Bhattacharya, Guzman and Smith (1998), our results does not require the marginal product of capital to exceed unity. Similarly, for tight money to be disinflationary, unlike Wallace (1984), we do not require that the marginal product of capital be less than unity.\(^4\)

The paper closest in spirit and form to our current endeavour is Espinosa and Russell (2001).\(^5\) They consider a model that is almost identical to ours except for the following features. First, and foremost, their specification of the monetary policy rule is different, more in line with other papers in this tradition including Miller and Sargent (1984) and Sargent and Wallace (1981), as opposed to Wallace (1984). Specifically, they assume that monetary policy is conducted by fixing the money growth rate and allowing the bonds-money ratio to adjust accordingly. Second, our specification of preferences and technology is more general; they focus on return-invariant saving and Cobb-Douglas technology throughout. Third, they use reserve requirements assumptions under which the return rates on capital and bonds are equal and then introduce “capital intermediation costs” to create a wedge between them. In contrast, we make assumption on reserve requirements that produce this wedge without the need for such costs. The last two differences are possibly trivial enough, but not the first.

Received wisdom from related overlapping generation models suggests that, under a fixed money growth rule (as in Sargent, 1987), only the equilibria exhibiting pleasant arithmetic are dynamically stable. Espinosa and Russell (2001) have confirmed this wisdom and gone on to raise concerns about the actual observability of stationary equilibria exhibiting unpleasant arithmetic. That is, they have tried to rule out the possibility of observing unpleasant arithmetic on stability grounds.\(^6\) Our results indicate that under their assumptions on preferences, but with a fixed bonds-money ratio rule, there is a unique low real-rate empirically relevant steady state that is dynamically stable, and which exhibits unpleasant arithmetic. Under more general preference (and production) assumptions, and under the fixed bonds-money ratio rule, all empirically relevant steady states are dynamically approachable, including those exhibiting unpleasant arithmetic.\(^7\) The upshot is that

\(^3\)In a model without production and capital, Espinosa and Russell (1998) show that under low interest rates, a small tightening of money is disinflationary if initially money is not too tight.

\(^4\)In fact, pleasant arithmetic is not a possibility in Bhattacharya, Guzman and Smith (1998) under their assumption that the real return to storage exceeds unity.

\(^5\)We thank two anonymous referees for drawing our attention to this paper.

\(^6\)Espinosa and Russell (2001) do not present a formal stability analysis in the more general return-dependent saving case. Numerical experiments however indicate that their stability results are robust to this generalization.

\(^7\)With return dependent (invariant) saving, the equilibrium law of motion for the capital stock is two (one) dimensional. The “strong” dynamic stability of the unique steady state in the latter case is replaced by the “weaker” dynamic approachability of all steady states that empirically relevant.
if the fixed bonds-money ratio rule formulation is realistic, then there is no presumption in favor of pleasant arithmetic steady states on stability grounds.

To further facilitate comparison with Espinosa and Russell (2001), we rework our model assuming return-invariant saving but with a fixed inflation rate policy rule. By means of numerical examples we show that multiple monetary stationary equilibria are now possible, and these may all have real interest rates less than unity. In the case there are exactly two equilibria, the case Espinosa and Russell focus on, we prove that the equilibrium with the lower (higher) interest rate exhibits pleasant (unpleasant) arithmetic. Moreover, the equilibrium exhibiting pleasant arithmetic is the only dynamically stable equilibrium. As such, just as Espinosa and Russell (2001) find, the only arithmetic one is likely to observe is pleasant.

In sum, the different monetary policy rules studied here produce sharply different predictions about the likely observability of unpleasant arithmetic. With a fixed bonds-money ratio rule, and return-invariant saving, it is quite possible for a low real-rate equilibrium to exhibit unpleasant arithmetic and be dynamically stable. With a fixed inflation rate rule, and return-invariant saving, it is not possible for a low real-rate equilibrium to exhibit unpleasant arithmetic and be dynamically stable. While our result strengthens the case for the observability of unpleasant arithmetic, the Espinosa and Russell result definitely weakens it.

The rest of the paper is organized as follows. Section 2 describes the model under a fixed bonds-money ratio rule. Section 3 defines a monetary equilibrium and explores its properties. Section 4 sets out the main results concerning the long-run effects of open-market operations. Section 5 sketches the model under a fixed inflation rate rule and establishes how the predictions for the likely observability of unpleasant arithmetic differs sharply across the two rules.

2 The Model

2.1 Environment

We consider an economy consisting of an infinite sequence of two-period lived overlapping generations, an initial old generation, and an infinitely-lived government. Let \( t = 1, 2, \ldots \) index time. At each date \( t \), a new generation comprised of \( N \) identical members appears.

There is a single final good produced using a standard neoclassical production function \( F(K_t, L_t) \)

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Espinosa and Russell (2001) study a fixed-money-growth rule, not a fixed-inflation rule. These rules are entirely equivalent in steady states, but not so all along the transition. The stability properties of steady states under the two rules are very similar.
where $K_t$ denotes the capital input and $L_t$ denotes the labor input at $t$. Let $k_t = K_t / L_t$ denote the capital-labor ratio (capital per young agent). Then, output per young agent at time $t$ may be expressed as $f(k_t)$ where $f(k_t) = F(K_t / L_t, 1)$ is the intensive production function. We assume that $f(0) = 0$, $f' > 0 > f''$, and that the usual Inada conditions hold. The final good can either be consumed in the period it is produced, or it can be stored to yield capital the following period. For reasons of analytical tractability, capital is assumed to depreciate 100% between periods.

Each agent is endowed with one unit of labor when young, and is retired when old. In addition, the initial old agents are each endowed with $M_1 > 0$ units of fiat currency and $k_1 > 0$ units of capital.

Let $c_{1t}$ ($c_{2t}$) denote the consumption of the final good by a representative young (old) agent born at $t$. All such agents have preferences representable by the utility function $U(c_{1t}, c_{2t})$ where $U$ is twice-continuously differentiable, strictly increasing, and strictly concave in its arguments.

Finally, the government has a constant net-of-interest deficit of $g \geq 0$ in each period. The government does not levy any direct taxes and hence finances this every period entirely by issuing bonds and money. Let $M_t$ denote the per capita stock of money outstanding at the end of period $t$, and $B_t$ denote the outstanding per capita supply of bonds (in nominal terms) where $M_0 = B_0 = 0$.

### 2.2 Markets

Young agents supply their labor endowment inelastically in competitive labor markets, earning a wage income of $\omega_t$ at time $t$ where

$$\omega_t \equiv \omega(k_t) = f(k_t) - k_t f'(k_t) \quad \forall t \geq 1. \quad (1)$$

In addition, capital is traded in competitive capital markets, and earns a gross real return of $q_{t+1}$ between $t$ and $t+1$ where,

$$q_{t+1} = f'(k_{t+1}). \quad (2)$$

It is assumed that private agents do not have direct access to productive capital presumably because of a minimum size restriction as in Freeman (1987). They may hold either bank deposits, $D_t$, or government bonds, $B_t$, and the latter are not intermediated. Let $p_t$ denote the time $t$ price level, $R_{t+1}^d$ denote the gross real interest rate on bank deposits, and $R_{t+1}^g$ be the gross real interest rate on government bonds. Denote $d_t \equiv D_t / N p_t$ and $b_t \equiv B_t / N p_t$. 

Each young agent born at date $t \geq 1$ chooses how much bonds and deposits to hold to maximize

$$U(c_{1t}, c_{2t})$$

subject to

$$c_{1t} + d_t + b_t \leq w_t$$

and

$$c_{2t} \leq R_{t+1} d_t + \rho_{t+1} b_t.$$  \hfill (3)

For both bonds and deposits to be held, $R_{t+1} = \rho_{t+1}$ must hold. Let $S_t = d_t + b_t$ denote savings of a young agent. Then the agent's problem is simply to choose $S_t$ to maximize $U \left[ \omega(k_t) - S_t, \rho_{t+1} S_t \right]$. Let

$$S(\omega(k_t), \rho_{t+1}) \equiv \text{arg max } U \left[ \omega(k_t) - S_t, \rho_{t+1} S_t \right].$$  \hfill (5)

The function $S$ summarizes an agent's optimal saving behavior.

At several points below, we will be assuming (for reasons of analytical simplicity) that saving is not too sensitive to a change in its return. Formally stated, we assume that the interest elasticity of saving is less than unity, or that

$$S_\rho \frac{\rho}{S} < 1.$$  \hfill (A.1)

We now turn to the determination of the interest on deposits (and bonds). As stated earlier, private agents cannot access the capital production technology directly. Banks arise to perform a simple intermediation function. They collect the individual deposits from all young agents and invest it in the capital production technology on their behalf. In return, private agents are promised a competitive return of $R_{t+1}^d$ per unit deposited. We assume that banks are subject to a currency reserve requirement in that for every unit of funds invested in the capital production technology, they are required by law to hold a fixed fraction $\lambda$ of that amount in the form of currency reserves. That is,

$$m_t \geq \lambda k_{t+1},$$  \hfill (A.2)

where $\lambda \in (0, 1)$. Let $k_{t+1}$ denote capital, and $m_t$ denote real reserves. Then, the banks' balance sheet requires $d_t = k_{t+1} + m_t$. If

$$q_{t+1} > \frac{p_t}{p_{t+1}}$$  \hfill (A.2)

The initial old agents face the following budget constraint: $c_{20} \leq q_1 k_1 + M_0/p_t$. Thus, at date 1, the initial old agents receive capital income, $q_1 k_1$, and nominal money, $M_0$, and they consume it all.

We have chosen to make this assumption solely to obtain sharper results. There is enough evidence to suggest that this assumption is empirically quite plausible. The consequences of abandoning this assumption are spelt out in Kudoh (2000).
holds, then capital dominates money in rate of return, and the reserve requirement (6) binds. In passing, note that (A.2) also implies that $f'(k_{t+1}) > \rho_{t+1}$. Henceforth, we exclusively focus on equilibria that satisfy (A.2).\textsuperscript{11}

Let $\phi \equiv \frac{1}{1+\lambda}$. Then $\phi$ represents the fraction of deposits held in the form of capital, while $(1-\phi)$ can be interpreted as the fraction of deposits required to be held as reserves. Then $(1-\phi)$ may be interpreted as a conventional reserve requirement. The real interest rate on bank deposits is then given by a weighted sum of returns to capital and money, \textit{i.e.},

\[
\rho_{t+1} = \phi q_{t+1} + (1-\phi) \left( \frac{p_t}{p_{t+1}} \right)
\]

must hold. It follows that if (A.2) holds, then $\rho_{t+1} > p_t/p_{t+1}$ holds too (\textit{i.e.}, in the equilibria we consider, bonds always dominate money in rate of return). It is the presence of a binding reserve requirement that creates a wedge between the return to capital and that on bonds.\textsuperscript{12} Since money pays no interest while bonds do, an open market purchase (\textit{i.e.}, selling more bonds) as an instrument of deficit finance essentially replaces a cheaper device with a more expensive one. We explore the consequences of this observation below.

2.3 The Government

The government finances a constant net-of-interest deficit of $g \geq 0$ every period by issuing one-period default-free bonds and by printing money. The government's flow budget constraint is

\[
p_t g = M_t - M_{t-1} + B_t - I_t B_{t-1},
\]

where $I_t$ is the gross nominal interest rate on bonds. Equation (8) states that the government finances its expenditures and interest obligations on outstanding government debt, from seignorage revenue earned by money creation and from the sale of new bonds. Using $m_t \equiv M_t/Np_t$, $b_t \equiv$

\textsuperscript{11}It is important to note that fiat money is valued in this setup solely because of the presence of a reserve requirement. In other words, if money is dominated in return and no legal restrictions (\textit{e.g.}, reserve requirements) are present, the demand for money would be zero. Since the current purpose is to study alternative modes of deficit finance, we restrict our attention only on equilibria where a positive demand for money (reserves) exists.

\textsuperscript{12}In Espinosa and Russell (1999, 2001), capital intermediation costs creates a similar wedge between the return to capital and the return to bonds. This particular difference between our models does not seem to create any major qualitative differences in the results.
Following Sargent and Wallace (1981), we assume that the fiscal authority in time 1 chooses the primary deficit $g$ for all $t$ and that it is the monetary authority that is forced to design its policies so that (9) is satisfied each period. Following Wallace (1984), Bhattacharya, Guzman, and Smith (1998), among others, we assume that the treasury fixes at date 1 the ratio of government debt to money, $\mu$, where

$$\mu \equiv \frac{B_t}{M_t} > 0 \quad \forall t \geq 1. \quad (10)$$

The central bank conducts its monetary policy by changing the nominal money stock at all future dates to ensure the ratio is always kept fixed at $\mu$. Henceforth, we will refer to this type of monetary policy as a fixed bond-money ratio rule. It may be convenient to think of variations in $\mu$ as permanent open market operations. We will sometimes interpret an increase in $\mu$ as a tight money policy. Finally, at date 1, the central bank also sets the reserve requirement $(1 - \phi)$.

3 Equilibrium

3.1 Characterization

We begin by formally defining a monetary competitive equilibrium for our economy.

**Definition 1** A binding monetary equilibrium is a set of sequences for allocations $\{S_t\}$, $\{b_t\}$, $\{k_t\}$, $\{m_t\}$, prices $\{g_t\}$, $\{\omega_t\}$, $\{p_t\}$, $\{\rho_t\}$, and the initial conditions $M_0 > 0$, $k_1 > 0$, $B_0 = 0$ such that

(a) factor markets clear, i.e., (1)-(2) hold, (b) asset market clears: $m_t + k_{t+1} + b_t = S(\omega_t, \rho_{t+1})$, $c) p_t \in \mathbb{R}_+$ for all $t$, (d) the government's budget constraints at date 1, $p_1 g + M_0 = M_1 + B_1$, and at dates $t > 1$, i.e., (9) hold, and (e) $f'(k_{t+1}) > p_t/p_{t+1} > 0$.

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$^{13}$Since $B_0 = 0$, the government budget constraint at date 1 is

$$g = \frac{M_1 - M_0}{p_1} + \frac{B_1}{p_1}$$

which may be rewritten as $p_1 g + M_0 = M_1 + B_1$. In equilibrium, this pins down the initial price level.

$^{14}$For a recent treatment of the "game of chicken" between the treasury and the central bank, see Barnett (2001).

$^{15}$Strictly speaking, Wallace (1984), as well as Espinosa and Russell (1998), study a rule that fixes the ratio of the nominal face value of bonds to the nominal money stock.

$^{16}$In Section 5, we will compare our results under this fixed $\mu$ rule with those obtained under a rule where the central bank sets the inflation rate and allows the bond-money ratio to reach the treasury's target of $\mu$ at some date in the future.

$^{17}$Bhattacharya and Haslag (2000) discuss issues of budget arithmetic in a model where the central bank conducts monetary policy by directly choosing the reserve requirement.
Noting that \( bt + dt = \left[ 1 + \frac{\mu(1 - \phi)}{\phi} \right] k_{t+1} \), we can rewrite the asset market clearing condition as
\[
[1 + \mu(1 - \phi)]k_{t+1} = \phi S(\omega(k_t), \rho_{t+1}). \tag{11}
\]

The equilibrium gross return on real balances (the inverse of the gross rate of inflation) for given \( \phi \) and \( g \) can now be determined as
\[
\frac{p_{t-1}}{p_t} = \frac{(1 + \mu)k_{t+1} - \mu \phi k_t f'(k_t) - \frac{\phi}{1 - \phi} g}{[1 + \mu(1 - \phi)]k_t}. \tag{12}
\]

Substituting (12) in (7), it is possible to write (7) as
\[
\rho_t = \frac{\phi k_t f'(k_t) + (1 - \phi)(1 + \mu)k_{t+1} - \phi g}{[1 + \mu(1 - \phi)]k_t}. \tag{13}
\]

(13) describes the equilibrium rate of return on government bonds as a function of the capital-labor ratio. Therefore, (11) and (13), along with the restrictions embedded in Definition 1 jointly constitute the equilibrium conditions for this economy. Together they determine the time path of the capital-labor ratio, \( \{k_t\}_{t=1}^\infty \), given \( k_1 \) and the government's prior pre-committed choices of \( \mu, \phi, M_1, \) and \( g \).

Note that in order to describe the entire price sequence \( \{p_t\}_{t=1}^\infty \), it is necessary to pin down the price level at date 1, which is an endogenous variable. The initial price level is given by
\[
p_1 = \frac{M_1}{m_1 + \beta_1 - g}, \tag{14}
\]
where \( m_1 + \beta_1 = (1 + \mu) k_2 \). Here, \( g \) is determined by the fiscal authority and \( M_1 \), the initial quantity of money, is determined by the monetary authority.

### 3.2 Steady State Equilibria

We first explore the issue of existence of stationary equilibria. In a steady state, \( k_t = k \) holds. Define \( A = 1 + \mu(1 - \phi) \). Then we may rewrite (11) and (13) as
\[
Ak = \phi S(\omega(k), \rho) \tag{15}
\]
and
\[
\rho = \rho(k) = 1 + \frac{\phi}{A} \left[ f'(k) - 1 - \frac{g}{k} \right], \tag{16}
\]
respectively. For future reference, note that
\[
\rho'(k) = \frac{\phi}{A} \left[ f''(k) + \frac{g}{k^2} \right]. \tag{17}
\]
and that $\rho'(k) < 0$ holds for sufficiently small values of $g$.

Define $R_m$ to be the stationary gross real return on money and $\pi \equiv 1/R_m$ to be the gross inflation rate. Then, it follows from (12) that

$$R_m = \frac{(1 + \mu) - \mu f'(k)}{A} - \frac{\phi g}{A k} \quad (18)$$

Combining (16) and (15) yields

$$k = \frac{\phi}{A} S(\omega(k), \rho(k)) \equiv \Omega(k). \quad (19)$$

The solution to (19) describes the steady state capital-labor ratio, $k^*$, where $k^* = \Omega(k^*)$. It is easy to verify that in a binding monetary equilibrium, $k^*$ must additionally satisfy

$$1 - \frac{\phi g}{(A - \phi)k^*} < f'(k^*) < \frac{(A - \phi)k^* - \phi g}{\mu(1 - \phi)k^*}. \quad (20)$$

For future reference, denote the slope of the function $\Omega(k)$ by $\Theta(k)$. As we shall see below, the size of $\Theta(k)$ is singularly important in determining the properties of various equilibria.

It is easy to check that

$$\Theta(k) = \frac{\phi}{A} \left[ S_\omega \omega'(k) + S_\rho \rho'(k) \right]. \quad (21)$$

Figure 1 illustrates several configurations of the function $\Omega(k)$. For $g = 0$, $\Omega(k)$ is concave and strictly increasing in $k$. In this case, as is well-known, there is a unique steady state if $k/\omega(k)$ is increasing in $k$. Similarly, if saving is return-invariant, i.e., $S_\rho = 0$, then there is necessarily an unique equilibrium (just as in Espinosa and Russell (1999)). Consider the more general case where $S_\rho > 0$ and $g > 0$. As $g$ increases, $\Omega(k)$ shifts down in a manner illustrated in Figure 1; multiple steady state equilibria are easily possible here. In contrast with the standard Diamond model with fiat money however, not all candidate stationary equilibria in our model may satisfy (20).

Lemma 1 If saving is interest-invariant, then the steady state equilibrium $k^*$ is necessarily unique, and is characterized by $\Theta(k^*) < 1$. If saving is increasing in its return and $g > 0$, then an unique equilibrium (characterized by $\Theta(k^*) \leq 1$) is possible; similarly, multiple equilibria (characterized by $\Theta(k^*) \leq 1$) with the same bonds-money ratio is also possible.

$^{18}$\(\Theta(k)\) may be also be written as a combination of various elasticities

$$\Theta(k) = \frac{\phi}{A} \left( \frac{k}{\omega} + \frac{\rho}{\omega} \right).$$

$^{19}$As is easily verified, $k/\omega(k)$ is increasing in $k$ if $f(k) = k^\alpha, \alpha \in (0, 1)$. However, for $f(k)$ specified as in (24) below, $k/\omega(k)$ is decreasing (increasing) in $k$ for small (large) values of $k$. See von Thadden (1999) for details.
3.2.1 Examples

We now explore the issue of existence by means of some illustrative numerical examples.

1. Suppose the production function and utility function is specified as follows:

\[ f(k) = ak^\alpha \]  \hspace{1cm} (22)

and

\[ U(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}. \]  \hspace{1cm} (23)

Suppose \( \alpha = 0.33, \phi = 0.9, a = 8, \beta = 0.96, \gamma = 0.97, \mu = 12, \) and \( g = 1.66. \) Then, the unique steady state capital stock is given by \( k^* = 1.10. \) That is, even with return-dependent saving, an unique steady state is possible.

2. Suppose we retain (23) but respecify the production function as follows:

\[ f(k) = (a_1k^\delta + a_2)\delta \]  \hspace{1cm} (24)

The parameters of the economy are as follows: \( a_1 = 0.11, a_2 = 0.25, \mu = 0.57, g = 0, \phi = 0.65, \gamma = 0.97, \beta = 0.89, \) and \( \delta = -0.71. \) Under this choice of parameterization, it can be checked that \( k^* = 0.12 \) and \( k^* = 0.31 \) are the two steady states.

3.3 Seigniorage

The government's ability to make use of currency and (possibly) bond seigniorage to raise revenue obviously faces certain endogenous restrictions. In particular, there may arise a tension between the inflation tax base and the inflation tax rate when it comes to raising a fixed amount of seigniorage. We turn to these issues below. First, we compute the total seigniorage raised in steady states, in the economy. From (9), it follows that the total seigniorage collected at a steady state \( k^* \) is given by

\[ \bar{H}(R_m, k; \mu) \equiv \lambda k^* \left[ 1 + \mu - AR_m - \mu \phi f'(k^*) \right]. \]  \hspace{1cm} (25)

Substitute (7) into (15) to obtain

\[ Ak^* = \phi S \left[ \omega(k^*), \phi f'(k^*) + (1 - \phi) R_m \right], \]  \hspace{1cm} (26)

which describes the relationship between \( k^* \) and \( R_m, \) for given \( \mu. \) From (26), it is easy to show that

\[ \frac{dR_m}{dk^*} = \frac{A \left[ 1 - \Theta(k^*) \right] + \phi \left[ \rho' (k^*) - \phi f''(k^*) \right]}{\phi \left[ q - \phi \right] S_P}. \]  \hspace{1cm} (27)
From (27), it is then apparent that if \( \Theta (k^*) < 1 \) holds, then \( dR_m/dk^* < 0 \) holds at any \( k^* \), given \( \mu \). Thus, for given \( \mu \), (26) implicitly defines \( k^* = h(R_m; \mu) \), where \( h'(R_m; \mu) > 0 \) if \( \Theta (k^*); < 1 \.

Then, we substitute \( k^* = h(R_m; \mu) \) into (25) to obtain the expression for total seigniorage:

\[
H(R_m; \mu) = \tilde{H}(R_m, h(R_m; \mu); \mu) = \lambda h(R_m; \mu) \left[ 1 + \mu - \lambda R_m - \mu \phi f'(h(R_m; \mu)) \right].
\]  

(28)

The right hand side of (28) is the conventional total seigniorage Laffer curve (drawn with the stationary return to money on the horizontal axis). For future reference, a steady state \( k^* \) is henceforth described to be on the "good side" of the total seigniorage Laffer curve if an increase in the inflation rate necessarily raises the volume of seigniorage revenue at that steady state.

**Proposition 1** On the \((H, R_m)\) space, a steady state \( k^* \) is on the "good side" of the total seigniorage Laffer curve, i.e., \( (\partial H(R_m; \mu)/\partial R_m < 0) \) holds if and only if \( \Theta (k^*) < 1 \).

Proposition 1 implies that an increase in the inflation rate necessarily raises the volume of seigniorage revenue only when \( \Theta (k^*) < 1 \) obtains. They are also the ones that are regarded widely as being the most empirically realistic because of the underlying relationship between the inflation rate and seigniorage revenue. With return-invariant saving, there is a unique steady state, and the seigniorage Laffer curve has only one side, the "good side". The intuition for this result is as follows. Ceteris paribus, an increase in the inflation rate, given a fixed inflation tax base (as is the case under return-invariant saving), increases currency seigniorage. At the same time, it reduces the overall return on bank deposits and hence, the return on government bonds. This fall in the interest rate on bonds reduces the government's interest expense. The overall effect is that the government's combined revenue rises.

Henceforth, we focus our attention only on those equilibria that are on the "good side" of the seigniorage Laffer curve. That is, we will maintain that

**Assumption 1** \( \Theta (k^*) < 1 \) holds.

### 3.4 Bond Seigniorage

In our analysis, the real interest rate is endogenous and may be greater or less than the growth rate of the economy. Equilibrium bond holdings are also endogenous. As such, the revenue from

\[20\] This is by no means the only way to draw a Laffer curve. Espinosa and Russell (2001) draw Laffer curves with the real interest rate on the horizontal axis while others present them with the inflation rate on the horizontal axis. Our formulation follows Sargent (1987).
the sale of bonds may go up or down when the interest rate changes. The revenue from the sale of bonds in steady states is given by

\[ s_b = b(\rho)(1 - \rho) = \mu\lambda k(1 - \rho(k)). \] (29)

For future reference, we collect some information about the volume of bond seigniorage in this economy in the lemma below.

Lemma 2 a) At steady state \( k^* \), the volume of bond seigniorage decreases with an increase in the interest rate on bonds if and only if

\[ \frac{1 - \rho(k^*)}{k^*\rho'(k^*)} \leq 1. \] (30)

b) Define

\[ \Gamma(k^*) \equiv \mu\rho'(k^*) - \frac{\phi}{1 - \phi(k^*)^2} g. \]

For small \( g \), \( \Gamma(k^*) < 0 \) holds. If \( \rho < 1 \) holds at a steady state \( k^* \), then

\[ \Gamma(k^*) < 0 \Leftrightarrow \frac{1 - \rho(k^*)}{k^*\rho'(k^*)} \leq 1. \] (31)

3.5 Dynamic Properties of Equilibria

As is clear from Lemma 1 above, multiple stationary equilibria are possible in our model. As such, it becomes crucial to know which steady state a particular economy is going to approach, given an arbitrary initial capital-labor ratio, \( k_1 \). In the appendix, we prove the following proposition describing the local stability properties of the stationary equilibria.

Proposition 2 Suppose saving is return-dependent. Then it is possible that multiple stationary equilibria (some characterized by \( \Theta(k^*) < 1 \) and others characterized by \( \Theta(k^*) > 1 \)) exist. Of these, only the equilibria characterized by \( \Theta(k^*) < 1 \) are saddles. Those characterized by \( \Theta(k^*) > 1 \) are likely to be sources. If however, with return-dependent saving, there is an unique equilibrium and \( \Theta(k^*) < 1 \) obtains at that equilibrium, then it is a saddle. \(^{21}\)

Corollary 1 If saving is return-invariant, \( k^* \) is unique and stable.

\(^{21}\) It is not possible to prove in general that with return-dependent saving, equilibria with \( \Theta(k^*) > 1 \) are sources (they cannot be sinks, for sure). Two comments are in order here. First, it seems that for valid equilibria with \( \Theta(k^*) > 1 \) to exist, it is necessary that \( k/\omega(k) \) be decreasing in \( k \) somewhere. A production function that satisfies this condition is (24). Second, using this production function and (23) where \( \beta \in (0, 1) \) and \( \gamma \in (0, 1) \), Bhattacharya (1990) proves that \( \phi > \left( \frac{1 + \beta}{2 + \beta} \right) \) and \( \gamma \in (0.5, 1] \) are sufficient to ensure that equilibria with \( \Theta(k^*) > 1 \) are sources.
Under return-dependent saving, steady states that have $\Theta(k^*) < 1$ are unstable in the sense that they cannot be approached from an arbitrary set of initial conditions. However, they are approachable from correctly chosen initial conditions that place the economy on the stable manifold or the saddle path. In this sense, steady states under return-dependent saving characterized by $\Theta(k^*) < 1$ are potentially dynamically approachable even though they are unstable. We think of “dynamic approachability” as a “weaker” notion of stability. For future reference, note that if saving is interest-invariant, then there is always a unique equilibrium, and it is necessarily stable.\textsuperscript{22}

For future reference, let us collect all our results in one place here. Thus far, we have shown that if monetary policy is conducted using a fixed bond-money rule, and saving is return-invariant, then there is an unique dynamically stable long-run equilibrium. At that equilibrium, increasing the money growth rate always raises revenue (across steady states) for the government. If saving is return-dependent, and we restrict ourselves to equilibria where an increase in the inflation rate necessarily raises seigniorage revenue, then only these equilibria (possibly one) are the only potentially dynamically approachable ones.

4 Open Market Operations

Following Sargent and Wallace (1981), Wallace (1984) and Espinosa and Russell (1999), we now investigate how changes in the bond-money ratio, $\mu$, affect the steady state equilibrium levels of the rate of inflation. As stated earlier, an increase in $\mu$ corresponds to a contractionary (and permanent) open market operation. In other words, tight money policies involve increases in $\mu$.

First, we investigate the effects of a change in $\mu$ on the steady state capital stock. Notice that (19) gives the relationship between $k^*$ and $\mu$, that is independent of $R_m$. Then, using (19), it is easily established that

$$\frac{dk^*}{d\mu} = \frac{(1 - \phi) \left( \frac{\phi}{\delta} (\rho - 1) - 1 \right) k^*}{A \left[ 1 - \Theta(k^*) \right]}$$

Lemma 3 Under Assumption (A.1), $dk/d\mu < 0$ at a steady state characterized by $\Theta(k^*) < 1$.

This result is crucial to much of what follows. What it says is that if monetary policy is conducted using a fixed bond-money rule, and saving is relatively insensitive to its return (Assumption A.1), then at any steady state on the good side of the total seigniorage Laffer curve, a marginal

\textsuperscript{22}To foreshadow, in Section 5 below, we show that were the central bank to follow a fixed inflation rate rule instead, then even with return-invariant saving and reasonable looking parameters, there are two steady states, one stable and the other unstable (though both are potentially dynamically approachable).
tightening of monetary policy reduces the long-run capital stock. Two effects are at work here. When the treasury increases the supply of bonds, agents buy up this debt only at a higher promised return on bonds. With return-invariant saving, a shift towards more bonds in private portfolios causes bank deposits get crowded out. At a given return on money, the bank is forced to lower its capital holdings (because of the reserve requirement) in order to raise the return on deposits so as to match up with the new higher return on bonds.

To study the effects of a marginal tightening on the steady state rate of inflation, transform (18) into an expression for the steady state inflation rate:

\[
\pi = \frac{[1 + \mu (1 - \phi)] k}{(1 + \mu) k - \mu \phi f''(k) - \frac{\phi}{1 - \phi} g}.
\]  

(33a)

At a steady state \(k^*\) characterized by \(\Theta (k^*) < 1\), (32) implicitly defines \(k^* = x(\mu)\), where \(x'(\mu) < 0\). Substitute \(k^* = x(\mu)\) into (33a) to obtain

\[
\pi = \frac{1 + \mu (1 - \phi)}{(1 + \mu) - \mu \phi f'(x(\mu)) - \frac{\phi}{1 - \phi} x(\mu)}.
\]  

(34)

Equation (34) gives us an expression for the gross inflation rate as a function of \(\mu\), provided that \(k^*\) is approachable. Then, the effect of an increase in \(\mu\) on the inflation rate is captured by:

\[
\frac{d\pi}{d\mu} = \frac{(\rho - 1) A + \left[\mu \phi f''(k) - \frac{\phi}{1 - \phi} g\right] A x'(\mu)}{\left[(1 + \mu) - \mu \phi f'(k) - \frac{\phi}{1 - \phi} k\right]^2}.
\]  

(35)

Using (32), and Lemma 2, we can simplify (35) to:

\[
\frac{1}{\pi^2} \frac{d\pi}{d\mu} = (\rho - 1) + \Gamma (k^*) \frac{dk^*}{d\mu}.
\]  

(36)

For sufficiently small values of \(g\), we know from Lemma 2 that \(\Gamma (k^*) < 0\) holds. It follows then that the sign of \(\frac{d\pi}{d\mu}\) crucially depends on whether the initial real interest rate, \(\rho \geq 1\) and on the sign of \(\frac{dk^*}{d\mu}\) (which in turn, is naturally connected to the stability properties of \(k^*\)).

**Proposition 3** Consider a steady state \(k^*\) characterized by \(\Theta (k^*) < 1\). At that steady state, (a) if \(\rho > 1\), then \(d\pi/d\mu > 0\) for all \(\mu\); (b) if \(\rho < 1\), then \(d\pi/d\mu\) may be either positive or negative.

**Corollary 2** Consider a steady state \(k^*\) characterized by \(\Theta (k^*) < 1\). At that steady state, suppose \(\rho < 1\). Then, a tight money policy engineered by a permanent open market operation raises the steady state inflation rate if and only if

\[
1 - \rho < \Gamma (k^*) \frac{dk^*}{d\mu}.
\]
Part (a) of Proposition 3 has a strong intuitive appeal. If the return on government bonds exceeds the growth rate of the economy (here, unity), and saving is relatively insensitive to its return, then all equilibria on the good side of the seigniorage Laffer curve (which are also potentially dynamically approachable) will exhibit unpleasant monetarist arithmetic. The intuition is clear. Suppose the treasury conducts an open market sale of new bonds. If the initial promised return on these bonds exceeds the growth rate of the economy, then, interest payments on outstanding debt will eventually outstrip the revenue from the fresh sale of these new bonds. The central bank eventually will have to oblige by printing money to cover the shortfall. It will succeed in its efforts only on the good side of the total seigniorage Laffer curve. Part (a) of Proposition 3 therefore extends Sargent and Wallace's (1981) classic result on unpleasant monetarist arithmetic to fairly general neoclassical economies of the Diamond (1965) variety.23

Since the publication of the Sargent and Wallace paper, a number of authors have cast doubts on the applicability of the unpleasant arithmetic by questioning whether the condition that real interest rates have to be higher than the growth rate of the economy hold for real-world economies. Darby (1984), and more recently, Espinosa and Russell (1998), have pointed out that for example that for post-war U.S., the after-tax real rate of return on government debt is near zero, while the average real growth rate of GDP has been around 3%. Miller and Sargent (1984) and later Bhattacharya, Guzman, and Smith (1998) went on to consider the possibility of resurrecting the unpleasant arithmetic by weakening the unrealistic high real rate condition. Part (b) of Proposition 3 is in that vein. It states that even if the return on bonds is less than the growth rate of the economy, it is still possible that a tight money policy may raise the long-run inflation rate. It bears emphasis here that if permanent open market operations have no real effects, i.e., \( \frac{dK}{du} = 0 \) obtains, then the necessary and sufficient condition for unpleasant arithmetic to hold would be \( \rho > 1 \). Therefore, in order to produce a condition for the unpleasant arithmetic, that is weaker than the Sargent-Wallace condition, it is necessary that the government's open market activity has real effects. From (32), it is clear that the presence of a unremovable reserve requirement is necessary for this to happen. In effect then, any weakening of the Sargent-Wallace condition that is achieved in the present paper is entirely attributable to this binding reserve requirement.

The intuition for Part (b) of Proposition 3 is as follows. When \( \rho < 1 \) obtains, the government raises a positive revenue from the sale of bonds. The question is whether following an open market

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23Bhattacharya (1996; Chapter 4) shows a similar result for an identical model with CRRA utility and CES preferences.
sale of debt, this bond seigniorage is large enough to finance the government's primary deficit without eventually necessitating an increase in currency seigniorage. Suppose, as in the experiment of Proposition 3 that the government sells new debt. On the one hand, this directly increases the bond seigniorage $b(1 - \rho)$; this is the "tax base effect" of Espinosa and Russell (1998). On the other hand, for reasons discussed earlier, this reduces the capital stock, and in turn increases the interest rate, which reduces the volume of bond seigniorage (the "tax rate effect" of Espinosa and Russell, 1998). On net, the latter effect outweighs the former. Additionally, the crowding out of capital reduces reserves, and hence reduces the inflation tax base. As such, on the good side of the total seigniorage Laffer curve, the central bank still has to make up the shortfall by raising the inflation rate.\(^{24}\)

Corollary 2 attempts to somewhat sharpen the requirements for obtaining unpleasant arithmetic in the case when $\rho < 1$ obtains. Notice that for small $g$, we know from Lemma 2 that $\Gamma (k^*) < 0$ holds. We also know from Lemma 2 that $\Gamma (k^*) < 0$ implies that the volume of bond seigniorage decreases with an increase in the bond interest rate at such a steady state. Additionally, since we are focusing our attention only on dynamically approachable steady states, it follows from Proposition 3 that $\frac{de^c}{du} < 0$ holds. Then, it is clear that the necessary and sufficient condition for unpleasant arithmetic is the one spelt out in Corollary 2.

It is useful to collect all the qualifiers needed to obtain unpleasant arithmetic in an economy where the real interest rate is below its growth rate. First, it is necessary that monetary policy has real effects; additionally, a marginal tightening of the monetary policy must reduce the capital stock. Without this, unpleasant arithmetic is obtained only if the real rate of return on bonds exceeds the growth rate of an economy. The next qualifier is that a higher interest rate on bonds reduces bond seigniorage. There are two additional effects to consider. One is that a tight money policy (higher reliance on bonds) increases bond seigniorage. The other is that, the tight money policy reduces the capital stock, raises the return on bonds, and thereby reduces bond seigniorage. The final qualifier needed is that the latter effect exceeds the former, which is the condition in Corollary 2. In plain English, this last condition requires the economy to be on the "bad" side of a bond-seigniorage Laffer curve (see (29)) drawn on the $(s_b, \mu)$ space.\(^{25}\) Parenthetically, it can be

\(^{24}\)Bhattacharya, Guzman, and Smith (1998) use a model similar to the one presented here to show that unpleasant arithmetic may hold in an economy with $\rho < 1$. The difference is that they use an endowment economy, while we work with a Diamond (1965) economy. The driving force of their result is the "tax base effect". Our structure allows a flexible return on capital and hence produces, in addition, a "tax rate effect" which reduces bond seigniorage (as well as currency seigniorage).

\(^{25}\)At such an equilibrium, a tightening of monetary policy (sale of bonds) causes the steady state revenue from bonds to fall. This case has been extensively studied in Espinosa and Russell (2001).
noted that a necessary condition required to obtain pleasant arithmetic when \( p < 1 \) is that the economy to be on the "good" side of a bond-seigniorage Laffer curve drawn on the \((s_b, \mu)\) space. This is exactly the type of condition that Miller and Sargent (1984) had in mind.

It seems possible that the effect on the inflation rate of a marginal tightening of monetary policy may therefore depend on the existing extent of monetary policy tightness. The next lemma formalizes all this.

**Lemma 4** Define \( \tilde{\mu} \) to be the solution to \((1 + \tilde{\mu})k(\tilde{\mu}) - \tilde{\mu} \phi k(\tilde{\mu}) f'(k(\tilde{\mu})) = \frac{\phi \mu}{1 - \phi} \). Then, at a steady state characterized by \( \Theta(k^*) < 1 \) and \( \rho < 1 \), for sufficiently small \( g \),

\[
\lim_{\mu \to 0} \frac{d\pi}{d\mu} < 0
\]

and

\[
\lim_{\mu \to 0} \frac{d\pi}{d\mu} > 0
\]

holds, where \( \frac{d\pi}{d\mu} \) is derived from (36). In other words, pleasant arithmetic obtains only for sufficiently low values of the bonds-money ratio.

How does our result compare to the one presented in Bhattacharya, Guzman, and Smith (1998). The crucial difference between our two models is that by virtue of their assumption of a linear-storage saving technology, the real interest rate is fixed. The driving force of their result is then the "tax base effect" of Espinosa and Russell (1998). A higher bond-money ratio through open market operations crowds-out deposits from the portfolio of private agents. This reduction in deposits causes a reduction in the volume of reserves held by banks. Consequently, the inflation tax base falls forcing the central bank to raise the inflation rate. In contrast, in our Diamond (1965) economy, an additional "interest rate effect" emerges, and this effect which is necessarily absent in the Bhattacharya, Guzman, and Smith (1998) framework, may reduce the volume of bond seigniorage (as well as currency seigniorage).

Dating back to Wallace (1984), researchers have studied the connection between the pleasantness of the arithmetic and the initial marginal product of capital at the equilibrium being studied. In this context, notice that in our model, as is clear from (16), when \( g > 0 \) holds, \( f'(k) > 1 \) does not necessarily imply that \( \rho > 1 \) holds. Thus Part (b) of Proposition 3 proves that it is possible for tight money to have inflationary consequences even when the return on bonds is lower and the marginal product of capital is higher than the growth rate of the economy.\(^{26}\) This has two important

\(^{26}\) For example, consider the specification of preferences and technology as described in Section 4.1 below. Let the
implications. One is that in contrast to Bhattacharya, Guzman, and Smith (1998), \( f'(k) > 1 \) is not required for tight money to be inflationary. In words, the marginal product of capital being greater than the growth rate is not necessary to obtaining unpleasant monetarist arithmetic. The other implication is that \( f'(k) < 1 \) is not required for tight money to be disinflationary.\(^{27}\) That is, for tight money to reduce the long-run inflation rate, it is not necessary that the marginal product of capital be less than the growth rate of the economy.

How does our result compare with those in Espinosa and Russell (1999, 2001)? Recall that Espinosa and Russell consider an alternative policy rule. In their setup, the central bank changes the money growth rate so as to satisfy the Treasury's target level of the bond-money ratio. In the section ahead, we consider a model that is identical to the one described above except that the monetary policy rule is almost identical to the one discussed in Espinosa and Russell. To foreshadow, what Espinosa and Russell (and we will) find is that with return-invariant saving, in cases where there are multiple (two) equilibria, and both the equilibria are characterized by real interest rates that are less than unity, it is the low (high) real-rate equilibrium that is dynamically stable (unstable) and exhibits pleasant (unpleasant) arithmetic, even though both are dynamically approachable. Since real interest rates in most real world economies are quite low, their result has the implication that the only equilibrium one can hope to observe in the real world will exhibit pleasant arithmetic. In other words, they rule out the possibility of unpleasant arithmetic on stability grounds. In sharp contrast, Proposition 3 along with Proposition 2 suggests that with return-invariant saving, an equilibrium with a low real interest rate, and which exhibits unpleasant arithmetic is stable, and hence potentially observable in the real world.

4.1 An example with return-invariant saving

In this subsection, we illustrate some of our main results thus far by focussing on an example with return-invariant saving. This will also facilitate the comparison with results obtained by Espinosa and Russell (1999) and others in the literature. We use the following specification: \( U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \), which (using (5)) implies that \( S = \beta \omega(k)/1 + \beta = \omega(k) \). Furthermore, assume parameters of the economy be given by \( f(k) = 4k^{0.35}, \phi = 0.97, g = 0.03, s = 0.55 \). Then, for \( \mu = 0.2 \), there is a unique steady state defined by \( k^* = 1.63, \rho^* = 0.98 \) and \( f'(k^*) = 1.01 \). At this steady state, an increase in \( \mu \) increases the inflation rate.

\(^{27}\)In Bhattacharya, Guzman, and Smith (1998), as in Wallace (1984) and others, tight money is necessarily disinflationary if the marginal product of capital is below unity.
\( f(k) = ak^\alpha, \alpha \in (0, 1) \). From (19), it follows then that

\[
k^* = \left[ \frac{s\phi(1-\alpha)}{1+\mu(1-\phi)} \right]^{1-\alpha}
\]

and therefore that \( k^* \) is unique. Also \( \rho(k^*) \leq 1 \) is possible. The law of motion for the capital stock is given by

\[
k_{t+1} = \frac{s\phi(1-\alpha)}{1+\mu(1-\phi)} k_t^2.
\]

from where it is clear that \( k^* \) is stable since \( \frac{\partial k_{t+1}}{\partial k_t} \bigg|_{k=k^*} = \alpha \in (0, 1) \). It is easy to check that \( \frac{\partial k^*}{\partial \mu} < 0 \) holds. The steady state inflation rate (using (??))

\[
\pi(\mu) = \frac{[1+\mu(1-\phi)]}{(1+\mu)^2 - \mu s \phi a (k^*)^{\alpha-1} \frac{\phi s}{1-\phi} k^*}
\]

Example 1 Let the parameters of the economy be given by \( f(k) = 4k^{0.35} \), \( \phi = 0.97 \), \( g = 0.03 \), \( s = 0.7 \). Then, for \( \mu = 1.1 \), there is a unique steady state, \( k^* = 2.28 \).\(^{28}\) As illustrated in Figure 2, a tight money policy engineered by an increase in the bonds-money ratio causes the inflation rate to fall in the range \( \mu \in (0, 4) \) and causes it to rise thereafter.

4.1.1 Transitional Dynamics

In this subsection, we study the behavior of inflation and the real interest rate on the transition path from one steady state to another by means of numerical experiments. The basic idea of these experiments is as follows. Take a set of parameters (including the bonds-money ratio) and find a unique monetary stationary equilibrium. Then set the stationary capital-labor ratio as the initial condition, and change the bonds-money ratio at date 1. Then keep track of the time paths for key endogenous variables.

The first example is intended to replicate Sargent and Wallace’s (1981) unpleasant monetarist arithmetic. The parameters for this example are: \( \phi = 0.9 \), \( \alpha = 0.2 \), \( a = 2 \), \( \beta = 0.97 \), \( g = 0.01 \), \( M_1 = 1 \). Initially set \( \mu = 9 \). Under these specifications, there is a unique monetary steady state equilibrium given by \( k^* = 0.29 \), \( \rho^* = 1.02 \), and \( \pi^* = 1.88 \). Now raise \( \mu \) from 9 to 10, given \( k_1 = k^* = 0.29 \) and \( M_1 = 1 \) as the initial conditions. The results are reported in Figures 3a. As is evident, \( k \) decreases over time, and \( \rho \) increases over time without a jump. Since we assume 100% depreciation, these transition paths are rather rapid. Since \( \rho > 1 \), bond seigniorage is negative for all \( t \). Note that the inflation rate jumps up at the time the new policy is implemented, and then

\(^{28}\)Similarly, for \( \mu = 7.12 \), there is a unique steady state, \( k^* = -1.78 \).
it declines over time until it reaches the new steady state. At the new steady state, $k^* = 0.27$, $\rho^* = 1.04$, and $\pi^* = 3.87$. The initial price level can be computed as $p_1 = 3.08$.

The next example is intended to explore the possibility of obtaining pleasant arithmetic in low real-interest rate economies. The parameters chosen are: $\phi = 0.9$, $\alpha = 0.2$, $a = 5$, $\beta = 1$, $g = 0.2$. The initial value of $\mu$ is 2 and $M_1 = 1$. Figures 3b shows the case where $\mu$ increases from 2 to 3 while Figure 3c presents the case where $\mu$ increases from 5 to 6.29 As suggested by Lemma 4, "pleasant" arithmetic obtains for these low (2 to 3) values of $\mu$. Note that the gross inflation rate decreases monotonically along the transition path, and bond seigniorage is positive at all dates. According to Figure 3c, however, as $\mu$ increases from 5 to 6, the inflation rate starts to rise. This is an illustration of unpleasant arithmetic in low real interest rate economies. The behavior of the inflation rate is similar to the one for high interest rates; inflation jumps up and then declines over time.

5 Fixed inflation rate rules

Thus far, we have considered a setup where the government conducts monetary policy by fixing the bonds-money ratio at the start of time and then choosing a nominal money stock then and all future dates that is consistent with its choice of the bonds-money ratio. In short, the principal instrument of monetary policy was the bonds-money ratio. Under this fixed bonds-money ratio rule, we derived the following result. If saving is return-invariant, then there is a unique monetary stationary equilibrium (if any) and it is dynamically stable. If the initial real interest rate is greater than unity, unpleasant arithmetic is observed. If the real interest rate is less than unity at that steady state, it is still possible for unpleasant arithmetic to be observed. Thus, real world economies that have low real interest rates may observe unpleasant arithmetic. This section examines the sharp contrast between this result and that obtained in another version of the model that is virtually identical to this one except that the monetary policy rule is different. In particular, we consider an alternative rule wherein the government fixes the inflation rate at date 1 and allows the bonds-money ratio to adjust gradually to its new level. This is very similar to the constant money growth rate policy rule described in Espinosa and Russell (2001). As they show, and we confirm, this change in the rule is enough to produce the prediction that the only arithmetic one would observe in low real-rate economies is of the pleasant type. For sake of brevity, we briefly sketch the model

29For $\mu = 2$, $k^* = 1.66$, $\rho^* = 0.66$, and $\pi^* = 1.68$. For $\mu = 3$, $k^* = 1.50$, $\rho^* = 0.72$, and $\pi^* = 1.53$. For $\mu = 5$, $k^* = 1.26$, $\rho^* = 0.80$, and $\pi^* = 1.84$. For $\mu = 6$, $k^* = 1.16$, $\rho^* = 0.84$, and $\pi^* = 2.48$. 22
and the main results below, and relegate most of the details to Bhattacharya and Kudoh (2001).

As stated above, the model presented below is identical to the one presented thus far except for the specification of monetary policy. Specifically, assume that the monetary authority sets the inflation rate at date 1,

\[ \frac{p_{t+1}}{p_t} = \Pi \geq 1, \quad \forall t \geq 1 \tag{37} \]

and implements it by changing the money growth rate according to

\[ \frac{M_{t+1}}{M_t} = \frac{M_{t+1}/p_{t+1}}{M_t/p_t} = \Pi \frac{m_{t+1}}{m_t} = \Pi \frac{k_{t+2}}{k_{t+1}} \tag{38} \]

where the last equality uses (6) at equality. Note that in a steady state equilibrium (and not otherwise), this inflation setting rule is identical to the constant money growth rule adopted in Espinosa and Russell (2001).

For ease of comparison, we too assume that saving is return-invariant. That is, \( S (\omega (k_t), \rho_{t+1}) = s \omega (k_t) \) where \( s \equiv \frac{\phi}{1 + \beta} \in (0, 1) \). Then, a binding monetary competitive equilibrium of this economy [using (7), (11) and (9)] is characterized by:

\[ m_t + k_{t+1} + b_t = s \omega (k_t) \tag{39} \]
\[ \rho_{t+1} = \phi f' (k_{t+1}) + \frac{1 - \phi}{\Pi} \tag{40} \]
\[ g = m_t - \frac{1}{\Pi} m_{t-1} + b_t - \rho_t b_{t-1} \tag{41} \]

and (38). Substitute (38) into (39) to obtain

\[ b_t = s \omega (k_t) - \frac{1}{\phi} k_{t+1}. \tag{42} \]

Substitute (39) and (40) into (41) to obtain

\[ g = \frac{1 - \phi}{\phi} k_{t+1} - \frac{1 - \phi}{\phi \Pi} k_t + b_t - \left[ \phi f' (k_t) + \frac{1 - \phi}{\Pi} \right] b_{t-1}. \tag{43} \]

Analogous to (11) and (13), equations (42) and (43) jointly constitute the equilibrium conditions for this economy.

In a steady state, (42) implies

\[ b = s \omega (k) - \frac{1}{\phi} k \equiv B(k) \tag{44} \]

and (43) may be rearranged to yield

\[ b = \frac{g - (1 - \frac{1}{\Pi}) \frac{1 - \phi}{\phi} k}{1 - \phi f' (k) - \frac{1 - \phi}{\Pi}} = \frac{g - (1 - \frac{1}{\Pi}) \frac{1 - \phi}{\phi} k}{1 - \phi} \equiv \Psi (k) \tag{45} \]
A valid monetary competitive equilibrium is a solution to (44) and (45) that satisfy \( b > 0, k > 0, \) and \( f'(k) > \frac{1}{11} \). We are principally interested in solutions where \( \rho < 1 \) obtains. Several configurations of the intersection of the \( b = B(k) \) and the \( b = \Psi (k) \) loci are possible; we are particularly interested in the configuration illustrated in Figure 4. Small values of the reserve requirement, intermediate values of the government deficit and the savings rate are conducive to producing this particular configuration.

**Example 2** Let the parameters of the economy be given by \( \Pi = 1.3, f(k) = 4k^{0.35}, \phi = 0.97, g = 0.03, \) and \( S = sw(k) \) where \( s = 0.7 \). Then, there are exactly two valid steady states; the high capital steady state has \( kh = 2.28 \) (\( \rho_l = 0.81 \)) and the low one has \( kl = 1.78 \) (\( \rho_h = 0.95 \)). The situation is illustrated in Figure 5.

It is important to note that the parameters of the economy outlined in Example 2 produce two values of the long-run bonds-money ratio (\( \mu = 7.11 \) and \( \mu = 1.1 \)). The latter coincides with the unique equilibrium presented in Example 1. In other words, a fixed money-growth rule does not uniquely define a equilibrium value for the bonds-money ratio. In general, it is the case that a fixed inflation rate or money-growth rule can easily produce multiple equilibria even with return invariant saving. This possibility does not exist with a fixed bonds-money ratio rule as we have seen in Section 4.1.

Having characterized the stationary equilibria, we now ask the question: what happens to real interest rates when the government follows a tight money policy? In the present setup, a tight money policy would imply a cut in the inflation rate at all dates. Again, the reason for this focus is the ease of comparison with the results in Espinosa and Russell (1999, 2001). In their terminology, the conventional wisdom is that of pleasant arithmetic which, under a fixed inflation rate rule, may be characterized as tight money policies causing the real interest rate to rise. Then, unpleasant arithmetic would naturally be defined as a situation where tight money policies precipitate a fall in the real interest rate. What Espinosa and Russell find is that in situations where there are two stationary equilibria both with low (less than unity) real interest rates, the one with the lower (higher) interest rate exhibits pleasant (unpleasant) arithmetic. Not surprisingly, given the ample similarity between this current setup and their paper, we find the same result which we simply state here for completeness.

**Lemma 5** A marginal tightening of the monetary policy achieved via a cut in the inflation rate causes the real interest rate to rise (fall) at \( \rho_l (\rho_h) \). In other words, a marginal tightening produces
pleasant arithmetic at the low real-rate equilibrium and produces unpleasant arithmetic at the high real-rate equilibrium.

As an illustration, consider the situation described in Example 2. If, ceteris paribus, II is cut to 1.2, it is easily verified that \( \rho_h \) falls to 0.94 and \( \rho_l \) rises to 0.83.

Finally, we undertake a quick study of the local stability properties of the two steady state equilibria. Recall, that the laws of motion are given by (42) and (43). Then, linearizing around a steady state \((k^*, b^*)\) yields the Jacobian matrix \(J\):

\[
J(k^*, b^*) = \begin{pmatrix}
-\left[1 - \frac{\phi}{\phi_l} + \phi f^{''}(k^*) b^* \right] + s\omega'(k^*) & -\rho^*
\frac{1}{\phi} \left[1 - \frac{\phi}{\phi_l} + \phi f^{''}(k^*) b^* \right] + \left(\frac{-1+\phi}{\phi} \right) s\omega'(k^*) & \frac{\phi}{\phi_l} \\
\end{pmatrix}
\]

A very likely scenario is the combination of a saddle and a sink as is illustrated in the example below.

Example 3 Let the parameters of the economy be the same as in Example 2 above. There are exactly two steady states, the high capital steady state has \( k_h = 2.28 \) \( (\rho_l = 0.81) \) and the low one has \( k_l = 1.78 \) \( (\rho_h = 0.95) \). Then,

\[
J(\rho_h) = \begin{pmatrix}
.61965 & -.9566 \\
-2.0093 & .98619 \\
\end{pmatrix}
\]

holds, with associated eigenroots 0.32774 and 1.2781 (indicating that \( \rho_h \) is a saddle). Similarly,

\[
J(\rho_l) = \begin{pmatrix}
.37445 & -.81672 \\
-1.3753 \times 10^{-2} & .84198 \\
\end{pmatrix}
\]

with associated eigenroots 0.35154, and 0.86488 (indicating that \( \rho_l \) is a sink). Hence the configuration is exactly as described in the phase diagram illustrated in Figure 6.

In other words, as is the case in Espinosa and Russell (1999), the low real rate equilibrium is dynamically stable while the higher real rate equilibrium is not (it is potentially dynamically approachable though). In conjunction with Lemma 5, it follows that the equilibrium exhibiting pleasant arithmetic is the only dynamically stable equilibrium. It is this last observation that Espinosa and Russell (2001) use to make the argument that real world economies with low real interest rates are likely to exhibit only pleasant arithmetic.\(^{30}\)

\(^{30}\)If the central bank is following a fixed-inflation rate rule, then it does not know which time path for \( k \) (and other real variables) to expect, because there is no unique path toward a steady state if it is a sink. But when it tries to tighten money, it is sure that pleasant arithmetic will obtain. On the other hand, if the central bank follows a fixed bonds-money ratio rule, it knows the unique equilibrium path to the steady state since it is a saddle, but then it no longer knows if pleasant or unpleasant arithmetic will obtain. This would be another way of characterizing the differences between our result and those obtained in Espinosa and Russell (2001).
5.1 Remarks

A quick consolidation of our results thus far and comparison across the two monetary policy rules is in order here. Consider the fixed bonds-money ratio rule version of our model. There we show the following. If saving is return-invariant, then there is a unique monetary stationary equilibrium (if any) and it is dynamically stable. If the initial real interest rate is greater than unity, unpleasant arithmetic is observed. If, as is the case in real world economies, the real interest rate is less than unity at that steady state, it is still possible for unpleasant arithmetic to be observed. That is, it is not possible to rule out unpleasant arithmetic in real world economies on stability grounds. Now consider the fixed inflation rate rule version of the model presented in Section 5. There we show the following. If saving is return-invariant, then there is likely to be multiple monetary stationary equilibria (if any) and these may all have real interest rates less than unity. The equilibrium with the lower (higher) interest rate exhibits pleasant (unpleasant) arithmetic. Moreover, the equilibrium exhibiting pleasant arithmetic is the only dynamically stable equilibrium. As such, it is possible to rule out unpleasant arithmetic in real world economies on stability grounds. The two monetary policy rules therefore produce sharply different predictions about the likely observability of unpleasant arithmetic.
References


Appendix

A Proof of Lemma 1

The proof readily follows from arguments that are standard and presented in detail in Galor and Ryder (1989). Also see Figure 1.

B Proof of Proposition 1

Differentiate (28) with respect to $R_m$ to obtain

$$\frac{\partial H}{\partial R_m} = \left[ \frac{g}{k} - \lambda \mu f''(k) \right] h'(R_m; \mu) - \lambda k A,$$

where we have used (25) to write $\lambda [1 + \mu - AR_m - \mu f'(k)] = g/k$. Substituting (27) into (46), we have

$$\frac{\partial H}{\partial R_m} = \frac{\left[ \frac{g}{k} - \lambda \mu f''(k) \right] \phi (1 - \phi) S_{\rho}}{A (1 - \Theta (k)) + \phi [\rho' (k) - \phi f''(k)] S_{\rho}} - \lambda k A.$$

Thus, $\partial H/\partial R_m < 0$ if

$$\lambda k A^2 (1 - \Theta (k)) > \left[ \frac{g}{k} - \lambda \mu f''(k) \right] \phi (1 - \phi) S_{\rho} - [\rho' (k) - \phi f''(k)] \phi k AS_{\rho} = 0,$$

where we have substituted (17) into the right-hand-side of (47). This proves that at a steady state characterized by $\Theta (k) < 1$, $\partial H/\partial R_m < 0$ at that steady state.

C Proof of Lemma 2

a) It is easily established (using (29)) that

$$\frac{\partial s_b}{\partial \rho} = \mu \lambda \left\{ \frac{1 - \rho (k^*)}{\rho' (k^*)} - k^* \right\}.$$

Then the rest is immediate.

b) For small $g$, $\rho' (k^*) < 0$ holds and hence $\Gamma (k^*) < 0$ holds. Substitute (17) into the expression for $\Gamma (k^*)$ to get the desired result.
D  Proof of Proposition 2

To study the local dynamical properties of this system, we linearize the system of nonlinear difference equations (11) and (13) around a steady state. First, to reduce notation we define

\[ \psi(k, \rho) \equiv A\rho - \phi f'(k) - \phi k f''(k). \]

Notice that since \( \rho = f'(k) + (1 - \phi) R_m \) in a binding steady state,

\[ \psi(k, \rho) \equiv \mu (1 - \phi) f'(k) + (1 - \phi) A R_m - \phi k f''(k) > 0. \]

Local dynamics of the system is described by the linear difference equations, \( X z_{t+1} = Y z_t \), where

\[
X = \begin{pmatrix}
A & -\phi S_p \\
(1 - \phi)(1 + \mu) & 0
\end{pmatrix}, \\
Y = \begin{pmatrix}
\phi S_p \omega'(k) & 0 \\
\psi(k, \rho) & A k
\end{pmatrix}, \\
\text{and } z_t = \begin{pmatrix}
k_t - k \\
\rho_t - \rho
\end{pmatrix}.
\]

The inverse of matrix \( X \) exists if \( S_p > 0 \). Then, we obtain \( z_{t+1} = (X^{-1}Y) z_t = J z_t \). It is straightforward to compute the Jacobian matrix:

\[
J = \frac{1}{(1 - \phi)(1 + \mu) \phi S_p} \begin{pmatrix}
\phi S_p \psi(k, \rho) & \phi A k S_p \\
-(1 - \phi)(1 + \mu) \phi S_p \omega'(k) + A \psi(k, \rho) & A^2 k
\end{pmatrix}.
\]

Let \( D(k) \) denote the determinant of \( J \) as a function of the steady state capital stock, and \( T(k) \) denote the trace of \( J \). Then it is easily checked that

\[
D(k) = \frac{\phi A k S_p \omega'(k)}{(1 - \phi)(1 + \mu) \phi S_p} > 0, \quad \text{and } T(k) = \frac{\phi S_p \psi(k, \rho) + A^2 k}{(1 - \phi)(1 + \mu) \phi S_p} > 0.
\]

It is therefore easily established that \( 1 + T(k) + D(k) > 0 \). Use \( \psi(k, \rho) = (1 - \phi)(1 + \mu) - A k \rho'(k) \) to obtain

\[
1 - T(k) + D(k) = -\frac{A^2 k [1 - \Theta(k)]}{(1 - \phi)(1 + \mu) \phi S_p},
\]

which is negative if and only if \( \Theta(k) < 1 \). This proves that all \( k^* \) characterized by \( \Theta(k^*) < 1 \) are saddles. The rest is immediate. 

E  Proof of Lemma 3

Follows straightforwardly from (32). 

F  Proof of Proposition 3

Follows straightforwardly from (36).
G Proof of Lemma 4

From (36), it follows that

\[
\frac{d\pi}{A_d\mu} = \frac{(\rho - 1) + \left[ \mu \phi_j''(k) - \frac{\phi_j'}{1 - \phi_j k^a} \right] \frac{dk^c}{d\mu}}{(1 + \mu) - \mu \phi_j'(k) - \frac{\phi_j}{1 - \phi_j k}} \]

(48)

Using Proposition 3, it follows that \( \frac{dk^c}{d\mu} < 0 \) holds. As \( \mu \to 0 \) and for small enough \( g \), the denominator of (48) remains finite, and the numerator of (48) is then clearly negative when \( \rho < 1 \) holds. As \( \mu \to \bar{\mu} \), the denominator goes to zero and hence the rest follows.
Figure 1. Steady state equilibria

Figure 2. Steady state inflation as a function of the bonds-money ratio (Example 1)
Figure 3a. Tight money raises inflation under $\rho > 1$.  

1. **Capital stock**
   - Time: 1, 3, 5, 7, 9
   - Values: 0.29, 0.285, 0.28, 0.275, 0.27

2. **Gross real interest**
   - Time: 1, 3, 5, 7, 9
   - Values: 1.045, 1.04, 1.035, 1.03, 1.025

3. **Inflation**
   - Time: 1, 3, 5, 7, 9
   - Values: 3.5, 3.45, 3.4, 3.35, 3.3

4. **Bond seignioro**
   - Time: 1, 3, 5, 7, 9
   - Values: -0.004, -0.006, -0.008, -0.01, -0.012, -0.014
Figure 3b. Tight money reduces inflation under $\rho < 1$. 
Figure 3c. Tight money raises inflation under $\rho < 1$. 

- **Capital stock**
  - Time: 1 to 9
  - Values: 1.26, 1.24, 1.22, 1.2, 1.18, 1.16, 1.14

- **Gross real interest**
  - Time: 1 to 9
  - Values: 0.85, 0.84, 0.83, 0.82, 0.81, 0.8

- **Inflation**
  - Time: 1 to 9
  - Values: 27, 25, 23, 21, 19, 17

- **Bond spread**
  - Time: 1 to 9
  - Values: 0.17, 0.16, 0.15, 0.14, 0.13, 0.12
Figure 4. Steady state equilibria under strict inflation targeting

Figure 5. Steady state equilibria under strict inflation targeting (Example 2)