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Radiation patterns for slotted cylinders of arbitrary cross section

Ralph Walter Klopfenstein

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RADIATION PATTERNS FOR SLOTTED CYLINDERS
OF ARBITRARY CROSS SECTION

by

Ralph Walter Klopfenstein

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Applied Mathematics

Approved:

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1954
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I. INTRODUCTION

A. Previous Investigations

Slotted antennas formed by making cuts in closed metallic surfaces have recently become of considerable importance in applications. A complete analysis of the radiation due to such an antenna would of necessity account for the exact configuration of the energy source, and a determination of the electromagnetic fields interior and exterior to the aforesaid metallic surface would be required. In practice, the cuts or slots are usually so shaped that reasonable assumptions as to the field distribution in the slot can be made. Analytically, the problem is then reduced to the solution of the exterior and interior boundary value problem of electromagnetic theory. As a rule energy sources are restricted to the interior of the closed metallic surface, and the method of excitation varies with the particular application.

In general, it can be said that there are two problems of fundamental importance to antenna theory. The first of these is the radiation problem which deals with the spatial distribution of fields at relatively large distances from the antenna. The second of these is the impedance problem
which deals with the relation between the voltage and current at the energy source. The radiation problem will be of major interest here, but information in regard to the impedance problem will be found through a determination of the external currents flowing on the radiating cylinder.

The radiation and diffraction properties of infinite cylinders has been a subject of continued and growing interest since World War II. A search of the literature revealed nineteen publications in this specific area, of which all but two carried dates within the last eight years.

A large portion of the work on this topic has dealt with slots in circular cylinders. Sinclair (13) determined the external field due to a narrow axial slot of infinite length in an infinite circular cylinder and presented computational results for the range of parameters of principal interest. Papas and King (8) continued the study and obtained additional results pertaining to the current distribution on the cylinder. They also presented more computational results.

Silvers and Saunders (10,11) considered a slot of arbitrary configuration in a circular cylinder with an arbitrary distribution of field across this slot. In particular, they have shown that the radiation pattern in the horizontal plane due to a narrow vertical axial slot of finite length with arbitrary voltage distribution along
its length is identical with that due to an infinite axial slot with uniform voltage distribution along its length. Haycock and Wiley (4) applied the results of Silvers and Saunders to obtain the radiation patterns and radiation conductances for several types of slotted circular cylinders.

Taylor (16) dealt with the problem of synthesis of horizontal radiation patterns due to arrays of axial slots on a circular cylinder.

All of the above work has been based on an expansion of the fields external to the circular cylinder in complete sets of circular wave functions. Special convergence difficulties arise in this type of solution when the cylinder radius is large in terms of wavelengths. This has led Bailin (1) to carry out computations on automatic digital computing machinery for several cases of cylinders of large radii. In addition, Bain (2) has presented a rather extensive experimental investigation of the radiation properties of circular cylinders of large radii.

Of more general interest than the slotted circular cylinder is the slotted elliptic cylinder. Until recently work in this connection has been hampered by the lack of suitable tables. The recent publication in 1951 of new tables of the elliptic wave functions (33) by the Computation Laboratory of the National Applied Mathematics Laboratories has stimulated interest in this topic, however.
The initial work in regard to radiating elliptic cylinders was apparently done by Carter, Martin, and Thorne (3) who treated the case of an infinite axial slot in an infinite elliptic cylinder. They derived the field expressions and presented some computational results. In addition, experimental results for a rather wide range of ellipse parameters were given.

Recently Wong (19) has obtained computational results in regard to the radiation conductance of axial and transverse slots in cylinders of elliptic cross section, and Rhodes (9) has considered the case of an infinite axial slot in a circular or elliptic cylinder excited by a travelling wave.

Taylor (17) has generalized the work of Silvers and Saunders (10,11) previously mentioned and obtained far field expressions for the radiation due to an arbitrary field distribution in an arbitrarily shaped slot on an infinite elliptic cylinder.

All of the work in regard to circular and elliptic cylinders suffers from rather serious convergence difficulties when the cylinder dimensions become large compared to wavelength. In this regard, Lax and Feshbach (6) have presented a very illuminating paper dealing with the two dimensional acoustical radiation problem for cylinders of arbitrary cross section. Although couched in the language of
acoustics, their work has its exact counterpart in problems of electromagnetic theory. They have obtained far field expansions which converge rapidly when the cylinder dimensions and radius of curvature are large compared to wavelength, and when the prescribed pressure or velocity distribution is relatively slow varying.

B. Statement of Problem and Brief Summary

The problem which is the subject of this investigation is concerned with a cylinder of infinite extent, the generators of which are parallel to the Z axis of a rectangular coordinate system. The cylinder is specified by a curve \( \sigma \) in the XY plane.

The curve \( \sigma \) is closed, has no multiple points, and has a continuous curvature. There exists a positive integer \( m \) such that every line parallel to the X or Y axis meets the curve in at most \( m \) points (31, pp. 85-86).

In connection with slotted antennas, the cylinder is considered to have an infinite aperture or slot parallel to the Z axis and of constant width as illustrated in Figures 1 and 2. The cylinder is assumed to be perfectly conducting, and, hence, the tangential electric field must be zero everywhere on the cylinder except in the slot. The slot is assumed to be excited in such a fashion that the electric field vector in the slot is independent of \( z \) in
Figure 1. Perspective Sketch of Slotted Cylindrical Antenna with an Infinite Axial Slot
Figure 2. Cross Sectional View of Slotted Cylindrical Antenna with an Infinite Axial Slot
both magnitude and phase, is parallel to the XY plane, and is tangential to the cylinder extended through the slot. It is further assumed that there are no energy sources in the infinite space exterior to the cylinder.

The electromagnetic field quantities which satisfy Maxwell's equations at every point external to the cylinder, which take on prescribed values on the cylinder, and which satisfy a radiation condition at infinity are to be determined.

In Chapter II a number of things related to the problem are discussed. In Chapter II.A the slotted antenna problem as described in the last paragraphs is reduced to a two dimensional boundary value problem in a single scalar quantity which satisfies the wave equation. The uniqueness of the solution to this two dimensional problem is established in Chapter II.B. Also an integral equation which the solution to this problem satisfies is derived in Chapter II.C, and an existence theorem can then be formulated from the theory of integral equations. It is shown in Chapter II.D that the solution of the slotted antenna problem is very closely related to a corresponding diffraction problem. In fact, the far field radiation characteristics of a slotted antenna can be obtained directly if the plane wave diffraction characteristics of the cylinder are completely known.
In order to determine current distributions and radiation patterns for particular types of radiating cylinders, it is necessary to solve the integral equation derived in Chapter II. It is ordinarily difficult to obtain such solutions in closed form, but approximate solutions are readily found. In Chapter III.A a number of topics of general interest connected with the problem are discussed while in Chapter III.B procedures for obtaining approximate solutions to the integral equation are reviewed with particular reference to the slotted antenna problem.

Specific applications to several types of slotted cylindrical antennas are described in Chapter IV. The main purpose of the work presented here is to illustrate the validity of the integral equations approach to the slotted antenna problem through direct application. Calculations have been carried out for circular and elliptic cylinders in Chapter IV.A and Chapter IV.B and the results have been compared to those obtained by known methods (3,13). An evaluation of various approximation methods is thus obtained. In Chapter IV.C the integral equation method has been applied to a slotted cylindrical antenna of square cross section. In addition, experimental radiation patterns have been obtained for the elliptic and square cases for comparison with the calculated radiation patterns. The integral equation has been solved explicitly for the circular case in Chapter
IV.A, and the result obtained agrees with that found through expansion in circular wave functions (8,13).

It might be noted at this point that the multiple slot case represents a trivial extension of the results for a single slot since the radiation characteristics of a multiple slot system can be obtained by the superposition of single slot solutions. This results from the fact that both the differential equation and the condition at infinity are linear and homogeneous.

C. Definitions of Symbols

Throughout this dissertation the time dependence of the field vectors has been represented by a factor $e^{-i\omega t}$; e.g.,

$$E(x,y,z,t) = \mathcal{R} E(x,y,z)e^{-i\omega t}. \quad (1.1)$$

As is conventional, the factor $e^{-i\omega t}$ as well as the symbol $\mathcal{R}$, which means "the real part of", has been suppressed in all of the equations which appear.

The symbolization which has been used is intended to conform to convention in the field of electromagnetic theory insofar as possible. All but the most commonplace symbols are defined as they are introduced in the text. A small amount of duplication in the symbolization exists, but this occurs only in widely removed topics and should lead
to no confusion. It was felt that this small amount of duplication was preferable to a large departure from the conventional.

A partial list of symbols is given below for convenience in reference:

\( i \) is the imaginary unit and satisfies the equation \( i^2 = -1 \).

\( t, \tau \) represent general values of the parameters in parametric expressions for closed curves. An exception is the use of \( t \) as the time variable on page 10.

\( \omega \) is the angular velocity in radians per second of a periodic function of time.

\( \mu \) is the magnetic permeability of an isotropic homogeneous medium.

\( \varepsilon \) is the dielectric permittivity of an isotropic homogeneous medium.

\( * \) denotes the complex conjugate of a given quantity, viz., \( w^* \).

\( k \) is defined by the expression \( k^2 = \omega^2 \mu \varepsilon \), or, alternatively, by \( k = 2\pi / \lambda \).

\( H_n^{(1)}(kr), H_n(kr), H_n \) are used interchangeably to represent the first kind of Hankel function of the nth order. Primes denote differentiation with respect to the argument.
$J_n(kr), J_n$ are used interchangeably to represent the first kind of Bessel function of the nth order. Primes again denote differentiation with respect to the argument.

$K(t,\tau)$ is used to signify the kernel of a Fredholm type integral equation of the second kind.

$v(P,Q)$ denotes a unit source function and is equal to $(1/4i)H_0^{(1)}(kr_{PQ})$ where $r_{PQ}$ is the distance between points $P$ and $Q$.

$U$ is a function which is proportional to the specified tangential electric field. It is defined in equations (2.12) and (2.14).

$f(t)$ denotes the inhomogeneous term in a Fredholm integral equation of the second kind. It is first used in equation (2.45).

$F(t)$ is a function used in connection with the geometry of the ellipse. It is defined in equation (4.40).
II. GENERAL THEORY

A. Formulation of Problem

It is considered in all that follows that the medium external to the cylinder is homogeneous and isotropic with zero conductivity, that the cylinder is a perfect conductor except at the slot, and that the constant electric field in the slot is independent of the $z$ coordinate. The statement of the problem as given in Chapter I.B leads to certain conclusions in regard to the behavior of the field quantities. The field quantities depend only on the coordinates in a plane perpendicular to the axis of the cylinder, and the surface currents on the cylinder have no $z$ component.

Since the vector potential of electromagnetic theory is expressible as an integral of the surface conduction currents (25, pp. 233-235), the vector potential can be written

$$\vec{A} = \hat{r}A_r(r,\theta) + \hat{\theta}A_\theta(r,\theta), \quad (2.1)$$

where circular cylindrical coordinates have been used, and $\hat{r}$ and $\hat{\theta}$ are unit vectors in the radial and azimuthal directions respectively.
Then the magnetic induction vector

\[ \mathbf{B} = \mu \mathbf{H} = \nabla \times \mathbf{A} \]

\[ = \hat{\mathbf{z}} \left[ \frac{1}{r} \frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \phi} \right] = \hat{\mathbf{z}} \mu H_z. \quad (2.2) \]

That is to say that at all points in space external to the cylinder, the magnetic field intensity vector has no component other than the \( z \) component.

With the time dependence which has been taken, Maxwell's equations for the space external to the cylinder can be written

\[ \nabla \cdot \mathbf{E} = 0, \quad (2.3) \quad \nabla \cdot \mathbf{H} = 0, \quad (2.5) \]

\[ \nabla \times \mathbf{E} = i \omega \mu \mathbf{H}, \quad (2.4) \quad \nabla \times \mathbf{H} = -i \omega \varepsilon \mathbf{E}. \quad (2.6) \]

Therefore, it follows from equations (2.2) and (2.6) that

\[ \mathbf{E} = \frac{1}{\omega \varepsilon} \left[ \hat{\mathbf{r}} \frac{\partial H_z}{\partial r} - \hat{\phi} \frac{\partial H_z}{\partial \phi} \right]. \quad (2.7) \]

Furthermore, it follows from equations (2.3)-(2.6) that \( H_z \) must satisfy the two dimensional scalar wave equation

\[ \nabla^2 H_z + k^2 H_z = 0, \quad (2.8) \]

where \( k^2 = \omega^2 \mu \varepsilon \).

It is required that on the cylinder the tangential component of \( \mathbf{E} \) take on prescribed values. If the curve \( \sigma \) is described by the equation...
the outward directed unit normal vector is

\[
\hat{n} = \frac{\nabla [r - g(\theta)]}{|\nabla [r - g(\theta)]|}, \quad (2.10)
\]

Then it follows that

\[
\hat{k}E_t = \hat{n}xE \\
= \frac{1}{\omega \varepsilon} \left[ \begin{array}{c}
-r \frac{\partial H_z}{\partial r} + \frac{r}{r} \frac{\partial H_z}{\partial \theta} \\
\sqrt{r^2 + r'^2}
\end{array} \right], \quad (2.11)
\]

but since

\[
\frac{\partial H_z}{\partial n} = \hat{n} \cdot \nabla H_z,
\]
equation (2.11) can be expressed as

\[
E_t = \frac{-1}{\omega \varepsilon} \frac{\partial H_z}{\partial n}. \quad (2.12)
\]

In summary, the problem of the slotted infinite cylinder has been reduced to the following boundary value problem. A function, \( u = H_z \), of two independent variables is sought which satisfies

\[
\nabla^2 u + k^2 u = 0, \quad (2.13)
\]
external to a specified closed curve $\sigma$, and whose normal derivative takes on prescribed values on the curve $\sigma$, that is,

$$\frac{\partial u}{\partial n} = U,$$  \hfill (2.14)

where $U$ is a known function.

In addition, the function $u$ is to satisfy the radiation condition (2.15, p. 193)

$$\lim_{r \to \infty} -\frac{1}{r} \left\{ \frac{\partial u}{\partial r} + iku \right\} = 0.$$  \hfill (2.15)

This last condition ensures that there are no sources at infinity.

B. Uniqueness Theorem

It is well known that the specification of the tangential field on a three-dimensional closed surface and the radiation condition uniquely determine the electromagnetic field at all points in space external to the surface (25, p. 486). It is believed, however, that the proof given below for the corresponding theorem for the two-dimensional problem has not been previously given. This theorem guarantees the uniqueness of the solution to the boundary value problem specified by equations (2.13), (2.14), and (2.15).

The hypotheses of the theorem are as follows:
1. \( \nabla^2 u + k^2 u = 0 \) in the exterior of some curve \( \sigma^- \), which may consist of several closed curves in the finite plane.

2. On \( \sigma^- \), \( \frac{\partial u}{\partial n} = U \), a prescribed function.

3. \( u \) has continuous second derivatives external to \( \sigma^- \).

4. \( \lim_{r \to \infty} \left\{ \frac{\partial u}{\partial n} - iku \right\} = 0 \), where \( r \) is the distance from any fixed point in the finite plane.

If \( u \) satisfying these four conditions exists, it is unique.

The terminology to be used in the proof is illustrated in Figure 3. Suppose that \( u_1 \) and \( u_2 \) satisfying the conditions of the theorem exist. Set

\[
\begin{equation}
    w = u_1 - u_2. 
\end{equation}
\]

Then the complex conjugate of \( w \),

\[
\begin{equation}
    w^* = u_1^* - u_2^*. 
\end{equation}
\]

The curve \( \Sigma \) is a circle which completely encloses \( \sigma^- \). If the curve \( \sigma = \Sigma + \sigma^- \), it is found by Green's theorem that

\[
\begin{equation}
    \int_{\Sigma} \left[ w \nabla^2 w^* - w^* \nabla^2 w \right] ds = \int_{\Sigma} \left[ w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right] ds \quad (2.16)
\end{equation}
\]
Figure 3. Sketch Illustrating the Terminology Used in Connection with the Uniqueness Theorem Proved in Chapter II.B
Due to hypothesis (1), the integrand of the left hand side of equation (2.18) vanishes identically in $S$. Thus, equation (2.18) becomes

$$\int_S \left[ w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right] \, ds + \int_S \left[ w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right] \, ds = 0. \quad (2.19)$$

Due to hypothesis (2) and the definition of $w$, the integrand of the first integral on the left of equation (2.19) vanishes identically, and, thus,

$$\int_S \left[ w \frac{\partial w^*}{\partial r} - w^* \frac{\partial w}{\partial r} \right] \, r \, ds = 0. \quad (2.20)$$

By hypothesis (3), $w$ and $\frac{\partial w}{\partial r}$ can be represented by Fourier series which converge absolutely and uniformly (27, p. 83). Thus,

$$w = \sum_{n=0}^{\infty} A_n(r) e^{i \omega}, \quad (2.21)$$

and

$$\frac{\partial w}{\partial r} = \sum_{n=0}^{\infty} B_n(r) e^{i \omega}. \quad (2.22)$$

Due to hypotheses (1) and (4), (25, p. 360),

$$A_n(r) = c_n H_n^1(kr), \quad (2.23)$$
and,

\[ B_n(r) = k \alpha H_n^{(1)'}(kr). \quad (2.24) \]

Due to the absolute convergence of series (2.21) and (2.22) their terms and terms in their product can be re-arranged without changing their convergent value. Thus,

\[ w \frac{\partial w^*}{\partial r} - w^* \frac{\partial w}{\partial r} = k \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* e^{i(n-m)e} \left[ H_n(1) H_m(2)' - H_n(1)' H_m(2) \right]. \quad (2.25) \]

The convergence of the series (2.25) is uniform with respect to \( e \) so that it may be integrated term by term. Upon integrating over \( \sum_\Sigma \) all terms vanish except those for which \( n = m \), and

\[ \int \left[ w \frac{\partial w^*}{\partial r} - w^* \frac{\partial w}{\partial r} \right] ds = k \sum_{n=-\infty}^{\infty} c_n c_m^* \left[ H_n(1) H_m(2)' - H_n(2)' H_m(1) \right] r \int_{-\pi}^{\pi} ds. \quad (2.26) \]

The expression in the brackets on the right hand side of equation (2.26) is the Wronskian for Bessel's differential equation, and it is equal to \(-4i/kr\) for every \( n \) and \( r \) (24, p. 321). Therefore,

\[ \int \left[ w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right] ds = -8i \sum_{n=-\infty}^{\infty} c_n c_m^*. \quad (2.27) \]
Hence, from equation (2.20)

\[ \sum_{n=0}^{\infty} c_n \alpha_n^* = 0, \quad (2.28) \]

and, therefore,

\[ c_n = 0, \quad (2.29) \]

for every \( n \).

Thus, \( w = 0 \) on \( \Sigma \), which has arbitrary radius. Since the solution to a boundary value problem described by an elliptic partial differential equation is analytic (24, pp. 47-48), the function \( w \) found above can be analytically continued from the largest circle completely enclosing \( \Sigma \) to every point external to \( \Sigma \). This completes the proof of the theorem.

C. Existence Theorem

Through Green's theorem, an integral equation whose solution is the solution of the slotted antenna problem can be formulated. Sternberg (14, 15) utilized the theory of integral equations to discuss the problem of the diffraction of a plane wave by a homogeneous isotropic cylinder of arbitrary cross section. The methods employed by him can be readily adapted to the slotted antenna problem. The
The integral equation developed here will be useful not only for establishing the existence of the solution, but it will also form the basis for the solution methods to be described in Chapter III.

The terminology to be used in the formulation of the integral equation is illustrated in Figure 4. The curve $\sigma$ represents the bounding curve of the infinite cylinder, the curve $\Sigma$ is a circle of arbitrarily large radius whose center is within the curve $\sigma$, while the curve $K$ is a circle of arbitrarily small radius with center at the point $P$. The point $L$ is an arbitrary point on the circle $\Sigma$, while the points $P$ and $Q$ are arbitrary points in the area enclosed between curves $\Sigma$ and $\sigma$. The area $S$ is bounded by the curve $C = \Sigma + \sigma + K$, and it is shown as the shaded area in Figure 4.

The function $u(P)$ is the solution to the slotted antenna problem and, thus, it satisfies equations (2.13), (2.14), and (2.15) which specify the problem. The function $v(P, Q)$ is defined as

$$v(P, Q) = \frac{1}{4\pi} \mathcal{H}_0^1(k r_{PQ}), \quad (2.30)$$

where $\mathcal{H}_0^1(k r_{PQ})$ is the zero order Hankel function of the first kind. Ordinarily $P$ is regarded as a fixed point while $Q$ is considered to be a variable point. The function $v(P, Q)$ satisfies equations (2.13) and (2.15) in $S$. 
Figure 4. Sketch Illustrating the Terminology Used in Connection with the Derivation of Integral Equation (2.45)
Application of Green's theorem shows that
\[
\int_S \left[ u \nabla^2 v - v \nabla^2 u \right] \, da = \int_C \left[ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right] \, ds. \quad (2.31)
\]

The left hand side of equation (2.31) vanishes since the integrand vanishes identically in \( S \). The line integral on the right can be expressed as the sum of three line integrals over the curves \( \Sigma_i \), \( \sigma^- \), and \( K \). In the limit as the radius of the circle \( K \) approaches zero, the line integral over \( K \) approaches \(-u(P)\) due to the simple logarithmic singularity of \( v(P,Q) \) as \( Q \) approaches \( P \) (24, p.49 and p.96). In the limit as the radius of the circle \( \Sigma_i \) approaches infinity the line integral over \( \Sigma \) vanishes due to the radiation condition which both \( u \) and \( v \) satisfy (10, pp. 262-263). Equation (2.31) can now be written

\[
-u(P) = \int_{\sigma^-} v(P,Q) \frac{\partial u(Q)}{\partial n} \, ds - \int_{\sigma^+} u(Q) \frac{\partial v(P,Q)}{\partial n} \, ds. \quad (2.32)
\]

It should be noted that in equation (2.32) the direction of positive normal is toward the interior of the area enclosed by \( \sigma \) (exterior of \( S \)). If the direction of the positive normal is reversed to conform with the previous usage in Chapter II.A, equation (2.32) becomes
\[ u(P) = \int_{\sigma} v(P,Q) \, ds - \int_{\sigma} u(Q) \frac{\partial v(P,Q)}{\partial n} \, ds, \]

\[ = F(P) - \int_{\sigma} u(Q) \frac{\partial v(P,Q)}{\partial n} \, ds, \tag{2.33} \]

where \( F(P) \) is a known function of position. Equation (2.33) comprises an integral equation whose solution is the solution of the slotted antenna problem.

It is to be noted that if the unknown function \( u(Q) \) is known everywhere on the cylinder, the integral equation (2.33) provides an explicit formula by which it can be calculated at all other points of space exterior to \( \sigma^- \). The solution problem thus reduces to a consideration of equation (2.33) when the point \( P \) approaches the curve \( \sigma^- \).

It will be shown in the following paragraphs that the solution of the boundary value problem described by equation (2.33) can be reduced to the solution of an integral equation which is the Fredholm type of the second kind with continuous kernel.

The situation with the point \( P \) restricted to the curve \( \sigma^- \) is illustrated in Figure 5. The kernel in equation (2.33) can be written

\[ \frac{\partial v(P,Q)}{\partial n} = \frac{\partial v(P,Q)}{\partial r_{PQ}} \frac{\partial r_{PQ}}{\partial n}. \tag{2.34} \]

Now, due to the definition of \( v(P,Q) \),
Figure 5. Sketch Illustrating the Situation when the Point P is Restricted to Lie on the Bounding Curve

Figure 6. Subdivision of the Bounding Curve into a Finite Number of Segments
\[ \frac{\partial v(P,Q)}{\partial r_{PQ}} = -\frac{k}{4\pi} H_1^{(1)}(kr_{PQ}), \]

\[ = \frac{1}{2\pi r_{PQ}} + \text{Higher order terms.} \quad (2.35) \]

In addition,

\[ \frac{\partial r_{PQ}}{\partial n} = \hat{r} \cdot \hat{n}. \quad (2.36) \]

It follows (37, p. 95) that

\[ \lim_{{P \to Q}} \frac{\partial v(P,Q)}{\partial n} = \frac{1}{4\pi R_Q}, \quad (2.37) \]

where \( R_Q \) is the radius of curvature of the curve \( \sigma \) at the point \( Q \). Therefore, the kernel of the equation (2.33) approaches a limit as \( P \) approaches \( Q \), and the kernel can be defined at \( Q \) so that it is continuous there. Hence, it is continuous everywhere on the curve \( \sigma \).

Since the function \( v(P,Q) \) has a simple logarithmic singularity as \( P \) approaches \( Q \), it follows (31, pp. 107-109) that \( F(P) \) is continuous for every continuous \( U \). To show that it is still continuous for every piecewise continuous \( U \) it is only necessary to divide the curve \( \sigma \) into a finite number of subdivisions on each of which \( U \) is continuous (See Figure 6). \( F(P) \) is then a finite sum of continuous functions, and, hence, it is still continuous.
It can be shown that both the right hand members of equation (2.33) are continuous when the point \( P \) lies in the region exterior to the curve \( \sigma^- \). However, it does not follow that they are continuous in the closed region consisting of the exterior of \( \sigma^- \) plus the curve \( \sigma^- \). In fact, the second member of the right hand side of equation (2.33) has a finite discontinuity as \( P \) attains the curve \( \sigma^- \) (31, p. 96) although \( F(P) \) remains continuous as \( P \) attains the curve \( \sigma^- \) (31, p. 109).

In what follows the second member on the right hand side of equation (2.33) will be denoted by \( w(P) \). \( w_e(P) \) denotes this function evaluated for a point \( P \) external to \( \sigma^- \), \( w_e(P_0) \) denotes the limit of \( w_e(P) \) as the point \( P \) approaches a point \( P_0 \) on the curve \( \sigma^- \), and \( w(P_0) \) denotes the function evaluated for a point \( P_0 \) on the curve \( \sigma^- \). A similar notation is used for \( F(P) \) and \( u(P) \). It has been shown (31, p. 96) that

\[
w_e(P_0) = w(P_0) - u(P_0), \tag{2.38}
\]

and (31, p. 109) that

\[
F_e(P_0) = F(P_0). \tag{2.39}
\]

Now for a point \( P \) external to the curve \( \sigma^- \) equation (2.33) can be written

\[
u_e(P) = F_e(P) - w_e(P), \tag{2.40}
\]
and as the point \( P \) approaches a point \( P_0 \) on the curve \( \sigma \),

\[
    u_0(P_0) = F_0(P_0) - w_0(P_0),
\]

(2.41)

which by equations (2.38) and (2.39) becomes

\[
    u_0(P_0) = F(P_0) - w(P_0) + u(P_0).
\]

(2.42)

It is required that \( u_0(P_0) = u(P_0) \), and when this requirement is introduced into equation (2.42) it becomes

\[
    u(P_0) = 2F(P_0) - 2w(P_0).
\]

(2.43)

Equation (2.43) is an integral equation whose solution yields the values of \( u(P) \) on the curve \( \sigma \).

If the parametric equations,

\[
    x = x(t),
    
    y = y(t),
    
    a \leq t \leq b,
\]

(2.44)

for the curve \( \sigma \) are introduced into the integral equation (2.43) it can be written

\[
    u(t) = f(t) + \int_a^b K(t, \tau) u(\tau) d\tau,
\]

(2.45)

where \( f(t) \) is continuous, and \( K(t, \tau) \) is continuous. This integral equation is the Fredholm type of the second kind.

An existence theorem will be formulated by establishing that plus one is not a characteristic constant of the integral
equation (2.45). If plus one is not a characteristic constant, then the Fredholm theory guarantees a continuous solution \( u(t) \) to equation (2.45) for every piecewise continuous prescribed function \( U \) (31, p. 41, p. 55). The continuation of this \( u(t) \) into the region external to \( \sigma \) through equation (2.33) furnishes the unique continuous solution to the boundary value problem.

For the purpose of establishing that plus one is not a characteristic constant, the homogeneous equation associated with equation (2.45) is considered, viz.,

\[
 u(t) = \int_{a}^{b} K(t, \tau) u(\tau) d\tau. \tag{2.46}
\]

This corresponds to setting the prescribed function \( U \equiv 0 \). Now if a solution \( u(t) \not\equiv 0 \) exists for this equation, it must have the property that when continued to the region external to \( \sigma \) through equation (2.33) it yields a function which is identically zero there. This is known from the uniqueness theorem proved in Chapter II.B.

But since \( u(P) \) is continuous as any point in \( S \) approaches a point \( P_0 \) on the curve, it follows that \( u(P_0) \equiv 0 \), and, hence, \( u(t) \equiv 0 \). Therefore, plus one is not a characteristic constant of the integral equation (2.45).

Q. E. D.

It is important to note that this result does not
deny the existence of non-radiating current distributions on the cylinder specified by the curve \( \sigma \). It simply says that such current distributions are not solutions of the integral equation (2.45). In fact, an integral equation to obtain such non-radiating current distributions could be derived in much the same manner as equation (2.45) was derived. If on page 29 \( u_0(p_0) \) was set equal to zero, the integral equation

\[
u(t) = -f(t) - \int_a^b K(t,\tau)u(\tau)d\tau,
\]

would have been obtained. This equation would yield only non-radiating current distributions and no others.

Before leaving this subject, it should be pointed out that the reference quoted in regard to the Fredholm theory of integral equations (31) treats only the case of real functions of real variables. But this boundary value problem involves complex functions of a real variable. The validity of the theorems quoted (31, p.41, p.55) for the complex case rest solely on the validity of Hadamard's Lemma (31, p. 28) for the case of complex elements, however, and Hadamard's original proof, as well as several others, was for the complex case (28, pp. 116-123). Thus, the existence theorem given above is not limited by this consideration.
D. Relation of Diffraction Problem to Radiation Problem

Through Green's theorem the solution of the slotted antenna problem specified by equations (2.13), (2.14), and (2.15) can be related directly to the solution of the corresponding diffraction problem. In fact, as will be indicated below, the solution of the corresponding diffraction problem is exactly the Green's function for the slotted antenna problem.

The terminology to be used here is illustrated in Figure 7. The function $u$ is the solution of the slotted antenna problem which satisfies equations (2.13), (2.14), and (2.15). The function $G$, on the other hand, satisfies equation (2.13) everywhere in $S$ as well as the radiation condition given by equation (2.15). On the curve $\sigma^-$, however,

$$\frac{\partial G}{\partial n} = 0. \tag{2.48}$$

In addition, $G$ has a unit source at the point $(r_0, \theta_0)$. A suitable function is

$$G = \frac{1}{4\pi} H_0^{(1)}(kR) + F(r, \theta), \tag{2.49}$$

where $F$ has no singularity at $(r_0, \theta_0)$, satisfies the wave equation and radiation condition, and is such that the boundary condition given by equation (2.48) is satisfied.
Figure 7. Sketch Illustrating the Terminology Used in Connection with the Derivation of Equation (2.51)
Application of Green's theorem yields

\[ \int_S (u \nabla^2 \sigma - \sigma \nabla^2 u) \, da = \int_C \left( u \frac{\partial \sigma}{\partial n} - \sigma \frac{\partial u}{\partial n} \right) \, ds, \quad (2.50) \]

where \( C = \Sigma + \sigma^- + K \). As in Chapter II.C the left hand side of equation (2.50) vanishes as does the line integral over \( \Sigma \) when its radius becomes indefinitely large. As the radius of the circle \( K \) approaches zero, the line integral over \( K \) approaches \(-u(P)\), and equation (2.50) can be written

\[ u(P) = - \int_{\sigma^-} \sigma \frac{\partial u}{\partial n} \, ds, \]

\[ = \int_{\sigma^-} \sigma U \, ds, \quad (2.51) \]

where the change of sign is due to the assumed positive direction of the normal in equation (2.14).

Equation (2.51) gives a direct formula by which the solution to the slotted antenna problem can be calculated when the solution to the corresponding diffraction problem is known. In fact, if the slot is extremely narrow, \( U \) may be assumed to be an impulse function, and equation (2.51) becomes a direct equality with no integration being required. This equation (2.51) is, of course, simply a statement of the reciprocity principle for the problem.
under consideration. That is, if a unit source on the cylinder gives rise to a magnetic field \( u(P) \) at the point P, then a unit source at the point P will cause a magnetic field (current flow) of \( u(P) \) to appear on the cylinder.

For most cross sections, the determination of the solution to the diffraction problem will be as difficult as a direct solution of the slotted antenna problem. It should be noted, however, that a consideration of plane wave diffraction, which corresponds to P receding to infinity, leads to a direct determination of the far field radiation pattern without an intermediate determination of the near field. This may yield an advantage in some cases.
III. SOLUTIONS SUITABLE FOR NUMERICAL COMPUTATIONS

A. General Considerations

The considerations of Chapter II were all of the nature of general theoretical considerations. The problem was formulated, and the existence and uniqueness of its solution was established. The formulas developed in Chapter II, however, are not suited to a direct numerical computation of the solution to the boundary value problem for most cross sections. It is true that a formally exact analytical solution could be sought through a determination of the eigenfunctions and eigenvalues for the integral equation (2.45). It is not likely, however, that this would be an economical procedure except in the case of the simplest cross sections, such as the circle for which case the solution is well known (13). It is the purpose of this chapter to discuss numerical methods for the approximate solution of the problem which will be applicable to a wide class of cross sections.

There are two quantities of primary interest in connection with the slotted antenna problem. The first of these is the current distribution on the cylinder while
the second is the far field radiation pattern. From the current distribution the input impedance of the antenna can be found, and in the case of a multiple slot antenna the mutual impedance of the various slots can be determined. On the other hand, the far field radiation pattern is of direct interest to the antenna designer as well.

The current distribution on the antenna can be obtained through a solution of the integral equation (2.45). There are a variety of approximate methods available for solutions to Fredholm type integral equations (30, pp. 444-461). The choice of an approximation method, in general, depends upon the characteristics of the problem under consideration. Some methods appropriate to the slotted antenna problem will be discussed in the second section of this chapter, and specific numerical applications will be given in Chapter IV.

Once the current distribution on the cylinder has been determined the fields at all external points in space may be obtained through equation (2.33). In particular, the far field radiation characteristics of the antenna can be found from this equation. For this purpose it will be convenient to replace \( v(P,Q) \) and its normal derivative by their asymptotic expansions (24, p. 100):

\[
v(P,Q) = A_{P} \, \frac{1}{r} \, \frac{1}{k} \, \frac{r}{P} \, Q,\]

and \[ \frac{\partial^2 v(P,Q)}{\partial n} = -ikAe^{-\alpha P}e^{i\kappa PQ} (\hat{n} \cdot \hat{u}), \tag{3.1} \]

where \( A \) is a constant, and \( \hat{u} \) is a unit vector directed from the point \( Q \) on the curve \( \sigma \) to the far field point \( P \). For approximation purposes the factor \( r_{PQ}^{-\frac{1}{2}} \) can be considered essentially constant over the curve \( \sigma \), and equation (2.33) becomes

\[ u(P) = A \int_{\sigma} e^{i\kappa PQ} ds + ik \int_{\sigma} e^{i\kappa PQ} (\hat{n} \cdot \hat{u}) ds, \tag{3.2} \]

where \( r_{PQ} \) denotes a mean value for \( r_{PQ} \).

For a single narrow slot located at a point \( Q_0 \) on the curve \( \sigma \), the function \( U \) may be taken as a unit impulse function, and equation (3.2) then becomes

\[ u(P) = A e^{i\kappa PQ_0} + ik \int_{\sigma} u(Q) e^{i\kappa PQ} (\hat{n} \cdot \hat{u}) ds \tag{3.3} \]

It is interesting to note in equations (3.2) and (3.3) that the first term represents an integration of the tangential electric field while the second term represents an integration of the surface currents on the cylinder. In most instances, the integrands in these equations will not be such that an integration in closed form can be
obtained so that the integrations will necessarily be performed by numerical methods.

In the event that only the far field radiation characteristics of the slotted antenna are needed, an alternative which exists to the procedure outlined above should be mentioned. As indicated in Chapter II.D the plane wave diffraction problem for a given cylinder cross section is intimately related to the slotted antenna problem. Equation (2.51) shows that in order to find the magnetic field at a far field point P due to a narrow slot at a point \( Q_0 \) on the curve \( \sigma \) one may evaluate the current at the point \( Q_0 \) due to a plane wave incident on the cylinder from the far field point P. The solution of the diffraction problem involves an integral equation quite similar to the integral equation (2.45) for the slotted antenna problem. In fact, the integral equation for the diffraction problem is identical with equation (2.45) except that \( f(t) \) is replaced by twice the value of the incident field at the curve \( \sigma \). This integral equation may be derived by methods similar to those employed in Chapter II.

Although leading directly to a determination of the far field, the procedure discussed in the previous paragraph has several drawbacks. In the first place, it is necessary to solve a separate integral equation for each direction in which it is desired to determine the far field. Secondly,
of course, no knowledge of the current on the radiating cylinder is obtained by this method. A very limited amount of numerical work seems to indicate that a higher degree of approximation is required in solutions of the integral equation for the diffraction problem than for that of the corresponding radiation problem.

In the applications to be given in Chapter IV, the integral equation (2.45) will be solved approximately to obtain current distributions, and the far field radiation characteristics will be found through numerical integration of equation (3.3).

B. Approximate Solutions for Fredholm Integral Equations

The linear integral equation (2.45) whose solution is sought here is a Fredholm type integral equation of the second kind, viz.,

\[ u(t) = f(t) + \int_{a}^{b} K(t, \tau)u(\tau)d\tau. \quad (3.4) \]

A distinguishing feature of the slotted antenna problem is that \( f(t) \) is a rapidly varying function for values of \( t \) which correspond to points near the slot location. Although \( f(t) \) is continuous for every piecewise continuous specified function \( U \) as indicated in Chapter II.6, it has a logarithmic singularity at the slot location in the event
that U is a unit impulse function which corresponds to the case of a vanishingly narrow slot. Since f(t) is equal to u(t) operated on by a bounded operator, it can be immediately inferred that the solution u(t) will also have a logarithmic singularity in the event that U is a unit impulse function. For narrow slots, in which case U is a narrow rectangular function, f(t) and u(t) remain continuous but nearly have a logarithmic singularity in the vicinity of the slot location. This feature of f(t) and u(t) limits the applicability of the various methods available for obtaining approximate solutions to Fredholm type integral equations.

Probably the most obvious method of obtaining approximate solutions to Fredholm type integral equations is that which depends on replacing the integral by an approximating sum through some rule of numerical integration such as Weddle's rule. A concise description of this method is given by Hildebrand (30, pp. 444-448). This procedure leads ultimately to a system of n linear algebraic equations for the functional values of u(t) at n points. It is sometimes considered that a disadvantage of this method is that the data obtained is in the form of numerical values for the unknown function at specified points rather than being in the form of an approximating expression which can be evaluated at any point desired. This does not appear to be a serious disadvantage in the present instance as these same
numerical values for \( u(t) \) can be used in the numerical
evaluation of the integrals of equations (3.2) and (3.3)
for the far field radiation pattern. In any event an
approximating expression could be designed from the point­
wise information in regard to the unknown function \( u(t) \).

Probably the most serious disadvantage of this method
in the case of the slotted antenna problem arises from the
unusual behaviour of \( f(t) \) and \( u(t) \). Since these functions
are nearly singular, numerical integration rules have diffi­
culty in handling the integral of equation (3.4). The end
result is that a large number of points must be used which
increases the labor necessary to obtain an approximate
solution to the problem. The applications of Chapter IV
illustrate this point.

When the kernel of the integral equation (3.4) is
 separable, i.e., expressible as a finite sum of terms each
of which involves the product of a function of \( t \) alone and
a function of \( r \) alone, the equation is exactly solvable
through the solution of a finite system of linear algebraic
equations (30, pp. 406-409). Although it is not expected
that such is the case in the present instance, it will be
possible to represent the kernel with an approximating
separable kernel which will lead to accurate values for
\( u(t) \) with a reasonable amount of labor.

When the kernel is replaced by its double Fourier
series expansion, viz.,

\[ K(t, \tau) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} A_{mn} e^{i(mt + n\tau)}, \quad (3.5) \]

it is seen that every term of the doubly infinite sum is expressed as a product of a function of \( t \) alone and a function of \( \tau \) alone. In this form, the kernel fails to be separable only because the sum is infinite instead of finite. If the integral equation is treated as if its kernel were separable, one is led formally to an infinite system of linear algebraic equations a solution of which would lead to a Fourier series expansion for the solution function \( u(t) \), viz.,

\[ u(t) = f(t) + \sum_{-\infty}^{\infty} C_m e^{imt}. \quad (3.6) \]

In this discussion, the general integration limits \((a,b)\) have been replaced by \((-\pi, \pi)\). This is no restriction since in any particular case the transformation to limits \((-\pi, \pi)\) can be accomplished through a linear change of variable.

A useful approximation procedure is obtained when the kernel is represented approximately by a finite Fourier sum of the type shown in equation \((3.5)\). The solution of a finite number of linear algebraic equations then leads to a finite Fourier sum for the solution function \( u(t) \).
The interesting feature of this approximation procedure is that the nearly singular property of $f(t)$ and $u(t)$ is not a fundamental limitation to the accuracy at a given stage of approximation. As is seen by equation (3.6) the nearly singular character of $f(t)$ is automatically reflected in $u(t)$. Thus, this approximation method is particularly suited to the slotted antenna problem.

The quantities needed to carry out the above outlined procedure are the leading Fourier coefficients for the kernel $K(t, \tau)$ and for $f(t)$. An analytical expression is available for both these quantities, but the expressions are not easily integrable. Thus, a direct integration for the Fourier coefficients is pretty much out of the question.

The leading Fourier coefficients for $f(t)$ can be obtained by numerical integration. The nearly singular nature of $f(t)$ poses some special problems, however. The usual numerical integration formulas are not particularly efficient in evaluating an integral of this type since they require a rather large number of points for a rapidly varying function such as $f(t)$. Of course, the required detail increases with the order of the Fourier coefficient due to the more rapid oscillation of the integrand. Likewise, numerical processes for determining Fourier coefficients which depend on assuming all the harmonics above a given order negligible are not especially good here either, since a nearly singular
function such as \( f(t) \) has Fourier coefficients which decrease relatively slowly in magnitude initially. An economical procedure for determining the leading Fourier coefficients of \( f(t) \) results if one calculates the values of \( f(t) \) at a reasonably small number of points, and then replaces \( f(t) \) by a function \( f_1(t) \) which consists of the straight line connection of the calculated points for \( f(t) \). The Fourier coefficients of \( f_1(t) \) can be evaluated readily by direct integration, and the leading Fourier coefficients of \( f_1(t) \) will coincide closely with those of \( f(t) \). The higher order coefficients of \( f_1(t) \) will be appreciably different from those of \( f(t) \), of course, since the higher order harmonics for \( f_1(t) \) will have to fill in corners in \( f_1(t) \) which do not exist in \( f(t) \).

A direct approach to the problem of obtaining the leading Fourier coefficients of \( K(t,\tau') \) would involve the numerical evaluation of double integrals. While feasible this is not particularly attractive from the point of view of the amount of labor required. A useful method is to postulate a finite double Fourier sum for \( K(t,\tau') \) and then require this postulated sum \( K_1(t,\tau') \) to pass through a number of calculated points for \( K(t,\tau') \). This will lead to a system of linear algebraic equations for the coefficients of \( K_1(t,\tau') \). Often certain symmetry properties of \( K(t,\tau') \) can be observed in advance and utilized to reduce
the amount of calculation required.

The approximation methods discussed in this chapter will be illustrated in detail in Chapter IV in connection with specific applications.
IV. APPLICATIONS

A. Circular Cylinder

While the slotted antenna problem in the case of a circular radiating cylinder has been solved completely through expansions in circular wave functions (8, 13), it will be considered here for two purposes. First it will be shown that the integral equation (2.45) can be solved explicitly for the case of the circular cylinder and that the solution so obtained agrees with that previously found. Secondly, numerical results will be obtained for a particular case of the circular cylinder by a number of approximation methods applied directly to the integral equation. An evaluation of the various approximation methods can then be made through a comparison with results calculated from the known direct solution.

A circular cylinder specified by the parametric equations

\[ x = a \cos t , \]
\[ y = a \sin t , \]  

(4.1)

is considered. The parameter \( t \) is the polar angle of a line segment connecting the origin with a point on the circle. The terminology to be used is illustrated in Figure 5.
Figure 8. Sketch Illustrating the Terminology Used in Connection with the Integral Equation Solution for a Slotted Circular Cylinder

\[ R = 2a \sin \frac{1}{2} |t - \tau| \]
\[ \hat{n} \cdot \hat{u} = \sin \frac{1}{2} |t - \tau| \]
\[ x = a \cos t \]
\[ y = a \sin t \]
\[ \psi = \frac{1}{2} [\pi - |t - \tau|] \]
Recalling the definition of \( v(P,Q) \) given in equation (2.30), the integral equation for the circular cylinder case can be written

\[
\left(\frac{21}{ka}\right)u(t) = \frac{1}{k} \int_{-\pi}^{\pi} H_0(kR) Ud\tau + \frac{1}{\pi a} \int_{-\pi}^{\pi} \frac{RH_1(kR)u(\tau)}{R} d\tau, \quad (4.2)
\]

where

\[
R = 2\sin\frac{\pi}{2} |t-\tau| . \quad (4.3)
\]

It is understood here and throughout this section that the first type of Hankel function is intended where the superscript is omitted.

The integral equation (4.2) can be simplified through the use of the addition theorem for Hankel functions (25, p. 374). Thus,

\[
H_0(kR) = \sum_{-\infty}^{\infty} J_m(ka)H_m(ka) \sin(t-\tau), \quad (4.4)
\]

and

\[
RH_1(kR) = -\frac{d}{dk} H_0(kR) \]

\[
= -a \sum_{-\infty}^{\infty} \left[ J_m^1(ka)H_m(ka) + J_m(ka)H_m^1(ka) \right] \sin(t-\tau) . \quad (4.5)
\]

With these substitutions the integral equation (4.2) becomes
From an identity involving the Wronskian associated with Bessel functions (24, p. 321) it is found that

\[
\frac{2i}{ka} u(t) = \sum_{-\infty}^{\infty} \frac{1}{k} J_m(ka) H_m(ka) e^{imt} \int_{-\pi}^{\pi} u(\tau) e^{-im\tau} d\tau
\]

\[
- \frac{1}{\pi} \left[ J_m^\prime(ka) H_m(ka) + J_m(ka) H_m^\prime(ka) \right] e^{imt} \int_{-\pi}^{\pi} u(\tau) e^{-im\tau} d\tau.
\]

(4.6)

Upon inserting this on the left hand side of equation (4.6) it can be written

\[
\pi \left[ J_m H_m^\prime - J_m^\prime H_m \right] u(t) = 2\pi \sum_{-\infty}^{\infty} e^{imt} \left\{ J_m H_m A_m / k \right. \\
- \frac{1}{\pi} \left[ J_m H_m^\prime + J_m^\prime H_m \right] B_m \}, \quad (4.8)
\]

where

\[
A_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\tau) e^{-im\tau} d\tau,
\]

and

\[
B_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\tau) e^{-im\tau} d\tau. \quad (4.9)
\]

In equation (4.8) the arguments of the Bessel functions, \(ka\), have been eliminated in the interest of brevity.
Now in equation (4.3) the $A_m$ are known, being the Fourier coefficients of the specified function $U$. The $B_m$ are to be determined, and they will completely specify the solution since they are the Fourier coefficients of the unknown function $u(t)$. If both sides of equation (4.3) are multiplied by $e^{-im\omega}$ and integrated from $-\pi$ to $\pi$, it is found that

$$\left[J_{mH_m} - J_{mH_m}'\right] B_m = \left(\frac{2}{k}\right) J_{mH_m} A_m - \left[J_{mH_m} - J_{mH_m}'\right] B_m,$$

and, therefore,

$$B_m = \frac{H_m(ka)}{kH_m'(ka)} A_m.$$  \hspace{1cm} (4.10)

Hence,

$$u(t) = \sum_{-\infty}^{\infty} \frac{H_m(1)(ka)}{kH_m'(1)(ka)} A_m e^{im\omega},$$

where the $A_m$ are the Fourier coefficients of the specified function $U$. Equation (4.12) furnishes the desired solution for the current distribution on the circular cylinder, and it agrees with previous results (8).

The next problem is to deduce the far field radiation pattern from the current distribution which has just been found. Although the far field could be found directly in
the case of the circular cylinder as the limiting form of equation (4.12), it may be of interest to apply equation (3.2) directly as must be done for other types of cylinder cross sections.

The terminology to be used here is illustrated in Figure 9. As indicated on the figure, for points at a great distance from the circular cylinder,

\[ r_{pq} = r_0 - a\cos(t-e), \]

and

\[ n\cdot\hat{u} = \cos(t-e). \]  

When these substitutions are made in equation (3.2), it can be written

\[ Cu(e) = \int_{-\pi}^{\pi} Ue^{-ik\cos(t-e)}\,dt \]

\[ + ik\int_{-\pi}^{\pi} u(t)e^{-ik\cos(t-e)}\cos(t-e)\,dt, \]  

where

\[ C = \left[ Ar_0^{-\frac{1}{2}} e^{ikr_0a} \right]^{-1}. \]

Now if the Fourier series expansions for \( U(t) \) and \( u(t) \) are inserted in equation (4.14), it becomes

\[ Cu(e) = \sum_{m=-\infty}^{\infty} A_m \left\{ \int_{-\pi}^{\pi} e^{-ik\cos(t-e)} e^{imt}\,dt \right\} \]

\[ + \frac{1}{H_m(ka)} \left\{ \int_{-\pi}^{\pi} e^{-ik\cos(t-e)} e^{imt}\cos(t-e)\,dt \right\}, \]  

(4.16)
Figure 9. Sketch Illustrating the Terminology Used in Finding the Far Fields for a Circular Cylinder Using Equation (3.2)
where the $A_m$ are the Fourier coefficients of $U(t)$, and the Fourier coefficients of $u(t)$ are obtained from equation (4.12).

The integrals involved in equation (4.16) can be expressed in terms of Bessel functions since they are related to the integral representations of Bessel functions (24, pp. 84-92). Thus,

$$\int_{-\pi}^{\pi} e^{-ik\cos(t-e)} e^{im\tau} d\tau = e^{ime} \int_{-\pi}^{\pi} e^{-ik\cos(t-e)} e^{im\tau} d\tau,$$

which is

$$= e^{ime} \int_{-\pi}^{\pi} e^{ik\cos w} e^{im(w-\pi)} dw$$

which is

$$= 2\pi e^{-im(\frac{\pi}{2})} J_m(ka)e^{ime}.$$

The changes of variable involved in equation (4.17) are $\tau = t - e$ and $w = \tau + \pi$. Due to the periodicity of the integrands the limits need not be changed. Differentiation of the identity above with respect to $ka$ leads to

$$\int_{-\pi}^{\pi} e^{-ik\cos(t-e)} e^{im\cos(t-e)} dt$$

$$= 2\pi e^{-i(m-1)\frac{\pi}{2}} J^i_m(ka)e^{ime}.$$

When the identities of equations (4.17) and (4.18) are substituted into equation (4.16), it becomes
\[ \text{Cu}(\varepsilon) = 2\pi \sum_{-\infty}^{\infty} A_m e^{im(\varepsilon - \frac{\pi}{2})} \left\{ J_m(ka) - \frac{H_m(ka)}{H_m^{\prime}(ka)} J_m^{\prime}(ka) \right\}. \] (4.19)

When the Wronskian given in equation (4.7) is introduced into equation (4.19) it is finally found that
\[ \text{Cu}(\varepsilon) = \frac{4i}{ka} \sum_{-\infty}^{\infty} \frac{A_m}{H_m^{(1)}(ka)} e^{im(\varepsilon - \frac{\pi}{2})}. \] (4.20)

Equation (4.20) furnishes the desired expression for the far field radiation pattern of a slotted circular cylinder, and it agrees with previous results. Thus, the integral equation (2.45) is directly solvable in the case of a circular cylinder, and the solution obtained agrees with the result obtained through an expansion in circular wave functions (8, 13).

Having shown that the integral equation (4.2) is solvable in terms of circular wave functions, it is now desired to apply several approximation methods directly to the integral equation in order to obtain an evaluation of these methods for subsequent application to non-circular cylinders. The particular case of the circular cylinder to be considered is shown in Figure 10. The dimensions were chosen as reasonably typical of slotted antenna applications.
Figure 10. Slotted Circular Cylinder
Table 1

Comparison of Results Obtained by Various Methods for Current Distribution on a Circular Cylinder

Circular Cylinder: \( ka = 0.5 \),

\( \alpha = 0.05 \) radians,

\( U = k/\alpha \), \(|t| < \alpha/2\),

\( = 0 \), \( \alpha/2 < |t| \leq \pi \).

Complex values of \( u(t) \) - Real part listed above imaginary part.

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<thead>
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1. Circular wave functions. Estimated accuracy: within three units in last decimal place.
2. Kernel approximation. First order.
4. Linear equations. 13 point. Weddle's rule.
5. Linear equations. 5 point. Simpson's rule.
6. Linear equations. 3 point. Simpson's rule.
$k\alpha = 0.5$

$\alpha = 0.05 \text{ RADIANS}$

$U = k/\alpha$, $|t| \leq \alpha/2$,

$= 0$, ELSEWHERE.

Figure 11. Complex Value of Current on Circular Cylinder
MAGNITUDE OF $u(t)$

ARGUMENT OF $u(t)$

$ka = 0.5$

$\alpha = 0.05$ RADIANS

$U = k/\alpha, |t| < \alpha/2, -\pi/2$

$= 0$, ELSEWHERE.

Figure 12. Complex Value of Current on Circular Cylinder
The current distribution on the circular cylinder was calculated by six different methods, and the results of those calculations are indicated in Table 1.

The values of current given in Column 1 of Table 1 were calculated from the exact series expression given by equation (4.12). The purpose of this calculation was to obtain accurate values of current in order that they might serve as a check on the various approximate solutions to the integral equation (4.2). In the case under consideration the series given by equation (4.12) takes the form

\[ u(t) = \frac{1}{2\pi} \left( \frac{H_0(0.5)}{H_0'(0.5)} + 2 \sum_{n=1}^{\infty} \frac{H_n(0.5)}{H_n'(0.5)} \frac{\sin\left(n/40\right)}{n/40} \cos nt \right) \]

In computing values of the series (4.21), the first twenty terms of the indicated sum were calculated from available tables (36, pp. 31-40, 46-67) to four decimal accuracy and their sum computed directly.

It is observed, however, that the series (4.21) is a very slowly converging series, especially for the case t = 0. In fact, for t = 0, it is only in virtue of the factor \( \frac{\sin(n/40)}{n/40} \) that the series converges at all. This factor is present due to the non-zero slot width. In the limit as the slot width approaches zero (U a unit impulse function) this factor is replaced by unity and the series (4.21) fails to converge at t = 0. u(t) then has a
logarithmic singularity at the point \( t = 0 \).

From the foregoing comments, it is clear that it is hardly valid to assume, per se, that the remainder after twenty terms of the series (4.21) is negligible. An estimate of this remainder was obtained, therefore, and applied as a correction to the computed values of \( u(t) \). The manner in which this estimate was made will be indicated in detail for the case \( t = 0 \). For \( t = 0 \)

\[
R_{20} = \frac{1}{\pi} \sum_{n=21}^{\infty} \frac{H_n(0.5)}{H_n(0.5)} \frac{\sin(n/40)}{n/40} . \tag{4.22}
\]

For large values of \( n \), the Hankel functions can be approximated by the leading terms of their series representations (24, pp. 92-97), and it is found that

\[
\frac{H_n(ka)}{H_n'(ka)} = -\frac{ka}{n} \left\{ 1 + \frac{2}{n(n-1)} \left( \frac{ka}{2} \right)^2 + \ldots \right\} . \tag{4.23}
\]

For the case under consideration, the second term of this expansion will be less than 0.03% of the first term for all values of \( n \) exceeding twenty. Therefore, the leading term of the expansion can be used with good accuracy.

With this substitution in equation (4.22) it is found that

\[
R_{20} \approx \frac{1}{150\pi} \sum_{n=21}^{\infty} \frac{\sin(n/40)}{(n/40)^2} . \tag{4.24}
\]
Since the individual terms here are slow varying functions of n, the value of this sum can be estimated by evaluating an integral which approximates it, viz.,

\[ \sum_{n=N}^{M} f(n) \approx \int_{N-\frac{1}{2}}^{M+\frac{1}{2}} f(n) \, dn . \quad (4.25) \]

Thus,

\[ R_{20} = \frac{1}{4\pi} \int_{0.5125}^{\infty} \frac{\sin x}{x^2} \, dx , \quad (4.26) \]

where the change of variable \( x = n/40 \) has been introduced. A single integration by parts yields

\[ \int_{x}^{\infty} \frac{\sin t}{t^2} \, dt = \frac{\sin x}{x} - Ci(x) , \quad (4.27) \]

where \( Ci(x) \) denotes the cosine integral function (35, p.3). It is thus found that \( R_{20} \approx -0.1771 \), a significant correction indeed to \( u(0) \).

A similar correction for the remainder has been applied to all the calculated values of \( u(t) \) in Column 1 of Table 1. For other values of \( t \) the remainder is not nearly so large as it is for \( t = 0 \), but it remains
significant. The correction applied after 23 terms of summation for all other angles varied from -0.0033 to -0.0043.

The imaginary parts of the \( u(t) \) converged much more rapidly than the real parts and required no correction. It is felt that the remainder evaluation is accurate, and that the major source of error in the \( u(t) \) in Column 1 of Table 1 is in the round-off error in the summation of the leading terms. On that basis, it is estimated that the real parts given are accurate to within three units in the last decimal and that the imaginary parts are accurate to within one unit in the last decimal.

The numerical values for \( u(t) \) given in Columns 2 through 6 of Table 1 are the result of various approximation methods applied directly to the integral equation (4.2). For the parameters of this example, the equation takes the form

\[
4iu(t) = f(t) + \int_{-\pi}^{\pi} H_1(\sin \frac{\theta}{2} |t-\tau|) \sin \frac{\theta}{2} |t-\tau| \, d\tau,
\]

where

\[
f(t) = \frac{1}{\alpha} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H_0(\sin \frac{\theta}{2} |t-\tau|) \, d\tau.
\]

Initially attention will be focused on the problem of obtaining values for \( f(t) \). This problem is conveniently
considered in two parts, that is, the problem of deter-
mining values of \( f(t) \) for \( t \) within the integration range
and the problem of determining values of \( f(t) \) for \( t \) out-
side the integration range.

For values of \( t \) well outside the integration range,
it is seen that \( f(t) \) will take on a mean value of the
integrand. Specifically,

\[
f(t) = H_0(\sin \frac{\pi}{2}|t|),
\]

\[
|t| \gg \frac{\alpha}{2},
\]  

(4.29)

seems like a natural approximation, the accuracy of which
would improve as the slot became narrower. The accuracy
of this approximation can be investigated by expanding the
integrand of \( f(t) \) in a Taylor series in \( \tau \) about the point
\( \tau = 0 \) and integrating the resulting series. The result of
this process is

\[
f(t) = H_0(x) + \frac{\alpha^2}{24} \left\{ xH_1(x) - (1-x^2)H_1'(x) \right\} + \ldots,
\]

where

\[
x = \sin \frac{\pi}{2}|t| ,
\]  

(4.30)

In the present instance, if it is assumed that two terms of
the expression given in equation (4.30) represent \( f(t) \)
exactly, it can be shown that the approximation of equation
(4.29) is accurate to four significant figures when the
magnitude of $t$ is greater than 15°.

As $t$ approaches zero, however, the approximation of equation (4.29) breaks down. This clearly must be so since the zero order Hankel function has a logarithmic singularity at $t = 0$ while it is known that $f(t)$ is continuous there. This situation can be handled by introducing for the zero order Hankel function in equation (4.28) its small argument approximation (35, p. 132)

$$H_0(z) \approx 1 + i \left[ \frac{2}{\pi} \left( \log z - 0.11593 \right) \right].$$

(4.31)

In the present case, it is found that

$$f(0) = 1.000 - 13.5001.$$  

(4.32)

Having disposed of the problem of evaluating $f(t)$ various methods of obtaining approximate solutions to the integral equation (4.28) will now be considered. The numerical results of Columns 4, 5, and 6 in Table 1 were obtained by replacing the integral by an approximating sum (30, pp. 444–448). This method is straightforward in application and will not be discussed in detail here. It is noted, however, that since it is known in advance that $u(t)$ is an even function, the required number of linear equations to be solved are halved.

In the computations leading to the results of Columns 5 and 6 in Table 1 Simpson's rule was used to approximate
the integral, and the number of linear equations to be solved were two and three respectively. It should be remembered, however, that these are linear equations with complex coefficients and unknowns so that the equivalent number of linear equations with real coefficients and unknowns was double, or four and six respectively. In the computations leading to the results of Column 4 in Table 1 Weddle’s rule was used to approximate the integral, and it was necessary to solve seven linear equations with complex unknowns and coefficients, or equivalently fourteen equations with real unknowns and coefficients.

The method of linear equations might be judged efficient if one had available computing machinery directly programmed for the solution of linear equations of large dimension. Even in this case, the labor involved in merely setting up the equations is quite formidable. The number of equations required to show sufficient detail in $u(t)$ will increase as the cylinder dimensions increase. The degree of approximation exhibited by the results of Column 4 in Table 1 probably is between that for the results of Column 2 and Column 3 which were obtained by a much simpler method.

The kernel of the integral equation (4.28) has several striking symmetry properties. In the first place, the kernel is symmetric in the usual sense, viz.,

$$K(t, \tau) = K(\tau, t) ,$$  \hspace{1cm} (4.33)
and in the second place, the kernel is a function of |t-\tau| only, viz.,

\[ K(t, \tau) = K(|t-\tau|). \tag{4.34} \]

The properties (4.33) and (4.34) are common to all cylinders of circular cross section. These properties are illustrated in Figure 13.

The numerical results of Columns 2 and 3 in Table 1 were obtained by introducing approximate separable kernels in equation (4.28) in place of the exact kernel. The choice of approximating kernels is considerably restricted by the symmetry properties (4.33) and (4.34).

The results of Column 3 were obtained by introducing the approximate kernel

\[ K_0(t, \tau) = K(0, \pi/2), \]

\[ = 0.23470 - 10.77321. \tag{4.35} \]

The corresponding expression obtained for u(t) was

\[ u(t) = \frac{1}{4\iota} \left[ f(t) - 0.5309 + 10.2199 \right]. \tag{4.36} \]

This relatively crude approximation for the kernel yields amazingly accurate results as may be seen by comparing the results of Column 3 of Table 1 with the accurate values of Column 1. The double integration for the average value of K(t, \tau) which might seem the best choice for K_0(t, \tau) has
LEGEND:

--- CONTOURS OF CONSTANT $K(t, \tau)$

--- BOUNDARY OF REGION CONSIDERED

Figure 13. Sketch Showing Behavior of Kernel of Integral Equation for Slotted Circular Cylinder Antenna
been avoided because of the computational labor required in the numerical evaluation of a double integral.

The results of Column 2 of Table 1 were obtained by introducing the approximate kernel

\[ K_1(t, \tau) = K(0, 0) + K(0, \pi) \sin \frac{1}{2}(t-\tau), \]

\[ = 0.22002 - 10.70892 \]

\[ - \left[ 0.22002 - 10.07230 \right] \cos(t-\tau). \]  

(4.37)

The corresponding expression obtained for \( u(t) \) was

\[ u(t) = \frac{1}{41} \left\{ f(t) + \left[ -0.51312 + 10.20708 \right] \right. \]

\[ + \left[ 0.17070 - 10.00045 \right] \cos t \} \] .  

(4.38)

This kernel approximation takes on the exact values of \( K(t, \tau) \) on the lines \( t-\tau = n\pi \) and approximates it elsewhere.

The solution of integral equations with separable kernels is straightforward and requires no comment here (30, pp. 406-411). The evaluation of the integrals involving \( f(t) \) was made through numerical integration formulas after \( f(t) \) had been determined at a number of points through equation (4.29) and (4.32). It is observed that while the kernel is indeterminate at \( t = \tau \), it is readily evaluated there either directly or through equation (2.37).
The results of this section may be summarized as follows: It has been shown that the integral equation (2.45) is directly solvable in the case of a circular cylinder, and that the solution obtained both for the current distribution and far field radiation pattern agrees with those previously obtained (6, 13). A number of approximate solution methods have been directly applied to the integral equation for a particular case of the slotted circular cylinder antenna. It appears that a finite Fourier approximation to the kernel yields accurate numerical results most economically.

B. Elliptic Cylinder

The problem of the slotted elliptic cylinder has been completely handled through an expansion in elliptic wave functions (3, 9, 19). A particular case of the slotted elliptic cylinder will be considered from the integral equation point of view in this section. This is done first of all because the elliptic cylinder furnishes an additional type of cross section for which solutions by the integral equation method can be checked against other methods of solution. There is an additional purpose to this particular application, however. Even with the new tables of elliptic wave functions which have recently been made available (33), computations of radiation patterns and current distributions
through expansions in elliptic wave functions are quite tedious involving considerably more computational labor than the corresponding calculations for circular cylinders. The integral equation furnishes an alternative method which is in general more economical from the computational point of view.

The terminology to be used in connection with the integral equation development in the case of the elliptic cylinder is indicated in Figure 14. The elliptic cylinder is specified by the parametric equations

\[ x = a \cos \theta, \]

and

\[ y = b \sin \theta. \] (4.39)

The relations given in Figure 14 follow directly from the ellipse geometry and the parametric representation (4.39). It may be noted that they reduce to the corresponding relations given for a circular cylinder in Figure 8 in the special case \( b = a \).

Utilizing these relations and the definition of \( v(P,Q) \) given in equation (2.30), the integral (2.45) takes the form

\[
\left(\frac{24}{k^2}\right) u(t) = \frac{1}{\kappa} \int_{-\pi}^{\pi} H_0(kR) UF(\tau) d\tau \\
+ \frac{1}{2\kappa} \int_{-\pi}^{\pi} R H_1(kR) u(\tau) \frac{F(0)d\tau}{F'[\frac{1}{2}(t+\tau)]^2},
\]
\[ x = a \cos t \]
\[ y = b \sin t \]

\[ R = 2a \sin \left( \frac{1}{2} |t - \tau| \right) F \left( \frac{1}{2} (t + \tau) \right), \]
\[ \hat{n} \cdot \hat{\mu} = \sin \left( \frac{1}{2} |t - \tau| \right) \frac{F(0)}{F(\tau) F \left( \frac{1}{2} (t + \tau) \right)}, \]

\[ dS = a F(\tau) d\tau, \text{ WHERE} \]
\[ F(\tau) = \sqrt{1 - e^2 \cos^2 \tau}, \]
\[ e^2 = 1 - \frac{b^2}{a^2} \]

Figure 14. Sketch Illustrating the Terminology Used in Connection with the Integral Equation Solution for a Slotted Elliptic Cylinder
where \( R = 2a \sin \left[ \frac{1}{2} \| t - \tau \| \right] F \left( t + \frac{\tau}{2} \right) \),

and \( F(t) = \sqrt{1 - e^2 \cos^2 t} \), \((4.40)\)

\( e \) being the eccentricity of the ellipse. It is seen that this equation is similar to that for the circular cylinder, equation \((4.3)\), although somewhat more complicated. It reduces to that equation when \( b \) is set equal to \( a \). The kernel retains the property of symmetry, \( K(t, \tau) = K(\tau, t) \), but no longer possesses the property of being a function of \( \| t - \tau \| \) alone.

When the current distribution on the elliptic cylinder has been found through the solution of the integral equation \((4.40)\), the far field radiation pattern can be found through application of equation \((3.2)\). The terminology to be used in the application of this equation is illustrated in Figure 15. The development is quite similar to that given for the circular cylinder leading to equation \((4.14)\). In this case,

\[ r_{PQ} = r_o - r(t) \cos(\psi - \phi), \]

and

\[ \hat{n} \cdot \hat{u} = \cos(\psi - \phi), \]

where

\[ r(t) = a F(t + \frac{\tau}{2}), \]

and

\[ \tan \psi = \frac{b}{a} \tan \phi. \] \((4.41)\)
Figure 15. Sketch Illustrating the Terminology Used in Finding the Far Fields for an Elliptic Cylinder Using Equation (3.2)

\[ x = a \cos t \]
\[ y = b \sin t \]

\[ \tan \psi = \frac{b}{a} \tan t \]

\[ r^2 = r_0^2 + r^2 - 2rr_0 \cos(\psi - \Theta) \]

FOR \( r_0 \gg r \)

\[ r_{PQ} = r_0 - r \cos(\psi - \Theta) \]

\[ \hat{n} \cdot \hat{\mu} = \cos(\psi - \Theta) \]
When these substitutions are made in equation (3.2), it can be written

\[ C u(e) = \int_{-\pi}^{\pi} u e^{-i(k \cos \theta e^{\cos t} + k \sin \theta e^{\sin t})} F(t) dt \]

\[ + ik \int_{-\pi}^{\pi} u(t) e^{-i(k \cos \theta e^{\cos t} + k \sin \theta e^{\sin t})} [F(0) \cos \theta e^{\cos t} + \sin \theta e^{\sin t}] dt, \]

(4.42)

where

\[ C = \left[ A^{r_0 \frac{1}{2} e^{ikr_0 a}} \right]^{-1}. \]

(4.43)

Equations (4.40) and (4.42) constitute the complete formulation of the slotted antenna problem for the elliptic cylinder by the integral equation method. The remainder of this section will be devoted to the solution of this problem for a particular case of the elliptic cylinder.

The special case of the slotted elliptic cylinder antenna to be considered is illustrated in Figure 16. This ellipse has a major axis of one-half wavelength and a minor axis of one-quarter wavelength with a narrow slot at one extreme of the major axis. Several types of calculations were carried out for this antenna. First of all, the current distribution on the cylinder was obtained through an approximate solution of the integral equation (4.40). Then the far field radiation characteristics were found through
Figure 16. Slotted Elliptic Cylinder
equation (4.42) wherein Weddle's rule was used to carry out the integrations numerically. For comparison, an alternate computation of the far field radiation pattern was made through the known solution in terms of elliptic wave functions (3, p. 10). Finally, an experimental determination of the far field radiation pattern was obtained. While the work to be described here is for a particular case of the elliptic cylinder, the methods used are directly applicable to any slotted elliptic cylinder antenna. Table 2 is a numerical tabulation of the results obtained through the various methods while Figures 17 through 19 give a graphical presentation of these same results.

For the case under consideration the integral equation (4.40) takes the form

\[(\delta i/\pi)u(t) = f(t) + \int_{-\pi}^{\pi} K(t, \tau)u(\tau)d\tau,\]

where

\[f(t) = \frac{1}{\alpha F(0)} \int_{-\pi/2}^{\pi/2} H_0(kR)F(\tau)d\tau,\]

and

\[K(t, \tau) = \frac{1}{2\pi} \frac{R H_1(kR)}{F\left[\frac{1}{2}(t+\tau)\right]^2}. \quad (4.44)\]

In equation (4.44)

\[kR = \pi \sin\left[\frac{1}{2}|t-\tau|\right] F\left[\frac{1}{2}(t+\tau)\right],\]

and

\[F(t) = \sqrt{1 - (3/4)\cos^2 t}. \quad (4.45)\]
Table 2

Comparison of Results Obtained by Various Methods for Current Distribution and Far Field of an Elliptic Cylinder

Elliptic Cylinder: \( k a = \pi/2, \ k b = \pi/4, \)

\( \alpha = 0.140 \) radians,

\( U = k/\alpha, \ |t| < \alpha/2, \)

\( = 0, \) elsewhere.

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<th>( t )</th>
<th>( u(t) ) - Current.</th>
<th>( u(\phi) ) - Far field.</th>
</tr>
</thead>
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<td>(-0.3338 )</td>
<td>(0.9930 )</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>(-0.0359 )</td>
<td>(0.9945 )</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>(-0.0907 )</td>
<td>(0.8992 )</td>
</tr>
<tr>
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<td>(-0.0609 )</td>
<td>(0.6274 )</td>
</tr>
<tr>
<td>( 5\pi/6 )</td>
<td>(-0.0076 )</td>
<td>(0.5739 )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>(-0.0148 )</td>
<td>(0.6521 )</td>
</tr>
</tbody>
</table>

Real part above imaginary part.

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<th>( e )</th>
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</thead>
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<td>(-0.3338 )</td>
<td>(0.9930 )</td>
<td>(13.55^\circ)</td>
<td>(1.0005)</td>
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<tr>
<td>( \pi/3 )</td>
<td>(-0.0359 )</td>
<td>(0.9945 )</td>
<td>(48.00^\circ)</td>
<td>(0.975)</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>(-0.0907 )</td>
<td>(0.8992 )</td>
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<td>(142.42^\circ)</td>
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<td>(0.598)</td>
</tr>
<tr>
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<td>(227.97^\circ)</td>
<td>(0.672)</td>
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</table>

Magnitude above phase angle.

2. Elliptic wave functions. Accuracy-four figures.
3. Integration of current distribution obtained from integral equation. Weddle's rule.
4. Experimental.
MAGNITUDE OF $u(t)$

ARGUMENT OF $u(t)$

$ka = \pi/2$, $kb = \pi/4$

$\alpha = 0.140$ RADIANS

$U = k/\alpha$, $|t| < \alpha/2$, $= 0$, ELSEWHERE.

Figure 17. Complex Values of Current on Elliptic Cylinder
Figure 18. Magnitude of Normalized Far Field of Elliptic Cylinder
Figure 19. Argument of Normalized Far Field of Elliptic Cylinder

ka = $\pi/2$, \( kb = \pi/4 \)

--- INTEGRATION OF CURRENT DISTRIBUTION OBTAINED FROM INTEGRAL EQUATION, WEDDLE'S RULE.

--- CALCULATED FROM ELLIPTIC WAVE FUNCTIONS.
On the line $t = T$ the limiting form of equation (4.44) is used so that

$$K(t, t) = -\frac{21}{\pi^2 F^2(t)}. \quad (4.46)$$

As in the case of the circular cylinder, the function $f(t)$ can be accurately approximated for points distant from the slot, which is located at $t = 0$, by a mean value of the integrand, viz.,

$$f(t) = H_0(kR_0),$$

where

$$kR_0 = \pi \sin \frac{3}{2} t |F(\frac{3}{2}t)|. \quad (4.47)$$

For points near the slot location, small argument approximations can be introduced in the integrand of the equation for $f(t)$, and, in this case,

$$f(0) = 1.000 - 12.55714 . \quad (4.48)$$

The behaviour of the kernel, $K(t, T)$, is illustrated in Figure 20, and its symmetry properties are indicated there. A finite double Fourier series expression is now sought which approximates the kernel. This expression when found can be used in the approximate solution of the integral equation (4.44). The symmetry properties of the kernel restrict the types of suitable trigonometric terms, and it is postulated that
Figure 20. Sketch Showing Behavior of Kernel of Integral Equation for Slotted Elliptic Cylinder
\[ K_1(t, \tau) = A_0 + A_1(\cos t + \cos \tau) + A_2\cos(t + \tau) \]
\[ + A_3\cos(t - \tau) + A_4(\cos 2t + \cos 2\tau) \]
\[ + A_5[\cos(2t + \tau) + \cos(2\tau + t)] \]
\[ + A_6[\cos(2\tau - t) + \cos(2t - \tau)] \]
\[ + A_7\cos 2(t + \tau) + A_8\cos 2(t - \tau). \] (4.49)

When \( K_1(t, \tau) \) is equated to \( K(t, \tau) \) at the thirteen points indicated on Figure 20, a system of thirteen linear equations in the \( A_0 \) through \( A_8 \) are obtained. Of these thirteen equations, seven are independent and by a solution of them the unknowns \( A_0, A_1, A_2, A_3, A_4, A_5 + A_6, \) and \( A_7 + A_8 \) can be determined. In this case, it is found that

\[ A_0 = 0.42434 - 10.32275, \]
\[ A_1 = 0, \]
\[ A_2 = 0.23726 - 10.49984, \]
\[ A_3 = -0.32956 - 10.08488, \]
\[ A_4 = -0.11864 + 10.09794, \]
\[ A_5 + A_6 = 0, \]
and \[ A_7 + A_8 = -0.09477 - 10.09898. \] (4.50)

The kernel \( K_1(t, \tau) \) so determined approximates the exact kernel \( K(t, \tau) \) everywhere and is exactly equal to it at the points indicated on Figure 20. The solution of the system of linear algebraic equations is not nearly so difficult as one might think at first glance since due to the selection of points at which \( K_1(t, \tau) \) is equated to \( K(t, \tau) \) the
elements of the coefficient matrix are all integers whose magnitude is equal to or less than two and almost one quarter of the elements are equal to zero. Neither is it necessary to write additional equations to determine \( A_5, A_6, A_7, \) and \( A_8 \) individually since only the sums indicated appear in the solution of the integral equation.

When the approximate separable kernel \( K_1(t, \tau) \) is introduced into the integral equation (4.44) it can be solved directly, and one is led to a system of two linear algebraic equations with complex unknowns and coefficients. The expression obtained for the current distribution is

\[
u(t) = \frac{n}{8} \left\{ f(t) - \left[ 0.1777 + 10.1398 \right] \\
- \left[ 0.2614 - 10.3227 \right] \cos \tau \\
+ \left[ 0.1191 + 10.0964 \right] \cos 2\tau \right\}. \tag{4.51}
\]

This is the approximate solution for the current distribution which has been sought, and from this expression the numerical values of Column 1 in Table 2 have been tabulated and the curves of Figure 17 have been plotted.

From this current distribution the far field radiation pattern for the elliptic cylinder antenna can be found through equation (4.42). Because of the nature of the specified tangential electric field \( U \) (narrow rectangular
function) the first term of this equation can be approximated by a mean value of its integrand. For this case, then, the equation (4.42) can be written

\[ C_1 u(\theta) = e^{-i\frac{\pi}{2}} \cos \theta + \int_{-\pi}^{\pi} u(t) e^{-i\frac{\pi}{4} (2\cos \theta \cos \phi + \sin \phi \sin \theta)} \left[ \cos \phi \cos \theta + 2 \sin \phi \sin \theta \right] dt, \]

where

\[ C = k F(0) C_1. \tag{4.52} \]

From this equation, the normalized values of the far field radiation pattern tabulated in Column 3 of Table 2 were calculated. In addition, they are shown graphically in Figures 18 and 19. The integration was performed numerically through the use of Weddle's rule.

For comparison, the far field radiation pattern was also calculated by the known result in terms of elliptic wave functions (3, p. 10). Although the formula given by Carter, et al., is for the limiting case of a narrow slot, it is readily generalized to the case of non-zero slot width. The computations were performed with the aid of the recently published tables of Mathieu functions computed by the National Bureau of Standards (33). The results of these calculations are given in Column 2 of Table 2 and are illustrated graphically in Figures 18 and 19.
Finally, an experimental model of the slotted elliptic cylinder antenna analyzed here was constructed and the magnitude of the far field radiation pattern obtained by actual measurement. A sketch of the experimental setup used is shown in Figure 21. Data was recorded at ten degree intervals, and a graphical plot of this data is shown in Figure 18. Reasonable precautions were taken to ensure that an accurate measure of the far field radiation pattern was obtained. The lack of symmetry in the measured pattern, to the extent to which it exists, is a measure of the imperfections of the experimental setup. Such lack of symmetry might come from reflections from external objects, such as buildings, or it might reflect the fact that the model cylinder is not perfectly symmetrical. In any event, it is felt that the measured pattern provides good verification of the theory presented in this section.

The experimental model was constructed by cutting wooden forms from plywood. These forms were in the form of ellipses with a major axis of 7.60 inches and a minor axis of 3.80 inches. Four of these forms were joined with strips of lath as vertical stiffeners, and the resulting structure was covered with a wire mesh to form a cylinder about 41 inches in length. A slot about \( \frac{4}{1} \) inch wide by 17 inches long was then cut in the wire mesh, and this slot was fed at the center by a length of flexible coaxial cable.
Figure 21. Sketch Showing Experimental Set Up Used in Obtaining Measured Values of the Far Field Radiation Pattern
Figure 22. Experimental Model of Slotted Elliptic Cylinder Antenna
Pattern measurements were made at a frequency of 776.4 mc. A photograph of the model antenna is shown in Figure 22.

In summary, it can be said that in this section the integral equation method has been applied to the elliptic cylinder with good success. The current distribution on a particular elliptic cylinder was calculated as well as the far field radiation pattern from the integral equation formulation. The far field radiation pattern obtained from the integral equation was compared to that obtained through a solution in terms of elliptic wave functions. A fair comparison was obtained. It is believed that the small difference that does exist between results calculated by the two methods is largely due to the application of Weddle's rule to evaluate the integral in equation (4.52) for the far field radiation pattern. An experimental radiation pattern was also obtained.

C. Square Cylinder

As a final application of the integral equation method of solving the slotted antenna problem, a slotted cylindrical antenna whose cross section is square will be analyzed. This type of cross section is of special interest since the solution is not obtainable through known methods as was the case with the previous applications. Also, special problems arise in connection with the corners of the square cross
section which do not occur for cross sections having continuous curvature.

The type of cross section being considered here is illustrated in Figure 23 where the length of a side of the square is 2a. In Figure 24 the parametric representation to be used in connection with the square cylinder is shown. Although the functions \( x(t) \) and \( y(t) \) cannot be conveniently expressed in closed form, their Fourier expansions are readily found. The Fourier expansions for \( x(t) \) and \( y(t) \) are given by

\[
x(t) = \left( \frac{4}{\pi} \right)^2 a \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi}{(2n+1)^2} \cos(2n+1)t, \\
y(t) = \left( \frac{4}{\pi} \right)^2 a \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi}{(2n+1)^2} \sin(2n+1)t.
\]

These Fourier expansions are rapidly convergent. The partial sums differ from the function most noticeably at the corners of the square. The curves represented by partial sums of the given Fourier series will be characterized by progressively smaller and more rapid oscillations about the sides of the square and a crowding into the corners. The curves represented by the first two partial sums of these series are illustrated in Figure 25.

Now it is these very corners which cause difficulty in the integral equation formulation of the slotted antenna
Figure 23. Infinite Cylinder of Square Cross Section
\[ x(t) = \left( \frac{4}{\pi} \right)^2 a \sum_{n=0}^{\infty} \frac{\cos(2n+1) \frac{\pi}{4}}{(2n+1)^2} \cos(2n+1)t \]

\[ y(t) = \left( \frac{4}{\pi} \right)^2 a \sum_{n=0}^{\infty} \frac{\sin(2n+1) \frac{\pi}{4}}{(2n+1)^2} \sin(2n+1)t \]

Figure 24. Parametric Representation of Square
Figure 25. Square Cylinder and its Fourier Approximations

$C$ — SQUARE CYLINDER

$C_0$ — $x = \left(\frac{4}{\pi}\right)^2 \frac{a}{\sqrt{2}} \cos t$

$C_1$ — $x = \left(\frac{4}{\pi}\right)^2 \frac{a}{\sqrt{2}} \left[\cos t - \frac{1}{9} \cos 3t\right]$

$y = \left(\frac{4}{\pi}\right)^2 \frac{a}{\sqrt{2}} \sin t$

$y = \left(\frac{4}{\pi}\right)^2 \frac{a}{\sqrt{2}} \left[\sin t + \frac{1}{9} \sin 3t\right]$
problem for a square cylinder. The kernel of the integral equation can be evaluated on the line \( t = \tau \) by means of equation (2.37). This equation states that on that line the kernel is proportional to the curvature of the curve \( \sigma \), or, correspondingly, inversely proportional to its radius of curvature. But at a corner the radius of curvature is zero, and the kernel becomes infinite. In the case of the square cylinder, the kernel is zero everywhere on the line \( t = \tau \) except at the corners where it is infinite. These remarks, of course, apply only to the line \( t = \tau \). For points \((t, \tau)\) removed from this line the kernel has well defined non-infinite values.

Thus, for the square cylinder the integral equations formulation leads to an integral equation whose kernel is singular at four points. This will be the case for any cross section having corners, and the number of singular points of the kernel will be equal to the number of corners. These singular points of the kernel cause not only theoretical difficulties, but also practical difficulties in obtaining approximate solutions to the integral equation.

One approach to this problem would be to obtain formal Fourier series expansions for the singular kernel which would converge to the value of the kernel at every point at which it was finite and would diverge infinitely at a singular point of the kernel. Examples of this technique
in one variable are the pseudo-Fourier expansions

\[ S(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imt}, \]

and

\[ \frac{1}{2}\log|t| = \log(\pi) - 1 - \sum_{n=1}^{\infty} \frac{\text{Si}(n\pi)}{n\pi} \text{coent}. \] (4.54)

The first of these, often used as a representation for the "impulse function", actually diverges at every point since the magnitude of the mth term does not approach zero as m approaches infinity. This series is, however, Cesaro summable to zero at every point in the closed interval \([-\pi, \pi]\) except at the point \(t = 0\) where it diverges infinitely. The second series converges at every point \(t \neq 0\) in \([-\pi, \pi]\) since it is an oscillating series whose nth term approaches zero, but it diverges infinitely at \(t = 0\).

Even after one has decided to use such a formal Fourier series expansion for the kernel, however, the problem of obtaining the coefficients of the expansion is not easily handled. The approach which was used in this instance was to use the kernel for a curve which approximated the square but which had continuous curvature. A natural choice for such approximating curves are the partial sums of the parametric Fourier series representations of the square given
in equation (4.53). In particular, the slotted antenna problem will be solved for the curve obtained by taking two terms of the parametric representation, and it will be asserted that the solution so obtained approximates that for the exact square. The curve to be used is shown in Figure 25 as $C_1$, and it will henceforth be designated as the quasi-square.

The quasi-square for which the solution of the slotted antenna problem will be obtained is shown in Figure 26. The parametric equations of the quasi-square are

$$x = A \left[ \cos t - \frac{1}{2} \cos 3t \right],$$

and

$$y = A \left[ \sin t + \frac{1}{2} \sin 3t \right],$$

where

$$A = \left( \frac{\pi}{2} \right)^2 \frac{a}{2},$$

(4.55)

and $a$ is the half-side of the corresponding square.

For the quasi-square the integral equation (2.45) can be written

$$\left( \frac{-31}{2ka} \right) u(t) = f(t) + \int_{-\pi}^{\pi} k(t, \tau) u(\tau) d\tau. \quad (4.56)$$

In this equation

$$k(t, \tau) = H_1(kR) \hat{n} \cdot \hat{u} \sqrt{\frac{10 + 6\cos^4 \tau}{16}}. \quad (4.57)$$

The meaning of $R$, $\hat{n}$, and $\hat{u}$ is shown in Figure 26.
Figure 26. Slotted Quasi-Square Cylinder
For the quasi-square

\[ R = 2A \left\{ \sin^2 \beta + \frac{\cos \beta \sin 3 \beta}{3} \cos 4 \gamma + \frac{1}{81} \sin^2 3 \beta \right\}, \]

where \[ \beta = \frac{1}{6}(t-T), \]
and \[ \gamma = \frac{1}{6}(t+T). \] (4.58)

The outward directed unit normal to the curve is given by the relation

\[ \frac{1}{A} \frac{ds}{d\tau} \hat{n}(\tau) = \hat{i} \left[ \cos \tau + \frac{1}{3} \cos 3 \tau \right] \]
\[ + \hat{j} \left[ \sin \tau - \frac{1}{3} \sin 3 \tau \right], \] (4.59)

where \[ \frac{ds}{d\tau} = \frac{A}{3} \sqrt{10 + 6 \cos 4 \tau}. \] (4.60)

The unit vector directed from \( P \) to \( Q \) is given by

\[ \frac{R}{A} \hat{r}(t, \tau) = \hat{i} \left\{ \cos \tau - \cos t - \frac{1}{9} [\cos 3 \tau - \cos 3 t] \right\} \]
\[ + \hat{j} \left\{ \sin \tau - \sin t + \frac{1}{9} [\sin 3 \tau - \sin 3 t] \right\}. \] (4.61)

The limit of \( K(t, \tau) \) as \( t \) approaches \( \tau \) can be obtained directly through the definitions given above, or it can be found from equation (2.37). Thus,

\[ K(t, t) = -\frac{9(1-\cos 4 t)}{2\pi kA(10-6 \cos 4 t)}. \] (4.62)
For the slotted antenna problem as illustrated in Figure 26, \( f(t) \) in the integral equation (4.56) is given by

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_0(kR) \sqrt{\frac{10+6\cos^2 \theta}{16}} d\tau. \tag{4.63}
\]

As in the case of the circular and elliptic cylinders, \( f(t) \) can be evaluated for points far removed from the slot by taking a mean value of the integrand. Thus,

\[
f(t) = H_0(kR), \quad |t| \gg \alpha/2. \tag{4.64}
\]

For points near the slot, the small argument approximation of \( H_0(kR) \) can be used [equation (4.31)] and the integration carried out directly. In the case \( k\alpha = 0.8 \),

\[
f(0) = 1.000 - 12.43442. \tag{4.65}
\]

The equation (4.56) through (4.65) constitute the complete integral equation formulation of the problem of obtaining the current distribution on a quasi-square cylinder excited by a narrow axial slot.

When the current distribution has been found through the solution of the integral equation (4.56), the far field radiation characteristics can be found through application of equation (3.2). Figure 27 illustrates the geometrical situation in the case of the quasi-square cylinder. The
\[ x = \left( \frac{a}{\pi} \right)^2 \frac{a}{\sqrt{2}} \left[ \cos t - \frac{1}{9} \cos 3t \right] \]
\[ y = \left( \frac{a}{\pi} \right)^2 \frac{a}{\sqrt{2}} \left[ \sin t + \frac{1}{9} \sin 3t \right] \]

\[ r = \sqrt{x^2 + y^2} \]

\[ \tan \psi = \frac{y}{x} \]

\[ r_{pq}^2 = r_0^2 + r^2 - 2rr_0 \cos(\psi - \Theta) \]

FOR \( r_0 \gg r \)

\[ r_{pq} = r_0 - r \cos(\psi - \Theta) \]

\[ \hat{n} \cdot \hat{u} = \cos(\psi - \Theta) \]

Figure 27. Sketch Illustrating the Terminology Used in Finding the Far Fields for a Quasi-Square Cylinder from Equation (3.2)
application of equation (3.2) is similar to the development leading up to equation (4.14) in the case of the circular cylinder and to equation (4.42) in the case of the elliptic cylinder. In the case of the quasi-square cylinder, one obtains

\[ C_\Omega(e) = \int_{-\pi}^{\pi} u(t)e^{-ikr(t)\cos(\psi - e)} \frac{ds}{dt} dt \]

\[ + ik \int_{-\pi}^{\pi} u(t)e^{-ikr(t)\cos(\psi - e)} \cos(\psi - e) \frac{ds}{dt} dt. \]

In this equation \( \frac{ds}{dt} \) is as given in equation (4.60), and

\[ r(t) = \sqrt{x^2 + y^2}, \]

\[ \tan \psi(t) = \frac{y(t)}{x(t)}, \]  \( \text{(4.67)} \)

x and y being the parametric expressions given in equation (4.55). This completes the formulation of the problem of finding the far field radiation characteristics of a quasi-square cylinder.

The results of this section were next applied to a particular case of the square cylinder for which \( ka = 0.6 \). The numerical values obtained are summarized in Table 3 and displayed graphically in Figures 22 through 30.

The current distribution values tabulated in Column 1 of Table 3 and shown graphically in Figure 28 were obtained
Table 3

Comparison of Results Obtained by Various Methods for Current Distribution and Far Field of a Square Cylinder

Square Cylinder: \( ka = 0.8 \)

\( \alpha = 0.109 \) radians,

\[ U = \frac{k}{\alpha}, \quad |t| < \frac{\alpha}{2}, \]

\( = 0, \) elsewhere.

\( u(t) \)-Current, \( u(\theta) \)-Normalized far field.

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Real part above imaginary part. Magnitude above phase angle.

   Eleven terms of double Fourier series.
2. Integration of current distribution obtained from integral equation for quasi-square. Weddle's rule.
3. Circular wave functions for circular cylinder which approximates square cylinder.
Figure 28. Complex Value of Current on Quasi-Square Cylinder

MAGNITUDE OF $u(t)$

$ka = 0.8, \alpha = 0.109$ RADIANS

$U = k/\alpha, \ |t| < \alpha/2,$

$= 0$, ELSEWHERE.

ARGUMENT OF $u(t)$
Figure 29. Magnitude of Normalized Far Field for Square Cylinder
Figure 30. Argument of Normalized Far Field for Square Cylinder
by an approximate solution of the integral equation (4.56) for the quasi-square. This was obtained by finding a partial double Fourier series expansion for the kernel \( K(t, \tau) \) which approximated it satisfactorily. Various properties of the kernel restrict the types of terms that need to be considered in the double Fourier sum. While it is not symmetric, the kernel does have the properties

\[
K(t, \tau) = K(t + \frac{1}{2} \pi, \tau + \frac{1}{2} \pi),
\]

(4.68)

for all values of \( t \) and \( \tau \), and

\[
K(\tau + b, \tau) = K(\tau - b, \tau),
\]

(4.69)

for \( \tau = n\pi/4 \), \( b \) being arbitrary. These properties eliminate all but certain types of Fourier terms from consideration, and it was postulated that

\[
K_1(t, \tau) = A_0 + A_1 \cos(t - \tau) + A_2 \cos(t - \tau)
+ A_3 \cos(t - \tau) + A_4 \cos(t - \tau)
+ A_5 \cos(t + \tau) + A_6 \cos(t + \tau) + A_7 \cos(t + \tau)
+ A_8 \cos(t + \tau) + A_9 \cos(t + \tau) + A_{10} \cos(t + \tau).
\]

(4.70)

The coefficients \( A_0 \) through \( A_{10} \) were determined by equating the postulated kernel \( K_1(t, \tau) \) to the actual values of \( K(t, \tau) \) at the eighty-one points.
t = mπ/4, m = -4,-3,...,3,4,
= nπ/4, n = -4,-3,...,3,4.

The kernel has equal values on many of these points so that this process leads to a system of ten independent linear algebraic equations in terms of the coefficients A_0 through A_{10}. In this case, it was found that

\[
\begin{align*}
A_0 &= 0.21706 - 10.25736, \\
A_1 &= -0.18483 - 10.06345, \\
A_2 &= -0.04857 + 10.01199, \\
A_3 &= 0.01321 - 10.03899, \\
2(A_4 + A_{10}) &= 0.00317 - 10.04267, \\
A_5 &= 0.06054 + 10.00050, \\
A_6 &= -0.09060 + 10.18824, \\
A_7 &= -0.00165 + 10.13815, \\
A_8 &= 0.01915 + 10.07836, \\
\text{and} & \quad A_9 = 0.01252 - 10.01477. \tag{4.71}
\end{align*}
\]

The kernel \(K_1(t, \tau)\) so determined is equal exactly to \(K(t, \tau)\) on the eighty-one points specified, and it approximates it elsewhere. It is not necessary to determine \(A_4\) and \(A_8\) individually since only their sum is needed in the approximate solution of the integral equation.

The approximate kernel \(K_1(t, \tau)\) was introduced into the integral equation (4.56). Since \(K_1(t, \tau)\) is separable, the resulting integral equation is directly solvable leading
to two systems of linear algebraic equations in two unknowns each with complex coefficients and unknowns. The resulting expression for the current distribution $u(t)$ is

$$u(t) = \frac{1}{1.635681} \left\{ f(t) + (-0.38921-10.12399) 
+ (0.17136 + 10.14431)\cos t 
+ (0.06036 - 10.11207)\cos 2t 
+ (0.04577 - 10.07512)\cos 3t 
+ (-0.02202 + 10.00014)\cos 4t \right\}.$$  

(4.72)

Values of this expression have been tabulated in Column 1 of Table 3 and plotted graphically in Figure 26.

Next, the far field radiation pattern of the slotted quasi-square cylinder was obtained through application of equation (4.66). In the present instance where $U$ is a narrow rectangular function (see Figure 26) the first integral of this expression can be approximated by a mean value of the integrand. The second integral was evaluated numerically by Weddle's rule, and the results of this computation are tabulated in Column 2 of Table 3 and shown graphically in Figures 29 and 30.

The quasi-square cylinder (shown as $C_1$ in Figure 25) is the second of a sequence of cross sections which might have been used. Successive ones would be found by taking
higher order partial sums of the Fourier representation (4.53) for the square cylinder. The first of this sequence of cross sections is a circle shown as $C_0$ in Figure 25. For comparison with the results obtained for the quasi-square cylinder, the far field radiation pattern for this circular cylinder was also computed. Equation (4.20) was used for this purpose, and the results are tabulated in Column 3 of Table 3 as well as being shown graphically in Figures 29 and 30.

Comparison of the two sets of values shows extremely close agreement considering the crudity of approximation involved in using a circle to approximate a square. In fact, the irregularity of the values shown for the quasi-square at $e$ equal to zero and $pi$ suggests that the values for the circular cylinder may be even better than those given for the quasi-square. It is felt that any major errors in the values for the quasi-square reside in the use of Weddle's rule to carry out the numerical integration of equation (4.66). In any event, a rapid convergence of the process of using successive curves which approximate the square seems to be indicated.

Finally, an experimental model of the square cylinder under consideration was constructed, and the magnitude of its far field radiation pattern was determined experimentally. A photograph of the model used is shown in Figure
31. It was constructed of aluminum sheet with corner brackets for stiffening. The square cylinder had a side of 6.10 inches, and a \( \frac{1}{4} \) inch slot approximately 30 inches long was cut down the center of one side. This slot was fed at the center by a length of flexible coaxial cable as shown in Figure 31. The experimental setup was identical with that for the elliptic cylinder previously discussed. A sketch of the arrangement was shown in Figure 21. Data was recorded at ten degree intervals of the azimuth angle, and a plot of this data is shown in Figure 29. As can be seen, a close agreement with calculated values was obtained.

In summary, it can be said that in this section the integral equation method has been applied successfully to slotted square cylinder antennas. The current distribution as well as the far field radiation pattern were computed for a particular case of the square cylinder. Good agreement with experimental results was obtained.
Figure 31. Experimental Model of Slotted Square Cylinder Antenna
V. CONCLUSIONS

This dissertation has been concerned with the general problem of finding the electromagnetic fields at all points external to an infinite cylinder of arbitrary cross section excited uniformly by an infinite axial slot. This is equivalent to a two-dimensional problem in which the normal derivative of a function is specified on an arbitrary closed curve, and it is desired to find the values of the function on the curve and at all points external to the curve.

It has been found that for a cylinder whose generating curve has continuous curvature and for which the specified tangential electric field is piecewise continuous there exists a unique continuous solution. The uniqueness property has been established through a partial differential equation formulation of the problem, while the existence has been established through a linear integral equation formulation of the problem with the aid of the Fredholm theory of integral equations.

The linear integral equation formulation of the boundary value problem forms the basis of a practical method of obtaining solutions to the slotted antenna problem for cylinders of arbitrary cross section. The method is rapidly convergent for cylinders having moderately small cross
sectional dimensions. Such dimensions are typical in practical antenna applications.

The quantities of primary interest in the slotted antenna problem are the current distribution on the cylinder and the far field radiation pattern. These quantities are readily found through the integral equations formulation and solution of the problem. This problem has been solved previously for the circular and elliptic cylinders through expansions in circular and elliptic wave functions. The expressions obtained for current distributions by these methods are very slowly convergent, however. The linear integral equation method undoubtedly has a very significant advantage over previous solutions in obtaining current distributions on circular and elliptic cylinders. It has the further advantage that it can be applied to find the current distributions and radiation patterns for slotted antennas of arbitrary cross section. Even in the case where the cross section of the cylinder has corners, which introduce singularities in the kernel of the integral equation, no insurmountable difficulties arise.
VI. BIBLIOGRAPHY

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VII. ACKNOWLEDGMENTS

The author takes this opportunity to express his sincere appreciation to

Dr. E. W. Anderson

for his assistance and guidance in the preparation of this dissertation. His continued encouragement and stimulating observations were directly responsible for the completion of this work.

In addition, the author would like to thank those members of the staff of the Mathematics Department and Physics Department of Iowa State College who assisted with their helpful conversations. Among these were Dr. H. D. Block, Dr. O. E. Langenhop, Dr. C. G. Maple, Dr. H. P. Thielman, and Dr. B. Vinograde of the Mathematics Department, and Dr. Carlson of the Physics Department.

The Radio Corporation of America generously made its facilities at the David Sarnoff Research Center, Princeton, New Jersey, available. It was through their cooperation that the experimental models were built and experimental verification obtained. Mr. Juri Tults took the experimental data displayed in Figures 16 and 29, and Mrs. M. A. Brockman performed the computations leading to the results tabulated in Column 4 of Table 1 and in Column 3 of Table 2.