A FOCUSED TRANSDUCER/SCATTERER MODEL FOR ULTRASONIC REFERENCE STANDARDS

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INTRODUCTION

In ultrasonic NDE, spherically focussed transducers are often used because they possess higher sensitivity and spatial resolution than conventional planar transducers [1]. A common reference standard setup that can be used to obtain the effective radius, focal length, and transduction efficiency of a spherically focussed probe is the immersion configuration shown in Figure 1 where a scatterer of simple shape (such as a sphere or cylinder) is placed on the axis of the transducer and the pulse-echo response measured. A model is developed here that allows the use of this measured response to obtain the important transducer parameters. The model is computationally efficient and the methods of characterization are directly analogous to those that have been successfully used for planar probes [2].

Figure 1 Reference scattering setup for the characterization of the parameters of a spherically focussed transducer.
THE MODEL

Planar transducers radiating into a fluid are normally modelled as a uniform piston source using Rayleigh-Sommerfeld theory [3]. O'Neil [4] has shown that the Rayleigh-Sommerfeld formulation can also be used for spherically focussed probes provided that the probe is not too tightly focussed. In this case, the incident wavefield pressure, $p^{inc}$, produced by the transducer is given by (Figure 1):

$$p^{inc} = -i\omega p v_o / 2\pi \int_S [\exp(ikr)/r] dS$$

where $\rho$ is the density of the fluid, $\omega$ is the frequency in rad/sec, and $v_o$ is the (uniform) velocity on the spherical surface $S$. At many wavelengths from the transducer surface, the spherical wave term in Eqn. (1) behaves locally like a plane wave. Therefore, the incident wave pressure in Eqn. (1) can be viewed as the superposition of suitably weighted plane waves over the transducer surface. Similarly, at many wavelengths from the scatterer, the scattered pressure wavefield, $p^{sca}$, due to a plane wave of unit amplitude can be calculated, once the far-field scattering amplitude $A(e_i; e_s)$, is known, i.e.

$$p^{sca} = A(e_i; e_s) \exp(ikr)/r_s$$

where $e_i$ and $e_s$ are the incident and scattered wave directions, respectively. If we combine Eqns. (1), (2) then by superposition the scattered pressure due to the focussed transducer can be written explicitly as

$$p^{sca} = -i\omega p v_o / 2\pi \int_S A(e_i; e_s) [\exp(ikr)/r] [\exp(ikrs)/r_s] dS$$

In the reception process, the quantity that usually is measured is the voltage versus time, $V_r(t)$, on an oscilloscope screen. If we let $V_r(\omega)$ be the frequency components of this received voltage, then we assume that $V_r(\omega)$ is proportional to the average radial velocity received, i.e.

$$V_r(\omega) = \beta(\omega) < v_R > / v_o$$

where $\beta(\omega)$ is an "efficiency" factor. The outward (from the transducer) radial velocity at many wavelengths from the scatterer can be calculated from $p^{sca}$ as

$$v_R = -\cos \alpha p^{sca} / \rho c$$

and the average radial velocity, $< v_R >$ is given by

$$< v_R > = 1/S \int_S v_R(r, \omega) dS$$

From Eqns. (3) - (5), therefore, we find

$$V_r(\omega) = ik\beta(\omega)/2\pi S \int_S \cos \alpha A(e_i; e_s)$$

$$[\exp(ikr)/r] [\exp(ikrs)/r_s] dS dS$$

\(926\)
where the unit vectors $\hat{e}_x; \hat{e}_y$ and the angle $\alpha$ are shown explicitly in Figure 2. Equation (7) is the basis for our spherically focussed transducer model. However, because of the double surface integrals present, this equation is not particularly suitable even for direct numerical evaluations. Fortunately, the area elements $dS, dS_s$ can be written in the forms [4]

$$dS = r dr d\phi / q_o$$

$$dS_s = r_s dr_s d\phi / q_o$$

where $q_o = 1 - z/R_o$, with $z$ being the axial location of the scatterer, $R_o$ is the geometrical focal length of the transducer, and $\phi$ and $\phi_s$ are angles measured in a plane perpendicular to the $z$-axis (Figure 3). In these coordinates

$$dS = \int_0^{2\pi} \int_0^{\pi} \exp(ikr) \exp(ikr_s) dr dr_s$$

where

$$<< A >\hat{e}_s > = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\pi} A d\phi d\phi_s$$

is the angular averaged far-field scattering amplitude of the scatterer and $r_s$ is the distance from the scatterer to the transducer edge. Although Eqn. (9) still contains four integrations, it is in a form considerably more convenient than Eqn. (7) since the angular averaged scattering amplitude is a quantity that can be calculated explicitly in a number of important special cases. For example, for a rigid cylinder of radius $b$ placed with one of its flat ends perpendicular to the transducer axis, we find, in the Kirchhoff approximation [5]:

$$<< A >\hat{e}_s > = -ikb \cos \theta_s / k^2 (\sin^2 \theta_s - \sin^2 \theta_s)$$

$$[k \sin \theta \tilde{J}_0(kb \sin \theta_s) \tilde{J}_1(kb \sin \theta_s) - k \sin \theta_s$$

$$- \tilde{J}_0(kb \sin \theta_s) \tilde{J}_1(kb \sin \theta_s)]$$

Similarly, for the front surface response of a rigid sphere of radius $b$ in the Kirchhoff approximation [5]:

$$<< A >\hat{e}_s > = b/2 \exp[-ikb\sqrt{\cos^2 \theta_s + \cos^2 \theta_s}]$$

$$- \tilde{J}_0(kb/\sqrt{\cos^2 \theta_s + \cos^2 \theta_s})$$

When these scattering amplitudes are slowly varying in $\theta, \theta_s$ (which is often the case in not too tightly focussed probes where $R_o >> a$), then

$$A(\hat{e}_x(\theta, \phi); \hat{e}_s(\theta_s, \phi_s)) = A_{b,s}$$

where $A_{b,s}$ is the "backscattered" scattering amplitude. Then the averages in Eqns. (11), (12) become
for the cylinder, and
\[ A_{b,z} = (b/2) \exp(-2ikb) \]  \hspace{1cm} (14b)
for the sphere, and the \( r, r_s \) integrations can be done explicitly (\( \cos \alpha = 1 \)) to yield
\[ V_s(\omega) = 2\pi \beta(\omega)A_{b,z}/ikq_s^2S[\exp(ikz) - \exp(ikr_s)]^2 \]  \hspace{1cm} (15)

In the paraxial approximation, Eqn. (15) is identical in form to that found by Thompson and Gray [6]. A particularly useful feature of Eqn. (15) is that it can serve as the basis for a straightforward method to obtain experimentally the effective radius and focal length of the transducer, as will be shown in the next section.

EFFECTIVE TRANSDUCER RADIUS AND FOCAL LENGTH

As Figure 4 illustrates schematically, the on-axis response of a scatterer in the wavefield of a focussed probe consists of a maximum response and a preceding minimum. If the locations of these maxima and minima on the axis, \( z_{\text{max}} \) and \( z_{\text{min}} \) respectively, are measured experimentally then estimates of both the effective transducer radius and focal length can be obtained as follows. First, from Eqn. (15) it can be shown that the location of the particular minimum shown in Figure 4 occurs at

\[ r_s = \frac{1}{2\beta S q_s^2} \]

Figure 2 Definition of unit vectors and geometric parameters in Eqn. (7).

Figure 3 Definition of the angles \( \phi, \theta, \theta_s \) and distances \( r, r_s, q_o \). Similar definitions hold for the angles \( \phi_s, \theta_s \) and the distance \( r_s \).
\[ z_{\min} = \frac{(a^2 + h^2) - \lambda^2}{2(\lambda + h)} = \frac{2R_o h - \lambda^2}{2(\lambda + h)} \]  \tag{16}

where \( \lambda \) is the wavelength and \( h = R_o - (R_o^2 - a^2)^{1/2} \). The maximum location satisfies

\[ \cos[k\delta/2] = 2\sqrt{(z_{\max} - h)^2 + a^2} \sin[k\delta/2]/(\delta + h)q_o kR_o \]  \tag{17}

where

\[ \delta = \sqrt{(z_{\max} - h)^2 + a^2 - z_{\max}} \]  \tag{18}

By placing Eqn. (16) into Eqn. (17), the latter equation can be rewritten in terms of \( R_o \) and measurable quantities only. Symbolically, we can rewrite Eqn. (17) in the form

\[ F[f, z_{\min}, z_{\max}, R_o] = 0 \]  \tag{19}

This non-linear equation can be solved iteratively for the best fit focal length, \( R_o \), from measured values of \( z_{\min} \) and \( z_{\max} \) at the frequency \( f \). Using this value of \( R_o \), we can then find the effective radius, \( a \), from Eqn. (16) which can be written in the equivalent form, for \( a \gg \lambda \), as

\[ a = \left[2R_o z_{\min} \lambda/(R_o - z_{\min})\right]^{1/2} \]  \tag{20}

Amin et al. [7] recently have proposed another non-linear fitting routine to determine the effective parameters \( R_o, a \). However, their method involves solving a rather complicated (and possibly ill-behaved) non-linear least squares problem and uses fitting data taken both on- and off-axis. In contrast, the above procedures uses only the discrete measured values, \( z_{\min} \) and \( z_{\max} \), and only requires the solution of Eqn. (19), which can be done with a simple root solver. Also, the above procedure is merely an extension of the methods previously used to determine the effective radius of unfocussed probes [8].

Table 1 shows the results of this method for a typical commercially available focussed probe (Panametrics V309, nominal radius = 0.25 inches, nominal focal length = 4.0 inches, "center" frequency - 5 MHz) using a 1/8" diameter steel sphere scanned along the transducer axis. Effective radius and focal length values were calculated at the nominal "center" frequency of the transducer (5 MHz) as well as at the "side" frequencies of 4 and 6 MHz. As

![Figure 4 Response of a spherically focussed transducer to an on-axis scatterer.](image-url)
can be seen from Table 1, there is some variation of these effective parameters with frequency. This same phenomena was noted previously by Amin et al. [7] for focussed probes and by Chivers et al. [8] for planar transducers.

**TRANSDUCER SYSTEM EFFICIENCY FACTOR**

To completely characterize a spherically focussed transducer in a given ultrasonic setup, it is necessary to also have a means to calculate the efficiency factor, $\beta(\omega)$. Once the equivalent parameters, $R_o$ and $\alpha$, are obtained, $\beta(\omega)$ can be found by placing a scatterer, such as a sphere, at the geometrical focus, i.e. $z = R_o$. Unfortunately, in Eqn. (9) (or, equivalently, Eqn. (15)), we cannot merely set $z = R_o$ since these expressions lead to indeterminant forms. However, if we introduce a new set of angular coordinates, $x = \cos \theta$, $y = \cos \phi$, in Eqn. (9), we find

$$V_r(\omega) = 2\pi k \beta(\omega)R_o^2/(S \int_1^\cos \theta \alpha << A >_{\phi} r r_z$$

$$\cdot \exp(ikr) \exp(ikr_z)dx dy (r + R_o q(x))(r_z + R_o q(y))$$

which is now well-behaved even at $z = R_o$ where we obtain

$$V_r(\omega) = 2\pi k \beta(\omega)R_o^2 \exp(2ikR_o) [<< A >_{\phi} A] / S$$

with

$$[A]_{\phi} = \int_0^\theta A \sin \theta d\theta$$

Again, if $A$ is slowly varying so that $A \sim A_{\phi,z}$, then Eqn. (22) simplifies even further and we have

$$V_r(\omega) = 2\pi k \beta(\omega)R_o^2 \exp(2ikR_o)(\cos \theta_z - 1)^2 A_{\phi,z} / S$$

Either Eqn. (22) or Eqn. (24) can be used directly to find $\beta(\omega)$ once $V_r(\omega)$ and $R_o, \alpha$ are measured since the remaining terms are easily calculated for simple scatterers and $\beta(\omega)$ then obtained by a division-like process using the concept of a Wiener filter [9]. Attenuation corrections can also be included in this process if they are needed. Although we have not carried out any efficiency calculations yet via these equations, they are similar to those we have previously used for planar probes [9] and should be straightforward to employ.

**CONCLUSIONS**

We have described the model of a spherically focussed transducer/scatterer combination that allows the effective radius, focal length, and efficiency factor of the probe to be determined. The experimental $R_o$ and $\alpha$ values were shown to be somewhat frequency dependent—a result that we are currently trying to understand with the use of more sophisticated models. Experimental efficiency factor calculations are also being conducted with this model. The calibration/standards methods derived from this model are simple and should be useful in a wide variety of practical applications.
Table 1. Effective radius and focal length calculations

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<th>Frequency (MHz)</th>
<th>a (in.)</th>
<th>R₀ (in.)</th>
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<tr>
<td>4</td>
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<td>3.49</td>
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<tr>
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REFERENCES