1955

Power of test procedures for certain incompletely specified random and mixed models

Helen Bozivich
Iowa State College

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Statistics and Probability Commons

Recommended Citation
Bozivich, Helen, "Power of test procedures for certain incompletely specified random and mixed models" (1955). Retrospective Theses and Dissertations. 12868.
https://lib.dr.iastate.edu/rtd/12868

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
NOTE TO USERS

This reproduction is the best copy available.

UMI®
INCOMPLETELY SPECIFIED RANDOM AND MIXED MODELS

by

Helen Bozivich

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY

Major Subject: Statistics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College

1955
INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.
# TABLE OF CONTENTS

I. INTRODUCTION
   A. Incompletely Specified Random Models 1
   B. Reduction of Mixed Models to Random Models 7

II. REVIEW OF LITERATURE 9

III. DERIVATION OF EXACT AND APPROXIMATE FORMULAS FOR POWER. COMPONENT OF VARIANCE MODEL 11
   A. Mathematical Formulation of the Pooling Procedure 11
   B. Integral Expressions for the Power 12
   C. Exact Formulas 15
      1. Series formulas 15
      2. Recurrence formulas 26
   D. Approximate Formulas for Large $n_1$ 36
   E. Theory of Reduction of Mixed Model to Random Model 39
   F. Application of Derived Formulas 42

IV. DISCUSSION OF POWER AND SIZE CURVES AND COMPARISON OF TEST PROCEDURES 45
   A. Type of Recommendations Attempted 45
   B. Size Curves 46
   C. Frequency of Pooling 50
   D. Power Curves 51

V. ILLUSTRATION OF RECOMMENDED PROCEDURES WITH PRACTICAL EXAMPLES 57
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Tests on Samples of Portland Cement</td>
<td>57</td>
</tr>
<tr>
<td>B. Porosity of Condenser Paper</td>
<td>59</td>
</tr>
<tr>
<td>VI. LITERATURE CITED</td>
<td>61</td>
</tr>
<tr>
<td>VII. ACKNOWLEDGMENTS</td>
<td>63</td>
</tr>
<tr>
<td>VIII. APPENDIX</td>
<td>64</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

A. Incompletely Specified Random Models

Let us assume the component of variance model

\[ x_{ijk} = \mu + a_i + b_{ij} + z_{ijk}, \quad (1) \]

where

\[ i = 1, 2, \ldots, q; \quad j = 1, 2, \ldots, r; \quad k = 1, 2, \ldots, s; \]

\( a_i \) is \( N(0, \sigma_a^2) \), \( b_{ij} \) is \( N(0, \sigma_b^2) \) and \( z_{ijk} \) is \( N(0, \sigma_z^2) \).

We wish to test a hypothesis concerning \( a_i \). If \( \sigma_b^2 \geq 0 \), then

\[ \begin{cases} x_{ijk} = \mu + a_i + b_{ij} + z_{ijk} & \text{for } \sigma_b^2 > 0, \\ x_{ijk} = \mu + a_i + z_{ijk} & \text{for } \sigma_b^2 = 0. \end{cases} \quad (2) \]

In this case (1) is said to be an incompletely specified model. If, however, \( \sigma_b^2 > 0 \), \( x_{ijk} = \mu + a_i + b_{ij} + z_{ijk} \)

and (1) is completely specified. Similarly, if \( \sigma_b^2 = 0 \),

\[ x_{ijk} = \mu + a_i + z_{ijk} \quad (4) \]

and again (1) is completely specified.

We wish to test the hypothesis \( H_0: \sigma_a^2 = 0 \) against the alternative \( H_1: \sigma_a^2 > 0 \). Now let us assume we have the completely specified model given by (3). Then \( \sigma_b^2 > 0 \) and we obtain the analysis of variance given in Table 1.
Table 1. Component of variance model with \( \sigma^2_b > 0 \).

Analysis of variance

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>d.f.</th>
<th>Mean square</th>
<th>Exp. mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between A</td>
<td>( n_3 = q-1 )</td>
<td>( V_3 )</td>
<td>( \sigma^2_z + s\sigma^2_b + r_s\sigma^2_a )</td>
</tr>
<tr>
<td>Between B within A</td>
<td>( n_2 = q(r-1) )</td>
<td>( V_2 )</td>
<td>( \sigma^2_z + s\sigma^2_b )</td>
</tr>
<tr>
<td>Within B</td>
<td>( n_1 = qr(s-1) )</td>
<td>( V_1 )</td>
<td>( \sigma^2_z )</td>
</tr>
</tbody>
</table>

Then it follows from the likelihood ratio principle that the appropriate test procedure is to calculate the test statistic

\[
F_0 = \frac{V_3}{V_2}
\]

and to reject \( H_0 \) if \( F_0 \geq F_a(n_3, n_2) \), where \( a \) is the prescribed level of significance. This test is the never pool test.

Next let us assume the completely specified model given by (4). Now the expected mean squares of Table 1 no longer include the \( \sigma^2_b \) component since \( \sigma^2_b = 0 \). Application of the likelihood ratio test procedure to this model for the test of \( H_0 \) gives us the test criterion

\[
F_0 = \frac{(n_1 + n_2)V_3}{n_1 V_1 + n_2 V_2}
\]

and the rule to reject \( H_0 \) if \( F_0 \geq F_a(n_3, n_1 + n_2) \). This gives us the always pool test.
Finally, we assume $\sigma^2_b \geq 0$, and hence are confronted with the incompletely specified model given by (2). Ordinarily this model (2) might be assumed when there exists some uncertainty as to whether $\sigma^2_b = 0$ or $\sigma^2_b > 0$. In such cases of incomplete specification attempts are often made to resolve the uncertainty by first testing the hypothesis that $\sigma^2_b = 0$. The model finally chosen and hence the final test (test of $H_0$) depend upon the outcome of this original test. When the original and final tests are performed on the same set of data, the original test is referred to as a preliminary test of significance. In our example the preliminary test becomes the test of $H'_0$: $\sigma^2_b = 0$ against $H'_1$: $\sigma^2_b > 0$. Again a likelihood ratio test procedure is available for this preliminary test. The statistic $F_o = V_2/V_1$ is calculated and $H'_0$ rejected at the level $a_1$ (usually different from $a$) if $F_o \geq F_{a_1}(n_2, n_1)$. If $H'_0$ is rejected the non-pooling test procedure indicated by (5) is used for the final test. If $H'_0$ is not rejected, the pooling procedure indicated by (6) is used for the final test. This entire testing procedure is called a sometimes pool procedure.

It should be noted that when the final test is carried out, the model is assumed to be completely specified, that is, to be either model (3) or model (4), according as the preliminary test is found to be significant or not significant, respectively.

The model (1) we have assumed is that of a hierarchal classification. Similar situations of incomplete specification and hence analogous test procedures arise for numerous other random models. For example, if we assume the model
\[ x_{ijk} = \mu + a_i + b_j + (ab)_{ij} + z_{ijk} \, , \]  

where \( i = 1, 2, \ldots, r; \ j = 1, 2, \ldots, s; \ k = 1, 2, \ldots, t; \ a_i \) is \( N(0, \sigma_a^2) \), 
\( b_j \) is \( N(0, \sigma_b^2) \), \((ab)_{ij}\) is \( N(0, \sigma_{ab}^2) \) and \( z_{ijk} \) is \( N(0, \sigma_z^2) \); we wish to test the hypothesis \( H_0: \sigma_a^2 = 0 \), against the alternative \( H_1: \sigma_a^2 > 0 \), but are uncertain as to whether \( \sigma_{ab}^2 = 0 \) or \( \sigma_{ab}^2 > 0 \); then we have a situation analogous to the one described above. Analogous situations of incompletely specified models also arise with the mixed model discussed in the following section.

Situations frequently arise in experimental design where the model is not completely specified. Furthermore, with the wider application of analysis of variance to operational research and to the study of routine data, analyses are often made of data which have not resulted from a designed experiment; and in these situations the model is often incompletely specified. In such cases of uncertainty, preliminary tests have often been used in the past as an aid to choosing an appropriate model.

Paul (1948) describes the following operational experiment on the sources of variability in the determination of the protein content of wheat by different laboratories. Out of a large population of laboratories in Canada, three were selected at random and each of them asked to analyze 10 subsamples of the same sample of wheat, making five protein determinations on each of two days. Let \( x_{ij} \) denote the protein determination from the \( j \)-th sample analyzed on the \( i \)-th day at the \( t \)-th laboratory. Then it is assumed that the data are adequately described
by the model

\[ x_{tij} = \mu + \mathcal{L}_t + d_{ti} + z_{tij} \]  

where the laboratory variables, \( \mathcal{L}_t \), the day variables, \( d_{ti} \), and the test variables, \( z_{tij} \), are assumed to be random samples from the respective normal populations \( N(0, \sigma_\mathcal{L}^2) \), \( N(0, \sigma_d^2) \) and \( N(0, \sigma_z^2) \). The analysis of variance based on the above model is shown below.

Table 2. Component of variance model example. Analysis of variance

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>d.f.</th>
<th>Mean square</th>
<th>Exp. mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between laboratories</td>
<td>( n_3 = 2 )</td>
<td>( V_3 )</td>
<td>( \sigma_3^2 = \sigma_z^2 + 5\sigma_d^2 + 10\sigma_\mathcal{L}^2 )</td>
</tr>
<tr>
<td>Between days within laboratories</td>
<td>( n_2 = 3 )</td>
<td>( V_2 )</td>
<td>( \sigma_2^2 = \sigma_z^2 + 5\sigma_d^2 )</td>
</tr>
<tr>
<td>Within days</td>
<td>( n_1 = 24 )</td>
<td>( V_1 )</td>
<td>( \sigma_1^2 = \sigma_z^2 )</td>
</tr>
</tbody>
</table>

The main interest of the experiment lies in testing whether the laboratories introduce any variability additional to that encountered within each laboratory, that is, in testing the null hypothesis whether \( \sigma_\mathcal{L}^2 = 0 \) against the alternative \( \sigma_\mathcal{L}^2 > 0 \). It is well known (and indicated by the above table) that the appropriate \( F \) statistic to use is \( V_3/V_2 \), unless \( \sigma_d^2 = 0 \), in which case we should pool \( V_2 \) and \( V_1 \) and use the statistic

\[
\frac{(n_1 + n_2)}{n_1 V_1 + n_2 V_2} \cdot \frac{V_3}{V_2}.
\]
In the present example the three mean squares $V_1$, $V_2$ and $V_3$, with which the pooling procedure is concerned, have arisen from what is commonly known as a hierarchical (or nested) analysis of variance with laboratories as primary units, days as secondary units and test results as the final observations. Moreover, we have assumed a component of variance model (Model II) consisting entirely of normal effect variates. We may summarize the consequences of such a model by three essential features:

(i) The three mean squares $V_i$ are independently distributed as
\[ \chi^2_{r} \sigma_i^2 / n_i \] where $\chi^2$ is the (central) $\chi^2$ statistic for $n_i$ degrees of freedom.

(ii) The main purpose of the analysis is to test the null hypothesis
\[ \sigma_3^2 = \sigma_2^2 \] against the alternative $\sigma_3^2 > \sigma_2^2$.

(iii) The error mean square $V_2$ has an expectation $\sigma_2^2$ which is greater than or equal to the expectation, $\sigma_1^2$, of the doubtful error mean square $V_1$, which may or may not be pooled.

It is clear that the above nested (or hierarchical) classification is not the only analysis of variance situation giving rise to the above conditions, (i), (ii) and (iii) which we take to define the Model II pooling procedure. There are numerous other analysis of variance tables resulting in the same test situation. As an example we may quote the two way classification with both factors random and cell repetition. Here $V_1$ would play the part of the within cell mean square while $V_2$ would be represented by the residual in the two-way analysis.
B. Reduction of Mixed Models to Random Models

The preceding section has been devoted to random models only. Another frequently occurring type of model is the mixed model in which one of the factors is fixed and the other factors are random, and the hypothesis of interest is concerned with the fixed factor. A typical example of an experiment giving rise to this type of model is a randomized block experiment in which $k$ rations are each fed to a pen of $m$ animals in each of $n$ replicates. Then a suitable model for these data is given by

$$x_{tij} = \mu + a_t + b_i + d_{ti} + e_{tij},$$

where the replicate variates $b_i$, error variates $d_{ti}$ and within pen error variates $e_{tij}$ are assumed to be random samples from the respective normal populations $N(0, \sigma^2_b)$, $N(0, \sigma^2_d)$ and $N(0, \sigma^2_z)$, while the ration means $a_t$ are fixed parameters. The analysis of variance based on this model is shown below.

Table 3. Mixed model example. Analysis of variance

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>d.f.</th>
<th>Mean square</th>
<th>Exp. mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between replicates</td>
<td>$n-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Between rations</td>
<td>$n_3 = k-1$</td>
<td>$V_3$</td>
<td>$\sigma^2_3 = \sigma^2_z + m\sigma^2_d + mn0(a)$</td>
</tr>
<tr>
<td>Reps x rations</td>
<td>$n_2 = (k-1)(n-1)$</td>
<td>$V_2$</td>
<td>$\sigma^2_2 = \sigma^2_z + m\sigma^2_d$</td>
</tr>
<tr>
<td>Within pens</td>
<td>$n_1 = nk(m-1)$</td>
<td>$V_1$</td>
<td>$\sigma^2_1 = \sigma^2_z$</td>
</tr>
</tbody>
</table>
Here $\Theta (a) = \Sigma (a_t - \bar{a})^2 (k-1)$.

The main purpose of the experiment is to test the equality of the ration means, that is, to test the hypothesis $H_0: \Theta (a) = 0$ against the alternative $H_1: \Theta (a) > 0$. The present situation is therefore identical with that of the component of variance model except that the treatment sum of squares, $n_3 V_3$, is distributed as $\chi^2 v_2^2$, where $\chi^2$ is the non-central $\chi^2$ statistic with $n_3$ degrees of freedom. However, we shall show that recommendations based upon the results obtained for the random model can be made for the present model. The main reasons for this are as follows:

(i) In the case of the null hypothesis of no treatment differences
the mixed model becomes identical with the random model; hence all our results on size curves are directly applicable.

(ii) In the case of the alternative hypothesis we can apply Patnaik's approximation for the non-central $\chi^2$ statistic, and, therefore, we can apply our power results to the present situation by making certain transformations.

---

*The appropriate non-centrality parameter will be given in Subsection E of Section III.*
situations where preliminary tests are performed for both homogeneity

Benett (1992) extended the studies of Mosteller and Kilgarae to

-test assuming unknown variance.

and moments of a pooled estimator of a mean based on a preliminary

in the variance estimation problem. He also derived the distribution

for the estimator obtained by the rule of procedure studied by Bancroft

Kilgarae (1990) derived the distribution function and the moments

in estimating a population mean.

of significance as an aid to deciding when to pool two sample means.

Mosteller (1984) has examined the effect of using preliminary tests

-Regression equation analysis.

due to the omission of several independent variables in the multiple

Later Bancroft (1990) incorporated on phases in estimates of variance

-z. to decide whether or not to retain the variable \( x_i \).

P being dependent upon a preliminary test of significance of \( p_i^2 \) made

\[
Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e
\]

the regression coefficient \( \beta_i \) in the model

In the same paper Bancroft studied the bias in the estimator \( \hat{p} \) of

test of the equality of two variances.

error of a variance estimator obtained after performing a preliminary

of Bancroft (1944), who investigated the bias, variance and mean square

The earliest work involving preliminary tests of significance was that

II. REVIEW OF LITERATURE
of variances and equality of means prior to estimating the mean or testing hypotheses about the mean. He derived the distribution functions, the biases, and the mean square errors for various cases, depending upon what assumptions were made concerning the parameters of the associated normal distributions.

Paul (1948) studied the size and power for the incompletely specified component of variance model described in Section I. However, he was able to express the size and power in closed form only for the case \( n_3 = 2 \), so that all comparisons made by him are restricted to that value of \( n_3 \).

A similar investigation was undertaken by Bechhofer (1951) for the incompletely specified linear hypothesis model.

The object of the present study is to provide the necessary extension of Paul's investigation to cover all of the important degrees of freedom combinations occurring in the analyses of variance under discussion. This extension was made possible by

(i) the development of the power integrals as series formulas, for even values of the degrees of freedom \( n_1, n_2 \) and \( n_3 \); 
(ii) the derivation of recurrence formulas for the power, for even values of \( n_1, n_2 \) and \( n_3 \); and 
(iii) the development of approximate formulas valid for large degrees of freedom, for even values of \( n_1, n_2 \) and \( n_3 \).
III. DERIVATION OF EXACT AND APPROXIMATE FORMULAS FOR POWER. COMPONENT OF VARIANCE MODEL

A. Mathematical Formulation of the Pooling Procedure

We now derive formulas for the power and size of the pooling procedure applied to the component of variance model described in Section I. Let us first state the procedure in mathematical terms.

We are given an analysis of variance table as shown below.

Table 4. Component of variance model. Analysis of variance

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Mean square</th>
<th>d.f.</th>
<th>Exp. mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>$V_3$</td>
<td>$n_3$</td>
<td>$\sigma_3^2$</td>
</tr>
<tr>
<td>Error</td>
<td>$V_2$</td>
<td>$n_2$</td>
<td>$\sigma_2^2$</td>
</tr>
<tr>
<td>Doubtful error</td>
<td>$V_1$</td>
<td>$n_1$</td>
<td>$\sigma_1^2$</td>
</tr>
</tbody>
</table>

We are interested in testing the hypothesis $H_0: \sigma_3^2 = \sigma_2^2$ against the alternative $H_1: \sigma_3^2 > \sigma_2^2$ when it is known that $\sigma_3^2 \geq \sigma_1^2$. We assume the sums of squares $n_i V_i$ are independently distributed as $\chi^2_i \sigma_i^2$ where $\chi^2_i$ is the central $\chi^2$ statistic based on $n_i$ degrees of freedom. The test procedure with sometimes pooling $V_2$ and $V_1$ is then as follows:
Reject $H_0$ if

either

\[ V_2/V_1 \geq F_{n_2, n_1}(a_1) \text{ and } V_3/V_2 \geq F_{n_3, n_2}(a_2) \]  

or

\[ V_2/V_1 \leq F_{n_2, n_1}(a_1) \text{ and } V_3/V \leq F_{n_3, n_1+n_2}(a_3) \]  

(9)

where $V = (n_1 V_1 + n_2 V_2)/(n_1 + n_2)$ and $F_{n_i, n_j}(a)$ is the upper $100a\%$ point of the $F$ distribution with numerator d.f. $= n_i$ and denominator d.f. $= n_j$.

The probability, $P$, of rejecting $H_0$, which in general is the power of the test procedure, is a function of the degrees of freedom, $n_1$, $n_2$ and $n_3$, the ratios, $\theta_{32} = \sigma_3^2/\sigma_2^2$ and $\theta_{21} = \sigma_2^2/\sigma_1^2$, and the levels of significance employed, $a_1$, $a_2$ and $a_3$. In the special case when $\theta_{32} = 1$ this power is equal to the size of the test, that is, to the type one error. In general the power $P$ is obtained as the sum of two components corresponding to the mutually exclusive alternatives headed by either and or in its definition above, namely,

\[ P_1 = \Pr \left\{ V_2/V_1 \geq F_{n_2, n_1}(a_1) \text{ and } V_3/V_2 \geq F_{n_3, n_2}(a_2) \right\}, \quad (10) \]

\[ P_2 = \Pr \left\{ V_2/V_1 \leq F_{n_2, n_1}(a_1) \text{ and } V_3/V \leq F_{n_3, n_1+n_2}(a_3) \right\}. \quad (11) \]

B. Integral Expressions for the Power

Definite integrals for $P_1$ and $P_2$ will now be derived. The joint density of $V_1$, $V_2$ and $V_3$ is given by
\[ c_1 V_1^{\frac{n_1}{2} - 1} V_2^{\frac{n_2}{2} - 1} V_3^{\frac{n_3}{2} - 1} \exp \left\{ -\frac{1}{2} \frac{n_1 V_1}{\sigma_1^2} + \frac{n_2 V_2}{\sigma_2^2} + \frac{n_3 V_3}{\sigma_3^2} \right\}, \]

where \( c_1 \) is a constant independent of \( V_1, V_2 \) and \( V_3 \). By introducing new variates,

\[ u_1 = \frac{n_2 V_2}{n_1 V_1}, \quad u_2 = \frac{n_3 V_3}{n_2 V_2}, \quad w = \frac{n_1 V_1}{n_3}, \]

and integrating out \( w \) as a gamma function, we obtain the joint density of \( u_1 \) and \( u_2 \) as

\[ f(u_1, u_2) = \frac{c_2 u_1^{\frac{n_2 + n_3}{2} - 1} u_2^{\frac{n_3}{2} - 1}}{1 \{ n_1 + n_2 + n_3 \}} \left( q_{21} q_{32} + q_{32} u_1 + u_1 u_2 \right), \]

where \( c_2 = \frac{n_1}{B(\frac{1}{2}, \frac{1}{2})} \frac{n_2 + n_3}{B(\frac{1}{2}, \frac{1}{2})} \frac{n_1 + n_2}{B(\frac{1}{2}, \frac{1}{2})} \), \( q_{21} = \frac{\sigma_2^2}{\sigma_1^2} \) and \( q_{32} = \frac{\sigma_3^2}{\sigma_2^2} \).

Let us now make the transformation

\[ u = \frac{u_1}{q_{21}}, \quad v = \frac{u_2}{q_{32}}. \]

The Jacobian of this transformation is \( q_{21} q_{32} \), and the joint density of \( u \) and \( v \) becomes

\[ f(u, v) = k u^{\frac{n_2 + n_3}{2} - 1} v^{\frac{n_3}{2} - 1} \exp \left\{ -\frac{1}{2} \frac{n_1 + n_2 + n_3}{2} \right\}, \]

\[ (1 + u + uv) \]
where
\[
k = \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) B\left(\frac{n_3}{2}, \frac{n_1+n_2}{2}\right)}.
\]  

(14)

The probability of rejecting the hypothesis \(H_0\) is obtained by integrating \(f(u, v)\) over the two ranges of variation of \(u\) and \(v\) which correspond to the two alternatives either and or of definition (9). These ranges are respectively given by

either
\[
\frac{u_1^0}{\theta_{21}} \leq u < \infty, \quad \frac{u_2^0}{\theta_{32}} \leq v < \infty;
\]

(15)
or
\[
0 \leq u \leq \frac{u_1^0}{\theta_{21}}, \quad \frac{u_3^0 (1 + \theta_{21}u)}{\theta_{32} \theta_{21}} \leq v < \infty;
\]

where
\[
u_1^0 = \frac{n_2}{n_1} F_{n_2, n_1}(a_1), \quad u_2^0 = \frac{n_3}{n_2} F_{n_3, n_2}(a_2),
\]

and
\[
u_3^0 = \frac{n_3}{n_1 + n_2} F_{n_3, n_1+n_2}(a_3).
\]

(16)

Hence the formulas for the two power components become

\[
P_1 = \int_a^\infty \int_0^\infty f(u, v) \, dv \, du,
\]

(17)

and

\[
P_2 = \int_0^{c(1+\theta_{21}u)} \int_0^a f(u, v) \, dv \, du,
\]

(18)
where \[ a = \frac{u_1^0}{q_{21}}, \quad c = \frac{u_3^0}{q_{21}q_{32}}, \quad \text{and} \quad d = \frac{u_2^0}{q_{32}}. \] (19)

C. Exact Formulas

1. Series formulas

From (13) and (17),

\[
P_1 = k \int_a^\infty \int_d^\infty \frac{\left(\frac{n_2 + n_3 - 1}{n_2} - 1\right) \frac{n_3 - 1}{n_3}}{v \frac{u}{u + v} \frac{n_1 + n_2 + n_3}{(1 + u + uv)^{n_1 + n_2 + n_3}}} \, dv \, du,
\]

where \[ k = \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) B\left(\frac{n_3}{2}, \frac{n_1 + n_2}{2}\right)}. \] (20)

Let \( y = \frac{v}{1 + u + uv} \); then it is easily verified that

\[
v = \frac{(1 + u)y}{1 - uy}, \quad dv = \frac{(1 + u)dy}{(1 - uy)^2},
\]

and

\[
P_1 = k \int_a^\infty \int_{x_5}^\infty \frac{\left(\frac{1}{y^2} - (1 - uy)^2\right) \frac{n_1 + n_2}{2} - 1}{(1 - uy)^2} \frac{\left(\frac{n_2 + n_3}{2} - 1\right) \frac{u}{u + v} \frac{n_1 + n_2 + n_3}{(1 + u + uv)^{n_1 + n_2 + n_3}}} \, dy \, du,
\]

where \( x_5 = \frac{d}{1 + u + du}. \) (21)

Let \( z = 1 - uy, \quad dz = -u \, dy. \)
Then
\[ P_1 = k \int_a^\infty \int_0^{x_6} \frac{n_1+n_2-1}{2} \frac{n_3-1}{2} \frac{n_2-1}{2} \frac{z}{(1-z)^{ \frac{1}{2} }} \frac{u}{(1 + u)^{ \frac{1}{2} }} \, dz \, du, \]

where
\[ x_6 = \frac{1 + u}{1 + u + du} \quad (22) \]
The binomial expansion of \((1-z)^{\frac{1}{2}}\) gives us
\[ P_1 = k \int_a^\infty \int_0^{x_6} \frac{n_2}{2} \frac{n_1+n_2-1}{2} f(z) \frac{u}{(1 + u)^{ \frac{1}{2} }} \, dz \, du, \]
where
\[ f(z) = z^{ \frac{n_1+n_2-1}{2} } - \left( \frac{n_3}{2} - 1 \right) z^{ \frac{n_1+n_2}{2} } + \left( \frac{n_3}{2} - 1 \right) z^{ \frac{n_1+n_2}{2} + 1 } \]

\[ - \left( \frac{n_3}{2} - 1 \right) z^{ \frac{n_1+n_2}{2} + 2 } + \ldots + (-1)^j \left( \frac{n_3}{2} - 1 \right) z^{ \frac{n_1+n_2}{2} + j - 1 } \]

\[ + \ldots + (-1)^j \frac{n_3}{2} z^{ \frac{n_1+n_2+n_3}{2} - 1 } . \]
The integration with respect to \( z \) gives us, upon simplification,
\[ P_1 = k \int_a^\infty g(u) \, du, \quad \text{where} \]
\[ g(u) = \frac{n_2}{2u} \frac{n_1+n_2-1}{2} - \frac{n_2}{2} \left( \frac{n_3}{2} - 1 \right) \frac{n_1+n_2}{2} - \frac{n_2}{2} \left( \frac{n_1+n_2+2}{2} \right) \frac{n_1+n_2}{2} \]
\[ (n_1+n_2)(1+u+du) \frac{n_1+n_2}{2} - (n_1+n_2+2)(1+u+du) \frac{n_1+n_2}{2} - 1 \]
\[
\begin{align*}
&\frac{\left(\frac{n_3}{2} - 1\right)}{2} + 2 \left(\frac{\frac{n_3}{2} - 1}{(1+u)^2 u}\right) \frac{n_2}{2} - 1 + \frac{2 \left(\frac{n_3}{2} - 1\right)}{3 (1+u) u} \frac{n_2}{2} - 1 \frac{n_2}{2} + 2 \\
&\frac{n_1+n_2}{2} + 2 (n_1+n_2+6)(1+u+du) \\
&\frac{n_1+n_2}{2} + 2 (n_1+n_2+4)(1+u+du)
\end{align*}
\]

Before proceeding to integrate these terms, let us consider a general case.

Let
\[
H = \int_{a}^{\infty} \frac{u^m}{(1+u+du)^n} \, du.
\]

Let us make the transformation
\[
y = \frac{1}{1+u+du} ;
\]
then
\[
u = \frac{1 - y}{(1+d)y} , \quad du = \frac{-1}{(1+d)y^2} \, dy .
\]

Upon simplification, we obtain
\[
H = \frac{1}{(1+d)^{m+1}} \int_{0}^{x} y^{n-m-2} (1-y)^m \, dy
\]
\[ B(n-m-1, m+1) I_{x_7} (n-m-1, m+1) \]
\[ = \frac{m+1}{(1+d)} \]
\[ \text{where} \quad x_7 = \frac{1}{1+a+ad} \]  

(24)

Now let us indicate the successive terms of \( P_1 \) as \( P_{11}, P_{12}, P_{13}, \ldots \).

\[ P_{1n} \]
\[ \frac{2k}{n_1+n_2} \int_a^\infty \frac{u}{\frac{n_2}{2} - 1} \frac{n_1+n_2}{2} \frac{n_1+n_2}{2} \frac{2k}{n_1+n_2} B(\frac{n_1}{2}, \frac{n_2}{2}) I_{x_7} (\frac{n_1}{2}, \frac{n_2}{2}) \]
\[ \frac{2k}{n_1+n_2} B(\frac{n_3}{2}, \frac{n_1+n_2}{2}(1+d) \frac{n_2}{2}) \]
\[ = \frac{n_1+n_2}{(\frac{n_2}{2}) B(\frac{n_3}{2}, \frac{n_1+n_2}{2}(1+d) \frac{n_2}{2})} \]

upon substitution of \( k \) from (14).

\[ P_{12} = c_{12} \int_a^\infty \frac{u}{\frac{n_1+n_2}{2} + 1} \frac{n_2}{2} \frac{n_1+n_2}{2} \frac{2k}{n_1+n_2} \]
\[ c_{12} \int_a^\infty \frac{u}{\frac{n_1+n_2}{2} + 1} \frac{n_1+n_2}{2} \frac{n_2}{2} \frac{2k}{n_1+n_2} \]
\[ \text{where} \quad c_{12} = \frac{\left( \frac{n_3}{2} - 1 \right) k}{\frac{n_1+n_2}{2} + 1} \]  

(25)
\[
P_{12} = \frac{c_{12} B(\frac{z}{2} + 1, \frac{z}{2}) I_{x/7}(\frac{z}{2} + 1, \frac{z}{2})}{(1+d) \frac{z}{2}} + \frac{c_{12} B(\frac{z}{2}, \frac{z}{2} + 1) I_{x/7}(\frac{z}{2} + 1, \frac{z}{2} + 1)}{(1+d) \frac{z}{2} + 1}
\]

\[
z = \frac{\begin{pmatrix} \frac{n_3}{z} - 1 \\ \frac{1}{z} \end{pmatrix}}{\begin{pmatrix} n_1 + n_2 \\ -\frac{n_1 + n_2}{z} \end{pmatrix}} \begin{pmatrix} n_1 n_2 \\ \frac{n_1 + n_2}{z} \end{pmatrix} \left[ B(\frac{z}{2} + 1, \frac{z}{2}) I_{x/7}(\frac{z}{2} + 1, \frac{z}{2}) \right] \]

from application of (23), and

upon substitution of \( c_{12} \) from (25) and of \( k \) from (14).

Using the procedure indicated in evaluating \( P_{11} \) and \( P_{12} \) formulas for the remaining terms can be similarly derived. In general, the \( j \)-th term \((1 \leq j \leq \frac{n_3}{z} - 1)\) is seen to be the product of two expressions,

\[
P_{j} = \left[ (-1)^{j-1} \begin{pmatrix} \frac{n_3}{z} - 1 \\ j - 1 \end{pmatrix} \right] \left[ \begin{pmatrix} n_1 + n_2 \\ -\frac{n_1 + n_2}{z} \end{pmatrix} \begin{pmatrix} n_1 n_2 \\ \frac{n_1 + n_2}{z} \end{pmatrix} \left(1 + \frac{d}{z}\right) \right]
\]

\[
x \sum_{r=0}^{j-1} \begin{pmatrix} \frac{n_1}{z} + j - 1 - r \\ \frac{n_2}{z} + r \end{pmatrix} I_{x/7}(\frac{z}{2} + j - 1 - r, \frac{z}{2} + r) \left(1 + \frac{d}{z}\right)^r
\]
Upon substituting for \( B\left(\frac{n_1}{z}, \frac{n_2}{z}\right), B\left(\frac{n_3}{z}, \frac{n_1+n_2}{z}\right) \) and \( B\left(\frac{n_1}{z}+j-1-r, \frac{n_2}{z}+r\right) \)
in each term and collecting terms, we obtain

\[
P_1 = \frac{1}{n_1+n_2} \left\{ \frac{n_3}{z} \left( \frac{n_1}{z} + \frac{n_2}{z} \right) \right\}
\]

\[
- \frac{(\frac{n_3}{z} - 1)}{n_1+n_2} \left[ \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 1, \frac{n_2}{z} \right) + \frac{n_2}{z} I_{x,7} \left( \frac{n_1}{z} + \frac{n_2}{z} + 1 \right) \right]
\]

\[
+ \frac{(\frac{n_3}{z} - 1)(\frac{n_3}{z} - 2)}{(n_1+n_2)(n_1+n_2+2)} \left[ \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 1, \frac{n_2}{z} + 2 \right) + \frac{n_2}{z} I_{x,7} \left( \frac{n_1}{z} + \frac{n_2}{z} + 2 \right) \right]
\]

\[
+ \frac{(\frac{n_3}{z} - 1)(\frac{n_3}{z} - 2)}{(n_1+n_2)(n_1+n_2+2)} \left[ \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 1, \frac{n_2}{z} + 2 \right) + \frac{n_2}{z} I_{x,7} \left( \frac{n_1}{z} + \frac{n_2}{z} + 2 \right) \right]
\]

\[
+ \frac{n_3}{3!(\frac{n_1+n_2}{z}+1)(\frac{n_1+n_2}{z}+2)(\frac{n_1+n_2}{z}+3)} \left[ \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 2, \frac{n_2}{z} + 3 \right) + \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 2, \frac{n_2}{z} + 3 \right) \right]
\]

\[
+ \frac{n_1}{3!(\frac{n_1+n_2}{z}+1)(\frac{n_1+n_2}{z}+2)(\frac{n_1+n_2}{z}+3)} \left[ \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 2, \frac{n_2}{z} + 3 \right) + \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 2, \frac{n_2}{z} + 3 \right) \right]
\]

\[
+ \frac{n_2}{3!(\frac{n_1+n_2}{z}+1)(\frac{n_1+n_2}{z}+2)(\frac{n_1+n_2}{z}+3)} \left[ \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 2, \frac{n_2}{z} + 3 \right) + \frac{n_1}{z} I_{x,7} \left( \frac{n_1}{z} + 2, \frac{n_2}{z} + 3 \right) \right]
\]

\[
+ \ldots \ldots \right\}.
\]
We now consider $P_2$. From (13) and (18),

$$P_2 = k \left\{ a \int_0^a \int_0^\infty \frac{u^{n_2+n_3-1} v^{n_3-1}}{(1+u+uv)^{(n_1+n_2+n_3)/2}} \, dv \, du \right\},$$

where $x_8 = \frac{c(1+\theta 21 u)}{u} = \frac{c + bu}{u}$,

where $b = \frac{u_3^o}{\theta 32}$. \hspace{1cm} (26)

Now let $z = \frac{1+u}{1+u+uv}$;

then $v = \frac{(1+u)(1-z)}{uz}$, \hspace{1cm} $dv = -\left(\frac{1+u}{uz^2}\right) \, dz$.

After performing the necessary simplification, we obtain

$$P_2 = k \left\{ a \int_0^a \int_0^{x_9} \frac{n_1+n_2}{z^{1/2}} - 1 \frac{n_3}{(1-z)^{1/2}} - 1 \frac{n_2}{\left(\frac{n_1+n_2}{2}\right)} \, dz \, du \right\},$$

where $x_9 = \frac{1+u}{(1+c+u(1+b))}$. \hspace{1cm} (27)

The binomial expansion of $(1-z)^{n_3-1}$ gives us

$$P_2 = k \left\{ a \int_0^a \int_0^{x_9} \frac{u^{n_2-1} f(z)}{(1+u)^{n_1+n_2}} \, du \, dz \right\},$$

(28)
where

\[
f(z) = z \frac{n_1 + n_2}{z^2 - 1} - \left( \frac{n_3}{z - 1} \right) \frac{n_1 + n_2}{z} + \left( \frac{n_3}{z^2 - 1} \right) \frac{n_1 + n_2}{z} + 1
\]

\[
+ \left( \frac{n_3}{z^2 - 1} \right) \frac{n_1 + n_2 + 2}{z} + \ldots + (-1)^j \left( \frac{n_3}{z^2 - 1} \right) \frac{n_1 + n_2}{z} + j - 1
\]

\[
+ \ldots + (-1)^j \left( \frac{n_3}{z^2 - 1} \right) \frac{n_1 + n_2 + n_3}{z} + 1
\]

The integration with respect to \( z \) gives us, upon simplification,

\[
P_2 = k \int_0^a h(u) \, du,
\]

where

\[
h(u) = \frac{n_2}{2u^2 - 1} \left( \frac{n_1 + n_2}{2} \right) \left[ 1 + c + u(1+b) \right] - \frac{n_1 + n_2}{2} \left( \frac{n_3}{z^2 - 1} \right) \frac{n_2}{u^2 - 1} + \frac{n_1 + n_2 + 2}{2} \left( \frac{n_3}{z^2 - 1} \right) \frac{n_2}{u^2 - 1} (1+u)^2
\]

\[
- \frac{n_1 + n_2 + 4}{2} \left( \frac{n_3}{z^2 - 1} \right) \frac{n_2}{u^2 + 2} + \frac{n_1 + n_2 + 6}{2} \left[ 1 + c + u(1+b) \right] - \frac{n_1 + n_2}{2} + 3
\]
\[ + \ldots + \frac{\left( \frac{n_3^2}{2} - 1 \right)}{2} \frac{n_2^2}{2} - 1 \frac{(1+u)^j}{u^2} \frac{n_1}{2} + j \left( n_1 + n_2 + 2j \right) \left[ 1 + c + u(1+b) \right] \]

\[ + \frac{\left( \frac{n_3^2}{2} - 1 \right)}{2} \frac{n_2^2}{2} - 1 \frac{n_3^2}{2} - 1 \frac{n_1}{2} + j \frac{n_1 + n_2 + n_3}{2} - 1 \left( n_1 + n_2 + n_3 - 2 \right) \left[ 1 + c + u(1+b) \right] \]

Let us now consider the general case

\[ G = \int_0^a \frac{u^m}{\left[ 1 + c + u(1+b) \right]^n} \, du . \]

If we make the transformation

\[ y = \frac{u(1+b)}{1+c+u(1+b)} , \]

then

\[ u = \frac{y(1+c)}{(1-y)(1+b)} , \quad du = \frac{(1+c) \, dy}{(1-y)^2 (1+b)} , \]

and it is easily verified that

\[ G = \frac{1}{(1+c)^{n-m-1}(1+b)^{m+1}} \int_0^{x_S} y^m (1-y)^{n-m-2} \, dy , \quad (30) \]
where
\[ x_8 = \frac{a(1+b)}{1+c+a(1+b)} \quad (31) \]

Therefore,
\[ G = \frac{B(m+1,n-m-1) \cdot x_8 \cdot (m+1,n-m-1)}{(1+b)^{m+1} \cdot (1+c)^{n-m-1}} \quad (32) \]

Now let the successive terms of (28) be denoted by \( P_{21}, \ P_{22}, \ P_{23}, \ldots, \ P_{2n_3} \).

From (32),
\[ P_{21} = \frac{2k}{n_1+n_2} \int_0^{\frac{a}{u}} \frac{u^{n_2-1}}{(1+c+u(1+b))^{n_1+n_2}} \, du \]
\[ = \frac{2k \cdot B(\frac{n_2}{2}, \frac{n_1}{2}) \cdot I_{x_8} \cdot (\frac{n_2}{2}, \frac{n_1}{2})}{(n_1+n_2)(1+b)(1+c)^{\frac{n_2}{2}}} \]
\[ = \frac{2 \cdot I_{x_8} \cdot (\frac{n_2}{2}, \frac{n_1}{2})}{(n_1+n_2) \cdot B(\frac{n_3}{2}, \frac{n_1+n_2}{2})(1+b)(1+c)^{\frac{n_2}{2}}} \]

after substituting for \( k \) from (14) and simplifying. The remaining terms of \( P_2 \) can be obtained similarly. The \( j \)-th term \((1 \leq j \leq n_3 - 1)\)
becomes the product of two expressions,

$$P_{2j} = \left[ \frac{(-1)^{j-1} \binom{n_3}{j-1}}{\frac{n_3}{2} - 1} \right] \left[ \frac{n_1 + n_2}{(z - j - 1)} B\left(\frac{1}{2}, \frac{n_1}{2}\right) B\left(\frac{1}{2}, \frac{n_2}{2}\right) \frac{n_3 + n_2}{2} \frac{n_1}{2} \frac{n_2}{(1 + b)^2} \frac{n_1}{(1 + c)^2} \right]$$

$$\times \left[ \sum_{r=0}^{j-1} \frac{j-1}{r} B\left(\frac{z}{2} + r, \frac{n_2}{2} + j - 1 - r\right) I_{x_8}\left(\frac{n_2}{2} + r, \frac{n_1}{2} + j - 1 - r\right) \frac{(1+b)^r}{(1+c)^{j-1-r}} \right].$$

After substituting for $B\left(\frac{1}{2}, \frac{n_1}{2}\right)$, $B\left(\frac{1}{2}, \frac{n_2}{2}\right)$ and $B\left(\frac{z}{2} + r, \frac{n_1}{2} + j - 1 - r\right)$ in each term and simplifying, we obtain

$$P_2 = \frac{1}{\frac{n_1 + n_2}{2} B\left(\frac{1}{2}, \frac{n_1 + n_2}{2}\right) (1 + b) (1 + c)^{j-1}} \left( \binom{n_1 + n_2}{j-1} \frac{n_1}{2} \frac{n_2}{2} \frac{n_3}{2} \frac{n_1 + n_2}{2} \frac{n_1}{2} \frac{n_2}{(1 + b)^2} \frac{n_1}{(1 + c)^2} \right)$$

$$= \frac{1}{\frac{n_1 + n_2}{2} B\left(\frac{1}{2}, \frac{n_1 + n_2}{2}\right) (1 + b) (1 + c)^{j-1}} \left\{ \binom{n_1 + n_2}{j-1} \left[ \frac{n_1}{2} \frac{n_2}{2} \frac{n_1}{2} + 1 \right] \frac{(1 + c)}{(1 + b)} + \frac{n_2}{2} \frac{n_2}{2} \frac{n_1 + 1}{2} \frac{n_1}{2} + 1 \right\}$$

$$= \frac{1}{\frac{n_1 + n_2}{2} B\left(\frac{1}{2}, \frac{n_1 + n_2}{2}\right) (1 + b) (1 + c)^{j-1}} \left\{ \binom{n_1 + n_2}{j-1} \left[ \frac{n_1}{2} \frac{n_1}{2} + 1 \right] \frac{(1 + c)}{(1 + b)} + \frac{n_2}{2} \frac{n_2}{2} \frac{n_1 + 1}{2} \frac{n_1}{2} + 1 \right\}$$

$$\times \frac{1}{\frac{n_1 + n_2}{2} B\left(\frac{1}{2}, \frac{n_1 + n_2}{2}\right) (1 + b) (1 + c)^{j-1}} \left\{ \binom{n_1 + n_2}{j-1} \left[ \frac{n_1}{2} \frac{n_1}{2} + 1 \right] \frac{(1 + c)}{(1 + b)} + \frac{n_2}{2} \frac{n_2}{2} \frac{n_1 + 1}{2} \frac{n_1}{2} + 1 \right\}$$

$$+ \frac{1}{\frac{n_1 + n_2}{2} B\left(\frac{1}{2}, \frac{n_1 + n_2}{2}\right) (1 + b) (1 + c)^{j-1}} \left\{ \binom{n_1 + n_2}{j-1} \left[ \frac{n_1}{2} \frac{n_1}{2} + 1 \right] \frac{(1 + c)}{(1 + b)} + \frac{n_2}{2} \frac{n_2}{2} \frac{n_1 + 1}{2} \frac{n_1}{2} + 1 \right\}$$
\[ \left\{ \begin{array}{c}
n_2 \frac{n_2}{z} \left( \frac{n_2}{z} + 1 \right) I_{x_8} \left( \frac{n_2}{z} + 1, \frac{n_1}{z} + 1 \right) \\
_1 \frac{n_1}{z} \left( \frac{n_1}{z} + 1 \right) I_{x_8} \left( \frac{n_1}{z} + 1, \frac{n_2}{z} + 1 \right)
\end{array} \right. \]

\[ \left\{ \begin{array}{c}
- \frac{n_1 + n_2}{z} \left( 1 + 3(\frac{n_1 + n_2}{z} + 1) \right) I_{x_8} \left( \frac{n_1 + n_2}{z} + 1, \frac{n_1}{z} + 1 \right) \\
- \frac{n_2}{z} \left( 1 + 3(\frac{n_1 + n_2}{z} + 1) \right) I_{x_8} \left( \frac{n_1 + n_2}{z} + 1, \frac{n_1}{z} + 1 \right)
\end{array} \right. \]

2. Recurrence formulas

From (13) and (17), we know that

\[ P_1 = k \int_a^\infty \int_u^{\infty} \frac{u^{n_2 + n_3 - 1}}{v^{n_1 + n_2 + n_3 - 1}} \, \frac{du}{(1 + u + uv)} \]  

Integrating by parts with respect to v, we obtain

\[ P_1(n_3) = k \int_a^\infty \frac{u^{n_2 + n_3 - 2}}{v^{n_1 + n_2 + n_3 - 1}} \left[ \frac{1}{(1 + u + uv)} \right]_0^{\infty} \, du \]
\[
\frac{n_3}{Z - 1} \int_0^\infty \int_0^\infty \frac{n_3}{Z - 1} \frac{n_2 + n_3}{Z - 2} \frac{d\nu}{1 + \nu + uv} \frac{d\mu}{1 + \mu + uv},
\]

\[
\frac{n_3}{Z - 1} \int_0^\infty \int_0^\infty \frac{n_2 + n_3}{Z - 2} \frac{d\nu}{1 + \nu + uv} \frac{d\mu}{1 + \mu + uv}.
\]  \hspace{1cm} (34)

\[
\frac{n_3}{Z - 1} \frac{(d)}{k} \int_0^\infty \frac{n_2 + n_3}{Z - 2} \frac{d\nu}{1 + \nu + uv} \frac{d\mu}{1 + \nu + uv}.
\]  \hspace{1cm} (35)

To integrate (34), which we designate by \( P_{11}(n_1, n_2, n_3) \), let

\[
u = \frac{1 - x}{z} \quad \frac{1}{1 + d}, \quad d\nu = -\frac{1}{(1 + d)z^2} dz.
\]

Then, upon simplification, we obtain

\[
P_{11}(n_3) = c_3 \int_0^x \frac{n_1}{Z - 1} \frac{n_2 + n_3}{Z - 2} \frac{dz}{1 - z}.
\]

where

\[
c_3 = \frac{\frac{n_3}{Z - 1}}{(d)} \frac{k}{k} \frac{n_2 + n_3}{Z - 2} \frac{1}{(1 + d)(1 + d)}.
\]  \hspace{1cm} (36)

and \( x_0 = \frac{1}{1 + a(1 + d)} \).  \hspace{1cm} (37)
It follows that

\[ P_{11}(n_3) = c_3 B\left(\frac{n_1}{2}, \frac{n_2+n_3}{2} - 1\right) I_{x_0} \left(\frac{n_1}{2}, \frac{n_2+n_3}{2} - 1\right) . \]  

(38)

Finally, by substituting for \( k \) from (14) and for \( c_3 \) from (36), and performing the necessary simplification, we obtain the recurrence formula for \( P_1(n_3) \),

\[ P_1(n_3) = \frac{n_3}{(d/2)^{n_3} - 1} \frac{n_1}{(d/2)^{n_1} - 1} \frac{n_2+n_3}{d^{n_2+n_3} - 1} \]  

\[ + P_1(n_3 - 2). \]  

(39)

For the special case \( n_3 = 2 \), it follows from (33) that

\[ P_1(2) = k \int_{d}^{\infty} \int_{a}^{\infty} \int_{0}^{\infty} \frac{n_2}{u} \frac{n_1+n_2}{(1+u+uv)^{\frac{n_2}{2}+1}} \]  

\[ dv \, du \]  

\[ = -c_4 \int_{a}^{\infty} \frac{n_2}{(1+d+du)^{\frac{n_2}{2}}} \]  

\[ du , \]  

(40)

where

\[ c_4 = \frac{2k}{n_1 + n_2} . \]  

(41)

It is easily verified that (40) is merely a special case of (34) with \( n_3 \) replaced by 2. Hence for the set of initial values at \( n_3 = 2 \),
If \( \frac{tk}{(1+dt)^2} \), we know that

\[
P_1(z) = \frac{I_{x_0}^{n_1 \frac{4}{z}, \frac{n_2}{z}}}{(1+d)^{\frac{2}{z}}} .
\]  

(42)

We now turn to \( P_2 \). From (13) and (18), we know that

\[
P_2 = k \int_a^\infty \int_0^\infty \frac{\frac{n_2+n_3}{z} - 1}{u} \frac{\frac{n_3}{z} - 1}{v} - \frac{n_1+n_2+n_3}{z} \frac{1}{u} \frac{1}{v} \frac{1}{(1+uv)^{\frac{2}{z}}} \, dv \, du .
\]  

(43)

If we make the transformation

\[
u = \frac{1}{w}, \quad v = y + cw, \quad \text{the Jacobian} \quad J = \frac{1}{w^2}, \quad \text{then, upon performing the necessary simplification, we obtain}
\]

\[
P_2 = \int_a^\infty \int_b^\infty \frac{\frac{n_3}{z} - 1}{w^2} - \frac{\frac{n_1}{z} - 1}{w^2} - \frac{1}{w} \frac{1}{(1+y+1+c)w} \, dy \, dw .
\]  

(44)

We now integrate by parts with respect to \( w \). Let

\[
u = \frac{1}{w} - \frac{n_1}{z} - 1 \quad \text{and} \quad dv = \frac{(1+c)}{[(1+y+1+c)w]^2} \, dw .
\]

It follows that
\[ du = \left[ \frac{n_1}{z - 1} \cdot \frac{n_3}{(y + cw)^2} - \frac{n_1}{(y + cw)^2} + \frac{n_1}{z - 2} \cdot \frac{n_3}{(y + cw)^2} \right] \, dw, \]

\[ v = - \frac{1 + y + (1+c)w}{\frac{n_1 + n_2 + n_3}{2} - 1}. \]

\[ P_2(n_1, n_3) = - c_6 \int_b^\infty \int_{\frac{1}{a}}^{\infty} \frac{n_1}{w^2 - 1} \cdot \frac{n_3}{(y + cw)^2} - \frac{n_1}{(y + cw)^2} + \frac{n_1}{\frac{n_1 + n_2 + n_3}{2} - 1} \int_{\frac{1+y+(1+c)w}{2}}^{\infty} \, dy \, dw \quad (45) \]

\[ + c_6 \int_{\frac{1}{a}}^{\infty} \int_b^\infty \frac{n_1}{w^2 - 1} \cdot \frac{n_3}{(y + cw)^2} - \frac{n_1}{(y + cw)^2} + \frac{n_1}{\frac{n_1 + n_2 + n_3}{2} - 1} \int_{\frac{1+y+(1+c)w}{2}}^{\infty} \, dy \, dw, \quad (46) \]

\[ + c_6 \int_{\frac{1}{a}}^{\infty} \int_b^\infty \frac{n_1}{(-\frac{1}{2} - 1) w^2 - 2} \cdot \frac{n_3}{(y + cw)^2} - \frac{n_1}{(y + cw)^2} + \frac{n_1}{\frac{n_1 + n_2 + n_3}{2} - 1} \int_{\frac{1+y+(1+c)w}{2}}^{\infty} \, dy \, dw, \quad (47) \]

where \[ c_6 = \frac{k}{\left( \frac{n_1 + n_2 + n_3}{2} - 1 \right)(1+c)}. \quad (48) \]
Let us designate the three terms of \( P_2 \) given by (45), (46) and (47) as \( P_{21}, P_{22} \) and \( P_{23} \), respectively. It follows that

\[
P_{21}(n_1, n_3) = c_7 \int_b^\infty \frac{(y + \frac{b}{u_0})^{n_3-1}}{(1+y+\frac{1+c}{a})^{n_1+n_2+n_3-1}} dy, \quad (49)
\]

where \( c_7 = \frac{c_6}{n_1^{\frac{2}{2}}-1} \), \( a \).

Now let

\[
z = \frac{y + \frac{b}{u_0}}{1 + \frac{1}{a} + \frac{b}{u_0}}.
\]

It follows that

\[
y + \frac{b}{u_0} = \frac{z}{1-z} \left( 1 + \frac{1}{a} \right), \quad dy = \frac{1 + \frac{1}{a}}{(1-z)^2} dz.
\]

It is easily verified now that

\[
P_{21}(n_1, n_3) = c_8 \int_{x_1'}^1 \frac{n_3}{z^{\frac{2}{2}}-1} \frac{n_1+n_2}{(1-z)^2} dz, \quad (50)
\]
where \( c_8 = \frac{c_7}{(1 + \frac{1}{a})^{\frac{n_1+n_2}{2}} - 1} \) \hspace{1cm} (51)

and \( x^i = \frac{b(1 + \frac{1}{u_1})}{1 + \frac{1}{a} + b(1 + \frac{1}{u_1^0})} \) .

Therefore, \( P_{21}(n_1, n_3) = c_9 \left[ 1 - I_{x^i_1} \left( \frac{n_3}{2}, \frac{n_1+n_2}{2} - 1 \right) \right] \) , \hspace{1cm} (52)

where \( c_9 = B(\frac{n_3}{2}, \frac{n_1+n_2}{2} - 1) \) \( c_8 \) . \hspace{1cm} (53)

Equation (52) may also be written as

\[ P_{21}(n_1, n_3) = c_9 \left( 1 + \frac{1}{a} \right) I_{x^i_1} \left( \frac{n_1+n_2}{2} - 1, \frac{n_3}{2} \right) \) , \hspace{1cm} (54)

where \( x_1 = 1 - x^i_1 = \frac{1 + \frac{1}{a}}{1 + \frac{1}{a} + b(1 + \frac{1}{u_1^0})} \) . \hspace{1cm} (55)

Substituting for \( c_9, c_8, c_7, c_6 \) and \( k \) from (53), (51), (50), (48) and (14), respectively, and simplifying, we obtain

\[ P_{21}(n_1, n_3) = \frac{\left( \frac{1}{a} \right) \left( \frac{n_1+n_2}{2} - 1 \right) I_{x_1} \left( \frac{n_1+n_2}{2} - 1, \frac{n_3}{2} \right)}{(1+c)(\frac{-n_2}{2} - 1)B(\frac{n_2}{2}, \frac{n_1+n_2}{2})(1 + \frac{1}{a})^{-\frac{n_1+n_2}{2} - 1}} \) . \hspace{1cm} (56)
Now let us consider the component

\[ P_{22}(n_1, n_3) = \frac{(n_3^2 - 1) c_6}{c} \int_{a}^{\infty} \int_{b}^{\infty} \frac{n_1^2 - 1}{w^2 (y+cw)^2} \frac{n_3^2 - 2}{y^2 \left(1 + y + (1+c)w\right)^2} - 1 \ dy \ dw. \]

(57)

Comparison of (57) and (43) shows us that

\[ P_{22}(n_1, n_3) = \frac{n_3^2 - 1}{k} c_6 c \]

\[ = \frac{n_3^2 - 1}{k} P_2(n_1, n_3 - 2). \]

Upon substitution for \( c_6 \) and \( k \) from (48) and (14), respectively, we obtain

\[ P_{22}(n_1, n_3) = \frac{c}{1+c} P_2(n_1, n_3 - 2). \]

(58)

Similarly,

\[ P_{23}(n_1, n_3) = \frac{n_1}{k} \frac{c_6}{c} \int_{a}^{\infty} \int_{b}^{\infty} \frac{n_1^2 - 2}{w^2 (y+cw)^2} \frac{n_3^2 - 1}{y^2 \left(1 + y + (1+c)w\right)^2 - 1} \ dy \ dw \]

\[ = \frac{n_1}{k} c_6 \]

\[ = \frac{n_1}{k} \frac{c_6}{k} \]

\[ = \frac{n_1}{k} \]

\[ = \frac{1+c}{1+c} P_2(n_1 - 2, n_3). \]

(59)

Upon substitution for \( c_6 \) and \( k \) as done above, we obtain

\[ P_{23}(n_1, n_3) = \frac{1}{1+c} P_2(n_1 - 2, n_3). \]

(60)
Combining the results obtained in (56), (58) and (60), we find that

\[
P_2(n_1, n_3) = \frac{\Gamma \left( \frac{n_1}{2} + \frac{n_2}{2} - 1, \frac{n_3}{2} \right)}{(1+c)(1+\frac{1}{a})^{\frac{n_1+n_2}{2}-1}} \cdot \frac{n_1+n_2}{2} \cdot \frac{n_1}{2} \cdot \frac{n_2}{2} \cdot \frac{n_3}{2} \cdot \frac{n_1+n_2}{2} - 1} \cdot B(\frac{n_1}{2}, \frac{n_2}{2})(1+c) \cdot P_2(n_1 - 1, n_3) + \frac{1}{1+c} P_2(n_1 - 2, n_3) \quad (61)
\]

Formula (61) requires two sets of initial values, a set for \( n_3 = 2 \) \[ P_2(n_1, 2) \] and a set for \( n_1 = 2 \) \[ P_2(2, n_3) \]. It is readily seen from (59) that \( P_{23}(2, n_3) = 0 \), and hence that

\[
P_2(2, n_3) = \frac{\Gamma \left( \frac{n_2}{2}, \frac{n_3}{2} \right)}{(1+c)(1+\frac{1}{a})^{\frac{n_2}{2}}} + \frac{1}{1+c} P_2(2, n_3 - 2) \quad (62)
\]

from simplification of (56) and (58).

For the case \( n_3 = 2 \),

\[
P_2(n_1, 2) = k_1 \int_{\frac{1}{a}}^{\infty} \int_{b}^{\infty} \frac{\frac{n_1}{w^2} - 1}{\left[1+y+(1+c)w\right]^\frac{n_1+n_2}{2} + 1} \, dy \, dw,
\]
where \[ k_1 = \frac{n_1 + n_2}{2B\left(\frac{x^2}{2}, \frac{y^2}{2}\right)}. \]

Integration with respect to \( y \) gives us

\[
P_2(n_1, 2) = \frac{k_1}{n_1 + n_2} \int_0^\infty \frac{n_1}{w^{2\frac{z}{z} - 1}} \frac{n_1 + n_2}{[1 + b + (1+c) w]^{\frac{n_1 + n_2}{z}}} \, dw.
\]

The transformation

\[
z = \frac{1 + b}{1 + b + (1+c) w}, \quad w = \left(\frac{1 - z}{z}\right) \left(\frac{1 + b}{1 + c}\right), \]

\[dw = \left(-\frac{1}{z^2}\right) \left(\frac{1 + b}{1 + c}\right) \, dz,
\]
gives us

\[
P_2(n_1, 2) = c_{11} \int_0^{x_2} \frac{n_2}{z^{2\frac{z}{z} - 1}} \frac{n_1}{(1 - z)^{\frac{n_1 + n_2}{z}}} \, dz.
\]

where \[ c_{11} = \frac{k_1}{n_1 + n_2} \frac{n_2}{n_1} \frac{n_1 + n_2}{[1 + b + (1+c) w]^{2}} \]

\[ (\frac{x^2}{2})(1+b)^{\frac{n_2}{2}} (1+c)^{\frac{n_1}{2}} \]
and \[ x_2 = \frac{a(1+b)}{1+c+a(1+b)} \] (65)

Evaluation of (26) and substitution for \( k_1 \) gives us

\[ P_2(n_1, z) = \frac{I_{x_2}^{n_2} n_1}{n_2 (1+b) (1+c) z} \] (66)

D. Approximate Formulas for Large \( n_1 \)

We now derive simpler approximate formulas. We first consider

\[ P_2 \text{ Writing } F_1 = F_{n_2, n_1}^1(a_1), \quad F_2 = F_{n_3, n_2}^1(a_2), \quad F_3 = F_{n_3, n_1+n_2}^1(a_3), \]

we have

\[ P_2 = \Pr \left\{ \frac{V_2}{V_1} \leq F_1 \text{ and } \frac{V_3}{V} \geq F_3 \right\} \]

As \( n_1 \to \infty \) both \( V_1 \to \sigma_1^2 \) and \( V \to \sigma_1^2 \) and, in the limit, the two ratios \( V_2/V_1 \) and \( V_3/V \) are independently distributed. It is therefore suggested that for large \( n_1 \) we use the approximation

\[ P_2 \approx \Pr \left\{ \frac{V_2}{V_1} \leq F_1 \right\} \Pr \left\{ \frac{V_3}{V} \geq F_3 \right\} \]

\[ \approx \left( 1 - I_{x_10}^1 \left( \frac{1}{Z} n_1, \frac{1}{Z} n_2 \right) \right) I_{x_11}^1 \left( \frac{1}{Z} (n_1+n_2), \frac{1}{Z} n_3 \right) \] (67)
where
\[ x_{10} = \frac{1}{(1 + v_{21}x(a_1))} \]

and
\[ x_{11} = \frac{(n_1 + n_2)}{(n_1 + n_2) + \frac{(n_2^2 + n_1^2)(1-x(a_3))}{v_{21}^2 v_{32}^2 x(a_3)}} \]

Next we turn to
\[ P_1 = \Pr\left\{ \frac{V_2}{V_1} \geq F_1 \text{ and } \frac{V_3}{V_2} \geq F_2 \right\}. \]

Here we could use a similar argument if we were to let \( n_2 \rightarrow \infty \). This limit would, however, not yield useful results. The important situation in pooling procedures is one in which \( n_2 \) is moderate or small, for otherwise nothing is gained by pooling \( V_2 \) with another mean square \( V_1 \).

Instead we use the well known normal approximation to \( \log V_1 \).

M. S. Bartlett and D. G. Kendall (1946) have shown that \( \log V_1 \) is approximately \( N(\log \sigma_1^2, \frac{2}{n_1 - 1}) \) provided that \( n_1 \) is not too small.

We decided to limit the use of this approximation to cases in which all three \( n_i \) are \( \geq 12 \). Writing
\[ u = \log V_2 - \log V_1 \quad \text{and} \quad z = \log V_3 - \log V_2, \]

it follows that the joint distribution of \( u \) and \( z \) is approximately bivariate normal with correlation coefficient
\[ \rho = -1 / \left\{ \left( 1 + \frac{n_2 - 1}{n_3 - 1} \right) \left( 1 + \frac{n_2 - 1}{n_1 - 1} \right) \right\}^{\frac{1}{2}}. \]
We may therefore employ the tables of the double probability integral of a bivariate normal surface of K. Pearson (1936), Tables VIII and IX. If \( x \) and \( y \) follow a bivariate normal distribution with both means equal to 0, correlation coefficient \( \rho \), and both standard deviations equal to unity, then these tables give the probabilities

\[
P(\rho (h, k) \text{ for } x \geq h \text{ and } y \geq k).
\]

In our case \( \rho \) is given by (69) and \( h \) and \( k \) by

\[
h = \frac{2z_{n_2, n_1}(a_1) - \log \theta_{31}}{\sqrt{\frac{2}{n_1-1} + \frac{2}{n_2-1}}}, \quad k = \frac{2z_{n_3, n_2}(a_2) - \log \theta_{32}}{\sqrt{\frac{2}{n_2-1} + \frac{2}{n_3-1}}},
\]

(70)

where \( z_{n_1, n_j}(a) \) is the upper 100\( a \)% point of Fishers' \( z \) distribution with numerator degrees of freedom \( n_i \) and denominator degrees of freedom \( n_j \). If the normal approximation is used for the distribution of \( \log V_i \), this same approximation may of course be employed to simplify \( h \) and \( k \) as follows: replace

\[
\frac{2z_{n_1, n_j}(a)}{\sqrt{\frac{2}{n_i-1} + \frac{2}{n_j-1}}}
\]

by the \( a \)% point of the normal distribution.
E. Theory of Reduction of Mixed Model to Random Model

Certain mixed models of analysis of variance were described in Section I. In this subsection we develop the distribution theory for these situations. No new formulas are required, as we shall show that the joint distribution of the three mean squares is, at least approximately, equal to that of the component of variance model. The exact specifications of the distribution for the mixed model being considered are as follows. (Primed parameters will be used to specify the parameters for the mixed model.)

(a) The error mean square $V_2$ and the doubtful error mean square $V_1$ are distributed as $\chi^2_i \sigma_i^2 / n_i^i (i = 1, 2)$, where $\chi^2_i$ is the central $\chi^2$ statistic with $n_i^i$ degrees of freedom. On the other hand, the treatment mean square $V_3$ is distributed as $\chi^2_{n_3^i} \sigma_3^i / n_3^i$, where $\chi^2_{n_3^i}$ is the non-central $\chi^2$ statistic with $n_3^i$ degrees of freedom and non-centrality parameter

$$\lambda = \frac{n_3^i \sigma_3^2 - n_1^i \sigma_2^2}{2\sigma_2^2} = \frac{n_3^i}{2} (\theta_{32} - 1) ,$$

where $\theta_{32} = \sigma_3^2 / \sigma_2^2$. $V_1$, $V_2$ and $V_3$ are independent.

(b) The main purpose of the analysis is to test the hypothesis

$H_0: \sigma_3^2 = \sigma_2^2$ against the alternative $H_1: \sigma_3^2 \neq \sigma_2^2$.

(c) The true error mean square, $V_2$, has an expectation $\sigma_2^2$ which is greater than or equal to the expectation, $\sigma_1^2$, of the doubtful error mean square.
ponding to \( a = 0.05 \).

of these parameters (these evaluations being restricted to cases cor-

\( \alpha \), \( \beta \), and \( \gamma \). This power has been evaluated for various combina-

as in \( g \) parameter function, the \( g \) parameters being \( u \), \( \nu \). \( \xi \), \( \zeta \), \( \Theta \), \( \Omega \).

probabilities given by (10) and (11) in Supposition B, and may be referred

The power for the random model is defined as the sum of the two

present mixed model.

suggests the possibility of applying the random model results to the

to a central \( \chi ^2 \) statistic, all three statistics are now central. Thus

Since the use of this approximation reduces the non-central \( \chi ^2 \) statistic

\[
\begin{align*}
\frac{\chi^2 + u}{\chi^2} + 1 &= C \\
\frac{\chi^2 + \xi u}{\chi^2} + \xi u &= \xi \chi
\end{align*}
\]

(17)

statistic based upon \( \xi \) degrees of freedom, where

\( \chi ^2 \) by Panical (1949), we replace \( \gamma\) by \( C \chi ^2 \). The replace \( \gamma \) without the central

In evaluating these probabilities we use the approximation first used

\[
\begin{align*}
\left\{ \text{if} \left( \frac{\chi^2 + \xi u}{\chi^2} - 1\right) \Lambda / \xi \Lambda \text{ and } \left( \frac{\chi^2 + \xi u}{\chi^2} - 1\right) \Lambda / \xi \Lambda \right\} \cdot x_d &= Z_d \\
\left\{ \left( \frac{\chi^2 + \xi u}{\chi^2} - 1\right) \Lambda / \xi \Lambda \text{ and } \left( \frac{\chi^2 + \xi u}{\chi^2} - 1\right) \Lambda / \xi \Lambda \right\} \cdot x_d &= 1_d
\end{align*}
\]

(21)

\( \text{components}. \)

The probability of rejecting \( H_0 \) is obtained as the sum of the two
We now return to the power for the mixed model as defined by (72) and (73), and compare this with the corresponding formulas (10) and (11) for modified values of the 8 parameters as indicated in Table 5.

Table 5. Modified parameters for random model corresponding to specified parameters for mixed model

<table>
<thead>
<tr>
<th>Specified parameters for mixed model</th>
<th>Modified parameters for random model</th>
</tr>
</thead>
<tbody>
<tr>
<td>n_1</td>
<td>n_1 = n'_1</td>
</tr>
<tr>
<td>n_2</td>
<td>n_2 = n'_2</td>
</tr>
<tr>
<td>n_3</td>
<td>n_3 = \nu'_3 = n'_3 + \frac{4\lambda^2}{n'_3 + 4\lambda}</td>
</tr>
<tr>
<td>a_1</td>
<td>a_1 = a'_1</td>
</tr>
<tr>
<td>a_2</td>
<td>a_2 = \text{Root of } F_{n'_3, n'_2}(a'<em>2) = F</em>{\nu'_3, n'_2}(a'_2)</td>
</tr>
<tr>
<td>a_3</td>
<td>a_3 = \text{Root of } F_{n'_3, n'_1+n'_2}(a'<em>3) = F</em>{\nu'_3, n'_1+n'_2}(a'_3)</td>
</tr>
<tr>
<td>\theta_{21} = \sigma^2_{21}/\sigma^2_1</td>
<td>\theta_{21} = \theta'_{21}</td>
</tr>
<tr>
<td>\theta_{32} = \frac{2\lambda + n'_3}{n'_3}</td>
<td></td>
</tr>
</tbody>
</table>

Entering the random model tables with these altered parameters we obtain the mixed model powers. It will be seen that when we deal with the size for the mixed model we have \lambda = 0 and hence \nu'_3 = n'_3, so that all primed parameters agree with those without primes. Thus our entire
size discussion to follow is directly applicable to the mixed model. On the other hand, the power evaluations, which refer to $\alpha_2 = \alpha_3 = .05$, will in general provide answers for larger values of $\alpha_2$ and $\alpha_3$, and these levels $\alpha_2$ and $\alpha_3$ will vary with $\lambda$. For a proper evaluation of power corresponding to a given pair of significance levels $\alpha_2$ and $\alpha_3$, say $\alpha_2 = \alpha_3 = .05$, a more extensive tabulation of (10) and (11) would be required.

F. Application of Derived Formulas

The recurrence formulas derived in Part D were used to construct master tables of $P_1$ and $P_2$. These master tables were constructed for

\[
\frac{n_2}{\lambda} = 5, \quad \frac{n_1}{\lambda} = 3(1)10 \quad \text{and} \quad \frac{n_3}{\lambda} = 1(1)6
\]

Also, tables were constructed for $\frac{1}{2} n_2 = 3$, in order that the effect of small error degrees of freedom could be better studied. However, the latter tables were confined to the values

\[
\frac{1}{2} n_3 = 1 \quad \text{and} \quad \frac{1}{2} n_1 = 8, 14 \quad \text{and} \quad 20,
\]

in order to save on computational expenses. The $P_1$ values were obtained by starting with the set of initial values from (42) and then using the recurrence relation (39), for grids of suitably selected values of $x_0$ and $d$. Similarly, tables of $P_2$ values were obtained by first computing the two sets of initial values from (62) and (66), and then using (61), for selected values of $\alpha$, $x_1$ and $x_2$. The values of the arguments used in the master
tables were selected so as to cover the ranges of variation of the values which would arise in the power function problems to be investigated. For examples of master tables see Tables 8 and 9.*

To compute the power component $P_1$ in a given problem from these master tables, for specified degrees of freedom $n_1$, $n_2$ and $n_3$ and levels of significance $a_1$, $a_2$, the values of the parameters $u_1^0$ and $u_2^0$ are computed from (16). From these values and those specified for $\Omega_{21}$ and $\Omega_{32}$ the corresponding values of $a$ and $d$ from (19) and hence the value of $x$ from (37) are computed. $P_1$ is then obtained by interpolation in the appropriate master table.

The procedure used to compute the component $P_2$ is similar. For specified degrees of freedom $n_1$, $n_2$ and $n_3$ and levels of significance $a_1$ and $a_3$, the values of the parameters $u_1^0$ and $u_3^0$ are computed from (16). From these values and those specified for $\Omega_{21}$ and $\Omega_{32}$ the corresponding values of $a$, $b$, and $c$ from (19) and (26) and hence the values of $x_1$ from (55) and $x_2$ from (65) are computed; $P_2$ is then obtained by interpolation in the appropriate master table. For the power computations that were actually made, interpolation with respect to $a$ was avoided by choosing values of $\Omega_{21}$ which would result in tabular values of $a$. This accounts for the decimal values of $\Omega_{21}$ found in Tables 13 - 29.

The computational procedures outlined above were performed for various combinations of $a_1$, $n_1$, $n_2$, and $n_3$, and for several values

*Tables 8-31 have been assembled in the Appendix.
of \( \theta_{32} \), for the case \( a_2 = a_3 = .05 \), primarily. In computing the \( P_1 \) components, three point Lagrangian interpolation was used with respect to \( d \); and linear interpolation with regard to \( x \), followed by a second difference correction. (Lagrangian interpolation coefficients were obtained from tables prepared by the National Bureau of Standards (1944)). For the \( P_2 \) evaluations, linear interpolation was used on the logarithms of the tabular values, followed by application of a second difference correction.

The \( P_1 \) interpolation was performed first on \( d \), then on \( x \). For the interpolation on \( d \), Lagrangian interpolation coefficients were preferred to the adjusted linear interpolation, because of the fact that, for the problems studied, the same \( d \) value (and hence the same interpolation coefficient) was used for a number of \( x \) values. On the other hand, for the remainder of the interpolation for \( P_1 \) and for all of the \( P_2 \) interpolation, the adjusted linear interpolation was less time consuming. It is believed that these interpolations gave results accurate to at least two decimals, and in most cases to three decimals.

The series formulas derived in Subsection (1) of Section C were used to compute special cases not covered by the computed master tables. These formulas also proved useful as independent checks for the recurrence computations.

The approximate formula (67) for \( P_2 \) derived in Subsection D is exact for \( n_1 = \infty \) and was found to be very effective for large \( n_1 \), large \( \theta_{32} \) and small \( \theta_{21} \), \( n_2 \) and \( n_3 \). Since the size refers to \( \theta_{32} = 1 \)
and arbitrary values of $\Theta_{21}$, it was necessary to correct the values computed for $n_1 = 60$ by an adjustment. This adjustment was obtained by evaluating the difference between the approximate value and the corresponding exact value of $P_2$ for $n_1 = 20$, and interpolating harmonically between this difference and that for $n_1 = \infty$ (where the difference is zero) to find the corresponding adjustment for $n_1 = 60$. 
IV. DISCUSSION OF POWER AND SIZE CURVES AND COMPARISON OF TEST PROCEDURES

A. Type of Recommendations Attempted

We have seen that the power of our test procedures depends upon the following 8 parameters: The degrees of freedom $n_1$, $n_2$ and $n_3$; the variance ratios $\theta_{21} = \frac{\sigma^2_1}{\sigma^2_2}$, $\theta_{32} = \frac{\sigma^2_2}{\sigma^2_3}$; and the levels of significance $\alpha_1$, $\alpha_2$, $\alpha_3$. Of these, the degrees of freedom $n_1$, $n_2$ and $n_3$ are completely determined by the analysis of variance table, while the variance ratios are generally unknown (except in the case of the size of the procedure, when $\theta_{32} = 1$). Any recommendations that are to be made must therefore be confined to the levels of significance, $\alpha_1$, $\alpha_2$, and $\alpha_3$. We shall here be primarily concerned with the type one error being in the vicinity of .05. Most of the discussion to follow will therefore be confined to test procedures in which $\alpha_2 = \alpha_3 = .05$, that is, to procedures in which the significance levels of both final tests are .05. However, the remaining parameter, $\alpha_1$, the level of significance for the preliminary test, is entirely at our disposal. In attempting recommendations, therefore, we shall be concerned with the choice of the level of $\alpha_1$; should $\alpha_1$ be, say, .05, .25, .50, or should we use what Paull (1948, p. 4) has called the borderline test, where $\alpha_1$ will be near .70 to .80? In choosing the level of $\alpha_1$, we shall consider
(i) the variation in the size of our test procedure as a function of the parameter $\theta_{21}$; and

(ii) a comparison of the power of our test procedure with that of the never pool test of the same size.

Both of these considerations were studied by Paull for his special cases. It will be seen, however, that when evaluating, as we do, the size and power for wider and more representative sets of degrees of freedom, much larger size disturbances (than those reported by Paull) occur for many of the sets considered; and that consequently we must modify his recommendations in certain respects in order to achieve size control, for an acceptable power.

B. Size Curves

The size of our test procedure does not equal the nominal level of .05, but varies about this level as $a_1$ and $\theta_{21}$ vary. Figures 1 to 15* give us examples of size curves, illustrating the variations in type one error with variation in $\theta_{21}$ for fixed values of the remaining parameters.

Note that as $\theta_{21}$ becomes large the size approaches .05; for, as $\theta_{21} \to \infty$ the preliminary test will almost certainly be significant, pooling will almost certainly not occur, and hence the final test will almost certainly be that of $V_3/V_2$, having a size of .05.

At the lower extreme, that is, at $\theta_{21} = 1$, the size is less than .05. The reason for this is more complex. The probabilities of pooling or not pooling, in accordance with the result of the preliminary test, are now

*All figures have been assembled in the Appendix.
1 - \( a_1 \) and \( a_1 \), respectively. But on the final test the probability of obtaining a significant result when not pooling is much smaller than \( (a_1)(.05) \) because in these situations \( V_2 \) is comparatively large. On the other hand the probability of \( V_3/V \) being significant when pooling is still about \( (1 - a_1)(.05) \). Thus, when \( \theta_{21} = 1 \), the total size is below \( (a_1)(.05) + (1 - a_1)(.05) \) = .05.

Of particular interest are the maximum size disturbance (the size peak) and the minimum value of the size; the latter occurs at \( \theta_{21} = 1 \).

We first consider the size peak. Referring to the size curves for a preliminary test carried out at the 5 per cent level, we note that the peak is usually very high. Clearly, a preliminary test carried out at this level will in many cases admit an unacceptable size disturbance. This is due to the fact that at this level the preliminary test will frequently admit pooling \( V_2 \) and \( V_1 \) when \( \sigma^2 \) is smaller than the true error mean square \( \sigma^2_2 \), and thereby increase the probability of type one error. We therefore seek a preliminary test in which pooling is admitted less readily; we next investigate the level \( a_1 = .25 \). At this level (see Figures 5-13) size control is considerably better, and in many cases the peaks do not go beyond .08. In fact, for \( n_3 = 2 \), the peak is usually below .07. It should be noted that, in general, the size peak increases as \( n_1 \) or \( n_2 \) increases or as \( n_2 \) decreases. *

It is of course quite arbitrary to specify any rules for maintaining an acceptable upper tolerance for the size peak, since what is considered acceptable is a matter of opinion. In using a nominal size of .05, if we stipulate that our size peak should not go much beyond 10 per cent,

---

*The nature of the dependence of the peak on \( n_1, n_2 \), and \( n_3 \) was obtained from observation of the curves, not from any analytic considerations.
then we find that even with the 25 per cent preliminary level, there are situations in which this upper limit is exceeded. Generally speaking, these unacceptable size peaks occur when

\[ n_3 \geq n_2 \]

and

\[ n_1 \geq 5n_2 \]  

(75)

(It should be noted that the occurrence of \( n_3 > n_2 \) is clearly rare.) This means that when the treatment degrees of freedom are greater than or equal to the error degrees of freedom we must be careful if at the same time the doubtful error degrees of freedom are greater than or equal to five times the true error degrees of freedom; or, briefly, we must be careful when pooling promises a large gain in the precision of the error estimate. In these situations a more conservative level of \( \alpha_1 \) would be appropriate. From a study of a number of size curves it appears that a preliminary test at the 50 per cent level will ensure adequate control of the size peak in these cases (see Figure 12).

Not only the size peak, but also the size minimum is affected by the level of the preliminary test. From theorems proved by Paull, (1948, Chapter 4) we know that the size of our test procedures is a minimum with respect to \( \theta_{21} \) at \( \theta_{21} \) equal to one, and that the lower bound for the size for this value of \( \theta_{21} \) is \((1-\alpha_1)(.05)\). These lower bounds are .0475, .0375 and .025 for \( \alpha_1 = .05, .25 \) and .50, respectively. For some of our curves the plotted minimum sizes are
situated very close to these lower bounds. For the borderline test, where the size is always less than .05, this lower bound lies approximately between .01 and .015. We have computed actual minimum size values for the borderline test for selected values of \( n_1 \), \( n_2 \) and \( n_3 \) (see Table 12). For small \( n_2 \) and \( n_3 \) these are very close to their lower bounds, irrespective of \( n_1 \). A person using this test should therefore remember that he may be using a test which has a considerably lower size than .05. The actual disturbance is of course small, but the proportional disturbance is considerable. However, since the borderline test size disturbance is a reduction rather than a size increase and is therefore on the conservative side, we are not attempting to make any definite rules as to when the experimenter should avoid the use of this test but merely remind him that large size disturbances occur when \( n_2 \) and \( n_3 \) are both small (\( \leq 6 \)).

Summarizing our considerations of size control, therefore, we have narrowed down our recommendable range of \( a_1 \) to \( a_1 \geq .25 \), with the reservations that in certain cases characterized by inequalities (75), \( a_1 = .25 \) would not be desirable, as it would admit too large a peak in the size curve; and that for very small values of \( n_2 \) and \( n_3 \) the experimenter may not wish to use the borderline test, as this would admit too low a size minimum.

The discussion thus far has been concerned with test procedures in which \( a_2 = a_3 = .05 \). A few special cases for \( a_2 = a_3 = .01 \) and \( a_1 = .25 \) have also been investigated. In all these situations, larger proportional
size disturbances than those found for \( a_2 = a_3 = .05 \) were experienced, even for cases which our rule would accept (see Figure 14).

C. Frequency of Pooling

We have been discussing the effect of increasing \( a_1 \) in order to achieve size control. It is obvious that for \( a_1 = 1 \), our preliminary \( F \) per cent point would be zero, and pooling would never occur. The question arises as to the relative frequency of pooling for the intermediate values of \( a_1 \) that we have been considering. When \( \theta_{21} = 1 \), the probability that \( V_2/V_1 \) exceeds \( F_{a_1} \{n_2, n_1\} \) is \( a_1 \), so that pooling occurs with relative frequency \( 1 - a_1 \). As \( \theta_{21} \) increases this frequency rapidly decreases, approaching the limit zero as \( \theta_{21} \) becomes infinite. Evaluations of these frequencies of pooling, which are summarized in Tables 10 and 11, show that, while for \( a_1 = .25 \) and small values of \( \theta_{21} \) pooling will occur in the majority of experiments, when \( a_1 = .50 \) the frequency is usually well below .50. This frequency of pooling is of course even smaller for the borderline test, where \( a_1 \) usually takes on values in the neighborhood of .7 to .8; when such large values of \( a_1 \) are employed, pooling occurs in only about twenty five per cent of all situations for which \( \theta_{21} = 1 \), and this pooling percentage rapidly decreases as \( \theta_{21} \) increases. While this property by itself cannot be regarded as a disadvantage of the borderline test, it is clear that, if this test were the only recommended one to the experimenter, he would hardly ever pool.
D. Power Curves

We now attempt a comparison of the power of our sometimes pool procedure with that of the never pool test. As is well known, any comparison of power of any two test procedures is a fair comparison if the two test procedures have the same size. We have seen that the size of our sometimes pool procedures is not at the constant level of .05, but varies about this, depending upon the parameter \( \theta_{21} \). The method of power comparison we have therefore adopted is as follows:

(i) Assume a fixed value of the parameter \( \theta_{21} \).

(ii) For this value of \( \theta_{21} \) evaluate the size of the sometimes pool test.

(iii) For this level of size evaluate the power curve of the never pool test; this power curve is then directly comparable with that of the sometimes pool test corresponding to the chosen value of \( \theta_{21} \).

For an illustration of this comparison see Figure 16. Here we have \( n_1 = 20 \), \( n_2 = 10 \), \( n_3 = 12 \), \( a_1 = a_2 = a_3 = .05 \), and a fixed value of \( \theta_{21} = 1.174 \) (\( \sigma_2^2 = 1.174 \sigma_1^2 \)). For this value of \( \theta_{21} \) the size of the sometimes pool test (see Figure 4) is .072; for this size we have evaluated the power curve of the never pool test, which is plotted against \( \theta_{32} \), using log scales in both cases, in Figure 16. In the same figure is shown the power curve for our sometimes pool procedure, which, in the present case, is always above that of the never pool test. This means that in the present example the power of the sometimes pool procedure is greater than that of the never pool test even though the latter is per-
mitted to have the same size (.072) as the former. Figures 17 and 18 give two similar comparisons of power curves corresponding to the same set of degrees of freedom, but to \( \theta_{21} = 1.789 \) and 2.866, respectively. The first of these shows intersecting power curves; the second shows a case in which the never pool test is more powerful. The three figures illustrate the fact that the sometimes pool procedure is more powerful for small \( \theta_{21} \) and less powerful for large \( \theta_{21} \). (This is a general property shared by all those sometimes pool procedures for which \( a_1 \) is smaller than the \( a_1 \) level for the borderline test, as was proved by Paull (1948, p. 55).) For various other combinations of \( n_1, n_2, n_3, a_1 (a_2 = a_3 = .05) \), \( \theta_{21} \) and \( \theta_{32} \), corresponding power comparisons are tabulated in Tables 13 to 31. In order to show more clearly the dependence of these power comparisons on \( \theta_{21} \), we have plotted in Figures 19 to 24 the difference between two corresponding power points against \( \theta_{21} \). Here each curve corresponds to a fixed value of \( \theta_{32} \), that is, that value of \( \theta_{32} \) at which the difference between the power ordinates of the power curves was taken. It will be seen, again, that for small \( \theta_{21} \) the differences are positive (the sometimes pool procedure is more powerful than the never pool test), while for larger \( \theta_{21} \) the position is reversed. As \( \theta_{21} \to \infty \) the difference tends to 0, since both procedures tend to the never pool test at the .05 level of significance. The transition from favorable to unfavorable power conditions generally occurs between \( \theta_{21} = 1.5 \) and \( \theta_{21} = 2.0 \). The magnitude of these power gains and losses increase with increasing \( n_3 \), or decreasing \( n_2 \) or increasing \( n_1 \).
The three particular Figures 19, 20 and 21 illustrate the effect of decreasing the per cent point of F for the preliminary test. In these figures corresponding power comparisons are given, respectively, for $a_1 = .05$, $F_1 = 2F_{.50}(n_2, n_1)$ and $a_1 = .25$. There is a general tendency for both power gains and power losses to diminish as the per cent point of F decreases, that is, as $a_1$ increases from values such as .05 through intermediate values such as .25 to the level of the borderline test (approximately $a = .70$ to .80). Here the gain in power has diminished further (see Tables 30 and 31) but the power losses have completely disappeared. In fact, a theorem by Paull (1948, p. 61) proves that the borderline test is always more powerful than the corresponding never pool test of the same size, although the power gain is small for large $\theta_{21}$. However, as we have seen (see Subsection B), this size is below the nominal level of .05. If we compare the borderline test power with that of the never pool test at the nominal level of .05, the former is always less powerful. It is likewise less powerful than our sometimes pool procedures for $a_1 = .50$ and $a_1 = .25$, which have, of course, a larger size.

We now attempt recommendations, considering the relative merits of the procedures at $a_1 = .25$, $a_1 = .50$ and $a_1 = .7$ to .8 (the borderline level). Unfortunately, these recommendations are somewhat subjective, since they are contingent upon what the experimenter may regard as a reasonable assumption concerning the parameter $\theta_{21}$.

(1) If the experimenter is reasonably certain that only small values of $\theta_{21}$ can be envisaged as a possibility, he is advised to use $a_1 = .25$
except in the cases (75), when he should use \( a_1 = .50 \), in order to ensure size control. Our Figures 19 to 24 show that the range of small values of \( \theta_{21} \), when the sometimes pool procedure gives a gain in power, is approximately between 1 and 1.5 to 2. An experimenter about to adopt this recommendation but not quite certain about his assumptions may wish to know the consequences which result from his adopting this procedure when, in fact, unknown to him, \( \theta_{21} \) is large. It is seen from Figures 1 to 14 that in such a situation he will still have control of the size of his test; in fact the size will be near .05 for large \( \theta_{21} \). All he loses (as is shown by Figures 17 to 24 and Tables 13 to 29) is the power of his test; this is a risk that he may well be prepared to take.

(ii) If, however, the experimenter can make no such assumption about \( \theta_{21} \), and wishes to guard against the possibility of power losses, he may then use the borderline test, which would ensure a power gain, although he must realize

(a) that for large \( \theta_{21} \) this gain would be very small;
(b) that for small \( \theta_{21} \) he would use a test procedure of a very much smaller size than \( a_1 = .05 \) (particularly when \( n_2 \) and \( n_3 \) are \( \leq 6 \)) and accordingly a test which is much less powerful than the never pool test of size .05. In fact, he may in these circumstances prefer not to pool at all.

It may be correctly argued that, in order to control the size peak, to advocate \( a_1 = .50 \) in the cases characterized by (75), and \( a_1 = .25 \) otherwise, introduces an artificial discontinuity in our recommenda-
tions. It would be quite feasible (although it would require a considerable effort in computation) to evaluate for any given triplet \( n_1, n_2 \) and \( n_3 \) that value of \( a_1 \) which results in a size peak of 0.10 exactly. Since this level of \( a_1 \) would depend on the degrees of freedom \( n_1, n_2 \) and \( n_3 \), it would be necessary to evaluate the associated per cent points of \( F \). For such recommendations to be useful this table of \( F_{a_1}(n_1, n_2) \) (which would be a large 3 parametric table with \( n_1, n_2 \) and \( n_3 \) as arguments) would have to be published. To encumber the experimenter with special tables for the preliminary \( F \)-test in addition to the standard \( F \) tables for the final \( F \)-tests appeared to us to be unnecessary, and the use of the published Merrington and Thompson (1943) 25% and 50% points of \( F \) preferable.

We should note here that a rule favored by Pau11 (1944, Chapter 6) advocating testing the ratio \( V_2/V_1 \) against \( 2F_{.50}(n_2, n_1) \) will not ensure adequate control of the size peak since \( 2F_{.50} > F_{.25} \) in general, and we have just seen that \( F_{.25} \) is sometimes too large and hence not always acceptable as a significance level for the preliminary test. Also, it would appear to us that no rule of the form \( V_2/V_1 \geq \) constant is very satisfactory, for with such a rule the frequency with which pooling occurs as well as the size vary considerably with the degrees of freedom \( n_1 \) and \( n_2 \).

Concerning recommendation 2, the experimenter would require knowledge of the precise level of \( a_1 \) for the borderline test, or, better still, the value of \( F \) associated with it. Paull (1944, p. 20) gives a
A simple formula from which the following is derived:

\[
F \text{ point for borderline test}
\]

\[
= \frac{n_1 \frac{F_{n_3, n_1+n_2}(a_3)}{(n_1+n_2)(F_{n_3, n_2}(a_2)-n_2 F_{n_3, n_1+n_2}(a_3))}}
\]

where \( F_{n_3, n_2}(a_2) \) represents the 100 \( a_2 \) per cent point of \( F \) with numerator d.f. \( n_3 \) and denominator d.f. \( n_2 \), and similar statements can be made for the other symbols.
V. ILLUSTRATION OF RECOMMENDED PROCEDURES
WITH PRACTICAL EXAMPLES

We now show, with the help of two selected examples, how the
recommended test procedures work in practical cases. Both of
these examples deal with data taken from actual industrial problems
and discussed in textbooks. In both cases there is some discussion
as to how to choose the valid error for the main test of significance.
Moreover, the examples are situations for which, in our opinion, a
suitable model is provided either by a component of variance model
or mixed model.

A. Tests on Samples of Portland Cement

O. L. Davies (1947, p. 90) gives data on the comparative break-
ing strength of Portland cement cubes. Each of three gaugers (mixers)
mixed the cement from which 12 cubes were made, and each of three
breakers (operators of the testing machine) tested the strength of four
cubes from each of the mixers. We have therefore a two-way classi-
fication with cell repetition. The analysis of variance of the breaking
strength values in 10 lb./sq. in. is set out below.

If we regard the breakers and gaugers as respective samples of
size three from larger populations of breakers and gaugers, we may
represent the above data by a component of variance model. Alterna-
vely, the three gaugers may be regarded as permanent key-personnel of
Table 6. Breaking strength of Portland cement. Analysis of variance

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>d.f.</th>
<th>Mean square</th>
<th>Exp. mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between breakers</td>
<td>2</td>
<td>12,530</td>
<td></td>
</tr>
<tr>
<td>Between gaugers</td>
<td>$n_3 = 2$</td>
<td>$V_3 = 4,482$</td>
<td>$\sigma_3^2$</td>
</tr>
<tr>
<td>Breakers x gaugers</td>
<td>$n_2 = 4$</td>
<td>$V_2 = 1,659$</td>
<td>$\sigma_2^2$</td>
</tr>
<tr>
<td>Within groups of 4 cubes</td>
<td>$n_1 = 27$</td>
<td>$V_1 = 2,746$</td>
<td>$\sigma_1^2$</td>
</tr>
</tbody>
</table>

the plant, and if the tests are concerned with the comparative performance of these particular three men, a mixed model would be appropriate.

Either case is covered by our recommendation made in Section IV, which would be applied as follows: It is a priori unlikely that $\theta_{21}$ is large (that $\sigma_2^2$ is much larger than $\sigma_1^2$), as this would mean that there is a large breaker x gauger interaction component adding to the cube to cube variation. This is very unlikely because the breaker's function was merely to operate the testing machine and read off the answers; further, as Davies (1947, p. 92) states, the breakers did not know whose gaugers' cubes they were testing. We may therefore decide (in accordance with our recommendation) to use either the 25 per cent or the 50 per cent $F$ point for our preliminary test, and, as $n_2 > n_3$, we should use the 25 per cent point. The observed $F$-ratio $V_2/V_1 = .604$ is clearly not significant. (In fact it would have been
not significant at the 50 per cent point of $F$ as well.) We therefore pool $V_2$ and $V_1$, compute

$$V = \frac{n_1 V_1 + n_2 V_2}{n_1 + n_2} = 2606,$$

and test $V_3/V = 1.72$ at the 5 per cent point of $F$ for 2, 31 degrees of freedom, for which the 5 per cent point is 3.31, so that our observed $F$ ratio is not significant. In the case of a completely random model the same procedure may be used for testing differences between breakers. Using the breaker mean square as $V_3$ and testing $V_3/V = 4.81$, we obtain a result which is significant. Differences between the testing personnel are therefore indicated, suggesting that a better standardization of the test procedure would be desirable. Indeed, Davies states that for these tests an old machine was in use, which would explain the variation of results with the tester operating it.

B. Porosity of Condenser Paper

H. A. Freeman (1942, p. 70) discusses data on the porosity of condenser paper. Three readings were made on each of 9 rolls from each of 3 lots. We now have an example of a hierarchical classification. The analysis of variance table is given below. Here it might be appropriate to regard the three lot means as fixed parameters, but the roll effects and measurement effects as respective random samples from larger populations of roll effect variates and measurement effect variates. If these assumptions are accepted, we have a mixed
Table 7. Porosity readings. Analysis of variance

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>d.f.</th>
<th>Mean square</th>
<th>Exp. mean square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between lots</td>
<td>$n_3 = 2$</td>
<td>$V_3 = 3.95$</td>
<td>$\sigma^2_3$</td>
</tr>
<tr>
<td>Between rolls within lots</td>
<td>$n_2 = 24$</td>
<td>$V_2 = 3.87$</td>
<td>$\sigma^2_2$</td>
</tr>
<tr>
<td>Between measurements within rolls</td>
<td>$n_1 = 54$</td>
<td>$V_1 = 0.78$</td>
<td>$\sigma^2_1$</td>
</tr>
</tbody>
</table>

A hierarchical model, to which our recommendations apply. Without making any assumptions concerning the magnitude of possible roll effects, that is, irrespective of our a priori assumptions concerning the magnitude of $\theta_{21}$, the preliminary test at the 25 per cent level is significant. In fact, it is significant at the 1 per cent level. Therefore we are not allowed to pool the within roll component. For the final test, then, we compute $V_3/V_2 = 1.021$ which is not significant at the 5 per cent level. The data indicate that variation in porosity is due largely to differences among rolls within lots, and, as the author suggests, attempts should be made to eliminate these differences.
VI. LITERATURE CITED


Tables of the incomplete Beta-function. London, Biometric Laboratory. 1933.
VII. ACKNOWLEDGMENTS

The writer wishes to express her sincere appreciation to Professor T. A. Bancroft and to Professor H. O. Hartley for their assistance and encouragement during the preparation of this thesis.

The research was sponsored by the Iowa State College Agricultural Experiment Station and the Wright Air Development Center.
VIII. APPENDIX
Figure 1. Size curves for $n_3 = 2$, $n_2 = 6$, $a_1 = a_2 = a_3 = .05$
Figure 2. Size curves for $n_3 = 2$, $n_2 = 10$, $a_1 = a_2 = a_3 = 0.05$
Figure 4. Size curves for $n_3 = 12$, $n_2 = 10$, $a_1 = a_2 = a_3 = .05$
Figure 5. Size curves for $n_3 = 2$, $n_2 = 6$, $a_1 = .25$, $a_2 = a_3 = .05$
Figure 6. Size curves for $n_3 = 2$, $n_1 = \infty$, $a_1 = .25$, $a_2 = a_3 = .05$
Figure 7. Size curves for $a = 2, a' = 10, a'' = 25$. 

$50 = a \neq a' \neq a''$
Figure 8. Size curves for $n_3 = 2$, $n_2 = 16$, $a_4 = .25$, $a_2 = a_3 = .05$
Figure 9. Size curves for $n_3 = 6$, $n_2 = 6$, $a_1 = .25$, $a_2 = a_3 = .05$
Figure 10. Data curves for $n^2 = 0$, $n^2 = 1$, $n^2 = 10$,
Figure 11. Size curves for $n_3 = 6$, $n_2 = 16$.

$a_1 = .25$, $a_2 = a_3 = .05$
Figure 13. Size curves for $n_3 = 12$, $n_2 = 16$, $a_1 = 0.25$, $a_2 = a_3 = 0.05$
Figure 14. Size curves for $n_3 = 12$, $n_1 = \infty$, $a_2 = a_3 = 0.1$.
Figure 15. Size curves for $n_2 = 2$, $n_1 = 20$, for the borderline test, for $a_2 = a_3 = .05$

(1) $n_2 = 10$, $a_1 = .68$

(2) $n_2 = 6$, $a_1 = .73$
Figure 16. Power curves for $n_3 = 12$, $n_2 = 10$, $n_1 = 20$.

$a_1 = a_2 = a_3 = 0.05$, $\theta_{21} = 1.174$
Figure 17. Power curves for $n_3 = 12$, $n_2 = 10$, $n_1 = 20$.

$a_1 = a_2 = a_3 = .05$, $\theta_{21} = 1.789$
Figure 18. Power curves for $n_3 = 12$, $n_2 = 10$, $n_1 = 20$, $a_1 = a_2 = a_3 = 0.05$, $\theta_{21} = 2.866$. 

(1) SOMETIMES POOL PROCEDURE

(2) NEVER POOL TEST
Figure 19. Power gain of the sometimes pool procedure over the never pool test of the same size for $n_1 = 20$, $n_3 = 2$, $n_2 = 6$, $a_1 = a_2 = a_3 = .05$
Figure 20. Power gain of the sometimes pool procedure over never pool test of the same size for

\[ n_1 = 20, \quad n_3 = 2, \quad n_2 = 6, \quad F_1 = 2 \quad F_{0.5}, \quad \alpha_2 = \alpha_3 = 0.05 \]
Figure 21. Power gain of the sometimes pool procedure over the never pool test of the same size for $n_1 = 20$, $n_3 = 2$, $n_2 = 6$, $a_1 = .25$, $a_2 = a_3 = .05$
In Figure 23, we see the power gain of the sometimes pool procedure over the
never pool test of the same size for n₁ = 20, n₂ = 2.

In Figure 22, we see the power gain of the sometimes pool procedure over the
never pool test of the same size for n₁ = 20, n₂ = 2.

\[ z_2 = 10, z_1 = 7 \]

\[ x = 50, e = 3, x = 50, e = 3 \]

\[ \theta = \theta \]

\[ \theta = \theta \]

\[ \theta = \theta \]

\[ \theta = \theta \]

\[ \theta = \theta \]

\[ \theta = \theta \]

\[ \theta = \theta \]

\[ \theta = \theta \]

\[ \theta = \theta \]
Figure 24. Power gain of the sometimes pool procedure over the never pool test of the same size for

\[ n_1 = 20, n_3 = 12, n_2 = 10, a_1 = a_2 = a_3 = 0.05 \]

\[ \theta_2 = 4 \]

\[ \theta_3 = 2 \]
Table 8. Master table for $P_1$, for $n_1 = 20$, $n_2 = 10$, $n_3 = 12$

<table>
<thead>
<tr>
<th>d</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.32*</td>
<td>0.0117</td>
<td>0.0184</td>
<td>0.0268</td>
<td>0.0353</td>
<td>0.0480</td>
</tr>
<tr>
<td>0.38</td>
<td>0.0361</td>
<td>0.0529</td>
<td>0.0721</td>
<td>0.0909</td>
<td>0.1166</td>
</tr>
<tr>
<td>0.44</td>
<td>0.0898</td>
<td>0.1224</td>
<td>0.1572</td>
<td>0.1893</td>
<td>0.2290</td>
</tr>
<tr>
<td>0.50</td>
<td>0.1868</td>
<td>0.2380</td>
<td>0.2886</td>
<td>0.3320</td>
<td>0.3790</td>
</tr>
<tr>
<td>0.56</td>
<td>0.3334</td>
<td>0.3986</td>
<td>0.4581</td>
<td>0.5052</td>
<td>0.5452</td>
</tr>
<tr>
<td>0.62</td>
<td>0.5187</td>
<td>0.5853</td>
<td>0.6413</td>
<td>0.6809</td>
<td>0.6985</td>
</tr>
<tr>
<td>0.68</td>
<td>0.7116</td>
<td>0.7643</td>
<td>0.8045</td>
<td>0.8280</td>
<td>0.8148</td>
</tr>
<tr>
<td>0.74</td>
<td>0.8702</td>
<td>0.8998</td>
<td>0.9197</td>
<td>0.9256</td>
<td>0.8846</td>
</tr>
<tr>
<td>0.80</td>
<td>0.9641</td>
<td>0.9739</td>
<td>0.9784</td>
<td>0.9724</td>
<td>0.9150</td>
</tr>
<tr>
<td>0.86</td>
<td>0.20*</td>
<td>0.0005</td>
<td>0.0004</td>
<td>0.0004</td>
<td>0.0002</td>
</tr>
<tr>
<td>0.32</td>
<td>0.0166</td>
<td>0.0154</td>
<td>0.0125</td>
<td>0.0066</td>
<td>0.0017</td>
</tr>
<tr>
<td>0.38</td>
<td>0.0519</td>
<td>0.0746</td>
<td>0.0378</td>
<td>0.0196</td>
<td>0.0110</td>
</tr>
<tr>
<td>0.44</td>
<td>0.1226</td>
<td>0.1085</td>
<td>0.0860</td>
<td>0.0441</td>
<td>0.0197</td>
</tr>
<tr>
<td>0.50</td>
<td>0.2333</td>
<td>0.2014</td>
<td>0.1575</td>
<td>0.0794</td>
<td>0.0293</td>
</tr>
<tr>
<td>0.56</td>
<td>0.3738</td>
<td>0.3143</td>
<td>0.2422</td>
<td>0.1200</td>
<td>0.0380</td>
</tr>
<tr>
<td>0.62</td>
<td>0.5207</td>
<td>0.4264</td>
<td>0.3237</td>
<td>0.1577</td>
<td>0.0443</td>
</tr>
<tr>
<td>0.68</td>
<td>0.6479</td>
<td>0.5177</td>
<td>0.3877</td>
<td>0.1859</td>
<td>0.0478</td>
</tr>
<tr>
<td>0.74</td>
<td>0.7379</td>
<td>0.5778</td>
<td>0.4279</td>
<td>0.2026</td>
<td>0.0493</td>
</tr>
<tr>
<td>0.80</td>
<td>0.7880</td>
<td>0.6088</td>
<td>0.4476</td>
<td>0.2102</td>
<td>0.0498</td>
</tr>
<tr>
<td>0.86*</td>
<td>0.8082</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*These were additional values subsequently computed to facilitate interpolation where necessary.
Table 9. Master table for $P^r_2 = \log_{10} (P_2 \cdot 10^7)$, for $n_1 = 20$, $n_2 = 10$, $n_3 = 2$ and $a = 1$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$P^r_2$</th>
<th>$x_2$</th>
<th>$P^r_2$</th>
<th>$x_2$</th>
<th>$P^r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.50</td>
<td>.38</td>
<td>1.9685</td>
<td>.53</td>
<td>2.6010</td>
<td>.68</td>
<td>3.7538</td>
</tr>
<tr>
<td>.60</td>
<td>.40</td>
<td>3.2227</td>
<td>.52</td>
<td>3.7339</td>
<td>.64</td>
<td>4.5595</td>
</tr>
<tr>
<td>.70</td>
<td>.40</td>
<td>4.2269</td>
<td>.50</td>
<td>4.6358</td>
<td>.60</td>
<td>5.2420</td>
</tr>
<tr>
<td>.80</td>
<td>.43</td>
<td>5.2036</td>
<td>.50</td>
<td>5.5054</td>
<td>.57</td>
<td>5.9042</td>
</tr>
<tr>
<td>.90</td>
<td>.46</td>
<td>6.0903</td>
<td>.50</td>
<td>6.2728</td>
<td>.54</td>
<td>6.4864</td>
</tr>
<tr>
<td>.95</td>
<td>.475</td>
<td>6.5072</td>
<td>.50</td>
<td>6.6250</td>
<td>.52</td>
<td>6.7275</td>
</tr>
<tr>
<td>.99</td>
<td>.495</td>
<td>6.9038</td>
<td>.503</td>
<td>6.9085</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.9999</td>
<td>.49995</td>
<td>6.9582</td>
<td>.50004</td>
<td>6.9587</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 10. Probability of pooling, $a_1 = .25$ and $a_1 = .50$

<table>
<thead>
<tr>
<th>$n_2$</th>
<th>$n_1$</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\theta_{21}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_1 = .25$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\infty$</td>
<td>.750</td>
<td>.603</td>
<td>.500</td>
<td>.426</td>
</tr>
<tr>
<td>4</td>
<td>$\infty$</td>
<td>.750</td>
<td>.536</td>
<td>.390</td>
<td>.227</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>.750</td>
<td>.522</td>
<td>.360</td>
<td>.184</td>
</tr>
<tr>
<td>6</td>
<td>$\infty$</td>
<td>.750</td>
<td>.485</td>
<td>.312</td>
<td>.144</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>.750</td>
<td>.475</td>
<td>.286</td>
<td>.108</td>
</tr>
<tr>
<td>10</td>
<td>$\infty$</td>
<td>.750</td>
<td>.407</td>
<td>.208</td>
<td>.061</td>
</tr>
<tr>
<td>16</td>
<td>20</td>
<td>.750</td>
<td>.432</td>
<td>.224</td>
<td>.058</td>
</tr>
<tr>
<td>16</td>
<td>$\infty$</td>
<td>.750</td>
<td>.321</td>
<td>.117</td>
<td>.018</td>
</tr>
</tbody>
</table>

$a_1 = .50$

| 10    | $\infty$ | .50  | .204 | .088 | .021 |


Table 11. Probability of pooling for the borderline test

<table>
<thead>
<tr>
<th>$n_3$</th>
<th>$n_2$</th>
<th>$n_1$</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>20</td>
<td>.152</td>
<td>.105</td>
<td>.080</td>
<td>.064</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>20</td>
<td>.225</td>
<td>.124</td>
<td>.078</td>
<td>.054</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>8</td>
<td>.277</td>
<td>.142</td>
<td>.081</td>
<td>.050</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>20</td>
<td>.268</td>
<td>.127</td>
<td>.069</td>
<td>.041</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>14</td>
<td>.320</td>
<td>.132</td>
<td>.060</td>
<td>.030</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>20</td>
<td>.317</td>
<td>.124</td>
<td>.054</td>
<td>.026</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>14</td>
<td>.238</td>
<td>.088</td>
<td>.037</td>
<td>.018</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>20</td>
<td>.228</td>
<td>.080</td>
<td>.032</td>
<td>.015</td>
</tr>
</tbody>
</table>
Table 12. Minimum size values for the borderline test

<table>
<thead>
<tr>
<th>$n_3$</th>
<th>$n_2$</th>
<th>$n_1$</th>
<th>Minimum size</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>20</td>
<td>.010</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>60</td>
<td>.009</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>20</td>
<td>.018</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>14</td>
<td>.024</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>20</td>
<td>.023</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>14</td>
<td>.030</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>20</td>
<td>.030</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>20</td>
<td>.017</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>20</td>
<td>.023</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
<td>20</td>
<td>.029</td>
</tr>
<tr>
<td>12</td>
<td>16</td>
<td>20</td>
<td>.030</td>
</tr>
</tbody>
</table>
Table 13. The power of the sometimes pool procedure and the never pool test of the same size, 
for \( n_1 = 8, n_2 = 6, n_3 = 2, a_1 = a_2 = a_3 = .05 \)

<table>
<thead>
<tr>
<th>( \theta_{21} )</th>
<th>Test</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 4 )</th>
<th>( 16 )</th>
<th>( 64 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.940* s.p.</td>
<td>.044</td>
<td>.175</td>
<td>.393</td>
<td>.782</td>
<td>.938</td>
<td></td>
</tr>
<tr>
<td>n.p.</td>
<td>.044</td>
<td>.143</td>
<td>.324</td>
<td>.723</td>
<td>.919</td>
<td></td>
</tr>
<tr>
<td>1.295 s.p.</td>
<td>.066</td>
<td>.217</td>
<td>.439</td>
<td>.799</td>
<td>.943</td>
<td></td>
</tr>
<tr>
<td>n.p.</td>
<td>.066</td>
<td>.191</td>
<td>.391</td>
<td>.768</td>
<td>.934</td>
<td></td>
</tr>
<tr>
<td>1.834 s.p.</td>
<td>.089</td>
<td>.249</td>
<td>.463</td>
<td>.803</td>
<td>.945</td>
<td></td>
</tr>
<tr>
<td>n.p.</td>
<td>.089</td>
<td>.235</td>
<td>.444</td>
<td>.799</td>
<td>.944</td>
<td></td>
</tr>
<tr>
<td>4.093 s.p.</td>
<td>.109</td>
<td>.248</td>
<td>.427</td>
<td>.773</td>
<td>.934</td>
<td></td>
</tr>
<tr>
<td>n.p.</td>
<td>.109</td>
<td>.270</td>
<td>.484</td>
<td>.820</td>
<td>.950</td>
<td></td>
</tr>
<tr>
<td>11.185 s.p.</td>
<td>.070</td>
<td>.175</td>
<td>.356</td>
<td>.741</td>
<td>.925</td>
<td></td>
</tr>
<tr>
<td>n.p.</td>
<td>.070</td>
<td>.200</td>
<td>.402</td>
<td>.775</td>
<td>.936</td>
<td></td>
</tr>
<tr>
<td>n.p.</td>
<td>.050</td>
<td>.156</td>
<td>.343</td>
<td>.737</td>
<td>.924</td>
<td></td>
</tr>
</tbody>
</table>

*In Tables 13-31, values of \( \theta_{21} < 1 \) have been inserted for interpolation purposes. These values do not of course arise in the analysis of variance models considered. For the same tables, s.p. represents the sometimes pool procedure and n.p. the never pool test.
Table 14. The power of the sometimes pool procedure and the never pool test of the same size, for \( n_1 = 14, n_2 = 6, n_3 = 2, a_1 = a_2 = a_3 = .05 \)

<table>
<thead>
<tr>
<th>( 0_{21} )</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>16</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0.834 )</td>
<td>s.p.</td>
<td>.034</td>
<td>.159</td>
<td>.378</td>
<td>.774</td>
<td>.937</td>
</tr>
<tr>
<td>n.p.</td>
<td>.034</td>
<td>.118</td>
<td>.285</td>
<td>.693</td>
<td>.908</td>
<td></td>
</tr>
<tr>
<td>( 1.220 )</td>
<td>s.p.</td>
<td>.065</td>
<td>.225</td>
<td>.449</td>
<td>.803</td>
<td>.945</td>
</tr>
<tr>
<td>n.p.</td>
<td>.065</td>
<td>.189</td>
<td>.387</td>
<td>.766</td>
<td>.933</td>
<td></td>
</tr>
<tr>
<td>( 1.860 )</td>
<td>s.p.</td>
<td>.102</td>
<td>.269</td>
<td>.475</td>
<td>.805</td>
<td>.945</td>
</tr>
<tr>
<td>n.p.</td>
<td>.102</td>
<td>.258</td>
<td>.470</td>
<td>.813</td>
<td>.948</td>
<td></td>
</tr>
<tr>
<td>( 5.083 )</td>
<td>s.p.</td>
<td>.102</td>
<td>.215</td>
<td>.388</td>
<td>.753</td>
<td>.929</td>
</tr>
<tr>
<td>n.p.</td>
<td>.102</td>
<td>.259</td>
<td>.472</td>
<td>.814</td>
<td>.949</td>
<td></td>
</tr>
<tr>
<td>n.p.</td>
<td>.050</td>
<td>.156</td>
<td>.343</td>
<td>.737</td>
<td>.924</td>
<td></td>
</tr>
</tbody>
</table>
Table 15. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1=14$, $n_2=10$, $n_3=2$, $a_1=a_2=a_3=.05$

<table>
<thead>
<tr>
<th>$Q_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.896</td>
<td>s.p.</td>
<td>.041</td>
<td>.179</td>
<td>.414</td>
<td>.695</td>
<td>.928</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.041</td>
<td>.158</td>
<td>.365</td>
<td>.652</td>
<td>.915</td>
</tr>
<tr>
<td>1.269</td>
<td>s.p.</td>
<td>.065</td>
<td>.229</td>
<td>.460</td>
<td>.723</td>
<td>.937</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.065</td>
<td>.212</td>
<td>.434</td>
<td>.704</td>
<td>.931</td>
</tr>
<tr>
<td>1.859</td>
<td>s.p.</td>
<td>.086</td>
<td>.257</td>
<td>.477</td>
<td>.729</td>
<td>.938</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.086</td>
<td>.253</td>
<td>.480</td>
<td>.736</td>
<td>.939</td>
</tr>
<tr>
<td>4.538</td>
<td>s.p.</td>
<td>.077</td>
<td>.213</td>
<td>.418</td>
<td>.690</td>
<td>.928</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.077</td>
<td>.236</td>
<td>.461</td>
<td>.723</td>
<td>.936</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.179</td>
<td>.393</td>
<td>.674</td>
<td>.922</td>
</tr>
</tbody>
</table>
Table 16. The power of the sometimes pool procedure and the never pool test of the same size, for \( n_1 = 20, n_2 = 6, n_3 = 2, a_1 = a_2 = a_3 = 0.05 \)

<table>
<thead>
<tr>
<th>( g_{21} )</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>16</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_{32} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.780</td>
<td>s.p.</td>
<td>0.078</td>
<td>0.147</td>
<td>0.366</td>
<td>0.768</td>
<td>0.935</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>0.078</td>
<td>0.101</td>
<td>0.256</td>
<td>0.668</td>
<td>0.900</td>
</tr>
<tr>
<td>1.188</td>
<td>s.p.</td>
<td>0.064</td>
<td>0.227</td>
<td>0.453</td>
<td>0.806</td>
<td>0.947</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>0.064</td>
<td>0.187</td>
<td>0.386</td>
<td>0.765</td>
<td>0.933</td>
</tr>
<tr>
<td>1.904</td>
<td>s.p.</td>
<td>0.109</td>
<td>0.278</td>
<td>0.480</td>
<td>0.804</td>
<td>0.944</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>0.109</td>
<td>0.271</td>
<td>0.485</td>
<td>0.821</td>
<td>0.951</td>
</tr>
<tr>
<td>3.247</td>
<td>s.p.</td>
<td>0.122</td>
<td>0.255</td>
<td>0.429</td>
<td>0.771</td>
<td>0.933</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>0.122</td>
<td>0.292</td>
<td>0.508</td>
<td>0.832</td>
<td>0.954</td>
</tr>
<tr>
<td>6.016</td>
<td>s.p.</td>
<td>0.087</td>
<td>0.192</td>
<td>0.367</td>
<td>0.744</td>
<td>0.928</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>0.087</td>
<td>0.233</td>
<td>0.442</td>
<td>0.780</td>
<td>0.944</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>0.050</td>
<td>0.156</td>
<td>0.343</td>
<td>0.737</td>
<td>0.924</td>
</tr>
</tbody>
</table>
Table 17. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 10$, $n_3 = 2$, $a_1 = a_2 = a_3 = .05$

<table>
<thead>
<tr>
<th>$\theta_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.174</td>
<td>s.p.</td>
<td>.061</td>
<td>.228</td>
<td>.459</td>
<td>.733</td>
<td>.946</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.061</td>
<td>.205</td>
<td>.426</td>
<td>.698</td>
<td>.929</td>
</tr>
<tr>
<td>1.789</td>
<td>s.p.</td>
<td>.091</td>
<td>.261</td>
<td>.480</td>
<td>.731</td>
<td>.938</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.091</td>
<td>.262</td>
<td>.489</td>
<td>.742</td>
<td>.941</td>
</tr>
<tr>
<td>2.866</td>
<td>s.p.</td>
<td>.094</td>
<td>.243</td>
<td>.448</td>
<td>.706</td>
<td>.932</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.094</td>
<td>.267</td>
<td>.495</td>
<td>.746</td>
<td>.942</td>
</tr>
<tr>
<td>9.058</td>
<td>s.p.</td>
<td>.057</td>
<td>.180</td>
<td>.394</td>
<td>.675</td>
<td>.924</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.057</td>
<td>.197</td>
<td>.416</td>
<td>.691</td>
<td>.927</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.179</td>
<td>.393</td>
<td>.674</td>
<td>.922</td>
</tr>
</tbody>
</table>
Table 18. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 10$, $n_3 = 12$, $a_1 = a_2 = a_3 = .05$

<table>
<thead>
<tr>
<th>$\theta_{21}$</th>
<th>Test</th>
<th>$\theta_{32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0.896</td>
<td>s.p.</td>
<td>.037</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.037</td>
</tr>
<tr>
<td>1.269</td>
<td>s.p.</td>
<td>.078</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.078</td>
</tr>
<tr>
<td>1.859</td>
<td>s.p.</td>
<td>.135</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.135</td>
</tr>
<tr>
<td>4.538</td>
<td>s.p.</td>
<td>.124</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.124</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
</tr>
</tbody>
</table>
Table 19. The power of the sometimes pool procedure and the never pool test of the same size, for \( n_1 = 20, n_2 = 10, n_3 = 12, a_1 = a_2 = a_3 = .05 \)

<table>
<thead>
<tr>
<th>( q_{21} )</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.802</td>
<td>s.p.</td>
<td>.023</td>
<td>.314</td>
<td>.817</td>
<td>.985</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.023</td>
<td>.168</td>
<td>.555</td>
<td>.946</td>
<td>1.000</td>
</tr>
<tr>
<td>1.174</td>
<td>s.p.</td>
<td>.072</td>
<td>.479</td>
<td>.870</td>
<td>.987</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.072</td>
<td>.350</td>
<td>.769</td>
<td>.985</td>
<td>1.000</td>
</tr>
<tr>
<td>1.789</td>
<td>s.p.</td>
<td>.152</td>
<td>.535</td>
<td>.824</td>
<td>.982</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.152</td>
<td>.529</td>
<td>.883</td>
<td>.995</td>
<td>1.000</td>
</tr>
<tr>
<td>2.866</td>
<td>s.p.</td>
<td>.178</td>
<td>.421</td>
<td>.734</td>
<td>.976</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.178</td>
<td>.572</td>
<td>.903</td>
<td>.996</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.050</td>
<td>.280</td>
<td>.702</td>
<td>.976</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Table 20. The power of the sometimes pool procedure and the never pool test of the same size, for \( n_1 = 8, n_2 = 6, n_3 = 2, F_1 = 2F_{.50}, \alpha_2 = \alpha_3 = .05 \)

<table>
<thead>
<tr>
<th>( Q_{21} )</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.995</td>
<td>s.p.</td>
<td>.041</td>
<td>.162</td>
<td>.369</td>
<td>.759</td>
<td>.932</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.041</td>
<td>.137</td>
<td>.315</td>
<td>.717</td>
<td>.917</td>
</tr>
<tr>
<td>1.457</td>
<td>s.p.</td>
<td>.059</td>
<td>.191</td>
<td>.394</td>
<td>.766</td>
<td>.933</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.059</td>
<td>.176</td>
<td>.371</td>
<td>.756</td>
<td>.930</td>
</tr>
<tr>
<td>2.220</td>
<td>s.p.</td>
<td>.072</td>
<td>.200</td>
<td>.392</td>
<td>.759</td>
<td>.930</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.072</td>
<td>.202</td>
<td>.405</td>
<td>.776</td>
<td>.937</td>
</tr>
<tr>
<td>3.556</td>
<td>s.p.</td>
<td>.062</td>
<td>.169</td>
<td>.352</td>
<td>.740</td>
<td>.924</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.062</td>
<td>.183</td>
<td>.380</td>
<td>.761</td>
<td>.932</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.156</td>
<td>.343</td>
<td>.737</td>
<td>.924</td>
</tr>
</tbody>
</table>
Table 21. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 6$, $n_3 = 2$, $F_1 = 2F_{.50}$, $a_2 = a_3 = .05$

<table>
<thead>
<tr>
<th>$Q_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>16</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.802</td>
<td>s.p.</td>
<td>.030</td>
<td>.142</td>
<td>.352</td>
<td>.755</td>
<td>.930</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.030</td>
<td>.108</td>
<td>.278</td>
<td>.679</td>
<td>.903</td>
</tr>
<tr>
<td>1.222</td>
<td>s.p.</td>
<td>.056</td>
<td>.195</td>
<td>.403</td>
<td>.775</td>
<td>.949</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.056</td>
<td>.170</td>
<td>.363</td>
<td>.750</td>
<td>.928</td>
</tr>
<tr>
<td>1.958</td>
<td>s.p.</td>
<td>.080</td>
<td>.216</td>
<td>.407</td>
<td>.767</td>
<td>.932</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.080</td>
<td>.218</td>
<td>.424</td>
<td>.788</td>
<td>.940</td>
</tr>
<tr>
<td>6.189</td>
<td>s.p.</td>
<td>.064</td>
<td>.166</td>
<td>.350</td>
<td>.739</td>
<td>.924</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.064</td>
<td>.186</td>
<td>.384</td>
<td>.764</td>
<td>.933</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.156</td>
<td>.343</td>
<td>.737</td>
<td>.924</td>
</tr>
</tbody>
</table>
Table 22. The power of the sometimes pool procedure 
and the never pool test of the same size, for 
n₁ = 14, n₂ = 10, n₃ = 2, F₁ = 2F₀.50, a₂ = a₃ = .05

<table>
<thead>
<tr>
<th>$\theta_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.957</td>
<td>s.p.</td>
<td>.043</td>
<td>.176</td>
<td>.407</td>
<td>.689</td>
<td>.927</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.043</td>
<td>.163</td>
<td>.372</td>
<td>.657</td>
<td>.917</td>
</tr>
<tr>
<td>1.401</td>
<td>s.p.</td>
<td>.063</td>
<td>.218</td>
<td>.441</td>
<td>.707</td>
<td>.931</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.063</td>
<td>.208</td>
<td>.430</td>
<td>.701</td>
<td>.930</td>
</tr>
<tr>
<td>2.135</td>
<td>s.p.</td>
<td>.075</td>
<td>.224</td>
<td>.436</td>
<td>.702</td>
<td>.931</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.075</td>
<td>.232</td>
<td>.457</td>
<td>.720</td>
<td>.935</td>
</tr>
<tr>
<td>3.420</td>
<td>s.p.</td>
<td>.068</td>
<td>.205</td>
<td>.412</td>
<td>.687</td>
<td>.928</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.068</td>
<td>.220</td>
<td>.443</td>
<td>.710</td>
<td>.932</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.179</td>
<td>.393</td>
<td>.674</td>
<td>.922</td>
</tr>
</tbody>
</table>
Table 23. The power of the sometimes pool procedure and the never pool test of the same size, for \( n_1 = 20, \ n_2 = 6, \ n_3 = 2, \ F_1 = 2F_{.50}, \ a_2 = a_3 = .05 \)

<table>
<thead>
<tr>
<th>( \theta_{21} )</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>16</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.843</td>
<td>s.p.</td>
<td>.031</td>
<td>.152</td>
<td>.367</td>
<td>.764</td>
<td>.934</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.031</td>
<td>.110</td>
<td>.272</td>
<td>.683</td>
<td>.905</td>
</tr>
<tr>
<td>1.350</td>
<td>s.p.</td>
<td>.065</td>
<td>.215</td>
<td>.422</td>
<td>.780</td>
<td>.937</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.065</td>
<td>.190</td>
<td>.389</td>
<td>.767</td>
<td>.934</td>
</tr>
<tr>
<td>2.304</td>
<td>s.p.</td>
<td>.089</td>
<td>.219</td>
<td>.402</td>
<td>.762</td>
<td>.931</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.089</td>
<td>.236</td>
<td>.446</td>
<td>.800</td>
<td>.944</td>
</tr>
<tr>
<td>4.269</td>
<td>s.p.</td>
<td>.075</td>
<td>.182</td>
<td>.361</td>
<td>.742</td>
<td>.924</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.075</td>
<td>.209</td>
<td>.414</td>
<td>.782</td>
<td>.939</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.156</td>
<td>.343</td>
<td>.737</td>
<td>.924</td>
</tr>
</tbody>
</table>
Table 24. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 10$, $n_3 = 2$, $F_1 = 2F_{.50}$, $a_2 = a_3 = .05$

<table>
<thead>
<tr>
<th>$\theta_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.966</td>
<td>s.p.</td>
<td>.044</td>
<td>.190</td>
<td>.416</td>
<td>.710</td>
<td>.941</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.044</td>
<td>.165</td>
<td>.375</td>
<td>.660</td>
<td>.918</td>
</tr>
<tr>
<td>1.473</td>
<td>s.p.</td>
<td>.071</td>
<td>.230</td>
<td>.453</td>
<td>.715</td>
<td>.934</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.071</td>
<td>.225</td>
<td>.449</td>
<td>.715</td>
<td>.934</td>
</tr>
<tr>
<td>2.359</td>
<td>s.p.</td>
<td>.080</td>
<td>.228</td>
<td>.435</td>
<td>.699</td>
<td>.931</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.080</td>
<td>.243</td>
<td>.469</td>
<td>.729</td>
<td>.937</td>
</tr>
<tr>
<td>7.456</td>
<td>s.p.</td>
<td>.057</td>
<td>.179</td>
<td>.393</td>
<td>.675</td>
<td>.922</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.057</td>
<td>.196</td>
<td>.415</td>
<td>.690</td>
<td>.927</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.179</td>
<td>.393</td>
<td>.674</td>
<td>.922</td>
</tr>
</tbody>
</table>
Table 25. The power of the sometimes pool procedure and the never pool test of the same size, for
\( n_1 = 14, n_2 = 10, n_3 = 12, F_1 = 2F_{.50}, a_2 = a_3 = .05 \)

<table>
<thead>
<tr>
<th>( \theta_{21} )</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.957</td>
<td>s.p.</td>
<td>.039</td>
<td>.354</td>
<td>.784</td>
<td>.986</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.039</td>
<td>.240</td>
<td>.657</td>
<td>.969</td>
<td>1.000</td>
</tr>
<tr>
<td>1.401</td>
<td>s.p.</td>
<td>.083</td>
<td>.434</td>
<td>.805</td>
<td>.981</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.083</td>
<td>.379</td>
<td>.791</td>
<td>.988</td>
<td>1.000</td>
</tr>
<tr>
<td>2.135</td>
<td>s.p.</td>
<td>.117</td>
<td>.412</td>
<td>.752</td>
<td>.977</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.117</td>
<td>.460</td>
<td>.846</td>
<td>.993</td>
<td>1.000</td>
</tr>
<tr>
<td>3.420</td>
<td>s.p.</td>
<td>.106</td>
<td>.329</td>
<td>.707</td>
<td>.976</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.106</td>
<td>.435</td>
<td>.831</td>
<td>.991</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.280</td>
<td>.702</td>
<td>.976</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Table 26. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 10$, $n_3 = 12$, $F_1 = 2F_{.50}$, $a_2 = a_3 = .05$

<table>
<thead>
<tr>
<th>$\theta_{21}$ Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.p. 0.966</td>
<td>.040</td>
<td>.383</td>
<td>.827</td>
<td>.985</td>
<td>1.000</td>
</tr>
<tr>
<td>n.p.</td>
<td>.040</td>
<td>.244</td>
<td>.662</td>
<td>.970</td>
<td>1.000</td>
</tr>
<tr>
<td>s.p. 1.473</td>
<td>.103</td>
<td>.479</td>
<td>.811</td>
<td>.981</td>
<td>1.000</td>
</tr>
<tr>
<td>n.p.</td>
<td>.103</td>
<td>.431</td>
<td>.828</td>
<td>.991</td>
<td>1.000</td>
</tr>
<tr>
<td>s.p. 2.359</td>
<td>.142</td>
<td>.406</td>
<td>.736</td>
<td>.976</td>
<td>1.000</td>
</tr>
<tr>
<td>n.p.</td>
<td>.142</td>
<td>.512</td>
<td>.874</td>
<td>.995</td>
<td>1.000</td>
</tr>
<tr>
<td>n.p.</td>
<td>.050</td>
<td>.280</td>
<td>.702</td>
<td>.976</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Table 27. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 6$, $n_3 = 2$, $a_1 = .25$, $a_2 = a_3 = .05$

<table>
<thead>
<tr>
<th>$q_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>16</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>s.p.</td>
<td>.038</td>
<td>.161</td>
<td>.368</td>
<td>.757</td>
<td>.930</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.038</td>
<td>.127</td>
<td>.314</td>
<td>.705</td>
<td>.913</td>
</tr>
<tr>
<td>1.5</td>
<td>s.p.</td>
<td>.060</td>
<td>.189</td>
<td>.385</td>
<td>.756</td>
<td>.930</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.060</td>
<td>.178</td>
<td>.373</td>
<td>.757</td>
<td>.930</td>
</tr>
<tr>
<td>2</td>
<td>s.p.</td>
<td>.068</td>
<td>.190</td>
<td>.377</td>
<td>.751</td>
<td>.928</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.068</td>
<td>.195</td>
<td>.396</td>
<td>.771</td>
<td>.935</td>
</tr>
<tr>
<td>3</td>
<td>s.p.</td>
<td>.068</td>
<td>.178</td>
<td>.361</td>
<td>.743</td>
<td>.926</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.068</td>
<td>.194</td>
<td>.394</td>
<td>.770</td>
<td>.935</td>
</tr>
<tr>
<td>5</td>
<td>s.p.</td>
<td>.058</td>
<td>.164</td>
<td>.348</td>
<td>.738</td>
<td>.924</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.058</td>
<td>.175</td>
<td>.369</td>
<td>.754</td>
<td>.930</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.156</td>
<td>.343</td>
<td>.737</td>
<td>.924</td>
</tr>
</tbody>
</table>
Table 28. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 10$, $n_3 = 12$, $a_1 = .25$, $a_2 = a_3 = .05$

<table>
<thead>
<tr>
<th>$Q_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s.p.</td>
<td>.013</td>
<td>.232</td>
<td>.708</td>
<td>.971</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.013</td>
<td>.111</td>
<td>.448</td>
<td>.911</td>
<td>1.000</td>
</tr>
<tr>
<td>1.045</td>
<td>s.p.</td>
<td>.039</td>
<td>.294</td>
<td>.757</td>
<td>.980</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.039</td>
<td>.240</td>
<td>.657</td>
<td>.969</td>
<td>1.000</td>
</tr>
<tr>
<td>1.593</td>
<td>s.p.</td>
<td>.075</td>
<td>.360</td>
<td>.727</td>
<td>.977</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.075</td>
<td>.358</td>
<td>.774</td>
<td>.986</td>
<td>1.000</td>
</tr>
<tr>
<td>2.552</td>
<td>s.p.</td>
<td>.082</td>
<td>.311</td>
<td>.704</td>
<td>.975</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.082</td>
<td>.377</td>
<td>.790</td>
<td>.988</td>
<td>1.000</td>
</tr>
<tr>
<td>8.066</td>
<td>s.p.</td>
<td>.050</td>
<td>.280</td>
<td>.699</td>
<td>.975</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.280</td>
<td>.702</td>
<td>.976</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Table 29: The power of the sometimes pool procedure and the never pool test of the same size, for \( n_1 = 20, n_2 = 10, n_3 = 12, a_1 = .25, a_2 = a_3 = .05 \)

<table>
<thead>
<tr>
<th>0.21</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.066</td>
<td>s.p.</td>
<td>.044</td>
<td>.354</td>
<td>.741</td>
<td>.979</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.044</td>
<td>.259</td>
<td>.679</td>
<td>.973</td>
<td>1.000</td>
</tr>
<tr>
<td>1.708</td>
<td>s.p.</td>
<td>.086</td>
<td>.360</td>
<td>.714</td>
<td>.976</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.086</td>
<td>.389</td>
<td>.799</td>
<td>.988</td>
<td>1.000</td>
</tr>
<tr>
<td>2.914</td>
<td>s.p.</td>
<td>.079</td>
<td>.292</td>
<td>.701</td>
<td>.975</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.079</td>
<td>.369</td>
<td>.784</td>
<td>.987</td>
<td>1.000</td>
</tr>
<tr>
<td>5.399</td>
<td>s.p.</td>
<td>.050</td>
<td>.280</td>
<td>.697</td>
<td>.975</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.050</td>
<td>.280</td>
<td>.702</td>
<td>.976</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Table 30. The power of the borderline sometimes pool procedure and the never pool test of the same size for $n_1 = 20$, $n_2 = 6$, $n_3 = 2$

<table>
<thead>
<tr>
<th>$Q_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>16</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>s.p.</td>
<td>.023</td>
<td>.117</td>
<td>.307</td>
<td>.722</td>
<td>.920</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.023</td>
<td>.087</td>
<td>.232</td>
<td>.646</td>
<td>.891</td>
</tr>
<tr>
<td>1.5</td>
<td>s.p.</td>
<td>.035</td>
<td>.140</td>
<td>.330</td>
<td>.732</td>
<td>.923</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.035</td>
<td>.121</td>
<td>.290</td>
<td>.697</td>
<td>.910</td>
</tr>
<tr>
<td>2</td>
<td>s.p.</td>
<td>.042</td>
<td>.152</td>
<td>.338</td>
<td>.735</td>
<td>.923</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.042</td>
<td>.137</td>
<td>.314</td>
<td>.716</td>
<td>.917</td>
</tr>
<tr>
<td>3</td>
<td>s.p.</td>
<td>.047</td>
<td>.156</td>
<td>.342</td>
<td>.736</td>
<td>.924</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.047</td>
<td>.149</td>
<td>.333</td>
<td>.730</td>
<td>.922</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.050</td>
<td>.156</td>
<td>.343</td>
<td>.737</td>
<td>.924</td>
</tr>
</tbody>
</table>
Table 31. The power of the borderline sometimes pool procedure and the never pool test of the same size for $n_1 = 20$, $n_2 = 10$, $n_3 = 2$

<table>
<thead>
<tr>
<th>$\theta_{21}$</th>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>s.p.</td>
<td>.030</td>
<td>.146</td>
<td>.362</td>
<td>.655</td>
<td>.917</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.030</td>
<td>.127</td>
<td>.321</td>
<td>.615</td>
<td>.904</td>
</tr>
<tr>
<td>1.5</td>
<td>s.p.</td>
<td>.041</td>
<td>.168</td>
<td>.384</td>
<td>.669</td>
<td>.921</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.041</td>
<td>.158</td>
<td>.366</td>
<td>.652</td>
<td>.916</td>
</tr>
<tr>
<td>2.0</td>
<td>s.p.</td>
<td>.046</td>
<td>.175</td>
<td>.390</td>
<td>.673</td>
<td>.922</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.046</td>
<td>.170</td>
<td>.382</td>
<td>.665</td>
<td>.919</td>
</tr>
<tr>
<td>3.0</td>
<td>s.p.</td>
<td>.049</td>
<td>.178</td>
<td>.393</td>
<td>.674</td>
<td>.922</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.049</td>
<td>.177</td>
<td>.391</td>
<td>.672</td>
<td>.921</td>
</tr>
<tr>
<td></td>
<td>n.p.</td>
<td>.05</td>
<td>.179</td>
<td>.393</td>
<td>.674</td>
<td>.922</td>
</tr>
</tbody>
</table>