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Overlapping Pfaffians with application to utility theory

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OVERLAPPING PFAPFIANS WITH APPLICATION
TO UTILITY THEORY

by

Clarence H. Lindahl

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

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1952
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I. INTRODUCTION

A. Meaning of the Solution of a Pfaffian Equation

1. Integrability

A Pfaffian form or Pfaffian is defined\(^1\) as any linear form in differentials \(dx_1, \ldots, dx_n\) where the coefficients are functions of the real variables \(x_1, \ldots, x_n\). The associated differential equation

\[ w = \sum_{i=1}^{n} x_i dx_i = 0, \quad X_i = X_i(x_1, \ldots, x_n) \quad (1.1) \]

will be referred to as a Pfaffian equation. Most of the Pfaffian equations encountered in elementary theory are either exact or can be made exact by multiplication with an integrating factor since they satisfy the conditions of integrability, of which a typical one is

\[ x_1 \left( \frac{\partial x_i}{\partial x_k} - \frac{\partial x_k}{\partial x_i} \right) + x_j \left( \frac{\partial x_k}{\partial x_i} - \frac{\partial x_i}{\partial x_k} \right) + x_k \left( \frac{\partial x_i}{\partial x_j} - \frac{\partial x_j}{\partial x_i} \right) = 0, \]

or

\[ x_1 a_{jk} + x_j a_{ki} + x_k a_{ij} = 0 \]

where

\[ a_{rs} = \frac{\partial x_r}{\partial x_s} - \frac{\partial x_s}{\partial x_r} \quad (1.2) \]

There are \(\binom{n}{2}\) or \(\frac{n(n-1)(n-2)}{6}\) of these conditions of which only \(\frac{1}{2}(n-1)(n-2)\) are independent.\(^2\) These are both necessary

---


and sufficient conditions that the Pfaffian equation be derived from a single integral (that is, if \( f(x_1, \ldots, x_n) = 0 \) is the integral, then \( df = 0 \) or \( vdf = 0 \) is the Pfaffian equation). The integrable case is quite special. Most Pfaffian equations that one might write satisfy only some or none of the conditions of integrability and the question arises as to the meaning of the solution of such an equation if there be any.

According to Forsyth, Euler declared that unless the integrability conditions be satisfied the equation was absurd and had no significance. However, Monge pointed out that the absurdity lies not in the supposition that the equation can have significance but in the inference that the solution consists of a single integral. For example, a Pfaffian equation in three variables, if the integrability condition is satisfied, belongs to two-dimensional surfaces or manifolds, but if the integrability condition is not satisfied the differential equation represents some property of space curves, which require two integrals for their full expression, and the differential equation is a consequence of the two equations. Thus the property represented by the above Pfaffian equation, \( w \), (when \( n=3 \)) is:

(a) a property common to a family of space curves defined by two equations when the integrability condition is not satisfied.

---

(b) a property common to a family of curves which can
be drawn on each member of a one-parameter family of surfaces
when the integrability condition is satisfied. The Pfaffian
equation is said to belong to the surfaces.

2. General case

Consider the equation (1.1) where the $X_i$ have continuous second derivatives in some common domain. If $w$ is
integrable there is an integral

\[ f(x_1, \ldots, x_n) = c, \]

where either $df = w$, or $df = uw$, $u$ being an integrating factor.
Since there is only one restriction on the variables, a
solution manifold (or solution space) of $n-1$ dimensions
exists. If the equation is not integrable one can find the
integrals\(^4\)

\[ f_1(x_1, \ldots, x_n) = c_1, \quad f_2(x_1, \ldots, x_n) = c_2, \ldots, \quad f_p(x_1, \ldots, x_n) = c_p, \]

such that

\[ w = r_1 df_1 + r_2 df_2 + \ldots + r_p df_p; \quad r_i = r_i(x_1, \ldots, x_n), (i=1, \ldots, p). \]

The solution manifold here is of $n-p$ dimensions. One
classical problem is to find integrals such that $p$ is as
small as possible and hence such that the solution manifold
is of the greatest possible dimension.

\(^4\)Goursat, op. cit., p. 3.
In 1815 Pfaff presented to the Academy of Berlin his classical memoir in which he stated the result that an integral equivalent of a Pfaffian equation containing $2n$ or $2n-1$ variables can always be constituted by a system of integrals, the number in the system being not greater than $n$. After that such men as Gauss, Jacobi, Lie, Frobenius, Darboux as well as many present day mathematicians have worked and are now working on related problems.\(^5\)

B. Canonical Pfaffians

Of special interest are the canonical forms to which the Pfaffian can be transformed. For example, if

$$x_1 \, dx_1 + x_2 \, dx_2 + x_3 \, dx_3 = 0$$

is not integrable it can be reduced to the canonical form

$$du_1 + u_2 \, du_3 = 0,$$

where $u_1$, $u_2$, and $u_3$ are independent functions of the original variables. This canonical equation may be satisfied in various ways as follows:

1. $u_1 = \text{const.}, \quad u_3 = \text{const.}$
2. $u_1 = \text{const.}, \quad u_2 = 0$,
3. $g(u_1, u_3) = 0, \quad u_2 \frac{\partial g}{\partial u_1} - \frac{\partial g}{\partial u_3} = 0$.

---

where (c) includes (b) but not (a).

The following sections of this chapter are a brief analysis of canonical forms, in particular Section B, 3 and 4, uses the algebraic approach of Frobenius. The bilinear covariant associated with the Pfaffian form will be described as well as a certain invariant integer (class number) whose existence is the necessary and sufficient condition for the transformation of one Pfaffian into another. The character of the canonical form is uniquely determined by the class number of the Pfaffian.

1. Reduction of the Pfaffian

The Pfaffian (1.1) can be written in any number of different forms by a change of variables:

\[ x_i = h_i(y_1, \ldots, y_n), \quad (i = 1, \ldots, n), \quad (1.3) \]

where the Jacobian \( \frac{D(h_1, \ldots, h_n)}{D(y_1, \ldots, y_n)} \) is not equal to zero in the domain where the solution is sought.

To a linear element (which, in the x-space, is defined to be a 2n-tuple or ensemble consisting of a point \( (x_1, \ldots, x_n) \) and a direction \( (dx_1, \ldots, dx_n) \) at the point) of the y-space corresponds, under the above point-transformation, a linear element of the x-space for which

\[ \quad \text{G. Frobenius. "Ueber das Pfaffsche Problem." Crelles Journal 82:230-315. 1877.} \]
\[ dx_1 = \frac{\partial h_1}{\partial y_1} dy_1 + \ldots + \frac{\partial h_1}{\partial y_n} dy_n, \quad (i=1, \ldots, n). \] (1.4)

Now applying the transformation to \( w \),

\[ \sum_{i=1}^{n} X_i \, dx_1 = Y_1 \, dy_1 + \ldots + Y_n \, dy_n, \]

where \( Y_1 = X_1 \frac{\partial h_1}{\partial y_1} + \ldots + X_n \frac{\partial h_n}{\partial y_1}, \quad (i=1, \ldots, n). \)

Further, the solution manifolds of the two Pfaffian equations

\[ w = \sum_{i=1}^{n} X_i \, dx_1 = 0 \quad \text{and} \quad w_0 = \sum_{i=1}^{n} Y_i \, dy_1 = 0 \]

correspond by the above point-transformation in such a way that the linear elements on the two solution manifolds (which are called linear integral elements since they verify the relations \( w=0 \) and \( w_0=0 \) respectively) also correspond by the transformations (1.3) and (1.4).

A sequence of transformations (1.3) can be chosen in such a way that any Pfaffian will have a simpler form. The transformation, (1.3), can be chosen such that

\[ y_1 = c_1, \quad y_2 = c_2, \quad \ldots, \quad y_{n-1} = c_{n-1} \]

represent a family of integral curves.\(^7\) \( w_0 \) must be satisfied when \( y_1, y_2, \ldots, y_{n-1} \) are replaced by these constants and consequently the coefficient \( Y_n \) will be zero. However, in some cases, the new form can be made to contain one less

\(^7\)Goursat, op. cit., p. 8.
variable than the original form except possibly for a common factor of all the coefficients. In other cases the new form can be rid of the variable $y_n$ except for its differential.

The following theorem is stated without proof:

Theorem I

A. The Pfaffian form

$$w = \sum_{i=1}^{n} x_i \, dx_i$$

can always be transformed by a convenient change of variables (1.3) into one of the three forms

(a) $y_n(y_1 \, dy_1 + \ldots + y_{n-1} \, dy_{n-1})$, 
(b) $y_1 \, dy_1 + \ldots + y_{n-1} \, dy_{n-1} + dy_n$, 
(c) $y_1 \, dy_1 + \ldots + y_{n-1} \, dy_{n-1}$, 

$$y_i = x_i(y_1, \ldots, y_{n-1}), \ (i = 1, \ldots, n-1).$$

B. The Pfaffian form

$$w = \sum_{i=1}^{n} x_i \, dx_i$$

can be transformed by (1.3) into one of the two following forms

(a) $z_1 \, dv_1 + \ldots + z_p \, dv_p + dv_{p+1}$, 
(b) $z_1 \, dv_1 + \ldots + z_p \, dv_p$,

where the functions $v_1, z_k$ constitute a system of independent

---

variables, that is, independent functions of the variables which appear in w.

Theorem IB follows from Theorem IA by induction and it is the forms (a) and (b) of Theorem IB that are the most useful, particularly when p has its minimum value. It can be proved (see section 3) that p has a unique minimum and it is very valuable to know a priori this minimum value. The latter can be determined from certain matrices associated with the bilinear covariant of the Pfaffian expression, described in the next section.

2. **Bilinear covariant**

The bilinear covariant may be derived as follows:

In the identity

\[ \sum_{i=1}^{n} x_i \, dx_i = \sum_{i=1}^{n} y_i \, dy_i \]

replace \( y_1, \ldots, y_n \) by functions of two independent variables u and v. Since

\( y_1 = y_1(u,v) \) and \( x_1 = x_1(u,v) \),

then

\[ dy_1 = \frac{\partial y_1}{\partial u} \, du + \frac{\partial y_1}{\partial v} \, dv \]

and

\[ dx_1 = \frac{\partial x_1}{\partial u} \, du + \frac{\partial x_1}{\partial v} \, dv. \]

Set the coefficients of du equal to each other and similarly for dv. Next, differentiate the first expression with respect to u. The resulting expression is
\[
\sum_{1 \leq k} a_{1k} \left( \frac{\partial x_1}{\partial u} \frac{\partial x_k}{\partial v} - \frac{\partial x_1}{\partial v} \frac{\partial x_k}{\partial u} \right) = \sum_{1 \leq k} b_{1k} \left( \frac{\partial y_1}{\partial u} \frac{\partial y_k}{\partial v} - \frac{\partial y_1}{\partial v} \frac{\partial y_k}{\partial u} \right).
\]

Multiply through by \( du \ dv \) and set

\[
dx_1 = \frac{\partial x_1}{\partial u} \ du, \ \delta x_1 = \frac{\partial x_1}{\partial v} \ dv; \ dy_1 = \frac{\partial y_1}{\partial u} \ du, \ \delta y_1 = \frac{\partial y_1}{\partial v} \ dv.
\]

The expression then becomes

\[
\sum_{1 \leq k} a_{1k}(dx_1 \ delta x_k - dx_k \ delta x_1) = \sum_{1 \leq k} b_{1k}(dy_1 \ delta y_k - dy_k \ delta y_1),
\]

or \( w^1 = \sum_{1 \leq k} a_{1k}(dx_1 \ delta x_k - dx_k \ delta x_1) = \sum_{1, k=1}^n a_{1k} dx_1 \ delta x_k. \)

\( w^1 \) is called the bilinear covariant of the original Pfaffian.

Two sets of differentials \((dx_1, \ldots, dx_n)\) and \((\delta x_1, \ldots, \delta x_n)\) are said to be in involution at the point \((x_1, \ldots, x_n)\) if they satisfy the relation \( w^1 = 0 \) there. Rewrite \( w^1 \) in matrix form:

\[
\begin{bmatrix}
 a_{11} & a_{12} & \ldots & a_{1n} \\
 a_{21} & a_{22} & \ldots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
 dx_1 \\
 \vdots \\
 dx_n
\end{bmatrix}
\]

\[
w^1 = (\delta x_1 \ \ldots \ \delta x_n)
\]

Now impose the conditions
\[
\begin{align*}
& \begin{cases}
\alpha \; dx_1 + \alpha_{12} \; dx_2 + \ldots + \alpha_{1n} \; dx_n = 0, \\
\alpha_{21} \; dx_1 + \alpha_{22} \; dx_2 + \ldots + \alpha_{2n} \; dx_n = 0, \\
\vdots \\
\alpha_{n1} \; dx_1 + \alpha_{n2} \; dx_2 + \ldots + \alpha_{nn} \; dx_n = 0.
\end{cases}
\end{align*}
\]

Then \(w^1 = 0\) regardless of the values of \(dx_1, \ldots, dx_n\).

Hence, \(dx_1, \ldots, dx_n\) is in involution with all other linear elements at the same point. Such an element is called a linear singular element.

\(S_1\) is a covariant system of \(w\) relative to any change of variables of type (1.3).\(^9\) It should be noted that the matrix of \(S_1\) is skew-symmetric since

\[a_{ik} + a_{ki} = 0, \; (i, k = 1, \ldots, n).\]

3. **Class of a Pfaffian**

The following theorem is stated without proof.\(^{10}\)

**Theorem II**

If the system \(S_1\) includes \(r\) linearly independent equations, these \(r\) equations are equivalent to a system of the form

\[df_1 = 0, \; df_2 = 0, \ldots, \; df_r = 0,\]

\(f_1, f_2, \ldots, f_r\) being \(r\) distinct functions of the variables \(x_1, \ldots, x_n.\)


\(^{10}\)Ibid., p. 27.
If the determinant of the coefficients of $S_1$, $|A|$, is not identically zero $S_1$ is equivalent to the $n$ equations
\[ dx_1 = 0, \; dx_2 = 0, \; \ldots, \; dx_n = 0. \]
If $S_1$ contains only $n-1$ linearly distinct equations (that is, $|A| = 0$) in $n$ variables it is solvable. If there are $r < n-1$ linearly distinct equations in $S_1$ adjoin $n-r-1$ equations where the coefficients are functions of $x_1, \ldots, x_n$. This again is solvable. Let
\[ f_1 = c_1, \; f_2 = c_2, \; \ldots, \; f_{n-1} = c_{n-1} \]
be the general integral of the system which depends on $n-1$ arbitrary constants. Now make a change of variables such that
\[ y_1 = f_1, \; y_2 = f_2, \; \ldots, \; y_{n-1} = f_{n-1}, \]
the last variable, $y_n$, remaining arbitrary. The given form $w$ is replaced by the new form
\[ w_0 = \sum_{1}^{n-1} Y_1 \, dy_1 \]
where the coefficient $Y_n$ can be given the value of zero or one. This type of transformation may have to be repeated several times until $w_0$ will have one of the two forms
\[ Y_1 \, dy_1 + Y_2 \, dy_2 + \ldots + Y_{2p} \, dy_{2p} + dy_{2p+1}, \tag{1.5} \]
or
\[ Y_1 \, dy_1 + Y_2 \, dy_2 + \ldots + Y_{2p} \, dy_{2p}, \]
where $2p = r$ and $Y_i = Y_i(y_1, \ldots, y_{2p}), (i=1, \ldots, 2p)$. 
The linear singular elements which are also integral elements verify equations of

\[ S_2 \left\{ \begin{aligned}
    a_{11} \, dx_1 + a_{12} \, dx_2 + \ldots + a_{1n} \, dx_n &= 0, \\
    x_1 \, dx_1 + x_2 \, dx_2 + \ldots + x_n \, dx_n &= 0,
\end{aligned} \right. \]

which is obtained by adjoining to the system \( S_1 \) the equation \( w = 0 \). The system \( S_2 \) is a covariant of the form \( w \) relative to any change of variables.

If the equation \( w = 0 \) is distinct from the equation of \( S_1 \), the system \( S_2 \) contains \( r+1 = 2p+1 \) linearly distinct equations. If \( w \) is transformed to the first equation of (1.5) then \( S_1 \) is equivalent to the system

\[ dy_1 = 0, \ldots, dy_{2p} = 0 \]

while the system \( S_2 \) becomes

\[ dy_1 = 0, dy_2 = 0, \ldots, dy_{2p} = 0, dy_{2p+1} = 0. \]

Hence, \( S_2 \) is also integrable since the last integral of this system can be obtained by a quadrature after the system \( S_1 \) has been integrated.

Let \( c \) be the number of distinct integrals of the system \( S_2 \), that is, the number of linearly independent equations of this system. The number \( c \) is called the class of the form of Pfaff. It is the minimum number of variables in which one can express \( w \) by a convenient change of variables. It was shown above that if \( S_2 \) contains \( 2p+1 \) or \( 2p \) distinct
equations the form \( w \) can be changed to one of the forms (1.5). In both cases the expression obtained depends only on \( c \) variables. It is impossible to obtain for \( w \) any other expression where there are less than \( c \) variables. For suppose that by a change of variables \( w \) takes on the form

\[
w_0 = \sum_{i=1}^{q} z_i dz_i + \cdots + z_q \, dz_q; \quad q < c, \quad z_i = z_i(z_1, \ldots, z_q), \quad (i=1, \ldots, q).
\]

Then the system \( S_2 \) corresponding to \( w_0 \) may not include more than \( q \) linearly distinct relations since it includes only \( q \) differentials. It cannot then be equivalent to the system \( S_2 \) which includes \( c > q \) distinct equations.

The same reasoning shows that if, by any means, one can obtain for \( w \) an expression where there are only \( c \) variables, these \( c \) variables are the integrals of the system \( S_2 \). One can, therefore, find any number of different expressions where there appear only \( c \) variables. However, any one of these expressions in \( c \) variables can be changed into any one of the others by a change of variables, and consequently is called a canonical form of \( w \).

4. Matrix representation and final canonical forms

Obtaining the class of \( w \) is equivalent to finding the rank of the matrix \( T_2 \) which comes from the system \( S_2 \):
In this thesis, however, the class of a Pfaffian will generally be found by the use of the skew-symmetric matrix \( B \) which is derived from \( T_2 \) by bordering on the left by the negative of the elements in the top row of \( T_2 \). Hence,

\[
T_2 = \begin{bmatrix}
X_1 & X_2 & \cdots & X_n \\
0 & a_{12} & \cdots & a_{1n} \\
-a_{12} & 0 & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
-a_{1n} & -a_{2n} & \cdots & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & X_1 & X_2 & \cdots & X_n \\
-X_1 & 0 & a_{12} & \cdots & a_{1n} \\
-X_2 & -a_{12} & 0 & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & & \vdots \\
-X_n & -a_{1n} & -a_{2n} & \cdots & 0
\end{bmatrix}
\]

It should be noted that

\[
A = \begin{bmatrix}
0 & a_{12} & \cdots & a_{1n} \\
-a_{12} & 0 & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
-a_{1n} & -a_{2n} & \cdots & 0
\end{bmatrix}
\]

is the matrix of the bilinear covariant and is skew-symmetric. It always has even rank \( 2p \). Hence, the rank of \( T_2 \) (which is the class \( c \) of the Pfaffian) is either \( 2p \) or \( 2p+1 \). In
the matrix B, if the rank of A is 2p the rank of B is
either 2p or 2p+2 since B is skew-symmetric. Therefore,
the class number can be derived by computing the mean of
the sum of the separate ranks of A and B. This will again
be either 2p or 2p+1 respectively.

It was stated earlier that the canonical forms listed
in Theorem IB are more useful, especially in the solution
of a Pfaffian equation than forms of the type of (1.5). It
can be proved\(\textsuperscript{11}\) that any Pfaffian of odd class, 2p+1 can
be written

\[ w_{2p+1} = du_{2p+1} + u_{2p} du_{2p-1} + u_{2p-2} du_{2p-3} + \ldots + u_{2p-2p} \ldots u_4 u_2 du_1, \]

or simply

\[ w_{2p+1} = z_1 dy_1 + z_2 dy_2 + \ldots + z_p dy_p + dy_{p+1}, \]  

\(1.6) \]

\(y_1, y_2, \ldots, y_{p+1}, z_1, z_2, \ldots, z_p\) forming a system of 2p+1
independent functions of the original variables. Similarly

\[ w_{2p} = z_1 dy_1 + z_2 dy_2 + \ldots + z_p dy_p, \]  

\(1.7) \]

\(y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_p\) forming a system of 2p
independent functions of the original variables.

Thus, any Pfaffian can be transformed to a form where
the number of variables is equal to the class. Since two

\(\textsuperscript{11}\) Ibid., pp. 41-42.
forms of the same class can always be changed to the same canonical form, one can always go from one to the other by a transformation of variables. Hence, the only invariant of the Pfaffian is the class of the form.

It may be noted that the two Pfaffian forms \( w \) and \( w_0 \) are related by the following matrix identity:

\[
B_0 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\vdots & & J' \\
0 & \cdots & \cdots & 0
\end{bmatrix}
\quad B = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & J \\
0 & \cdots & \cdots & 0
\end{bmatrix},
\]

where \( J \) is the Jacobian matrix of (1.3),

\[
B_0 = \begin{bmatrix}
0 & Y_1 & \cdots & Y_n \\
-X_1 & & & \\
\vdots & & A_0 & \\
-X_n & & & &
\end{bmatrix}
\]

and \( A_0 \) is the matrix of the bilinear covariant of \( w_0 \).

C. Illustration

1. Method of Frobenius

(a) Determination of class of Pfaffian:

If

\[
w = x_2 \, dx_1 + x_3 \, dx_2 + x_1 \, dx_3,
\]

then

\[
w^{1} = (\delta x_1 \, \delta x_2 \, \delta x_3) \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{bmatrix} \begin{bmatrix}
dx_1 \\
dx_2 \\
dx_3
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
0 & x_2 & x_3 & x_1 \\
-x_2 & 0 & 1 & -1 \\
-x_3 & 0 & 1 & 0 \\
-x_1 & 1 & -1 & 0
\end{bmatrix},
\]

rank of \(A = 2\), rank of \(B = 4\),
therefore, class of \(w = \frac{4+2}{2} = 3\).

(b) Solution of the Pfaffian equation, \(w = 0\):

Since
\[
S_1 \begin{cases}
\text{dx}_2 - \text{dx}_3 = 0, \\
-\text{dx}_1 + \text{dx}_3 = 0, \\
\text{dx}_1 - \text{dx}_2 = 0,
\end{cases}
\]

therefore
\[
f_1 = x_2 - x_1 = c_1, \quad f_2 = x_3 - x_1 = c_2
\]
are two independent integrals of \(S_1\).
Let \(y_1 = x_2 - x_1, \quad y_2 = x_3 - x_1,\)

hence
\[
x_1 = x_3 - y_2, \quad x_2 = y_1 - y_2 + x_3,
\]

and \(w\) becomes
\[
w_0 = -y_1 \, dy_2 + d\left(\frac{3x_3^2}{2} + \frac{y_2^2}{2} + y_1 \, x_3 - 2x_3 \, y_2\right),
\]
or
\[
w_0 = d(x_2 \, x_3 + \frac{1}{2} \, x_1^2) + (x_1 - x_2) \, d(x_3 - x_1).
\]

This is the canonical form
\[
w_3 = du_1 + u_2 \, du_3
\]
whose solutions are:
(a) \( x_3 - x_1 = \text{const.}, \quad x_2 x_3 + \frac{1}{2} x_1^2 = \text{const.} \)

(b) \( x_1 - x_2 = 0 \), \( x_2 x_3 + \frac{1}{2} x_1^2 = \text{const.} \)

(c) \( g(u_1, u_3) = 0 \), \( u_2 \frac{\partial g}{\partial u_1} - \frac{\partial g}{\partial u_3} = 0 \)

D. Grassmann Calculus of Pfaffian Forms

Another method for the determination of the class of a Pfaffian form is the use of the algebra invented by Grassmann and used, in particular, by Cartan. Rules for multiplication and differentiation of differential forms are needed in order to use this method. They are (roughly):

(a) The multiplication of two distinct differential elements is non-commutative and obeys the law

\[ dx_1 \ dx_j = - \ dx_j \ dx_1, \]

and any differential raised to an integral power greater than one is zero.

(b) The multiplication of two variables or of a variable and a differential is commutative.

(c) The rules of ordinary calculus can be applied in the differentiation of differential forms provided before each differentiation the factor to be differentiated, \( X_i \),


\(^{13}\) Élie Cartan. Les systèmes différentiels extérieurs. Actualités Scientifiques et Industrielles 994. 1945.
is placed in the leading position and \( dx_1 \) treated as a constant. Using these rules the class of a Pfaffian form can be determined as follows:

<table>
<thead>
<tr>
<th>Class</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2p</td>
<td>( w(1)p \neq 0, \ w(1)p \ w = 0 )</td>
</tr>
<tr>
<td>2p+1</td>
<td>( w(1)p \ w \neq 0, \ w(1)p+1 = 0 ).</td>
</tr>
</tbody>
</table>

1. **Example of the method of Grassmann for determination of class**

If

\[
w = x_2 \, dx_1 + x_3 \, dx_2 + x_1 \, dx_3
\]

then

\[
w(1) = dx_2 \, dx_1 + dx_3 \, dx_2 + dx_1 \, dx_3,
\]

or

\[
w(1) = -dx_1 \, dx_2 - dx_2 \, dx_3 + dx_1 \, dx_3.
\]

Hence

\[
w(1)w = -(x_1 + x_2 + x_3) \, dx_1 \, dx_2 \, dx_3,
\]

and

\[
w(1)^2 = 0.
\]

Therefore

\[
p+1 = 2,
\]

\[
p = 1,
\]

\[
2p+1 = 3 \text{ = class of } w.
\]

(Note: The solution of the Pfaffian equation is omitted because of the amount of additional Grassmann theory required.)
E. Objective and Nature of Results

The general objective of this thesis is to consider the properties of those Pfaffian forms for which the bilinear covariant matrix, $A$, has a special structure. Most of the results concern the case where the matrix has one super-diagonal stripe. This particular structure usually is a consequence of the choice of variables which appear in the coefficients of the Pfaffians treated in this thesis. The principal results depend on properties of overlapping Pfaffian chains (defined in Chapter II).

Chapter II first deals with the rank of any one-stripe matrix $A$ as well as with the rank of the matrix which results after bordering by a row and column of functions which are the coefficients of the Pfaffian. Some results concerning common class of overlapping Pfaffians are then deduced.

In Chapter III is considered the case where a sub-set of the integrability conditions is imposed upon the overlapping sub-Pfaffian chains of length three. It is proved that these conditions imply exactness when the coefficients, in a restricted number of variables, are polynomials of the first and second degree or general functions under the specific restriction given in the corollary to Theorem III in Chapter III.
Chapter IV is concerned with overlapping chains of length greater than three and with chains of fluctuating class. Also, results are obtained in the case where non-adjacent variables are used in the coefficients.

A type of Pfaffian in which the variables form a complete cycle is analyzed in Chapter V.

In the last chapter properties of the above types of Pfaffians are applied to utility theory in economics and indicate that if enough rationality is assumed to imply integrability in a certain "local sense," then the imposition of this "local" rationality implies rationality in the large.

F. Designation of Properties of Functions

In order to facilitate the statement of theorems, corollaries, and lemmas certain properties of functions and families of functions will be defined and designated by $P_1$. They are:

$P_1$: Functions being considered have continuous second order partial derivatives in some common domain (simply connected open set),

$P_2$: Functions being considered are of type $P_1$ and not identically zero.
II. THE ONE-STRIPE CASE

In this chapter a general study is made of the Pfaffians whose bilinear covariant matrices have the fairly simple structure described in Section B below. Almost any deviation from this case presents difficulties the author has not yet overcome. The particular problem treated is that of the possibility of common class for the sub-Pfaffians defined in Section A, as well as the implications of such an assumption for the Pfaffian itself. A less general but more detailed analysis of this problem for common class at most two is reserved for Chapter III.

A. Overlapping Pfaffian Chains

From the Pfaffian

\[ w = \sum_{j=1}^{n} X_j \, dx_j, \]

\[ X_j = X_j(x_1, \ldots, x_n) \] of type \( P_1 \) in a common domain, take a particular sub-Pfaffian

\[ w_s = X_1 \, dx_1 + \ldots + X_k \, dx_k, \]

where \( k \geq 1 \) and every term from 1 to \( k \), inclusive, appears. \( w_s \) will be called a Pfaffian chain of \( w \). Extend \( w_s \) by one
neighboring term first to the right and then to the left giving the two additional Pfaffian chains

\[ w_{r} = x_1 \, dx_1 + \ldots + x_k \, dx_k + x_{k+1} \, dx_{k+1}, \]

and \[ w_{x} = x_{1-1} \, dx_{1-1} + x_1 \, dx_1 + \ldots + x_k \, dx_k. \]

The active variables, as distinguished from parameters, in the Pfaffian chains \( w_{r} \) and \( w_{x} \) will be recognized by the respective sets of differentials. All other variables in \( w_{r} \) and \( w_{x} \) will be considered passive variables or parameters.

The matrix associated with \( w_{r} \) is

\[
\begin{bmatrix}
0 & x_1 & x_{1+1} & \ldots & x_{k+1} \\
-x_1 & 0 & a_{1,1+1} & \ldots & a_{1,k+1} \\
-x_{1+1} & -a_{1,1+1} & 0 & \ldots & a_{1+1,k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-x_k & -a_{1,k} & \ldots & 0 & a_{k,k+1} \\
-x_{k+1} & -a_{1,k+1} & \ldots & -a_{k,k+1} & 0
\end{bmatrix},
\]

where

\[ a_{rs} = \frac{\partial x_r}{\partial x_s} - \frac{\partial x_s}{\partial x_r}, \]

and the class of \( w_{r} \) is the arithmetic mean of the ranks of the entire matrix and the bilinear covariant matrix obtained by deleting the top row and the left-hand column (Chapter I, B4). The question arises as to the conditions under which
$w_{s_r}$ and $w_{s_1}$ have the same class. The bilinear covariant matrix of $w_{s_r}$ will be designated by $A_i$ while the corresponding bordered matrix will be called $B_i$ ($A_{i-1}$ and $B_{i-1}$ are similarly defined for $w_{s_1}$). A few simple illustrations of common class one (exactness) follow. It is easily seen that in the two cases $X_j = \text{constant}$ and $X_j = X_j(x_j)$ the ranks of $A_i$ and $A_{i-1}$ are zero and those of $B_i$ and $B_{i-1}$ are two giving a common class number of one. Again, $X_j = x^{j-1} x_j + x_j x_j x_{j+1}$, which is somewhat less trivial, gives a common class number of one. These are very special conditions and can, of course, be liberalized.

B. Restriction of Bilinear Covariant Matrix to a One-Stripe Matrix

By the imposition of suitable restrictions on the coefficients of the Pfaffian $w$, a bilinear covariant can be obtained whose matrix has non-zero elements only in the stripes just above and below the principal diagonal. This will be called a one-stripe matrix. A study will now be made of the rank of this matrix, first the unbordered $A$ and then the bordered $B$. The bordered matrix is, of course, skew-symmetric. Following is the unbordered matrix $A$:
1. Rank of the one-stripe bilinear covariant matrix

The following Lemma can be stated:

**Lemma I**

In the Pfaffian form

\[ w = \sum_{j=1}^{n} x_j \, dx_j; \quad a_{rs} = 0, \quad |r-s| \geq 2. \]

(a) If \( n \) is even, a necessary and sufficient condition for the rank of \( A \) to be full, that is, equal to the number of elements in the stripe plus one, is that there exists at least one point at which the elements \( a_{12}, a_{34}, \ldots, a_{n-1,n} \) do not simultaneously equal zero.

(b) If \( n \) is odd and if the condition on \( a_{rs} \) of part (a) is true for some \( n-1 \) order principal minor then the rank of \( A \) is equal to the number of elements in the stripe (that is, \( n-1 \)), and conversely.

Proof:

(a) By inspection it can be seen that
\[ |A| = (a_{12} a_{34} \ldots a_{n-1,n})^2. \]

This is a special case of the value of the determinant of a skew-symmetric matrix of even order.\(^1\)

(b) Since \(n\) is odd and skew-symmetric, \(|A| = 0\). However, there exists an \(n-1\) order principal minor whose determinant is not zero. On the other hand, if the rank of \(A\) is \(n-1\), then it is well known that there exists a non-vanishing \(n-1\) order principal minor.

Hereafter in this chapter it will be assumed that all the elements in the stripe satisfy the conditions in the above lemma.

2. Rank of the bordered one-stripe matrix

Consider the bordered matrix,

\[
B = \begin{pmatrix}
0 & x_1 & x_2 & x_3 & \ldots & x_n \\
-x_1 & 0 & a_{12} & 0 & \ldots & 0 \\
-x_2 & -a_{12} & 0 & a_{23} & 0 & \ldots \\
-x_3 & 0 & -a_{23} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-x_{n-1} & \ldots & 0 & a_{n-1,n} \\
-x_n & 0 & \ldots & -a_{n-1,n} & 0
\end{pmatrix}
\]

Bordering a skew-symmetric in the above manner will not decrease the rank but may increase it. By Lemma I if the order of $A$ is even, the rank of $A$ is full and even, hence the rank of $B$ equals the rank of $A$ since $B$ is then skew-symmetric and of odd order. Therefore, the class of the Pfaffian chain of even length is even and equal to the rank of its bilinear covariant matrix which in turn equals the number of elements in the stripe plus one. Thus, the following theorem has been proved.

**Theorem I**

In the Pfaffian form

$$w = \sum_{j=1}^{n} X_j \, dx_j,$$

$a_{rs}, |r-s| = 1$, of type as in Lemma I; $a_{rs} = 0, |r-s| \geq 2$, if $d(k - 1 + 2)$ is an even number (the lengths of the Pfaffian chains, $w_{s1}$ and $w_{s2}$) then $w_{s1}$ and $w_{s2}$ have a common class number which is $d$.

If the order of $A$ is odd the question to be answered is: Does bordering add to the rank or not? Is the top row, for example, a linear combination of the other rows of the matrix? To show the type of condition implied when the top border is a linear combination of the other rows of $B$, take a chain of length three:

$$X_1 \, dx_1 + X_{i+1} \, dx_{i+1} + X_{i+2} \, dx_{i+2},$$
where

\[
B_1 = \begin{bmatrix}
0 & X_1 & X_{1+1} & X_{1+2} \\
-X_1 & 0 & a_{1,1+1} & 0 \\
-X_{1+1} & -a_{1,1+1} & 0 & a_{1+1,1+2} \\
-X_{1+2} & 0 & -a_{1+1,1+2} & 0
\end{bmatrix}
\]

If bordering does not add to the rank of $A_1$ then

\[
|B_1| = (X_1 a_{1+1,1+2} + X_{1+2} a_{1,1+1})^2 = 0,
\]
or

\[
X_1 a_{1+1,1+2} + X_{1+2} a_{1,1+1} = 0.
\]

This may be written:

\[
X_1 a_{1+1,1+2} + X_{1+1} a_{1+2,1} + X_{1+2} a_{1,1+1} = 0,
\]

where the middle term can be inserted since in the one-stripe matrix $a_{1+2,1} = 0$. Hence, this is the integrability condition of a Pfaffian chain of length three.

For a chain of length five the corresponding condition is

\[
X_1 a_{1+1,1+2} a_{1+3,1+4} + X_{1+2} a_{1+1,1+3} a_{1+4,1+5} + X_{1+3} a_{1+1,1+2} a_{1+4,1+5} = 0.
\]

This condition can easily be read from the $B_1$ matrix by deleting the odd-numbered columns and even-numbered rows and expanding the determinant of the remaining elements by the elements of the first row.
The latter is the condition stated above.

The following general case will now be proved as an easy consequence of the identity used in the proof of Lemma Ia.

**Theorem II**

If, in the w of Lemma I, n is odd, then w has class n-1 if and only if

$$X_1 a_{23} a_{45} \ldots a_{n-1,n} + X_2 a_{12} a_{45} \ldots a_{n-1,n} + \ldots$$

$$+ X_n a_{12} a_{34} \ldots a_{n-2,n-1} = 0.$$ 

Proof:

Since n is odd the order of the bordered matrix, B, is even. The determinant of B is a perfect square function
of its elements and is given by

$$ (X_1 \sqrt{P_1} + X_2 \sqrt{P_2} + \ldots + X_n \sqrt{P_n})^2, $$

where $P_1$ is a principal minor obtained from $A$ by striking out the row and column which correspond to $X_1$ in $B$. It is easily verified that these minors are alternately zero.

**Corollary I**

If the chains of Theorem I are of odd length $d$ then they have a common class number which is $d-1$ if and only if the condition of Theorem II holds for each chain.

**Proof:**

By Theorem II each has a class number of $d-1$.

**Corollary II**

The whole Pfaffian of Theorem I (consisting of the two chains together, $w_{s_1}$ and $w_{s_2}$) is of odd length $d+1$ and has class $d$ or $d+1$, according as the condition of Theorem II holds or does not hold.

**Proof:**

If the condition of Theorem II holds the common rank of $A$ and $B$ is $d$, and hence the class is $d$. If the condition does not hold the respective ranks of $A$ and $B$ are $d$ and $d+2$. Therefore, the class is $d+1$.

**Corollary III**

The whole Pfaffian of Corollary I is of even length
d+1 and has class d+1.

Proof:

Since A is of even order and of full rank d+1, the bordering does not add to the rank. Hence, the class is d+1.

G. Canonical Form as a One-Stripe Case

In Chapter I was described a canonical form of the Pfaffian w. This form is, respectively, for odd and even class,

\[ w_{2p+1} = z_1 dz_2 + z_3 dz_4 + \ldots + z_{2p-1} dz_{2p} + dz_{2p+1}, \]

and \[ w_{2p} = z_1 dz_2 + z_3 dz_4 + \ldots + z_{2p-1} dz_{2p}. \]

The variables, respectively, form a system of 2p+1 and 2p independent functions of the original variables. \( w_{2p+1} \) and \( w_{2p} \) can be rewritten in the following form by using zero coefficients such that each variable appears in a differential:

\[ w_{2p+1} = 0 \cdot dz_1 + z_1 dz_2 + 0 \cdot dz_3 + z_3 dz_4 + \ldots + 0 \cdot dz_{2p-1} + z_{2p-1} dz_{2p} + dz_{2p+1}, \]

\[ w_{2p} = 0 \cdot dz_1 + z_1 dz_2 + 0 \cdot dz_3 + z_3 dz_4 + \ldots + 0 \cdot dz_{2p-1} + z_{2p-1} dz_{2p}. \]

The bilinear covariant matrices, \( A_{2p+1} \) and \( A_{2p} \), have the following forms:
It is interesting to note that both of these matrices are one-stripe; however, half or just less than half of the $X_j$'s are zero. It is easily seen that if $w$ is in canonical form, then it is impossible to have constant class, for $w_{sX}$ is always even class and $w_{sr}$ odd class if $c$ (the class of $w$) is odd. On the other hand, if $c$ is even then the class of $w_{sr}$ is two more than the class of $w_{sX}$. One may also note that $A_{2p+1}$ and $A_{2p}$ are canonical forms of
constant and real skew-symmetric matrices.

D. Restriction of Variables in the Coefficients of a Pfaffian

A bilinear covariant whose matrix is one-stripe may be obtained even from a Pfaffian whose coefficients are functions of all the variables $x_1, \ldots, x_n$. An example of this is given in Appendix A. However, in order to obtain some manageable expressions as well as to accommodate applications, in this thesis the coefficients will generally be restricted to functions of only three variables, so chosen as to give a bilinear covariant whose matrix is one-stripe. Thus, generally in this thesis

$$X_j = X_j(x_{j-1}, x_j, x_{j+1}),$$

where $X_j$ will usually be subject to one of the restrictions $P_1, P_2$.

E. Cyclo-symmetry

First consider an additional special restriction in the form of the coefficients, namely:

$$X_i = f(x_{i-1}, x_i, x_{i+1}) \quad \text{and} \quad X_{i+1} = f(x_i, x_{i+1}, x_{i+2}),$$

c.s. \quad \text{c.s.}

that is $X_i \rightarrow X$ (where $\rightarrow$ means "goes into" or "transforms into"),
by the transformations

\[ x_{i-1} \to x_i, \quad x_i \to x_{i+1}, \quad x_{i+1} \to x_{i+2}, \quad \text{for all } i. \]

This type of transformation will be called cyclo-symmetric.

\( w_s, w_{s_r}, w_{s_2} \) will be respectively:

\[
    w_s = f(\alpha_{i-1}, x_i, x_{i+1}) \ dx_i + \ldots + f(x_{k-1}, x_k, \alpha_{k+1}) \ dx_k,
\]

(Note: \( x_{i-1} \) and \( x_{k+1} \) are parameters and the letters \( \alpha_{i-1} \) and \( \alpha_{k+1} \) are used, but when the Pfaffian chain \( w_s \) is extended to the right \( \alpha_{k+1} \) becomes the variable \( x_{k+1} \) again and when extended to the left \( \alpha_{i-1} \) becomes the variable \( x_{i-1} \) again.)

\[
    w_{s_r} = f(\alpha_{i-1}, x_i, x_{i+1}) \ dx_{i+1} + \ldots + f(x_{k-1}, x_k, x_{k+1}) \ dx_k + f(x_k, x_{k+1}, \alpha_{k+2}) \ dx_{k+1},
\]

\[
    w_{s_2} = f(\alpha_{i-2}, x_{i-1}, x_i) \ dx_{i-1} + f(x_{i-1}, x_i, x_{i+1}) \ dx_i + \ldots + f(x_{k-1}, x_k, \alpha_{k+1}) \ dx_k.
\]

According to the above scheme of clustering the variables in groups of three or less, it can be seen that there are possibilities of non-zero elements only in the stripes just above and below the principal diagonal of the skew-symmetric matrix \( A_i \); and, in fact, these elements have the same general structure; that is,

\[
    \frac{\partial x_i}{\partial x_{i+1}} - \frac{\partial x_{i+1}}{\partial x_i} \quad \text{c.s.} \quad \frac{\partial x_{i+1}}{\partial x_{i+2}} - \frac{\partial x_{i+2}}{\partial x_{i+1}}.
\]
This means that consecutive elements in the stripe are related by the cyclo-symmetric transformation. Hence, since the coefficients are cyclo-symmetric and the corresponding elements of the $A_1$ matrices are cyclo-symmetric, the $B_1$ matrices must have the same structure. Therefore, the Pfaffian chains of length $d$ overlapping in $d-1$ consecutive terms must have the same class. That is, the arithmetic mean of the ranks of $A_1$ and $B_1$ matrices is the same in each case. This proves:

**Theorem III**

The cyclo-symmetry of the $X_j$'s is a sufficient condition for common class of the overlapping Pfaffian chains, $w_{g_1}$ and $w_{g_2}$, of any given length $d$.

It can easily be seen that if the condition of cyclo-symmetry is imposed on the coefficients for chains of length $d$, where the bilinear covariant matrix is one-stripe, the rank of $A_1$ is zero or $d-1$ or $d$. The corresponding rank of $B_1$ is two or $d+1$ or $d$. Hence, the class will be one or $d$. Therefore, with $d \geq 3$ it is impossible for the class of the chain to be exactly two (which means integrable but not exact). Hence, the theorem:

**Theorem IV**

If in the Pfaffian chain

$$w = \sum_{J=n_1}^{n_2} X_j \, dx_j, \quad (n_2 - n_1) \geq 2$$

the condition of cyclo-symmetry

\[ x_j = f(x_{j-1}, x_j, x_{j+1}), \quad x_{j+1} = f(x_j, x_{j+1}, x_{j+2}) \]

is imposed, class two is impossible. Hence, any two chains of length \( d = n_2 - n_1 + 1 \), overlapping or not, cannot have a common class of two.

F. Common Class Less Than or Equal to Two

For overlapping chains of length two the class may be one or two since there are only two active variables. Any set of overlapping chains may have a common class of two if and only if the ranks of the \( B_1 \) and \( A_1 \) matrices are each equal to two. The question then arises concerning the possibility of finding successive integrating factors such that each of a sequence of overlapping chains will be integrable but not exact. In the cyclo-symmetric case it is impossible to find such integrating factors when the length of chains is \( d \geq 3 \). It should be noted that it was only necessary to use a single chain. In the general case one must clearly use more than one chain and the possibility of common class two is, of course, not obvious. The next chapter is devoted to the proof of theorems concerning this possibility.
III. INTEGRABILITY OF CHAINS OF LENGTH THREE

This chapter will deal with the one-stripe case where the variables in the coefficients are restricted to three (see Chapter II, Section D). Two is clearly a crucial class in Pfaffian theory. The purpose of this chapter is to discover what is implied when integrability is imposed upon overlapping Pfaffian chains of length three. Other lengths are considered in Chapter IV. Generally it turns out that the imposition of the above condition implies exactness of the whole Pfaffian consisting of the chains under question.

A. Differential Equations Satisfied by Integrating Factors

**Lemma I**

If from all Pfaffian chains of length three,

\[ w_{i+t} = \sum_{j=i+t}^{i+t+k} x_j \, dx_j, \]

\( n_1, n_2 \) integers and \( n_1 \leq i+t \leq n_2 - 2; \ k = 0, 1, 2 \), of the Pfaffian form,

\[ w = \sum_{j} x_j \, dx_j; \ n_1 \leq j \leq n_2, \ (n_2 - n_1) \geq 7, \]
where \( X_j = X_j(x_{j-1}, x_j, x_{j+1}) \) are of type \( P_1 \), two overlapping chains of length three,

\[
\omega_1 = \sum_{k=0}^{2} x_{1+k} \, dx_{1+k} \quad \text{and} \quad \omega_{1+1} = \sum_{k=0}^{2} x_{1+1+k} \, dx_{1+1+k}, \quad (3.1)
\]

can be chosen which satisfy the conditions of integrability, the systems of partial differential equations for the respective integrating factors, \( u_1 \) and \( u_{1+1} \), are\(^2\)\(^3\):

(a) \( x_1 \, \frac{\partial u_1}{\partial x_{1+2}} - x_{1+2} \, \frac{\partial u_1}{\partial x_1} = 0, \)

(b) \( x_1 \, \frac{\partial u_1}{\partial x_{1+1}} - x_{1+1} \, \frac{\partial u_1}{\partial x_1} = u_1 \, a_{1,1+1}, \quad (3.2) \)

(c) \( x_{1+1} \, \frac{\partial u_1}{\partial x_{1+2}} - x_{1+2} \, \frac{\partial u_1}{\partial x_{1+1}} = u_1 \, a_{1+1,1+2}, \)

\(^1\)In this case \( u_1 = \frac{1}{v_1} \) and \( u_{1+1} = \frac{1}{v_{1+1}} \) where \( v_1 \) and \( v_{1+1} \) are the usual integrating factors and not identically zero.

\(^2\)\(^3\)While the general form of these systems is well known (as indicated by E. L. Ince. Ordinary differential equations. London, Dover Publications Inc. 1926. p. 28) it is desirable to derive them directly, due to the nature of the coefficients and of the integrating factors, since the expressions which arise in this case are used in later theory. Therefore, the converse will not be proved.
(a') \( x_{i+1} \frac{\partial u_{i+1}}{\partial x_{i+1}} - x_{i+2} \frac{\partial u_{i+1}}{\partial x_{i+1}} = 0 \),

(b') \( x_{i+1} \frac{\partial u_{i+1}}{\partial x_{i+2}} - x_{i+2} \frac{\partial u_{i+1}}{\partial x_{i+1}} = u_{i+1} a_{i+1, i+2}, \quad (3.3) \)

(c') \( x_{i+2} \frac{\partial u_{i+1}}{\partial x_{i+3}} - x_{i+3} \frac{\partial u_{i+1}}{\partial x_{i+2}} = u_{i+1} a_{i+2, i+3} \).

Any pair of non-zero solutions, \( u_i \) and \( u_{i+1} \), will be integrating factors of the respective chains.

Proof:

Consider the two overlapping chains (3.1). Assume these chains to be integrable but not necessarily exact. Hence, there exists successive integrating factors, \( u_i \) and \( u_{i+1} \), such that the chains (3.1) may be written,

\[ u_i \overline{x}_i \, dx_i + u_i \overline{x}_{i+1} \, dx_{i+1} + u_i \overline{x}_{i+2} \, dx_{i+2}, \]

and

\[ u_{i+1} \overline{x}_{i+1} \, dx_{i+1} + u_{i+1} \overline{x}_{i+2} \, dx_{i+2} + u_{i+1} \overline{x}_{i+3} \, dx_{i+3}, \]

where

\[ \overline{x}_i \, dx_i + \overline{x}_{i+1} \, dx_{i+1} + \overline{x}_{i+2} \, dx_{i+2}, \]

and

\[ \overline{x}_{i+1} \, dx_{i+1} + \overline{x}_{i+2} \, dx_{i+2} + \overline{x}_{i+3} \, dx_{i+3}, \]

are exact. For these forms the following obvious relationships can be written

(a) \( x_i = u_i \overline{x}_i \),

(b) \( x_{i+1} = u_i \overline{x}_{i+1} = u_{i+1} \overline{x}_{i+1} \),

(c) \( x_{i+2} = u_i \overline{x}_{i+2} = u_{i+1} \overline{x}_{i+2} \),

(d) \( x_{i+3} = u_{i+1} \overline{x}_{i+3} \).

The exactness conditions are
(a) \( \frac{\partial x_i}{\partial x_{i+1}} = \frac{\partial x_{i+1}}{\partial x_i} \),

(b) \( \frac{\partial x_i}{\partial x_{i+2}} = \frac{\partial x_{i+2}}{\partial x_i} \),

(c) \( \frac{\partial x_{i+1}}{\partial x_{i+2}} = \frac{\partial x_{i+2}}{\partial x_{i+1}} \).

Due to the conditions on \( x_j = x_j(\mathbf{x}_{j-1}, x_j, x_{j+1}) \), the integrability conditions on the chains (3.1) may be written in shortened form, namely:

(a) \[ a_{1,1+1,1+2} + x_{1+2} u_{1,1+1} = 0, \]

(b) \[ x_{1+1} a_{1+2,1+3} + x_{1+3} a_{1+1,1+2} = 0. \]

By differentiating relationships (3.2), a and b, it follows that

\[ \frac{\partial x_i}{\partial x_{i+1}} = u_i \frac{\partial x_i}{\partial x_{i+1}} + \frac{\partial u_i}{\partial x_i}, \quad \frac{\partial x_{i+1}}{\partial x_i} = u_i \frac{\partial x_{i+1}}{\partial x_i} + x_{i+1} \frac{\partial u_i}{\partial x_i}. \]

Subtracting the latter expressions one obtains:

\[ a_{1,1+1} = u_i \left( \frac{\partial x_i}{\partial x_{i+1}} - \frac{\partial x_{i+1}}{\partial x_i} \right) + x_{i+1} \frac{\partial u_i}{\partial x_{i+1}} - x_{i+1} \frac{\partial u_i}{\partial x_i}. \]

Assumed here from the theory of Chapter I, Section B, 3 which consequently yields a derivation different from that in most textbooks.
But by (3.5), a, and (3.4), a and b,

\[ a_{1,1+1} = \bar{x}_{1} \frac{\partial u_1}{\partial x_{1+1}} - \bar{x}_{1+1} \frac{\partial u_1}{\partial x_{1}} = \frac{1}{u_1}(x_{1,1} \frac{\partial u_1}{\partial x_{1+1}} - x_{1+1} \frac{\partial u_1}{\partial x_{1}}). \] (3.7)

Also from (3.4), b and c, it follows that

\[ \frac{\partial x_{1+1}}{\partial x_{1+2}} = u_1 \frac{\partial x_{1+1}}{\partial x_{1+2}} + \bar{x}_{1+1} \frac{\partial u_1}{\partial x_{1+2}}, \quad \frac{\partial x_{1+2}}{\partial x_{1+1}} = u_1 \frac{\partial x_{1+2}}{\partial x_{1+1}} + \bar{x}_{1+2} \frac{\partial u_1}{\partial x_{1+1}}. \]

Subtracting the latter expressions one obtains:

\[ a_{1+1,1+2} = u_1 \left( \frac{\partial x_{1+1}}{\partial x_{1+2}} - \frac{\partial x_{1+2}}{\partial x_{1+1}} \right) + \bar{x}_{1+1} \frac{\partial u_1}{\partial x_{1+2}} - \bar{x}_{1+2} \frac{\partial u_1}{\partial x_{1+1}}. \]

But by (3.5), c, and (3.4), b and c,

\[ a_{1+1,1+2} = \bar{x}_{1+1} \frac{\partial u_1}{\partial x_{1+2}} - \bar{x}_{1+2} \frac{\partial u_1}{\partial x_{1+1}} = \frac{1}{u_1}(x_{1+1} \frac{\partial u_1}{\partial x_{1+2}} - x_{1+2} \frac{\partial u_1}{\partial x_{1+1}}). \] (3.8)

Now substitute (3.8) and (3.7) in (3.6), a. This gives:

\[ \frac{1}{u_1} \left[ x_1 \left( \frac{\partial u_1}{\partial x_{1+2}} - \frac{\partial u_1}{\partial x_{1+1}} \right) + x_{1+2} \left( \frac{\partial u_1}{\partial x_{1+1}} - \frac{\partial u_1}{\partial x_{1}} \right) \right] = 0, \]

which, after simplification becomes:

\[ \frac{x_{1+1}}{u_1} \left( x_{1+1} \frac{\partial u_1}{\partial x_{1+2}} - x_{1+2} \frac{\partial u_1}{\partial x_{1}} \right) = 0. \] (3.9)

Therefore, \( u_1 \) is a solution of the system (3.2).

Proof of the converse will not be given (see footnote 2).

In like manner \( u_{1+1} \) satisfies (3.3). Also, if \( u_{1-1} \) is the integrating factor for \( w_{1-1} \) the three partial differential equations it satisfies are recorded for reference:
\[
\begin{align*}
X_{i+1} \frac{\partial u_{i-1}}{\partial x_{i+1}} - X_i \frac{\partial u_{i-1}}{\partial x_i} &= 0, \\
X_{i+1} \frac{\partial u_{i-1}}{\partial x_i} - X_i \frac{\partial u_{i-1}}{\partial x_{i+1}} &= u_{i-1} a_{i-1,i}, \quad (3.10) \\
X_{i} \frac{\partial u_{i+1}}{\partial x_{i+1}} - X_{i+1} \frac{\partial u_{i+1}}{\partial x_i} &= u_{i+1} a_{i,i+1}.
\end{align*}
\]

B. Choice of Integrating Factors

The above equations for the integrating factor become much simpler if one makes a special assumption on the \( u_{i+1} \)'s such as making \( u_1 \) for example, free of \( x_1 \) and \( x_{i+2} \). It will turn out that this restriction is not nearly as severe as one might expect. The first result concerning such integrating factors follows.

1. Conditions on the \( a_{i,j} \)

**Theorem I**

If \( u_1 \) and \( u_{i+1} \) are respective integrating factors of

\[
\begin{align*}
w_i &= \sum_{k=0}^{2} X_{i+k} \, dx_{i+k} \quad \text{and} \quad w_{i+1} = \sum_{k=0}^{2} X_{i+1+k} \, dx_{i+1+k}, \quad (3.11) \\
X_j &= X_j(x_{j-1}, x_j, x_{j+1}) \text{ are of type } P_2
\end{align*}
\]

such that

\( u_1 \) is free of \( x_1 \) (and therefore of \( x_{i+2} \)) and \( u_{i+1} \) is free of \( x_{i+1} \) (and therefore of \( x_{i+3} \)), then
\( a_{1,1+1} \) is free of \( x_{1+2} \),
\( a_{i+1,1+2} \) is free of \( x_{1} \) and \( x_{1+3} \), \hspace{1cm} (3.12)
\( a_{i+2,1+3} \) is free of \( x_{1+1} \).

Conversely, if the above conditions are satisfied and \( w_{1} \)
and \( w_{i+1} \) are integrable then there exists a \( u_{1} \) free of \( x_{1} \)
and a \( u_{i+1} \) free of \( x_{1+1} \).

Proof:

Consider the three partial differential equations (3.2)
algebraically. The matrix of coefficients and augmented
matrix are respectively:

\[
M_{c} = \begin{bmatrix}
-X_{1+2} & 0 & X_{1} \\
-X_{i+1} & X_{1} & 0 \\
0 & -X_{i+2} & X_{i+1}
\end{bmatrix}, \hspace{1cm} M_{a} = \begin{bmatrix}
-X_{1+2} & 0 & X_{1} & 0 \\
-X_{i+1} & X_{1} & 0 & u_{1} a_{i,1+1} \\
0 & -X_{i+2} & X_{i+1} & 0 \\
0 & -X_{i+2} & X_{i+1} & u_{1} a_{i+1,1+2}
\end{bmatrix}
\]

\( |M_{c}| = 0 \) but it is easy to find second order minors whose
determinants are non-zero. For \( M_{a} \) each minor of order three
which includes the last column has a determinant, a factor
of which is

\[ X_{1} a_{i+1,1+2} + X_{i+2} a_{1,1+1} \].

This expression is the left-hand side of the integrability
condition (3.6), \( a \), which is assumed to be satisfied by
the hypothesis. The system of partial differential equa-
tions is then a consistent system in the partial derivatives
with the common rank of the matrix and augmented matrix equal to two. Hence, one of the unknowns may be assigned at pleasure, and the others will be uniquely determined. \( \frac{\partial u_1}{\partial x_1} \) will be chosen as the unknown. This is possible since the coefficient matrix of the remaining unknowns, \( \frac{\partial u_1}{\partial x_1+1} \) and \( \frac{\partial u_1}{\partial x_1+2} \), is of rank two using, for example, equations a and b of system (3.2). Hence,

\[
\frac{\partial u_1}{\partial x_1+2} = \frac{x_1+2}{x_1} \frac{\partial u_1}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_1+1} = \frac{x_1+1}{x_1} \frac{\partial u_1}{\partial x_1} + \frac{u_1}{x_1} \frac{a_1,1+1}{x_1}.
\]

Now assume \( \frac{\partial u_1}{\partial x_1} = 0 \), that is, \( u_1 \) does not contain \( x_1 \). Also by Lemma I \( \frac{\partial u_1}{\partial x_1+2} = 0 \) so \( u_1 \) does not contain \( x_1+2 \) either. Therefore, \( u_1 \) is a function of \( x_1+1 \) and possibly of the non-active variables, \( x_1-1 \) and \( x_1+3 \). This means that among all the integrating factors \( u_1 \) which might be chosen, the one that is the function of only one active variable is picked. However, this \( u_1 \), free of \( x_1 \) and \( x_1+2 \) and non-trivial, must satisfy the partial differential equations (3.2). Equation b becomes

\[
x_1 \frac{\partial u_1}{\partial x_1+1} = u_1 a_1,1+1. \tag{3.13}
\]

Differentiate (3.13) with respect to \( x_1+2 \). The result is
0 = u_1 \frac{\partial a_{1,1+1}}{\partial x_{1+2}}.

Since \(u_1\) is assumed to be non-zero then

\[
\frac{\partial a_{1,1+1}}{\partial x_{1+2}} = 0. \tag{3.14}
\]

Now use equation c of (3.2) which becomes

\[
-x_{1+2} \frac{\partial u}{\partial x_{1+1}} = u_1 a_{1+1,1+2}. \tag{3.15}
\]

Differentiate (3.15) with respect to \(x_1\). The result is

\[
0 = u_1 \frac{\partial a_{1+1,1+2}}{\partial x_1}
\]

Since \(u_1\) is assumed to be non-zero then

\[
\frac{\partial a_{1+1,1+2}}{\partial x_1} = 0. \tag{3.16}
\]

Therefore,

\[
a_{1,1+1} \text{ must be free of } x_{1+2},
\]

and \(a_{1+1,1+2}\) must be free of \(x_1\). \tag{3.17}

It should be noted here that the equations resulting from b and c of (3.2) by choosing \(u_1\) free of \(x_1\) (and hence of \(x_{1+2}\)) are, when solved for \(\frac{1}{u_1} \frac{\partial u_1}{\partial x_{1+1}}\),

\[
\frac{1}{u_1} \frac{\partial u_1}{\partial x_{1+1}} = \frac{a_{1,1+1}}{x_1} \quad \text{and} \quad \frac{1}{u_1} \frac{\partial u_1}{\partial x_{1+1}} = -\frac{a_{1+1,1+2}}{x_{1+2}}.
\]
Since the left-hand-sides of the above equations are identical expressions in \( u_1 \), the right-hand-sides must be the same in order that \( u_1 \) is the same for both equations. However, it can be seen that if the right-hand-sides are set equal to each other, and simplified, the resulting expression is \( (3.6) \), \( a \), which is assumed true in the hypothesis.

In the same way but using the other system, \( (3.3) \), \( u_{1+1} \) will be chosen which imply conditions corresponding to those of \( (3.17) \). They are:

\[
u_{1+1} = u_{1+1}(x_1, x_{1+2}, x_{1+4})
\]

implies

\[
a_{1+1,1+2} \text{ free of } x_{1+3},
\]

and

\[
a_{1+2,1+3} \text{ free of } x_{1+1}.
\]

The converse is true since all statements down to \( (3.14) \) and \( (3.16) \) are equivalent statements. Hence, the proof is completed.

Conditions similar to \( (3.17) \) and \( (3.18) \) may be written for any of the neighboring chains. For example, using the system \( (3.10) \) one may choose

\[
u_{1-1} = u_{1-1}(x_{1-2}, x_1, x_{1+2}),
\]

which implies

\[
a_{1-1,1} \text{ free of } x_{1+1},
\]

and

\[
a_{1,1+1} \text{ free of } x_{1-1}.
\]
Recapitulating, it seemed reasonable and convenient to try to choose the $u_{i+t}$'s as indicated above. In order to choose the integrating factors in that manner it was necessary and sufficient that they satisfy the simplified partial differential equations. The necessary and sufficient conditions that the simplified partial differential equations be satisfied are those on $a_{rs}$ stated above, (3.12).

2. Linear and quadratic coefficients

Corollary I

If the $x_j$ of Theorem I are linear functions of $x_{j-1}, x_j, x_{j+1}$ the conditions of Theorem I are always satisfied, hence the integrability of $w_1$ and $w_{i+1}$ is equivalent to the existence of a $u_1$ free of $x_1$ (and $x_{i+2}$) and a $u_{i+1}$ free of $x_{i+1}$ (and $x_{i+3}$).

Proof:

Take

$$X_1 = A_7 x_{1-1} + A_8 x_1 + A_9 x_{1+1} + A_{10},$$
$$X_{i+1} = B_7 x_1 + B_8 x_{i+1} + B_9 x_{i+2} + B_{10},$$
$$X_{i+2} = C_7 x_{i+1} + C_8 x_{i+2} + C_9 x_{i+3} + C_{10},$$
$$X_{i+3} = D_7 x_{i+2} + D_8 x_{i+3} + D_9 x_{i+4} + D_{10}.$$

Then

$$a_{1,i+1} = A_9 - B_7,$$
$$a_{i+1,i+2} = B_9 - C_7,$$
$$a_{i+2,i+3} = C_9 - D_7.$$
Corollary II

If the $X_j$ of Theorem I are quadratic functions of
$x_{j-1}, x_j, x_{j+1}$ the conditions of Theorem I are satisfied
if and only if the terms $x_j x_{j+2}$ do not appear in $X_{1+1}$ and
$X_{1+2}$. Hence, if $w_1$ and $w_{1+1}$ are integrable the non-
appearance of the terms $x_j x_{j+2}$ in $X_{1+1}$ and $X_{1+2}$ is equiva-
lent to the existence of a $u_1$ free of $x_1$ (and $x_{1+2}$) and
a $u_{1+1}$ free of $x_{1+1}$ and $(x_{1+3})$.

Proof:

Take

$$
X_1 = A_1 x_{1-1}^2 + A_2 x_1^2 + A_3 x_{1+1}^2 + A_4 x_{1-1} x_1
+ A_5 x_{1-1} x_{1+1} + A_6 x_1 x_{1+1} + A_7 x_{1-1}
+ A_8 x_1 + A_9 x_{1+1} + A_{10},
$$

$$
X_{1+1} = B_1 x_1^2 + B_2 x_{1+1}^2 + B_3 x_{1+2}^2 + B_4 x_1 x_{1+1}
+ B_5 x_1 x_{1+2} + B_6 x_{1+1} x_{1+2} + B_7 x_1
+ B_8 x_{1+1} + B_9 x_{1+2} + B_{10},
$$

$$
X_{1+2} = C_1 x_{1+1}^2 + C_2 x_{1+2}^2 + C_3 x_{1+3}^2 + C_4 x_{1+1} x_{1+2}
+ C_5 x_{1+1} x_{1+3} + C_6 x_{1+2} x_{1+3} + C_7 x_{1+1}
+ C_8 x_{1+2} + C_9 x_{1+3} + C_{10},
$$

$$
X_{1+3} = D_1 x_{1+2}^2 + D_2 x_{1+3}^2 + D_3 x_{1+4}^2 + D_4 x_{1+2} x_{1+3}
+ D_5 x_{1+2} x_{1+4} + D_6 x_{1+3} x_{1+4} + D_7 x_{1+2}
+ D_8 x_{1+3} + D_9 x_{1+4} + D_{10}.
$$
Then
\[ a_{1,1+1} = A_5 x_{1-1} + (A_6 - B_1 x_1 + (2A_3 - B_4) x_{1+1} \]
\[ - B_5 x_{1+2} + (A_9 - B_7), \]
\[ a_{1+1,1+2} = B_5 x_1 + (B_6 - 2C_1 x_{1+1} + (2B_3 - C_4) x_{1+2} \]
\[ - C_5 x_{1+3} + (B_9 - C_7), \]
\[ a_{1+2,1+3} = C_5 x_{1+1} + (C_6 - 2D_1 x_{1+2} + (2C_3 - D_4) x_{1+3} \]
\[ - D_5 x_{1+4} + (C_9 - D_7). \]

C. Implication of Exactness from the Assumption of Integrability on Three or More Chains

1. Three overlapping chains in the quadratic case

Theorem II

Given the three overlapping Pfaffian chains of length three
\[ w_{i-1} = \sum_{l=0}^{2} x_{i-1+k} \, dx_{i-1+k}, \quad w_1 = \sum_{l=0}^{2} x_{i+k} \, dx_{i+k}, \quad w_{i+1} \]
\[ = \sum_{l=0}^{2} x_{i+1+k} \, dx_{i+1+k}, \]

where the \( X_j \) are quadratic functions of \( x_{j-1}, x_j, x_{j+1} \). If these Pfaffian chains satisfy the conditions of integrability, the further condition that the coefficient of the term \( x_j \, x_{j+1} \) in \( X_{i+1} \) is non-zero forces at least two identically-
zero coefficients among $X_{1-1}$, $X_1$, $X_{1+2}$, $X_{1+3}$.

Proof:

Take $X_j$ and $a_{r3}$ as listed in Corollary II of Theorem I plus corresponding expressions for $X_{1-1}$ and $a_{1-1,1}$. The three integrability conditions

$$X_{1-1} a_{1,1+1} + X_{1+1} a_{1-1,1} = 0,$$
$$X_1 a_{1+1,1+2} + X_{1+2} a_{1,1+1} = 0,$$
$$X_{1+2} a_{1+2,1+3} + X_{1+3} a_{1+1,1+2} = 0,$$

must be identically satisfied. If $B_3$ is assumed different from zero the following results are obtained:

If $A_3 \neq 0$, then $X_{1-1} = 0$;
if $A_3 = 0$, then $X_1 = 0$;
if $C_1 \neq 0$, then $X_{1+3} = 0$;
if $C_1 = 0$, then $X_{1+2} = 0$.

Corollary

For the chains $w_{1-1}$, $w_1$, and $w_{1+1}$ of Theorem II with the $X_j$ of type $P_2$, there exist, in light of Corollary II of Theorem I, integrating factors of the type described in Theorem I.

Proof:

The assumption that the $X_j$ have property $P_2$ rules out the appearance of terms of type $x_j x_{j+2}$.

5See Appendix B.
2. Six overlapping chains in the general case

Theorem III

Let the conditions of integrability be imposed on the separate Pfaffian chains

$w_{1-2}, w_{1-1}, w_1, w_{1+1}, w_{1+2}, w_{1+3},$

where

$x_j = x_j(x_{j-1}, x_j, x_{j+1})$ are of type $P_2$.

Then when $u_{1+t}$ can be chosen free of $x_{1+t}$ the Pfaffians are exact. Hence, such overlapping Pfaffian chains cannot have a common class number less than or equal to two, that is, be integrable, unless the Pfaffian form of length eight is itself exact.

Proof:

The results which occur when the integrating factors are chosen as indicated in Theorem I will be further investigated. It will be recalled that $u_1$ and $u_{1+1}$ were chosen such that

$$u_1 = u_1(x_{1-1}, x_{1+1}, x_{1+3}),$$
$$u_{1+1} = u_{1+1}(x_1, x_{1+2}, x_{1+4}).$$

(3.20)

From the relations (3.4), b and c, take

$$X_{1+1} = \frac{u_1}{u_{1+1}} X_{1+1} \text{ and } X_{1+2} = \frac{u_1}{u_{1+1}} X_{1+2}.$$  (3.21)
Substitute these values in the exactness condition (3.5), a', which becomes:

\[
\frac{u_1}{u_{1+1}} \frac{\partial \bar{x}_{1+1}}{\partial x_{1+2}} + \bar{x}_{1+1} \frac{\partial (u_1)}{\partial x_{1+2}} = \frac{u_1}{u_{1+1}} \frac{\partial \bar{x}_{1+2}}{\partial x_{1+1}} + \bar{x}_{1+2} \frac{\partial (u_1)}{\partial x_{1+1}},
\]

or

\[
\frac{u_1}{u_{1+1}} \left( \frac{\partial \bar{x}_{1+1}}{\partial x_{1+2}} - \frac{\partial \bar{x}_{1+2}}{\partial x_{1+1}} \right) + \bar{x}_{1+1} \frac{\partial (u_1)}{\partial x_{1+2}} = \bar{x}_{1+2} \frac{\partial (u_1)}{\partial x_{1+1}}.
\]

Since \( \frac{\partial \bar{x}_{1+1}}{\partial x_{1+2}} - \frac{\partial \bar{x}_{1+2}}{\partial x_{1+1}} = 0 \), and \( \frac{\bar{x}_{1+1}}{\bar{x}_{1+2}} = \frac{x_{1+1}}{x_{1+2}} \), one may write

\[
\frac{x_{1+1}}{x_{1+2}} = \frac{\partial (u_1)}{\partial x_{1+1}} = \frac{1}{u_{1+1}} \frac{\partial u_1}{\partial x_{1+1}} = \frac{u_1+1}{u_1+1} \frac{\partial u_1}{\partial x_{1+1}},
\]

or

\[
\frac{\partial u_1}{\partial x_{1+1}} x_{1+2} = - \frac{\partial u_{1+1}}{\partial x_{1+2}} x_{1+1} \tag{3.22}
\]

Consider the variables which are contained in each member of this identity. In the left member of (3.22) the only variables which can appear are \( x_{1-1}, x_{1+1}, x_{1+2}, x_{1+3} \) while on the right-hand-side only the variables \( x_1, x_{1+1}, x_{1+2}, x_{1+4} \) can appear. In particular, the left member may contain
$x_{i-1}$ while the right member may not. Therefore, in order that (3.22) hold the ratio \( \frac{\partial u_1}{\partial x_1+1} \) must be free of $x_{i-1}$.

A similar argument may be used to show that the ratio \( \frac{\partial u_{i+1}}{\partial x_{i+2}} \) must be free of $x_{i+4}$. Now write the companion expression for (3.22) which can be derived in exactly the same way but from $w_{i-1}$ and $w_1$:

\[
\frac{\partial u_{i-1}}{\partial x_1} \frac{x_{i+1}}{u_{i-1}} = \frac{\partial u_1}{\partial x_{i+1}} \frac{x_1}{u_1} \quad (3.23)
\]

Since

\[
u_{i-1} = u_{i-1}(x_{i-2}, x_1, x_{i+2})
\]

it can be shown as above that \( \frac{\partial u_{i-1}}{\partial x_{i+1}} \) is free of $x_{i+3}$ and \( \frac{\partial u_1}{\partial x_1} \) is free of $x_{i-2}$. To complete the proof, use the three pairs of chains $(w_{i-2}, w_{i-1})$, $(w_{i+1}, w_{i+2})$ and $(w_{i+2}, w_{i+3})$. The corresponding integrating factors are by assumption:

\[
u_{i-2} = u_{i-2}(x_{i-3}, x_{i-1}, x_{i+1})
\]

\[
u_{i+2} = u_{i+2}(x_{i+1}, x_{i+3}, x_{i+5})
\]

\[
u_{i+3} = u_{i+3}(x_{i+2}, x_{i+4}, x_{i+6})
\]

The identities corresponding to (3.22) and (3.23) and accompanying results are as follows:
\[
\frac{\partial u_{1-2}}{\partial x_{1-1}} \frac{u_{1-2}}{u_{1-1}} = - \frac{\partial u_{1-1}}{\partial x_{1}} \frac{u_{1-1}}{u_{1-1}} \quad \text{and} \quad \frac{\partial u_{1-1}}{\partial x_{1}} \frac{u_{1-1}}{u_{1-1}} \quad \text{is free of} \quad x_{1+2}, \quad (3.24)
\]

\[
\frac{\partial u_{1+1}}{\partial x_{1+2}} \frac{u_{1+1}}{u_{1+2}} = - \frac{\partial u_{1+2}}{\partial x_{1+3}} \frac{u_{1+2}}{u_{1+2}} \quad \text{is free of} \quad x_{1}, \quad (3.25)
\]

\[
\frac{\partial u_{1+2}}{\partial x_{1+3}} \frac{u_{1+2}}{u_{1+3}} = - \frac{\partial u_{1+3}}{\partial x_{1+4}} \frac{u_{1+3}}{u_{1+2}} \quad \text{is free of} \quad x_{1+1}. \quad (3.26)
\]

The following conclusions can then be drawn:

From (3.22) and (3.23), \( x_{1+1} = x_{1+1} (x_{1+1}, x_{1+2}) \);

from (3.23) and (3.24), \( x_{1+1} = x_{1+1} (x_{1}, x_{1+1}) \);

from (3.22) and (3.23), \( x_{1+2} = x_{1+2} (x_{1+1}, x_{1+2}) \);

from (3.25) and (3.26), \( x_{1+2} = x_{1+2} (x_{1+2}, x_{1+3}) \).

A sample argument will be given for the above conclusions using \( x_{1+1} \):

Note that \( \frac{\partial u_{1-1}}{\partial x_{1+1}} \frac{u_{1-1}}{u_{1-1}} \) is free of \( x_{1+2} \) by (3.24).

Therefore, if the left member of (3.23) is to contain an \( x_{1+2} \) it must be in \( x_{1+1} \), but the right member does not contain an \( x_{1+2} \) so \( x_{1+1} = x_{1+1} (x_{1}, x_{1+1}) \). Then from
\[
\frac{\partial u_{i+1}}{\partial x_{i+2}} \text{ is free of } x_1. \text{ Hence, if the right member of (3.22) contains an } x_1 \text{ it must be in the } X_{i+1}, \text{ but the left member does not contain an } x_1 \text{ so } X_{i+1} = X_{i+1}(x_{i+1}, x_{i+2}).
\]

In order to reconcile the two statements,
\[
X_{i+1} = X_{i+1}(x_{i+1}). \tag{3.27}
\]

In like manner
\[
X_{i+2} = X_{i+2}(x_{i+2}).
\]

Now consider the integrability conditions (3.6) for \(w_1\) and \(w_{i+1}\). From (3.27) \(a_{i+1}, i+2 = 0\) forcing \(a_1, i+1 = a_{i+2}, i+3 = 0\) because of the conditions assumed on the \(X_j\). This condition of exactness is forced on each neighboring chain in turn until all the \(a_{rs}\) are zero and hence the Pfaffian form of length eight is exact.

**Corollary**

Under the conditions of integrability of the six Pfaffian chains of Theorem III, if the \(a_{rs}\) can be chosen such that

- \(a_{1-2}, i-1\) is free of \(x_1\),
- \(a_{i-1}, i+1\) is free of \(x_{i-2}\) and \(x_{i+1}\),
- \(a_{i, i+1}\) is free of \(x_{i-1}\) and \(x_{i+2}\),
- \(a_{i+1, i+2}\) is free of \(x_1\) and \(x_{i+3}\),
- \(a_{i+2, i+3}\) is free of \(x_{i+1}\) and \(x_{i+4}\),
- \(a_{i+3, i+4}\) is free of \(x_{i+2}\) and \(x_{i+5}\),
- \(a_{i+4, i+5}\) is free of \(x_{i+3}\).
then the corresponding Pfaffian chain of length eight is exact. In particular this applies to the quadratic case in Theorem II.

Proof:

See Theorems I and III.

D. Formal Description of the General Problem

The essential aspects of the problem are perhaps best brought out by abstracting the above elements in a set theoretical sense. The following terms will now be defined:

(a) \( S \) is the set whose elements are \( x_j \, dx_j \) \((n_1 \leq j \leq n_2)\), that is, the individual summands of some Pfaffian.

(b) A sub-set of \( S \) is the collection \( \{ x_{j_1} \, dx_{j_1}, \ldots, x_{j_r} \, dx_{j_r} \} \)

where the \( j_i \)'s are some selection of the \( j \)'s.

(c) A neighborhood of \( x_j \, dx_j \) is any sub-set containing \( x_j \, dx_j \).

(d) A chain containing \( x_j \, dx_j \) is a neighborhood of \( x_j \, dx_j \) (say \( x_{j_1} \, dx_{j_1}, \ldots, x_{j_r} \, dx_{j_r} \)) such that the \( j_i \)'s are a consecutive sequence of integers and contain \( j \).

(e) Class of the chain or of \( S \) is the usual class of the Pfaffian of which the elements of the chain or the elements of \( S \) are summands.

(f) The \( m \)-local class \( c(m,j) \) of \( S \) at \( x_j \, dx_j \) is the maximum class of all chains of length less than or equal to \( m \) which contain \( x_j \, dx_j \).
(g) The m-local class \( c(m, S) \) of \( S \) is the maximum \( r_j \) (for all \( j \)).

As an illustration of the above concepts, if one defines class \( c_f \) of a neighborhood \( \{ X_j f, dx_j f \} \) as the class of the associated Pfaffian, then the usual integrability condition for \( S \) can be stated as follows:

\( S \) is integrable if and only if \( c_f \leq 2 \) for all \( f \).

Also, if one considers the conditions of Theorem III on the \( X_j \) and the \( u_{i+t} \), then the claims of the theorem are:

\( c(3,j) \leq 2 \), where \( 1-2 \leq j \leq 1+5 \) implies \( c(3,3) = 1 \).

The next chapter considers some other cases of the general problem implied by the above discussion, namely, what can be said when \( m \) is greater than three and \( c \) subject to special restrictions.
IV. LONGER CHAINS AND FLUCTUATING CLASS

In Chapter III the length of chains was limited to three. This chapter will treat chains of length greater than three using various restrictions on the variables in the $X_j$. Possibilities of fluctuating class, mainly between one and two, for the Pfaffian chains with constant class for the associated Pfaffian equation will be discussed, as well as Pfaffian chains where the $X_j$ are functions of three, or less, non-adjacent variables. Thus, this chapter takes up some questions suggested by the material in Chapter II.

A. Assumption of Integrability on a Chain of Length Greater than Three

If one considers the integrability of any Pfaffian of length $n$ there are, as stated in Chapter I, $1/6 \ n(n-1)(n-2)$ integrability conditions. One of these, for $n$ greater than three, is:

$$X_1 a_{1+1,1+3} + X_{i+1} a_{1+3,i} + X_{i+3} a_{1,i+1} = 0.$$  

If it is again assumed that $X_j = X_j(x_{j-1}, x_j, x_{j+1})$ then

$$a_{1+1,1+3} = a_{1+3,1} = 0.$$  

Hence, if $X_j$ is not identically zero then $a_{1,i+1}$ must equal zero if the condition is
to be satisfied. It is well known\(^1\) that the conditions of integrability can be found by taking all the determinants of the third order derived from the B matrix by associating the first row with any other two rows. Hence, write the B\(_4\) matrix for a chain of length d:

\[
W_1 = \sum_{j=1}^{i+k} x_j \, dx_j,
\]

\[1 \leq i \leq n-d+1; \quad k = 0, 1, \ldots, d-1 \quad (d > 2);
\]

\[
B = \begin{bmatrix}
0 & X_1 & X_1+1 & \ldots & X_{i+h-1} & X_{i+h} & \ldots & X_d \\
-X_1 & 0 & a_{1,i+1} & \ldots & & & & \\
-X_{i+1} & -a_{1,i+1} & 0 & \ldots & & & & \\
-X_{i+2} & 0 & -a_{1,i+1,i+2} & \ldots & & & & \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \\
-X_{i+h-1} & 0 & & & & a_{i+h-1,i+h} & \ldots & X_d \\
-X_{i+h} & -a_{i+h-1,i+h} & 0 & \ldots & \ddots & \ddots & \ddots & \\
-X_{i+h+1} & 0 & \ldots & \ddots & \ddots & \ddots & \ddots & \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \\
-X_{d-1} & 0 & \ldots & \ddots & \ddots & \ddots & \ddots & \\
-X_d & 0 & \ldots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\]

It is desired to prove any element in the stripe, say \(a_{i+h-1,i+h}\), equal to zero if the integrability conditions (1.2) are assumed. Choose the second order principal minor

\(^1\)Goursat, op. cit., p. 38.
containing \( a_{i+h-1,i+h} \). This involves \( X_{i+h-1} \) and \( X_{i+h} \) in the top row. Then take any \( X_j \), in particular \( X_{i+1} \), where \( i+1 \) is not adjacent to either \( i+h-1 \) or \( i+h \). This is always possible if \( d \) is greater than three. The minor array is:

\[
M = \begin{bmatrix}
X_{i+1} & X_{i+h-1} & X_{i+h} \\
0 & 0 & a_{i+h-1,i+h} \\
0 & -a_{i+h-1,i+h} & 0
\end{bmatrix},
\]

thus \( |M| = X_{i+1} a_{i+h-1,i+h} = 0 \) is one of the integrability conditions as indicated above. Hence, if \( X_{i+1} \) is not identically zero then \( a_{i+h-1,i+1} \) must be zero and \( w_1 \) is exact. Hence, integrability implies exactness for this chain and the following theorem has been proved:

**Theorem I**

All Pfaffian chains of length greater than three where

\[
X_j = X_j(x_{j-1}, x_j, x_{j+1}) \quad \text{and} \quad X_j \quad \text{is of type } P_2,
\]

are exact if they satisfy the conditions of integrability.

---

**B. Pfaffian Chains of Fluctuating Class**

1. **Pfaffian equations**

While Pfaffians may have either even or odd class, Pfaffian equations may be restricted to odd class. This is easily seen from the canonical form (1.7) which is of even class. Set (1.7) equal to zero and divide through by \( z_1 \neq 0 \). The result is:
\[ dy_1 + z_1^{-1} dy_2 + \ldots + z_p^{-1} dy_p = 0, \]

where \( z_1^{-1} = \frac{z_{j+1}}{z_1} \) and the total number of \( y_1 \) and \( z_j^{-1} \) is odd. Again the \( y_1 \) and \( z_j^{-1} \) are independent functions of the original variables. Hence, any Pfaffian form of even class can have its class lowered by one if the form is set equal to zero. Therefore, in considering the common class of Pfaffian chains it appears possible that if the individual chains were each set equal to zero a constant class might be achieved even though the Pfaffian chains differed in class by one, the higher class in each case being even. For example, chains of class \( m \) (odd) and \( m+1 \) might occur which when set equal to zero have a common class of \( m \).

2. Fluctuations between one and two

Consider the case where \( m = 1 \). The \( B \) matrix is assumed to be of the form:

\[
B = \begin{bmatrix}
0 & X_1 & X_2 & x_3 & \ldots & X_d & X_{d+1} & X_{d+2} & \ldots & X_{n-1} & X_n \\
-X_1 & 0 & 0 & 0 \\
-X_2 & 0 & 0 & 0 \\
-X_3 & 0 & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & -X_d & 0 & 0 & a_{d,d+1} \\
& & -X_{d+1} & -a_{d,d+1} & 0 & \ddots \\
& & -X_{d+2} & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots & \ddots \\
-\ldots & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & -X_n & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & -X_{n-1} & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & -X_n & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]
where $a_d, d+1 \neq 0$.

The first chain of length $d$ is characterized by $B_1$ (the square matrix in the upper left-hand corner of order $d+1$) and is of class one. In order for the second chain, characterized by $B_2$, which contains the elements $a_d, d+1$, to have class two it is necessary and sufficient that all the bordering elements $X_j (j=1, 2, ..., d-1)$ shall be zero. This is proved in the following manner. If the rank of $B_2$ is two the matrix must have two linearly independent rows. These rows must be the two which contain $a_d, d+1$ and $-a_d, d+1$, or rows $d+1$ and $d+2$. The top row must be a linear combination of rows $d+1$ and $d+2$ but it contains $X_2, ..., X_{d-1}$ as elements which correspond to zero elements in the rows $d+1$ and $d+2$. Hence, $X_2 = X_3 = ... = X_{d-1} = 0$. Conversely, if $X_2 = X_3 = ... = X_{d-1} = 0$ all the rows above rows $d+1$ and $d+2$ are linear combinations of those two linearly independent rows except possibly the top row.

But $B_2$ is skew-symmetric and cannot have odd rank. Hence, the top row must be a linear combination of rows $d+1$ and $d+2$. The third chain likewise has class two if and only if $X_3 = ... X_{d-1} = X_{d+2} = 0$. This can be continued until the $(d+2)$nd chain which can be made to have class one if all the elements of the $A_{d+2}$ matrix are zero. Then this cycle can be repeated by placing another non-zero element $a_{2d+1, 2d+2}$ in the stripe.
If each of these chains was set equal to zero the canonical forms of class two would be of type $z_1 dz_2$ which when set equal to zero and divided through by $z_1 \neq 0$ indicates class one for the Pfaffian equation. Hence, there would be a common class of one for the Pfaffian equations even though a fluctuating class (between one and two) for the Pfaffian chains. The successive chains would be:

$$\begin{align*}
X_1 & \frac{dx_1}{dx_1} + X_d \frac{dx_d}{dx_d}, \\
X_d & \frac{dx_d}{dx_d} + X_{d+1} \frac{dx_{d+1}}{dx_{d+1}}, \\
& \vdots \\
X_d & \frac{dx_d}{dx_d} + X_{d+1} \frac{dx_{d+1}}{dx_{d+1}}, \\
x_{d+1} & \frac{dx_{d+1}}{dx_{d+1}} + x_2 \frac{dx_2}{dx_2}, \\
& \vdots \\
x_{2d} & \frac{dx_{2d}}{dx_{2d}} + x_{2d+1} \frac{dx_{2d+1}}{dx_{2d+1}}, \\
& \vdots \\
x_{2d} & \frac{dx_{2d}}{dx_{2d}} + x_{2d+1} \frac{dx_{2d+1}}{dx_{2d+1}},
\end{align*}$$

and so on where the 1st, (d+1)st, (2d+1)st, ..., (rd+1)st chains are exact and all others of class two. Hence, the following theorem has been proved:

**Theorem II**

If any Pfaffian chain

$$w_1 = \sum_{j=1}^{1+k} X_j \frac{dx_j}{dx_j}, \ X_j \ \text{of type P}_1,$$

$1 \leq i \leq d-1; \ k = 0, 1, \ldots, d-1,$

of the Pfaffian form

$$w = \sum_j X_j \frac{dx_j}{dx_j},$$
where \( n_1 \leq j \leq n_2 \) and the \( A \) matrix is one-stripe, is of class one and the next chain of higher class, the necessary and sufficient condition that the next chain be of class two is that the \( d-2 \) inside coefficients of the first chain be zero. Hence, when \( X_j \) is of type \( P_2 \) then regardless of the nature of the \( a_{ij} \) if the Pfaffian is locally of class two then all the Pfaffian chains are exact or all the Pfaffian chains are of class two.

C. Non-adjacent Variables in the Coefficients

Up to this point the usual restriction on the variables in the coefficients has been to limit \( X_j \) to functions of the three clustered variables \( x_{j-1}, x_j, x_{j+1} \). Now consider the case where

\[
X_j = X_j(x_{j-m}, x_j, x_{j+m}), \quad m > 1.
\]

As a simple illustration take \( m \) equal to two. The matrix \( B \) which characterizes such a Pfaffian is:

\[
B = \begin{bmatrix}
0 & X_1 & X_2 & X_3 & X_4 & \ldots & X_{n-1} & X_n \\
-X_1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
-X_2 & 0 & 0 & 0 & 0 & a_{24} & \ldots & \\
-X_3 & 0 & 0 & 0 & a_{13} & 0 & 0 & \ldots & \\
-X_4 & 0 & 0 & a_{24} & 0 & 0 & \ldots & \ldots & a_{n-2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-X_{n-1} & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 & 0 \\
-X_n & 0 & \ldots & -a_{n-2,n} & 0 & \ldots & \ldots & 0 & 0 \\
\end{bmatrix}
\]
It will be noted that the bilinear covariant matrix $A$ is of the same nature as the one-stripe matrix except that the super-diagonal stripe is raised. This matrix and similar deviations will be called modified one-stripe matrices. The following conclusions can be read from the $B$ matrix by inspection:

(a) All chains of length two are exact or of class one.

(b) For any chain of length three to have class two it is necessary that the middle coefficient of the chain be zero.

Next consider the more general case where the $B$ matrix is:

$$
B = 
\begin{bmatrix}
0 & X_1 & X_2 & \cdots & X_m & X_{m+1} & X_{m+2} & \cdots & X_{n-1} & X_n \\
-X_1 & 0 & 0 & \cdots & 0 & a_{1,m+1} & 0 & \cdots & 0 & 0 \\
-X_2 & 0 & 0 & \cdots & 0 & 0 & a_{2,m+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-X_m & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{m,n} & \vdots \\
-X_{m+1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \vdots & \vdots \\
-X_{m+2} & 0 & -a_{2,m+2} & 0 & \cdots & 0 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-X_{n-1} & 0 & 0 & \cdots & -a_{m,n} & 0 & 0 & \cdots & 0 & 0 \\
-X_n & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots
\end{bmatrix}
$$

By noting the structure of the above matrix the following theorem can be stated:
Theorem III

If \( X_j \) equals \( X_j(x_{j-m}, x_j, x_{j+m}) \) and is of type \( P_1 \), then

(a) All chains of length 2, 3, ..., \( m \) are exact.

(b) For any chain of length \( m+q \) (\( q < m \)) to have class two it is necessary that it have \( m-q \) zero coefficients.

Proof:

Non-zero elements in the \( A \) matrix occur only in the \( m \)th stripe above (and below) the principal diagonal. Hence, the number of coefficients \( X_j \) which are in columns where all the other elements are zero depends on \( m \) and the length of the chain. It can be seen from the above matrices that necessary conditions for chains to be of class two are that:

(a) chains of length \( m+1 \) have \( m-1 \) zero coefficients,

(b) chains of length \( m+2 \) have \( m-2 \) zero coefficients,

and in general

(c) chains of length \( m+q \) have \( m-q \) zero coefficients \( (q < m) \).

Corollary

If \( X_j \) is of type \( P_2 \) it is impossible for any chain of length \( m+q \) (\( q < m \)) to have class two and accordingly impossible for overlapping chains of length less than or equal to \( m+q \) to have a common class of two.
D. Remarks on Asymmetric Triples of Variables

As an illustration of what might happen if the three permissible variables in $X_j$ appear in some asymmetric fashion, consider now a case where the variables in $X_j$ occur periodically as follows:

$$w_1 = X_1(x_{i-1}, x_i, x_{i+1}) \, dx_i + X_{i+1}(x_i, x_{i+1}, x_{i+3}) \, dx_{i+1} + X_{i+2}(x_{i+1}, x_{i+2}, x_{i+5}) \, dx_{i+2} + X_{i+3}(x_{i+2}, x_{i+3}, x_{i+7}) \, dx_{i+3} + X_{i+4}(x_{i+3}, x_{i+4}, x_{i+5}) \, dx_{i+4} + X_{i+5}(x_{i+4}, x_{i+5}, x_{i+7}) \, dx_{i+5} + X_{i+6}(x_{i+5}, x_{i+6}, x_{i+9}) \, dx_{i+6} + X_{i+7}(x_{i+6}, x_{i+7}, x_{i+11}) \, dx_{i+7} + X_{i+8}(x_{i+7}, x_{i+8}, x_{i+9}) \, dx_{i+8} + X_{i+9}(x_{i+8}, x_{i+9}, x_{i+11}) \, dx_{i+9} + \ldots + X_k \, dx_k.$$ 

$$B_1 = \begin{bmatrix} 0 & X_1 & X_{i+1} & X_{i+2} & X_{i+3} & X_{i+4} & \cdots \\
-X_1 & 0 & a_{1,1+1} & 0 & 0 & 0 & \\
X_{i+1} & -a_{1,1+1} & 0 & a_{1+1,1+2} & a_{1+1,1+3} & 0 & \\
-X_{i+2} & 0 & -a_{1+1,1+2} & 0 & a_{1+2,1+5} & 0 & \\
-X_{i+3} & 0 & -a_{1+1,1+3} & -a_{1+2,1+3} & 0 & a_{1+3,1+4} & \\
-X_{i+4} & 0 & 0 & 0 & -a_{1+3,1+4} & 0 & \\
-X_{i+5} & 0 & 0 & -a_{1+2,1+5} & 0 & -a_{1+4,1+5} & \\
-X_{i+6} & 0 & 0 & 0 & 0 & 0 & \\
-X_{i+7} & 0 & 0 & 0 & -a_{1+3,1+7} & 0 & \\
-X_{i+8} & 0 & 0 & 0 & 0 & 0 & \\
-X_{i+9} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$
The above case is easier to handle if the \( x_{j-1} \) in each \( x_j \) does not appear. Making this latter assumption the pattern which evolves is as follows:

\[
\begin{bmatrix}
0 & X_1 & X_{1+1} & X_{1+2} & X_{1+3} & X_{1+4} & \ldots \\
-X_1 & 0 & a_{1,1+1} & 0 & 0 & 0 & \\
-X_{1+1}-a_{1,1+1} & 0 & 0 & a_{1+1,1+3} & 0 & \\
-X_{1+2} & 0 & 0 & 0 & 0 & 0 & \\
-X_{1+3} & 0 & -a_{1+1,1+3} & 0 & 0 & 0 & \\
-X_{1+4} & 0 & 0 & 0 & 0 & 0 & \\
-B = & -X_{1+5} & 0 & -a_{1+2,1+5} & 0 & -a_{1+4,1+5} & \\
-X_{1+6} & 0 & 0 & 0 & 0 & 0 & \\
-X_{1+7} & 0 & 0 & -a_{1+3,1+7} & 0 & \\
-X_{1+8} & 0 & 0 & 0 & 0 & 0 & \\
-X_{1+9} & 0 & 0 & 0 & 0 & 0 & \\
\vdots & & & & & & \\
\end{bmatrix}
\]

For chains of length three, assuming the \( a_{rs} \neq 0 \) (except as indicated in the matrix),

\[
w_{1+k} = X_{1+k} dx_{1+k} + X_{1+k+1} dx_{1+k+1} + X_{1+k+2} dx_{1+k+2},
\]

the following conclusions can be drawn:

(a) \( w_1 \) is class three unless \( X_{1+2} = 0 \), in which case it is of class two,

(b) \( w_{1+1} \) is of class three unless \( X_{1+2} = 0 \), in which case it is of class two,

(c) \( w_{1+2} \) is exact or of class one,

(d) \( w_{1+3} \) is of class three unless \( X_{1+3} = 0 \), in which case it is of class two,
and so on. One may thus conclude:

If none of the coefficients, $X_{i+2}, X_{i+3}, X_{i+6}, X_{i+7}, X_{i+10}, X_{i+11}$, are zero the class numbers vary in the following manner:

$$3, 3, 1, 3, 3, 1, 3, 3, 3, 1, \ldots;$$

If the above $X_{i+k}$ are each zero then the class numbers vary as follows:

$$2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, \ldots.$$
V. CYCLIC CHAINS

Up to this point Pfaffian chains have been considered which are part of a Pfaffian of finite, or even infinite, length. The purpose of this chapter is to analyze, in the locally integrable case (that is, \( c(3,j) \leq 2 \)), Pfaffians in which the chains form cycles, as defined below, in a natural way and in which local integrability is imposed on sub-chains of the cyclic chains. A Pfaffian chain of the cyclic or circular type is defined as follows:

\[
\begin{align*}
  w_j &= \sum_{j=1}^{k} X_j \, dx_j; \quad X_j = X_j(x_j, x_{j+1}), \quad (j \leq k); \\
  X_k &= X_k(x_k, x_1).
\end{align*}
\]

(5.1)

Its \( B_1 \) matrix is:

\[
B_1 = \begin{bmatrix}
0 & X_1 & X_{1+1} & X_{1+2} & \cdots & X_k \\
-X_1 & 0 & a_{1,1+1} & 0 & \cdots & a_{1,k} \\
-X_{1+1} & -a_{1,1+1} & 0 & a_{1+1,1+2} & \cdots & 0 \\
-X_{1+2} & 0 & -a_{1+1,1+2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-X_k & -a_{1k} & 0 & \cdots & 0 & -a_{1-1,k} \\
\end{bmatrix}
\]
A. Cycles of Length Four

Consider the case \( d=4 \), hence of four overlapping Pfaffian sub-chains of length three which make up a cycle. For simplification of notation let \( i=1 \). The sub-chains are

\[
\begin{align*}
X_1(x_1, x_2) &\ d x_1 + X_2(x_2, x_3) \ d x_2 + X_3(x_3, x_4) \ d x_3, \\
X_2(x_2, x_3) &\ d x_2 + X_3(x_3, x_4) \ d x_3 + X_4(x_4, x_1) \ d x_4, \\
X_3(x_3, x_4) &\ d x_3 + X_4(x_4, x_1) \ d x_4 + X_1(x_1, x_2) \ d x_1, \\
X_4(x_4, x_1) &\ d x_4 + X_1(x_1, x_2) \ d x_1 + X_2(x_2, x_3) \ d x_2.
\end{align*}
\]

(5.2)

The respective local integrability conditions, in this case, simplify to

\[
\begin{align*}
x_1 \ a_{23} + x_3 \ a_{12} &= 0, \\
x_2 \ a_{34} + x_4 \ a_{23} &= 0, \\
x_3 \ a_{41} + x_1 \ a_{34} &= 0, \\
x_4 \ a_{12} + x_2 \ a_{41} &= 0.
\end{align*}
\]

(5.3)

1. Linear case

The possibility of inexact integrability will now be checked for the sub-chains of the above cyclic form where the \( X_j \) are linear functions of the indicated variables.

Let

\[
\begin{align*}
X_1 &= A_4 \ x_1 + A_5 \ x_2 + A_6, \\
X_2 &= B_4 \ x_2 + B_5 \ x_3 + B_6, \\
X_3 &= C_4 \ x_3 + C_5 \ x_4 + C_6, \\
X_4 &= D_4 \ x_4 + D_5 \ x_1 + D_6.
\end{align*}
\]
then

\[ a_{12} = A_5, \quad a_{23} = B_5, \quad a_{34} = C_5, \quad a_{41} = D_5. \]

Using the integrability conditions and the assumption that the \( X_j \) are of type \( F_2 \), one observes from (5.3) that if the cycle is not already exact then none of the \( a_{rs} \) are zero. Hence, from identities (5.3), it follows that \( X_1, X_2, X_3, X_4 \) must each be a constant. This contradicts the assumption on the \( a_{rs} \). Therefore, the \( a_{rs} \) are all zero and the cycle is exact.

2. Quadratic case

Now let the \( X_j \) be quadratic functions of the indicated variables:

\[ X_1 = A_1 x_1^2 + A_2 x_2^2 + A_3 x_1 x_2 + A_4 x_1 + A_4 x_2 + A_6, \]
\[ X_2 = B_1 x_2^2 + B_2 x_3^2 + B_3 x_2 x_3 + B_4 x_2 + B_5 x_3 + B_6, \]
\[ X_3 = C_1 x_3^2 + C_2 x_4^2 + C_3 x_3 x_4 + C_4 x_3 + C_5 x_4 + C_6, \]
\[ X_4 = D_1 x_4^2 + D_2 x_4^2 + D_3 x_4 x_1 + D_4 x_4 + D_5 x_1 + D_6. \]  

Then

\[ a_{12} = 2A_2 x_2 + A_3 x_1 + A_5, \]
\[ a_{23} = 2B_2 x_3 + B_3 x_2 + B_5, \]
\[ a_{34} = 2C_2 x_4 + C_3 x_3 + C_5, \]
\[ a_{41} = 2D_2 x_1 + D_3 x_4 + D_5. \]  

(5.5)
Assume the $X_j$ of type $P_2$ and consider the integrability conditions (5.3). Since $a_{23}$ cannot be zero without implying exactness for the whole cycle it must be concluded that $a_{23}$ is non-zero in the inexact case. In the latter case, at least one of the three $B_2$, $B_3$, $B_5$ must be non-zero. The results (see Appendix C) of assuming any one of these to be non-zero are as follows:

- If $B_2 \neq 0$ then $X_1 = 0$,
- if $B_3 \neq 0$ then $X_1 = 0$,
- if $B_2 = B_3 = 0$ and $B_5 \neq 0$ then $X_1 = 0$.

Hence, assuming $a_{23}$ non-zero presents a contradiction. Therefore, $a_{23}$ must equal zero. From (5.3) and the assumption that the $X_j$ are of type $P_2$, one must conclude that $a_{12} = a_{34} = a_{41} = 0$ and the cyclic chain is exact. The following theorem has then been proved:

**Theorem I**

If in the four overlapping sub-chains of length three of the cyclic chain

$$X_1(x_1, x_{1+1}) \, dx_1 + X_{1+1}(x_{1+1}, x_{1+2}) \, dx_{1+1} + X_{1+2}(x_{1+2}, x_{1+3}) \, dx_{1+2} + X_{1+3}(x_{1+3}, x_1) \, dx_{1+3},$$

where the $X_j$ are of type $P_2$,

the $X_j$ are quadratic functions of the indicated variables, 
local integrability implies exactness.
3. An inexactlly integrable sub-chain

In the proof of the above theorem any one of the sub-chains (5.2) turns out to be exact, independently of any other, when $X_j$ is a polynomial function of at most degree two. To show that this is not a trivial result an example will be given where the $X_j$ are non-polynomial functions and the first chain, in particular, is inexactlly integrable.

Let $X_1 = X_1(x_1, x_2) = e^{x_2 + f(x_1)}$,  
$X_2 = X_2(x_2, x_3) = g(x_2) + h(x_3)$,  
$X_3 = X_3(x_3, x_4) = -h'(x_3)$,

then the integrability condition of (5.3) becomes

$$e^{x_2 + f(x_1)} h'(x_3) - h'(x_3) e^{x_2 + f(x_1)} = 0.$$ 

It should be noted that if $X_4$ is permitted to equal zero an inexactlly integrable cycle is obtained. In fact, in this example this choice of $X_4$ is the only possible one. On the other hand, one notes in the linear case above that $X_4 = 0$ implies exactness.

B. Cycles of Length Greater Than Four

Going back to the case where the $X_j$ are polynomial functions of the first or second degree it can be seen that
every cycle of the form (5.1), where the $X_j$ are of type $P_2$ and $d(= k+1) > 4$, is exact. For, the integrability conditions needed are the following $n-2$ conditions:

\[ X_1 \ a_{i+1,1+2} + X_{i+2} \ a_{i,1+1} = 0, \]
\[ X_{i+1} \ a_{i+2,1+3} + X_{i+3} \ a_{i+1,1+2} = 0, \]
\[ \ldots \ldots \ldots \ldots \ldots \]
\[ X_{k-3} \ a_{k-2,k-1} + X_{k-1} \ a_{k-3,k-2} = 0, \]
\[ X_{k-2} \ a_{k-1,k} + X_{k} \ a_{k-2,k-1} = 0. \]

Hence, if $a_{i+1,1+2} = 0$ then

\[ a_{i,1+1} + a_{i+2,1+3} = \ldots = a_{k-3,k-2} = a_{k-2,k-1} = a_{k-1,k} = 0 \]

and the whole chain is exact. Since in the case $d=4$ each sub-chain turns out exact independently of the other sub-chains, the longer cycle has the desired property and the following theorem has been proved:

**Theorem II**

If in the $d$ overlapping sub-chains of length three of the cyclic chain (5.1) where

\[ d > 4, \ X_j \text{ are of type } P_2, \]

the $X_j$ are quadratic functions of the indicated variables, local integrability implies exactness.
C. Cycles of Length Greater Than Seven

If the cyclic chain is of length greater than seven it is easy to see that the conditions of the corollary to Theorem III of Chapter III on the $a_{rg}$ are satisfied. Hence, the chain is exact even though the $X_j$ are general functions of the indicated variables. Hence, the theorem:

Theorem III

If in the $d$ overlapping sub-chains of length three of the cyclic chain $(5.1)$ where

$$d > 7, \ X_j \text{ are of type } P_2,$$

the $X_j$ are otherwise general functions of the indicated variables, local integrability implies exactness.

D. Note on Cyclic Chains of Length Three

The integrability condition for the cyclic chain of length three may be satisfied by polynomial coefficients without implying exactness as shown by the following example:

If $w_1 = (-x_1+x_2+1)dx_1 + (-x_2+x_3+1)dx_2 + (-x_3+x_1-2)dx_3$, then $a_{12} = 1, \ a_{31} = 1, \ a_{23} = 1$, and the integrability condition becomes

$$(-x_1+x_2+1)\cdot 1 + (-x_2+x_3+1)\cdot 1 + (-x_3+x_1-2)\cdot 1 = 0.$$
The example of Section A, 3 is another illustration of the above type but using general coefficients.

E. Summary

The above results show that the cyclic chain (5.1) reacts to the assumption of local integrability in much the same way as the chains considered in Chapter III even though the $B_1$ matrix differs in structure. The number of integrability conditions needed are $d-2$ of the total number of $\frac{1}{6} d(d-1)(d-2)$ such conditions of which $\frac{d}{2}(n-1)(n-2)$ are independent. To emphasize the economy in integrability conditions used the following brief tabulation is shown:

<table>
<thead>
<tr>
<th>Length of chain</th>
<th>Number of independent integrability conditions</th>
<th>Number of integrability conditions used</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>8 or more</td>
<td>21 or more</td>
<td>6 or more</td>
</tr>
</tbody>
</table>
VI. APPLICATION OF PFAFFIANS TO UTILITY THEORY

One of the fields to which Pfaffian theory may be applied is that of utility theory in economics. In this chapter a possible approximate model is suggested to which some of the theorems of this thesis can be applied. After setting up hypotheses certain implications are drawn from the results obtained in earlier chapters.

A. The Utility Function

1. Consumer satisfaction

The following statement of the utility problem is adopted from a text in economics:¹

One of the general methods of theoretical economics is the discussion of economic situations with respect to their desirability. This has necessitated the construction of an artificial function used as a measure of desirability, sometimes called utility (Jevons) or ophelimity (Pareto), of desirability (Walras). But in accepting these theories as general we encounter some difficulties . . . and there is in some of the authors who have propounded such theories rather a serious confusion of ideas.

Such authors interpret the situation in the following way. They assume that they are dealing with two aspects of the material of economics, one

subjective, the other objective. On the one hand they consider that the acts which are investigated in economics are devised for the attainment of some kind of good — for the pleasure of ownership or of prospective consumption, for the sake of doing one's duty and the resultant moral satisfaction, for power, for glory, for the sake of activity itself, or for revenge. Such pleasures, satisfactions and vanities obviously are not directly measurable. On the other hand, there are actual quantities of commodities and money, and the money values of services and rights. The authors with whom we are concerned, however, affirm that the use of mathematics need not be confined merely to this second set of entities, but may also be applied to the order relations among the subjective "quantities." This last statement is indeed incontestable provided such order relations can be established. The properties of inequality, equality, variation, continuous or discontinuous change, and so on, may be supposed to generate these order relations.

We may, perhaps without making any existential assumptions, denote by \( S \) the non-measurable hedonistic value (pleasure or satisfaction) for a person in quantities \( x_1, x_2, ..., x_n \) of various commodities. Then we invent a sort of comparison function \( U(x_1, x_2, ..., x_n) \) which is to increase, decrease or be constant with \( S \), that is, if one collection of quantities \( x_1, ..., x_n \) is preferable to another, the value of \( U(x_1, ..., x_n) \) in the first case is to be greater than in the second; it will be still more convenient if the changes of \( U \) are also greater or less according as the changes in \( S \) are greater or less. The function \( U(x_1, x_2, ..., x_n) \) may be called a utility function and may be regarded as a scale function for the hedonistic value. If it can be constructed it will not be unique, since evidently \( kU \) or \( \tan U \) or many another function of \( U \) will serve as well as \( U \), but any such function can be used as a mathematical tool to replace \( S \).

How is such a function \( U \) to be constructed? We consider an individual (1) with amounts \( (x_1, x_2, ..., x_n) \) of various commodities. With some assortments of small changes \( dx_1, dx_2, ..., dx_n \), his "satisfaction" will increase and, with others, diminish; the intermediate changes where satisfaction remains constant will be given by an equation
\[ X_1(x_1, \ldots, x_n)dx_1 + X_2(x_1, \ldots, x_n)dx_2 + \ldots + X_n(x_1, \ldots, x_n)dx_n = 0. \]  \hspace{1cm} (6.1)

The left hand member is supposed to be positive if satisfaction increases, and negative if satisfaction decreases. The utility function \( U(x_1, \ldots, x_n) \) is a function which satisfied the relations:

- \( dU(x_1, \ldots, x_n) = 0 \)
- \( dU(x_1, \ldots, x_n) > 0 \)
- \( dU(x_1, \ldots, x_n) < 0 \)

according as the left hand member of (1) is zero, positive or negative.

With the above discussion of the construction of a utility function in mind it is easy to see why the integrability problem has played such a large part in the development of utility theory. It is clear that in general there is no one utility function that will satisfy (6.1).

Paul A. Samuelson, in a recent article,\(^2\) gives a historical survey of the integrability issue. One may say that the question of the validity of assuming integrability in utility theory is as yet not answered.

2. Is the consumer rational?

In utility theory it may be correct to consider the distinction between integrability and non-integrability as a distinction between rational and irrational behavior on the part of the consumer. To quote Samuelson\(^3\):


\(^3\)Ibid.,pp. 363-4.
Thus far I have sidestepped the problem of non-integrability by assuming it away. I have implicitly assumed that the consumer does have a consistent ordinal preference field; that he can always tell of two situations either that A is "better than" B, or B better than A, or that they are indifferent; and that his preferences among three or more situations are transitive in the sense that "if A is better than B and B better than C, then A is better than C," etc. In short I have assumed that indifference surfaces do exist. But need they?

After discussion of various theories he\(^4\) concludes:

Observation of reality must be the decisive test as to which hypothesis is the more fruitful -- or whether neither is very fruitful.

In this chapter still another theory is ventured whose implication is that the consumer is rational.

B. Possible Approximate Model for Simple Utility Theory

1. Definitions

(a) A consumer is said to be rational if he acts according to his best interests in an economic transaction.

(b) Assuming the variables \(x_i\) of the problem to be real and at most denumerable, two kinds of ordering are considered:

1) A sequential order not necessarily with beginning or end: \(\ldots, x_{i-1}, x_i, x_{i+1}, \ldots\);

\(^4\)Ibid., p. 376.
(2) Finite cyclic orders: $x_1, x_{i+1}, \ldots, x_k, x_i$.

(c) Grouping the variables $x_i$ in harmony with an ordering above means selecting subsets of adjacent variables, for instance $(x_{i-1}, x_i, x_{i+1})$ in (1) or $(x_k, x_i, x_{i+1})$ in (2).

2. General hypotheses

(a) A consumer will be rational in an economic transaction if the number of variables involved is sufficiently (but not trivially) restricted.

(b) A consumer cannot only order the variables of the above transaction and therefore group them in harmony with such an order but there exists an ordering such that he can deal rationally within the groups provided the size of the groups is less than four.

3. Specific hypotheses

(a) The economic transaction is of the "satisfaction" type, hence the variables are quantities of commodities and the mathematical equivalent of the problem is the analysis of a utility Pfaffian

$$w = \sum_{j=n_1}^{n_2} X_j \, dx_j.$$
(b) The $X_1$ of the utility Pfaffian involve at the most three variables, specifically three adjacent variables centered at $x_1$ from the order of hypothesis (1)b.

(c) Rationality means integrability in a "local" sense; namely that connected Pfaffian chains, generally of length three, are all integrable.

4. Discussion of hypotheses

Suppose a consumer has sequentially ordered his commodity space in accordance with 2(b). Then the grouping in harmony with the ordering is reflected first by $3(b)$ in $X_j = X_j(x_{j-1}, x_j, x_{j+1})$ and secondly by $3(c)$ in the integrability of

$$w_i = \sum_{j=1}^{i+2} X_j \, dx_j.$$ 

The same general procedure may be used for cyclic ordering.

If one considers the utility Pfaffian, of finite length, for the consumer it is certainly not obvious that the local integrability implicit in the above model implies integrability of the whole Pfaffian (that is, integrability in the large). The results of the preceding few chapters may be applied to answer the question which thus arises. This is done in the next section.
C. Principal Implication

To consider the applications of the results obtained in previous chapters it would be well to restate, in a general way, with the above model in mind the principal consequence of Theorems II and III of Chapter III and Theorems I, II and III of Chapter V. It is:

The consumer's utility Pfaffian in general is not only integrable (in the large), but is exact.

The implication is that the behavior of the consumer in the large is rational in general. Exceptions correspond to violation of the hypotheses of the theorems listed above or to a violation of the basic hypotheses of this chapter.
VII. BIBLIOGRAPHY


VIII. ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Dr. Bernard Vinograde for his encouragement and help during the preparation of this thesis and also to acknowledge gratefully the helpful advice given by Dr. Gerhard Tintner on the economic application.
A. Example of a Pfaffian which has a One-stripe Bilinear Covariant Matrix without Assuming Local Variables

If

\[ w = (A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 + A_5x_5 + A_6x_6 + A_7x_7 + A_8x_8 + \ldots + A_nx_n)dx_1 + (B_1x_1 + B_2x_2 + B_3x_3 + B_4x_4 + B_5x_5 + B_6x_6 + B_7x_7 + B_8x_8 + \ldots + B_nx_n)dx_2 + (A_3x_1 + C_2x_2 + C_3x_3 + C_4x_4 + C_5x_5 + C_6x_6 + C_7x_7 + C_8x_8 + \ldots + C_nx_n)dx_3 + (A_4x_1 + B_4x_2 + D_3x_3 + D_4x_4 + D_5x_5 + D_6x_6 + D_7x_7 + D_8x_8 + \ldots + D_nx_n)dx_4 + (A_5x_1 + B_5x_2 + C_5x_3 + E_4x_4 + E_5x_5 + E_6x_6 + E_7x_7 + E_8x_8 + \ldots + E_nx_n)dx_5 + (A_6x_1 + B_6x_2 + C_6x_3 + D_6x_4 + F_5x_5 + F_6x_6 + F_7x_7 + F_8x_8 + \ldots + F_nx_n)dx_6 + (A_7x_1 + B_7x_2 + C_7x_3 + D_7x_4 + E_7x_5 + G_7x_6 + G_7x_7 + G_8x_8 + \ldots + G_nx_n)dx_7 + (A_8x_1 + B_8x_2 + C_8x_3 + D_8x_4 + E_8x_5 + F_8x_6 + H_7x_7 + H_8x_8 + \ldots + H_nx_n)dx_8 + \ldots + (A_nx_1 + B_nx_2 + C_nx_3 + D_nx_4 + E_nx_5 + F_nx_6 + G_nx_7 + H_nx_8 + \ldots + Z_nx_n)dx_n. \]

then

\[ a_{rs} \neq 0 \text{ where } |r-s| = 1, \ a_{rs} = 0 \text{ where } |r-s| > 1. \]
B. Results of Assuming $B_5$ Different from Zero in Theorem II of Chapter II

To simplify the notation it will be set equal to one.

Let

$$x_0 = R_1x_1^2 + R_2x_0^2 + R_3x_1^2 + R_4x_{-1}x_0 + R_5x_{-1}x_1 + R_6x_0x_1$$
$$+ R_7x_{-1} + R_8x_0 + R_9x_1 + R_{10},$$

$$x_1 = A_1x_0^2 + A_2x_1^2 + A_3x_2^2 + A_4x_0x_1 + A_5x_0x_2 + A_6x_1x_2$$
$$+ A_7x_0 + A_8x_1 + A_9x_2 + A_{10},$$

$$x_2 = B_1x_1^2 + B_2x_2^2 + B_3x_3^2 + B_4x_1x_2 + B_5x_1x_3 + B_6x_2x_3$$
$$+ B_7x_1 + B_8x_2 + B_9x_3 + B_{10},$$

$$x_3 = C_1x_2^2 + C_2x_3^2 + C_3x_4^2 + C_4x_2x_3 + C_5x_2x_4 + C_6x_3x_4$$
$$+ C_7x_2 + C_8x_3 + C_9x_4 + C_{10},$$

$$x_4 = D_1x_3^2 + D_2x_4^2 + D_3x_5^2 + D_4x_3x_4 + D_5x_3x_5 + D_6x_4x_5$$
$$+ D_7x_3 + D_8x_4 + D_9x_5 + D_{10},$$

then

$$a_{01} = R_5x_{-1} + (R_6-2A_1)x_0 + (2R_3-A_4)x_1-A_5x_2 + (R_9-A_7),$$

$$a_{12} = A_5x_0 + (A_6-2B_1)x_1 + (2A_3-B_4)x_2 + B_5x_3 + (A_9-B_7),$$

$$a_{23} = B_5x_1 + (B_6-2C_1)x_2 + (2B_3-C_4)x_3 - C_5x_4 + (B_9-C_7),$$

$$a_{34} = C_5x_2 + (C_6-2D_1)x_3 + (2C_3-D_4)x_4 - D_5x_5 + (C_9-D_7).$$

The three integrability conditions used for $w_0$, $w_1$ and $w_2$ are respectively:
(a) \( x_0 a_{12} + x_2 a_{01} = 0 \),  
(b) \( x_1 a_{23} + x_3 a_{12} = 0 \),  
(c) \( x_2 a_{34} + x_4 a_{23} = 0 \).

The values of (7.1) and (7.2) will be substituted in (7.3) and some of the terms written in Tables 1, 2 and 3 respectively.

**Table 1. Some Terms from a of (9.3)**

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( x_0^3 )</td>
<td>( R_2 A_5 )</td>
</tr>
<tr>
<td>2. ( x_0 x_1^2 )</td>
<td>( R_3 A_5 + R_6(A_6-2B_1)+B_2(R_6-2A_1) )</td>
</tr>
<tr>
<td>3. ( x_1 x_1^2 )</td>
<td>( R_5(A_6-2B_1) + R_5B_1 )</td>
</tr>
<tr>
<td>4. ( x_1 x_1 x_2 )</td>
<td>( R_5(2A_3-B_4) + R_5B_4 )</td>
</tr>
<tr>
<td>5. ( x_0 x_1 x_2 )</td>
<td>( R_6(2A_3-B_4) + B_6(R-2A_1) )</td>
</tr>
<tr>
<td>6. ( x_0^2 x_3 )</td>
<td>( -R_1 B_5 )</td>
</tr>
<tr>
<td>7. ( x_0 x_3^2 )</td>
<td>( -R_2 B_5 )</td>
</tr>
<tr>
<td>8. ( x_1^2 x_3 )</td>
<td>( -R_3 B_5 + B_5(2R_3-A_4) )</td>
</tr>
<tr>
<td>9. ( x_1 x_0 x_3 )</td>
<td>( -R_4 B_5 )</td>
</tr>
<tr>
<td>10. ( x_1 x_2^2 )</td>
<td>( R_2 B_2 )</td>
</tr>
<tr>
<td>11. ( x_1 x_3^2 )</td>
<td>( R_3 B_3 )</td>
</tr>
<tr>
<td>12. ( x_1 x_2 x_3 )</td>
<td>( R_5 B_6 )</td>
</tr>
<tr>
<td>13. ( x_0 x_2^2 )</td>
<td>( B_2(R_6-2A_1) )</td>
</tr>
<tr>
<td>14. ( x_0 x_3^2 )</td>
<td>( B_3(R_6-2A_1) )</td>
</tr>
</tbody>
</table>
**Table 1. (Continued)**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$x_0 x_2 x_3$</td>
<td>:</td>
<td>$B_6(R_6-2A_1)$</td>
</tr>
<tr>
<td>16</td>
<td>$x_0 x_1$</td>
<td>:</td>
<td>$R_9 A_5 + R_8(A_6-2B_1) + R_6(A_9-B_7) + B_7(R_6-2A_1)$</td>
</tr>
<tr>
<td>17</td>
<td>$x_0$</td>
<td>:</td>
<td>$R_{10} A_5 + R_8(A_9-B_7) + B_{10}(R_6-2A_1)$</td>
</tr>
<tr>
<td>18</td>
<td>$x_{-1} x_1$</td>
<td>:</td>
<td>$R_7(A_6-2B_1) + R_5(A_9-B_7) + R_5 B_7$</td>
</tr>
<tr>
<td>19</td>
<td>$x_1$</td>
<td>:</td>
<td>$R_9(A_6-2B_1) + R_3(A_9-B_7) + B_7(2R_3-A_4) + B_1(R_9-A_7)$</td>
</tr>
<tr>
<td>20</td>
<td>$x_1$</td>
<td>:</td>
<td>$R_{10}(A_6-2B_1) + R_9(A_9-B_7) + B_{10}(2R_3-A_4) + B_7(R_9-A_7)$</td>
</tr>
<tr>
<td>21</td>
<td>$x_{-1} x_2$</td>
<td>:</td>
<td>$R_7(2A_3-B_4) + R_5 B_8$</td>
</tr>
<tr>
<td>22</td>
<td>$x_0 x_2$</td>
<td>:</td>
<td>$R_8(2A_3-B_4) + R_8(R_6-2A_1)$</td>
</tr>
<tr>
<td>23</td>
<td>$x_1 x_2$</td>
<td>:</td>
<td>$R_9(2A_3-B_4) + R_8(2R_3-A_4) - A_5 B_7 + B_4(R_9-A_7)$</td>
</tr>
<tr>
<td>24</td>
<td>$x_2$</td>
<td>:</td>
<td>$R_{10}(2A_3-B_4) - A_5 B_{10} + B_8(R_9-A_7)$</td>
</tr>
<tr>
<td>25</td>
<td>$x_{-1} x_3$</td>
<td>:</td>
<td>$-R_7 B_5 + R_5 B_9$</td>
</tr>
<tr>
<td>26</td>
<td>$x_0 x_3$</td>
<td>:</td>
<td>$-R_8 B_5 + B_9(R_6-2A_1)$</td>
</tr>
<tr>
<td>27</td>
<td>$x_3$</td>
<td>:</td>
<td>$-R_{10} B_5 + B_9(R_9-A_7)$</td>
</tr>
<tr>
<td>28</td>
<td>$x_{-1}$</td>
<td>:</td>
<td>$R_7(A_9-B_7) + R_5 B_{10}$</td>
</tr>
<tr>
<td>29</td>
<td>const.</td>
<td>:</td>
<td>$R_{10}(A_9-B_7) + B_{10}(R_9-A_7)$</td>
</tr>
<tr>
<td>30</td>
<td>$x_2^2$</td>
<td>:</td>
<td>$-A_5 B_8 + B_2(R_9-A_7)$</td>
</tr>
<tr>
<td>31</td>
<td>$x_2 x_3$</td>
<td>:</td>
<td>$-A_5 B_9 + B_6(R_9-A_7)$</td>
</tr>
<tr>
<td>32</td>
<td>$x_3^2$</td>
<td>:</td>
<td>$B_3(R_9-A_7)$</td>
</tr>
</tbody>
</table>
Table 2. Some Terms from b of (9.3)

1 \( x_0^2 x_1 \) : \( A_1 B_5 \)
2 \( x_1^3 \) : \( A_2 B_5 \)
3 \( x_1 x_2^2 \) : \( A_3 B_5 + A_6(B_6-2C_1) + C_1(A_6-2B_1) \)
4 \( x_0 x_1^2 \) : \( A_4 B_5 \)
5 \( x_0 x_1 x_2 \) : \( A_5 B_5 + A_4(B_6-2C_1) \)
6 \( x_1^2 x_2 \) : \( A_6 B_5 + A_2(B_6-2C_1) \)
7 \( x_2^3 \) : \( A_3(B_6-2C_1) + C_1(2A_3-B_4) \)
8 \( x_2^2 x_3 \) : \( A_3(2B_3-C_4) + C_4(2A_3-B_4) - B_5 C_1 \)
9 \( x_2 x_3^2 \) : \( C_2(2A_3-B_4) - B_5 C_4 \)
10 \( x_2 x_3 x_4 \) : \( C_6(2A_7-B_4) - B_5 C_5 \)
11 \( x_3^3 \) : \( -B_5 C_2 \)
12 \( x_3 x_4^2 \) : \( -B_5 C_3 \)
13 \( x_2^2 x_4 \) : \( -B_5 C_6 \)
14 \( x_0 x_1 \) : \( A_7 B_5 + A_4(B_9-C_7) \)
15 \( x_1^2 \) : \( A_8 B_5 + A_2(B_9-C_7) \)
16 \( x_1 x_2 \) : \( A_9 B_5 + A_8(B_6-2C_1) + A_6(B_9-C_7) + C_7(A_6-2B_1) \)
17 \( x_1 \) : \( A_{10} B_5 + A_8(B_9-C_7) + C_{10}(A_6-2B_1) \)
18 \( x_2^2 \) : \( A_9(B_6-2C_1) + A_3(B_9-C_7) + C_7(2A_3-B_4) + C_1(A_9-B_7) \)
19 \( x_2 \) : \( A_{10}(B_6-2C_1) + A_9(B_9-C_7) + C_{10}(2A_3-B_4) + C_7(A_9-B_7) \)
Table 2. (Continued)

<table>
<thead>
<tr>
<th></th>
<th>Expression</th>
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</thead>
<tbody>
<tr>
<td>20</td>
<td>$x_2x_3 : A_9(2B_3-C_4) + C_8(2A_3-B_4) - B_5C_7 + C_4(A_9-B_7)$</td>
</tr>
<tr>
<td>21</td>
<td>$x_3 : A_{10}(2B_3-C_4) - B_5C_{10} + C_8(A_9-B_7)$</td>
</tr>
<tr>
<td>22</td>
<td>$x_4 : -A_{10}C_5 + C_9(A_9-B_7)$</td>
</tr>
<tr>
<td>23</td>
<td>const. : $A_{10}(B_9-C_7) + C_{10}(A_9-B_7)$</td>
</tr>
<tr>
<td>24</td>
<td>$x_3^2 : -B_5C_8 + C_2(A_9-B_7)$</td>
</tr>
<tr>
<td>25</td>
<td>$x_3x_4 : -B_5C_9 + C_6(A_9-B_7)$</td>
</tr>
<tr>
<td>Term</td>
<td>Description</td>
</tr>
<tr>
<td>------</td>
<td>-------------</td>
</tr>
<tr>
<td>1</td>
<td>$x_1 x_2^2 : B_5(C_6 - 2D_1) + B_5 D_1$</td>
</tr>
<tr>
<td>2</td>
<td>$x_1^2 x_4 : B_1(2C_3 - D_4)$</td>
</tr>
<tr>
<td>3</td>
<td>$x_2^2 x_4 : B_2(2C_3 - D_4)$</td>
</tr>
<tr>
<td>4</td>
<td>$x_2^3 x_4 : B_3(2C_3 - D_4) + D_4(2B_3 - C_4) - C_5 D_1$</td>
</tr>
<tr>
<td>5</td>
<td>$x_1 x_2 x_4 : B_4(2C_3 - D_4)$</td>
</tr>
<tr>
<td>6</td>
<td>$x_2 x_3 x_4 : B_6(2C_3 - D_4) + D_4(B_6 - 2C_1)$</td>
</tr>
<tr>
<td>7</td>
<td>$x_1 x_5 : -B_1 D_5$</td>
</tr>
<tr>
<td>8</td>
<td>$x_2 x_5 : -B_2 D_5$</td>
</tr>
<tr>
<td>9</td>
<td>$x_2^2 x_5 : -B_3 D_5 + D_5(2B_3 - C_4)$</td>
</tr>
<tr>
<td>10</td>
<td>$x_1 x_2 x_5 : -B_4 D_5$</td>
</tr>
<tr>
<td>11</td>
<td>$x_2 x_3 x_5 : -B_6 D_5 + D_5(B_6 - 2C_1)$</td>
</tr>
<tr>
<td>12</td>
<td>$x_1 x_4^2 : B_5 D_2$</td>
</tr>
<tr>
<td>13</td>
<td>$x_1 x_5^2 : B_5 D_3$</td>
</tr>
<tr>
<td>14</td>
<td>$x_1 x_4 x_5 : B_5 D_6$</td>
</tr>
<tr>
<td>15</td>
<td>$x_1 x_2 : B_7 C_5 + B_4(C_9 - D_7)$</td>
</tr>
<tr>
<td>16</td>
<td>$x_2^2 : B_8 C_5 + B_2(C_9 - D_7)$</td>
</tr>
<tr>
<td>17</td>
<td>$x_2 x_3 : B_9 C_5 + B_8(C_6 - 2D_1) + B_6(C_9 - D_7) + D_7(B_6 - 2C_1)$</td>
</tr>
<tr>
<td>18</td>
<td>$x_2 : B_{10} C_5 + B_8(C_9 - D_7) + D_{10}(B_6 - 2C_1)$</td>
</tr>
<tr>
<td>19</td>
<td>$x_3^2 : B_9(C_6 - 2D_1) + B_3(C_9 - D_7) + D_7(2B_3 - C_4) + D_1(B_9 - C_7)$</td>
</tr>
</tbody>
</table>
Table 3. (Continued)

<table>
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<tr>
<th>20 x1</th>
<th>: B_{10}(C_6-2D_4) + B_9(C_9-D_7) + D_{10}(2B_3-C_4) + D_7(B_9-C_7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 x_1x_4</td>
<td>: B_7(2C_3-D_4) + B_5D_6</td>
</tr>
<tr>
<td>22 x_2x_4</td>
<td>: B_8(2C_3-D_4) + D_8(B_6-2C_1)</td>
</tr>
<tr>
<td>23 x_3x_4</td>
<td>: B_9(2C_3-D_4) + D_8(2B_3-C_4) - C_5D_7 + D_4(B_9-C_7)</td>
</tr>
<tr>
<td>24 x_4</td>
<td>: B_{10}(2C_3-D_4) - C_5D_{10} + D_8(B_9-C_7)</td>
</tr>
<tr>
<td>25 x_1x_5</td>
<td>: -B_7D_5 + B_5D_9</td>
</tr>
<tr>
<td>26 x_2x_5</td>
<td>: -B_8D_5 + D_9(B_6-2C_1)</td>
</tr>
<tr>
<td>27 x_3x_5</td>
<td>: -B_9D_5 + D_9(2B_3-C_4) + D_5(B_9-C_7)</td>
</tr>
<tr>
<td>28 x_5</td>
<td>: -B_{10}D_5 + D_9(B_9-C_7)</td>
</tr>
<tr>
<td>29 x_1^2</td>
<td>: B_1(C_9-D_7)</td>
</tr>
<tr>
<td>30 x_1</td>
<td>: B_7(C_9-D_7) + B_5D_{10}</td>
</tr>
<tr>
<td>31 const.</td>
<td>: B_{10}(C_9-D_7) + D_{10}(B_9-C_7)</td>
</tr>
</tbody>
</table>
Since the equations of (9.3) are identically true, each of the above coefficients must be zero. The results of assuming $R_5 \neq 0$ are tabulated below. It should be noted that whenever a horizontal line is drawn all results above it may be used to draw further conclusions. References to all tables as well as specific coefficients are given.

Table 4. Results Assuming $R_5 \neq 0$

<table>
<thead>
<tr>
<th>Line</th>
<th>Result</th>
<th>Line</th>
<th>Result</th>
<th>Line</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$R_1 = 0$</td>
<td>1</td>
<td>$A_1 = 0$</td>
<td>12</td>
<td>$D_2 = 0$</td>
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<tr>
<td>7</td>
<td>$R_2 = 0$</td>
<td>2</td>
<td>$A_2 = 0$</td>
<td>13</td>
<td>$D_3 = 0$</td>
</tr>
<tr>
<td>9</td>
<td>$R_4 = 0$</td>
<td>4</td>
<td>$A_4 = 0$</td>
<td>14</td>
<td>$D_4 = 0$</td>
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<td></td>
<td></td>
<td>5</td>
<td>$A_5 = 0$</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>6</td>
<td>$A_6 = 0$</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>11</td>
<td>$C_2 = 0$</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>12</td>
<td>$C_3 = 0$</td>
<td></td>
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<td></td>
<td></td>
<td>13</td>
<td>$C_4 = 0$</td>
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<tr>
<td>8</td>
<td>$R_3 = 0$</td>
<td>9</td>
<td>$C_4 = 0$</td>
<td>1</td>
<td>$D_1 = 0$</td>
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<td>$C_5 = 0$</td>
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<tr>
<td>2</td>
<td>$R_6 A_1 = 0$</td>
<td>3</td>
<td>$A_3 B_5 - 2 B_1 C_1 = 0$</td>
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<td>$B_1 D_4 = 0$</td>
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<tr>
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<td>7</td>
<td>$A_3 B_6 - B_4 C_1 = 0$</td>
<td>3</td>
<td>$B_2 D_4 = 0$</td>
</tr>
<tr>
<td>4</td>
<td>$R_5 A_3 = 0$</td>
<td>8</td>
<td>$2 A_3 B_3 - B_5 C_1 = 0$</td>
<td>4</td>
<td>$B_3 D_4 = 0$</td>
</tr>
<tr>
<td>5</td>
<td>$R_5 A_3 = 0$</td>
<td>14</td>
<td>$A_7 = 0$</td>
<td>5</td>
<td>$B_4 D_4 = 0$</td>
</tr>
<tr>
<td>10</td>
<td>$R_5 B_2 = 0$</td>
<td>15</td>
<td>$A_8 = 0$</td>
<td>6</td>
<td>$C_1 D_4 = 0$</td>
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</table>
### Table 1

<table>
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<td>11</td>
<td>$R_5B_5=0$</td>
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<tr>
<td>12</td>
<td>$R_5B_6=0$</td>
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<tr>
<td>13</td>
<td>$R_6B_2=0$</td>
</tr>
<tr>
<td>14</td>
<td>$R_6B_3=0$</td>
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<tr>
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<td>$R_6B_6=0$</td>
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### Table 2

<table>
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<td>24</td>
<td>$C_8=0$</td>
</tr>
<tr>
<td>25</td>
<td>$C_9=0$</td>
</tr>
<tr>
<td>8</td>
<td>$B_2D_5=0$</td>
</tr>
<tr>
<td>9</td>
<td>$B_3D_5=0$</td>
</tr>
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<td>10</td>
<td>$B_4D_5=0$</td>
</tr>
<tr>
<td>11</td>
<td>$C_1D_5=0$</td>
</tr>
</tbody>
</table>

### Table 3

1. $A_9B_5-2B_1C_7=0$
2. $A_{10}B_5-2B_1C_{10}=0$
3. $A_9B_6-A_9C_1+A_3B_9+A_3C_7-B_4C_7-B_7C_1=0$
4. $A_{10}B_6-2A_{10}C_1+A_9B_9+2A_3C_{10}-B_4C_{10}-C_7B_7=0$
5. $2A_9B_3-B_5C_7=0$
6. $2A_{10}B_3-B_5C_{10}=0$
7. $A_9C_7=0$
8. $A_{10}B_9-A_{10}C_7+A_9C_{10}-B_7C_{10}=0$

### Table 1

<table>
<thead>
<tr>
<th>line</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$-2R_6B_1+R_6A_9=0$</td>
</tr>
<tr>
<td>17</td>
<td>$R_8A_9-R_8B_7+R_6B_{10}=0$</td>
</tr>
<tr>
<td>18</td>
<td>$-2R_7B_1+R_5A_9=0$</td>
</tr>
<tr>
<td>19</td>
<td>$R_9B_1=0$</td>
</tr>
<tr>
<td>20</td>
<td>$-2R_{10}B_1+R_9A_9=0$</td>
</tr>
<tr>
<td>21</td>
<td>$2R_7A_3-R_7B_4+R_5B_8=0$</td>
</tr>
<tr>
<td>22</td>
<td>$2R_8A_3-R_8B_4+R_6B_8=0$</td>
</tr>
<tr>
<td>23</td>
<td>$R_9A_3=0$</td>
</tr>
<tr>
<td>24</td>
<td>$2R_{10}A_3-R_{10}B_4+R_9B_8=0$</td>
</tr>
<tr>
<td>25</td>
<td>$-R_9B_5+R_5B_9=0$</td>
</tr>
<tr>
<td>26</td>
<td>$-R_8B_5+R_6B_9=0$</td>
</tr>
</tbody>
</table>

### Table 3

<table>
<thead>
<tr>
<th>line</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$B_4D_7=0$</td>
</tr>
<tr>
<td>16</td>
<td>$B_2D_7=0$</td>
</tr>
<tr>
<td>17</td>
<td>$C_1D_7=0$</td>
</tr>
<tr>
<td>18</td>
<td>$-B_8D_7+B_6D_{10}-2C_1D_{10}=0$</td>
</tr>
<tr>
<td>19</td>
<td>$B_3D_7=0$</td>
</tr>
<tr>
<td>20</td>
<td>$2B_3D_{10}-C_7D_7=0$</td>
</tr>
<tr>
<td>21</td>
<td>$-B_7D_4+B_5D_8=0$</td>
</tr>
<tr>
<td>22</td>
<td>$-B_8D_4+B_6D_8-2C_1D_8=0$</td>
</tr>
<tr>
<td>23</td>
<td>$2B_3D_8-C_7D_4=0$</td>
</tr>
<tr>
<td>24</td>
<td>$-B_{10}D_4+B_9D_8-C_7D_8=0$</td>
</tr>
<tr>
<td>25</td>
<td>$-B_7D_5+3D_{10}D_9=0$</td>
</tr>
</tbody>
</table>
Since $A_3$ and $C_1$ are key coefficients they will first be assumed to be non-zero and then to be zero. There is no interrelationship in the identities used and hence this will give all possible combinations.

### Table 1

<table>
<thead>
<tr>
<th>line</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td>$-R_10 B_5 + R_2 B_9 = 0$</td>
</tr>
<tr>
<td>28</td>
<td>$R_7 A_7 - R_6 B_7 + R_5 B_{10} = 0$</td>
</tr>
<tr>
<td>29</td>
<td>$R_10 A_9 - R_9 B_7 + R_8 B_{10} - A_7 B_{10} = 0$</td>
</tr>
<tr>
<td>30</td>
<td>$R_7 B_2 = 0$</td>
</tr>
<tr>
<td>31</td>
<td>$R_9 B_6 = 0$</td>
</tr>
<tr>
<td>32</td>
<td>$R_9 B_3 = 0$</td>
</tr>
</tbody>
</table>

### Table 3

<table>
<thead>
<tr>
<th>line</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>$-B_6 B_5 + B_6 B_9 - B_1 B_9 = 0$</td>
</tr>
<tr>
<td>27</td>
<td>$2B_2 D_9 - C_7 D_5 = 0$</td>
</tr>
<tr>
<td>28</td>
<td>$-B_10 D_5 + B_9 D_9 - C_7 D_9 = 0$</td>
</tr>
<tr>
<td>29</td>
<td>$B_1 D_7 = 0$</td>
</tr>
<tr>
<td>30</td>
<td>$-B_7 D_7 + B_5 D_{10} = 0$</td>
</tr>
<tr>
<td>31</td>
<td>$-B_{10} D_7 + B_9 D_{10} - C_7 D_{10} = 0$</td>
</tr>
</tbody>
</table>

---

1If none of the three ($R_5$, $R_6$, $R_9$) is non-zero then $X_0 = 0$.

2If none of the three ($D_4$, $D_5$, $D_7$) is non-zero then $X_4 = 0$. 
<table>
<thead>
<tr>
<th>Table 1</th>
<th>Table 2</th>
<th>Table 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>line</td>
<td>result</td>
<td>line</td>
</tr>
<tr>
<td>16</td>
<td>$R_6A_9 = 0$</td>
<td>23</td>
</tr>
<tr>
<td>18</td>
<td>$R_5A_9 = 0$</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>$R_9A_9 = 0$</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>$A_9 = 0$</td>
<td></td>
</tr>
</tbody>
</table>

Conclusion from Table 4:

Either $X_0 = 0$ or $X_1 = 0$,
and either $X_4 = 0$ or $X_3 = 0$.

These zero coefficients may be paired $(X_0, X_4)$, $(X_0, X_3)$, $(X_1, X_4)$, or $(x_1, x_3)$.

C. Results of Assuming One of $B_2$, $B_3$, $B_5$

Non-zero in Theorem I of Chapter V

The values of $(5.4)$ and $(5.5)$ will be substituted in the first integrability condition of $(5.3)$ and some of the terms written in Table 1.
Table 1

1 $x_1^2 x_3$ : $2A_1 B_2$
2 $x_2^2 x_3$ : $2A_2 B_2$
3 $x_1 x_2 x_3$ : $2A_3 B_2$
4 $x_1^2 x_2$ : $A_1 B_3$
5 $x_2^3$ : $A_2 B_3$
6 $x_1 x_2^2$ : $A_3 B_3$
7 $x_1 x_3$ : $2A_4 B_2 + A_3 C_4$
8 $x_2 x_3$ : $2A_5 B_2 + 2A_2 C_4$
9 $x_1 x_2$ : $A_4 B_3 + A_3 B_5$
10 $x_2^2$ : $A_5 B_3 + A_2 B_5$
11 $x_1^2$ : $A_1 B_5$
12 $x_3$ : $2A_6 B_2 + A_5 C_4$
13 $x_2$ : $A_6 B_3 + A_5 B_5 + 2A_2 C_6$
14 $x_1$ : $A_4 B_5 + A_3 C_6$
15 const. : $A_6 B_5 + A_5 C_6$

Since the equations (5.3) are identically true, each of the above coefficients must be zero. The results of assuming $B_2$, $B_3$, or $B_5$ non-zero will be tabulated below. Reference to specific coefficients is given.
Table 2

<table>
<thead>
<tr>
<th>B₂ ≠ 0</th>
<th>B₃ ≠ 0</th>
<th>B₂=0, B₃=0, B₅≠0</th>
</tr>
</thead>
<tbody>
<tr>
<td>line</td>
<td>result</td>
<td>line</td>
</tr>
<tr>
<td>1</td>
<td>A₁ = 0</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>A₂ = 0</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>A₃ = 0</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>A₄ = 0</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>A₅ = 0</td>
<td>10</td>
</tr>
<tr>
<td>12</td>
<td>A₆ = 0</td>
<td>13</td>
</tr>
</tbody>
</table>

Conclusion from Table 2:

If any one of the three cases, B₂ ≠ 0 or B₃ ≠ 0 or B₂ = 0, B₃ = 0, B₅ ≠ 0, occur then X₁ = 0.