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On quadratic estimates of variance components

Franklin A. Graybill

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UMI®
ON QUADRATIC ESTIMATES OF VARIANCE COMPONENTS

by

Franklin A. Graybill

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of

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1952
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I. INTRODUCTION AND REVIEW OF THE LITERATURE

A mathematical model frequently used to specify statistical observations is the additive or linear model. For example, if our data are arranged into \( n_a \) distinct A classes and each A class is divided into \( n_b \) distinct B classes each containing \( n_c \) elements, the additive model is

\[
y_{ijk} = \mu + a_i + b_{ij} + c_{ijk}
\]

This is called the two-fold nested classification model. \( y_{ijk} \) is the \( k^{th} \) observation in the \( j^{th} \) B class within the \( i^{th} \) A class. \( \mu \) is a constant which is common to all observations, \( a_i \) is common to all observations in the \( i^{th} \) A class, \( b_{ij} \) is common to all observations in the \( j^{th} \) B class which is within the \( i^{th} \) A class, and \( c_{ijk} \) is an individual term attached to each observation.

The elements \( a_i, b_{ij}, c_{ijk} \) may be classified into one of the following two types: (1) The \( a_i \) and \( b_{ij} \) are constants and \( c_{ijk} \) is a random variable with mean zero and variance \( \sigma_c^2 \). (2) The \( a_i, b_{ij}, \) and \( c_{ijk} \) are independent random variables with mean zero and variances \( \sigma_a^2, \sigma_b^2, \) and \( \sigma_c^2 \) respectively.

In type (1) the problem which concerns us is the estimation of certain linear functions of \( \mu, a_i, \) and \( b_{ij} \), and the estimation of \( \sigma_c^2 \). A solution to this problem is given by the Markoff theorem. While this solution is quite satisfactory for estimating linear functions of the constants \( \mu, a_i, \) and \( b_{ij} \), the only property it claims in estimating \( \sigma_c^2 \) is unbiasedness. The estimate of \( \sigma_c^2 \) given by the Markoff theorem is a quadratic function of the observations, and Hsu (5) has given
conditions for which this estimate has minimum variance in the class of
unbiased quadratic estimates. Hsu has also given conditions for which
the estimate of the variance of the random term has minimum variance
in the class of unbiased quadratic estimates for any linear model which
is the sum of any number of constant components plus a random component.

If we assume a distribution for the $c_{ijk}$, then we can get the
maximum likelihood estimate for $\sigma^2$. However, this method does not tell
us how the variance of this estimate compares with the variance of
estimates obtained by other methods for finite sized samples.

In type (2) the problem is the estimation of the variance components
$\sigma_a^2$, $\sigma_b^2$, $\sigma_c^2$. Gump (2) gives methods for estimating $\sigma_a^2$, $\sigma_b^2$, and
$\sigma_c^2$ by making an analysis of variance table, equating expected to observed
mean squares and using the solutions to these equations as the estimates.
These estimates do not depend on any assumptions concerning the distribu-
tion of the $a_i$, $b_{ij}$, or $c_{ijk}$. These estimates are quadratic functions
of the observations and are unbiased, but very little has been said
about the size of the variance of these estimates relative to estimates
given by other methods of estimation. Hammersly (4) has given explicit
expression for the variance of the estimate of the variance components
for the one-fold classification. Tukey (7) has given an explicit
expression for the variance of the estimate of variance components for
various linear models including the one-fold classification.

Another model frequently used is the cross-classification model
where our data are divided into $n_a$ $A$ classes and $n_b$ $B$ classes according
to the model
\[ Y_{ij} = \mu + a_i + b_j + \epsilon_{ij} \]

This model may also be classified into two types. (1) The \( a_i \) and \( b_j \) are constants and \( \epsilon_{ij} \) is a random variable with mean zero and variance \( \sigma^2 \). (2) The \( a_i \), \( b_j \), and \( \epsilon_{ij} \) are independent random variables with means zero and variances \( \sigma_a^2 \), \( \sigma_b^2 \) and \( \sigma_c^2 \) respectively.

The problem which concerns us in type (1) is the estimation of certain linear functions of \( \mu \), \( a_i \), and \( b_j \), and the estimation of \( \sigma^2 \). Again the Markoff theorem is quite satisfactory but it says nothing about the variance of the estimate of \( \sigma^2 \). However, as noted above, Ney has given conditions under which the estimate of \( \sigma^2 \) given by the Markoff theorem has minimum variance in the class of unbiased quadratic estimators.

The problem which concerns us in type (2) is the estimation of \( \sigma_a^2 \), \( \sigma_b^2 \) and \( \sigma_c^2 \), and some specification of the variance of these estimates. Crump (1) gives a method for estimating these variance components which depends on equating observed and expected mean squares in an analysis of variance table. This method gives quadratic estimates which are unbiased, but nothing is stated as to the size of the sampling variances of these estimates relative to other estimates.

From many points of view it seems desirable to estimate variance components by quadratic functions of the observations. For example, it is possible to examine the variance of specified quadratic functions of the observations, whereas it is extremely difficult to examine the variance of arbitrary functions of the observations. A restriction on the type of estimating functions enables one to make definite statements on
the reliability of the estimating functions without a complete specification of the distributions of the random variables.

The purpose of this thesis is to find, under various linear models, estimates of variance components which are quadratic, unbiased, invariant under translation, and which have less variance than any other estimator with these properties. Also in cases where such estimators are not obtainable we shall establish the variances, relative to some standard, of various methods of estimating variance components.
II. GENERAL NESTED CLASSIFICATION WHEN THE NUMBERS IN THE SUBCLASSES ARE EQUAL.

A. Notation and Definitions

Throughout this thesis we will denote scalar random variables by

\[ y_{i_1i_2\ldots i_k} \text{ and } x_{i_1i_2\ldots i_k} \]

where \( i_j = 1, 2, \ldots, n_j \). Also \( N, P, \) and \( S \) will denote symmetric matrices of dimension \( n \times n_j \times n_j \).

\[ j=1 \quad j=1 \quad j=1 \]

Definition 1. By an ordered vector \( Y' \) (dimension \( 1 \times \prod_{j=1}^{k} n_j \)) we will mean a row vector constructed from the set of scalar random variables \( y_{i_1i_2\ldots i_k} \) in the following way:

(a) Let \( S'_{1i} \) be the row vector consisting of the random variables \( y_{i_1i_2\ldots i_k} \) (arranged in any order) whose first subscript is \( i \). Then form the vector

\[ X_1' = (S'_{11}, S'_{12}, \ldots, S'_{1n_1}) \]

(b) Now choose any \( S'_{1i} \). Let \( S'_{ij} \) be the row vector consisting of all the random variables \( y_{i_1i_2\ldots i_k} \) from \( S'_{1i} \) whose second subscript is \( j \). Then replace \( S'_{1i} \) by

\[ (S'_{1i1}, S'_{1i2}, \ldots, S'_{1in_j}) \]

Do this for each \( S'_{1i} \) in \( X_1' \) and label the resulting vector \( X_2' \).

(c) Now from \( X_2' \) pick any \( S'_{1j} \). Let \( S'_{1je} \) be the row vector consisting of all the random variables \( y_{i_1i_2\ldots i_k} \) in \( S'_{1j} \) whose
third subscript is e. Then replace $S'_{ij}$ by

\[(S'_{i1j}, S'_{i2j}, \ldots, S'_{im_j}) \]

Do this for each $S'_{ij}$ in $X_2'$ and label the resulting vector $X_3'$. 

(d) Continue this process until we arrive at the vector $X_k'$. This is the vector we shall call $Y'$. 

Definition 2. By a \( t \)-ordered sub vector $Y'_{i_1 i_2 \ldots i_t}$ \((t \leq k)\) we will mean a row vector which is constructed from the set of scalar random variables $y_{i_1 i_2 \ldots i_t}$ by choosing the subset consisting of only those random variables whose first $t$ subscripts are $i_1 i_2 i_3 \ldots i_t$ respectively, and from this subset construct an ordered vector as in definition 1. There are obviously $\prod_{j=1}^{t} n_j$ distinct $t$ ordered sub vectors. Their elements are mutually exclusive and they exhaust $Y'$. 

Definition 3. By a first partition of a matrix $M$ we will mean a partition of $M$ into $n_1 \times n_2$ matrix blocks each of dimension $n_j \times n_k$ $j=2$, $i=2$. $M_{p_1 q_1}^{p_2 q_2}$ refers to the first partition block in the $p_1$ row and $q_1$ column of $M$. 

\[
M = \begin{pmatrix}
M_{11}^1 & M_{12}^2 & \ldots & M_{1n_1}^1 \\
M_{21}^1 & M_{22}^2 & \ldots & M_{2n_1}^1 \\
\vdots & \vdots & \ddots & \vdots \\
M_{n_11}^1 & M_{n_12}^2 & \ldots & M_{n_1n_1}^1
\end{pmatrix}
\]

By a second partition of $M$ we will mean a partition of each block of the first partition (of each $M_{q_1 p_1}^{p_2 q_2}$) into $n_2 \times n_2$ blocks each of dimension $n_j \times n_k$. $M_{q_1 q_2}^{p_1 p_2}$ refers to the matrix block in row $p_2$ and column $j=3$, $i=3$. 

q_2 of N(q_1) . By a t partition of N we will mean a partition of each block of the t-1 partition into n_t blocks each of dimension 
\[ P_1 \ldots P_t \]
\[ q_1 q_2 \ldots q_t \]
\[ j=t+1 \]
\[ j=t+1 \]
\[ \begin{array}{cccc}
 1 & 2 & \cdots & n_j \\
 n_j & n_j & \cdots & n_j \\
 \end{array} \]

M(P_{j+1}^{P_1 P_2 \cdots P_{t-1}}) refers to the matrix block in row j+1 and column q_t of M(P_{j+1}^{P_1 P_2 \cdots P_{t-1}}) which in turn refers to the matrix block in row q_{t-1} and column q_t of M(q_1 q_2 \cdots q_{t-2}) and so forth. If we are interested in referring only to a general block in the t th partition we will write this as N(q_1 q_2 \cdots q_{t-1}) . Obviously N(q_1 q_2 \cdots q_{t-1}) is a scalar.

Also \[ N(q_1 q_2 \cdots q_t) = N(P_1 P_2 \cdots P_t) \]

\[ Y' MY \]

can also be written as

\[ \sum_{q_1} \sum_{P_1} Y_{P_1}^T N(q_1) Y_{q_1} \]

(1.2)

or more generally as

\[ \sum_{i=1}^t \sum_{p_1}^{n_i} \sum_{q_1}^{n_i} Y_{P_1 P_2 \cdots P_t}^T N(q_1 q_2 \cdots q_t) Y_{q_1 q_2 \cdots q_t} \]

(1.3)

This will be shortened and written as

\[ \sum_{p_1 \cdots q_1}^{p_1 \cdots} \sum_{q_1 \cdots q_t}^{q_1 \cdots t} Y_{p_1 \cdots t}^T N(q_t) Y_{q_t} \]

(1.4)

where p_1 \cdots t means sum over p_1 = 1,\ldots,n_1; p_2 = 1,\ldots,n_2; \ldots; p_t = 1,\ldots,n_t

q_1 \cdots t means sum over q_1 = 1,\ldots,n_1; q_2 = 1,\ldots,n_2; \ldots; q_t = 1,\ldots,n_t

Now the first row of summation indication (ie p_1 \cdots t) will always be as given, but if we want to sum p_j = q_j we will replace the j opposite
q with a zero. If we wish to sum \( p_j \neq q_j \) we will replace the \( j \) opposite \( q \) with a theta. For example \( p_{1, 2, 3} \) \( q_{0, 0, 0} \) means the range of summation is \( p_1 = q_1 = 1, 2, \ldots n_1; p_2 \neq q_2 = 1, 2, \ldots n_2; p_3, q_3 = 1, 2, \ldots n_3 \). As an example consider \( Y' = Y \)

\[
\begin{pmatrix}
Y_{111} \\
Y_{112} \\
Y_{121} \\
Y_{122} \\
Y_{211} \\
Y_{212} \\
Y_{221} \\
Y_{222}
\end{pmatrix}
= 
\begin{pmatrix}
\begin{array}{cccc}
11 & 12 \\
21 & 22 \\
31 & 32 \\
41 & 42 \\
51 & 52 \\
61 & 62 \\
71 & 72 \\
81 & 82 \\
\end{array}
& \begin{array}{cccc}
13 & 14 \\
23 & 24 \\
33 & 34 \\
43 & 44 \\
53 & 54 \\
63 & 64 \\
73 & 74 \\
83 & 84 \\
\end{array}
& \begin{array}{cccc}
15 & 16 \\
25 & 26 \\
35 & 36 \\
45 & 46 \\
55 & 56 \\
65 & 66 \\
75 & 76 \\
85 & 86 \\
\end{array}
& \begin{array}{cccc}
17 & 18 \\
27 & 28 \\
37 & 38 \\
47 & 48 \\
57 & 58 \\
67 & 68 \\
77 & 78 \\
87 & 88 \\
\end{array}
\end{pmatrix}
\]

Now

\[
\sum_{i_1 = 1}^{n_1} \sum_{j_1 = 1}^{n_1} Y'_{i_1 j_1} M_{i_1 j_1} = \sum_{i_2 = 1}^{n_2} \sum_{j_2 = 1}^{n_2} Y'_{i_2 j_2} M_{i_2 j_2}
\]

where the summation refers to \( i_1 = j_1 = 1, 2; i_2 \neq j_2 = 1, 2 \) since \( n_1 = n_2 = n_3 = 2 \). This gives

\[
\begin{align*}
Y'_{111} & \begin{pmatrix}
13 & 14 \\
23 & 24 \\
31 & 32 \\
41 & 42 \\
51 & 52 \\
61 & 62 \\
71 & 72 \\
81 & 82 \\
\end{pmatrix}
\quad & \begin{pmatrix}
Y_{121} \\
Y_{122} \\
Y_{211} \\
Y_{212} \\
Y_{221} \\
Y_{222}
\end{pmatrix}
= \\
Y_{112} & \begin{pmatrix}
15 & 16 \\
25 & 26 \\
35 & 36 \\
45 & 46 \\
55 & 56 \\
65 & 66 \\
75 & 76 \\
85 & 86 \\
\end{pmatrix}
\quad & \begin{pmatrix}
Y_{111} \\
Y_{112} \\
Y_{121} \\
Y_{122} \\
Y_{211} \\
Y_{212} \\
Y_{221} \\
Y_{222}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
& + \begin{pmatrix}
Y_{211} & \\
Y_{221} & \\
Y_{212} & \\
Y_{222} & \\
\end{pmatrix}
\begin{pmatrix}
75 & 76 \\
65 & 66 \\
57 & 58 \\
47 & 48 \\
\end{pmatrix}
\quad & \begin{pmatrix}
Y_{121} \\
Y_{122} \\
Y_{211} \\
Y_{212} \\
Y_{221} \\
Y_{222}
\end{pmatrix}
\end{align*}
\]

\[
= Y'_{11} M_{11} Y_{12} + Y'_{12} M_{12} Y_{11} + Y'_{21} M_{21} Y_{22} + Y'_{22} M_{22} Y_{21}
\]
Definition 4. \( \mathbf{a} \) will denote a column vector with each element equal to unity. The dimension will generally be obvious from its position in the discussion.

Definition 5. We will define \( \mathbf{N}_t = a_{t_1} a_{t_2} \ldots a_{t_k} \) if \( t \leq k \)
\[ \mathbf{N}_t = 1 \] if \( t > k \)

Definition 6. We will now give the definition of Model I. Let the random variable \( Y_{1_12 \ldots 1_k} \) be given as

\[ Y_{1_12 \ldots 1_k} = \mu + a_{1_1}^{(1)} + a_{1_12}^{(2)} + a_{1_121_3}^{(3)} + \ldots + a_{1_12 \ldots 1_k}^{(k)} \] (1.5)

where \( \mu \) is a constant and the \( a_i \)'s are independent random variables with the following properties:

(a) \( \mathbb{E} a_{1_12 \ldots 1_t}^{(t)} = 0 \)

(b) Variance \( \text{Var} a_{1_12 \ldots 1_t}^{(t)} = \sigma_t^2 \)

(c) \( \mathbb{E} \left[ a_{1_12 \ldots 1_t}^{(t)} \right]^4 = \nu_{4t} < \infty \)

This is the general balanced, nested or hierarchical classification and will be referred to as Model I. Now the \( t \) ordered sub vector

\[ Y_{1_12 \ldots 1_t} = \mathbf{a} \mu + A_{1_t}^{(1)} + A_{1_t}^{(2)} + \ldots + A_{1_t}^{(k)} \] (1.7)

where \( A_{1_t}^{(p)} \) is a vector consisting of all the elements \( a_{1_12 \ldots 1_p}^{(p)} \) in the same order as they occur in \( Y_{1_12 \ldots 1_t} \). Obviously \( A_{1_t}^{(p)} = a_{1_12 \ldots 1_p}^{(p)} \). if \( p \leq t \). If we consider the \( a_i \)'s as fixed constants and run the conventional analysis of variance and equate expected to observed mean squares, we get upon solution for \( \sigma_t^2 \) (\( t < k \)): 
\[ s_t = \sum_{i_1 i_2 \ldots i_t} \left( \frac{y_{i_1} y_{i_2} \ldots y_{i_t}}{H_t (n_t - 1) H_{t+1}} \right) - \sum_{i_1 i_2 \ldots i_{t+1}} \left( \frac{y_{i_1} y_{i_2} \ldots y_{i_{t+1}}}{H_t (n_{t+1} - 1) H_{t+1}} \right) \]

where \( y_{i_1 i_2 \ldots i_p} \ldots \) is equal to \( \sum_{i_k \ldots i_{p+1}} \frac{y_{i_1 i_2 \ldots i_k}}{H_{p+1}} \).

**Definition 7.** \( s_t \) will be called the analysis of variance estimate of \( s_t \) for \( t < k \).

\( s_t \) is a quadratic function of the observations and hence can be written as

\[ s_t = Y' SY \]

where \( S \) is as follows:

(a) \( S(i_1 i_2 \ldots i_{t-1} i_t) = \frac{1}{H_t H_{t+1} (n_t - 1)} \phi \phi' \)

if \( i_p = j_p \) for \( p = 1, 2, \ldots , t - 1; \ i_t \neq j_t \).

\( \phi \) is of dimension \( n \times n_j \times 1 \).

(b) \( S(i_1 i_2 \ldots i_t i_{t+1}) = \frac{1}{H_t H_{t+1} (n_{t+1} - 1)} \phi \phi' \)

if \( i_p = j_p \) for \( p = 1, 2, \ldots , t ; \ i_{t+1} \neq j_{t+1} \).

\( \phi \) is of dimension \( k \times n_j \times 1 \) if \( j = t + 2 \).
(c) All other blocks are identically equal to zero.

Definition 6. S will be called the analysis of variance matrix for estimating \( \sigma^2_t \) \((t < k)\).

Definition 9. By the best quadratic unbiased estimate of \( \sigma^2_t \) which is independent of \( \mu \) and of \( \sigma^2_j \) \((j=1,2,\ldots,t-1)\) we will mean a quadratic form \( Q_t \) which satisfies the following:

(a) \( \mathbb{E} Q_t = \sigma^2_t \) is unbiased

(b) \( \text{Var} Q_t \) independent of \( \mu \) and \( \sigma^2_j \) \((j=1,2,\ldots,t-1)\)

(c) \( \text{Var} Q_t \leq \text{Var} Q_t^2 \) where \( Q_t^2 \) is any other quadratic form which satisfies (a) and (b).

B. The Main Theorem

Theorem: Let the scalar random variable \( Y_{i_1 i_2 \ldots i_k} \) be given by

Model I. The estimate of \( \sigma^2_t \) \((t=1,2,\ldots,k-1)\) which has the properties as in definition 9 is given by the analysis of variance.

The proof requires several pages. Let \( Q_t \) be the best unbiased quadratic estimate of \( \sigma^2_t \) \((t < k)\) that is

\[
Q_t = Y' M Y
\]

where \( M \) is any arbitrary symmetric matrix subject only to conditions (a), (b) and (c) of definition 9. We must give explicit expression to each element of \( M \). By 1.4 we can write 2.1 as

\[
Q_t = \sum_{i_1 i_2 \ldots i_{t-1}} Y'_{i_{t-1}} M_{i_{t-1} j_{t-1}} Y_{j_{t-1}}
\]

(2.2)

which by 1.7 becomes
\[
\sum_{1 \leq j_1 \leq \ldots \leq j_t \leq \ldots \leq \ldots \leq j_{t-1} \leq \ldots \leq \ldots \leq j_{t-1}} \left[ (\mu_1^{(1)} + \ldots + \mu_1^{(k)}) J_{j_{t-1}}^{(1)} (\mu_2^{(1)} + \ldots + \mu_2^{(k)}) \right] \frac{t_{j_{t-1}}^{(1)}}{t_{j_{t-1}}^{(1)}} \frac{t_{j_{t-1}}^{(2)}}{t_{j_{t-1}}^{(2)}} \ldots \frac{t_{j_{t-1}}^{(t-1)}}{t_{j_{t-1}}^{(t-1)}} \frac{t_{j_{t-1}}^{(t)}}{t_{j_{t-1}}^{(t)}} \frac{t_{j_{t-1}}^{(t+1)}}{t_{j_{t-1}}^{(t+1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \right] (2.3)
\]

\[
= \sum_{1 \leq j_1 \leq \ldots \leq j_t \leq \ldots \leq \ldots \leq j_{t-1} \leq \ldots \leq \ldots \leq j_{t-1}} \left[ (\mu_1^{(1)} + \ldots + \mu_1^{(k)}) J_{j_{t-1}}^{(1)} (\mu_2^{(1)} + \ldots + \mu_2^{(k)}) \right] \frac{t_{j_{t-1}}^{(1)}}{t_{j_{t-1}}^{(1)}} \frac{t_{j_{t-1}}^{(2)}}{t_{j_{t-1}}^{(2)}} \ldots \frac{t_{j_{t-1}}^{(t-1)}}{t_{j_{t-1}}^{(t-1)}} \frac{t_{j_{t-1}}^{(t)}}{t_{j_{t-1}}^{(t)}} \frac{t_{j_{t-1}}^{(t+1)}}{t_{j_{t-1}}^{(t+1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \right] (2.4)
\]

\[
= \sum_{1 \leq j_1 \leq \ldots \leq j_t \leq \ldots \leq \ldots \leq j_{t-1} \leq \ldots \leq \ldots \leq j_{t-1}} \left[ (\mu_1^{(1)} + \ldots + \mu_1^{(k)}) J_{j_{t-1}}^{(1)} (\mu_2^{(1)} + \ldots + \mu_2^{(k)}) \right] \frac{t_{j_{t-1}}^{(1)}}{t_{j_{t-1}}^{(1)}} \frac{t_{j_{t-1}}^{(2)}}{t_{j_{t-1}}^{(2)}} \ldots \frac{t_{j_{t-1}}^{(t-1)}}{t_{j_{t-1}}^{(t-1)}} \frac{t_{j_{t-1}}^{(t)}}{t_{j_{t-1}}^{(t)}} \frac{t_{j_{t-1}}^{(t+1)}}{t_{j_{t-1}}^{(t+1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \right] (2.5)
\]

Now the var \( Q_t = E [Q_t^2] - (E Q_t)^2 = E Q_t^2 - \sigma_t^2 \) by condition 2.9(a).

Now the only terms in var \( Q_t \) involving \( \sigma_{t-1} \) are

\[
= \sum_{1 \leq j_1 \leq \ldots \leq j_t \leq \ldots \leq \ldots \leq j_{t-1} \leq \ldots \leq \ldots \leq j_{t-1}} \left[ (\mu_1^{(1)} + \ldots + \mu_1^{(k)}) J_{j_{t-1}}^{(1)} (\mu_2^{(1)} + \ldots + \mu_2^{(k)}) \right] \frac{t_{j_{t-1}}^{(1)}}{t_{j_{t-1}}^{(1)}} \frac{t_{j_{t-1}}^{(2)}}{t_{j_{t-1}}^{(2)}} \ldots \frac{t_{j_{t-1}}^{(t-1)}}{t_{j_{t-1}}^{(t-1)}} \frac{t_{j_{t-1}}^{(t)}}{t_{j_{t-1}}^{(t)}} \frac{t_{j_{t-1}}^{(t+1)}}{t_{j_{t-1}}^{(t+1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \right] (2.5)
\]

But it will be shown by 2.16 (with \( p = k \)) that for unbiasedness we must have

\[
E \sum_{1 \leq j_1 \leq \ldots \leq j_t \leq \ldots \leq \ldots \leq j_{t-1} \leq \ldots \leq \ldots \leq j_{t-1}} (\mu_1^{(1)} + \ldots + \mu_1^{(k)}) J_{j_{t-1}}^{(1)} (\mu_2^{(1)} + \ldots + \mu_2^{(k)}) \frac{t_{j_{t-1}}^{(1)}}{t_{j_{t-1}}^{(1)}} \frac{t_{j_{t-1}}^{(2)}}{t_{j_{t-1}}^{(2)}} \ldots \frac{t_{j_{t-1}}^{(t-1)}}{t_{j_{t-1}}^{(t-1)}} \frac{t_{j_{t-1}}^{(t)}}{t_{j_{t-1}}^{(t)}} \frac{t_{j_{t-1}}^{(t+1)}}{t_{j_{t-1}}^{(t+1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \ldots \frac{t_{j_{t-1}}^{(t+k-1)}}{t_{j_{t-1}}^{(t+k-1)}} \right] (2.6)
\]

Therefore the second term of 2.5 is zero. We will now consider the first term of 2.5 which upon expansion becomes
\[
\sum_{j \leq \ldots \leq t-1} (A^{(t-1)}_{i_j}) \cdot M^{(t-1)}_{j_{t-1}} (A^{(t-1)}_{j_{t-1}}) \cdot \sum_{p \leq \ldots \leq t-1} (A^{(t-1)}_{k_p}) M^{(t-1)}_{p_{t-1}} (A^{(t-1)}_{p_{t-1}}) \\
= k \sum_{i_1 \leq \ldots \leq t-1} \sum_{p_1 \leq \ldots \leq t-1} \sum_{q_1 \leq \ldots \leq t-1} a^{(t-1)}_{i_1} a^{(t-1)}_{p_1} a^{(t-1)}_{q_1} M^{(t-1)}_{j_{t-1}} M^{(t-1)}_{j_{t-1}} M^{(t-1)}_{j_{t-1}} \phi \\
\text{Now } E a^{(t-1)}_{i_1} a^{(t-1)}_{p_1} a^{(t-1)}_{q_1} = 0 \text{ unless } i_j = p_j \text{ for all } j = 1, 2, \ldots, t-1 \\
= \sigma^2_{t-1} \text{ if } i_j = p_j \text{ for all } j = 1, 2, \ldots, t-1 . \\
\text{Also} \\
E (A^{(k)}_{i_t}) (A^{(k)}_{j_t}) = \text{the zero matrix unless } i_j = q_j, \text{ for } j = 1, 2, \ldots, t-1 \\
= \sigma^2_k \text{ I if } i_j = q_j \text{ for all } j = 1, 2, \ldots, t-1 . \\
\text{So, 2.5 becomes} \\
k \sigma^2_{t-1} \sigma^2_k \sum_{i_1 \leq \ldots \leq t-1} \sum_{j_1 \leq \ldots \leq t-1} \phi^{(t-1)}_{M_{i_{t-1}}} \phi^{(t-1)}_{M_{j_{t-1}}} . \\
(2.8)
\text{But by condition 9(b) the coefficient of } \sigma^2_{t-1} \sigma^2_k \text{ in 2.8 must be} \\
\text{identically zero. But this is the sum of a sum of squares. Hence each} \\
\text{element under the summation sign must be identically zero. That is} \\
\text{to say} \\
\phi^{(P_{t-1})} = \phi^{(P_{q_1 q_2 \ldots q_{t-1}})} = \text{zero vector for every combination of} \\
(2.9) \\
p_1, q_i = 1, 2, \ldots, n_i; i = 1, 2, \ldots, t-1 \\
\text{and}
\( \mathbf{N}(p_1 p_2 \cdots p_{t-1}) \phi = \text{zero vector for the same range of its subscripts.} \) (2.10)

Making these substitutions into the first two terms of 2.4 we get

\[
Q_t = \sum_{i12\ldots t-1} (\lambda_1^{(i)} + \ldots + \lambda_k^{(i)}) \mathbf{N}(i_{t-1}) (\lambda_1^{(j)} + \ldots + \lambda_k^{(j)})
\]

\[j12\ldots t-1 \]

(2.11)

This can also be written

\[
Q_t = \sum_{i12\ldots s} (\lambda_1^{(i)} + \ldots + \lambda_k^{(i)}) \mathbf{N}(i_s) (\lambda_1^{(j)} + \ldots + \lambda_k^{(j)})
\]

for \( s = 1, 2, 3, \ldots k \) .

(2.12)

Now 2.9 gives

\[
\phi \mathbf{N}(p_1 p_2 \cdots p_s) = \text{zero vector for } s = 1, 2, \ldots t-1.
\]

(2.13)

Similarly 2.10 gives

\[
\mathbf{N}(p_1 p_2 \cdots p_s) \phi = \text{zero vector for } s = 1, 2, \ldots t-1.
\]

(2.14)

Now for condition D.9(a), i.e. unbiasedness we must have

\[
\mathbb{E} Q_t = \sigma^2_t
\]

Using 2.3 this becomes

\[
\mathbb{E} \sum_{p, q=1}^k \sum_{i12\ldots s} \lambda_1^{(p)} \mathbf{N}(i_s) \lambda_1^{(q)} = \sigma^2_t
\]

(2.15)

which may be written

\[
\mathbb{E} \sum_{p=1}^k \sum_{i12\ldots p} \lambda_1^{(p)} \mathbf{N}(i_p) \lambda_1^{(p)} = \sigma^2_t
\]
\[
\begin{align*}
&= \sum_{j=0}^{\infty} A_j(t) A_j(t) + \sum_{p=1}^{k} \sum_{p \neq t} \sum_{j=0}^{\infty} A_j(p) A_j(p)A_j(p) = c^t_t \\
&= c^t_t \sum_{j=0}^{\infty} \delta_j M(j, t) \delta + \sum_{p=1}^{k} c^t_p \sum_{p \neq t} \sum_{j=0}^{\infty} \delta_j M(j, p) \delta = c^t_t.
\end{align*}
\]

So we must have
\[
\sum_{j=0}^{\infty} \delta_j M(j, t) \delta = 1 \quad (2.17)
\]
\[
\sum_{p=1}^{k} c^t_p \sum_{p \neq t} \sum_{j=0}^{\infty} \delta_j M(j, p) \delta = 0 \quad \text{for } p = 1, 2, \ldots, k; \ s \neq t \quad (2.18)
\]

Let us set
\[
M = P + S \quad (2.19)
\]
where \(M\) is an arbitrary matrix such that \(Q_t\) satisfies D.9, \(S\) is the matrix given in D.7, and \(P\) is then given by 2.19. The problem is to give explicit expression to each element of \(P + S\) (of \(M\)). Since \(S\) is known, the problem can be reduced to finding a matrix \(P\) such that \(P + S = M\) satisfies D.9. We shall now find the restrictions that 2.17 and 2.18 imposes on \(P\). From 2.17 we get
\[
\sum_{j=0}^{\infty} \delta_j M(j, t) \delta = \sum_{j=0}^{\infty} \delta_j \left[ S(j, t) + P(j, t) \right] \delta = 1
\]
\[ \sum_{j_0 \ldots t} \phi \cdot S(j_t) \phi + \sum_{j_0 \ldots t} \phi \cdot P(j_t) \phi = 1 \]

\[ \sum_{j_0 \ldots t} \phi \cdot S(j_{t+1}) \phi + \sum_{j_0 \ldots t} \phi \cdot P(j_t) \phi = 1 \]

Using D.7 this becomes

\[ \sum_{j_0 \ldots t} \phi \cdot \left[ \phi \cdot \frac{1}{N_1 N_{t+2} (n_0 - 1)} \right] \phi + \sum_{j_0 \ldots t} \phi \cdot P(j_t) \phi = 1 \]

\[ \frac{N_{t+2} (n_0 - 1)}{N_1 N_{t+2} (n_0 - 1)} \sum_{j_0 \ldots t} \phi \cdot P(j_t) \phi = 1 \]

\[ = 1 + \sum_{j_0 \ldots t} \phi \cdot P(j_t) \phi = 1 \]

So 2.17 imposes the following condition on P

\[ \sum_{j_0 \ldots t} \phi \cdot P(j_t) \phi = 0 . \]  \hspace{1cm} (2.20)

2.18 becomes, for the appropriate \( s \)'s,

\[ \sum_{j_0 \ldots s} \phi \cdot P(j_t) \phi = \sum_{j_0 \ldots s} \phi \cdot P(j_t) \phi + \sum_{j_0 \ldots s} \phi \cdot S(j_t) \phi = 0 . \]

\[ s = t+1, \ldots, k \]
But from D.7 we get
\[ s_{i_1 i_2 \ldots i_{t+1}} = 0 \quad \text{for} \quad i_q = j_q \quad q=1, 2, \ldots, t+1 \]

So 2.18 becomes
\[ \sum_{i \leq \ldots \leq s} \phi P_j^i \phi = 0 \quad \text{for} \quad s = t+1, t+2, \ldots, k \quad (2.21) \]

Also from 2.13 we get
\[ \sum_{i \leq \ldots \leq s} \phi M_j^i \phi = \sum_{i \leq \ldots \leq s} \phi P_j^i \phi + \sum_{i \leq \ldots \leq s} \phi S_j^i \phi = 0 \quad \text{for} \quad s = 1, 2, \ldots, t-1 \quad (2.22) \]

We shall now show that
\[ \sum_{i \leq \ldots \leq s} \phi S_j^i \phi = 0 \quad \text{if} \quad s = 1, 2, \ldots, t-1 \quad (2.23) \]

To verify 2.23 we note that \( \phi S_{j_{t-1}}^i \phi = 0 \) (by D.7) and that the \( s \) partitions \( (s < t-1) \) are made up of blocks of the \( t-1 \) partition.

So from 2.20, 2.21, 2.22, and 2.23 we get
\[ \sum_{i \leq \ldots \leq s} \phi P_j^i \phi = 0 \quad \text{for} \quad s = t, t+1, \ldots, k. \quad (2.24) \]

and
\[ \phi P_j^i \phi = 0 \quad \text{for} \quad s = 1, 2, \ldots, t-1. \quad (2.25) \]

From 2.24 we get
\[ \sum_{i \leq \ldots \leq s} \phi P_j^i \phi = \sum_{i \leq \ldots \leq s, s+1} \phi P_j^{i+1} \phi \]
\[
\sum_{j00...0} \sum_{j12...s} \phi'_{j_{s+1}} \phi + \sum_{j00...0} \sum_{j12...s} \phi'_{j_{s+1}} \phi \\
= \sum_{j12...s} \phi'_{j_{s+1}} \phi = 0 \text{ for } z = t-1, t, \ldots, y, y = t-1, \ldots, k \quad (2.26)
\]

Now let us consider \( \mathbb{E} Q_t^s \). From 2.19 we get

\[
\mathbb{E} Q_t^s = \mathbb{E} \left[ \sum_{j12...s} \left( \sum_{p=t}^{k} A_{j_z}^{(p)} \right) \left( P_{j_z}^{(s)} + S_{j_z}^{(s)} \right) \left( \sum_{q=t}^{k} A_{j_z}^{(q)} \right) \right]^2 
\]

\[
= \mathbb{E} \left[ \sum_{j12...s} \left( \sum_{p=t}^{k} A_{j_z}^{(p)} \right) \phi'_{j_{s+1}} \left( \sum_{q=t}^{k} A_{j_z}^{(q)} \right) \right]^2 
\]

\[
+ \mathbb{E} \left[ \sum_{j12...y} \left( \sum_{p=t}^{k} A_{j_y}^{(p)} \right) S_{j_y}^{(s)} \left( \sum_{q=t}^{k} A_{j_y}^{(q)} \right) \right]^2 
\]

\[
+ 2 \mathbb{E} \sum_{j12...s} \sum_{j12...y} \sum_{p,q,r,s=t}^{k} (A_{j_z}^{(p)} P_{j_z}^{(s)} A_{j_z}^{(q)} A_{j_y}^{(r)} S_{j_y}^{(s)} A_{j_y}^{(s)}) \quad (2.27)
\]

\[
= I + Y + T 
\]

We will examine the last term, \( T \), of 2.27 in detail. Now \( p, q, r, s=t, t+1, \ldots, k \) and it is clear that if any one of these subscripts is distinct from the other three then the last term of 2.27 becomes zero. So we need to consider only four cases:

1. \( p = q \neq r = s \)
2. \( p = r \neq q = s \)
(3) \( p = s \neq q = r \)  
(4) \( p = q = r = s \).

Let us consider these in turn. For case (1) \( T \) is composed of terms such as

\[
\sum_{i_1 \leq \ldots \leq s, j_1 \leq \ldots \leq s} (A^{(P)}_{i_1}) (J^{(S)}_{j_1}) (A^{(P)}_{i_s}) \sum_{v_1 \leq \ldots \leq y, m_1 \leq \ldots \leq y} (A^{(R)}_{v_1}) (S^{(Y)}_{m_1}) (A^{(R)}_{v_y}) p \neq r = t, t+1, \ldots, k
\]

\[
= \sum_{i_1 \leq \ldots \leq p, j_1 \leq \ldots \leq p} \sum_{v_1 \leq \ldots \leq r, m_1 \leq \ldots \leq r} \phi^{(P)}_{i_1} \phi^{(S)}_{j_1} \phi^{(P)}_{i_p} \phi^{(S)}_{j_p} \phi^{(R)}_{v_1} \phi^{(Y)}_{m_1} \phi^{(R)}_{v_r} \phi^{(Y)}_{m_r}
\]

but \( p \) must be greater than or equal to \( t \). So from 2.24 this expression becomes zero. So for case (1) of 2.25 \( T = 0 \).

Now let us consider case (2). For this assumption \( T \) is composed of terms such as

\[
\sum_{i_1 \leq \ldots \leq s, j_1 \leq \ldots \leq s} (A^{(P)}_{i_1}) (J^{(S)}_{j_1}) (A^{(Q)}_{i_s}) \sum_{v_1 \leq \ldots \leq y, m_1 \leq \ldots \leq y} (A^{(P)}_{v_1}) (S^{(Y)}_{m_1}) (A^{(Q)}_{v_y}) p \neq q = t, t+1, \ldots, k
\]

Without loss of generality we can assume \( q > p \). So putting \( s = y = q \) this becomes

\[
\sum_{v_1 \leq \ldots \leq q} \sum_{i_1 \leq \ldots \leq q} \sum_{v_1 \leq \ldots \leq q} \sum_{j_1 \leq \ldots \leq q} \sum_{m_1 \leq \ldots \leq q} \phi^{(P)}_{i_1} \phi^{(S)}_{j_1} \phi^{(P)}_{i_q} \phi^{(S)}_{j_q} \phi^{(R)}_{v_1} \phi^{(Y)}_{m_1} \phi^{(R)}_{v_q} \phi^{(Y)}_{m_q}
\]
\[
= \sigma_p \sigma_q \sum_{12 \ldots p, p+1 \ldots q} \sum_{00 \ldots 0, p+1 \ldots q} \phi p(j_q^q) \phi \cdot \phi s(j_q^q) \phi
\]

\[\text{(2.29)}\]

This can be written as

\[
\sigma_p \sigma_q \sum \phi p(j_q^q) \phi \cdot \phi s(j_q^q) \phi \quad \text{for } q > p = t, t+1, \ldots k \quad (2.30)
\]

where the summation is

\[
i_s = v_s = 1,2, \ldots n_s \quad (s = 1,2, \ldots p) \\
j_s = 1,2, \ldots n_s \quad (s = 1,2, \ldots q) \\
i_s, v_s = 1,2, \ldots n_s \quad (s = p+1, \ldots q)
\]

(2.31)

But by (2.7), (2.29) is zero unless \( v_s = v_s \) \( (s = 1,2, \ldots t-1) \), so (2.29) becomes equal to (2.30) where the range of summation is

\[
i_s = v_s = j_s = 1,2, \ldots n_s \quad (s = 1,2, \ldots t-1) \\
i_s = v_s = 1,2, \ldots n_s \quad (s = t, t+1, \ldots p) \\
j_s = 1,2, \ldots n_s \quad (s = t, t+1, \ldots q) \\
i_s, v_s = 1,2, \ldots n_s \quad (s = p+1, \ldots q)
\]

(2.32)

This range of summation is equivalent to

\[
i_s = v_s = 1,2, \ldots n_s \quad (s = 1,2, \ldots p) \\
j_s = v_s = 1,2, \ldots n_s \quad (s = 1,2, \ldots t-1) \\
j_s \neq v_s = 1,2, \ldots n_s \quad (s = t+1, \ldots q) \\
j_s = 1,2, \ldots n_s \quad (s = t+1, \ldots q) \\
i_s, v_s = 1,2, \ldots n_s \quad (s = p+1, \ldots q)
\]

(2.33)

plus
\[ \begin{align*}
    i_s &= v_s = 1, 2, \ldots, n_s \\
    j_s &= v_s = 1, 2, \ldots, n_s \\
    j_s &= 1, 2, \ldots, n_s \\
    i_s, v_s &= 1, 2, \ldots, n_s \\
\end{align*} \] (s = 1, 2, \ldots, p) \\
\[ \begin{align*}
    j_s &= 1, 2, \ldots, n_s \\
    j_s &= 1, 2, \ldots, n_s \\
    j_s &= v_s = 1, 2, \ldots, n_s \\
    j_{t+1} &= v_{t+1} = 1, 2, \ldots, n_{t+1} \\
    j_s &= 1, 2, \ldots, n_s \\
    i_s, v_s &= 1, 2, \ldots, n_s \\
\end{align*} \] (s = t+2, \ldots, q) \quad (2.34) \\
plus \\
\[ \begin{align*}
    i_s &= v_s = 1, 2, \ldots, n_s \\
    j_s &= v_s = 1, 2, \ldots, n_s \\
    j_{t+1} &= v_{t+1} = 1, 2, \ldots, n_{t+1} \\
    j_s &= 1, 2, \ldots, n_s \\
    i_s, v_s &= 1, 2, \ldots, n_s \\
\end{align*} \] (s = p+1, \ldots, q) \quad (2.35) \\

Now from 2.34 we note that the range of summation is \( v_s = j_s \) s = 1, 2, \ldots, t+1. But from D.7 we get

\[ S(j_1^2 \cdots j_{t+1}^2, j_{t+2} \cdots j_q) = 0 \]

So 2.29 is equivalent to 2.30 summed over the range defined by 2.33

plus the summation over 2.35.

Let us consider the range defined by 2.35. We see that

\( v_s = j_s \) s = 1, 2, \ldots, t; \( v_{t+1} \neq j_{t+1} \); \( v_s \) independent for s = t+2, \ldots, q.

For every combination of these subscripts we have from D.7

\[ \phi ^ s (j_q) \phi = \text{constant} \]

Also

\[ \sum \phi ^ s (j_q) \phi = \text{constant} = c_1 \]

for \( v_s, j_s = 1, 2, \ldots, n_s \); s = p+1, \ldots, q. So when 2.30 is summed over 2.35 we get
\[
\sum_{i12...t, t+1, \ldots q} c_1^{n t} q^{n q} \sum_{j00...t+1, \ldots q} \phi^i_{P(j)} q^j = 0 \]

But this is zero by 2.24.

Finally let us consider 2.30 when summed over the range defined by 2.33. Since \( v_{s} = j_{s} \) \( s = 1, 2, \ldots t-1 \); \( v_{t} \neq j_{t} \) and \( v_{s}, j_{s} \) independent for \( s = t+1, \ldots q \) we have from 2.7

\[
\sum_{j_{q}} \phi^{s}_{P(j)} q = \text{constant}
\]

and

\[
\sum_{j_{q}} \phi^{s}_{P(j)} q = \text{constant} = c_{2}
\]

for these subscripts. So when 2.30 is summed over 2.33 we get

\[
\sum_{i12...t-1, t, \ldots q} c_1^{n t} q^{n q} \sum_{j00...t+1, \ldots q} \phi^i_{P(j)} q^j = 0
\]

But this is zero by 2.24. So we have shown that 2.29 is zero and hence \( T = 0 \) for case (2) of 2.28. Now \( T = 0 \) for case (3) due to the symmetry with case (2).

Now let us consider case (4). Under this assumption \( T \) is composed of terms such as

\[
\sum_{i12...u} \sum_{i12...y} \sum_{j00...x} \sum_{j00...y} \sum_{p=t+1, \ldots k} x_{i_{12...u}} x_{j_{00...x}} x_{p} \phi^i_{P(j)} q^j s_{p} = 0
\]

(2.36)
This is obviously zero if any one or more subscripts on any a is distinct from the corresponding subscripts of the other three a's. So we are left with only four cases to consider

(1) \( i_v = j_w \); \( v = 1,2,\ldots \ p \); \( v \neq v_w \) for some \( w \).

(2) \( i_v = v_w \); \( j_w = v_w \); \( v = 1,2,\ldots \ p \); \( w \neq j_w \) for some \( w \).

(3) \( i_v = v_w \); \( j_w = v_w \); \( v = 1,2,\ldots \ p \); \( w \neq j_w \) for some \( w \). \hspace{1cm} (2.37)

(4) \( i_v = j_w = v_w = m_v \); \( v = 1,2,\ldots \ p \).

We will consider these 4 cases in turn. For case (1) (2.36) becomes

\[
\sigma_p^h \sum_{v_1, \ldots, v_p}^{i_v} \phi P(i_p) \phi \cdot \phi S(v_p) \phi \text{ where } i_v \neq v_w \text{ for at least one value of } v, p = t, t+1, \ldots, k
\]

This can be written

\[
\sigma_p^h \sum_{v_1, \ldots, v_p}^{i_v} \phi P(i_p) \phi \cdot \phi S(v_p) \phi
\]

\[
- \sigma_p^h \sum_{v_1, \ldots, v_p}^{i_v} \phi P(i_p) \phi \cdot \phi S(v_p) \phi
\]

where the summation on \( i_v \) and \( v_w \) are independent for all \( v \) in the first term. The first term becomes zero by 2.24. The second term becomes

\[
- \sigma_p^h \phi S \left( \frac{i_1, i_2, \ldots, i_p}{v_1, v_2, \ldots, v_p} \right) \phi \sum_{j_0, \ldots, j_0}^{i_1, i_2, \ldots, i_p} \phi P(i_p) \phi \hspace{1cm} (2.38)
\]

since by D.7 \( \phi S \left( \frac{i_1, i_2, \ldots, i_p}{v_1, v_2, \ldots, v_p} \right) \phi \) is constant for every fixed \( p; \ p=t, t+1, \ldots, k. \)

So 2.38 is zero by virtue of 2.24. Therefore \( T = 0 \) for case (1) of 2.37.
Now we will investigate case (2). For this case 2.36 becomes

\[ \sigma_p^h \sum_{j12...p} \phi P(p) \phi \cdot \phi S(p) \phi \] where \( i_v \neq j_v \) for at least one value of \( v \). \( p = t, t+1, ... k \)

which reduces to

\[ \sigma_p^h \sum_{j00...0} \phi P(p) \phi \cdot \phi S(p) \phi \] \hspace{1cm} (2.39)

\[ - \sigma_p^h \sum_{j12...p} \phi P(p) \phi \cdot \phi S(p) \phi \] \hspace{1cm} (2.40)

where \( i_v \) and \( j_v \) range independently over all \( v = 1, 2, ... p \). The last term is zero by above. We will study the first term

\[ \sigma_p^h \sum \phi P(p) \phi \cdot \phi S(p) \phi \] \hspace{1cm} (2.41)

where the range of summation is

\[ p = t, t+1, ... k; \]
\[ i_v, j_v = 1, 2, ... n_v \] \( (v = 1, 2, ... k) \).

This range of summation is equivalent to using the fact that
\[ \phi S(p) \phi = 0 \] unless \( i_z = j_z \) for \( z = 1, 2, ... t-1 \)

\[ i_z = j_z = 1, 2, ... n_z \] \( (z = 1, 2, ... t-1) \)
\[ i_k \neq j_k = 1, 2, ... n_k \] \hspace{1cm} (2.41)

\[ i_z, j_z = 1, 2, ... n_z \] \( (z = t+1, ... p) \)
\[ i_s = j_s = 1, 2, \ldots, n_s \ (s = 1, 2, \ldots, t) \]
\[ i_{t+1} \neq j_{t+1} = 1, 2, \ldots, n_{t+1} \]
\[ i_s, j_s = 1, 2, \ldots, n_s \ (s = t+2, \ldots, p) \]  \hspace{1cm} (2.42)

plus
\[ i_s = j_s = 1, 2, \ldots, n_s \ (s = 1, 2, \ldots, t+1) \]
\[ i_s, j_s = 1, 2, \ldots, n_s \ (s = t+2, \ldots, p) \]  \hspace{1cm} (2.43)

But 2.40 summed over 2.41, 2.42, and 2.43 is zero by reasoning exactly as in showing that 2.30 was zero by summing over 2.33, 2.34, and 2.35. So 2.39 is zero and hence \( T = 0 \) for case (2) of 2.37.

Now case (3) is zero due to its symmetry with case (2).

For case (4) 2.36 becomes

\[ \mu_{hp} \sum_{\mathbf{i}} \phi_{i_j} \left( \frac{1}{p} \right) \phi_{i_j} \left( \frac{1}{p} \right) \phi_{s(p)} \phi_{s(p)} \]

which is zero by 2.35. So we have shown that \( T = 0 \) if \( S \) is defined as in 2.7 and for any \( P \) which satisfies 2.21, 2.25, and 2.26. But these conditions on \( P \) are exactly the conditions which make \( Q_p \) unbiased and independent of \( \mu \) and \( \sigma_j^2 \) \((j=1, 2, \ldots, t-1)\). So under the conditions of (a) and (b) of 2.9 the \( \beta Q_p \) becomes

\[ \text{E} \left\{ \sum_{i12 \ldots z} \left( \sum_{p=\mathbf{i}} A(p) \right) \left( \frac{1}{i_j} \right) \left( \frac{1}{s(p)} \right) \left( \sum_{q=i} A(q) \right) \right\}^2 \]

\[ + \ E \left\{ \sum_{i12 \ldots y} \left( \sum_{p=\mathbf{i}} A(p) \right) \left( \frac{1}{y} \right) \left( \frac{1}{s(y)} \right) \left( \sum_{q=i} A(q) \right) \right\}^2 \]  \hspace{1cm} (2.44)
Since this is the expected value of the sum of two squared terms and since $S$ is a fixed matrix and $P$ is variable, the unconditional minimum of $H Q_t^2$ would be to let $P = 0$. Since $P = 0$ is consistent with (a) and (b) of D.9 i.e. with 2.24, 2.25, and 2.26 we see that a sufficient condition for 2.44 to be a minimum subject to conditions 2.24, 2.25 and 2.26 is that $P = 0$. But from 2.19 we thus get $N = S$. This proves the theorem.

By expanding the second term of 2.44 we compute the variance of $Q_a^2$.

\[
\text{var } Q_a = \frac{2(n_t n_{t+1}-1)}{NN_{t+1} (n_t-1)(n_{t+1}-1)} 2 \sum_{p=t+1}^{k-1} \sum_{q=p+1}^{k} N_{p+1} N_{q+1} \sigma^2_p \sigma^2_q + \sum_{p=t+1}^{k} N_{p+1} \sigma^2_p \\
+ \frac{k}{N(n_t-1)} \sum_{q=t+1}^{k} N_{q+1} \sigma^2_q + \frac{N_{t+1}}{N} \left[ \mu_{4t} - 3s_t^4 \right] + \frac{2s_t}{N(n_t-1)} \sigma_t^4 .
\]

(2.45)
III. THE ONE-WAY CLASSIFICATION WITH UNEQUAL NUMBERS IN THE SUBCLASSES.

A. Definitions and Notations

Definition 1. Let the following model be given

\[ y_{i_1 i_2} = \mu + a_{i_1} + b_{i_1 i_2} \quad i_2 = 1,2, \ldots n_{i_1} \quad i_1 = 1,2, \ldots n > 2 \]

where \( \mu \) is a constant, and \( a_{i_1} \) and \( b_{i_1 i_2} \) are independent random variables with mean zero, variance \( \sigma_a^2 \) and \( \sigma_b^2 \) respectively and with fourth moment \( \Theta_a \sigma_a^4 \) and \( \Theta \sigma_b^4 \) respectively. This will be called Model III.

Definition 2. If \( \Theta_a = 3 \) and \( \Theta = 3 \) we will call the above Model IV.

We will use the following notation

\[ N = \sum_{i_1=1}^{n} n_{i_1} \quad r = N - \sum_{i_1=1}^{n} n_{i_1} a_{i_1} \quad t = \sum_{i_1=1}^{n} n_{i_1} a_{i_1}^2 - N \]

If we consider the \( a_{i_1} \) as constant, we may make an analysis of variance (6), equate expected and observed mean squares, and solve for \( \sigma_b^2 \) and \( \sigma_a^2 \). These solutions will be referred to as the analysis of variance estimates. These estimates are quadratic functions of the observations and are unbiased.

Definition 3. Let the matrix \( M(N \times N) \) be given by

\[ M_{(i_1 i_2)} = \frac{(N-1)(N-n_{i_1})}{r(N-n_{i_1})} a_{i_1}^{-1} \quad \text{if} \ i_1 = j_1; i_2 = j_2 \]
\[ M_{j_1 j_2} = \frac{N(N-1) - n}{r(N-n)} \frac{1}{\pi} \text{ if } i_1 = j_1, i_2 \neq j_2 \]

\[ M_{j_1 j_2} = -\frac{1}{\pi} \text{ if } i_1 \neq j_1. \]

It can be shown that \( M \) is the matrix of the quadratic form which is given by the analysis of variance estimate of \( \sigma^2 \). Hence \( M \) will be called the analysis of variance matrix for estimating \( \sigma^2 \).

**Definition 4.** Let \( Q_a = Y^T P Y \) be a quadratic estimate for \( \sigma_a^2 \) such that

1. \( E Q_a = \sigma_a^2 \), i.e. unbiasedness.
2. \( \text{Var} Q_a \) is independent of \( \mu \).
3. \( \text{Var} Q_a \leq \text{Var} Q_a^* \) where \( Q_a^* \) is any other quadratic form satisfying 1., 2.

If an estimate satisfies these three conditions we will call it the best unbiased quadratic estimate of \( \sigma_a^2 \).

**B. Investigation**

The situation for Model III and IV is much more complex than for Model I and II. This is, of course, due to the non-symmetry of the number of observations in the groups. It can be shown that if \( Q_a \) satisfies D.4 then \( P \) will be a function of \( \sigma^2 \) and \( \sigma_a^2 \). Since \( \sigma^2 \) and \( \sigma_a^2 \) are not known this is not very useful. So we will approach this problem from a slightly different point of view. Consider

\[ Q_a = Y^T P Y = \sum_{j_1}^N \left( \mu \phi + A_{j_1} + B_{j_1} \right)^T \left( \mu \phi + A_{j_1} + B_{j_1} \right). \]  \hspace{1cm} (3.1)
Also

\[ E Q_a = E Y P Y = E \left[ \sum_{j1} \mu^2 \phi P(j_1) \phi + \sum_{j1} A_{i1} P(j_1) A_{j1} + \sum_{j1} B_{i1} P(j_1) B_{j1} \right] \]

since the expected values of all cross products are zero. Thus we have

\[ E Q_a = \mu^2 \phi P \phi + \sigma^2 \sum_{j0} \phi P(j_1) \phi + \sigma^2 \sum_{j0} \phi P(j_1 j_2) \phi \]. (3.2)

Now in the variance of \( Q_a \) we will get the term

\[ E \sum_{j1} \mu \phi P(j_1) B_{j1} \sum_{m1} \mu \phi P(m_1) B_{m1} = \mu^2 \sigma^2 \sum_{k1} \sum_{j0} \phi P(j_1) P(k_1) \phi \]

\[ = \mu^2 \sigma^2 \sum_{m1} \left[ \sum_{j1} \phi P(j_1) \right] \left[ \sum_{k1} \phi P(k_1) \right] \]. (3.3)

But for unbiasedness we must have 3.2 equal to \( \sigma_a^2 \). Also for the variance of \( Q_a \) to be independent of \( \mu^2 \) we must have 3.3 equal to zero.

So we must have

(a) \( \phi P \phi = 0 \)

(b) \( \sum_{j0} \phi P(j_1) \phi = 1 \)

(c) \( \sum_{j0} \phi P(j_1 j_2) \phi = 0 \)

(d) \( \phi P = 0 \).
Now \( \text{var} \ Q_a = E \left( Q_a^2 - \sigma_a^4 \right) \)

\[
= E \left\{ \sum_{j_1} A_{1j_1} P_{j_1} B_j + 2 \sum_{j_1} A_{1j_1} P_{j_1} B_j \sum_{k_1} A_{2k_1} P_{k_1} B_{k_1} \right\}^2 - \sigma_a^4
\]

\[
= E \sum_{j_1} A_{1j_1} P_{j_1} B_j \sum_{k_1} A_{2k_1} P_{k_1} B_{k_1} + E \sum_{j_1} A_{1j_1} P_{j_1} B_j \sum_{m_1} A_{2m_1} P_{m_1} B_{m_1}
\]

\[
+ 2E \sum_{j_1} A_{1j_1} P_{j_1} B_j \sum_{j_2} B_{j_1} P_{j_1j_2} B_{j_2} \sum_{m_1} A_{2m_1} P_{m_1} B_{m_1}
\]

since the expected value of the other cross products are zero.

By using the methods of section one this becomes

\[
\sigma_a^4 + 2\sigma_a^6 \sum_{j_1} \left[ \phi P_{j_1} \phi \right]^2 - 3\sigma_a^4 \sum_{j_0} \left[ \phi P_{j_0} \phi \right]^2
\]

\[
+ 4 \sigma_a^4 \sum_{j_1} \left[ \phi P_{j_1} \phi \right] \left[ \phi P_{j_1} \phi \right] \left[ \phi P_{j_1} \phi \right] \left[ \phi P_{j_1} \phi \right]
\]

\[
- 3 \sigma_a^4 \sum_{j_0} \left[ \phi P_{j_0} \phi \right] \left[ \phi P_{j_0} \phi \right],
\]

\[
+ \sigma_a^4 \sum_{j_0} \left[ \phi P_{j_0} \phi \right]^2 - \sigma_a^4
\]
So the variance of $Q_a$ under Model III becomes

$$
\sigma^4_a \left[ (\theta_a - 3) \sum_{j1} 1 \left( \phi P(j1) 1\phi \right)^2 + 2 \sum_{j1} 1 \left( \phi P(j1) 1\phi \right)^2 \right] 
+ \sigma^4_a \left[ (\theta - 3) \sum_{j12} 1 \left( \phi P(j1) 1\phi \right)^2 + 2 \sum_{j12} 1 \left( \phi P(j1) 1\phi \right)^2 \right] 
+ 4 \sigma^2_a \sum_{j1} 1 \left[ \phi P(j1) 1\phi \right] \left[ \phi P(j1) 1\phi \right] X $$

Under Model IV this becomes

$$
\text{var } Q_a = 2\sigma^4_a \sum_{j1} 1 \left( \phi P(j1) 1\phi \right)^2 + 2\sigma^4_a \sum_{j12} 1 \left( \phi P(j1) 1\phi \right)^2 
+ 4 \sigma^2_a \sum_{j1} 1 \left[ \phi P(j1) 1\phi \right] \left[ \phi P(j1) 1\phi \right] X $$

Unless otherwise stated, what follows is under the assumption of Model IV. In order to expand $\text{var } Q_a$ in more detail we will use the symbols $P_{km}^{ij}$ to represent the element in the $k^{th}$ row and $m^{th}$ column of the first partition block which is in the $i^{th}$ row and $j^{th}$ column.

So let

$$
G = \sum_{j1} 1 \left[ \phi P(j1) 1\phi \right] \left[ \phi P(j1) 1\phi \right] = \sum_{ij} \left( \sum_{km} P_{km}^{ij} \right)^2 
$$

$$
H = \sum_{j12} 1 \left[ \phi P(j1) 1\phi \right] \left[ \phi P(j1) 1\phi \right] = \sum_{ijkm} \left( P_{km}^{ij} \right)^2, $$

(3.7)
\[
I = \sum_{ij} \left( \phi \phi^{-1} \right)_{ij} \left( \phi \phi^{-1} \right) = \sum_{ij} \left( \sum_{qkm} p_{kq} p_{mq} \right).
\]

Obviously condition 3.4(d) contains 3.4(a) so we will use only \(N + 2\) side conditions. These are not all independent and we will reduce them to fewer conditions. We will show that the \(p_{km}\) have certain symmetrical properties and hence we can reduce the number of unknowns in our matrix \(P\).

We shall use the well known theorem that the minimum of the quantity
\[
\sum_{i=1}^{n} a_i^2 \text{ subject to the condition } \sum_{i=1}^{n} a_i = \text{constant, is}
\]

\[
a_i = \frac{1}{q} \sum_{i=1}^{q} a_i = \bar{a}.
\]

Now let us assume that \(p_{km}^{ij}\) is the general element of the matrix \(P\) which minimizes the variance of \(Q_a\) subject to the conditions 3.4. So by 3.7 we have minimum \(\text{var } Q_a = 2 \sigma_a^4 \sigma_b + 4 \sigma_a^2 \sigma^2 + 2 \sigma_b^2 \). We shall show that this minimum variance of \(Q_a\) is not increased if the following substitutions are made:

\[
p_{km}^{ij} = \frac{1}{n_1 n_2} \sum_{km} p_{km}^{ij} = b_{ij} \text{ if } i \neq j
\]

\[
p_{km}^{ii} = \frac{1}{n_1 (n_1 - 1)} \sum_{km} p_{km}^{ii} = c_i \text{ if } k \neq m
\]

\[
p_{kk}^{ii} = \frac{1}{a_i} \sum_{j} p_{jj}^{ii} = d_i
\]

Now the term \(c\) is unaffected by the substitution 3.8. The term \(b\) either remains constant or is reduced by the substitution 3.8. This is evident when the above theorem is applied. Let us consider the term \(I\).
\[ I = \sum_{ij} \left( \sum_{k=q}^{l} p_{i,j,k}^2 \right) = \sum_{i} \sum_{j} \sum_{k} \left( \sum_{n} p_{n,k}^2 \right)^2. \]

I is the sum of squares of the total of each row of each first partition block of the matrix \( F \). Let \( a_1, a_2, \ldots, a_p \) be the row totals of a certain off-diagonal block. Now if each element in the block is replaced by the mean of the whole block as given in 3.8, then the \( \sum_{i=1}^{p} a_i \) is not affected and \( \sum_{i=1}^{p} a_i^2 \) is not increased. So by similar substitutions into all off-diagonal blocks we do not increase I. Similarly, substituting 3.8 into diagonal blocks does not increase I. Also it is evident that the substitutions 3.8 are consistent with the side conditions 3.4. However we see that \( y'Fy = 0 \) now reduces to a district side conditions.

Under the substitutions 3.8 the side conditions 3.4 become

\[(a) \sum_{i=1}^{n} n_i a_i + \sum_{i=1}^{n} n_i (a_i - 1) C_i = \sum_{i,j=1}^{n} n_i n_j b_{ij} = 0 \]

\[(b) \sum_{i=1}^{n} \left[ n_i (a_i - 1) C_i + n_i d_i \right] = 1 \]

\[(c) \sum_{i=1}^{n} n_i d_i = 0 \]

\[(d) (a_j - 1) C_j + d_j + \sum_{i=1}^{n} n_i b_{ij} = 0 \quad j = 1, 2, \ldots, n. \]

To find the minimum of variance \( Q \) we will use the method of Lagrange multipliers. Let \( \lambda_i \) \((i=1, 2, \ldots, n+2)\) be the Lagrange multipliers. Then using (3.8) the function which is to be unconditionally minimized is given by (we replace \( g_p \) by \( \sum g_p g_p \)) for \( \ldots, n; g_1 \) by \( h g_1 \) and \( g_2 \) by \( h g_2 \) for
\[ F = 2c_1^h \left[ \sum_{i=1}^{n} \left( \frac{n_1(n_1-1)}{2} + a_1 d_1 \right) \right]^2 + \sum_{i=1}^{n} \left( a_1 a_j b_{ij} \right)^2 \]
\[ + 2c_1^h \left[ \sum_{i=1}^{n} n_1(n_1-1) c_1^2 + \sum_{i=1}^{n} n_1 d_1^2 + \sum_{i=1}^{n} n_1 n_j b_{ij} \right] \]
\[ + 4c_1^s c_2 \left[ \sum_{i=1}^{n} n_1 n_j^2 b_{ij}^2 + \sum_{i=1}^{n} \left( n_1 d_1^2 + n_1(n_1-1) c_1^2 + 2n_1(n_1-1)d_1 c_1 \right) \right] \]
\[ + 8 \sum_{i=1}^{n} n_1 d_1 \right] d_j + (n_j-1) c_j + \sum_{i=1}^{n} n_1 b_{ij} \right] . \]

Differentiating and simplifying we get

\[ \frac{\partial F}{\partial d_1} (i=1,2,\ldots,n) = d_1 \left( n_1 c_1 + 2c_2 c_1 c_1 + c_1 \right) \]
\[ + c_1 \left[ c_1^2(n_1-1) (n_1 c_1^2 + 2c_2^2) \right] + 2c_1 + 2c_2 = 0 \]

\[ \frac{\partial F}{\partial c_1} (i=1,2,\ldots,n) = d_1 \left( n_1 c_1^2 + 2c_2 c_1 c_1 \right) \]
\[ + c_1 \left[ c_1^2(n_1-1) (n_1 c_1^2 + 2c_2^2) + c_1^4 \right] + 2c_1 + 2c_2 = 0 \]

\[ \frac{\partial F}{\partial b_{ij}} (i=1,2,\ldots,n) = b_{ij} \left[ n_1 n_j c_1^2 + c_1^2 (n_1 + n_j) + c_1^4 \right] + 2c_1 + 2c_2 = 0 \]

To get the matrix \( P \) we must solve these equations in conjunction with

equations 3.9. If the \( n_1 \) are not equal, the elements of the matrix \( P \) will
be functions of $\sigma_a$ and $\sigma_a^2$. Hence the matrix $M$ given in D.3 will not
satisfy these equations unless all the $n_i$ are equal. So the analysis
of variance estimates do not give the best quadratic unbiased estimates
for estimating $\sigma_a$ under Model IV.

Although we will never know $\sigma_a$ and $\sigma_a^2$, we may have some prior
indication as to the approximate value of $W = \sigma_a^2/\sigma^2$. Even though
the analysis of variance does not give the best quadratic unbiased
estimate of $\sigma_a^2$ we will show that it gives very good estimates for certain
ranges of $W$. Let $B$ represent the variance of the best quadratic unbiased
estimate of $\sigma_a^2$. $B$ is a function of $\sigma_a^2$ and $\sigma^2$. Let $A$ represent the
variance of the analysis of variance method of estimating $\sigma_a^2$. Then the
true efficiency $E$ of the analysis of variance method of estimating $\sigma_a^2$
is

$$E = \frac{B}{A}.$$  

To find $B$ we must first solve equations 3.11. These are in general very
complicated and not adaptable to easy solution. So instead of finding
$E$, we will find a lower bound $L_1$ for the true efficiency $E$. This will be
done by finding a lower bound, $L_1$, for $B$. Let 3.6 be written

$$\text{var } q_a = \sigma_a^4 \cdot K_1 + \sigma_a^2 \sigma^2 K_2 + \sigma^4 K_3$$  \hspace{1cm} (3.12)

where

$$K_1 = 2 \sum_{i1}^n (\phi_{j1}^{i1})^2.$$
\[ k_2 = 4 \sum_{j1} \left[ \phi' P(j1) \right] \left[ \phi' P(j1) \right]^T \]  \quad (3.13)

\[ k_3 = 2 \sum_{j12} \left( \phi' P(j1) \phi \right)^2 \]  

To find \( B \) we must minimise 3.12 under the restrictions imposed by 3.9.

To get \( B_L \) we will minimise \( k_1 \) under conditions 3.9, then minimise \( k_2 \) under conditions 3.9, and finally minimise \( k_3 \) under conditions 3.9.

We may then substitute these minimal values of \( k_1 \), \( k_2 \), and \( k_3 \) into 3.12 and call this our lower bound \( B_L \). There will be one exception. We will minimise \( k_3 \) subject to conditions 3.9 \((a), (b) \) and \((c)\) only, since imposing, in addition, condition 3.9 \((d)\) leads to a very cumbersome and lengthy expression. The solution for \( k_3 \) is certainly not greater than the minimum using all the conditions of 3.9. Hence \( B_L \) is a lower bound. Let \( C_1 \), \( C_2 \), and \( C_3 \) denote the minimal values of \( k_1 \), \( k_2 \), and \( k_3 \) respectively.

We then have

\[ B_L = \sigma_a^4 C_1 + \sigma_a^2 \sigma_c^2 C_2 + \sigma_c^4 C_3 \]  \quad (3.14)

Now let us find var \( Q_a \) \((=A)\) using the analysis of variance estimate. After substituting D.3 into 3.6 and simplifying we get

\[ A = 2\sigma_a^4 \left[ \frac{N(N \sum n_i^2 - 2 \sum n_i^3) + (\sum n_i^2)^2}{r^2} \right] \]

\[ + 4 \sigma_a^2 \sigma_c^2 \frac{N}{r} + 2 \sigma_c^2 \left( \frac{N^2 (N-1)}{r^2 (N-2)} \right) \]  \quad (3.15)
We must now find the \( C_i \).

\( C_i \) is the minimum of

\[
\sum_{j=1}^{n} (\phi_{j}^{i})^2 \]

imposing the restrictions 3.9. We shall use equation 3.10 neglecting the coefficients of \( \sigma_a \), \( \sigma_b \) and \( \sigma_i \). After differentiating and simplifying we get

\[
\frac{\partial F}{\partial d_i} = \sigma_i n_i d_i + \sigma_i n_i (n_i - 1) C_i + \varepsilon_i + 2 \varepsilon_i + 2 = 0
\]

\[
\frac{\partial F}{\partial C_i} = \sigma_i n_i d_i + \sigma_i n_i (n_i - 1) C_i + \varepsilon_i + 2 \varepsilon_i + 2 = 0
\]

\[
\frac{\partial F}{\partial b_{ij}} = \sigma_i n_i n_j b_{ij} + \varepsilon_i + 2 \varepsilon_i + 2 = 0
\]

A solution is

\[
\begin{align*}
d_i &= 0 \\
C_i &= \frac{1}{n_i (n_i - 1)} \\
b_{ij} &= -\frac{1}{n(n-1) n_i n_j}
\end{align*}
\]

(3.16)

We shall show that this is an absolute minimum. The problem is to find the absolute minimum of

\[
2\sigma_i n_i \sum_{j=1}^{n} \left( \phi_{j}^{i} \right)^2 = 2\sigma_i \sum_{i=1}^{n} \left( n_i (n_i - 1) C_i + n_i d_i \right)^2 + \sum_{i,j=1}^{n} (n_i n_j b_{ij})^2
\]

(3.17)
subject to the conditions 3.9. Let us assume that a solution is

\[ d_i = p_i \]
\[ C_i = q_i + \frac{1}{n \cdot n_i (n_i - 1)} \]
\[ b_{ij} = r_{ij} - \frac{1}{n(n-1) \cdot n_i n_j} . \]

Where the \( d_i, C_i \) and \( b_{ij} \) satisfy 3.9 and they minimise 3.17. Equations 3.9 impose the conditions

\[ \sum_{i=1}^{n} n_i p_i = \sum_{i=1}^{n} n_i (n_i - 1) q_i = \sum_{i,j=1}^{n} n_i n_j r_{ij} = 0 \]  

(3.19)
on the \( p_i, q_i \) and \( r_{ij} \).

Substituting our assumed solution into 3.17 we get

\[ 2^{a_4^{\frac{1}{2}}} \left[ \sum_{i=1}^{n} n_i (n_i - 1) \left[ q_i + \frac{1}{n \cdot n_i (n_i - 1)} \right] + n_i p_i \right]^2 \]
\[ + \sum_{i,j=1}^{n} \left( \frac{n_i n_j}{n(n-1)n_i n_j} - \frac{1}{n(n-1)} \right)^2 \]
\[ = 2^{a_4^{\frac{1}{2}}} \left[ \sum_{i=1}^{n} \left( \frac{1}{n} + \left[ n_i (n_i - 1) q_i + n_i p_i \right] \right)^2 \right. \]
\[ + \sum_{i,j=1}^{n} \left( \frac{n_i n_j r_{ij}}{n(n-1)} \right)^2 \]
\[ = 2^{a_4^{\frac{1}{2}}} \left[ \sum_{i=1}^{n} \frac{1}{n^2} + \sum_{i=1}^{n} \left( n_i (n_i - 1) q_i + n_i p_i \right) \right]^2 \]
\[ + \sum_{i,j=1}^{n} n_i^2 n_j^2 r_{ij}^2 - 2 \sum_{i,j=1}^{n} \frac{n_i n_j r_{ij}}{n(n-1)} + \sum_{i,j=1}^{n} \frac{1}{n^2(n-1)^2} \right]. \]
But by conditions 3.19 we see that

\[
\sum_{i=1}^{n} \left[ n_i (n_i - 1) q_i + a_i p_i \right] = 0
\]

and also

\[
-2 \sum_{i,j=1}^{n} \frac{n_i n_j r_{ij}}{s(n-1)^2} = 0.
\]

So 3.20 reduces to

\[
2\sigma_a^2 \left[ \sum_{i=1}^{n} \frac{1}{n^2} + \sum_{i=1}^{n} \left( n_i (n_i - 1) q_i + a_i p_i \right)^2 + \sum_{i,j=1}^{n} n_i n_j s r_{ij} a_i a_j + \sum_{i,j=1}^{n} \frac{1}{s(n-1)^2} \right].
\]

This is the sum of four positive terms. So the minimum is obtained if

\( p_i = q_i = r_{ij} = 0. \) This is also consistent with the conditions 3.19.

Hence 3.16 gives an absolute minimum of

\[
2\sigma_a^2 \sum_{j=1}^{n} \left[ \phi^2 (j) \right] \frac{1}{\phi^2}
\]

subject to the side conditions 3.9.

After substituting \( p_i = q_i = r_{ij} = 0 \) in 3.20 we get

\[
2\sigma_a^2 \left[ \sum_{i=1}^{n} \frac{1}{n^2} + \sum_{i,j=1}^{n} \frac{1}{s(n-1)^2} \right] = 2\sigma_a^2 \left[ \frac{1}{n} + \frac{1}{s(n-1)} \right] = \frac{2\sigma_a^2}{n-1}.
\]

So

\[
c_1 = \frac{2}{n-1}.
\] (3.21)
$C_2$ is the minimum of

$$\sum_{j=1}^{\alpha} \left( \phi \frac{\partial}{\partial \gamma_j} \right) \left( \phi \frac{\partial}{\partial \gamma_j} \right)' \gamma$$

subject to conditions 3.9. We now use the equation 3.10 neglecting the coefficients of $\sigma_i^h$ and $\sigma_i^b$. After differentiating and simplifying we get

$$\frac{\partial Y}{\partial d_i} = 2 \sigma_i^a \sigma_i^a \left[ d_i + (n_i - 1) c_i \right] + g_i + g_2 + 2g_{i+2} = 0$$

$$\frac{\partial Y}{\partial c_i} = 2 \sigma_i^a \sigma_i^a \left[ d_i + (n_i - 1) c_i \right] + g_i + 2g_{i+2} = 0$$

$$\frac{\partial Y}{\partial b_{ij}} = \sigma_i^a \sigma_i^a (n_i b_{ij} + n_j b_{ij}) + g_{i+2} + g_{j+2} = 0$$

A solution is the analysis of variance matrix given in D.3. We shall show that this solution is an absolute minimum. The problem is to minimize

$$h \frac{\sigma_i^a}{\sigma_i^a} \left[ \sum_{i=1}^{n} \left( n_i d_i^a + n_i (n_i - 1)^2 c_i^a + 2n_i (n_i - 1) c_i d_i \right) + \sum_{i,j=1}^{n} n_i n_j b_{ij} \right]$$

(3.22)

subject to the conditions 3.9. Let us assume a solution

$$d_i = \frac{h(n-1)(n-n_i)}{r(n-n_i) + p}$$

(3.23)

$$c_i = \frac{h(n-1) - n_i (n-n_i)}{r(n-n_i) n_i} + q_i$$

$$b_{ij} = -\frac{1}{r} + \frac{r_{ij}}{r}$$

Substituting these into equations 3.9 we find that the side conditions
\[ \sum_{i=1}^{n} n_i p_i = \sum_{i=1}^{n} n_i(q_i - 1)q_i = \sum_{i \neq j} n_i n_j r_{ij} = p_i + (n_i - 1)q_i + \sum_{j \neq i}^{n} n_j r_{ij} = 0 . \]  
\[ (3.24) \]

Putting 3.23 into 3.22 we get

\[ 4\sigma^2 a = \left( \sum_{i=1}^{n} n_i \left( \frac{(n_i - 1)(n_i - 2)}{r n_i(n_i - 1)} + p_i + (n_i - 1) \frac{n_i(n_i - 2)}{r n_i(n_i - 1)} + (n_i - 1)q_i \right) \right)^2 \]

\[ + \sum_{i \neq j}^{n} n_i n_j \left( - \frac{1}{r} + r_{ij} \right)^2 \]  
\[ (3.25) \]

After simplification this becomes

\[ 4\sigma^2 a = \left( \sum_{i=1}^{n} n_i \left[ p_i + (n_i - 1)q_i \right] \right)^2 + \sum_{i=1}^{n} \frac{n_i(n_i - 1)}{r^2} + \sum_{j \neq i}^{n} \frac{n_i n_j}{r^2} \]

\[ + \sum_{j \neq i}^{n} n_i n_j r_{ij} - \frac{2}{r} \sum_{i \neq j}^{n} n_i n_j r_{ij} - \frac{2}{r} \sum_{i=1}^{n} n_i^2 \left[ p_i + (n_i - 1)q_i \right] \]

\[ + \frac{2n}{r} \sum_{i=1}^{n} n_i \left[ p_i + (n_i - 1)q_i \right] \}

The last three terms are zero by 3.24. So 3.25 becomes

\[ 4\sigma^2 a = \left( \sum_{i=1}^{n} n_i \left[ p_i + (n_i - 1)q_i \right] \right)^2 + \sum_{j \neq i}^{n} \frac{n_i n_j}{r} \]

\[ \sum_{i=1}^{n} \frac{n_i(n_i - 1)}{r^2} + \sum_{j \neq i}^{n} \frac{n_i n_j}{r^2} \]

Since all the terms are positive an absolute minimum is achieved if

\[ p_i = q_i = r_{ij} = 0. \]  
These values are consistent with the side conditions
3.2h. So putting these values into 3.23 we see that the analysis of variance matrix D.3 gives an absolute minimum to

\[ h_0^2 \sigma_a^2 \sum_{j1} \begin{bmatrix} \phi_P(j_1) \end{bmatrix}^2 \end{bmatrix}^T \]

subject to the side conditions 3.9. Putting \( p_1 = q_1 = r_{ij} = 0 \) into 3.26 we see that this minimum is

\[ h_0^2 \sigma_a^2 \frac{N}{P} \]

So

\[ c_2 = \frac{h_0}{P} \]

(3.27)

\( c_3 \) is the minimum of

\[ \sum_{j12} (\phi_P(j_12) \phi - \phi_P(j_12))^2 \]

subject to conditions 3.9 (a), (b) and (c). We now use the equation 3.10 neglecting the coefficients of \( \sigma_a^2 \), \( \sigma^b \) and \( \sigma_a^k \). We will take \( \sigma_{p+2} = \sigma \) (p=1,2,...,n) which excludes the n conditions \( \phi_P = 0 \), but includes the condition \( \phi_P \phi = 0 \). After differentiating and simplifying we get

\[ \frac{\partial F}{\partial \phi_{11}} = \sigma_{11} \phi_{11} + \phi_{1} + \phi_{2} = 0 \]

\[ \frac{\partial F}{\partial \phi_{11}} = \sigma_{11} \phi_{11} + \phi_{1} = 0 \]

\[ \frac{\partial F}{\partial b_{1j}} = b_{1j} \phi_{1j} + \phi = 0 \]
A solution is
\[
d_i = 0 \\
c_i = \frac{1}{t} \\
b_{ij} = -\frac{1}{r}.
\] (3.29)

We shall show that this solution gives an absolute minimum. The problem is to find an absolute minimum of
\[
2\sigma_i \sum_{j=1}^{k+1} \left[ \phi (p_{i,j}^{1,2}) \phi \right]^2
= 2\sigma_i \left[ \sum_{i=1}^{n} n_i (n_i - 1) c_i^2 + \sum_{i=1}^{n} n_i d_i^2 + \sum_{i=j=1}^{n} n_i n_j b_{ij}^2 \right].
\]

Subject to the conditions 3.9 (a), (b) and (c). Let us assume the solution
\[
d_i = p_i \\
c_i = \frac{1}{t} + q_i \\
b_{ij} = -\frac{1}{r} + r_{ij}.
\] (3.30)

Substituting these into 3.9 (a), (b) and (c) we get the conditions
\[
\sum_{i=1}^{n} n_i p_i = \sum_{i=1}^{n} n_i (n_i - 1) q_i = \sum_{i,j=1}^{n} n_i n_j r_{ij} = 0.
\] (3.31)

Substituting 3.30 into 3.29 we get
\[
2\sigma_i \left[ \sum_{i=1}^{n} \frac{n_i (n_i - 1)}{t^2} (\frac{1}{t} + q_i)^2 + \sum_{i=1}^{n} n_i p_i^2 + \sum_{i,j=1}^{n} n_i n_j (-\frac{1}{r} + r_{ij})^2 \right]
= 2\sigma_i \left[ \sum_{i=1}^{n} \frac{n_i (n_i - 1)}{t^2} + \frac{2}{t} \sum_{i=1}^{n} n_i (n_i - 1) q_i + \sum_{i=1}^{n} n_i (n_i - 1) q_i^2 + \sum_{i=1}^{n} n_i p_i^2 \right].
\]
\[ + \frac{1}{r^3} \sum_{i \neq j = 1}^{n} n_i n_j - \frac{2}{r} \sum_{i \neq j = 1}^{n} n_i n_j r_{ij} + \frac{n}{r \sum_{i \neq j = 1}^{n} n_i n_j r_{ij}^2} \].

But by 3.31 this reduces to

\[ 2c \left[ \sum_{i=1}^{n} \frac{n_i(n_i-1)}{t^2} + \frac{1}{r^2} \sum_{i \neq j = 1}^{n} n_i n_j \right. \]

\[ + \sum_{i=1}^{n} n_i(n_i-1) q_i^2 + \sum_{i=1}^{n} n_i p_i^2 + \sum_{i \neq j = 1}^{n} n_i n_j r_{ij}^2 \]. \quad (3.32)

This achieves its absolute minimum if \( p_1 = q_1 = r_{ij} = 0 \).
These values also satisfy the side conditions 3.31 to the absolute minimum of

\[ 2c \left[ \phi P(\frac{j}{l_1}) \phi \right] \]

subject to 3.9 (a), (b) and (c) is achieved by the values given in 3.26.

Substituting \( p_1 = q_1 = r_{ij} = 0 \) into 3.32 and simplifying we get the minimum to be

\[ 2c \left[ \frac{N(N-1)}{rt} \right. \]

So
\[ q_j = \frac{2N(N-1)}{rt} \]. \quad (3.33)

So
\[ \rho_L = \sigma_a \left. \frac{2}{n-1} + \frac{4}{n-1} \right] \sigma_a \left. \frac{N}{r} \right] + \frac{1}{r} \frac{2N(N-1)}{rt} \]. \quad (3.34)

and
\[ L_1 = \frac{w^2}{n - 1} + 2w \frac{H}{r} + \frac{H(N-1)}{rt} \]

\[ w^2 \left[ \frac{H \left( \sum n_1^2 - 2 \sum n_1^3 \right) + \left( \sum n_1^2 \right)^2}{r^2} \right] + 2w \frac{H}{r} + \frac{H^2 (N-1) (n-1)}{r^2 (N-n)} \]

(3.35)

So if we have an experiment and have some priori knowledge of \( W \) then we can substitute into 3.35 and get a lower bound on the efficiency of the analysis of variance method of estimation.

If \( \sigma_a^2 \) times the minimum \( n_1 \) is large relative to \( 2 \sigma^2 \) then in 3.11 we can replace

\[ n_1 \sigma_a^2 + 2 \sigma_a^2 \sigma^2 + \sigma^4 \by n_1 \sigma_a^2 \]

replace

\[ n_1 \sigma_a^2 + 2 \sigma^2 \] by \( n_1 \sigma_a^2 \)

and replace

\[ n_1 (n_1-1) \sigma_a^2 + \sigma^4 \] by \( n_1 (n_1-1) \sigma_a^2 \)

then 3.11 becomes the same as the equations for finding the minimum of

\[ \sum_{i=1}^{\Gamma} \left[ \frac{1}{\sigma_i} \frac{1}{\sigma_i^2} \Phi \right] \]

Thus we would expect the efficiency of this method to be quite good when \( W \) times the minimum \( n_1 \) is large relative to \( 2 \). The matrix giving the minimum of the variance of \( Q_a \) for this case is given by 3.16. By substituting these values into 3.6 we get
\[ \text{var } q_a = \frac{2a^4}{n-1} + 4q_a^3 c^3 \left[ \frac{1}{n(n-1)} \sum_{i=1}^{n} \frac{1}{n_i} \right] \]

\[ + 2a^4 \left[ \frac{1}{n^3} + \frac{n}{n(n-1)} \sum_{i=1}^{n} \frac{1}{n_i} + \frac{1}{n^2(n-1)^2} \sum_{i<j=1}^{n} \frac{1}{n_i n_j} \right] \]

and a lower bound \( L_2 \) on the efficiency of this method of estimating \( q_a^2 \) is

\[ L_2 = \frac{n^2}{n-1} + 2n \frac{1}{n-1} + \frac{n(n-1)}{n} \sum_{i=1}^{n} \frac{1}{n_i} + \frac{1}{n^2(n-1)^2} \sum_{i<j=1}^{n} \frac{1}{n_i n_j} \]

To find what the solutions 3.16 are in terms of the observations \( Y_{ij} \) let us form

\[ \sum_{i} \sum_{j} r_{i} Y_{ij}^2 + \sum_{i} s_{i} \left( \sum_{j} Y_{ij} \right)^2 - \left( \sum_{i} \sum_{j} t_{i} Y_{ij} \right)^2 \]

(3.37)

where the \( r_i, s_i \) and \( t_i \) are to be found such that 3.37 is given by

\[ Y' \underline{PT} . \]

(3.38)

Where \( P \) is the matrix given in 3.16. We will equate coefficients of the \( Y_{ij} \) in 3.37 and 3.35. We get

Coefficient of \( Y_{11}^2 \): \( r_1 + s_1 - n_1 = 0 \)

Coefficient of \( Y_{1j} Y_{1k} \): \( s_1 + n_1 = \frac{1}{n_1 (n_1-1)} \)

Coefficient \( Y_{ij} Y_{ik} \): \( -n_1 n_k = \frac{1}{n(n-1) n_i n_k} \)
Solving these equations we get

\[ \tau_i = -\frac{1}{n_i n_i(n_i-1)} \]

\[ \sigma_i = \frac{n_i}{n(n-1)} n_i a_{n_i} (n_i-1) \]

\[ t_i = \frac{1}{n_i n(n-1)} \]

Putting these solutions into 3.37 we thus get the solutions 3.16 in terms of the $T_{1j}$.

So if we have an experiment and we want to estimate $\sigma^2_a$, we would pick all the $n_i$ equal if possible. If this is not possible then we compute $L_1$ and $L_2$ and get the lower bounds on the efficiency of using the analysis of variance and of using the method given in 3.37. Then on the basis of $L_1$ and $L_2$ we decide which method to use. It is believed that one of these two methods will give estimates of $\sigma^2_a$ which is more than 80 per cent efficient in most cases for which $W$ is not too small, say greater than one-tenth.

Below are some calculations showing the lower bounds of the efficiency of the two methods of estimating $\sigma^2_a$ for various values of $n_i$ and $W$. The top number in each cell is the efficiency using the analysis of variance, the lower number is the efficiency using the method given in 3.37.
<table>
<thead>
<tr>
<th>n = 3; ( n_1 = 3,5,12 )</th>
<th>10</th>
<th>5</th>
<th>2</th>
<th>1</th>
<th>.5</th>
<th>.1</th>
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<td></td>
<td>89.4</td>
<td>89.6</td>
<td>90.3</td>
<td>90.8</td>
<td>90.9</td>
<td>79.8</td>
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<tr>
<td></td>
<td>99.4</td>
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<td>97.1</td>
<td>93.9</td>
<td>87.3</td>
<td>59.3</td>
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<tr>
<td>n = 5, ( n_1 = 7,12,13,20,25 )</td>
<td>89.1</td>
<td>89.5</td>
<td>89.9</td>
<td>90.4</td>
<td>91.3</td>
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<tr>
<td></td>
<td>99.3</td>
<td>99.5</td>
<td>98.9</td>
<td>97.8</td>
<td>95.6</td>
<td>82.1</td>
</tr>
<tr>
<td>n = 10; ( n_1 = 6,6,7,9,15 )</td>
<td>79.1</td>
<td>79.3</td>
<td>79.9</td>
<td>80.6</td>
<td>81.6</td>
<td>76.5</td>
</tr>
<tr>
<td>16,19,23,26,30</td>
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<td>99.0</td>
<td>97.3</td>
<td>94.1</td>
<td>86.9</td>
<td>44.5</td>
</tr>
<tr>
<td>n = 5, ( n_1 = 8,10,13,15,21 )</td>
<td>92.6</td>
<td>92.7</td>
<td>92.9</td>
<td>93.0</td>
<td>92.7</td>
<td>95.1</td>
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<td></td>
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<td>97.7</td>
<td>98.3</td>
<td>96.1</td>
<td>76.2</td>
</tr>
<tr>
<td>n = 3, ( n_1 = 20,200,300 )</td>
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<td>62.2</td>
<td>65.5</td>
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<td>95.6</td>
<td>80.0</td>
</tr>
</tbody>
</table>
IV. THE GENERAL BALANCED CROSS CLASSIFICATION WITH NORMALITY ASSUMPTIONS.

A. Definitions and Notations

Definition 1. Let the following model be given

\[ y_{i_1 i_2 \ldots i_k} = \mu + a_{i_1} + a_{i_2} + \ldots + a_{i_k} + e_{i_1 i_2 \ldots i_k} i_1,2, \ldots, n_j \]

where \( \mu \) is a constant and the \( e_{i_1 i_2 \ldots i_k} \) and \( a_{i_p} \) (\( p = 1,2, \ldots k \)) are independent random variables. Also \( e_{i_1 i_2 \ldots i_k} \) is distributed normally with mean zero and variance \( \sigma^2 \). \( a_{i_p} \) is distributed normally with mean zero and variance \( \sigma^2_{i_p} \) (\( p = 1,2, \ldots k \)). This will be called model V.

We will use the following notation

\[ N = n_{i_1} \quad N_j = \sum_{i=1}^{k} n_{i_1} \quad N_0 = 0 \]

\[ N_{t-1} - (t-3) = a_t \quad t = 1,2, \ldots, k+1 \]

\[ N_t - (t-1) = b_t \quad t = 1,2, \ldots, k \]

\[ N = b_{k+1} \]

Definition 2. Let \( Y \) be an ordered vector as defined in section II.

Let \( Z = (z_1) \) be a vector of dimension \( N \times 1 \) such that

\[ Z = CY \]

where \( C = (c_{ij}) \) an orthogonal transformation such that \( Z \) has the following properties
(a) $z_n \equiv 1, 2, \ldots, n$ is distributed normally
(b) $z_1 = \nu \Rightarrow i = 1, 2, \ldots, n$
(c) $z_{i+1} = 0 \Rightarrow i = 1, 2, \ldots, n$.
(d) The following relationships in the analysis of variance are satisfied:

\[
\begin{align*}
\text{Source} & & \Sigma & & \Sigma & & \Sigma & & \Sigma & & \Sigma & & \Sigma \\
\text{class} & & a_{11} & & a_{12} & & a_{13} & & \ldots & & a_{1k} & & a_{1N} \\
& & a_{21} & & a_{22} & & a_{23} & & \ldots & & a_{2k} & & a_{2N} \\
& & \vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
& & a_{k1} & & a_{k2} & & a_{k3} & & \ldots & & a_{kk} & & a_{kN} \\
& & \Sigma & & \Sigma & & \Sigma & & \ldots & & \Sigma & & \Sigma \\
& & & & & & & & & & & \text{total} \\
& & & & & & & & & & & \text{except} \ y \\
& & & & & & & & & & & \frac{y^2}{2} \\
\end{align*}
\]

where

\[
\begin{align*}
S, S & = \frac{\sum a_{ij} y}{n} \\
S, S & = \frac{\sum a_{ij} (y - \bar{y})^2}{n} \\
S, S & = \frac{\sum a_{ij} y^2}{n} \\
S, S & = \frac{\sum a_{ij} (y - \bar{y})^2}{n} \\
S, S & = \frac{\sum a_{ij} y^2}{n} \\
\end{align*}
\]
The matrix $C$, such that $Z$ has the properties stated above, exists. This can be seen as follows. Let us construct a matrix $D = (d_{ij})$ as follows. Let $d_{ij} = \frac{1}{N} \quad (j = 1, 2, \ldots, N)$. Now consider the ordered vector $Y'$. Divide the $N$ elements of $Y'$ into $n_b$ sets of $N/n_b$ elements each by putting into set number one all elements whose $j^{th}$ subscript equals one; put into the second set all elements whose $j^{th}$ subscript is two etc. Now construct an orthogonal $n_b \times n_b$ matrix such that all elements in the first row are equal. Call the elements of this matrix $u_{ij}$. Now in the ordered vector $Y'$ replace every element whose $j^{th}$ subscript equals $j$ by $u_{ij}$. This is the second row of $D$. Then replace every element whose $j^{th}$ subscript equals $j$ by $u_{ij}$. This is the third row of $D$. Continue this until we have exhausted $u_{ij}$. We now have $n_b$ rows of $D$. Do this for every $t = 1, 2, \ldots, k$. We will then have constructed $k - (k-1)$ rows of $D$. Then fill in the remaining rows of $D$ so that they are orthogonal to the first $k - (k-1)$ rows and are orthogonal among themselves. Then normalize each row. The resulting matrix is the matrix $C$. By the structure of $C$ we see that

$$z_j = \sum_{i_1 i_2 \ldots i_k} c_{j t_1} q_{i_1 i_2 \ldots i_k} \quad \text{for } j = a_1, \ldots, b_k$$

where

$$\sum_{i_t} c_{j i_t} q_{i_1 i_2 \ldots i_k} = 0.$$

It is a well known fact that $\sum_{j=a_1}^{b_k} z_j^2 = \sum_{i_t} (G_{i_1 i_2 \ldots i_k} - \overline{G})^2$. So (a) is satisfied.
Also
\[ E \sum_{i} s_{i} s_{j} = E \sum_{i} \sum_{j} \gamma_{i} \gamma_{j} = \sum_{i} \sum_{j} \gamma_{i} \gamma_{j} = \sum_{i} \sum_{j} \gamma_{i} \gamma_{j} = 0 \quad i \neq j. \]

So (a) is satisfied.

(a), (b) and (c) are obviously satisfied by C.

It is also well known that
\[ E \left( \sum_{i} \gamma_{i} \right)^{2} = E \sum_{i} \gamma_{i}^{2} = \frac{N(n-1)}{n} \sigma_{\gamma}^{2}, \]

and
\[ E \left[ \text{remainder sum of squares} \right] = E \sum_{i=k+1}^{N} s_{i}^{2} = N - K + (k-1) \sigma_{\gamma}^{2} \]

So the analysis of variance estimate of \( \sigma_{\gamma}^{2} \) is
\[ \frac{n_{1}}{N} \left[ \frac{\sum_{i=1}^{N} \gamma_{i}^{2}}{N-1} \right] = \frac{N_{1}}{N} \left[ \frac{K_{1}}{K-1} \right] = \sum_{i=k+1}^{N} s_{i}^{2} \]

Definition 3. The matrix \( N \) will be called the analysis of variance matrix for estimating \( \sigma_{\gamma}^{2} \) where \( N = (n_{ij}) \) is defined
\[ n_{ij} = \begin{cases} 0 & i \neq j \\ 0 & i = j \end{cases} \]

\[ n_{ii} = \frac{n_{1}}{N(n_{1}-1)} \quad i = 2, 3, \ldots, N_{1} = n_{1} \]
\[ m_{ii} = 0 \quad \text{for } i = N_1 + 1, N_1 + 2, \ldots, N_k - k + 1 \]

\[ m_{ii} = \frac{n_1}{N(N_1 N_k + k - 1)} \]

To estimate \( \sigma^2_t \) there is no loss of generality if we permute the sub-
scripts on \( y_{i_1 \ldots i_k} \) until \( i_t \) is the first subscript and hence estimate \( \sigma^2_{1t} \).

**Definition.** Let \( Q = Z \cdot P \cdot Z \) be a quadratic estimate of \( \sigma^2_{1t} \) such that

(a) \( E \cdot Q = \sigma^2_{1t} \) i.e. unbiasedness.

(b) \( \text{var} \cdot Q \) is independent of \( \mu_t \).

(c) \( \text{var} \cdot Q \leq \text{var} \cdot Q' \) where \( Q' \) is any other quadratic estimate of \( \sigma^2_{1t} \)

which satisfies (a) and (b).

If an estimate satisfies conditions (a), (b) and (c) we will call it a best
quadratic unbiased estimate of \( \sigma^2_{1t} \).

**B. Theorems and Discussions.**

**Theorem 1.** Let the scalar random variable \( y_{i_1 \ldots i_k} \) be given by

model V. Then the best quadratic unbiased estimate of \( \sigma^2_{1t} (t = 1, 2, \ldots, k) \)

is given by the analysis of variance.

Let

\[ Q_t = Z \cdot P \cdot Z \quad (4.5) \]

be a quadratic estimate of \( \sigma^2_{1t} \). We will show that if \( Q_t \) satisfies the

conditions of D.4 then \( P \) must be equal to the matrix \( N \) defined in B.3. We
explained above that estimating \( \sigma^2 \) was perfectly general. To satisfy

conditions D.4(a) we must have
\[ E Z^T P Z = E \sum_{i,j} z_i z_j p_{ij} = E \sum_j z_j^2 p_{jj}. \tag{4.6} \]

Now \( E z_i^2 = E H y \ldots = H (\mu^2 + \text{some function of } \sigma^2, \sigma_1^2, \ldots, \sigma_k^2) = \mu^2 + NK \)
where \( K \) is some function of \( \sigma^2, \sigma_1^2, \ldots, \sigma_k^2 \). So using this fact together
with 4.2 and 4.3 we get

\[
E Z^T P Z = p_{11} (\mu^2 + NK) + \sum_{i=2}^{b_1} (\sigma^2 + \frac{H}{n_1} \sigma_1^2) p_{i1} + \sum_{i=n_2}^{b_2} (\sigma^2 + \frac{H}{n_2} \sigma_2^2) p_{ii} + \ldots + \sum_{i=n_k}^{b_k} (\sigma^2 + \frac{H}{n_k} \sigma_k^2) p_{ii} + \sum_{i=n_{k+1}}^{n} \sigma^2 p_{ii}. \tag{4.7} \]

This must equal \( \sigma_1^2 \). So we must have

(a) \( p_{11} = 0 \)

(b) \( \sum_{i=2}^{b_1} p_{i1} = \frac{n_1}{n} \)

(c) \( \sum_{i=n_j}^{b_j} p_{i1} = 0 \quad j = 2, 3, \ldots k \)

(d) \( \sum_{i=n_{k+1}}^{n} p_{i1} = -\frac{n_1}{n} \).

Let us examine \( E q_{1}^2 \).

\[ E q_{1}^2 = E(Z^T P Z)^2 = E \left[ \sum_{i,j} z_i z_j p_{ij} \right]^2 \]

\[ = E \left[ \sum_i p_{ii} z_i^4 + 2 \sum_{i \neq j} p_{ij} z_i^2 z_j^2 + \sum_i \sum_{j \neq k} p_{ij} p_{ik} z_i z_j z_k + \sum_{i \neq j} p_{ii} p_{jj} z_i^2 z_j^2 + \sum_{i \neq j \neq k \neq \ell} p_{ij} p_{jk} z_i z_j z_k z_\ell \right]. \tag{4.9} \]
\[
\Pi_N \left( \frac{2\omega_f}{\lambda} + \frac{(1-\eta)\omega_f}{\lambda_u} \right) = \frac{E_H}{\lambda_u} \] 

\[
\Pi_N \frac{2\omega_f}{\lambda} + \Pi_N \frac{2\omega_f}{\lambda} = \Pi_d \frac{2\omega_f}{\lambda} 
\]

\( o \) and \( \eta \) produce

\( \Pi_N (\eta) \) to \( \Pi_N (\eta) \) must have

\( \Pi_N (\eta) \) or \( \Pi_N (\eta) \) and zero. When the beam is zero, the second term of \( \eta \) is zero. The initial and boundary values of \( \xi \) are

\( \xi = \xi_{10} \) and \( \xi = \xi_{10} \) or \( \xi = \xi_{10} \) and have

\( \xi < 1 \) (1)}
So we get
\[
\sum_{i=2}^{N} a_{ii} = 0 \quad \text{(4.14)}
\]

Also from 4.8(c) it is evident that
\[
\sum_{i=s_j}^{b_j} a_{ii} = 0 \quad j = 2,3,\ldots k. \quad \text{(4.15)}
\]

Finally
\[
\sum_{i=n_{k+1}}^{N} \sum_{i=n_{k+1}}^{N} a_{ii} + \sum_{i=n_{k+1}}^{N} a_{ii} = 0
\]

Putting in \(a_{ii}\) and using 4.8(a) we have
\[
- \frac{n_1}{N} = \sum_{i=n_{k+1}}^{N} \frac{n_1}{N(N-H_{k+1})} + \sum_{i=n_{k+1}}^{N} a_{ii}
\]

Or
\[
\sum_{i=n_{k+1}}^{N} a_{ii} = 0 \quad \text{(4.16)}
\]

So \(S\) must satisfy the side conditions
\[
\begin{align*}
(a) & \quad a_{ij} = 0 \quad i \neq j \\
(b) & \quad a_{11} = 0 \\
(c) & \quad \sum_{i=2}^{N} a_{ii} = \sum_{i=n_{k+1}}^{N} a_{ii} = \sum_{i=2}^{N} a_{ii} = 0 \\
& \quad j = 2,3,\ldots k.
\end{align*}
\]

Substituting 4.11 into \(Q_1\) we have
\[ E Q_1^2 = E(Z'PZ)^2 = E \left[ Z'(M + X)Z \right]^2 \]
\[ = E(Z'MZ)^2 + 2 E(Z'MZ)(Z'SZ) + E(Z'SZ)^2. \quad (4.18) \]

Consider the cross product term
\[ T = 2E (Z'MZ)(Z'SZ) = 2E \left( \sum_{ij} s_i s_j e_{ij} \right) \left( \sum_{qr} s_q s_r e_{qr} \right) \]
\[ = 2E \sum_{ijqr} s_i s_j s_q s_r e_{ij} e_{qr}. \quad (4.19) \]

If any one of the subscripts \( ijqr \) is distinct from the other three it is evident that \( T = 0 \). So we have four cases to consider
\[
\begin{align*}
(a) & \quad i = j \neq q = r \\
(b) & \quad i = q \neq j = r \\
(c) & \quad i = r \neq j = q \\
(d) & \quad i = j = q = r.
\end{align*}
\]

But by 4.17(a) we have \( T = 0 \) for cases (b) and (c) since \( e_{qr} = 0 \) \( r \neq q \).

For case (a) we get
\[ T = 2E \sum_{i \neq r=2} s_i^2 s_r^2 e_{ii} e_{rr}. \quad (4.21) \]

By 4.2 we get \( E s_i^2 \) equals a constant, \( c_i \) say, for the cases
\( i = a_j, a_j + 1, \ldots, b_j \) and \( j = 1, 2, \ldots, k+1 \). So
\[ T = 2 \sum_{i=2}^N E s_i^2 e_{ii} \sum_{r=2}^N E s_r^2 e_{rr} \]
\[ = 2 \sum_{i=2}^N E s_i^2 e_{ii} \left[ \sum_{r=2}^N E s_r^2 e_{rr} - E s_i^2 e_{ii} \right] \]
\begin{align*}
&= 2 \sum_{i=2}^{N} \bar{z}_i \bar{z}_i \sum_{j=1}^{k+1} \frac{b_j}{r=a_j} \bar{z}_r \bar{z}_r - 2 \sum_{j=1}^{k+1} \frac{b_j}{i=a_j} (\bar{z}_i \bar{z}_i) \bar{z}_{ii} \bar{z}_{ii}.
\end{align*}

But from D.3 we have \( m_{ii} \) equals a constant, say, for \( i=a_j, a_{j+1}, \ldots, b_j \) and \( j=1, 2, \ldots, k+1 \). So \( T \) becomes

\begin{align*}
T &= 2 \sum_{i=2}^{N} \bar{z}_i \bar{z}_i \sum_{j=1}^{k+1} \frac{b_j}{r=a_j} \bar{z}_r \bar{z}_r - 2 \sum_{j=1}^{k+1} \frac{b_j}{i=a_j} \bar{z}_{ii} \bar{z}_{ii}.
\end{align*}

But these terms are both zero by 4.17(c). So \( T = 0 \) for case 4.20(a).

For case 4.20(d) we have

\begin{align*}
T &= 2 \sum_{i=2}^{N} \bar{z}_i \bar{z}_i \sum_{j=1}^{k+1} \frac{b_j}{r=a_j} \bar{z}_r \bar{z}_r = 6 \sum_{i=2}^{N} (\bar{z}_i \bar{z}_i) \bar{z}_{ii} \bar{z}_{ii} \\
&= 6 \sum_{j=1}^{k+1} \frac{b_j}{r=a_j} \bar{z}_{ii} \bar{z}_{ii} = 0 \text{ by 4.17(c).}
\end{align*}

So \( T = 0 \) for case 4.20(d). Thus the cross product term of 4.18 is zero.

So we have

\[
\mathbb{E} Q_1^2 = \mathbb{E} (Z' M Z)^2 \neq \mathbb{E} (Z' S Z)^2.
\]

The minimum value \( \mathbb{E} Q_1^2 \) can have is when \( S = 0 \). This is consistent with the side conditions 4.17. So the minimum variance of \( Q_1 \) is given by the matrix

\[
F = M.
\]

This proves the theorem.
Theorem 2. Let \( y_1 y_2 \ldots y_k \) be given by model V. The best quadratic unbiased estimate of \( \sigma^2 \) is given by the analysis of variance.

Let the best quadratic unbiased estimate of \( \sigma^2 \) be given by

\[
Q = Z' P Z.
\]

We want to show that \( P = K \) where \( K \) is defined as

\[
k_{ii} &= \frac{1}{\sum_{i=1}^{N} s_i^2} \quad i = 1, 2, \ldots, N \\
k_{ij} &= 0 \quad \text{otherwise.}
\]

This matrix \( K \) is the analysis of variance matrix as can be seen by using

4.3. Now for unbiasedness we must have

\[
E Q = E Z' P Z = E \sum_{i=1}^{N} \sum_{j=1}^{N} s_i s_j p_{ij} = E \sum_{i=1}^{N} s_i^2 p_{ii} = \frac{1}{N} \sum_{i=1}^{N} (\sigma^2 + \text{some function of } \sigma_i) \]

\[
= \left[ \mu^2 + \text{some function of } \sigma^2 \text{ and } \sigma_i^2 \right] p_{ii} + \frac{b_1}{N} \left( \sigma^2 + \frac{N}{n_1} \sigma_1^2 \right) p_{ii} + \frac{b_2}{N} \left( \sigma^2 + \frac{N}{n_2} \sigma_2^2 \right) p_{ii} + \ldots + \frac{b_k}{N} \left( \sigma^2 + \frac{N}{n_k} \sigma_k^2 \right) p_{ii} + \sigma^2 \sum_{i=1}^{N} p_{ii} = \sigma^2.
\]
So we must have

(a) \( p_{11} = 0 \)

\[
\begin{align*}
(b) & \quad \sum_{i=1}^{b} p_{ii} = 0 \quad j = 1, 2, \ldots, k \\
(c) & \quad \sum_{i=k+1}^{b} p_{ii} = 1.
\end{align*}
\]

Now

\[
E q^2 = E\left( \sum_{ij} z_1^2 z_j^2 p_{ij}\right)^2
\]

\[
= E\left[ 2 \sum_{i \neq j=2}^{N} z_1^2 z_j^2 p_{ij}^2 + \sum_{i=2}^{N} p_{ii}^2 z_1^2 + \sum_{i \neq j=2}^{N} p_{ii} p_{jj} z_1^2 z_j^2 \right].
\]

It is evident from 4.25 that to minimize \( E q^2 \) we will take

\[
p_{ij} = 0 \quad (i \neq j).
\]

This is also consistent with the restrictions of unbiasedness.

So the problem reduces to finding the minimum of the variance of

\[
q = E q^2 - \sigma^2 = E (Z P Z)^2 - \sigma^2
\]

subject to conditions 4.24 and 4.26.

Let us assume

\[
P = K + H
\]

where \( K \) is given by 4.22 and \( H \) is a matrix such that \( P \) satisfies 4.24, 4.26 and minimizes the variance of \( Q \). In view of the side conditions on \( P \) we see that \( H \) must satisfy the side restrictions.

(a) \( h_{11} = 0 \)

(b) \( h_{ij} = 0 \quad i \neq j \)
(c) \[ \sum_{i=a_j}^{b_j} h_{ii} = 0 \quad i = 1, 2, \ldots k \]

(d) \[ \sum_{i=a_{k+1}}^{b_{k+1}} h_{ii} = 0 \]  \hspace{1cm} (4.28)

So we have

\[ \mathbb{E}w^2 = \mathbb{E}(Z' P Z)^2 = \mathbb{E} \left[ Z' (K + H) Z \right]^2 = \mathbb{E} \left[ Z' K Z + Z' H Z \right]^2 \]

\[ = \mathbb{E}(Z' K Z)^2 + 2\mathbb{E}(Z' K Z)(Z' H Z) + \mathbb{E}(Z' H Z)^2. \]  \hspace{1cm} (4.29)

We will examine the term

\[ T = 2\mathbb{E}(Z' K Z)(Z' H Z) = 2\mathbb{E} \left( \sum_{ij} s_i s_j k_{ij} \right) \left( \sum_{qr} s_q s_r h_{qr} \right). \]  \hspace{1cm} (4.30)

By 4.22 and 4.26(a), (b) this reduces to

\[ T = 2\mathbb{E} \left( \sum_{i=a_{k+1}}^{b_i} z_i^2 k_{ii} \right) \left( \sum_{j=2}^{b_j} z_j^2 h_{jj} \right) \]

\[ = 2 \mathbb{E} \left( \sum_{i=a_{k+1}}^{b_i} z_i^2 k_{ii} \right) \left( \sum_{j=1}^{b_j} \sum_{q=a_j}^{b_i} z_q^2 h_{qq} + \sum_{r=a_{k+1}}^{b_j} z_r^2 h_{rr} \right). \]

By 4.2 we see that \( z_i^2 = c_j \) for \( i = a_j, \ldots b_j \) and \( j = 1, 2, \ldots k+1 \).

So 4.30 becomes

\[ T = 2\mathbb{E} \left( \sum_{i=a_{k+1}}^{b_i} z_i^2 k_{ii} \right) \left( \sum_{j=1}^{b_j} c_j \sum_{q=a_j}^{b_i} h_{qq} + \sum_{r=a_{k+1}}^{b_j} z_r^2 h_{rr} \right). \]

Using 4.28 (c) this reduces to

\[ T = 2\mathbb{E} \left( \sum_{i=a_{k+1}}^{b_i} z_i^2 k_{ii} \right) \left( \sum_{j=2}^{b_j} z_j^2 h_{jj} \right) \]
Using \( h.22 \) we get

\[
T = \sum_{i=1}^{N} h_{ii} + \sum_{i=j}^{N} \frac{2h}{H-k} \sum_{i=j}^{N} h_{jj} .
\]

But both terms are zero by \( h.26(3) \). So \( T = 0 \) and

\[
E \bar{q}^2 = H(\bar{x} \bar{z} \bar{z})^2 + H(\bar{z} \bar{z} \bar{z})^2 .
\]

So this is an absolute minimum if \( H = 0 \). This is consistent with the side conditions \( h.26 \). Thus theorem 2 is proved.

Theorem 3. Let \( Y_1, Y_2, \ldots, Y_k \) be given by model \( V \). Let

\[
L = \sum_{i=1}^{k} \varepsilon_i \sigma_i^2 + g \sigma^2
\]

where \( g \) and \( \varepsilon_i \) \((i=1, 2, \ldots, k)\) are constants independent of the model. Then the best quadratic unbiased estimate of \( L \) is given by

\[
\sum_{i=1}^{k} \varepsilon_i \sigma_i^2 + g \sigma^2
\]

where \( \sigma^2 \) and \( \sigma_i^2 \) \((i=1, 2, \ldots, k)\) are respectively the best quadratic unbiased estimate of \( \sigma^2 \) and \( \sigma_i^2 \) \((i=1, 2, \ldots, k)\). Using theorem 1 and D.3 we see that

\[
\sum_{i=1}^{k} \varepsilon_i \sigma_i^2 + g \sigma^2 \sum_{i=1}^{k} \varepsilon_i \frac{n_i}{H} \left[ \frac{\sum_{i=1}^{k} s_i \sum_{i=k+1}^{N} s_i}{H-k} \right] + g \sum_{i=1}^{k+1} s_i
\]

(4.31)
If we denote this estimate of \( L \) by
\[
\bar{Z} = \overline{Z}
\]
then \( R \) is equal to

(a) \( r_{11} = 0 \)

(b) \( r_{ij} = 0 \) \( i \neq j \) (4.32)

(c) \( r_{jj} = \frac{\xi_i n_1}{N(n_1 - 1)} \) \( j = a_1, a_1 + 1, \ldots, b_i ; i = 1, 2, \ldots, k \)

(d) \( r_{jj} = -\left[ \sum_{i=1}^{k} \frac{\xi_i n_1}{N} - \delta \right] \cdot \frac{1}{N^k - n - 1} \) \( j = a_k + 1, \ldots, N \).

So the problem is to find a quadratic estimate of \( L \) which is (a) unbiased (b) independent of \( \mu \) and (c) has minimum variance. Let \( Q_L = \bar{Z}' P \bar{Z} \) be this quadratic estimate. If we can show that \( P = \bar{Z} \), the theorem is proved. For unbiasedness we must have

\[
E_Q = E \bar{Z}' P \bar{Z} = E \sum_{j=1}^{n} s_j z_j P_{1j} = E \sum_{i=1}^{N} s_i a_{i+1} P_{11}
\]

\[
= (\mu^2 + \text{some function of } \sigma^2, a_i \ i = 1, 2, \ldots, k) P_{11}
\]

\[
+ \sum_{i=2}^{b_1} (\sigma_i^2 + \frac{N}{n_1} a_i^2) P_{11}
\]

\[
+ \sum_{i=2}^{b_2} (\sigma_i^2 + \frac{N}{n_2} a_i^2) P_{11} + \cdots + \sum_{i=b_k}^{b_k} (\sigma_i^2 + \frac{N}{n_k} a_i^2) P_{11} + \sum_{i=b_k+1}^{b_k} \sigma_i^2 P_{11} = \bar{L}.
\]

So we must have

(a) \( p_{11} = 0 \)

(b) \( \sum_{i=a_j}^{b_j} p_{11} = \frac{\xi_j n_1}{N} \) \( j = 1, 2, \ldots, k \)
(c) \[ \sum_{i=2}^{N} p_{ii} = g \] or \[ \sum_{i=a_{k+1}}^{N} p_{ii} = g - \sum_{j=1}^{k} \frac{E_{i} n_{i}}{N}. \] (4.33)

Now

\[ E Q_{L}^{2} = E\left( \sum_{i,j=1}^{N} s_{i} s_{j} p_{ij}\right)^{2} = E \left[ 2 \sum_{i,j=2}^{N} s_{i} s_{j} p_{ij}^{2} + \sum_{i=2}^{N} s_{i}^{2} p_{ii}^{2} + \sum_{i,j=2}^{N} s_{i} s_{j} p_{ii} p_{jj} \right]. \] (4.34)

To minimize \( E Q_{L}^{2} \) we must obviously have

\[ p_{ij} = 0 \quad i \neq j. \] (4.35)

So we get

\[ E Q_{L}^{2} = E \left( \sum_{i=2}^{N} s_{i}^{2} p_{ii}^{2} + \sum_{i,j=2}^{N} s_{i} s_{j} p_{ii} p_{jj} \right) = E \left[ \sum_{i=2}^{N} p_{ii} s_{i}^{2} \right]. \] (4.36)

The problem is to minimize 4.36 subject to conditions 4.33 and 4.35. Let us assume the solution is

\[ P = R + U \] (4.37)

where \( R \) is the matrix given in 4.32 and \( U \) is any arbitrary matrix such that

\[ E(ZPZ)^{2} \]

is a minimum subject to 4.33 and 4.35.

For \( P \) to satisfy conditions 4.33 and 4.35 we must have

(a) \[ \sum_{i=2}^{N} p_{ii} = 0 \]

(b) \[ \sum_{i=a_{j}}^{N} p_{ii} = 0 \quad j = 1, 2, \ldots k \]
(c) \( \sum_{i=2}^{N} u_{1i} = 0 \)

(d) \( u_{ij} = 0 \) \( i \neq j \).  \hspace{1cm} (4.35)

From (4.36) we get

\[
\sum \mathbf{e}_{i}^{2} = \mathbb{E} \left[ \sum_{i=2}^{N} \left( r_{1i} + u_{1i} \right) s_{i}^{2} \right] - \sum_{i=2}^{N} r_{ii} s_{i}^{2} + \sum_{j=2}^{N} \mathbb{E} \left[ \sum_{i=2}^{N} u_{ij} \right] s_{j}^{2} + \mathbb{E} \left[ \sum_{i=2}^{N} u_{ii} s_{i}^{2} \right] - \sum_{i=2}^{N} r_{ii} s_{i}^{2} + \sum_{j=2}^{N} \mathbb{E} \left[ \sum_{i=2}^{N} u_{ij} \right] s_{j}^{2} + \mathbb{E} \left[ \sum_{i=2}^{N} u_{ii} s_{i}^{2} \right].
\]

(4.39)

Let us consider the cross product term

\[
T = 2 \mathbb{E} \sum_{i=2}^{N} r_{1i} U_{jj} s_{1i}^{2} s_{j}^{2} + \mathbb{E} \sum_{i=2}^{N} r_{ii} U_{1j} s_{i}^{2} s_{j}^{2}.
\]

(4.40)

Now \( r_{1i} = \frac{s_{i}}{N(a_{j} - 1)} = \delta_{ij} \) for \( i = a_{j}, \ldots, b_{j}; j = 1, 2, \ldots, k+1 \).

(4.41)

Also \( \mathbb{E} s_{i}^{2} = \) constant, say \( c_{j} \) for \( i = a_{j}, \ldots, b_{j}; j = 1, 2, \ldots, k+1 \).

So

\[
T = 6 \sum_{j=1}^{k+1} \delta_{ij} c_{j} \sum_{i=a_{j}}^{b_{j}} U_{1i} + 2 \sum_{i=2}^{N} \mathbb{E} s_{i}^{2} r_{1i} \left( \sum_{j=2}^{N} \mathbb{E} s_{j}^{2} U_{1j} - \mathbb{E} s_{i}^{2} U_{1i} \right).
\]

The first term is zero by (4.35). The second term can be written

\[
2 \sum_{i=2}^{N} \mathbb{E} s_{i}^{2} r_{1i} \sum_{j=1}^{k+1} \delta_{ij} \sum_{q=a_{j}}^{b_{j}} U_{1j} - 2 \sum_{j=1}^{k+1} \delta_{ij} c_{j} \sum_{i=a_{j}}^{b_{j}} U_{1i}
\]

which is zero by (4.35).
So $T = 0$ and

$$
E Q_L^2 = E(Z' U Z)^2 + E(Z' R Z).
$$

This is an absolute minimum if $U \equiv 0$. Since this is also consistent with the side conditions 4.35 this proves that $P = R$ is the matrix that minimizes variance $Q_L$ subject to the side conditions 4.33 and 4.35. Thus theorem 3 is proved.
V. SUMMARY

Quadratic estimates of variance components for various linear models were investigated.

For the general balanced nested classification with no specific distributions assumed, it was shown that the quadratic estimate which was unbiased and which had minimum variance was given by the analysis of variance method of estimating the variance components.

For the one fold classification with unequal numbers the model is

$$y_{ij} = \mu + a_i + b_{ij}$$

where $E a_i^2 = \sigma_a^2$; $E b_{ij}^2 = \sigma^2$.

The best quadratic unbiased estimate of $\sigma_a^2$ depends on $\sigma^2$ and $\sigma_a^2$. Under the above model with the additional assumptions $E a_i^3 = 3 \sigma_a^4$ and $E b_{ij}^3 = 3 \sigma^4$ the quadratic estimation of $\sigma_a^2$ was investigated. Two methods of estimation were given, and a method of determining a lower bound on the efficiency of each method was given if $w = \sigma_a^2/\sigma^2$ is known. It is believed in most cases when $w > .01$, that one of these two methods will give an efficiency above 50 per cent.

For the general balanced cross classification with normality assumptions it was shown that the best quadratic unbiased estimate of the variance components is given by the analysis of variance method of estimating variance components. It was also shown that the best quadratic unbiased estimate of any linear combination of variance components is given by the same linear combination of the best quadratic unbiased
estimates of the individual variance components.
VI. LITERATURE CITED


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