Periodic solutions of Duffing's equation with forcing term containing first and third harmonics

Fowler Redford Yett
Iowa State College

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PERIODIC SOLUTIONS OF DUFFING'S EQUATION WITH FORCING TERM CONTAINING FIRST AND THIRD HARMONICS

by

Fowler Redford Yett

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY Major Subject: Applied Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College

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I. INTRODUCTION

A. Classification of Periodic Solutions

The differential equation

\[ \ddot{x} + C\dot{x} + f(x) = F \cos wt, \]

which is commonly referred to as the Duffing equation when \( f(x) = Ax + Bx^3 \), has been studied by many workers. Some of these who should be mentioned are Hayashi (1), McLachlan (4), Stoker (5), and Morris E. Levenson (3).

Many physical problems lead to this type of equation, for example, the hunting of synchronous electrical machines and alternating current circuits with saturable iron core inductances. The main questions of interest in studying the equation are the conditions under which periodic solutions exist.

The periodic solutions of the equation are classified into four types which are the harmonic, the subharmonic, the ultraharmonic, and the ultra-subharmonic. The significance of these terms is explained in the book by Stoker; however, a brief discussion similar to his is presented here. If in the Duffing equation \( C = B = 0 \), the following linear differential equation results:

\[ \ddot{x} + Ax = F \cos wt, \quad A > 0, \quad w^2 \neq A. \]

For this, the linear case without damping, all the solutions
may be obtained from solutions satisfying initial conditions of the type \( x(0) = E, x(0) = 0 \). The solution of (1.2) satisfying these conditions is

\[
x = G \cos A^{1/2} t - \frac{F}{w^2 - A} \cos wt
\]

where

\[
G = E + \frac{F}{w^3 - A}.
\]

The solution (1.3) of (1.2) with the specified initial conditions is periodic only in the following cases:

a) \( G = 0 \),

b) \( G \neq 0, w = n A^{1/2}, n \) any integer except 1, 0,

c) \( G \neq 0, w = \frac{1}{n} A^{1/2}, n \) any integer except 1, 0,

d) \( G \neq 0, w = \frac{p}{q} A^{1/2}, p \) and \( q \) relative prime integers \( \neq 1 \).

In case (a) the solution has the period \( 2\pi/w \) which is the same as the period of the forcing term, \( F \cos wt \). This is called a harmonic oscillation. In this case the free oscillation, which is the periodic solution when \( F = 0 \), does not contribute.

In case (b) the solution has the least period \( \frac{2\pi}{A^{1/2}} = n \frac{2\pi}{w} \), that is, it has a period which is \( n \) times the period of the forcing term. In terms of frequency of the forcing term, and for this reason the solution in this case is called a subharmonic of order \( 1/n \), or simply a subharmonic.
In case (c) the solution has the least period $2\pi/w$ which is the same as that of the forcing term. The difference between this case and case (a) is that here the free oscillation contributes a higher harmonic component to the periodic response. The solution in case (c) is called an ultraharmonic oscillation of order $n$.

In case (d) the least period of the solution is $\frac{2\pi}{w}$, that is, $p$ times the period of the forcing term. The distinction between this case and case (b) is that here the free oscillation has the period $\frac{2\pi}{A^{1/2}} = \frac{1}{q} \left( \frac{2\pi}{w} \right)$. That is, the response has a period which is $q$ times that of the free oscillation rather than the same as that of the free oscillation. The solution in case (d) is referred to as an ultraharmonic oscillation.

When $f(x)$ is nonlinear in equation (1.1) it is not always possible to distinguish between the harmonic and the ultraharmonic or between the subharmonic and the ultraharmonic. In the work of Levenson and also in this thesis the distinction is handled in the following way. The periodic solutions are found for equation (1.1) by a perturbation procedure, that is, by an expansion of the solution in powers of a small parameter which appears in the equation. When this parameter equals zero (1.1) reduces to (1.2). The periodic solution reduces to (1.3) and the periodic solution of the
nonlinear equation is classified according to the case which occurs in the limiting process.

B. Review of Literature

A discussion of some of the work which was performed by the four previously mentioned authors will now be given. Hayashi made use of approximations and gave a descriptive and detailed, though non-rigorous, account of the various sub-harmonic oscillations that result when \( f(x) = A_0 + A_1x + A_2x^2 + \cdots \). He stated that when \( f(x) = A_1x + A_2x^3 \), the only sub-harmonic that will occur is the one of order \( 1/3 \). When \( f(x) = A_1x + A_2x^2 + A_3x^3 \), the subharmonic of order \( 1/2 \) predominates although a discrepancy was found between experimental and approximated values. After he considered \( f(x) = A_1x + A_5x^5 \), and found subharmonics of order \( 1/3 \) and \( 1/5 \), he deduced that the presence of \( A_nx^n \) in \( f(x) \) is a sufficient condition for the occurrence of a subharmonic of order \( 1/n \) and that subharmonics of order \( 1/n \) do not occur when the highest power of \( f(x) \) is less than \( n \). He displayed a large amount of data among which he showed that the region of stability for subharmonics of order \( 1/3 \) is much larger for \( f(x) = A_1x + A_3x^3 \) than for \( f(x) = A_1x + A_5x^5 + \cdots + A_{2n-1}x^{2n-1} + \cdots \) which has a region of stability that is only a thin shell compared to the stable region of the former.
the discussion that follows this equation is the excellent work of
S. S. W. and the results obtained in equation (2.27) of this
paper. The damping coefficient must be. The above results com-
plete. The amplitude of the forcing term on the response, the smal-
ler the damping coefficient, the smaller the response. This means that the positive damping coefficient

\[ \frac{v}{\sqrt{2}} \leq c \leq 3/2 \]

where \( v/\sqrt{2} \) is the amplitude of the
forcing term, \( c \) is the damping coefficient, and \( v \) is the
amplitude of the forcing term. When the damping is introduced into the above equation, \( c \geq \frac{1}{2} \).

When no subharmonic oscillation can exist with \( v = \sqrt{2} \), the
forcing term must be greater than \( 1/2 \times \frac{1}{2} \). When \( v = 0 \) and \( T + \pi \), we must
round that the subharmonic of order \( 1/3 \) can occur only when
the damping is introduced into the above equation, \( c \geq \frac{1}{2} \).

When we round that the subharmonic of order \( 1/3 \) can occur only when
the damping is introduced into the above equation, \( c \geq \frac{1}{2} \).

He let \( j(x) = ax + bx + cx^2 \) and
out damping. He let \( j(x) = (x) \) and
book to the discussion of the forcing equation with and with-

Stoker devoted one chapter and part of others in his
ultrashort monos and ultrasonic monos for the
the determined response curves and subharmonic regions for the
forcing term, \( c \geq \frac{1}{2} \). Also, in the case \( j(x) = x^2 \),
with respect to the square of the amplitude of the solution

\[ \frac{v}{\sqrt{2}} \leq c \leq 3/2 \]

with respect to the square of the forcing term

For symmetrical or unsymmetrical nonlinearity when the de-

Hawkins showed that the harmonic oscillation is stable
are not too large, the subharmonic of order $1/3$ exists provided
when damping was introduced, the force that if \( B > 0 \) and \( C > 0 \)
subharmonic of order $1/3$ exists when \( w = \frac{1}{3} \). The
approximate solution as \( x = \frac{1}{3} \cos \frac{1}{3} t \) + \( A \cos \frac{1}{3} t \)
harmonic of order $1/3$ exists, and therefore, he took the
assumed that \( A/\nu > 1/2 \), \( v > 0 \). \( B' = 0 \), then a sub-
when Motion theory considered the Duffing equation, he
methods device

...data for a series, and the series that result from approximation
when \( w/\nu \) is not a rational number the difficulty of small
when \( w/\nu \) is rational, however, that \( w/\nu \)

\[
x(t) = A(t) + B(t) + C(t)
\]
solutions could be obtained for the differential equation
Motion theory mentioned that the existence of periodic
harmonics of the Duffing equation with and without damping
harmonic force. He also determined stability region for
present, no periodic motion results while there is an ex-
under different forms to the deduced that when damping is
After Motion considered the Duffing equation in

...the Duffing equation without damping
harmonic solutions but obtained a first approximation for
...did not give conditions for the existence of
\[ w^a > \frac{9(A + 21/16 A^a)}{1-16C/3B A^a} \]. These results correspond to the results mentioned above by Stoker and to the results in equation (2.27) and the discussion which follows that equation, with the exception that \( A_1 \) does not enter the other results.

McLachlan considered the Duffing equation with two forcing terms under certain assumptions (\( w^a_0 \gg w^a_1 \gg A \), \( F_0 \gg F_1 \), \( C \) small enough to be neglected but sufficient to extinguish transients, and the conditions for subharmonics to exist are not satisfied) and found an approximate solution by first assuming the solution in the form of the solution of the linear equation.

Hayashi, Stoker, and McLachlan made use of the iteration process in solving the approximate periodic solutions. This method assumes a first approximate solution and after the first approximation is used to obtain the second approximation, the second approximation is used to obtain the third approximation, and so forth. Usually the second approximation seemed to be sufficient.

In many instances the authors assumed the periodic solution to be in the form of a Fourier series and used enough terms from this series to obtain the desired approximation by determining the Fourier coefficients from the nonlinear relations which resulted from substituting into the equation.
Stoker also used the perturbation method in which the desired quantities are developed in powers of some small parameter in the differential equation. The coefficients of the expansions are usually determined by a sequence of linear equations.

Levenson proved the existence of the four types of periodic solutions mentioned above for the Duffing equation without damping. He also proved the existence of sub-harmonics of order $1/3$ for the Duffing equation with damping. The method used by Levenson to establish the existence of the periodic solutions of the Duffing equation is the one used in this thesis; and, therefore, will not be explained here. The results that he obtained will be commented on in the appropriate places in the body of the thesis. Levenson did not consider the stability of the periodic solutions.

C. Object of Thesis

The purpose of this thesis is to show the existence of the subharmonic, ultraharmonic, and ultra-subharmonic solutions of the nonlinear differential equation

$$
(1.4) \quad \ddot{x} + Ax + Bx^3 + C\dot{x} = F_1 \cos(wt + \phi_1) + F_2 \cos(3wt + \phi_2),
$$

$$
A > 0, \ x(t_0) = E, \ \dot{x}(t_0) = 0
$$

and to show under what conditions these solutions are stable.
Also the existence of harmonic solutions and their stability are shown for (1.4) when \( F_3 = 0 \). The conditions under which periodic solutions will exist are shown for subharmonics of order 1/3 and ultraharmonics of order 2, 5, 7, and 9. A procedure is outlined to show how the initial conditions, \( E \) and \( t_0 \), may be determined when the other parameters in (1.4) are known in order to insure the existence of subharmonics of order 1/3. Similar methods could be used to obtain the corresponding conditions that would give rise to the other periodic solutions.

The phase angle \( \phi_1 \) is introduced into the \( F_1 \cos(\omega t + \phi_1) \) term because the periodic oscillations which satisfy the initial conditions in (1.4) will, in general, be out of phase with the forcing function. The phase angle \( \phi_2 \) is introduced because of the phase difference that may exist between the two forcing terms. Since only the odd harmonics of the forcing function occur in most physical problems (which is notably true in electrical systems), it is natural to extend the Duffing equation by adding the next odd harmonic to the forcing function in (1.1) to obtain (1.4). It would be of interest to consider the extension of the problem by adding successive odd harmonics to the forcing function and to seek a further generalization of the effect studied here. However, the calculations would become very involved and it is probable that numerical approximations would be necessary.
II. EXISTENCE OF PERIODIC SOLUTIONS

A. General Periodicity Conditions

1. Development of equations for periodicity conditions.

This section will be devoted to the development of two equations which must be satisfied to insure the existence of periodic solutions of (1.4). These two equations will be referred to as the periodicity conditions. Before working with the four types of periodic solutions singly the general theory will be carried forward to the place where the conditions of periodicity require specific values.

The substitutions

\[
(2.1) \quad \theta + \alpha = wt + \phi_1 \text{ such that } \theta = 0 \text{ when } t = t_0, \quad t_0
\]

\[
= \frac{a - \phi_2}{w}, \quad Ax = F_1 \rho, \quad \frac{v^2}{A} = v^2, \quad 3a - 3\phi_1 + \phi_3
\]

\[
= \beta, \quad \varepsilon = \frac{BF_3}{A}, \quad k = \frac{CA}{F_2}, \quad \frac{F_2}{F_1} = r, \quad F_1 + 0
\]

will put equation (1.4) in the following dimensionless form. Throughout this thesis the prime denotes differentiation with respect to \( \theta \).

\[
(2.2) \quad v^2\rho'' + \rho + \varepsilon\rho'^2 + k\varepsilon v\rho' = \cos(\theta + \alpha) + r \cos(3\theta + \beta)
\]

\[
\rho(\theta = 0) = \frac{EA}{F_2} = M, \quad \rho'(\theta = 0) = 0.
\]
The phase angle $\alpha$ is introduced in (2.1) in order that the initial condition could be specified at $\theta = 0$ and still have the slope of the solution curve zero at this value when damping is present. The phase angle $\beta$ is a function of the phase difference between the two forcing terms.

Equation (2.2) may be transformed to the following integral equation. This is conveniently accomplished by use of Laplace transforms.

\[(2.3) \quad \rho(\theta) = a \cos \frac{\theta}{v} + b \sin \frac{\theta}{v} + c \cos(\theta + \alpha) + g \cos(3\theta + \beta)\]

\[- \frac{v}{\xi} \int_{0}^{\theta} (\rho^3(\phi) + kv \rho^4(\phi)) \sin \frac{\theta - \phi}{v} \, d\phi\]

where $a = M + \frac{\cos \alpha}{v^a - 1} + \frac{r \cos \beta}{9v^a - 1}$,

$b = - \frac{v \sin \alpha}{v^a - 1} - \frac{3vr \sin \beta}{9v^a - 1}$,

$c = - \frac{1}{v^a - 1}$, $g = - \frac{r}{9v^a - 1}$, $v \neq 0, 1, \frac{1}{3}$.

Since $\rho'(\theta)$ will be needed several times, the equation for it is written with the same symbols as in (2.3).
\[ \rho'(\theta) = -\frac{a}{V} \sin \frac{\theta}{V} + \frac{b}{V} \cos \frac{\theta}{V} - c \sin(\theta + \alpha) \]
\[ - 3g \sin(3\theta + \beta) \]
\[ - \frac{e}{V^3} \int_{0}^{\theta} (\rho^3(\phi) + k \rho'(\phi)) \cos \frac{\theta - \phi}{V} d\phi. \]

From equation (2.2) it is observed that the right hand side is a periodic function of \( \theta \) with period \( 2\pi \). Thus if \( \rho(\theta) \) is a solution, then \( \rho(\theta) = \rho(\theta + 2m\pi) \) is also, where \( m \) is some integer. If \( \rho(0) = \rho(0) \) and \( \rho'(0) = \rho'(0) \), then, since the solutions are unique, it follows that \( \rho(\theta) \equiv \rho(\theta) \). Thus \( \rho(\theta + 2m\pi) \equiv \rho(\theta) \equiv \rho(\theta) \), that is, \( \rho(\theta) \) has the period \( 2m\pi \). Thus, a necessary and sufficient condition that \( \rho(\theta) \) be a periodic solution of (2.2) with the period \( 2m\pi \) is that \( \rho(0) = \rho(0) \) and \( \rho'(0) = \rho'(0) \), or in terms of \( \rho \) itself, \( \rho(2m\pi) = \rho(0) \) and \( \rho'(2m\pi) = \rho'(0) \).

Now the solution \( \rho(\theta) \) in (2.3) depends upon the values of the parameters in (2.2) and this is made explicit when the solution is written as \( \rho(\theta, M, \varepsilon, v, k, r, \alpha, \beta) \). When the initial conditions in (2.2) are used, it is seen that the necessary and sufficient conditions that \( \rho(\theta) \) have the period \( 2m\pi \) are the following:

\[ \rho(2m\pi, M, \varepsilon, v, k, r, \alpha, \beta) = M, \]
\[ \rho'(2m\pi, M, \varepsilon, v, k, r, \alpha, \beta) = 0. \]
All the parameters, including $\theta$, in (2.2) enter analytically; therefore, they enter analytically in (2.4). When $\varepsilon = 0$ equations (2.4) have a solution, in fact, they may have many solutions; however, this difficulty will be discussed later. Conditions under which the equations (2.4) have a solution form the main results of this thesis. These conditions are determined at the parameter value $\varepsilon = 0$ and the implicit function theorem is then used to establish the existence of a solution for $\varepsilon \neq 0$ in a neighborhood of $\varepsilon = 0$. By a solution at $\varepsilon = 0$ is meant that some two of the parameters in (2.4) are expanded in powers of $\varepsilon$ and the first coefficient in each of these expansions which is not prescribed will be solved for in terms of the other parameters. The other parameters may have arbitrary values.

For $\varepsilon = 0$, the equation (2.2) reduces to

\[ v^n \rho^n + \rho = \cos(\theta + \alpha) + r \cos(3\theta + \beta) \]

and the solution is

\[ \rho(\theta = 0) = M, \quad \rho'(\theta = 0) = 0; \]

\[ \rho(\theta) = a \cos \frac{\theta}{v} + b \sin \frac{\theta}{v} + c \cos(\theta + \alpha) \]

\[ + g \cos(3\theta + \beta), \]

where $a, b, c,$ and $g$ are the same as the symbols in (2.3).
If $\rho(\theta)$ is to be periodic for a range of values of $\varepsilon$ near $\varepsilon = 0$, then in particular $\rho(\theta)$ must be periodic when $\varepsilon = 0$. By observation from (2.6) it is necessary that either $\frac{1}{\sqrt{v}}$ be a rational number when $a^2 + b^2 \neq 0$ or that $a = b = 0$. In any case, for $\rho(\theta)$ to be periodic for $\varepsilon \neq 0$ the parameters $M, v, k, r, a, \beta$ must be considered functionally related to each other and to $\varepsilon$ so that they reduce to quantities which will cause $\rho(\theta)$ to be periodic when $\varepsilon = 0$.

From the above paragraph it is seen that two cases must be considered in order to have periodic solutions when $\varepsilon = 0$.

The first case is

\[(2.7) \quad v = v_0 + v_1 \varepsilon + v_2 \varepsilon^2 + \cdots \]

and

\[a^2_0 + b^2_0 \neq 0 \]

where $v_0 = \frac{p}{q}$ is a rational number. The second case is

\[(2.8) \quad a = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \cdots \]

\[b = b_0 + b_1 \varepsilon + b_2 \varepsilon^2 + \cdots \]

and

\[a_0 = b_0 = 0.\]

The discussion of this last case will be deferred for the present. In the first case when $\varepsilon = 0$, all solutions $\rho(\theta)$ in (2.6) have a period $2\pi k$ that is equal to the lowest
common multiple of $2\pi, 2\pi/3,$ and $2\pi/q$. This means that for $\varepsilon = 0$ equations (2.4) are satisfied identically in the parameters; and, therefore, the implicit function theorem cannot be used until this difficulty is removed.

The difficulty here is analogous to that encountered in consideration of the equation $\varepsilon(x - y) = 0$. This equation requires $y = x$ if $\varepsilon \neq 0$, but $x$ and $y$ may have any finite values when $\varepsilon = 0$. To circumvent this difficulty it turns out that both equations in (2.4) must be divided by $\varepsilon$. Accordingly, two new functions, $\Phi$ and $\Psi$, are introduced and equations (2.4) are written as

\begin{align*}
\Phi(2m,M,\varepsilon,v,k,r,a,\beta) &= \frac{1}{\varepsilon}(\rho(2m,M,\varepsilon,v,k,r,a,\beta)-M)=0, \\
\Psi(2m,M,\varepsilon,v,k,r,a,\beta) &= \frac{1}{\varepsilon}(\rho(2m,M,\varepsilon,v,k,r,a,\beta)-0)=0.
\end{align*}

Equations (2.9) determine the same implicit relations among the parameters $M, \varepsilon, v, k, r, a, \beta$ when $\varepsilon \neq 0$ as do equations (2.4). However, now $\Phi$ and $\Psi$ must be defined for $\varepsilon = 0$. This will be done by a limiting process.

If the value of the parameters in the original differential equation (1.4) are specified (that is, $A, B, C, F_1, F_2, \phi_1, \phi_2, w$), then from (2.1), $\varepsilon, v, k, r$ are determined. Since $\beta$ is determined from $a, \phi_1$, and $\phi_2$, this leaves $M$ and $\alpha$ to be determined such that one of the four types of periodic solutions will occur. It would be desirable to
obtain $M$ and $a$ explicitly as

$$ M = M(\varepsilon, v, k, r, \beta) $$
$$ a = a(\varepsilon, v, k, r, \beta). $$

However, it develops that $M$ and $a$ are multiple-valued functions of the other parameters and it happens in Case 1 above that it is simpler to solve (2.9) in the following form:

$$(2.10a) \quad k = f(M, \varepsilon, r, a, \beta) = k_0 + k_1 \varepsilon + \cdots,$$
$$ v = g(M, \varepsilon, r, a, \beta) = v_0 + v_1 \varepsilon + \cdots$$
$$ = v_0 + \varepsilon v_0 + \varepsilon (v_0 + v_1 \varepsilon + \cdots)$$

where $M, r, a, \beta$ are arbitrary. From these $M$ and $a$ can be determined as functions of the other parameters.

In Case 2 where $r = 0$ is prescribed (consequently $\beta$ does not enter) and where $M$ and $k$ have arbitrary values, it is convenient to solve for $v$ and $a$ as follows:

$$(2.10b) \quad v = F(M, \varepsilon, k) = v_0 + v_1 \varepsilon + \cdots = v_0 + \varepsilon v_0 + \varepsilon (v_0 + v_1 \varepsilon + \cdots)$$
$$ a = G(M, \varepsilon, k) = a_0 + a_1 \varepsilon + \cdots = a_0 + Z \varepsilon$$
$$ = \varepsilon (Z_0 + Z_1 \varepsilon + \cdots)$$

and then to solve (2.10b) for $M$ and $a$ as functions of $v, \varepsilon, k$. It is observed from the definitions following (2.3) that in this case ($r=0$) if $b_0 = 0$, then $a_0 = n \varepsilon$ and since
when \( \varepsilon = 0 \) there is no damping, \( n \) will be taken as zero.
Thus \( a_0 = 0 \) throughout this thesis in Case 2. It is also observed that if \( a_0 = 0 \), then \( v_0 = (1 - 1/M)^{1/2} \) where \( M < 0 \) or \( M > 1 \).

The following notation is used in this thesis. Parameters which are expanded in powers of \( \varepsilon \) will be denoted by a subscript zero when evaluated at \( \varepsilon = 0 \), and if some other parameter is a function of such parameters, then it will also carry a subscript zero for \( \varepsilon = 0 \). For example, \( a \) is a function of \( v \) and when the value of \( v \) at \( \varepsilon = 0 \) is denoted by \( v_0 \), then the value of \( a \) at \( \varepsilon = 0 \) will be denoted by \( a_0 \). Parameters with arbitrary values will carry no subscripts.

The remainder of this section is devoted to obtaining the value of \( \bar{F} \) and \( \bar{Q} \) in (2.9) when \( \varepsilon = 0 \). For the discussion of Case 1 the following notation is convenient:

\[
\bar{F}(2m\pi, M, \varepsilon, v, k, r, a, \beta) = P(\varepsilon, v_0, V, k)
\]

and

\[
\bar{Q}(2m\pi, M, \varepsilon, v, k, r, a, \beta) = Q(\varepsilon, v_0, V, k)
\]

where the parameters \( M, r, a, \beta \) have arbitrary values. For Case 2 (the harmonic case) the following notation is used

\[
\bar{F}(2m\pi, M, \varepsilon, v, k, r, a, \beta) = P_h(\varepsilon, v_0, V, Z)
\]

and

\[
\bar{Q}(2m\pi, M, \varepsilon, v, k, r, a, \beta) = Q_h(\varepsilon, v_0, V, Z)
\]
where \( M, k \) have arbitrary values and \( r = a_o = 0 \) and, therefore, \( \beta \) does not enter. The reason for this notation is that for \( \varepsilon = 0 \) the functions \( \overline{F} \) and \( \overline{Q} \) depend on the first degree term in \( v \) as well as on the choice of \( v_o \). A similar remark in the harmonic case applies to \( a \). From the form of \( \rho(\theta) \) in (2.3) and the definitions of \( a, b, c, g \) which follow (2.3), the functions \( \overline{F} \) and \( \overline{Q} \) may now be written as

Case 1, \( v_o = \frac{p}{q} \) and \( a^2 + b^2 \neq 0; \)

\[
(2.11a) \quad P(\varepsilon, v_o, V, k) = \frac{1}{\varepsilon}(a \cos \frac{2\pi}{v} - M + c \cos a + g \cos \beta)
\]

\[
+ \frac{1}{\varepsilon}(b \sin \frac{2\pi}{v}) - \frac{1}{v} \int_0^{2\pi} (\rho^s(\phi) + kv \rho(\phi)) \sin \frac{2\pi - \phi}{v} \, d\phi
\]

\[
= \frac{a}{\varepsilon} \left( \cos \frac{2\pi}{v} - 1 \right) + \frac{b}{\varepsilon} \sin \frac{2\pi}{v}
\]

\[
- \frac{1}{v} \int_0^{2\pi} (\rho^s(\phi) + kv \rho^t(\phi)) \sin \frac{2\pi - \phi}{v} \, d\phi,
\]

\[
Q(\varepsilon, v_o, V, k) = \frac{1}{\varepsilon}(a \sin \frac{2\pi}{v} + b \cos \frac{2\pi}{v} - c \sin a - 3g \sin \beta)
\]

\[
- \frac{1}{v^2} \int_0^{2\pi} (\rho^s(\phi) + kv \rho^t(\phi) \cos \frac{2\pi - \phi}{v} \, d\phi)
\]

\[
= \frac{a}{\varepsilon} \left( - \frac{1}{v} \sin \frac{2\pi}{v} \right) + \frac{b}{\varepsilon v} (\cos \frac{2\pi}{v} - 1)
\]

\[
- \frac{1}{v^2} \int_0^{2\pi} (\rho^s(\phi) + kv \rho^t(\phi)) \cos \frac{2\pi - \phi}{v} \, d\phi,
\]
Case 2, \(a_0 = b_0 = 0, \ r = 0, \ v_0 = (1 - 1/M)^{1/2}\) where \(M < 0\) or \(M > 1\):

\[
(2.11b) \quad P_h(\epsilon, v_0, v, Z) = \frac{a}{\epsilon} \left( \cos \frac{2\pi}{v} - 1 \right) + \frac{b}{\epsilon} \sin \frac{2\pi}{v} - \frac{1}{v} \int_0^{2\pi} (\rho^3(\phi) + kv \rho'(\phi)) \sin \frac{2\pi - \phi}{v} \, d\phi,
\]

\[
Q_h(\epsilon, v_0, v, Z) = \frac{a}{\epsilon} \left( -\frac{1}{v} \sin \frac{2\pi}{v} \right) + \frac{b}{\epsilon v} \left( \cos \frac{2\pi}{v} - 1 \right) - \frac{1}{v^2} \int_0^{2\pi} (\rho^3(\phi) + kv \rho'(\phi)) \cos \frac{2\pi - \phi}{v} \, d\phi.
\]

In the first case, where \(v \to v_0 = \frac{P}{q}\) as \(\epsilon \to 0\), the ratios

\[
\frac{1}{\epsilon} \left( \cos \frac{2\pi}{v} - 1 \right) \text{ and } \frac{1}{\epsilon} \left( \sin \frac{2\pi}{v} \right)
\]

are indeterminate forms which may be evaluated by L'Hospital's rule. Thus, in Case 1

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \cos \frac{2\pi}{v} - 1 \right) = \lim_{\epsilon \to 0} \left( -\sin \frac{2\pi}{v} \right) \left( -\frac{2\pi}{v^2} \right) \frac{dv}{dc} = (\sin \frac{2\pi}{v_0} \frac{2\pi}{v_0^3} v_1)
\]

and

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \sin \frac{2\pi}{v} \right) = \lim_{\epsilon \to 0} \left( \cos \frac{2\pi}{v} \right) \left( -\frac{2\pi}{v^2} \right) \frac{dv}{dc} = (\cos \frac{2\pi}{v_0} \frac{2\pi}{v_0^3} v_1).
\]
When (2.10b) and the definitions following (2.3) are used with $M$ and $k$ prescribed the limit of the ratios $\frac{a}{\epsilon}$ and $\frac{b}{\epsilon}$ may be evaluated in a similar fashion. Thus

$$\lim_{\epsilon \to 0} \frac{a}{\epsilon} = \lim_{\epsilon \to 0} \left[ \frac{-2v \cos a}{(v^2 - 1)^2} (V) + \frac{-\sin a}{v^2 - 1} (Z) \right]$$

$$= - \frac{2v_0 v}{(v_0^2 - 1)^2} = - \frac{2v_0 v_1}{(v_0^2 - 1)^2},$$

$$\lim_{\epsilon \to 0} \frac{b}{\epsilon} = \lim_{\epsilon \to 0} \left[ \frac{(v^2 - 1) \sin a - 2v \sin a}{(v^2 - 1)^2} (V) + \frac{-v \cos a}{v^2 - 1} (Z) \right]$$

$$= - \frac{v_0 a_1}{v_0^2 - 1}.$$

In both cases $\rho(\theta) = \rho_0(\theta) + \rho_1(\theta) \epsilon + \rho_2(\theta) \epsilon^2 + \cdots$ and $\rho(\theta) \to \rho_0(\theta)$ as $\epsilon \to 0$ where from (2.6)

$$(2.12) \quad \rho_0(\theta) = a_0 \cos \frac{\theta}{v_0} + b_0 \sin \frac{\theta}{v_0} + c_0 \cos (\theta + \alpha)$$

$$+ g_0 \cos (3\theta + \beta).$$

Thus, the proper definitions of $P$ and $Q$ when $\epsilon = 0$ are the following equations:

Case 1:

$$(2.13) \quad P(0, v_0, v_1, k_0) = \frac{2\pi k}{v_0} v_1 (a_0 \sin \frac{2\pi k}{v_0} - b_0 \cos \frac{2\pi k}{v_0}).$$
\[- \frac{1}{v_0} \int_0^{2\pi} \left( \rho_0^s(\phi) + k_0 v_0 \rho_0^s(\phi) \right) \sin \frac{2\pi - \phi}{v_0} \, d\phi, \]

\[Q(0, v_0, v_1, k_0) = \frac{2\pi}{v_0} v_1 \left( a_0 \cos \frac{2\pi}{v_0} + b_0 \sin \frac{2\pi}{v_0} \right) \]

\[- \frac{1}{v_0} \int_0^{2\pi} \left( \rho_0^s(\phi) + k_0 v_0 \rho_0^s(\phi) \right) \cos \frac{2\pi - \phi}{v_0} \, d\phi, \]

Case 21:

\[(2.14) \quad P_h(0, v_0, v_1, a_1) = - \frac{2v_0 v_1}{(v_0^2 - 1)^2} \left( \cos \frac{2\pi}{v_0} - 1 \right) \]

\[- \frac{v_0 a_1}{v_0^2 - 1} \sin \frac{2\pi}{v_0} - \frac{1}{v_0} \int_0^{2\pi} \left( \rho_0^s(\phi) + k v_0 \rho_0^s(\phi) \right) \sin \frac{2\pi - \phi}{v_0} \, d\phi, \]

\[Q_h(0, v_0, v_1, a_1) = \frac{2v_1}{(v_0^2 - 1)^2} \sin \frac{2\pi}{v_0} \]

\[- \frac{a_1}{v_0^2 - 1} \left( \cos \frac{2\pi}{v_0} - 1 \right) - \frac{1}{v_0} \int_0^{2\pi} \left( \rho_0^s(\phi) + k v_0 \rho_0^s(\phi) \right) \cos \frac{2\pi - \phi}{v_0} \, d\phi. \]

The evaluation of the integrals in (2.13) and (2.14) is made easier by the substitution of \( \phi = v_0 t \) which changes (2.12) to
\( (2.15) \quad \rho_o(v_o t) = a_o \cos t + b_o \sin t \)

\[ + c_o \cos(v_o t + \alpha) + g_o \cos(3v_o t + \beta). \]

Moreover, since it happens that \( m = p \) in Case 1, the value of \( \sin \frac{2\pi m}{v_o} = 0 \) and \( \cos \frac{2\pi m}{v_o} = 1 \) for \( v_o = \frac{p}{q} \). Therefore, equations (2.13) and (2.14) may be written as follows:

Case 1:

\( (2.16a) \quad P(0, v_o, v_1, k_o) = -\frac{2\pi}{v_o} a_o v_1 \)

\[ + \int_0^{\frac{2\pi}{v_o}} (\rho_o^s(v_o t) + k_o v_o \rho_o^t(v_o t)) \sin(t - \frac{2\pi}{v_o}) dt, \]

\( (2.17a) \quad Q(0, v_o, v_1, k_o) = \frac{2\pi}{v_o} a_o v_1 \)

\[ - \frac{1}{v_o} \int_0^{\frac{2\pi}{v_o}} (\rho_o^s(v_o t) + k_o v_o \rho_o^t(v_o t)) \cos(t - \frac{2\pi}{v_o}) dt, \]

Case 2:

\( (2.16b) \quad P_\lambda(0, v_o, v_1, a_l) = \frac{h v_o v_1}{(v_o - 1)^s} \sin^s \frac{m\pi}{v_o} \)
\[-\frac{v_0^{a_1}}{v_0^{a-1}} \sin \frac{2\pi}{v_0} + \int_0^{2\pi} p_o^s(v_0 t) + kv_o \rho_0^f(v_0 t)\sin(t - \frac{2\pi}{v_0})dt,\]

\[(2.17b) \quad Q_h(0, v_0, v_1, a_1) = \frac{2v_3}{(v_0-1)^2} \sin \frac{2\pi}{v_0} + \frac{2a_1}{v_0-1} \sin^2 \frac{\pi}{v_0}\]

\[-\frac{1}{v_0} \int_0^{2\pi} p_o^s(v_0 t) + kv_o \rho_0^f(v_0 t)\cos(t - \frac{2\pi}{v_0})dt\]

where the identity \(2 \sin^2 \frac{\pi}{v_0} = 1 - \cos \frac{2\pi}{v_0}\) has been used.

The values of the integrals that involve \(p_o^s(v_0 t)\) in (2.16a,b) are in Table 1 and the corresponding values for (2.17a,b) are in Table 2. The values of the integrals that involve \(\rho_0^f(v_0 t)\) in (2.16a,b) are in equation (2.18) and the corresponding values for (2.17a,b) are in equation (2.19). These tables and equations appear in the next section.

The equations (2.16a,b) and (2.17a,b) are the starting point then for establishing the existence of solutions of
for $\varepsilon \neq 0$, and hence the existence of periodic solutions of (2.2) or (1.4).

2. **Evaluation of the required integrals.**

In this section a presentation is made of the computations and values of the integrals involved in equations (2.16a,b) and (2.17a,b). For convenience, however, the subscript zero is dropped on all the parameters except in the title of the tables and the beginning of equations (2.18) and (2.19). In these last equations $k$ has a subscript for Case 1 but not for Case 2, and because of this, $k$ does not have a zero subscript under the integral sign.

The following four formulas are useful in evaluating the necessary integrals:

(Formula 1) \[
\int_0^{2\pi} \sin(et + d) \sin(t - \frac{2\pi}{v}) dt
\]

\[
= \frac{1}{e-1} \sin(\frac{e\pi}{v}) \cos(\frac{(e+1)\pi}{v}) + d)
\]

\[- \frac{1}{e+1} \sin(\frac{(e+1)\pi}{v}) \cos(\frac{e\pi}{v} + d), \ e \neq 1
\]

or

\[
= \frac{e\pi}{v} \cos(\frac{2\pi}{v} + d) - \frac{1}{2} \sin \frac{2\pi}{v} \cos d, \ e = 1.
\]
(Formula 2) \[ \int_{0}^{\frac{2\pi}{v}} \cos(\epsilon t + d)\sin(t - \frac{2\pi}{v})dt \]

\[ = \frac{1}{e+1} \sin(\frac{\pi}{v}(e+1))\sin(\frac{\pi}{v}(e-1) + d) \]

\[ - \frac{1}{e-1} \sin(\frac{\pi}{v}(e-1))\sin(\frac{\pi}{v}(e+1) + d), \ e \neq 1 \]

or

\[ = \frac{1}{2} \sin \frac{2\pi}{v} \sin d - \frac{\pi}{v} \sin \frac{2\pi}{v} + d), \ e = 1. \]

(Formula 3) \[ \int_{0}^{\frac{2\pi}{v}} \sin(\epsilon t + d)\cos(t - \frac{2\pi}{v})dt \]

\[ = \frac{1}{e+1} \sin(\frac{\pi}{v}(e+1))\sin(\frac{\pi}{v}(e-1) + d) \]

\[ + \frac{1}{e-1} \sin(\frac{\pi}{v}(e-1))\sin(\frac{\pi}{v}(e+1) + d), \ e \neq 1 \]

or

\[ = \frac{1}{2} \sin \frac{2\pi}{v} \sin d + \frac{\pi}{v} \sin \frac{2\pi}{v} + d), \ e = 1. \]
(Formula 4) \[ \int_{0}^{\frac{2\pi}{v}} \cos(\varepsilon t + d)\cos(\varepsilon - \frac{2\pi}{v})dt \]

\[ = \frac{1}{\varepsilon + 1} \sin(\frac{\pi}{v}(\varepsilon + 1))\cos(\frac{\pi}{v}(\varepsilon - 1) + d) \]

\[ + \frac{1}{\varepsilon - 1} \sin(\frac{\pi}{v}(\varepsilon - 1))\cos(\frac{\pi}{v}(\varepsilon + 1) + d), \varepsilon \neq 1 \]

or

\[ = \frac{1}{2} \sin \frac{2\pi}{v} \cos d + \frac{\pi}{v} \cos(\frac{2\pi}{v} + d), \varepsilon = 1. \]

Throughout this thesis, as may be observed from the definitions following (2.3), the values \( v = 0, 1, \frac{1}{3} \) are excluded for all values of \( \varepsilon \) and in particular for \( \varepsilon = 0 \) and no further notation of this will be made.

The integral containing \( \rho_0(v_0 t) \) in equations (2.16a, b) is evaluated by use of (2.15) and the first two of the preceding formulas.

(2.18) \[ \int_{0}^{\frac{2\pi}{v_0}} \left( kv_0 \rho_0(v_0 t)\sin(t - \frac{2\pi}{v_0})dt \right) \]

\[ = k \int_{0}^{\frac{2\pi}{v}} (-a \sin t + b \cos t - cv \sin vt - 3gv \sin(3vt + \beta)) \]

\[ \cdot (\sin(t - \frac{2\pi}{v})dt) \]
\[
= k \left\{ -a \left( \frac{2\pi}{v} \cos \frac{2\pi}{v} - \frac{1}{2} \sin \frac{2\pi}{v} \right) - b \left( \frac{\pi}{v} \sin \frac{2\pi}{v} \right) \\
+ 2v \sin \frac{\pi}{v} \left[ \frac{c}{v^2 - 1} (v \cos \frac{\pi}{v} \cos \alpha - \sin \frac{\pi}{v} \sin \alpha) \\
+ \frac{3e}{9v^2 - 1} (3v \cos \frac{\pi}{v} \cos \beta - \sin \frac{\pi}{v} \sin \beta) \right] \right\}
\]

The integral containing \( \rho_o'(v_o t) \) in equations (2.17a,b) is evaluated by use of (2.15) and (Formula 3) and (Formula 4) above.

\[
(2.19) \int_0^{2\pi} \frac{2\pi}{v_o} k v_o \rho_o'(v_o t) \cos \left( t - \frac{2\pi}{v_o} \right) dt
\]

\[
= k \left\{ -\frac{4\pi}{v} \sin \frac{2\pi}{v} + b \left( \frac{1}{2} \sin \frac{2\pi}{v} + \frac{\pi}{2} \cos \frac{2\pi}{v} \right) \\
+ 2v \sin \frac{\pi}{v} \left[ \frac{c}{v^2 - 1} (v \sin \frac{\pi}{v} \cos \alpha + \cos \frac{\pi}{v} \sin \alpha) \\
+ \frac{3e}{9v^2 - 1} (3v \sin \frac{\pi}{v} \cos \beta + \cos \frac{\pi}{v} \sin \beta) \right] \right\}
\]

When \( \rho_o^3(v_o t) \) is expanded, twenty terms result. These terms will be numbered and any particular term can be determined from its coefficient so that numbers 6a and 6b in
\[
\begin{align*}
(a + (\xi + \Delta) \frac{\Delta}{\pm 0}) \times s((\xi + \Delta) \frac{\Delta}{\pm 0}) \times s \frac{\xi + \Delta}{t} &= \\
(a + (\xi + \Delta) \frac{\Delta}{\pm 0}) \times s((\xi + \Delta) \frac{\Delta}{\pm 0}) \times s \frac{\xi + \Delta}{t} &= + \\
(a + (\xi + \Delta) \frac{\Delta}{\pm 0}) \times s((\xi - \Delta) \frac{\Delta}{\pm 0}) \times s \frac{\xi - \Delta}{t} &= - \\
(a + (\xi - \Delta) \frac{\Delta}{\pm 0}) \times s((\xi - \Delta) \frac{\Delta}{\pm 0}) \times s \frac{\xi - \Delta}{t} &= \}
\end{align*}
\]

The following sum

implying the following (formulae above are needed, the integral is converted

\[
\left[(a + \xi(2-\xi))(\cos^2 + a + \xi(2+\xi)) \cos^2 \cos a \right] \frac{\theta}{t} =
\]

\[
(a + a \cos a) \frac{\theta}{t}
\]

When the following 

steps necessary in the construction of Table 1 and Table 2

This integration will be carried out to demonstrate the

\[
\begin{align*}
\frac{\Delta}{\pm 0} - \int_0^\infty \cos \cos t \cos \Delta t \cos \frac{\Delta}{\pm 0} \text{d}t.
\end{align*}
\]

Table 1 refer to
\[ + \frac{1}{v-1} \sin\left(\frac{\pi}{v}(v-1)\right)\sin\left(\frac{\pi}{v}(v-3) + \alpha\right) \]

\[ - \frac{1}{v-3} \sin\left(\frac{\pi}{v}(v-3)\right)\sin\left(\frac{\pi}{v}(v-1) + \alpha\right) \].

This sum reduces to the value numbered 6a, \( v \neq 3 \), in Table 1.

If the limit as \( v \to 3 \) is taken of the last term above as follows:

\[ \lim_{v \to 3} \frac{1}{v-3} \sin\left(\frac{\pi}{v}(v-3)\right) \sin\left(\frac{\pi}{v}(v-1) + \alpha\right) \]

\[ = \frac{\pi}{3} \sin\left(\frac{2\pi}{3} + \alpha\right) \],

the sum above for \( v = 3 \) reduces to the value 6b, \( v = 3 \), in Table 1.
Table 1

Evaluation of \( \int_0^{2\pi} \rho_0^s(v_0 t) \sin \left( t - \frac{2\pi}{v} \right) dt \)

1. \( \frac{g^s}{4} \sin \frac{2\pi}{v} \left( -\frac{3\pi}{4} - \frac{1}{4} \sin \frac{4\pi}{v} \right) \)

2. \( \frac{b^s}{4} \cos \frac{2\pi}{v} \left( \frac{3\pi}{v} + \frac{1}{4} \sin \frac{4\pi}{v} \right) - \frac{1}{2} \left( 3 + \cos \frac{4\pi}{v} \right) \sin \frac{2\pi}{v} \)

3. \( \frac{a^s}{2} \sin \frac{\pi}{v} \left[ \frac{3 \cos a}{v^2-1} \sin \frac{\pi}{v} + \frac{1}{9v^2-1} \sin \frac{\pi}{v} \cos 3a + 3v \cos \frac{\pi}{v} \sin 3a \right] \)

\( \left. + \frac{3v \sin a}{v^2-1} \cos \frac{\pi}{v} \right] \)

4a. \( v \neq \frac{1}{9} \) \( \frac{g^s}{2} \sin \frac{\pi}{v} \left[ \frac{3}{9v^2-1} \left( \sin \frac{\pi}{v} \cos \beta + 9v \cos \frac{\pi}{v} \sin \beta \right) \right] \)

\( + \frac{1}{81v^2-1} \left( \sin \frac{\pi}{v} \cos 3\beta + 9v \cos \frac{\pi}{v} \sin 3\beta \right) \)
\[ \frac{3}{4} a^b \left[ \cos \frac{2\pi}{v} \left( \frac{\pi}{v} - \frac{1}{4} \sin \frac{\pi}{v} \right) + \frac{1}{2} \sin \frac{2\pi}{v} \left( \cos \frac{\pi}{v} - 1 \right) \right] \]

\[ \frac{3}{2} a^c \left[ \frac{1}{v^2 - 1} \sin \frac{\pi}{v} \left( 2 \sin \frac{\pi}{v} \cos \alpha + 2v \cos \frac{\pi}{v} \sin \alpha + \sin \frac{3\pi}{v} \cos \alpha \right) ight. 
\left. - \frac{v}{v^2 - 1} \sin \frac{\pi}{v} \cos \frac{3\pi}{v} \sin \alpha - \frac{1}{v^2 - 1} \sin \frac{3\pi}{v} \left( 3 \sin \frac{\pi}{v} \cos \alpha - v \cos \frac{\pi}{v} \sin \alpha \right) \right] \]

\[ \frac{3}{4} a^c \left[ \frac{1}{4} \sin \frac{\pi}{3} \left( 2 \sin \left( \frac{2\pi}{3} + \alpha \right) - \sin \alpha \right) \right. 
\left. + \sin \frac{2\pi}{3} \left( \frac{1}{2} \sin \alpha - \sin \left( \frac{\pi}{3} + \alpha \right) \right) - \frac{\pi}{3} \sin \left( \frac{2\pi}{3} + \alpha \right) \right] \]
7. \( \frac{3}{2} a^2 g \left[ \frac{1}{9v^2-1} \sin \frac{\pi}{v} (2 \sin \frac{\pi}{v} \cos \beta + 6v \cos \frac{\pi}{v} \sin \beta + \sin \frac{3\pi}{v} \cos \beta) \right. \\
- \left. \frac{3v}{9v^2-1} \sin \frac{\pi}{v} \cos \frac{3\pi}{v} \sin \beta - \frac{1}{3v^2-3} \sin \frac{3\pi}{v} (\sin \frac{\pi}{v} \cos \beta - v \cos \frac{\pi}{v} \sin \beta) \right] \\
8. \frac{3}{4} ab^2 \sin \frac{2\pi}{v} \left( - \frac{\pi}{v} + \frac{1}{4} \sin \frac{4\pi}{v} \right) \\
9. \frac{3}{2} ac^2 \sin \frac{2\pi}{v} \left( - \frac{\pi}{v} + \frac{v}{2v^2-2} \sin \alpha \right) \\
10. \frac{3}{4} ag^2 \sin \frac{2\pi}{v} \left( - \frac{2\pi}{v} + \frac{3v}{9v^2-1} \sin 2\beta \right) \\
11a. v \neq 3, \frac{3}{2} b^3 c \left[ \frac{1}{v^2-1} \sin \frac{\pi}{v} \left( 2 \sin \frac{\pi}{v} \cos \alpha + 2v \cos \frac{\pi}{v} \sin \alpha - \sin \frac{3\pi}{v} \cos \alpha \right) \right. \\
+ \frac{v}{v^2-1} \sin \frac{\pi}{v} \cos \frac{3\pi}{v} \sin \alpha + \frac{1}{v^2-9} \sin \frac{3\pi}{v} (3 \sin \frac{\pi}{v} \cos \alpha - v \cos \frac{\pi}{v} \sin 2\alpha) \right]
Table 1, (Continued)

11b.  \( v = 3, \frac{3}{4} b^a c \left( \frac{1}{4} \sin \frac{4\pi}{3} \left(2 \sin \left(\frac{2\pi}{3} + a\right) + \sin a\right) \right) \)

\[ - \sin \frac{2\pi}{3} \left( \sin \left(\frac{4\pi}{3} + a\right) + \frac{1}{2} \sin a \right) + \frac{\pi}{3} \sin \left(\frac{2\pi}{3} + a\right) \]

12.  \( \frac{3}{2} b^a g \left[ \frac{1}{9v^2-1} \sin \frac{\pi}{v} \left( 2 \sin \frac{\pi}{v} \cos \beta + 6v \cos \frac{\pi}{v} \sin \beta - \sin \frac{3\pi}{v} \cos \beta \right) \right] \)

\[ + \frac{3v}{9v^2-1} \sin \frac{\pi}{v} \cos \frac{3\pi}{v} \sin \beta + \frac{1}{3v^2-3} \sin \frac{3\pi}{v} \left( + \sin \frac{\pi}{v} \cos \beta - v \cos \frac{\pi}{v} \sin \beta \right) \]

13.  \( \frac{3}{4} b^c \left[ \frac{2\pi}{v} \cos \frac{2\pi}{v} + \sin \frac{2\pi}{v} \left( \cos \frac{2\pi}{v^2-1} - 1 \right) \right] \]

14.  \( \frac{3}{4} b^g \left[ \frac{2\pi}{v} \cos \frac{2\pi}{v} + \sin \frac{2\pi}{v} \left( \cos \frac{2\pi}{9v^2-1} - 1 \right) \right] \)
Table 1, (Continued)

15a. \( v = \frac{1}{5}, \frac{3}{2} c^a g \sin \frac{mx}{v} \left( \frac{2}{9v^2 - 1} \left[ \sin \frac{mx}{v} \cos \beta + 3v \cos \frac{mx}{v} \sin \beta \right] \right. \\

\left. + \frac{1}{25v^2 - 1} \left[ \sin \frac{mx}{v} \cos (2a + \beta) + 5v \cos \frac{mx}{v} \sin (2a + \beta) \right] \right) \\

\left. + \frac{1}{v^2 - 1} \left[ \sin \frac{mx}{v} \cos (-2a + \beta) + v \cos \frac{mx}{v} \sin (-2a + \beta) \right] \right) \\

15b. \( v = \frac{1}{5}, \frac{3}{4} c^a g (-5mx \sin (2a + \beta)) \)
Table 1. (Continued)

16a. \( v = \frac{1}{5}, \frac{1}{7}, \frac{3}{2} \) \( cg^2 \sin \frac{m\pi}{v} \left( \frac{2}{v^2-1} \sin \frac{m\pi}{v} \cos \alpha + \frac{v}{v^2-1} \sin \frac{m\pi}{v} \sin \alpha \right) \)

\[ + \frac{1}{49v^2-1} \left[ \sin \frac{m\pi}{v} \cos (\alpha + 2\beta) + 7v \cos \frac{m\pi}{v} \sin (\alpha + 2\beta) \right] \]

\[ + \frac{1}{25v^2-1} \left[ \sin \frac{m\pi}{v} \cos (-\alpha + 2\beta) + 5v \cos \frac{m\pi}{v} \sin (-\alpha + 2\beta) \right] \]

16b. \( v = \frac{1}{5}, \frac{3}{4} \) \( cg^2 (-5m\pi \sin (-\alpha + 2\beta)) \)

16c. \( v = \frac{1}{7}, \frac{3}{4} \) \( cg^2 (-7m\pi \sin (\alpha + 2\beta)) \)
| Table 1, (Continued) |

17a. \( v \neq 3, \ 3abc \left[ \frac{1}{v^2-1} \sin \frac{3\pi}{v} (-\cos \frac{3\pi}{v} \cos \alpha - v \sin \frac{3\pi}{v} \sin \alpha) \right. \\
\left. + \frac{1}{v^2-9} \sin \frac{3\pi}{v} \left( 3 \cos \frac{3\pi}{v} \cos \alpha + v \sin \frac{3\pi}{v} \sin \alpha \right) \right] \\

17b. \( v = 3, \ \frac{3}{2} \ abc \left[ -\frac{\pi}{3} \cos \left( \frac{2\pi}{3} + \alpha \right) + \frac{\cos \alpha}{4} \left( \sin \frac{4\pi}{3} + 2 \sin \frac{2\pi}{3} \right) \right] \\

18. \ 3abg \left[ \frac{1}{9v^2-1} \sin \frac{\pi}{v} (-\cos \frac{3\pi}{v} \cos \beta - 3v \sin \frac{3\pi}{v} \sin \beta) \right. \\
\left. + \frac{1}{3v^2-3} \sin \frac{3\pi}{v} \left( \cos \frac{\pi}{v} \cos \beta + v \sin \frac{\pi}{v} \sin \beta \right) \right] \\

19a. \( v \neq \frac{1}{2}, \ 3acg \sin \frac{2\pi}{v} \left( \frac{v}{4v^2-1} \sin (\alpha + \beta) + \frac{v}{2v^2-2} \sin (-\alpha + \beta) \right) \)
Table 1, (Continued)

<table>
<thead>
<tr>
<th>19b.</th>
<th>$v = \frac{1}{2}, \frac{3}{2} \cos (-2\pi x \sin (\alpha + \beta))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20a.</td>
<td>$v \pm \frac{1}{2}, \frac{3}{2} \cos \sin \frac{2\pi x}{v} \left(\frac{1}{4v^2 - 1} \cos (\alpha + \beta) + \frac{1}{v^2 - 1} \cos (-\alpha + \beta)\right)$</td>
</tr>
<tr>
<td>20b.</td>
<td>$v = \frac{1}{2}, \frac{3}{2} \cos (-2\pi x \cos (\alpha + \beta))$</td>
</tr>
</tbody>
</table>
Table 2

Evaluation of
\[ \int_0^{\frac{2\pi}{v_0}} \rho_0^5 (v_0 t) \cos(t - \frac{2\pi}{v_0}) dt \]

1. \( \frac{a^3}{4} \left[ \frac{1}{2} \sin \frac{2\pi}{v} (3 + \cos \frac{4\pi}{v}) + \cos \frac{2\pi}{v} (\frac{3\pi}{4} + \frac{1}{4} \sin \frac{4\pi}{v}) \right] \)

2. \( \frac{3b^3}{4} \sin \frac{2\pi}{v} (\frac{\pi}{v} - \frac{1}{4} \sin \frac{4\pi}{v}) \)

3. \( \frac{c^3}{2} \sin \frac{\pi}{v} \left[ \frac{-3}{v-1} (\cos \frac{\pi}{v} \cos \alpha - v \sin \frac{\pi}{v} \sin \alpha) \right] \)

\[ - \frac{1}{9v^3-1} \left( \cos \frac{\pi}{v} \cos 3\alpha - 3v \sin \frac{\pi}{v} \sin 3\alpha \right) \]
<table>
<thead>
<tr>
<th>4a. ( \mathbf{v} \pm \frac{1}{9} \cdot \frac{g^3}{2} \sin \frac{\mu \pi}{v} \left[ \frac{-3}{9v^2-1} \left( \cos \frac{\mu \pi}{v} \cos \beta - 3v \sin \frac{\mu \pi}{v} \sin \beta \right) \right] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4b. ( \mathbf{v} = \frac{1}{9} \cdot \frac{g^3}{4} \left( 9 \mu \pi \cos 3\beta \right) )</td>
</tr>
<tr>
<td>5. ( \frac{3}{4} \cdot a^2 \cdot b \cdot \sin \frac{2\mu \pi}{v} \left( \frac{\mu \pi}{v} + \frac{3}{4} \sin \frac{\mu \pi}{v} \right) )</td>
</tr>
<tr>
<td>6a. ( \mathbf{v} \pm 3 \cdot \frac{3}{2} \cdot a^2 \cdot c \left[ \frac{-1}{v^2-1} \sin \frac{\mu \pi}{v} \left( 2 \cos \frac{\mu \pi}{v} \cos \alpha - 2v \sin \frac{\mu \pi}{v} \sin \alpha + \cos \frac{3\mu \pi}{v} \cos \alpha \right) \right] )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>4a. ( \mathbf{v} \pm \frac{1}{9} \cdot \frac{g^3}{2} \sin \frac{\mu \pi}{v} \left[ \frac{-3}{9v^2-1} \left( \cos \frac{\mu \pi}{v} \cos \beta - 3v \sin \frac{\mu \pi}{v} \sin \beta \right) \right] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4b. ( \mathbf{v} = \frac{1}{9} \cdot \frac{g^3}{4} \left( 9 \mu \pi \cos 3\beta \right) )</td>
</tr>
<tr>
<td>5. ( \frac{3}{4} \cdot a^2 \cdot b \cdot \sin \frac{2\mu \pi}{v} \left( \frac{\mu \pi}{v} + \frac{3}{4} \sin \frac{\mu \pi}{v} \right) )</td>
</tr>
<tr>
<td>6a. ( \mathbf{v} \pm 3 \cdot \frac{3}{2} \cdot a^2 \cdot c \left[ \frac{-1}{v^2-1} \sin \frac{\mu \pi}{v} \left( 2 \cos \frac{\mu \pi}{v} \cos \alpha - 2v \sin \frac{\mu \pi}{v} \sin \alpha + \cos \frac{3\mu \pi}{v} \cos \alpha \right) \right] )</td>
</tr>
</tbody>
</table>
Table 2, (Continued)

6b. \( v = 3, \ \frac{3}{4} a^c \left[ \frac{1}{4} \sin \frac{4\pi}{3}(2\cos(\frac{2\pi}{3} + \alpha) + \cos \alpha) \right. \]
\[ + \sin \frac{2\pi}{3}(\frac{1}{2} \cos \alpha + \cos(\frac{4\pi}{3} + \alpha)) \left] + \frac{a^c}{4} \frac{4\pi}{3} \cos(\frac{2\pi}{3} + \alpha) \right. \]

7. \( \frac{3}{2} a^g \left[ \frac{-1}{9v^2-1} \sin \frac{\pi}{v}(2\cos \frac{\pi}{v} \cos \beta - 6v \sin \frac{\pi}{v} \sin \beta + \cos \frac{3\pi}{v} \cos \beta) \right. \]
\[ - \frac{3v}{9v^2-1} \sin \frac{\pi}{v} \sin \frac{3\pi}{v} \sin \beta - \frac{1}{3v^2-3} \sin \frac{3\pi}{v} (\cos \frac{\pi}{v} \cos \beta + v \sin \frac{\pi}{v} \sin \beta) \]

8. \( \frac{3}{8} ab^a \left[ \sin \frac{2\pi}{v} (1 - \cos \frac{4\pi}{v}) + \cos \frac{2\pi}{v} (\frac{2\pi}{v} - \frac{1}{2} \sin \frac{4\pi}{v}) \right. \]

9. \( \frac{3}{4} ac^a (\sin \frac{2\pi}{v} (1 - \cos \frac{2\pi}{v^2-1}) + \frac{2\pi}{v} \cos \frac{2\pi}{v}) \)
10. \( \frac{2}{4} ag^2 \sin \left( \frac{2\pi}{v} \left( 1 - \cos \frac{2\beta}{9v^2-1} \right) + \frac{2\pi}{v} \cos \frac{2\pi}{v} \right) \)

11a. \( v = 3, \frac{3}{2} b^2 c \left[ \frac{1}{v^2-1} \sin \frac{\pi}{v} \left( -2 \cos \frac{\pi}{v} \cos \alpha + 2v \sin \frac{\pi}{v} \sin \alpha + \cos \frac{3\pi}{v} \cos \alpha \right) \right. \)

\[ + \frac{v}{v^2-1} \sin \frac{\pi}{v} \sin \frac{3\pi}{v} \sin \alpha + \frac{1}{v^2-9} \sin \frac{3\pi}{v} \left( 3 \cos \frac{\pi}{v} \cos \alpha + v \sin \frac{\pi}{v} \sin \alpha \right) \]

11b. \( v = 3, \frac{3}{4} b^2 c \left[ \sin \frac{2\pi}{3} \left( \cos \left( \frac{\pi}{3} + \alpha \right) - \frac{1}{2} \cos \alpha \right) \right. \)

\[ + \frac{1}{4} \sin \frac{\pi}{3} \left( 2 \cos \left( \frac{2\pi}{3} + \alpha \right) - \cos \alpha \right) - \frac{\pi}{3} \cos \left( \frac{2\pi}{3} + \alpha \right) \]

12. \( \frac{3}{2} b^2 g \left[ \frac{1}{9v^2-1} \sin \frac{\pi}{v} \left( -2 \cos \frac{\pi}{v} \cos \beta + 6v \sin \frac{\pi}{v} \sin \beta + \cos \frac{3\pi}{v} \cos \beta \right) \right. \)

\[ + \frac{3v}{9v^2-1} \sin \frac{\pi}{v} \sin \frac{3\pi}{v} \sin \beta + \frac{1}{3v^2-3} \sin \frac{3\pi}{v} \left( \cos \frac{\pi}{v} \cos \beta + v \sin \frac{\pi}{v} \sin \beta \right) \]
Table 2, (Continued)

13. \( \frac{3}{4} bc^2 \sin \frac{2\pi}{v} \left( \frac{2\pi}{v} + \frac{v}{v^2 - 1} \sin 2\alpha \right) \)

14. \( \frac{3}{4} bg^2 \sin \frac{2\pi}{v} \left( \frac{2\pi}{v} + \frac{3v}{9v^2 - 1} \sin 2\beta \right) \)

15a. \( v \neq \frac{1}{2}, \frac{3}{2} \cos g \sin \frac{\pi}{v} \left[ \frac{2}{9v^2 - 1} \left( \cos \frac{\pi}{v} \cos \beta - 3v \sin \frac{\pi}{v} \sin \beta \right) \right. \)

\[ - \frac{1}{25v^2 - 1} \left( \cos \frac{\pi}{v} \cos(2\alpha + \beta) - 5v \sin(2\alpha + \beta) \sin \frac{\pi}{v} \right) \]

\[ - \frac{1}{v^2 - 1} \left( \cos \frac{\pi}{v} \cos(-2\alpha + \beta) - v \sin(-2\alpha + \beta) \sin \frac{\pi}{v} \right) \]

15b. \( v \neq \frac{1}{2}, \frac{3}{4} \cos g \left( \frac{5\pi}{v} \cos(2\alpha + \beta) \right) \)
Table 2, (Continued)

\[ 16a. \ v = \frac{1}{5} \cdot \frac{1}{7} \cdot \frac{3}{2} \cdot c g^2 \sin \frac{\pi r}{v} \left[ \frac{2}{v^2 - 1} \left( \cos \frac{\pi r}{v} \cos \alpha - v \sin \frac{\pi r}{v} \sin \alpha \right) \right. \]
\[ \left. - \frac{1}{25v^2 - 1} \left( + \cos \frac{\pi r}{v} \cos(-\alpha + 2\beta) - 5v \sin(-\alpha + 2\beta) \sin \frac{\pi r}{v} \right) \right] \]
\[ - \frac{1}{49v^2 - 1} \left( \cos \frac{\pi r}{v} \cos(\alpha + 2\beta) - 7v \sin(\alpha + 2\beta) \sin \frac{\pi r}{v} \right) \]

\[ 16b. \ v = \frac{1}{5} \cdot \frac{3}{4} c g^2 (5\pi r \cos(-\alpha + 2\beta)) \]

\[ 16c. \ v = \frac{1}{7} \cdot \frac{3}{4} c g^2 (7\pi r \cos(\alpha + 2\beta)) \]

\[ 17a. \ v = 3, 3a b c \left[ \frac{1}{v^2 - 1} \sin \frac{\pi r}{v} \left( - \sin \frac{3\pi r}{v} \cos \alpha + v \cos \frac{3\pi r}{v} \sin \alpha \right) \right. \]
\[ \left. + \frac{1}{v^2 - 9} \sin \frac{3\pi r}{v} \left( -3 \sin \frac{\pi r}{v} \cos \alpha + v \cos \frac{\pi r}{v} \sin \alpha \right) \right] \]
Table 2, (Continued)

17b. $v = 3, \frac{3}{2} \: \text{abc} \left[ \frac{1}{4} \sin a (\sin \frac{4\pi}{3} - 2 \sin \frac{2\pi}{3}) - \frac{\pi}{3} \sin (\frac{2\pi}{3} + a) \right]$

18. $3\text{abg} \left[ \frac{1}{9v^2-1} \sin \frac{\pi}{v} (- \sin \frac{3\pi}{v} \cos \beta + 3v \cos \frac{3\pi}{v} \sin \beta) \right.$

$+ \frac{1}{3v^2-3} \sin \frac{3\pi}{v} \left( - \sin \frac{\pi}{v} \cos \beta + v \cos \frac{\pi}{v} \sin \beta \right) \right]$

19a. $v \neq \frac{1}{2}, \frac{3}{2} \: \text{acg} \sin \frac{2\pi}{v} (- \frac{1}{4v^2-1} \cos (a + \beta) - \frac{1}{v^2-1} \cos (-a + \beta))$

19b. $v = \frac{1}{2}, \frac{3}{2} \: \text{acg} (2\pi \cos (a + \beta))$

20a. $v \neq \frac{1}{2}, \frac{3}{2} \: \text{bcg} \sin \frac{2\pi}{v} (\frac{2v}{4v^2-1} \sin (a + \beta) + \frac{v}{v^2-1} \sin (-a + \beta))$

20b. $v = \frac{1}{2}, \frac{3}{2} \: \text{bcg} (-2\pi \sin (a + \beta))$
The preceding tables were constructed in order to facilitate the computations of the periodicity conditions in the various special cases. In the next sections, the periodicity conditions for these special cases are discussed.

B. Periodicity Conditions for Subharmonic Solutions

Subharmonics have frequencies that are 1/n times the frequency of the forcing function where n is an integer greater than one. The least period of a subharmonic solution of order 1/n will be 2πn. This requires m = n in (2.16a) and (2.17a). From equation (2.12) it is observed that \( v_0 \) must equal \( n \) when \( t = 0 \) in order for the solutions to have the least period of \( 2\pi \), thus Case 1 of the general periodicity conditions, \( v_0 = \frac{p}{q} \), applies.

When the values from the indicated tables and equations are substituted in (2.16a) and (2.17a), the periodicity conditions become (where \( 8L = a_o^2 + b_o^2 + 2c_o^2 + 2g_o^2 \))

\[
(2.20) \quad n \neq 3
\]

\[
\frac{v_o}{\pi} P(0,n,v_1,k_o) = -2b_ov_1 - nac_0k_0 + 6nb_0L = 0,
\]

\[
\frac{v_o}{\pi} Q(0,n,v_1,k_o) = 2a_ov_1 - nb_0k_0 - 6na_0L = 0,
\]
(2.21) \( n = 3 \)

\[
\frac{3}{\pi} P(0, 3, v_1, k_0) = -2b_0 v_1 - 3a_0 k_0 + 18b_0 L
\]

\[-\frac{9}{4} c_0 (a_0^2 - b_0^2) \sin \alpha - \frac{9}{2} a_0 b_0 c_0 \cos \alpha = 0,\]

\[
\frac{9}{\pi} Q(0, 3, v_1, k_0) = 2a_0 v_1 - 3b_0 k_0 - 18a_0 L
\]

\[-\frac{9}{4} c_0 (a_0^2 - b_0^2) \cos \alpha + \frac{9}{2} a_0 b_0 c_0 \sin \alpha = 0\]

where the second term in each equation comes from equation (2.18) or (2.19) and the terms after the second term come from Table 1 or Table 2.

Equations (2.20) yield the following solutions:

(2.22a) \( (a_0^2 + b_0^2) k_0 = 0 \),

\[
v_1 = 3nL = \frac{3n}{8} (A_{1/n}^2 + \frac{2}{(n^2 - 1)^2} + \frac{2n^2}{(9n^2 - 1)^2})
\]

where \( A_{1/n} = a_0^2 + b_0^2 \) is the amplitude of the subharmonic oscillation of order \( 1/n \) as can be seen from (2.12) and the definitions of \( c_0 \) and \( g_0 \) which follow (2.3) have been used.

The Jacobian is

\[
J(\frac{P_0 Q_0}{k_0, v_1}) = -\frac{2}{3} (a_0^2 + b_0^2) + 0
\]

for this case, (see (2.7)).
Thus \( k_0 = 0 \) which indicates that in this case \( k \) may vanish when \( \varepsilon \neq 0 \) as well; however, it is not known whether \( k \) must vanish identically or not. Levenson encountered this same difficulty and conjectured that the damping coefficient, \( C \), in (1.4) would have to be taken proportional to \( B \) raised to some power greater than one to prove the existence of subharmonics when \( n \neq 3 \) with some (nonzero) damping. The implication is that as \( B \) tends to zero in (1.4) then \( C \) must also tend to zero in order for subharmonics of order \( 1/n \), \( n \neq 3 \), to exist. It is still an open question whether \( C \) must be identically zero or not for such subharmonics to exist.

If the value of \( v_1 \) in (2.22a) is substituted in (2.10a), a first approximation can be obtained for \( \bar{v} \) which will be referred to as \( \bar{v} \).

\[
(2.22b) \quad \bar{v} = n + \varepsilon \frac{3n}{8} \left( A_{1/n} + \frac{2}{(n^2 - 1)^2} + \frac{2n}{(9n^2 - 1)^2} \right).
\]

In the introduction, it was mentioned that Hayashi had conjectured that a sufficient condition for the existence of subharmonics of order \( 1/n \) was that \( n \) be equal to the highest power of \( x \) in (1.1). In accordance with this conjecture, the presence of the \( x^3 \) in (1.4) should result in the existence of subharmonics of order \( 1/3 \). This
is found to be true. The remainder of the work on subharmonics will be restricted to the case where \( n = 3 \).

The solutions of equation (2.21) are

\[
(2.23) \\
\begin{align*}
    k_0 &= -\frac{3\text{co}}{4(a_0^2+b_0^2)}(a_0(a_0^2-3b_0^2)\sin a+b_0(3a_0^2-b_0^2)\cos a), \\
    v_1 &= 9L + \frac{9\text{co}}{8(a_0^2+b_0^2)}(-b_0(3a_0^2-b_0^2)\sin a+a_0(a_0^2-3b_0^2)\cos a).
\end{align*}
\]

The value of the Jacobian is

\[
J(\frac{P,Q}{k_0,v_1}) = -(a_0^2 + b_0^2) \neq 0,
\]

and the existence of the subharmonics of order 1/3 is established.

If the value of \( v_1 \) from (2.23) is substituted in (2.10a), a first approximation can be obtained for \( v \) which will be referred to as \( \bar{v} \).

\[
(2.24) \\
\bar{v} = 3 + 9\epsilon L + \frac{9\epsilon\text{co}}{8(a_0^2+b_0^2)}(-b_0(3a_0^2-b_0^2)\sin a+a_0(a_0^2-3b_0^2)\cos a).
\]

If all terms that do not contain trigonometric functions in (2.24) are transferred to the left hand side and the resulting equation is squared and added to the square of \( k_0 \) in (2.23), the phase angle \( a \) is eliminated.
(2.25)  
\[ \left[ \frac{v - 3 - 9\varepsilon L}{3\varepsilon/2} \right]^2 + k_o^2 \]

\[ \frac{1}{4(a_o^2 + b_o^2)} \]  
\[ (b_o^2(3a_o^2 - b_o^2) + a_o^2(a_o^2 - 3b_o^2)) \]

\[ = \frac{9}{16} c_o^2 (a_o^2 + b_o^2). \]

The amplitude of the subharmonic term in the first approximation to the solution will be designated as \( A_{1/3} \). Clearly \( A_{1/3} = (a_o^2 + b_o^2)^{1/2} \) as can be seen from (2.12). If (2.25) is solved for \( \bar{v} \), response curves can be drawn in the \( A_{1/3} \), \( \bar{v} \)-plane for fixed values of the parameters \( k_o \) and \( r \). Since \( c_o = -1/8 \) and \( g_o = -r/80 \), the equation of these curves is

(2.26)  
\[ \bar{v} = 3\left(1 + \varepsilon \left(\frac{3}{8} A_{1/3} + \frac{3}{256} + \frac{3r^2}{25600} + \frac{1}{2} \left(-\frac{9}{1024} A_{1/3} - k_o^2 \right)^{1/2}\right)\). \]

It is noted that for \( \varepsilon \geq 0 \), \( \bar{v} \geq 3 \) or for \( \varepsilon < 0 \), \( \bar{v} < 3 \). An important observation from (2.26) is that subharmonics of order 1/3 exist if \( 9A_{1/3}^2/1024 > k_o^2 \). This inequality insures the existence of subharmonics of order 1/3 when \( k_o \neq 0 \). Levenson found these same results when \( r \neq 0 \).

The work involved in obtaining the values for \( M \) and \( \alpha \) requires that the relationship between \( \bar{v} \) and \( A_{1/3} \) be
determined by means of graphs or tables. As an illustration, graphs have been drawn for \( \epsilon = 0.1, |r| < 5 \) and \( |r| = 100 \) for the two values \( k_0 = 0 \) and \( k_0 = 0.1 \) in Figure 1. All subsequent work is based on the assumption that the required graphs or tables are available. It is assumed that the parameters in the differential equation (1.4) and consequently (2.2), excluding \( A \) and \( a, \) are known (see the discussion following (2.9)) and that \( \nu = \overline{\nu} \) and \( k \neq k_0. \) Thus \( \overline{\nu} \) and \( k_0 \) are known as well as \( r \) and \( \epsilon \) in equation (2.26). Consequently, \( A_1/a \) is determined.

The following substitutions will facilitate the remaining work.

(2.27)

\[
R = -\frac{4k_0}{3c_0} A_1^2/a, \quad N = -\frac{8}{9c_0} A_1^2/a (\nu_1 - 9L), \quad a - z = s,
\]

\[
a_0/A_1/a = \sin \gamma, \quad a_0(a_0^2 - 3b_0^2)/A_1^2/a = \sin z,
\]

\[
b_0/A_1/a = \cos \gamma, \quad b_0(3a_0^2 - b_0^2)/A_1^2/a = \cos z.
\]

From the last four of equations (2.27) it can be seen that

\[
\sin z = \sin \gamma(\sin^2 \gamma - 3\cos^2 \gamma) = -\sin 3\gamma,
\]

\[
\cos z = \cos \gamma(3\sin^2 \gamma - \cos^2 \gamma) = -\cos 3\gamma.
\]

Therefore, it is seen that

(2.28)

\[3\gamma = z + (2n-1)\pi.\]
Figure 1. Approximate Response Curves for Subharmonics of Order $1/3$
When (2.23) is substituted in the first two equations in (2.27) the following result is obtained.

(2.29)

\[ \frac{R}{A_{1/3}} = \sin z \sin \alpha + \cos z \cos \alpha = \cos(\alpha-z) = \cos s, \]

\[ \frac{N}{A_{1/3}} = \cos z \sin \alpha - \sin z \cos \alpha = \sin(\alpha-z) = \sin a. \]

The angle \( s \) would thus be determined since \( R, N, \) and \( A_{1/3} \) are known. From the definition of \( b_o \) following (2.3)

\[ b_o = -3/8 \sin \alpha - 9/80 r \sin \beta \]

and from (2.27) and (2.28)

\[ b_o = A_{1/3} \cos \left[ (z+(2n-1)\pi)/3 \right]. \]

Therefore, these two quantities may be equated as follows

(2.30)

\[ A_{1/3} \cos \left[ (z+(2n-1)\pi)/3 \right] = -3/8 \sin \alpha - 9/80 r \sin \beta. \]

By the substitution of values from (2.1) and (2.27) equation (2.30) may be written as

(2.31) \[ A_{1/3} \cos \left[ (z+(2n-1)\pi)/3 \right] = -3/8 \sin(z+s) \]

\[ - 9/80 r \sin(3z + 3s - 3\phi_1 + \phi_3). \]

Equation (2.31) can be solved for angle \( z \) and thus \( \alpha \) from the relation \( \alpha = z + s \). With \( \alpha \) and \( z \) known \( M \) can be obtained from the definition of \( a_o \) which follows (2.3) as follows:

(2.32) \[ a_o = A_{1/3} \sin \gamma = A_{1/3} \sin(z+(2n-1)\pi)/3 \]

\[ = M+(1/8)\cos \alpha + (r/80)\cos(3\alpha + 3\phi_1 - \phi_3). \]
The values of $M$ and $a$ that are obtained above will specify the first approximation to the initial condition and phase angles in (2.2) and thus the initial conditions in (1.4) which give rise to subharmonic solutions of order $1/3$.

C. Periodicity Conditions for Ultraharmonic Solutions

Ultraharmonics have the same period as the forcing function but contain components that have frequencies $n$ ($n>1$) times the frequency of the forcing function or rather for (1.4) $n$ times the frequency of the fundamental. Ultraharmonics of order $n$ thus require $m = 1$ in (2.16a) and (2.17a) and from (2.12) it is observed that $v_0$ must equal $1/n$.

The appropriate values of the functions $P$ and $Q$ will thus be $P = P(0,1/n,v_1,k_0)$ and $Q = Q(0,1/n,v_1,k_0)$ for ultraharmonics. Thus with the above values and with the values from the indicated tables and equations, the periodicity conditions become (as before $8L = a_0^2 + b_0^2 + 2c_0^2 + 2s_0^2$)

\[(2.33)\quad n \neq 2,5,7,9\]

\[\frac{P}{n\pi} = 2nb_0v_1 + a_0k_0 - 6b_0L = 0,\]

\[\frac{Q}{n^2\pi} = 2na_0v_1 - b_0k_0 - 6a_0L = 0.\]
\[(2.34)\quad n = 2\]

\[-\frac{P}{2\pi} = 4b_0v_1 + a_0k_0 - 6b_0L + \frac{3}{2}c_0g_0(a_0\sin(\alpha + \beta) + b_0\cos(\alpha + \beta)) = 0,\]

\[\frac{Q}{4\pi} = 4a_0v_1 - b_0k_0 - 6a_0L - \frac{3}{2}c_0g_0(a_0\cos(\alpha + \beta) - b_0\sin(\alpha + \beta)) = 0,\]

\[(2.35)\quad n = 5\]

\[-\frac{P}{5\pi} = 10b_0v_1 + a_0k_0 - 6b_0L + \frac{3}{4}c_0g_0(c_0\sin(2\alpha + \beta) + g_0\sin(2\beta - \alpha)) = 0,\]

\[\frac{Q}{25\pi} = 10a_0v_1 - b_0k_0 - 6a_0L - \frac{3}{4}c_0g_0(c_0\cos(2\alpha + \beta) + g_0\cos(2\beta - \alpha)) = 0,\]

\[(2.36)\quad n = 7\]

\[-\frac{P}{7\pi} = 14b_0v_1 + a_0k_0 - 6b_0L + \frac{3}{4}c_0g_0^2 \sin(\alpha + \beta) = 0,\]

\[\frac{Q}{49\pi} = 14a_0v_1 - b_0k_0 - 6a_0L - \frac{3}{4}c_0g_0^2 \cos(\alpha + \beta) = 0,\]

\[(2.37)\quad n = 9\]

\[-\frac{P}{9\pi} = 18b_0v_1 + a_0k_0 - 6b_0L + \frac{1}{4}g_0^3 \sin 3\beta = 0,\]

\[\frac{Q}{81\pi} = 18a_0v_1 - b_0k_0 - 6a_0L - \frac{1}{4}g_0^3 \cos 3\beta = 0.\]

The solutions of equations (2.33) are

\[(2.38a)\quad k_0 = 0,\]

\[v_1 = 3L/n\]
which shows that $k$ vanishes when $\epsilon$ vanishes. The parameter $k$ may vanish identically in $\epsilon$ but no evidence of this is shown here. The Jacobian for the general case is shown to have a nonzero value later. The particular cases for $n = 2, 5, 7, \text{ and } 9$ are now considered.

The solutions of (2.34) through (2.37) are

(2.38b) $n = 2$,

$$k_0 = \frac{-3\epsilon g_0}{2(a_0^2 + b_0^2)} \left( (a_0^2 - b_0^2)\sin(a + \beta) + 2a_0 b_0 \cos(a + \beta) \right),$$

$$v_1 = \frac{3}{2} - \frac{3\epsilon g_0}{8(a_0^2 + b_0^2)} \left( 2a_0 b_0 \sin(a + \beta) - (a_0^2 - b_0^2) \cos(a + \beta) \right),$$

(2.39) $n = 5$,

$$k_0 = \frac{-3\epsilon g_0}{4(a_0^2 + b_0^2)} \left( a_0 \sin(2a + \beta) + b_0 \cos(2a + \beta) \right),$$

$$-\frac{3\epsilon g_0}{4(a_0^2 + b_0^2)} \left( a_0 \sin(2\beta - a) + b_0 \cos(2\beta - a) \right),$$

$$v_1 = \frac{3}{5} - \frac{3\epsilon g_0}{40(a_0^2 + b_0^2)} \left( b_0 \sin(2a + \beta) - a_0 \cos(2a + \beta) \right),$$

$$-\frac{3\epsilon g_0}{40(a_0^2 + b_0^2)} \left( b_0 \sin(2\beta - a) - a_0 \cos(2\beta - a) \right),$$

(2.40) $n = 7$,

$$k_0 = \frac{-3\epsilon g_0}{4(a_0^2 + b_0^2)} \left( a_0 \sin(a + 2\beta) + b_0 \cos(a + 2\beta) \right),$$

$$v_1 = \frac{3}{7} - \frac{3\epsilon g_0}{56(a_0^2 + b_0^2)} \left( b_0 \sin(a + 2\beta) - a_0 \cos(a + 2\beta) \right).$$
(2.41) \( n = 9, \)

\[
k_0 = \frac{-e_0}{4(a_0^2 + b_0^2)} (a_0 \sin 3\beta + b_0 \cos 3\beta),
\]

\[
v_1 = \frac{L}{3} - \frac{e_0^3}{72(a_0^2 + b_0^2)} (b_0 \sin 3\beta - a_0 \cos 3\beta).
\]

The Jacobian for all of the equations (2.33) through (2.37) is \( J(\frac{P, Q}{k_0, v_1}) = -2n(a_0^2 + b_0^2) \neq 0. \)

The remarks following equation (2.22) concerning the existence of subharmonics in the general case \( n \neq 3, \) apply here to ultraharmonics for the general case \( n \neq 2, 5, 7, 9; \) and thus the existence of ultraharmonics in general is an open question for nonzero damping.

In the special cases \( n = 2, 5, 7, 9 \) the values \( k_0, v_1, \) and \( J(\frac{P, Q}{k_0, v_1}) \) are not zero; therefore, the implicit function theorem can be used to assure the existence of functions \( k(\epsilon) \) and \( v(\epsilon) \) which satisfy the periodicity conditions above. This insures the existence of ultraharmonics of order 2, 5, 7, 9 for (1.4) when \( \epsilon \) is sufficiently small. However, if the parameters in the original differential equation (1.4) are specified, it will be shown that only certain ones of these will occur. The conditions under which they occur in terms of the original parameters will now be examined as was done earlier in the case of subharmonics of order \( 1/3. \)
If the values of \( v \) in (2.38a) through (2.41) are substituted in (2.10a), a first approximation can be obtained for \( v \) which will be referred to as \( \overline{v} \) in each case as follows:

(2.42a) \( n = 2, 5, 7, 9 \)

\[
\overline{v} = \frac{1}{n} + \frac{3\varepsilon}{8n} \left( a_0^2 + b_0^2 + \frac{2n^4}{(n^2-1)^2} + \frac{2n^4 r^2}{(n^2-9)^2} \right),
\]

(2.42b) \( n = 2 \),

\[
\overline{v} = 1/2 + 3\varepsilon L/2
\]

- \( \frac{3\varepsilon c_0 a_0^8}{8(a_0^2+b_0^2)} \left( 2a_0 b_0 \sin(a+\beta) - (a_0^2-b_0^2) \cos(a+\beta) \right),
\]

(2.43) \( n = 5 \),

\[
\overline{v} = 1/5 + 3\varepsilon L/5
\]

- \( \frac{3\varepsilon c_0 a_0^8}{40(a_0^2+b_0^2)} \left( b_0 \sin(2a+\beta) - a_0 (2a+\beta) \right),
\]

(2.44) \( n = 7 \),

\[
\overline{v} = 1/7 + 3\varepsilon L/7
\]

- \( \frac{3\varepsilon c_0 a_0^8}{56(a_0^2+b_0^2)} \left( b_0 \sin(a+2\beta) - a_0 \cos(a+2\beta) \right),
\]
(2.45) \( n = 9, \)

\[
\overline{v} = \frac{1}{9} + 3\varepsilon L/9
\]

\[
- \frac{g_0^3}{72(a_0^2+b_0^2)} (b_0 \sin 3\beta - a_0 \cos 3\beta).
\]

If all terms that do not contain trigonometric functions in (2.42) through (2.45) are transferred to the left hand side and this result is multiplied by \(2n/\varepsilon\) and then squared and added to the square of \(k_0\) in (2.38) through (2.41), the phase angles \(\alpha\) and \(\beta\) are eliminated or the number of trigonometric functions is reduced as follows.

(2.46) \( n = 2, \)

\[
\overline{v} = \frac{1}{2} + 3\varepsilon L/2 \pm (\varepsilon/4)(9c_0^2g_0^2/4-k_0^2)^{1/2},
\]

(2.47) \( n = 5, \)

\[
\overline{v} = \frac{1}{5} + 3\varepsilon L/5
\]

\[
\pm \frac{\varepsilon}{10}\frac{9c_0^2g_0^2}{16(a_0^2+b_0^2)} (c_0^2 + g_0^2 + 2c_0g_0 \cos(3\alpha-\beta)) - k_0^2)^{1/2},
\]

(2.48) \( n = 7, \)

\[
\overline{v} = \frac{1}{7} + 3\varepsilon L/7 \pm (\varepsilon/14)(9c_0^2g_0^2/16(a_0^2+b_0^2) - k_0^2)^{1/2},
\]

(2.49) \( n = 9, \)

\[
\overline{v} = \frac{1}{9} + 3\varepsilon L/9 \pm (\varepsilon/18)(g_0^2/16(a_0^2+b_0^2) - k_0^2)^{1/2}.
\]
Since $c_0 = \frac{n^2}{n^2-1}$ and $g_0 = \frac{n^2 r}{n^2-9}$, the conditions for the existence of ultraharmonics are obtained from equations (2.46) through (2.49) as follows:

$n = 2$, $(8r/5)^2 \geq k_0^2$,

$n = 5$, $(25/16)^6 (r/6)^2 (4+9r^2+12r \cos(3\alpha-\beta))/A_5^2 \geq k_0^2$,

$n = 7$, $(49/16)^6 (r/5)^4 /A_7^4 \geq k_0^2$,

$n = 9$, $(81r/72)^6 /16A_9^6 \geq k_0^2$.

where $A_n^2 = a_0^2 + b_0^2$ is the square of the amplitude of the ultraharmonic oscillation of order $n$ (see (2.12)).

Since the condition for the existence of ultraharmonics of order 2 does not contain the amplitude, it will be considered separately. When (2.46) is solved for $A_2^2$ the result is

$$A_2^2 = \frac{8}{3e} (2\sqrt{\nu} - 1) - 2(c_0^2 + g_0^2) + \frac{2}{3} (9c_0^2 g_0^2 - 4k_0^2)^{1/2}.$$  

Consider $\epsilon, k_0, r$ fixed and the condition for the existence of the ultraharmonic of order 2 satisfied. Then $A_2$ on either branch of the above curve will be zero for some frequency $\nu$ and will increase with increasing frequency for $\epsilon > 0$.

For $\epsilon < 0$ the two amplitudes increase with decreasing frequency to maximums at $\nu = 0$.  

The equations (2.47), (2.48), (2.49) for ultraharmonics of order 5,7,9 are similar to each other because the square root term contains the inverse of the amplitude. Again consider $\varepsilon, k_0, r$ fixed and the conditions for existence satisfied. Then it is seen from the three equations just mentioned that the amplitude will have a maximum value which may be found by setting the square root term equal to zero. Thus the amplitudes of the ultraharmonics of order 5,7,9 are bounded. If $\varepsilon > 0$, the frequencies larger than the frequency corresponding to the maximum amplitude will cause the amplitudes to be very small. If $\varepsilon < 0$, the frequency can be increased high enough again to cause the amplitudes to be very small.

Levenson obtained a similar result to (2.42a) for $n \neq 3$ when (1.4) is considered with $C = 0$ and $F_3 = 0$. His result then does not contain the symbols $b, g, a, \beta$ and thus in his work $a_0 = M + n^2/n^2 - 1)$. Since $F_3$ was not present in his work, $v_0 = 1/3$ was not excluded and he obtained the following value for $v^2$ in this instance.

\[ n = 3, \; v^2 = \frac{1}{9} + \frac{\varepsilon}{12}(M^2 - \frac{9M}{4} + \frac{213(M-1)}{8(8M-9)}), \; M \neq \frac{9}{8}. \]
D. Periodicity Conditions for Ultra-Subharmonic Solutions

Ultra-subharmonics have periods that are p times the period of the forcing function and that are q times the period of the free oscillation. Thus \( m = p \) in (2.16a) and (2.17a) and from (2.12) it is observed that \( v_0 = p/q \) where p and q are relatively prime integers and neither is one.

The corresponding values of P and Q are

\[
P = P(0, p/q, v_1, k_0) \quad \text{and} \quad Q = Q(0, p/q, v_1, k_0).
\]

The periodicity conditions in this case are

\[
(2.50) \quad \frac{P}{Q} = -2b_0v_1/p - a_0k_0 + 6b_0L = 0
\]

\[
\frac{P}{Q/q^2} = 2a_0v_1/p - b_0k_0 - 6a_0L = 0.
\]

The solution of (2.38) in terms of \( k_0 \) and \( v_1 \) follows:

\[
k_0 = 0,
\]

\[
v_1 = 3pL/q.
\]

The Jacobian is

\[
J\left(\frac{P, Q}{k_0, v_1}\right) = 2q(a_0^2 + b_0^2)/p \neq 0.
\]

Since \( k_0 = 0 \), the same remarks following (2.22) for general subharmonics \( n \neq 3 \) apply here. Again the first approximation for \( v \) can be obtained by putting \( v_1 \) in (2.10a).

This approximation will be referred to as \( \overline{v} \).

\[
\overline{v} = p/q + 3ep(a_0^2 + b_0^2 + 2q^4/(p^2 - q^2)^2 + 2q^4r^2/(p^2 - q^2))/8q
\]
which checks with Levenson when the conditions with which he worked are imposed as stated at the end of ultraharmonics. No special case occurs here.

E. Periodicity Conditions for Harmonic Solutions

Harmonics are obtained when the frequency of the solution is the same as the frequency of the driving force and the solution has no component of any other frequency. Thus \( m = 1 \) in the equations that express the periodic conditions. From (2.12) it is observed that \( a_0, b_0, \) and \( g_0 \) must equal zero when \( \varepsilon = 0 \) to have harmonic solutions. Since \( a_0 = b_0 = 0 \), the harmonics are considered under Case 2 of the general periodic conditions.

Since no component of any other frequency is allowed in the solution, it was thought advisable to have a similar condition on the forcing function. Thus in (1.14) the second forcing term was omitted which means that in the discussion of harmonics \( F_3 = 0 \) which from (2.1) causes \( r = 0 \) and from the definitions following (2.3) \( g = 0 \). Moreover, \( \beta \) does not enter in \( a \) or \( b \) then either.

It is observed from the definitions following (2.3) that if \( b_0 = 0 \) then \( a_0 = 0 \), and if \( a_0 = 0 \), then \( v_0 = (1-1/M)^{1/2} \) for \( M < 0 \) or \( M > 1 \), or equivalently \( M = -1/(v_0^2 - 1) \). The subscript zero will be omitted from \( M \) and \( k \) since in the discussion of harmonics they may have any prescribed values.
and the periodicity conditions will be solved for \( v_1 \) and \( a_1 \) as is described earlier. (See equation (2.10b) and the accompanying discussion.)

The appropriate values of the \( P_h \) and \( Q_h \) functions will then be \( P_h = P_h(0,v_0,v_1,a_1) \) and \( Q_h(0,v_0,v_1,a_1) \).

When the above values and the indicated values from the tables and equations are substituted in (2.16b) and (2.17b), the periodicity conditions become

\[
(2.51) \quad (v_0^2 - 1)P_h = \frac{k v_0 v_1}{v_0^2 - 1} \sin \frac{\pi}{v_0} - v_0 a_1 \sin \frac{2\pi}{v_0} \\
+ k v_0 c_0 \sin \frac{2\pi}{v_0} + \frac{(1 + v_0^2 - 2)}{9 v_0^2 - 1} c_0^3 \sin \frac{\pi}{v_0} = 0
\]

\[
\frac{(v_0^2 - 1)}{2} Q_h = \frac{v_1}{v_0^2 - 1} \sin \frac{2\pi}{v_0} + a_1 \sin \frac{\pi}{v_0} \\
- k v_0 c_0 \sin \frac{\pi}{v_0} + \frac{(7 v_0^2 - 1)}{2 v_0 (9 v_0^2 - 1)} c_0^3 \sin \frac{2\pi}{v_0} = 0.
\]

Since \( \sin \frac{\pi}{v_0} \) is a common factor in both equations in (2.51), it is necessary to consider two cases as follows:

Case 1. \( \sin \frac{\pi}{v_0} \neq 0 \) or \( v_0 \neq 1/\sqrt{w} \)

where \( W \) is any integer except 0, 1, 3 as these values are excluded throughout this thesis.
Case 2. \( \sin \frac{\pi}{v_o} = 0 \) or \( v_o = 1/\omega \).

Consideration of Case 2 will be deferred until later.

In Case 1 equation (2.51) may be solved for \( a_1 \) and as follows:

\[
(2.52) \quad a_1 = v_o c_o \kappa = - \frac{v_o k}{v_o^2 - 1}
\]

\[
v_1 = \frac{(7v_o^2 - 1)(v_o^2 - 1)}{2v_o(9v_o^2 - 1)} \quad c_o^3 = \frac{(7v_o^2 - 1)}{2v_o(v_o^2 - 1)(9v_o^2 - 1)}
\]

where the value \( c_o = - \frac{1}{v_o^2 - 1} \) has been used.

The Jacobian \( J(\Phi_{1h}, \Psi_{1h}) = - \frac{v_o^2 \sin^2 \frac{\pi}{v_o}}{v_o^2 - 1} \) is not zero in this case and, therefore, the existence of harmonics is assured by the application of the implicit function theorem. The conditions under which the harmonics will exist are specified by \( a \) and \( v \). The first approximation of these values will be called \( \bar{a} \) and \( \bar{v} \) and from \((2.10b)\) and \((2.52)\) these become

\[
(2.53) \quad \bar{a} = - \frac{\epsilon v_o k}{v_o^2 - 1} = \epsilon \Omega k (1 - 1/M)^{1/2}
\]

\[
\bar{v} = v_o + \frac{\epsilon (7v_o^2 - 1)}{2v_o(v_o^2 - 1)(9v_o^2 - 1)} = (1 - 1/M)^{1/2} + \frac{\epsilon M^2 (6M - 7)}{2(1 - 1/M)^{1/2}(8M - 9)}
\]

The above value \( \bar{v} \) was obtained by Levenson for the Duffing equation without damping. Since he considered \( k = 0 \), he did not have \( a \) in his work.
Case 2 will now be considered and similar results obtained. In effect, \( \sin \frac{\pi}{v_o} \) can be divided out in this case also if an argument used by Levenson is applied. Let \( S = \sin(\pi/v_o) \) where \( v_o \) is such that \( S \) is small and positive. This can be accomplished by the following substitution:

\[
(2.54) \quad v_o = 1/(W + \Delta)
\]

where \( \Delta \) is to be taken positive or negative according as \( W \) is even or odd. Also \( |\Delta| \leq \Delta_1 \) where \( \Delta_1 \) is sufficiently small and positive.

In order to effect the division by \( \sin \frac{\pi}{v_o} \), it is necessary to carry out certain limiting processes. The following functions are introduced:

\[
(2.55) \quad F_h = \frac{P_h(\varepsilon, v_o, v_1, a_1)}{S + \varepsilon^2}, \\
\quad C_h = \frac{Q_h(\varepsilon, v_o, v_1, a_1)}{S + \varepsilon^2}.
\]

By the choice of \( \Delta \) the denominator \( S + \varepsilon^2 \neq 0 \) for \( 0 < |\Delta| \leq \Delta_1 \); therefore, \( F_h \) and \( C_h \) are analytic functions of \( \Delta \) and \( \varepsilon \) in the neighborhood of \( \Delta = 0 \) and \( \varepsilon = 0 \). The conditions \( F_h = 0 \) and \( C_h = 0 \) are, in effect, the same periodicity conditions as before. They must be defined for \( \varepsilon = 0 \), however. When the limits \( \lim_{\varepsilon \to 0} \frac{P_h(\varepsilon, v_o, v_1, a_1)}{S + \varepsilon^2} \) and \( \lim_{\varepsilon \to 0} \frac{Q_h(\varepsilon, v_o, v_1, a_1)}{S + \varepsilon^2} \)
are taken it can be seen from (2.51) that the same equations are obtained without the factor \( \sin \frac{\pi}{v_0} \) as follows:

\[
(2.56) \quad (v_0^2 - 1) F_h = \frac{1}{v_0} v_0 v_1 \sin \frac{\pi}{v_0} - 2v_0 a_1 \cos \frac{\pi}{v_0} 
+ 2kv_0 c_0 \cos \frac{\pi}{v_0} + \frac{(14v_0^2 - 2)}{9v_0^3 - 1} c_0^3 \sin \frac{\pi}{v_0} = 0
\]

\[
(\frac{1}{2}) \frac{v_0^2 - 1}{2} \frac{\pi}{v_0} = \frac{2v_1}{v_0^2 - 1} \cos \frac{\pi}{v_0} + a_1 \sin \frac{\pi}{v_0} 
- kv_0 c_0 \sin \frac{\pi}{v_0} + \frac{(7v_0^2 - 1)}{v_0(9v_0^2 - 1)} c_0^3 \cos \frac{\pi}{v_0} = 0.
\]

However, the periodicity conditions (2.56) now hold only under the following two cases.

Case 2a. \( W \) is an even integer \( \neq 0 \). When \( v_0 \) is replaced by \( 1/(W+\Delta) \) in (2.56) it represents the periodicity conditions for the range of values \( 1/W > v_0 > 1/(W+\Delta_1) \).

Case 2b. \( W \) is an odd integer \( \neq 1,3 \). Then (2.56) represents the periodicity conditions for the range of values

\( 1/(W-\Delta_1) > v_0 > 1/W \) where \( v_0 \) is replaced by \( 1/(W-\Delta) \) in (2.56).

When (2.56) is solved for \( a_1 \) and \( v_1 \) the same results in (2.51) are obtained and the Jacobian is

\[
J(\frac{F_h, Q_h}{a_1, v_1}) = -4v_0/(v_0^3 - 1)
\]

which is not zero. Therefore, the existence of harmonics is established for these two cases also.
III. STABILITY OF PERIODIC SOLUTIONS

A. General Stability Conditions

If when a periodic oscillation is subjected to a small disturbance it returns to its original state essentially unaffected, it is referred to as a stable periodic oscillation. If some small disturbance causes the periodic oscillation to be considerably altered relative to its original state, it is said to be unstable. This is stated more precisely by means of inequalities as follows. The solution \( \rho(\theta) \) is said to be stable if for every \( \epsilon > 0 \), there exists a \( \Delta(\epsilon) > 0 \) such that if

\[
(3.1) \quad |\rho(\theta_0) - \bar{\rho}(\theta_0)| < \Delta, \text{ and } |\rho'(\theta_0) - \bar{\rho}'(\theta_0)| < \Delta,
\]

then for any \( \theta > \theta_0 \)

\[
(3.2) \quad |\rho(\theta) - \bar{\rho}(\theta)| < \epsilon, \text{ and } |\rho'(\theta) - \bar{\rho}'(\theta)| < \epsilon,
\]

where the bar designates a neighboring solution curve. The disturbance is expressed by slightly different initial conditions from those giving the periodic solution; therefore, the solution of (2.2) with slightly altered initial conditions will be considered as \( \bar{\rho} \) and the periodic solutions of Part II as \( \rho \). General stability conditions on the periodic solutions discussed in Part II will be obtained analogous to the general periodicity conditions already obtained.
Let $f, h$ be real parameters and let $\tilde{p}(\theta, \epsilon, v, k, r, a, \beta, f, h)$ be the solution of (2.2) with initial conditions $\tilde{p}(0) = f$, $\tilde{p}'(0) = h$, that is,

\begin{equation}
\tilde{p}(0, \epsilon, v, k, r, a, \beta, f, h) = f,
\end{equation}

\begin{equation}
\tilde{p}'(0, \epsilon, v, k, r, a, \beta, f, h) = h.
\end{equation}

For $\theta = 2m\pi$, $\tilde{p}$ and $\tilde{p}'$ assume values which will be denoted by $\tilde{T}$ and $\tilde{T}'$. This is made precise as follows:

\begin{equation}
\tilde{T}(\epsilon, v, k, r, a, \beta, f, h) = \tilde{p}(2m\pi, \epsilon, v, k, r, a, \beta, f, h),
\end{equation}

\begin{equation}
\tilde{T}'(\epsilon, v, k, r, a, \beta, f, h) = \tilde{p}'(2m\pi, \epsilon, v, k, r, a, \beta, f, h).
\end{equation}

As in the preceding sections, two cases will be considered along with the appropriate parameters as analytic functions of $\epsilon$ which satisfy the periodicity conditions (2.4). As in (2.10a) and (2.10b), the following functions are defined:

Case 1, $v_0 = \frac{p}{q}$, $a_0^2 + b_0^2 \neq 0$:

\begin{equation}
k = k(\epsilon) = f(M, \epsilon, r, a, \beta) = k_0 + k_1 \epsilon + \cdots,
\end{equation}

\begin{equation}v = v(\epsilon) = g(M, \epsilon, r, a, \beta) = v_0 + v_1 \epsilon + \cdots
\end{equation}

\begin{equation}= v_0 + \epsilon v = v_0 + \epsilon (v_0 + v_1 \epsilon + \cdots),
\end{equation}

Case 2, $a_0 = b_0 = 0$, $v_0 = (1-1/M)^{1/2}$ for $M < 0$ or $M > 1$, $r \neq 0$:
Similar to the discussion preceding equations (2.11a,b), the following notation is convenient:

Case 1: \[ \bar{T}(\varepsilon, v, k, r, a, \beta, f, h) = T(\varepsilon, v(\varepsilon), k(\varepsilon), f, h) \]
and
\[ \bar{T}'(\varepsilon, v, k, r, a, \beta, f, h) = T'(\varepsilon, v(\varepsilon), k(\varepsilon), f, h) \]

Case 2: \[ \bar{T}(\varepsilon, v, k, r, a, \beta, f, h) = T_h(\varepsilon, \varepsilon, a(\varepsilon), f, h) \]
and
\[ \bar{T}'(\varepsilon, v, k, r, a, \beta, f, h) = T'_h(\varepsilon, \varepsilon, a(\varepsilon), f, h) \].

For these functions equations (2.4) are satisfied identically and in terms of \( \bar{T} \) and \( \bar{T}' \) these identities may be written as

Case 1:
\[ (3.6a) \quad \bar{T}(\varepsilon, \varepsilon, k(\varepsilon), M, 0) \equiv M, \]
\[ \bar{T}'(\varepsilon, \varepsilon, k(\varepsilon), M, 0) \equiv 0. \]

Case 2:
\[ (3.6b) \quad \bar{T}_h(\varepsilon, \varepsilon, a(\varepsilon), M, 0) \equiv M, \]
\[ \bar{T}'_h(\varepsilon, \varepsilon, a(\varepsilon), M, 0) \equiv 0. \]

The initial conditions \( f \) and \( h \) will be taken close to the initial conditions for the periodic solution, the stability of which is being determined. Accordingly, for both cases, \( u \) and \( y \) are defined by the equations
(3.7) \[ f = M + u, \]
\[ h = 0 + y, \]
where \( u \) and \( y \) will be taken small (See condition (3.1)) In a similar manner \( U \) and \( Y \) are defined by the equations
(3.8) \[ \bar{T} = M + U, \]
\[ \bar{T}' = 0 + Y, \]

The functions \( \bar{T} \) and \( \bar{T}' \) are analytic in all the parameters and can be expanded with respect to \( f \) and \( h \) in the neighborhood of \( f = M \) and \( h = 0 \), that is, \( \bar{T} \) and \( \bar{T}' \) may be expanded in powers of \( f - M = u \) and \( h - 0 = y \). In this expansion \( \bar{T} \) and \( \bar{T}' \) and their partial derivatives will be evaluated at \((\epsilon, v(\epsilon), k(\epsilon), M, 0)\) in Case 1 and at \((\epsilon, v(\epsilon), a(\epsilon), M, 0)\) in Case 2.

For convenience of notation, the evaluation of the partial derivatives will be designated as at \( f = M \) and \( h = 0 \). With the understanding that \( \bar{T} \) and \( \bar{T}' \) stand for either Case 1 or Case 2 the expansion may be written as follows:

\[
(3.9) \quad M + U = M + \frac{\partial \bar{T}}{\partial f} \bigg|_{f=M, h=0} u + \frac{\partial \bar{T}}{\partial h} \bigg|_{f=M, h=0} y + \cdots
\]

\[
Y = 0 + \frac{\partial \bar{T}'}{\partial f} \bigg|_{f=M, h=0} u + \frac{\partial \bar{T}'}{\partial h} \bigg|_{f=M, h=0} y + \cdots
\]

where the dots refer to higher order terms in \( u \) and \( y \).

When \( u \) and \( y \) are sufficiently small, the higher order terms may be neglected and equations (3.9) may be written
as the following matrix equation:

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix} = K \begin{bmatrix}
  u \\
  v
\end{bmatrix}
\]

where the elements of \( K \) are the partial derivatives in (3.9). The values of the partial derivatives will be expanded in powers of \( \varepsilon \) about \( \varepsilon = 0 \) and only the first two terms in the expansion will be significant if \( \varepsilon \) is sufficiently small. Thus \( K \) will be expressed by

\[
K = \begin{bmatrix}
  a_{11} + \varepsilon b_{11} & a_{12} + \varepsilon b_{12} \\
  a_{21} + \varepsilon b_{21} & a_{22} + \varepsilon b_{22}
\end{bmatrix} = \bar{A} + \varepsilon \bar{B}.
\]

The double subscripts on \( a \) and \( b \) in (3.11) will differentiate these symbols from the other meanings of \( a \) and \( b \).

In order to evaluate the partial derivatives in (3.9) the solution and its derivative at \( \theta = 2\pi \) with initial conditions (3.3) are given below. (See equation (2.21))

\[
T = \bar{a} \cos \frac{2\pi}{v(\varepsilon)} + \bar{B} \sin \frac{2\pi}{v(\varepsilon)} + \bar{c} \cos \alpha + \bar{g} \cos \beta
\]

\[
-\frac{\varepsilon}{v(\varepsilon)} \int_0^{2\pi} \left( \rho^3(\phi) + kv(\varepsilon)\rho^2(\phi) \right) \sin \frac{2\pi}{v(\varepsilon)} \phi d\phi
\]

where

\[
\bar{a} = a + \frac{\cos \alpha}{v^3(\varepsilon)-1} + \frac{r \cos \beta}{9v^3(\varepsilon)-1}, \quad \bar{c} = -\frac{1}{v^2(\varepsilon)-1}
\]

\[
\bar{B} = v(\varepsilon)h - \frac{v(\varepsilon)\sin \alpha}{v^3(\varepsilon)-1} - \frac{3r \bar{v} \sin \beta}{9v^3(\varepsilon)-1}, \quad \bar{g} = -\frac{r}{9v^3(\varepsilon)-1}
\]
\[
\overline{T} = -\frac{a}{v(\epsilon)} \sin \frac{2\pi}{v(\epsilon)} + \frac{b}{v(\epsilon)} \cos \frac{2\pi}{v(\epsilon)} - c \sin a - 3 \frac{c}{v(\epsilon)} \sin \beta 
\]

\[
-\frac{e}{\nu(\epsilon)} \int_0^{2\pi} \left( -\rho^3(\phi) + k(v(\epsilon)) \rho'(\phi) \right) \cos \frac{2\pi - \phi}{v(\epsilon)} d\phi.
\]

The bars on a and b in (3.12) refer to the fact that they are functions of different initial conditions than in (2.2) as well as functions of \(v(\epsilon)\) in each case and also \(c(\epsilon)\) in Case 2. The values \(\overline{a}\) and \(\overline{b}\) will equal \(a_0\) and \(b_0\) respectively when \(f = M, h = 0, \) and \(\epsilon = 0\). The bars on c and g denote that they are functions of \(v(\epsilon)\). In Case 1, \(k\) is to be designated as \(k(\epsilon)\) in (3.12) and in Case 2, \(c\) is to be denoted by \(c(\epsilon)\) as can be seen from equations (3.6a) and (3.6b).

The method by which the first element in \(K\), \(a_{11} + \epsilon b_{11}\), is obtained will now be shown in detail. From (3.9) it is seen that the first element in \(K\) is \(\frac{\partial T}{\partial f}\bigg|_{f=M, h=0}\). The value of this is obtained from (3.12) as follows:

\[
(3.13) \quad \left. \frac{\partial T}{\partial f} \right|_{f=M, h=0} = \left[ \left. \frac{\partial T}{\partial a} \right|_{h=0} \left. \frac{\partial a}{\partial f} \right|_{f=M} \right] = \left. \frac{\partial T}{\partial a} \right|_{f=M, h=0} = \left. \frac{\partial T}{\partial \epsilon} \right|_{f=M, h=0}
\]

\[
= \cos \frac{2\pi}{v(\epsilon)} - \frac{e}{v(\epsilon)} \frac{\partial}{\partial \epsilon} \int_0^{2\pi} \left( -\rho^3(\phi) + k(v(\epsilon)) \rho'(\phi) \right) \sin \frac{2\pi - \phi}{v(\epsilon)} d\phi
\]

\[
= a_{11} + \epsilon b_{11} + \epsilon^2 c_{11} + \cdots
\]
From (3.13) $a_{11}$ in the two cases is evaluated at $\epsilon = 0$ with the use of (3.5a,b) as follows:

Case 1:

\begin{equation}
(3.14a) \quad a_{11} = \cos \frac{2\pi}{v(\epsilon)} \bigg|_{\epsilon=0} = \cos \frac{2\pi}{v_0} = 1,
\end{equation}

Case 2, $m = 1$:

\begin{equation}
(3.14b) \quad a_{11} = \cos \frac{2\pi}{v_0}.
\end{equation}

From (3.5a) and (3.12) it follows that

\begin{equation}
b_{11} = \left[ \frac{\partial}{\partial \epsilon} \left[ \frac{\partial T}{\partial \zeta} \right] \right]_{\epsilon=0} = \left[ -\sin \frac{2\pi}{v(\epsilon)} \left( -\frac{2\pi}{v^2(\epsilon)} \right)(v) \right]_{\epsilon=0}

- \left[ \frac{1}{v(\epsilon)} \frac{\partial}{\partial \zeta} \right] \left[ \frac{2\pi}{v(\epsilon)} \left( \rho^s(\phi) + k(\epsilon)v(\epsilon)\rho^t(\phi) \right) \sin \frac{2\pi - \phi}{v(\epsilon)} \ d\phi \right]_{\epsilon=0}.
\end{equation}

Thus in Case 1

\begin{equation}
(3.15a) \quad b_{11} = \frac{2\pi v_1}{v_0^2} \sin \frac{2\pi}{v_0}

- \left[ \frac{1}{v_0} \frac{\partial}{\partial a_0} \right] \left[ \frac{2\pi}{v_0} \left( \rho_0^s(\phi) + k_0 v_0 \rho_0^t(\phi) \right) \sin \frac{2\pi - \phi}{v_0} \ d\phi \right]

= \frac{\partial}{\partial a_0} \int_0^{2\pi} \frac{v_0}{v_0} \left( \rho_0^s(v_0 t) + k_0 v_0 \rho_0^t(v_0 t) \right) \sin(t - \frac{2\pi}{v_0}) dt.
\end{equation}
In Case 2, \( a_0 = 0 \) but the derivative with respect to \( a_0 \) may be computed formally as if \( a_0 \neq 0 \) and then \( a_0 \) replaced by zero. A similar remark pertains to \( b_0 \) in the work that follows. Thus from (2.18) and number 9 in Table 1, the value of \( b_{11} \) from above becomes in the second case

\[
(3.15b) \quad b_{11} = \pi \left( \frac{2v_1}{v_0^2} - \frac{3v_0}{2v_0} + \frac{k}{2\pi} \right) \sin \frac{2\pi}{v_0} - \frac{k}{v_0} \cos \frac{2\pi}{v_0}
\]

In Case 1 the partial derivative in \( b_{11} \) may be obtained by taking the indicated partial derivatives of the entries in Table 1 and of equation (2.18) or from the periodicity conditions. Similarly, the values of \( b_{12}, b_{21}, b_{22} \) may be obtained easily from the periodicity conditions for Case 1. Since these values depend on the value of \( v_0 \), the equations (3.14a,b) and (3.15a,b) will be the starting point for obtaining the first element of \( K \) in the consideration of the stability of the four types of periodic solutions. Similar equations which follow will be the starting point for obtaining the other three elements of \( K \) for the four types of periodic solutions.

The element \( a_{12} + \varepsilon b_{12} \) of \( K \) is obtained from the following:

\[
(3.16) \quad \frac{\partial T}{\partial h} \bigg|_{f=M, h=0} = v(\varepsilon) \sin \frac{2\pi}{v(\varepsilon)}
\]
\[
- \epsilon \frac{\partial}{\partial \epsilon} \int_0^{2\pi} (\bar{p}^3(\phi) + kv(\epsilon)\bar{p}'(\phi)) \sin \left( \frac{2\pi \epsilon}{v(\epsilon)} \right) d\phi
\]

\[= a_{1a} + \epsilon b_{1a} + \cdots.\]

As in the evaluation of the first element of \( K \), the values in (3.16) become in Case 1

(3.17a) \[a_{1a} = 0,\]

(3.17b) \[b_{1a} = - \frac{2\pi \epsilon v_1}{v_0}\]

\[+ v_0 \frac{\partial}{\partial b_0} \int_0^{2\pi} \frac{2\pi}{v_0} (\rho_0^3(v_0 t) + k_0 v_0 \rho_0'(v_0 t)) \sin \left( \frac{2\pi \epsilon}{v_0} \right) dt,\]

and in Case 2

(3.17c) \[a_{1a} = v_0 \sin \frac{2\pi}{v_0},\]

(3.17d) \[b_{1a} = (v_1 - \frac{3v_0^2 v_0^2 - 2}{4(v_0^2 - 1)} - \pi k) \sin \frac{2\pi}{v_0} + \pi \left( \frac{3v_0^2}{2} - \frac{2v_1}{v_0} \right) \cos \frac{2\pi}{v_0}\]

where in Case 2 (2.18) and number 13 in Table 1 were used to obtain the value of the term which contains the integral in \( b_{1a} \).

The element \( a_{21} + \epsilon b_{21} \) of \( K \) is obtained from the following:

(3.18) \[\frac{\partial T!}{\partial f} \bigg|_{r=M} = - \frac{1}{v(\epsilon)} \sin \frac{2\pi \epsilon}{v(\epsilon)}\]
\[-\frac{\epsilon}{v^2(\epsilon)} \frac{3}{2\pi} \int_0^{2\pi} (p^3(\phi) + kv(\epsilon)p'_1(\phi)) \cos \frac{2m\pi - \phi}{v(\epsilon)} d\phi\]

\[= a_{21} + \epsilon b_{21} + \ldots.\]

As in the evaluation of the first element of $K$, the values in (3.18) become in Case 1

(3.19a) \[a_{21} = 0,\]

\[b_{21} = \frac{2m\pi v_1}{v_o},\]

\[-\frac{1}{v_o} \frac{3}{2\pi} \int_0^{2\pi} (p^3(v_0t) + k\rho_0v_0\rho_0'(v_0t)) \cos(t - \frac{2m\pi}{v_0}) dt,\]

and in Case 2

(3.19b) \[a_{21} = -\frac{1}{v_o} \sin \frac{2\pi}{v_o},\]

\[b_{21} = \left(\frac{v_1}{v_o} + \frac{nk}{v_o^2} - \frac{3c_0^2(v_o^2 - 2)}{4v_0(v_o^2 - 1)}\right) \sin \frac{2\pi}{v_o} + \pi \left(\frac{2\pi}{v_o^2} - \frac{3c_0^2}{2v_o^2}\right) \cos \frac{2\pi}{v_o},\]

where in Case 2 (2.19) and number 9 in Table 2 were used to obtain the value of the term which contains the integral in $b_{21}$. 
The element $a_{22} + \epsilon b_{22}$ of $K$ is obtained from the following:

\[(3.20) \quad \frac{\partial T'}{\partial h} \bigg|_{r=M} = \cos \frac{2\pi n}{v(t)}

- \frac{\epsilon}{v(t)} \frac{\partial}{\partial b} \int_0^{2\pi} \left( (\rho_c^3(\phi)+kv(t)(\rho_c^1(\phi))) \cos \frac{2\pi n}{v(t)} \right) d\phi

= a_{22} + \epsilon b_{22}.

As in the evaluation of the first element of $K$, the values in (3.20) become for Case 1

\[(3.21a) \quad a_{22} = 1,

b_{22} = -\frac{\partial}{\partial b} \int_0^{2\pi} \frac{v_0}{v(t)} (\rho_c(\phi)+kv(t)\rho_c'(\phi)) \cos(t - \frac{2\pi n}{v_0}) dt

where the value of the integral is obtained from the periodicity conditions, and in Case 2

\[(3.21b) \quad a_{22} = \cos \frac{2\pi}{v_0},

b_{22} = \left( \frac{2\pi v_1}{v_0^2} - \frac{k}{2} - \frac{3\pi c_o^2}{v_0} \right) \sin \left( \frac{2\pi}{v_0} \right) - \frac{k}{v_0} \cos \frac{2\pi}{v_0}

where (2.19) and number 13 of Table 2 were used to obtain the value involving the integral in $b_{22}$.
It is proved by Langenhop (2) that if the characteristic roots of \( K = \bar{A} + \varepsilon \bar{B} \) in (3.10) are less than one in magnitude then the periodic solution \( \rho \) is stable. The solution is unstable if any characteristic root is greater than one in absolute value. The stability conditions will be obtained for two cases as were the periodicity conditions in Part II with the exception that the condition \( v_o = 1/W \) where \( W \) is an integer \( \neq 0,1,3 \) in the harmonic case will be considered with the first case in stability.

\[ (3.22a) \]

Case SI, \( v_o = p/q, \bar{A} = I \), the identity matrix,

\[ (3.22b) \]

Case SII, \( v_o \neq 1/W \), where \( W \) is an integer.

It is also proved by Langenhop that if \( \bar{A} = I \), then for \( \varepsilon > 0 \) but sufficiently small the characteristic roots of \( K \) will be less than one (greater than one) in absolute value if the characteristic roots of \( \bar{B} \) have negative real parts (positive real parts). Since the characteristic roots of \( \bar{B} \) are a solution of a quadratic equation, their real parts will be negative if the trace of \( \bar{B}, \bar{A} \), is negative and the determinant of \( \bar{B}, D \), is positive.

In Case SII the characteristic roots of \( K \) are obtained from (3.10) as follows:
\[(3.23) \quad (a_{11} + \varepsilon b_{11} - \lambda)(a_{22} + \varepsilon b_{22} - \lambda) - (a_{12} + \varepsilon b_{12})(a_{21} + \varepsilon b_{21}) = 0.\]

From (3.14b) and (3.21b) it is observed that \(a_{11} = a_{22}\); therefore the resulting quadratic equation is

\[(3.24) \quad \lambda^2 - (2a_{11} + \varepsilon(b_{11} + b_{22}))\lambda + a_{11}^2 - a_{12}a_{21} + \varepsilon(a_{11}(b_{11} + b_{22}) - a_{12}b_{21} - a_{21}b_{12}) + \varepsilon^2(b_{11}b_{22} - b_{12}b_{21}) = 0.\]

The roots of (3.24) will be shown to be complex conjugates. To this end, the discriminant \(R\) is reduced to the following:

\[(3.25) \quad R = 4a_{12}a_{21} + 4\varepsilon(a_{12}b_{21} + a_{21}b_{12}) + \varepsilon^2((b_{11} + b_{22})^2 - 4(b_{11}b_{22} - b_{12}b_{21})).\]

Since \(a_{12}a_{21} = -\sin^2 \frac{2\pi}{v_0}\) for \(v_0 \frac{1}{1/W}\), \(R\) is always negative for \(\varepsilon\) sufficiently small and the roots of \(K\) are complex conjugates. Since the roots of \(K\) are complex conjugate in Case SII, the magnitude of either root is the constant term in (3.24). The condition for stability in Case SII is then that the constant term in (3.24) be less than one in absolute value. Since \(a_{11}^2 - a_{12}a_{21} = 1\) as can be seen from equations (3.14b) through (3.21b), this condition reduces to

\[a_{11}(b_{11} + b_{22}) - a_{12}b_{21} - a_{21}b_{12} \leq 0 \quad \text{for} \quad \varepsilon \geq 0.\]

The conditions for stability in both cases are stated explicitly as follows:
Case SI: \( X = b_{11} + b_{22} < 0 \) for \( \epsilon < 0 \),
\[
D = b_{11}b_{22} - b_{12}b_{21} > 0,
\]
(3.26b)
Case SII:
\[
a_{11}X - a_{12}b_{21} - a_{21}b_{12} < 0 \text{ for } \epsilon > 0.
\]

Applications of the stability conditions to the periodic solutions will be given in the next sections.

B. Stability of Subharmonic Solutions of Order 1/3

The values needed for the stability conditions (3.26a) are obtained from (3.15a) through (3.21a) with the use of (2.21).

The equation for \( X \) in (3.26a) then becomes
\[
X = \frac{3}{2} \pi (a_0 b_0 - a_0 c_0 \sin \alpha - b_0 c_0 \cos \alpha - 2k_0/3)
\]
\[
- \frac{3}{2} \pi (a_0 b_0 - a_0 c_0 \sin \alpha - b_0 c_0 \cos \alpha + 2k_0/3)
\]
\[
= - 2\pi k_0.
\]

Therefore, \( X \leq 0 \) for \( \epsilon > 0 \) if \( k_0 \) has the same algebraic sign as \( \epsilon \).

The equation for \( D \) in (3.26a) becomes
\[ D = \left( \frac{3}{2}a_0b_0 - a_0c_0 \sin \alpha - b_0c_0 \cos \alpha - 2k_0/3 \right) \]

\[ \cdot \left( -\frac{3}{2}a_0b_0 - a_0c_0 \sin \alpha - b_0c_0 \cos \alpha + 2k_0/3 \right) \]

\[ - \left( \pi (-2v_1 + \frac{9}{2}(4L + b_0^2 + b_0c_0 \sin \alpha - a_0c_0 \cos \alpha)) \right) \]

\[ \cdot \left( \pi \left( \frac{2}{3}v_1 - \frac{1}{2}(4L + a_0^2 + a_0c_0 \cos \alpha - b_0c_0 \sin \alpha) \right) \right). \]

Divide the above equation by \( \pi \) and make the substitution (see (2.24))

\[ \frac{\delta A_1^{2}}{2}(v_1-9L) = -b_0(3a_0^2-b_0^2) \sin \alpha + a_0(a_0^2-3b_0^2) \cos \alpha. \]

Then when the square is completed on \( (v_1-9L) \) the result is

\[ (3.28) \quad D/\pi = k_0^2 + \frac{4}{9}(v_1-9L)^2 - \frac{9}{4} c_0 A_1^{2}/s \]

\[ - \frac{9}{2} \frac{2}{3}(v_1-9L))A_1^{2}/s. \]

From (3.23) it is seen that

\[ k_0^2 + \frac{4}{9}(v_1-9L)^2 = \frac{9}{16} c_0 A_1^{2}/s. \]

When this value is substituted in (3.28) and the substitution

\[ \frac{2}{3}(v_1-9L) = \pm (9c_0 A_1^{2}/s/16 - k_0^2)^{1/2} \]


which is obtained from (3.23) also is used, the result is

\[ D/\pi = \frac{9}{2} A_1^2/\varepsilon \left( \frac{3}{8} c_0^2 + \left( 9 c_0^2 A_1^2 / a - k_0^2 \right)^{1/2} \right). \]

From (3.26a) it is seen that the condition for stability is that \( D > 0 \). From the last equation, this is true when

\[ (9 A_1^2 / a / 1024 - k_0^2)^{1/2} > 3/8^s \]

where the value \( c_0^2 = 1/64 \) has been substituted. From (3.29) it is seen that the positive square root in (3.30) corresponds to the negative square root in (2.26) or the upper branch of the curves in Figure 1 when \( \varepsilon > 0 \). However, only that portion of the upper branch is stable which satisfies the condition (3.30). The equality point of this condition is also obtained by finding the vertical tangent to the curves, that is, by equating to zero the derivative of (2.26) with respect to \( A_1 / a \). Therefore, stability occurs for frequencies that are greater than the frequency where this vertical tangent occurs and for the larger amplitude when \( \varepsilon \) and \( k_0 \) are both positive. For \( \varepsilon < 0 \) the curves in Figure 1 would bend to the left instead of to the right as shown. Then the positive radical in (3.30) corresponds to the negative radical in (2.26) that is again the upper branch of the response curves. Again the condition (3.30) could be obtained by finding the vertical tangent. The subharmonics of order 1/3 are stable
for frequencies less than the frequency where the vertical tangent occurs and for the larger amplitude when $\epsilon$ and $k_o$ are both negative.

C. Stability of Ultraharmonic Solutions of Order 2, 5, 7, 9

The stability of ultraharmonics of order 2 is considered first. From the stability condition (3.26a) and (3.14a), (3.21a) the value of $X$ becomes

$$X = 3\pi(a_o b_o - c_o g_o \sin(a+\beta) - 2k_o/3)$$

$$- 3\pi(a_o b_o - c_o g_o \sin(a+\beta) + 2k_o/3)$$

$$= - \frac{4\pi k_o}{3}.$$ 

Therefore, $X \leq 0$ for $\epsilon \geq 0$ if $\epsilon$ and $k_o$ have the same algebraic sign.

The condition that $D > 0$ will be concluded from the following equations. The following values are obtained from (3.14a) through (3.21a).

$$D = (3\pi(a_o b_o - c_o g_o \sin(a+\beta) - 2k_o/3)$$

$$- (-3\pi(a_o b_o - c_o g_o \sin(a+\beta) + 2k_o/3)$$

$$- (-\frac{4\pi v_1 + \frac{3\pi}{2}(4L + b_o^2 - c_o g_o \cos(a+\beta)))}$$

$$- (16\pi v_1 - 6\pi(4L + a_o^2 + c_o g_o \cos(a+\beta)))$$
Divide this equation by \(9\pi^2\) and from (2.38b) make the substitution

\[- \frac{8}{3} A_2^{a} \left(v_1 - 3L/2\right)\]

\[= c_0e_0\left(2a_0b_0 \sin(a+\beta)-(a_0^2-b_0^2)\cos(a+\beta)\right)\]

where \(A_n^a\) in this section represents \(a_0^a + b_0^a\) and \(n = 2,5,7,\) or 9 here. When the three required terms from the above equation in D are grouped together, the square of \((v_1-3L/2)\) is obtained and the equation yields

\[
\frac{D}{9\pi^2} = \frac{6h}{9}(v_1-3L/2)^2 + \frac{4}{9} k_o^2 \]

\[- \frac{16}{3} A_2^{a} (v_1-3L/2) - c_0e_0^2.\]

By squaring and adding the equations in (2.38) the substitutions

\[16(v_1-3L/2)^2 + k^2 = 9c_0^2e_0^2/4\]

and

\[v_1-3L/2 = \frac{1}{4}(9c_0^2e_0^2/4-k_o^2)^{1/2} = \pm \frac{1}{4}((8r/5)^2-k_o^2)^{1/2}\]

are obtained which reduce the D equation to

\[
\frac{D}{9\pi^2} = - \frac{1}{3} A_2^{a} \left(\pm((8r/5)^2-k_o^2)^{1/2}\right).\]
Then \( D > 0 \) for \( \varepsilon > 0 \) when the negative sign of the radical in (2.46) is used and \((8r/5)^s > k_o^s\) which is the condition for existence established in Part II. The negative sign of the radical corresponds to the top branch of the response curve in the \( \omega, \varphi \)-plane and is stable for frequencies greater than the frequency where \( \omega_a = 0 \). For \( \varepsilon < 0 \) the amplitude which corresponds to the negative sign of the radical in (2.38b) is stable. It is also represented by the upper branch of the corresponding response curves.

As can be seen from (3.14a) through (3.21a) the values needed for \( X \) and \( D \) are the same for ultraharmonics of order \( n = 5, 7, 9 \) when \( n \) replaces these numbers. Thus the stability conditions (3.26a) will be carried out by using \( n \) with the understanding that only these three numbers apply.

\[
X = \frac{3n\pi}{2} (a_0 b_0 - 2k_0/3) - \frac{3n\pi}{2} (a_0 b_0 + 2k_0/3)
\]

\[
= - 2\pi k_o.
\]

Therefore, \( X \leq 0 \) for \( \varepsilon \geq 0 \) when \( \varepsilon \) and \( k_o \) have the same algebraic sign.

\[
D = \left( \frac{3n\pi}{2} (a_0 b_0 - 2k_0/3) \right) \left( - \frac{3n\pi}{2} (a_0 b_0 + 2k_0/3) \right)
\]

\[
- \left( \frac{3\pi}{2} (4L + b_o^s - 4n\nu_1/3) \right) \left( - \frac{3n\pi}{2} (4L + a_o^s - \frac{4n\nu_1}{3}) \right).
\]
they will be used in the three cases if $57.9$

The values which follow (2.9) are repeated here as

\[
\left[ \frac{\sqrt[10]{(x-0.002\cos 90^0 + 0.003\cos 90^0)} \frac{16\pi}{1}}{\frac{10}{1}} \right] \frac{57.9}{20}
\]

\[
((9-0.002\cos 90^0 + 0.003\cos 90^0) \frac{16\pi}{1} = n(2.9)9/10
\]

reduces to the following:

\[
\left( \frac{n}{\pi - 1/\Lambda} \right) = \frac{100}{\pi - 1/\Lambda}
\]

and from this, the substitution of the value of the result can be seen in (2.9)

II adding the equations in (2.9), as was done in part

form the substitutions which are obtained by substitution

III of those of order 5 are considered first. The

for $n = 57.9$

0 $x$ 57.9

Equation (2.9) will be used to obtain the condition

\[
(\frac{n}{\pi - 1/\Lambda})^{1/\eta} \frac{\eta}{1/\eta} -
\]

\[
(0 + \frac{n}{\pi - 1/\Lambda})^{1/\eta} = 0^{1/\eta} (2.9)
\]

is the result (2.9) square of the square of the result

the three required terms

in the numerators of order 2, the three required terms

when this equation is multiplied by $/11/2$ and, as before

-55-
\[ c_0 = \frac{n^2}{(n^2-1)} \quad \text{and} \quad g_0 = \frac{n^2}{(n^2-9)}. \]

With these values and the above equation the condition that \( D > 0 \) is given by

\[(3.32) \quad (25/16)^6 (r/6)^2 (4+9r^2 + 12r \cos(3\alpha-\beta)) \]

\[ > \pm \frac{3A_0^2}{2A_0^2} \left[ (25/16)^6 (r/6)^2 (4+9r^2 + 12r \cos(3\alpha-\beta)) / A_0^2 - k_0^2 \right]^{1/2}. \]

Since the stability conditions for \( n = 5, 7, 9 \) are similar, the next paragraph is devoted to a general discussion.

The response curves in the cases \( n = 5, 7, 9 \) all have a similar shape which can be described as a curve with a peak which is bent to one side or another (to the right for \( \varepsilon > 0 \) and to the left for \( \varepsilon < 0 \)). The \( \overline{v} \) axis is divided into three sections \( \overline{v} < \overline{v}_1, \overline{v}_1 < \overline{v} < \overline{v}_n, \) and \( \overline{v}_n < \overline{v}. \)

In the first and last sections, the graph of \( A_n \) versus \( \overline{v} \) is single-valued but for \( \overline{v}_1 < \overline{v} < \overline{v}_n \) there are three \( A_n \)'s for each \( \overline{v} \). The condition (3.32) can be shown to be equivalent to the statement that the \( \overline{v} \) for any \( A_n \) in either the first or last sections (that is, \( \overline{v} < \overline{v}_1 \) or \( \overline{v} > \overline{v}_n \)) gives a stable solution while for \( \overline{v} \) between \( \overline{v}_1 \) and \( \overline{v}_n \) the largest and smallest \( A_n \)'s give stable solutions but the middle value of \( A_n \) gives an unstable solution. These facts can be shown by consideration of \( d\overline{v}/dA_n^2 \) on the rightmost
branch of the $A_n$ versus $\nabla$ curve. When $dV/dA_n^2 < 0$ on this branch then the condition (3.32) is satisfied.

By squaring and adding the equations in (2.40) it is found that

$$(14(v_1 - 3L/7))^2 + k_0^2 = g_0^2 g_5^2/16A_7^2$$

and from this the value of $(v_1 - 3L/7)$ is obtained. When these values are substituted in (3.31) the result is

$$4D/9(49)r^2 = g_0^2 g_5^2/4A_7^2$$

$$- \frac{28}{3} A_7^2 (\pm \frac{1}{14} (g_0^2 g_5^2/16A_7^2 - k_0^2)^{1/2})^2.$$  

It is seen from this equation that $D > 0$ when

$$\left(\frac{49}{40}\right)^2 \left(\frac{49r}{40}\right)^4$$

$$> \frac{8}{3} A_7^2 ((49/16)^6 (r/5)^4/A_7^2 - k_0^2)^{1/2}.$$  

The discussion of this condition is found in the preceding general discussion when $n = 7$.

When the equations in (2.41) are squared and added the result is

$$(18(v_1 - 3L/9))^2 + k_0^2 = g_0^2/16A_9^2$$

from which the value of $(v_1 - 3L/9)$ can be obtained. These
values then reduce (3.31) to the following equation:

\[ 4D/9(81)\pi = e_0^6/36A_9^8 \]

\[-12A_9^3(\pm \frac{1}{18}(e_0^6/16A_9^8 - k_0^8)^{1/4}). \]

The condition that \( D > 0 \) is then

\[ (81r/72)^6 > \pm 24A_9^4((81r/72)^6/16A_9^8 - k_0^8)^{1/4}. \]

For the discussion of this condition see the remarks following the similar condition for ultraharmonics of order 5.

D. Stability of Harmonic Solutions

As stated previously the stability of harmonics is considered under two cases. The first case in which \( v_o \) is in the neighborhood of \( 1/w, w \neq 0,1,3 \), will be discussed first as it has the same stability conditions as the preceding periodic solutions, Case SI. The values needed here are obtained from (3.26a) and (3.14b) through (3.21b).

\[ X = -\frac{\pi k}{v_0} - \frac{\pi k}{v_0} = -\frac{2\pi k}{v_0}. \]

Therefore, the condition that \( X \leq 0 \) for \( \varepsilon \geq 0 \) is met when \( \varepsilon \) and \( k \) have the same algebraic sign.
\[ D = \frac{\pi^2 k^2}{v_0^2} - \pi^2 \left( -\frac{6c_0^2}{4v_0^2} + \frac{6c_0}{v_0^2} - \frac{1}{v_0^2} \right) \]

\[ = \frac{\pi^2 k^2}{v_0^2} + \pi^2 \left( \frac{3c_0^2}{2v_0^2} - \frac{2v_1}{v_0^2} \right)^2 \]

which is always positive. Therefore, in the harmonic case when \( v_0 = \frac{1}{W}, W \neq 0,1,3, \) the oscillation is stable if \( \varepsilon \) and \( k \) have the same algebraic sign.

The stability condition for the harmonics when \( v_0 = \frac{1}{W} \) is considered next. When the values from (3.14b) through (3.24b) are substituted in (3.26) the stability condition is

\[ a_{11}X - a_{12}b_{11} - a_{11}b_{12} = -\frac{2\pi k}{v_0} \leq 0 \text{ for } \varepsilon \geq 0. \]

This is true when \( \varepsilon \) and \( k \) have the same algebraic sign.

Therefore, all harmonic solutions are stable if \( \varepsilon \) and \( k \) have the same algebraic sign.
IV. SUMMARY

The Duffing equation with damping has been considered when the forcing function is composed of a harmonic term of frequency \( w \) and a higher harmonic term of frequency \( 3w \). The existence of subharmonics of order 1/3 and ultraharmonics of order 2, 5, 7, 9 was shown. The existence of other subharmonics, ultraharmonics, and ultra-subharmonics when damping is present was left as an open question since the first term, \( k_0 \), of the expansion of the damping coefficient, \( k \), in powers of \( \varepsilon \) is zero. A procedure was outlined by which the initial conditions could be determined when the other parameters in the Duffing equation are known in order to insure the existence of subharmonics of order 1/3. A similar procedure could be used to obtain the initial conditions that would insure the existence of ultraharmonics or harmonics. The existence of harmonics for the Duffing equation with damping but with only the first harmonic forcing term was shown.

Stability conditions were established for the four types of periodic solutions. The special stability conditions were worked out for harmonics, subharmonics of order 1/3, and ultraharmonics of order 2, 5, 7, 9.
V. LITERATURE CITED


VI. ACKNOWLEDGMENT

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