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Analytical study of dynamic loads on elastically supported slabs

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UMI
ANALYTICAL STUDY OF DYNAMIC LOADS ON ELASTICALLY SUPPORTED SLABS

by

Frank Edward Bortle

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

Major Subject: Applied Mathematics

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1949
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INTRODUCTION

Previous Investigations

The general subject of analysis of plates has been of interest to mathematicians for over a century. Lagrange, Navier, Poisson, Kirchhoff, Kelvin, Tait, Boussinesq, and others have contributed to the general theory of the equilibrium of plates. They did not treat the problem of a plate supported on a subgrade.

The first investigation of a thin plate resting on an elastic foundation was made in 1884 by H. Hertz (6). He published a solution for the deflection of an infinite plate resting on an elastic medium where the upward pressure on the bottom of the plate was proportional to the deflection. This assumption of foundation pressure being proportional to deflection has been included in most analyses made since. Extensions to the Hertz method of solution were introduced by Föppl (4) a few years later. He applied the solution to an infinite circular plate with a load at the center. A different method of solving specialized problems of finite circular plates on elastic foundations has been treated by Schleicher (11). He made use of Bessel, Hankel and Neumann functions.

The problem of the rectangular plate, or the infinite
plate in rectangular coordinates, was studied by Happel (5). He obtained a solution by setting up the expression for energy change involved in loading the plate. He then expressed the deflection as the sum of a double infinite series, each term of which approximately satisfied the boundary conditions at the edge of the plate. He obtained the values of the coefficients for as many terms as was desirable by substituting the expression for the deflection into the energy equation, then minimized this equation.

In recent years Westergaard (15) has treated both the infinite plate, and the infinite plate strip with various edge conditions. He set up the solution of Lagrange's plate equation as a function involving a Fourier series. Hogg (7) treated the foundation as a semi-infinite elastic body and made use of the general equations of elasticity to determine the subgrade reaction. Roll (8) was concerned with a determination of the deflection and stresses in plates supported by a subgrade and in the nature and distribution of the reaction pressure. He considered four types of subgrade.

The above authors considered only static loads. In 1931 Schmidt (12), and in 1932 Reissner (10) published papers in which the load was a dynamic load. The rectangular plates were not resting on elastic foundations. In 1943 Dörr (3) published a paper in which he considered a dynamic load moving along an elastically supported beam. At the beginning of his
paper he made a point of the fact that he was using the Zimmerman hypothesis (also known as the Winkler hypothesis), that is, there existed only a linear relationship between the deflection of the plate and the reaction of the subgrade. He also emphasized the fact that he did not consider the possibility of energy transfer from the beam to the subgrade. He developed an expression for the deflection \( y = y(x,t) \) that satisfied the differential equation

\[
EI \frac{\partial^4 y}{\partial x^4} + cy + m \frac{\partial^2 y}{\partial t^2} = \delta(x - vt)
\]

where \( c \) is the spring constant of the subgrade per unit of length; \( m \) the mass of the beam; \( EI \) the bending strength of the beam; \( v \) the velocity of the load; \( x,y,t \) the coordinates of the system and the time; and \( \delta(x - vt) \) a function with the following properties:

\[
\delta(x - vt) = \begin{cases} 
0 & \text{when } x \not= vt, \\
\infty & \text{when } x = vt,
\end{cases}
\]

\[
\lim_{\varepsilon \to 0} \int_{\sqrt{vt} - \varepsilon}^{\sqrt{vt} + \varepsilon} \delta(x-vt) \, dx = 1.
\]

His expressions for the deflection involved trigonometric functions and are valid only for steady state conditions, that is, as \( t \to \infty \). He derived expressions for the deflection both in front of and behind the load. The form of these expressions
depended upon the velocity at which the point load was moving. He had one expression that was valid when this velocity was less than $\sqrt{4\sigma E I/m^2}$ and another when it was larger.

Statement of Problem

In the following investigation expressions for the deflection caused by a load moving along an elastically supported plate will be derived. The plate will have pinned edges, that is, boundary conditions will be imposed so that there will be no deflection along the edges of the plate and the moments at the edges will be zero. The initial conditions imposed will be that at time $t = 0$, the deflection of the plate will be zero and the plate will be at rest.

Formal equations for the deflection caused by a block load, a line load, and a point load moving across a finite plate with a constant velocity will be set up. The finite plate will be expanded to an infinite strip.

Expressions for the deflection caused by a point load moving along this strip will be developed in more detail. The method of attack will be analogous to that of Dörr in his beam problem. The expressions for the deflection, both in front of and behind the load, will be expressed as infinite series.

The differential equation to be satisfied in all these problems is
where $w$, the deflection, is a function of the coordinates $x$ and $y$ and time $t$. The equation, \[ D \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^2 w}{\partial x \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] + \rho \frac{\partial^2 w}{\partial t^2} + kw = f(x,y,t) \]
is Lagrange's plate equation. There is a derivation of it in "Theory of Plates and Shells" by Timoshenko (14). The function $f(x,y,t)$ represents the applied load. The $\rho \frac{\partial^2 w}{\partial t^2}$ term has been inserted to take care of the dynamic load. The $kw$ term is the reaction of the subgrade. The symbols will be defined in a later section.

**Assumptions**

The following assumptions are made:

1. There is a linear relationship between the reaction of the subgrade and the deflection of the plate.
2. The deflection is small in comparison with the thickness of the plate.
3. The subgrade reacts as a bed of vertical springs, with no interaction between the springs.
4. The plate and subgrade are in contact at every point.
5. The load is at all times in contact with the plate.
6. The load is normal to the plate.
7. The plate is horizontal and the load is vertical.
8. The plate and subgrade are elastic, homogeneous, and isotropic.
9. There is no shear between the plate and the foundation.

10. The vertical compressive stress between layers of the plate due to the load is small in comparison to the bending stresses.

11. A straight line normal to the neutral plane of the plate remains straight after the plate is bent.

Definition of Symbols

The symbols used are as follows:

\( a \) = dimension of the plate parallel to the \( x \)-axis.
\( b \) = dimension of the plate parallel to the \( y \)-axis.
\( D \) = rigidity modulus \( \frac{Eh^3}{12(1-\nu)} \) (flexural rigidity).
\( D_1 \) = \( D/P \).
\( E \) = Young's Modulus.
\( f(x,y,t) \) = load on plate.
\( h \) = plate thickness.
\( k \) = modulus of foundation.
\( k_1 \) = \( k/D \).
\( p_o \) = intensity of load per unit area.
\( t \) = time.
\( v \) = velocity of moving load.
\( w \) = vertical deflection.
\( W_x = \frac{\partial W}{\partial x} \).
\( W_{xy} = \frac{\partial^2 W}{\partial x \partial y} \).
\(x, y\) = coordinate axes.

\(\beta = \frac{mn}{b}\).

\(\lambda^4 = \frac{ps^2 + k}{D}\).

\(p\) = mass density of plate.

\(\nu\) = Poisson's Ratio.

\(\omega = \frac{mn}{a}\).

\(\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\).

Other symbols will be defined as they are introduced.
FORMAL SOLUTION OF PROBLEMS OF FINITE PLATE UNDER MOVING LOAD

Three types of load will be considered in this section; a block load, line load, and point load. Each will move with a constant velocity \( v \).

The differential equation to be satisfied is

\[
D \left[ \nabla^4 w(x,y,t) \right] + P_{tt} w + kw = f(x,y,t) \quad (1)
\]

where the function \( f(x,y,t) \) is the load applied to the plate. The edges of the plate are to be pinned; the plate is to be motionless and have no deflection at time \( t = 0 \). Therefore, the following edge and initial conditions must be imposed on \( w \) (see Figure 1):

\[
\begin{align*}
    w(x,y,t) &= 0 \quad \text{if} \ x = 0, a, \quad \text{if} \ y = 0, b, \\
    w_{xx}(x,y,t) &= 0 \quad \text{if} \ x = 0, a, \\
    w_{yy}(x,y,t) &= 0 \quad \text{if} \ y = 0, b, \\
    w(x,y,t) \text{ and its time derivative} &= 0 \quad \text{at} \ t = 0.
\end{align*}
\]

Block Load

First apply a block load of width \( 2d \) and length \( c \) having
Fig. 1 Block Load on Finite Plate.
an intensity per unit area of $p_0$. This load moves with a constant velocity $v$ parallel to the $x$-axis (see Figure 1).

The center of the base of the load moves along the line $y = y_0$. The sides of the load remain parallel to the coordinate axes. This load can be expressed as

$$ f(x, y, t) = \frac{p_0}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{st^{-sx/v} - s(x+c)/v}{s} \, ds $$

(3)

$$ \chi \sum_{n=1}^{\infty} \frac{4}{nm} \sin \beta y_0 \sin \beta d \sin \beta y, $$

where $\beta = \pi n/b$. The expression

$$ \frac{1}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{st^{-sx/v} - (x+c)/v}{s} \, ds $$

is the Laplace Transform of a unit function of length $c$.
(see Figure 2) moving with a velocity $v$. The expression

$$\sum_{n=0}^{\infty} \frac{4}{n\pi} \sin \beta y \sin \beta d \sin \beta y$$

is a Fourier expansion of the function

$$f(y) = 0, \quad 0 < y < y_c - d,$$

$$f(y) = 1, \quad y_c - d < y < y_c + d,$$

$$f(y) = 0, \quad y_c + d < y < b,$$

$$f(y) = -f(-y), \text{ and } f(y+2b) = f(y).$$

Assume that the solution of equation (1) is of the form

$$w(x,y,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} W(x,y,s)e^{st} \, ds, \quad (4)$$

where $W(x,y,s)$ is the Laplace transform of $w(x,y,t)$. If equation (4) is substituted in equation (1), we have

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left[ \partial_t W(x,y,s) + (\rho s^2 + k)W \right] e^{st} \, ds =$$

$$\frac{p_c}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} \left[ e^{-sx/v} - e^{-s(x+c)/v} \right] s^{-1} ds \sum_{n=0}^{\infty} \frac{4}{n\pi} \sin \beta y \sin \beta d \sin \beta y. \quad (5)$$
The conditions imposed here are that \( w(x,y,0) = w_t(x,y,0) = 0 \), that is, at \( t = 0 \) there is no deflection in the plate and the plate is motionless.

From equation (5) we get the relationship

\[
\nabla^4 w + \lambda^4 w = \frac{p_s e}{sD} \left( 1 - e^{-s \omega/v} \right) \sum_{n} \frac{4}{n \omega} \sin \beta y \sin \beta d \sin \beta y,
\]

where \( \lambda^4 = \frac{p_s + k}{D} \).

On substituting the Fourier Series expansion for \( e^{-s \omega/v} \) into equation (6), the equation can be expressed as

\[
\nabla^4 w + \lambda^4 w = \frac{2p_s (1 - e^{-s \omega/v})}{sD} \sum_{n} \frac{4}{n \omega} \sin \beta y \sin \beta d \sin \beta y,
\]

where \( \omega = \frac{mn}{a} \).

Now let

\[
W(x,y,s) = \sum_{m} \sum_{n} A_{mn}(s) \sin \omega x \sin \beta y.
\]

Equation (8) will satisfy the following conditions:

\[
W(0,y,s) = W(a,y,s) = W(x,0,s) = W(x,b,s) = 0,
\]
\[ W^{(0,y,s)} - W^{(x,0,s)} - W^{(x,b,s)} - 0, \quad (9) \]

\[ W^{y}(0,y,s) - W^{y}(s,y,s) - W^{y}(x,0,s) - W^{y}(x,b,s) = 0, \]

so that the remaining boundary conditions will be satisfied. These are the conditions that must be satisfied if the four edges of the plate are simply supported, that is, there is no deflection along the edges and the moments at the edges are zero.

Now

\[ \nabla^4 W = W^{xxxx} + 2W^{xxy} + W^{yyy} \]

\[ = \sum_{m} \sum_{n} A_{mn}(s)(\omega^2 + \beta^2)^2 \sin \omega x \sin \beta y. \quad (10) \]

On substituting the expression for \( W(x,y,s) \) from equation (8) into equation (7), equation (7) can be expressed as

\[ A_{mn}(s)[(\omega^2 + \beta^2)^2 + \lambda^4] \sin \omega x \sin \beta y = \]

\[ \frac{3p_0\omega (1-e^{-sc/v}) [1 - (-1) e^{-ss/v}]}{n\nu sDa(s + \omega^2 v^2)} \sin \omega x \sin \beta y \sin \beta d \sin \beta y. \quad (11) \]

From equation (11)
When this expression for $A_{mn}(s)$ is substituted in equation (8) and this new expression for $W(x,y,s)$ is substituted in equation (4), the expression for the deflection takes the form

$$w(x,y,t) = \frac{1}{2\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} \sum_{m} \sum_{n} X$$

$$\frac{8p_o \omega(1-e) \left[1-(1) e^{m \frac{-sc}{v}}\right] \sin \beta_y \sin \beta_d \sin \omega x \sin \beta y ds}{nus\pi(s^2 + \omega^2 v^2)\left[(\omega^2 + \beta^2)^2 + \lambda^4\right]}.$$

This is a formal expression for the deflection when a block load moves with constant velocity along a finite plate that is resting on an elastic subgrade. If the $s$-integration is carried out, the deflection will be expressed as a double infinite series.

**Line Load**

$$-\frac{sc}{v}$$

If the function $e^{-\frac{sc}{v}}$ is expanded in a power series,

$$e^{-\frac{sc}{v}} = 1 - \frac{sc}{v} + \frac{s^2 c^2}{2v^2} - \frac{s^3 c^3}{6v^3} + \ldots.$$

(14)

Using this expansion, $\frac{p_o(1 - \frac{e}{s})}{s}$ can be expressed as
\[
\frac{P_0(1-e^{-sc/v})}{s} = \frac{P_0c}{v} - \frac{P_0c(sc)}{2v^3} + \frac{P_0c(sc)^2}{6v^5} + \ldots \quad (15)
\]

Therefore if \( P_0c \to \overline{P} \) as \( c \to 0 \),

\[
\frac{P_0(1-e^{-sc/v})}{s} \to \frac{\overline{P}}{v} \cdot (16)
\]

In order to get an expression for the deflection caused by a line load of width 2d moving along the plate, substitute equation (15) in equation (13) and then let \( c \to 0 \). If this substitution is made and the limiting operation performed, the deflection function becomes

\[
w(x,y,t) = \frac{1}{2\pi i} \sum_{m} \sum_{n} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \chi
\]

\[
\frac{\overline{P}}{v} \left[ (s + \omega^2 + \beta^2 + \lambda^2) \right]^{-1} \sin \beta y \sin \beta d \sin \omega x \sin \beta y \, ds
\]

\[
\overline{P} = \frac{m - sa/v}{m \pi a D (s^2 + \omega^2 + \beta^2 + \lambda^2)} \cdot (17)
\]

where \( \overline{P} \) is the linear density of the line load. This is a formal expression for the deflection when a line load moves with constant velocity along a finite plate that is resting on an elastic subgrade. If the \( s \)-integration is performed, the deflection will be expressed as a double infinite series.
Point Load

The load can be changed to a point load by letting
\( 2Fd \to P \) as \( d \to 0 \), where \( P \) is the density of the point load. If this limiting operation is performed in equation (17), the expression for the deflection caused by a point load is

\[
w(x,y,t) = \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{m - sa/v}{\mu a D(s^2 + \omega^2 v^2) \left( (\omega^2 + \beta^2)^2 + \lambda^2 \right)} \right) \sin \beta y \sin \omega x \sin \beta y \, ds.
\]

Equation (18) is a formal expression for the deflection when a point load moves with constant velocity along a finite plate that is resting on an elastic subgrade. If the \( s \)-integration is carried out, the deflection will be expressed as a double infinite series.
COMPLETE SOLUTION OF PROBLEM OF SEMI-INFINITE PLATE
UNDER A MOVING POINT LOAD

A study will be made in this section of the deflections caused by a point load moving along a semi-infinite plate.

Expression of the Deflection Function in Terms of a Double Integral

The plate of finite dimensions can be expanded into a semi-infinite strip in the $x$-direction by letting $a \to \infty$. In equation (18) set $\pi/a = h$ so that $\omega = mn/s = mh$. Next use the Lemma: If $f(x)$ satisfies a Hölder condition for $0 \leq x \leq \infty$, that is if $|f(x+Ax) - f(x)| \leq H|Ax|^{\gamma}$ for $0 \leq \gamma \leq 1$, and if $|f(x)| \leq \phi(x)$ where $\phi(x)$ is a monotone decreasing function of $x$, and $\int_0^\infty \phi(x) \exists$, then for $h > 0$, $\sum_{m} hf(mh)$ converges and

$$\lim_{h \to 0} \sum_{m} hf(mh) = \int_{0}^{\infty} f(x)dx.$$ Further, if $f(x)$ depends on a parameter $t$, and $H$ and $\phi(x)$ are independent of $t$, then the limit exists uniformly in $t$. If

* The proof of this Lemma is in Appendix A.
these substitutions are made and the limit is carried out, the expression for the deflection caused by a point load moving along the line \( y = y_0 \) of a semi-infinite strip can be expressed as

\[
w(x,y,t) = \frac{1}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{-\infty}^{\infty} \sum_{n} \frac{2P\nu \beta \sin \alpha x \sin \beta y \sin \beta y_0 \, d\alpha \, d\beta}{n^{2}D(s + \alpha^{2}v^{2})[(\alpha^{2} + \beta^{2})^{2} + \lambda^{2}]} \, ,
\]

(19)

where \( \alpha \) is a variable of integration.

This expression is an even function of \( \alpha \), therefore

\[
w(x,y,t) = \frac{1}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{-\infty}^{\infty} \sum_{n} \frac{2P\nu \beta (e^{-i\alpha x} - e^{i\alpha x}) \sin \beta y \sin \beta y_0 \, d\alpha \, d\beta}{n^{2}D2i(s + \alpha^{2}v^{2})[(\alpha^{2} + \beta^{2})^{2} + \lambda^{2}]} \, ,
\]

(20)

\[
= -\frac{1}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{-\infty}^{\infty} \sum_{n} \frac{P\nu \beta e^{i\alpha x} \sin \beta y \sin \beta y_0 \, d\alpha \, d\beta}{n^{2}D(s + \alpha^{2}v^{2})[(\alpha^{2} + \beta^{2})^{2} + \lambda^{2}]} \, ,
\]

\[
+ \frac{1}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{-\infty}^{\infty} \sum_{n} \frac{P\nu \beta e^{-i\alpha x} \sin \beta y \sin \beta y_0 \, d\alpha \, d\beta}{n^{2}D(s + \alpha^{2}v^{2})[(\alpha^{2} + \beta^{2})^{2} + \lambda^{2}]} \, ,
\]

(21)

Now let \( \alpha = -z \) in the second term on the right of equation (21).
\[ w(x,y,t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n} \frac{iux}{\pi n D(s^2 + \alpha^2 v^2)} \left[ \frac{\sin \beta y \sin \beta y_0}{\left( \alpha^2 + \beta^2 \right)^2 + \lambda^2} \right] e^{i(s^2 + \alpha^2 v^2)} ds dv \]

Finally, the expression for the deflection caused by a point load moving along the line \( y = y_0 \) of a semi-infinite strip can be expressed as

\[ w(x,y,t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n} \frac{iux}{\pi n D(s^2 + \alpha^2 v^2)} \left[ \frac{\sin \beta y \sin \beta y_0}{\left( \alpha^2 + \beta^2 \right)^2 + \lambda^2} \right] e^{i(s^2 + \alpha^2 v^2)} ds dv \]

(22)

**First Integration**

If \( L \) denotes the semicircle \(|u| = R\) in the upper half-plane, then

\[ w(x,y,t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n} \frac{iux}{\pi n D(s^2 + \alpha^2 v^2)} \left[ \frac{\sin \beta y \sin \beta y_0}{\left( \alpha^2 + \beta^2 \right)^2 + \lambda^2} \right] e^{i(s^2 + \alpha^2 v^2)} ds dv \]

\[ = 0 \]

because the conditions of Jordan's Lemma* are satisfied. Using

---

this relationship, equation (22) can be expressed as

\[
 w(x,y,t) = -\frac{1}{\nu} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{e^{-st}}^{\infty} \sum_{n=0}^{\infty} \frac{iax}{\sin \beta y \sin \beta \omega} \text{d} \omega \text{d} y \text{d} t
\]

\[
 w(x,y,t) = -\frac{1}{\nu} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{C}^{\infty} \sum_{n=0}^{\infty} \frac{iax}{\sin \beta y \sin \beta \omega} \text{d} \omega \text{d} y \text{d} t
\]

\[
 w(x,y,t) = -\frac{1}{\nu} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{C}^{\infty} \sum_{n=0}^{\infty} \frac{iax}{\sin \beta y \sin \beta \omega} \text{d} \omega \text{d} y \text{d} t
\]

where \( C \) denotes the contour that includes the semi-circle \(|a| = \infty\) in the upper half-plane as described above, and the real axis.

Consider next the integral

\[
 I = \int_{C} \frac{iax}{\text{ae}} \frac{\text{da}}{(a + \frac{s^2}{v^2})[\left(a + \beta\right)^2 + \lambda^2]} \]

\[
 I = \int_{C} \frac{iax}{\text{ae}} \frac{\text{da}}{(a-a_1)(a-a_2)(a-a_3)(a-a_4)(a-a_5)(a-a_6)} , \quad (24)
\]

where

\[
 a_1 = \frac{is}{v},
\]

\[
 a_2 = -\frac{is}{v} = -a_1,
\]

\[
 a_3 = i\sqrt{\beta^2 - i\lambda^2},
\]

\[
 a_4 = -i\sqrt{\beta^2 - i\lambda^2} = -a_3,
\]

\[
 a_5 = i\sqrt{\beta^2 + i\lambda^2},
\]

\[
 a_6 = -i\sqrt{\beta^2 + i\lambda^2} = -a_5,
\]

(25)
These poles are located as shown in Figure 3.

\[ I = 2\pi i \left( \sum \text{residues of the integrand at its poles within } C \right) \]

Using this theorem, the value of \( I \) is

\[ I = 2\pi i \left( \frac{ia_1 x}{(a_1-a_2)(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_1-a_6)} + \frac{ia_3 x}{(a_3-a_1)(a_3-a_2)(a_3-a_4)(a_3-a_5)(a_3-a_6)} \right) \]

---

Using the relations in equation (25), equation (26) can be expressed as

\[
I = \pi i \left[ \frac{i a_1 x}{e^{(a_1^2 - a_3^2)(a_1^2 - a_5^2)}} - \frac{i a_3 x}{e^{(a_1^2 - a_3^2)(a_3^2 - a_5^2)}} \right]
\]

or

\[
I = \frac{\pi i}{2 \left( \frac{s^2}{v^2 - \beta^2} + \lambda^4 \right)} \left[ 2 e^{-s x / v - \sqrt{\beta^2 - 1} \lambda^2 x} (1 - \frac{s^2}{v^2 - \beta^2}) - e^{-\sqrt{\beta^2 + 1} \lambda^2 x} (1 + \frac{s^2}{v^2 - \beta^2}) \right].
\]

The expression for \( I \) in equation (27) permits the deflec-
tion function, as expressed in equation (23), to be written as

\[ w(x,y,t) = -i \int_{-\infty}^{\infty} e^{\sigma t} \sum_{n} \frac{\beta \sin \beta y \sin \beta y_0}{\sin \beta y} ds \times \frac{1}{2\pi \sqrt{D\left[\left(\frac{s^2}{v^2} - \beta^2\right)^2 + \lambda^4\right]}} \]

\[ \times \left[ 2e^{-sx/v} - e^{-\sqrt{\beta^2 - \frac{\lambda^2 x}{i}} \left(1 - \frac{s^2}{\frac{v^2}{\lambda^2}}\right)} \right. \]

\[ \left. - e^{-\sqrt{\beta^2 + \frac{\lambda^2 x}{i}} \left(1 + \frac{s^2}{\frac{v^2}{\lambda^2}}\right)} \right]. \quad (28) \]

This is the expression for the deflection after the \( \alpha \)-integration has been completed.

**Second Integration**

The expression for \( w(x,y,t) \), equation (28), can be expressed as the sum of three integrals.

\[ w(x,y,t) = I_1 + I_2 + I_3, \quad (29) \]

where

\[ I_1 = -i \int_{-\infty}^{\infty} \sum_{n} \frac{v^3 \beta \sin \beta y \sin \beta y_0 e^{-s(x/v - t)}}{n^2 D\left[\left(s^2 - \frac{v^2 \beta^2}{\lambda^2}\right)^2 + \lambda^4\right]} ds \quad (30) \]
\[ I_a = i \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n} \frac{\nu^3 \beta \sin \beta y \sin \beta y e^{-\sqrt{\beta^2 - i\lambda^2}x + st}}{2\nu \pi \sqrt{D \left( (s^2 - \nu \beta^2)^2 + \nu^4 \right)}} \left(1 - \frac{s^2 - \beta^2}{\nu^2 - i\lambda^2}\right) ds, \tag{31} \]

and
\[ I_s = i \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n} \frac{\nu^3 \beta \sin \beta y \sin \beta y e^{-\sqrt{\beta^2 + i\lambda^2}x + st}}{2\nu \pi \sqrt{D \left( (s^2 - \nu \beta^2)^2 + \nu^4 \right)}} \left(1 + \frac{s^2 - \beta^2}{\nu^2 + i\lambda^2}\right) ds. \tag{32} \]

The integration of Integral \( I_a \)

Equation (30) can be expressed in the form
\[ I_1 = -i \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n} \frac{\nu^3 \beta \sin \beta y \sin \beta y e^{-s^2 \left( \frac{x}{\nu} - t \right)}}{\nu \pi \sqrt{D \left( s^4 + \nu^2 \left( \frac{\rho^2}{D} - 2\beta^2 \right) s + \nu^4 \left( \beta + \frac{k}{D} \right) \right)}} ds, \tag{33} \]

where
\[ s_1 = \pm i \sqrt{s \left( \frac{\rho^2}{D} - 2\beta^2 \right) \left[ 1 + \sqrt{1 - \frac{4\beta^2 + k}{(\rho^2/2 - 2\beta^2)^2}} \right] \}, \tag{34} \]

\[ s_2 = -i \sqrt{s \left( \frac{\rho^2}{D} - 2\beta^2 \right) \left[ 1 - \sqrt{1 - \frac{4\beta^2 + k}{(\rho^2/2 - 2\beta^2)^2}} \right] \}. \]
\[ v < \sqrt{\frac{2D\beta}{P}} \]

**Figure 4**

\[ v = \sqrt{\frac{2D\beta}{P}} \]

**Figure 5**

\[
\sqrt{\frac{2D\beta^2}{P}} < v < \sqrt{\frac{2D}{P} (\beta^2 + \beta^4 + \frac{k}{D})}
\]

**Figure 6**

\[
v = \sqrt{\frac{2D}{P} (\beta^2 + \beta^4 + \frac{k}{D})}
\]

**Figure 7**

\[
v > \sqrt{\frac{2D}{P} (\beta^2 + \beta^4 + \frac{k}{D})}
\]

**Figure 8**
\( s_3 = -s_1 \) and \( s_4 = -s_2 \).

The location of these four poles depends on the value of the velocity \( v \). If \( v > \sqrt{2D_1\beta^2} \), where \( d_1 = D/p \), the expression \( \left( \frac{p v^2}{D} - 2\beta^2 \right) \) is positive. If \( v \geq \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \), where \( k_1 = k/D \), the expression \( 1 - \frac{4(\beta + k_1)}{(p v D - \beta^2)^2} \) is either positive or zero. Figures 4 to 8 show the location of the poles for different values of \( v \).

The integration of equation (33) can be performed by employing a contour where \( \sigma \) is of such value that all the poles will fall to the left of the contour. The contour is closed by a circle to the left as shown in Figure 9. This contour will enclose all the poles for every value of the velocity \( v \). Using the residue theorem, \( I_1 \) can be expressed as

\[ \text{Figure 9} \]
\[ I_1 = 2\pi \sum_{n} \frac{v \beta \sin \beta y \sin \beta y_e}{n^2D} \left[ \frac{-s_1(\frac{x}{v} - t)}{(s_1-s_2)(s_1-s_3)(s_1-s_4)} \right. \\
+ \frac{-s_2(\frac{x}{v} - t)}{(s_2-s_1)(s_2-s_3)(s_2-s_4)} + \frac{-s_3(\frac{x}{v} - t)}{(s_3-s_1)(s_3-s_2)(s_3-s_4)} \right. \\
\left. + \frac{-s_4(\frac{x}{v} - t)}{(s_4-s_1)(s_4-s_2)(s_4-s_3)} \right] \]

or

\[ I_1 = \sum_{n} \frac{v \beta \sin \beta y \sin \beta y_e}{n^2D} \left[ \frac{-s_1(\frac{x}{v} - t)}{s_1(s_1^2-s_2^2)} - \frac{-s_2(\frac{x}{v} - t)}{s_2(s_1^2-s_2^2)} \right. \\
\left. - \frac{-s_3(\frac{x}{v} - t)}{s_3(s_1^2-s_2^2)} + \frac{-s_4(\frac{x}{v} - t)}{s_4(s_1^2-s_2^2)} \right] \]

when the relations \( s_1 = -s_3 \), and \( s_2 = -s_4 \) are substituted.

The value of the integral \( I_1 \) is therefore

\[ I_1 = \sum_{n} \frac{v \beta \sin \beta y \sin \beta y_e}{n^2D(s_1^2-s_2^2)} \left[ -s_1(\frac{x}{v} - t) \right. \\
\left. + s_2(\frac{x}{v} - t) \right] \\
- e^{s_1(\frac{x}{v} - t)} \]

\[ \left[ -s_2(\frac{x}{v} - t) \right. \\
\left. + s_2(\frac{x}{v} - t) \right] \]

\[ = \frac{-s_2(\frac{x}{v} - t) - s_2(\frac{x}{v} - t)}{s_2} \]

\[ \left(36\right) \]
valid for $x < vt$, and

$$0 < v < \sqrt{2D_1}\beta^2,$$

$$\sqrt{2D_1}\beta^2 < v < \sqrt{2D_1(\beta^2 + \beta^4 + k_1)},$$

$$v > \sqrt{2D_1(\beta^2 + \beta^4 + k_1)}.$$

$$I_1 = 0,$$  \hspace{1cm} (37)

valid for $x > vt$ and for all values of $v$.

When $v = \sqrt{2D_1}\beta^2$, all four poles are located at the origin so that $s_1 = s_2 = s_3 = s_4 = 0$. Equation (33) becomes

$$I_1 = -i \int_{\sigma-i \infty}^{\sigma+i \infty} \sum_{n=1}^{\infty} \frac{v P \sin \beta y \sin \beta y_0 e^{-s(X/V - t)}}{n \pi D (s - s_1)^4} ds.$$

Now

$$\frac{d^3}{ds^3} e^{-s(X/V - t)} = -(\frac{X}{V} - t)^3 e^{-s(X/V - t)}.$$

Using the residue theorem,

$$I_1 = 2\pi i (-i) \sum_{n=1}^{\infty} \frac{v P \sin \beta y \sin \beta y_0}{n \pi D} \left[ -\frac{(\frac{X}{V} - t)^3}{3!} e^{-0(X/V - t)} \right].$$
\[
I_1 = \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n} \frac{v^3 \beta \sin \beta y \sin \beta y_0 e^{-s(x - vt)}}{n \pi D (s - s_1)^2(s - s_3)^2} \, ds - s(\frac{x}{v} - t) \]

Now
\[
\frac{d}{ds} \left[ -s(\frac{x}{v} - t) \right] = \frac{e}{{(s - \sqrt{\lambda \frac{2\pi}{v}}) - \sqrt{\lambda \frac{2\pi}{v}}}} - \frac{[2s - \sqrt{\lambda \frac{2\pi}{v}}]}{D \sqrt{\lambda \frac{2\pi}{v}}} - s(\frac{x}{v} - t),
\]

valid when \( x < vt \) and \( v = \sqrt{2D_1\beta^2} \).

When \( v = \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \), the poles are located as shown in Figure 7, and

\[
s_1 = s_2 = -s_3 = -s_4 = \frac{\sqrt{\frac{2D_1\beta^2}{D} - 2\beta^2}}{2}.
\]
and

\[
\frac{d}{ds} \left[ \frac{-s(\frac{x}{v} - t)}{(s^2 - v^2\beta^2) + i\lambda^2v^2} \right] = \exp \left( s\left( \frac{x}{v} - t \right) \right)
\]

(40)

\[
\frac{(s^2 - v^2\beta^2 + i\lambda^2v^2)(t - \frac{x}{v}) - \left[ 2s + \frac{i v^2 p s}{D\lambda} \right]}{[s(s^2 - v^2\beta^2) + i\lambda^2v^2]^2} \exp \left( s\left( \frac{x}{v} - t \right) \right)
\]

When \( v = \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \),

\[
s = -\frac{\nu u^2}{2D} - 2\beta^2
\]

\[
\frac{s^2}{v^2} = -\frac{p v^2}{2D} + \beta^2,
\]

\[
\frac{v^2}{2D} = \beta^2 + \sqrt{\beta^4 + k_1}
\]

(41)

\[
[1 \lambda^2]_{s = s_1 = s_2} = \left( \frac{s_1}{v^2} - \beta \right) = -\frac{p v^2}{2D} = -\left( \beta^2 + \sqrt{\beta^4 + k_1} \right)
\]

\[
[1 \lambda^2]_{s = s_3 = s_4} = -\left( \frac{s_3}{v^2} - \beta \right) = \frac{p v^2}{2D} = \left( \beta^2 + \sqrt{\beta^4 + k_1} \right)
\]

Equation (39) can be evaluated using the residue theorem.

If the relations in equations (40) and (41) are used,

---

* See Appendix B.
\[ I_1 = 2\pi(1 - i) \sum_n \frac{v^3 \beta \sin \beta y \sin \beta y_0}{\pi^2 D} \]

\[ \chi \left[ \frac{(s_1^2 - s_1^2 + \beta^2 - \frac{v^2}{2D})(t - \frac{s_1}{v}) - (2s_1 + \frac{v^3 \beta^3}{D\pi^2})}{(s_1 - v^2 \beta - \frac{v^2}{2D})} e^{-s_1(\frac{s_1}{v} - t)} \right] \]

\[ + \frac{(s_3^2 - s_3^2 + \beta^2 - \frac{v^2}{2D})(t - \frac{s_3}{v}) - (2s_3 + \frac{v^3 \beta^3}{D\pi^2})}{(s_3 - v^2 \beta - \frac{v^2}{2D})} e^{-s_3(\frac{s_3}{v} - t)} \]

\[ = 2\pi \sum_n \frac{v^3 \beta \sin \beta y \sin \beta y_0}{\pi^2 D} \left[ \frac{-\frac{v^2}{D}(t - \frac{s_1}{v}) - 4s_1}{\rho^2 \beta^2 - s} e^{-s_1(\frac{s_1}{v} - t)} \right. \]

\[ + \frac{-\frac{v^2}{D}(t - \frac{s_3}{v}) + 4s_3}{\rho^2 \beta^2 - s} e^{s_3(\frac{s_3}{v} - t)} \right] \]

valid for \( x < vt \) and \( v = \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \).

The integration of Integrals \( I_2 \) and \( I_3 \)

The \( I_2 \) integral, equation (31), is

\[ I_2 = \int_{s_1^2 - i\infty}^{s_1^2 + i\infty} \sum_n \frac{v^3 \beta \sin \beta y \sin \beta y_0 e^{-\sqrt{\beta^2 - i\lambda^2} x + st}}{2\pi^2 D[(s^2 - v^2\beta^2)^2 + v^4 \lambda^4]} \left(1 - \frac{s^2 - \beta^2}{i\lambda^2}\right) ds. \]

This integrand has only two poles \( s_3 \) and \( s_4 \), defined in
equation (34), and two branch points $s_5 = \pm i\sqrt{k/\rho}$ and $s_6 = -i\sqrt{k/\rho}$. The cuts are to the left of the imaginary axis.

The "saddlepoint method" will be used to evaluate this integral. This method is applicable for large positive values of $t$. The saddlepoint is a point where the real part of $(s - \frac{X}{t}\sqrt{\beta^2 - i\lambda^2})$ decreases most rapidly from its maximum value. Saddlepoints are found by setting the first derivative of $(s - \frac{X}{t}\sqrt{\beta^2 - i\lambda^2})$ equal to zero. There are six of these saddlepoints.

\[
\frac{d}{ds}(s - \frac{X}{t}\sqrt{\beta^2 - i\lambda^2}) = 0
\]

or

\[
(\sqrt{\beta^2 - i\lambda^2})(\lambda^2) + i\frac{Xps}{2dt} = 0.
\]

Choose a path $C_1$ (see Figure 10) in the $s$-plane so that the value of the integral, for large $t > 0$, is determined in the neighborhood of a single point $s_s$, a saddlepoint, plus the residue at the poles to the right of this path. The unshaded

---


Fig. 10 Contour in \( s \)-Plane.
area of Figure 9 is that area where the real part of the exponent, \((s - \frac{x}{t}\sqrt{\beta^2 - i\lambda^2})t\), is negative. The integral has no value, for large values of \(t\), when the contour is taken through the unshaded portion. It is necessary to evaluate the integral only in the neighborhood of \(s_s\).

This saddlepoint \(s_s\) moves up and down the imaginary axis, its location depending upon the value of \(x/t\). It approaches \(-\infty\) as \(x/t \to \infty\) and approaches the branch point \(s_0 = -i\sqrt{k/p}\) as \(x/t \to 0\). The saddlepoint is always below this branch point.

Only the deflections in the neighborhood of the point load are of interest, so set

\[ x = \delta t + z. \] (44)

In order to get a value for the deflection under the point load \((x = vt)\), set \(\delta = v\) and \(z = 0\). This \(z\) is a measurement, in the \(x\)-direction, measured from the point load.

Label \(I_s\) as \(N_s\) while working with that part of the contour that is in the neighborhood of \(s_s\). Then

\[ N_s = \int_{C_s} \sum_n^\infty \left( s - \frac{\delta t + z}{t}\sqrt{\beta^2 - 1\lambda^2} \right) t \quad e^{G(s)ds} \] (45)

\[ = \int_{C_s} \sum_n^\infty \left( s - \delta\sqrt{\beta^2 - 1\lambda^2} - z\sqrt{\beta^2 - 1\lambda^2} \right) t \quad e^{G(s)ds}, \]
where

\[ k = \frac{iv \beta \sin \beta y \sin \beta \tau}{2 \pi \lambda}, \]

and

\[ G(s) = \frac{s^2 - \beta^2}{(s - \sqrt{v \beta^2})^2 + v^2 \lambda^2}. \]

It can be shown* that the value of this \( N \) integral is zero. The value of \( I_2 \), equation (31), can be found then by multiplying the sum of the residues at the poles to the right of the contour \( C_1 \) by \( 2\pi i \). The particular poles to the right of this contour depend upon the value of \( v \). This gives rise to five different cases.

**Case I:** \( v < \sqrt{2D_1 \beta^2} \). When \( v < \sqrt{2D_1 \beta^2} \), only one pole, \( s_* \), is to the right of the contour so,

\[ I_2 = \]

\[ \int_{\sigma - \infty}^{\sigma + \infty} \sum_{\nu} \frac{v \beta \sin \beta y \sin \beta \nu e^{(s - \frac{x}{\nu} \beta^2 - \frac{1}{\nu} \lambda^2)t}}{2 \pi \nu^2 D [(s - \nu \beta^2)^2 + v^2 \lambda^2]} (1 - \frac{s^2 - \beta^2}{v^2 - \beta^2}) ds \]

* See Appendix C.
\begin{align*}
\sum_{n}^{\infty} \frac{v \beta \sin \beta y \sin \beta y_n e^{(s_4 - \frac{x}{t}) (s_4 - i \lambda^2)y_n}}{2m^2 D(s_4 - s_1)(s_4 - s_2)(s_4 - s_3)}
\end{align*}

\begin{align}
X (1 - \frac{\frac{\lambda^2}{s^4} - \beta^2}{[i \lambda^2]})
\end{align}

The relations

\begin{align}
[i \lambda^2]_{s=s_1} &= \frac{s_1^2}{v^2} - \beta^2, \\
[i \lambda^2]_{s=s_2} &= \frac{s_2^2}{v^2} - \beta^2, \\
[i \lambda^2]_{s=s_3} &= -\frac{s_3^2}{v^2} + \beta^2, \\
[i \lambda^2]_{s=s_4} &= -\frac{s_4^2}{v^2} + \beta^2
\end{align}

are easily established. *

Using the relations of equation (34) and the relations of equation (47), \( I_2 \) can be expressed as

\begin{align}
I_2 = -\sum_{n}^{\infty} \frac{v \beta \sin \beta y \sin \beta y_n e^{(s_4 - \frac{x}{t}) (s_4 - \frac{1}{v})}}{mnD(s_4 - s_1)(s_4 - s_2)(s_4 - s_3)}
\end{align}

\[ * \] See Appendix B.
The same procedure can be followed in evaluating \( I_3 \).

\[
I_3 = 2\pi i \sum_{n=1}^{\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} v^3 \beta \sin \beta y \sin \beta y_o e^{-(\sqrt{\beta^2 + \lambda^2} - \sqrt{\beta^2 + \lambda^2}) x + s\lambda} ds \left(1 + \frac{v^2 - \beta^2}{\lambda^2} \right) ds.
\]

The poles of this integrand are \( s_1 \) and \( s_2 \). The same branch points \( s_5 \) and \( s_6 \) appear. Only the pole \( s_1 \) is to the right of the contour.

\[
I_3 = 2\pi i \sum_{n=1}^{\infty} v \beta \sin \beta y \sin \beta y_o e^{(s_1 - \frac{X}{v} s_1) t} \frac{(s_1 - \frac{X}{v} s_1) t}{\lambda^2}.
\]

Case II: \( v = \sqrt{2D_1 \beta^2} \). When \( v = \sqrt{2D_1 \beta^2} \), the four poles are located at the origin. In this case the contour \( C_1 \) passes to the right of all the poles, therefore,

\[
I_2 = 0,
\]
\[
I_3 = 0.
\]
Case III: \( \sqrt{2D_1 \beta^2} < v < \sqrt{2D_1 (\beta^2 + \sqrt{\beta^4 + k_1})} \). In evaluating \( I_3 \), for this range of the velocity, only one pole, \( s_3 \), is to the right of the contour \( C_1 \). Therefore

\[
I_3 = 2\pi i \sum_{n} \frac{v^3 P\beta \sin \beta y \sin \beta y_e e^{(s_3 - \frac{x}{t} \frac{s_3}{v})}}{2\pi \omega D(s_3 - s_1)(s_3 - s_2)(s_3 - s_4)}
\]

(2)

Only one pole, \( s_3 \), is to the right of the contour when evaluating \( I_3 \) for this range of the velocity.

\[
I_3 = 2\pi i \sum_{n} \frac{v^3 P\beta \sin \beta y \sin \beta y_e e^{(s_3 - \frac{x}{t} \frac{s_3}{v})}}{2\pi \omega D(s_3 - s_1)(s_3 - s_2)(s_3 - s_4)}
\]

(2)

\[
= \sum_{n} \frac{v^3 P\beta \sin \beta y \sin \beta y_e}{\pi \omega D} \frac{s_3(x - vt)}{s_3(s_1^2 - s_2^2)}.
\]

(51)

Case IV: \( v = \sqrt{2D_1 (\beta^2 + \sqrt{\beta^4 + k_1})} \). In this case the poles of the integrand coincide and are located on the imaginary axis (Figure 7). The saddlepoint, \( s_3 \), will coincide with this double pole for some value of \( x \). The equation of the saddlepoints, equation (43), is
If, in equation (43), \( x \) is replaced by \( \overline{y}t \), where \( \overline{y} \) is the value of \( y \) where the saddlepoint coincides with the pole \( s_a \),

\[
\frac{s_a^a}{v} (\beta - s_a^a) = \frac{\overline{y} \cdot s_a^a}{2D},
\]

or

\[
\overline{y} = \frac{1}{v} (\beta - s_a^a) \frac{2D}{p} = \frac{2D\beta}{pv} + \frac{2D}{pv^2} \frac{\nu (\nu^2 - 2\beta^2)}{2} = v,
\]

when the relations of equation (41) are used to simplify the equation.

The contours chosen for the integral \( I_a \) when \( v = \sqrt{2D_1(\beta^2 + \sqrt{\beta^2 + k_1})} \) are shown in Figures 11 and 12.

\[
\gamma \geq \overline{y} + \varepsilon \quad \text{Figure 11}
\]
\[
\gamma \leq \overline{y} - \varepsilon \quad \text{Figure 12}
\]
Using the residue theorem,

\[ I_2 = 0, \quad (53) \]

valid when \( \gamma \geq \gamma + \varepsilon \), or \( x \geq (\gamma + \varepsilon)t \), as there are no poles to the right of the contour.

\[
I_2 = \int_{-i\infty}^{i\infty} \sum_{n=1}^{\infty} \frac{s \beta \sin \beta y \sin \beta y_o \ e^{(s - \frac{x}{t} \sqrt{v^2 - 1 \lambda^2})t}}{2n\pi D[(s^2 - v^2 \beta^2) + v^2 \lambda^2] \lambda^2} \frac{(s - \frac{x}{t} \sqrt{v^2 - 1 \lambda^2})t}{1 - \frac{s^2 - \beta^2}{i\lambda^2}} ds
\]

\[
= \int_{-i\infty}^{i\infty} \sum_{n=1}^{\infty} \frac{s \beta \sin \beta y \sin \beta y_o \ e^{(s - \frac{x}{t} \sqrt{v^2 - 1 \lambda^2})t}}{2n\pi D[(s^2 - v^2 \beta^2) + iv^2 \lambda^2] \lambda^2} \frac{(s - \frac{x}{t} \sqrt{v^2 - 1 \lambda^2})t}{1 - \frac{s^2 - \beta^2}{i\lambda^2}} ds. \quad (54)
\]

The first derivative,

\[
\frac{d}{ds} \left[ \frac{(s - \frac{x}{t} \sqrt{v^2 - 1 \lambda^2})t}{\lambda^2} \right] = \quad (55)
\]
\[ \lambda^s \left[ 1 + \frac{ix_s^p}{2tD \lambda^s \sqrt{\beta - i \lambda^s}} \right] - \frac{\rho_s}{D \lambda^s} \left( s - \frac{x}{t} \sqrt{\beta - i \lambda^s} \right) t \]

is needed to find the residue at this double pole.

If the relations of equation (41) are used to simplify the expressions,

\[ I_s = -2\pi i \sum_{n=1}^{\infty} \frac{v \beta \sin \beta y \sin \beta y_s}{2\pi D} \left[ \frac{\lambda^s t + \frac{ix_s^p}{2D} - \frac{\rho_s}{D \lambda^s}}{-(i \lambda^s)^2} \right] e^{\left( s - \frac{x}{t} \sqrt{\beta - i \lambda^s} \right) t} \]

\[ = 2\pi \sum_{n=1}^{\infty} \frac{v \beta \sin \beta y \sin \beta y_s}{2\pi D} \left[ \frac{i \lambda^s t - \frac{ix_s^p}{2D} + \frac{\rho_s}{D \lambda^s}}{(i \lambda^s)^2} \right] e^{\left( s - \frac{x}{t} \sqrt{\beta - i \lambda^s} \right) t} \]

\[ = 2\pi \sum_{n=1}^{\infty} \frac{v \beta \sin \beta y \sin \beta y_s}{2\pi D} \left[ \frac{\rho_s}{\beta} \left( \frac{x}{v} \right) - 4s_1 \right] e^{\frac{s_1(\frac{x}{v} - t)}{\rho_s^2 \beta}} , \] (56)

valid when \( \gamma \leq \bar{\gamma} - \epsilon \), or \( x \leq (\bar{\gamma} - \epsilon)t \).

Figures 13 and 14 show the contours used in evaluating \( I_s \).

The values of \( I_s \) are found to be

\[ I_s = 0, \] (57)
valid when \( \gamma \geq \frac{\gamma}{\nu} + \epsilon \), \( x \geq (\frac{\gamma}{\nu} + \epsilon)t \), and

\[
I_3 = 2\pi \sum_{n} \frac{P_v \beta \sin \beta y \sin \beta \nu}{\nu n \pi D} \left[ \frac{\beta^4}{y^4} (t - \frac{x}{\nu}) + 4s_1 \right] e^{-s_1\left(\frac{t}{\nu} - t\right)}, \tag{58}
\]

valid when \( \gamma \leq \frac{\gamma}{\nu} - \epsilon \), or \( x \leq (\frac{\gamma}{\nu} - \epsilon)t \).

\[\gamma \geq \frac{\gamma}{\nu} + \epsilon\]
**Figure 13**

\[\gamma \leq \frac{\gamma}{\nu} - \epsilon\]
**Figure 14**

**Case V**: \( v > \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \). When \( v > \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \),

the poles of the integrand of \( I_3 \) are located as shown in

**Figure 8**. The saddlepoint, \( S_3 \), will coincide with a pole at two different values of \( x \). If, in equation (43), \( x \) is replaced by \( \gamma_1 t \), where \( \gamma_1 \) is the value of \( \gamma \) when the saddlepoint coincides with the pole \( S_3 \).
\[
\frac{s_3^2 (\beta - \frac{s_3}{v^2})}{v} = \gamma_1 \frac{s_3}{2D},
\]
\[
\gamma_1 = \frac{1}{v} (\beta - \frac{s_3}{v^2})^{2D}.
\]
\[
= \frac{2D \beta}{v^3} - \frac{2D}{\rho v^3} s_3^2.
\]
\[
= \frac{2D \beta}{v^3} + \frac{2D}{\rho v^3} \left( \frac{\rho v^2}{\rho v^3} - 2\beta \right) \left[ \frac{1}{\left( \frac{\rho v^2}{\rho v^3} - 2\beta \right)^2} \right]
\]
\[
= v + (v - \frac{2D \beta}{\rho v}) \sqrt{1 - \frac{4(\beta^2 + k_1)}{\left( \frac{\rho v^2}{\rho v^3} - 2\beta \right)^2}}.
\]

In the same manner it can be found that
\[
\gamma_2 = v - (v - \frac{2D \beta}{\rho v}) \sqrt{1 - \frac{4(\beta^2 + k_1)}{\left( \frac{\rho v^2}{\rho v^3} - 2\beta \right)^2}},
\]

where \( \gamma_2 \) is the value of \( \gamma \) when the saddlepoint coincides with the pole \( s_4 \).

The contours chosen, in this case, for the integration of \( I_2 \) are shown in Figures 15, 16, and 17.

Using the residue theorem,
\[
I_2 = 0,
\]
when \( \gamma \geq \gamma_1 + \epsilon \).
Figure 15

\[ y > T_y + e \]

Figure 16

\[ y < T_y - e \]

Figure 17

\[ y > Y_n \]

\[ y < Y_n \]

\[ y > Y_n \]

\[ y < Y_n \]
\[
\frac{(s_4 - \frac{x}{t} s^2 t)}{t (s_4 - s_1)(s_4 - s_2)(s_4 - s_3)} + \sum_{n} \frac{v \beta \sin \beta y \sin \beta y e^{\frac{2\pi}{y}}}{n \mu (s_1 \sin \beta y - s_2 \sin \beta y e^{\frac{2\pi}{y})}} \left[ \frac{s_1(x - vt)}{s_1} + \frac{s_2(x - vt)}{s_2} \right],
\]

(63)

when \( y \leq y_2 - \epsilon \),

\( y \geq \epsilon \).

The integral \( I_3 \) can be evaluated in a similar manner.

The contours chosen for the integrations are shown in Figures 18, 19, and 20.

\( \gamma \geq \gamma_1 + \epsilon \)  
**Figure 18**  
\( \gamma \geq \gamma_2 + \epsilon \)  
**Figure 19**  
\( \gamma \leq \gamma_2 - \epsilon \)  
**Figure 20**

\( I_3 = 0, \)  

(64)

when \( y \geq \gamma_1 + \epsilon \).
Expressions for the Deflection

Expressions for the deflection can now be found by properly combining equations derived in the previous section. The symbol \( w_\infty(x,y,t) \) indicates that the function approaches validity as time increases indefinitely, that is, only after the load has been in contact with the plate for a long period of time.

Equation (29) states that

\[
  w_\infty(x,y,t) = I_1 + I_2 + I_3. \quad (67)
\]

The five cases of the previous section must be considered separately.
Case I

when \( v < \sqrt{2D_1\beta^2} \), and \( x < vt \), equations (36), (48), and (49) give values for \( I_1 \), \( I_2 \), and \( I_3 \) respectively.

\[
\begin{align*}
\omega_\infty(x, y, t) &= \sum_{n=1}^{\infty} \frac{v^3 \beta \sin \beta y \sin \beta y_o}{nmD(s_1^2 - s_2^2)} \left[ s_1 \left( \frac{x}{v} - t \right) - \frac{s_1 \left( \frac{x}{v} - t \right)}{s_1} \right] \\
&\quad - \frac{s_2 \left( \frac{x}{v} - t \right) - \frac{s_2 \left( \frac{x}{v} - t \right)}{s_2}}{s_2} \\
&\quad - \frac{s_1 \left( \frac{x}{v} - t \right) - \frac{s_1 \left( \frac{x}{v} - t \right)}{s_1}}{s_1}
\end{align*}
\]

\( (68) \)

The expressions \( \frac{s_1}{v} \) and \( \frac{s_2}{v} \) can be separated into their real and imaginary components:

\[
\frac{s_1}{v} = (f - ig), \quad (69)
\]

\*See Appendix D.
\[
\frac{s_a}{v} = -(f + ig), \quad (69)
\]
where
\[
f = \sqrt{\beta + k_1} \sqrt{\frac{1}{2}\left(1 - \sqrt{\frac{\rho v^2 - 2 \beta^2}{4(\beta^2 + k_1)}}\right)},
\]
and
\[
g = \sqrt{\beta + k_1} \sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{\rho v^2 - 2 \beta^2}{4(\beta^2 + k_1)}}\right)}.
\]

From these relations
\[
\frac{s_1 s_a}{v^2} = -(f^2 + g^2) = -\sqrt{\beta^2 + k_1},
\]
and
\[
\frac{s_1 s_2}{v^2} = -41fg. \quad (70)
\]

Using the relations in equations (69) and (70),
\[
w_\infty(x, y, t) = -\sum_{n, \ell} \frac{\rho \sin \beta y \sin \beta_y}{nmD(41fg)\sqrt{\beta^2 + k_1}} \left[(-f - ig) e^{(f - ig)(x - vt)} + \right]
\]
so that

$$w_\infty(x,y,t) = \sum_{n} P\beta \sin \beta y \sin \beta y_0 e^{\frac{f(x-\nu t)}{2n\pi \sqrt{\beta^4 D^2 + kD}}} [-f \sin \sigma(x-\nu t)$$

$$+ g \cos \sigma(x-\nu t)] .$$

This expression is valid when $x < \nu t$ and $0 < \nu < \sqrt{2D_1 \beta^2}$. It is the equation of the "trailing wave."

When $x > \nu t$, equation (37) gives $I_1 = 0$. Under these conditions
Using the expressions in equations (69) and (70), this expression for the deflection can be expressed as

\[ w_\infty(x,y,t) = - \sum_{n} \frac{P \beta \sin \beta y \sin \beta y_o}{2\pi D(4\pi f \beta A^2 + kD)} \left[ \left( f + ig \right)(x-\tau t) \right. \]

\[ - \left( f + ig \right)(x-\tau t) \]

\[ \left. - \frac{f(x-\tau t)}{2\pi D(4\pi f \beta A^2 + kD)} \right] \]
so that

\[
w_\infty(x, y, t) = \sum_{n=1}^{\infty} \frac{P\beta \sin \beta y \sin \beta y_0 e^{-(x-\nu t)}}{2n\pi f\sqrt{\nu^2 - D^2 + kD}} \left[ f \sin g(x-\nu t) + g \cos g(x-\nu t) \right].
\]  

This expression is valid when \( x > vt \) and \( 0 < \nu < \sqrt{2D_k \beta} \). It is the equation of the "leading wave."

**Case II**

When \( x < vt \), and \( \nu = \sqrt{2D_1 \beta} \),

\[
I_1 = -\sum_{n=1}^{\infty} \frac{P\beta \sin \beta y \sin \beta y_0 (x - \nu t)^3}{3n\pi D}, 
\]

\( I_2 = 0, \)

and

\( I_3 = 0. \)

Using these equations, the deflection is

\[
w_\infty(x, y, t) = -\sum_{n=1}^{\infty} \frac{P\beta \sin \beta y \sin \beta y_0 (x - \nu t)^3}{3n\pi D}. 
\]

This is the equation of the "trailing wave."

When \( x > vt \), \( I_1 = I_2 = I_3 = 0 \), equations (37) and (50),
so that

\[ w_\infty(x,y,t) = 0. \quad (74) \]

This is the equation of the "leading wave."

**Case III**

When \( \sqrt{2D\beta_x} < v < \sqrt{2D_1(\beta_x^2 + \beta_y^2 + k_1^2)} \), and \( x < vt \),

\[
I_1 = \sum_{n} \frac{v \beta_x \sin \beta_y \sin \beta_x}{\rho \nu D(s_1^2 - s_2^2)} \left[ e^{-s_1 \frac{x}{v} - t} - e^{s_1 \frac{x}{v} - t} \right] \frac{s_1 - s_2}{s_2} \quad (36)
\]

\[
I_2 = \sum_{n} \frac{v \beta_x \sin \beta_y \sin \beta_x}{\rho \nu D(s_1^2 - s_2^2)} \frac{s_1(x-vt)}{s_1} \quad (51)
\]

and

\[
I_3 = \sum_{n} \frac{v \beta_x \sin \beta_y \sin \beta_x}{\rho \nu D(s_1^2 - s_2^2)} \frac{-s_2(x-vt)}{s_2} \quad (52)
\]

Combining these three expressions,

\[
w_\infty(x,y,t) = \sum_{n} \frac{v \beta_x \sin \beta_y \sin \beta_x}{\rho \nu D(s_1^2 - s_2^2)} \left[ e^{-s_1 \frac{x}{v} - t} - e^{s_1 \frac{x}{v} - t} \right] \frac{s_1 - s_2}{s_2} \quad (53)
\]
\[
- \frac{s_2(x-vt)}{v} - \frac{s_2(x-vt)}{v} + \frac{s_1(x-vt)}{v} + \frac{s_2(x-vt)}{v}
\]

\[
= \sum_{n=1}^{\infty} \frac{\beta \sin \beta y \sin \beta y_n}{n \\pi (s_1^2 - s_2^2) s_1 s_2} \left[ \frac{s_2}{v} e^{-\frac{s_1(x-vt)}{v}} + \frac{s_1}{v} e^{-\frac{s_2(x-vt)}{v}} \right]. \quad (75)
\]

It can be shown that for this range of values of $v$,
\[
\frac{s_1}{v} \quad \text{and} \quad \frac{s_2}{v}
\]

can again be separated into their real and imaginary components:

\[
\frac{s_1}{v} = -f + ig,
\]

\[
\frac{s_2}{v} = f + ig,
\]

where again

\[
f = \sqrt{\beta^2 + k_1} \left\{ \frac{1}{2} \left( 1 - \sqrt{\frac{\beta^2}{\beta^2 - 2\beta^2}} \right) \right\},
\]

and

\[
g = \sqrt{\beta^2 + k_1} \left\{ \frac{1}{2} \left( 1 + \sqrt{\frac{\beta^2}{\beta^2 - 2\beta^2}} \right) \right\}.
\]
From these relations

\[ \frac{s_1 s_2}{v^2} = -(f^2 + g^2) = -\sqrt{\beta^4 + k_1^2}, \]

and

\[ \frac{s_1^2 - s_2^2}{v^2} = -4ifg. \]

Using the relations in equations (76) and (77),

\[
w_\infty(x,y,t) = \sum_n \frac{\beta \sin \beta y \sin \beta y_0}{n D (4ifg) \sqrt{\beta^4 + k_1^2}} [(f + ig)(x - vt) + (f + ig)(x - vt)]
\]

\[
= \sum_n \frac{\beta \sin \beta y \sin \beta y_0}{2nmf \sqrt{\beta^4 + kD}} \left[ f \frac{e^{-(i\gamma \gamma_0 - i\gamma \gamma_0) e^{-(i\gamma \gamma - i\gamma \gamma)}}}{21} + e^{-(i\gamma \gamma - i\gamma \gamma) e^{-(i\gamma \gamma - i\gamma \gamma)}} \right],
\]

so that

\[
w_\infty(x,y,t) = \sum_n \frac{\beta \sin \beta y \sin \beta y_0 \sin \beta y_0 \sin \beta y_0}{2nmf \sqrt{\beta^4 + kD}} \left[ -f \sin g(x - vt) + \right.
\]

\[
\left. + e^{-(i\gamma \gamma - i\gamma \gamma) e^{-(i\gamma \gamma - i\gamma \gamma)}} \right].
\]
This is the equation of the "trailing wave," valid when \( x < vt \) and \( \sqrt{2D_1\beta^2} < v < \sqrt{2D_1(\beta^2 + \beta_1^2 + k_1^2)} \).

When \( x > vt \), equation (37) gives \( I_1 = 0 \); and \( I_2 \) and \( I_3 \) have the values as in equations (51) and (52), so that

\[
w_{\infty}(x,y,t) = \sum_{n=0}^{\infty} \frac{P\beta \sin \beta y \sin \beta y_o}{nmD(4\beta^2 + k_1^2)} \left[ s_2 \frac{s_1(x-vt)}{e^v} + \frac{s_1}{v} e^{-\frac{s_2}{v}(x-vt)} \right].
\]

Using the expressions in equations (76) and (77),

\[
w_{\infty}(x,y,t) = \sum_{n=0}^{\infty} \frac{P\beta \sin \beta y \sin \beta y_o}{nmD(4\beta^2 + k_1^2)} \left[ (f + ig)e^{-f+ig}(x-vt) + (-f + ig)e^{-(f+ig)(x-vt)} \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{P\beta \sin \beta y \sin \beta y_o}{2nmD\sqrt{\beta^2 + kD^2}} \left[ \frac{e^{-f(x-vt)}}{e^{\frac{ig(x-vt)}{2} - \frac{-ig(x-vt)}{2}}} \right. \\
\left. + \frac{e^{ig(x-vt)}}{2} - \frac{e^{-g(x-vt)}}{2} + g \frac{e^{\frac{ig(x-vt)}{2}} + e^{-\frac{-ig(x-vt)}{2}}}{2} \right]
\]
so that

$$w_\infty(x,y,t) = \sum_{n \neq 0} \frac{P\beta \sin \beta y \sin \beta y_0}{2\pi \sqrt{\beta_0 \Delta^2 + kD}} \left[ f \sin g(x-\nu t) - \frac{\nu^2}{2\pi \sqrt{\beta_0 \Delta^2 + kD}} \right].$$

(79)

This is the equation of the "leading wave," valid when $x > \nu t$, and $\sqrt{2D_1 \beta_0^2} < \nu < \sqrt{2D_1 (\beta_0^2 + \beta_k^2 + k_1)}$.

Case IV

When $\nu = \sqrt{2D_1 (\beta_0^2 + \beta_k^2 + k_1)}$, and $x < \nu t$, equation (42) gives

$$I_1 = 2\pi \sum_{n \neq 0} \frac{\nu P\beta \sin \beta y \sin \beta y_0}{2\pi \sqrt{\beta_0 \Delta^2 + kD}} \left[ \frac{-\nu^4}{2\pi \sqrt{\beta_0 \Delta^2 + kD}} (t - \frac{\nu}{\nu} - 4s_1 - s_1(\frac{\nu}{\nu} - t) \right]$$

$$+ \frac{-\nu^4}{2\pi \sqrt{\beta_0 \Delta^2 + kD}} (t - \frac{\nu}{\nu} + 4s_1 - s_1(\frac{\nu}{\nu} - t) \right].$$

(42)

$$I_2 = 2\pi \sum_{n \neq 0} \frac{\nu P\beta \sin \beta y \sin \beta y_0}{2\pi \sqrt{\beta_0 \Delta^2 + kD}} \left[ \frac{-\nu^4}{2\pi \sqrt{\beta_0 \Delta^2 + kD}} (t - \frac{\nu}{\nu} - 4s_1) \right] s_1(\frac{\nu}{\nu} - t)$$

(56)

when $\gamma \leq \nu - \epsilon$, or $x \leq (\nu - \epsilon)t$. 
\[ I_\alpha = 2\pi \sum_{n} \frac{v^3 P \sin \beta y \sin \beta y_e}{n a D} \left[ \frac{v^4 (t - \frac{X}{V}) + 4s_1}{\rho^2 v D^2 - \beta^2} \right] e^{-s_1 \left( \frac{X}{V} - t \right)} \]

when \( \gamma \leq \bar{y} - \varepsilon \), or \( x \leq (\bar{y} - \varepsilon)t \).

Adding these three equations,

\[ w_\infty (x, y, t) = 0, \] (80)

when \( x \leq (\bar{y} - \varepsilon)t \) or \( x \leq (v - \varepsilon)t \) and \( v = \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \).

Equations (37), (53) and (57) show that the values of \( I_1 \), \( I_2 \), and \( I_3 \) are all zero when \( x \geq (\gamma + \varepsilon)t \) or \( x \geq (v + \varepsilon)t \). Therefore

\[ w_\infty (x, y, t) = 0, \] (81)

when \( x \geq (v + \varepsilon)t \) and \( v = \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \).

Case V

When \( v > \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \), and \( x \leq (\gamma_2 - \varepsilon)t + z \), equations (36), (63) and (66) give the proper values of \( I_1 \), \( I_2 \), and \( I_3 \) respectively. Adding these expressions,

\[ w_\infty (x, y, t) = \sum_{n} \frac{v^3 P \sin \beta y \sin \beta y_e}{n a D(s_1^2 - s_2^2)} \left[ \frac{-S_1(x-vt)}{v} \frac{S_1(x-vt)}{e^v} \right] s_1 \]
\[- \frac{s_3(x-vt)}{v} \frac{s_5(x-vt)}{v} \frac{s_1(x-vt)}{s_1} - \frac{s_5(x-vt)}{s_5} \]
\[- \frac{s_1(x-vt)}{e} \frac{s_3(x-vt)}{e} \frac{s_1(x-vt)}{s_1} + \frac{s_5(x-vt)}{s_5} \]

or

\[w_\infty(x,y,t) = 0, \quad (82)\]

when \( x \leq (y_2 - \epsilon)t + z \).

When \((y_2 - \epsilon)t + z < x < vt\), equations (36), (62) and (65) give the values of \(I_1, I_2,\) and \(I_3\). Adding these expressions,

\[w_\infty(x,y,t) = \sum_{n=1}^{\infty} \frac{s_5(x-vt)}{v} \frac{s_3(x-vt)}{s_3} \frac{s_1(x-vt)}{s_1} \frac{r}{n} \frac{\sin \theta(y) \sin \theta(y)}{\sin \theta(y) \sin \theta(y)} \left[ - \frac{s_1(x-vt)}{v} - \frac{s_3(x-vt)}{s_3} \right] \]
\[- \frac{s_3(x-vt)}{v} - \frac{s_5(x-vt)}{s_5} + \frac{s_1(x-vt)}{s_1} - \frac{s_1(x-vt)}{s_1} \]

\[= \sum_{n=1}^{\infty} \frac{r}{n} \frac{\sin \theta(y) \sin \theta(y)}{\sin \theta(y) \sin \theta(y)} \left[ \frac{s_3(x-vt)}{v} - \frac{s_5(x-vt)}{s_5} \right]. \quad (83)\]
Now let

$$s_1 = \sqrt{\frac{\rho v^2}{(D - 2\beta)^2}} \left[ 1 + \sqrt{1 - \frac{4(\beta^2 + k_1)}{\rho v^2}} \right]$$ \hspace{1cm} (84)

and

$$s_2 = \sqrt{\frac{\rho v^2}{(D - 2\beta)^2}} \left[ 1 - \sqrt{1 - \frac{4(\beta^2 + k_1)}{\rho v^2}} \right]$$ \hspace{1cm} (85)

so that

$$\frac{s_1}{v} = ig_1 \text{ and } \frac{s_2}{v} = ig_2.$$ \hspace{1cm}

Then

$$\frac{s_1^2 - s_2^2}{v^2} = \left(\frac{\rho v^2}{D - 2\beta} \right) \sqrt{1 - \frac{4(\beta^2 + k_1)}{\rho v^2}} = f_1.$$ \hspace{1cm} (86)

If the expressions in equations (84), (85) and (86) are used, equation (83) can be written as

$$w_\infty(x, y, t) = \sum_{n}^{\infty} \frac{\rho \beta \sin \beta y \sin \beta y}{n \pi D f_1} \left[ \frac{ig_2(x-\nu t) - ig_2(x-\nu t)}{i g_2(x-\nu t)} \right].$$
or

\[ w_\infty(x, y, t) = \sum_{n} 2P\beta \sin \beta y \sin \beta y_0 \frac{\sin s_2(x-vt)}{\nu n Df_1 g_2} \sin s_2(x-vt) \]  \hspace{1cm} (87)

when \((\gamma_2 + \varepsilon)t + z \leq x < vt\).

When \(vt > x \geq (\gamma_1 - \varepsilon)t + z\), equations (37), (62) and (65) give the values of \(I_1, I_2,\) and \(I_3\). Adding these expressions,

\[ w_\infty(x, y, t) = \sum_{n} \frac{3P\beta \sin \beta y \sin \beta y_0}{\nu n D(s_1^2 - s_2^2)} \left[ 0 + \frac{s_1(x-vt)}{s_1} \right] \]

\[ - \frac{s_1(x-vt)}{s_1} \]

\[ = \sum_{n} \frac{P\beta \sin \beta y \sin \beta y_0}{\nu n D(s_1^2 - s_2^2)} \left[ \frac{s_1(x-vt)}{s_1} - e \frac{s_1(x-vt)}{s_1} \right]. \]  \hspace{1cm} (88)

If the relations in equations (84), (85) and (86) are used, equation (88) can be written as

\[ w_\infty(x, y, t) = \sum_{n} \frac{2P\beta \sin \beta y \sin \beta y_0}{\nu n Df_1 g_1} \sin s_1(x-vt), \]  \hspace{1cm} (89)

when \(vt > x \geq (\gamma_1 - \varepsilon)t + z\).
When \( x \geq (\gamma_1 + \varepsilon)t + z \), equations (37), (61) and (64) show that \( I_1 \), \( I_2 \) and \( I_3 \) are each zero, therefore

\[
W_\infty(x, y, t) = 0, \quad (90)
\]
when \( x \geq (\gamma_1 + \varepsilon)t + z \).

Summary of the Expressions for the Deflection

Listed below are the expressions for the deflection when a point load is moving with a constant velocity along a semi-infinite plate that is resting on an elastic subgrade.

If \( 0 < v < \sqrt{2D_\beta} \),

\[
W_\infty(x, y, t) = \sum_{n=1}^{\infty} \frac{P \sin \beta y \sin \beta y_0 \, e^{-f(x-vt)}}{2bf_0 \sqrt{\beta^2 D^2 + kD}} \left[-f \sin g(x-vt) + g \cos g(x-vt) \right], \quad (71)
\]
when \( x < vt \), and

\[
W_\infty(x, y, t) = \sum_{n=1}^{\infty} \frac{P \sin \beta y \sin \beta y_0 \, e^{-f(x-vt)}}{2bf_0 \sqrt{\beta^2 D^2 + kD}} \left[f \sin g(x-vt) + g \cos g(x-vt) \right], \quad (72)
\]
when $x > vt$, where $f$ and $g$ are defined in equation (59).

If $v = \sqrt{2D_1 \beta^2}$,

$$w_\infty(x,y,t) = -\sum_n \frac{\text{P} \sin \beta y \sin \beta y_0}{3bD} (x - vt)^3, \quad (73)$$

when $x < vt$, and

$$w_\infty(x,y,t) = 0, \quad (74)$$

when $x > vt$.

If $\sqrt{2D_1 \beta^2} < v < \sqrt{2D_1 (\beta^2 + \sqrt{\beta^4 + k_1})}$,

$$w_\infty(x,y,t) = \sum_n \frac{\text{P} \sin \beta y \sin \beta y_0 e^{-f(x-vt)}}{2bf_0 \sqrt{\beta^4 \Delta + kD}} [-f \sin g(x-vt)$$

$$+ g \cos g(x-vt)], \quad (78)$$

when $x < vt$, and

$$w_\infty(x,y,t) = \sum_n \frac{\text{P} \sin \beta y \sin \beta y_0 e^{-f(x-vt)}}{2bf_0 \sqrt{\beta^4 \Delta + kD}} [f \sin g(x-vt)$$

$$+ g \cos g(x-vt)], \quad (79)$$

when $x > vt$, where $f$ and $g$ are defined in equation (76).
If \( v = \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \),

\[ w_\infty(x,y,t) = 0, \quad (80) \]

when \( x \leq (v - \epsilon)t \), and

\[ w_\infty(x,y,t) = 0, \quad (81) \]

when \( x \geq (v + \epsilon)t \).

If \( v > \sqrt{2D_1(\beta^2 + \sqrt{\beta^4 + k_1})} \),

\[ w_\infty(x,y,t) = 0, \quad (82) \]

when \( x \leq (\gamma_2 - \epsilon)t + z \),

\[ w_\infty(x,y,t) = \sum_{n}^{\infty} \frac{2 \text{Ps} \sin \beta y \sin \beta y_n}{b \text{D}_1 \gamma_2} \sin \gamma_2(x - vt), \quad (87) \]

when \( (\gamma_2 + \epsilon)t + z \leq x < vt \),

\[ w_\infty(x,y,t) = \sum_{n}^{\infty} \frac{2 \text{Ps} \sin \beta y \sin \beta y_n}{b \text{D}_1 \gamma_1} \sin \gamma_1(x - vt), \quad (89) \]

when \( vt > x \geq (\gamma_1 - \epsilon)t + z \), and

\[ w_\infty(x,y,t) = 0, \quad (90) \]
when \( x \geq (\gamma_1 + \varepsilon)t + z \), where \( s_1 \) is defined in equation (84), \( g_2 \) in equation (85), \( f_1 \) in equation (86), \( \gamma_1 \) in equation (59), and \( \gamma_2 \) in equation (60).
Discussion of the Problem

Three types of dynamic loads were considered in this thesis. First a block load, moving with a constant velocity, was applied to an elastically supported rectangular plate of finite dimensions. A formal expression for the deflection of the plate caused by this load was derived. The block load was then shrunk to a line load through a limiting process. A formal expression for the deflection caused by this line load was derived. Next the line load was shrunk, through another limiting process, to a point load. A formal expression for the deflection caused by this point load was then derived. No attempt was made to determine explicit values of the deflection in these three expressions.

The finite plate was next expanded into a semi-infinite strip by letting the length of the plate, in the x-direction, increase without limit. A point load moving with a constant velocity was applied to the strip. Lörr (3), in his paper, had a point load moving with a constant velocity v along a beam that was resting on an elastic foundation. The problem in this paper was developed by a method that was analogous to the method used by Lörr. It was necessary that the deflection function be a function of two linear dimensions and time.
instead of just one linear dimension and time. There was one distinct difference; Dörr found only one critical value of the velocity while two were encountered in this paper.

Dörr found that when \( x < vt \) and the velocity was less than his critical velocity, the deflection could be expressed as

\[
y_\infty = \frac{a(x-vt)}{4\sqrt{cEI}} [-a \sin b(x-vt) + b \cos b(x-vt)],
\]

where

\[
a = \sqrt{\frac{c}{EI}} \sqrt{\frac{1}{2} \left( 1 - \frac{m^2 v^4}{4cEI} \right)},
\]

and

\[
b = \sqrt{\frac{c}{EI}} \sqrt{\frac{1}{2} \left( 1 + \frac{m^2 v^4}{4cEI} \right)}.
\]

In this paper, when \( x < vt \) and the velocity was less than the first critical velocity, the deflection function was

\[
w_\infty(x,y,t) = \sum_{n=1}^{\infty} \frac{P_n \sin \beta y \sin \beta y_0 e^{f(x-vt)}}{2bf \sqrt{\beta^4 D^2 + KD}} [f(x-vt) - f \sin g(x-vt)] + g \cos g(x-vt)],
\]

where
\[
\begin{align*}
\phi &= \sqrt{\beta + k_1} \sqrt{\frac{1}{3} \left( 1 - \sqrt{\frac{(\beta^2 - 2\beta)^2}{4(\beta^4 + k_1)}} \right)}, \\
\gamma &= \sqrt{\beta + k_1} \sqrt{\frac{1}{3} \left( 1 + \sqrt{\frac{(\beta^2 - 2\beta)^2}{4(\beta^4 + k_1)}} \right)}.
\end{align*}
\]

On the other side of the point load, when \( x > vt \), Dörr found the deflection to be

\[
y_\infty = \frac{e}{4WeIL} a \sin b(x-vt) + b \cos b(x-vt),
\]

where \( a \) and \( b \) are defined above. In this paper the deflection function is

\[
w_\infty(x,y,t) = \sum_{n} \frac{F \sin \beta_y \sin \beta_y e^{-f(x-vt)}}{2bfg \sqrt{\beta^4 + kD}} [f \sin g(x-vt) + g \cos g(x-vt)].
\]

After passing his critical velocity, Dörr has the following expressions for the deflection:
\begin{align*}
y_\infty &= 0, & \text{for } x \leq (\lambda_2 - \varepsilon)t + z, \\
y_\infty &= -\frac{\sin b_2(x-\nu t)}{\sqrt{\frac{v}{\nu^2} b_2}}, & \text{for } (\lambda_2 + \varepsilon)t + z \leq x \leq \nu t, \\
y_\infty &= -\frac{\sin b_1(x-\nu t)}{\sqrt{\frac{v}{\nu^2} b_1}}, & \text{for } \nu t \leq x \leq (\lambda_1 - \varepsilon)t + z, \\
y_\infty &= 0, & \text{for } x \geq (\lambda_1 + \varepsilon)t + z, 
\end{align*}

where

\begin{align*}
b_1 &= \sqrt{\frac{mv^2}{2EI}} \left(1 + \sqrt{1 - \frac{4EI}{m^2v^4}}\right), \\
b_2 &= \sqrt{\frac{mv^2}{2EI}} \left(1 - \sqrt{1 - \frac{4EI}{m^2v^4}}\right), \\
a &= 2\sqrt{\frac{m^2v^4}{4EI}} - 1.
\end{align*}

In this paper, after passing the second critical velocity, the deflection functions were

\begin{align*}
w_\infty(x,y,t) &= 0, & \text{for } x \leq (y_2 - \varepsilon)t + z, \\
w_\infty(x,y,t) &= \sum_{1}^{\infty} \frac{2\nu \sin \beta y \sin \beta y_2}{b_1 b_2} \sin \frac{\beta}{b_1} \sin \frac{\beta}{b_2} (x-\nu t),
\end{align*}
for \((\gamma_2 + \varepsilon)t + z \leq x < vt\),

\[
w_\infty(x,y,t) = \sum_{n=1}^{\infty} \frac{2\pi \sin \beta y \sin \beta y_n}{bD \pi_1} \sin \pi_1(x-vt),
\]

for \(vt > x \geq (\gamma_1 - \varepsilon)t + z\),

\[
w_\infty(x,y,t) = 0,
\]

for \(x \geq (\gamma_1 + \varepsilon)t + z\),

where

\[
\pi_1 = \sqrt{\frac{V^2 - \beta^2}{2}} \left[ 1 + \sqrt{1 - \frac{4(\beta^4 + k_1)}{(V^2 - \beta^2)^2}} \right],
\]

\[
\pi_2 = \sqrt{\frac{V^2 - \beta^2}{2}} \left[ 1 - \sqrt{1 - \frac{4(\beta^4 + k_1)}{(V^2 - \beta^2)^2}} \right],
\]

\[
f_1 = -(\frac{V^2}{D} - 2\beta^2) \sqrt{1 - \frac{4(\beta^4 + k_1)}{(V^2/D - 2\beta^2)^2}}.
\]

These are the only conditions where comparison of expressions can be made. They coincide as to form. The infinite series are convergent with one exception. Equation
\( w_\infty(x, y, t) = - \sum_{n=1}^{\infty} \frac{P \sin \beta y \sin \beta y}{3bD} (x - vt)^3, \)

seems to be a divergent series. This is the expression for the deflection when \( x < vt \) and at the critical velocity \( v = \sqrt{2D_1 \beta^2} \). The same method of attack was used in deriving this equation as was used in deriving the other equations. Possibly a limiting process of some kind would result in a useful expression. The deflection functions on both sides of this critical velocity were identical. When \( x > vt \), the deflection was found to be zero at this critical velocity.

**Possible Extensions**

Further work could be done on this problem at several specific points. The deflection at the critical velocity, \( v = \sqrt{2D_1 \beta^2} \), needs further study. The deflection directly under the point load was not studied for any value of the velocity. The deflections at the points \( x = \gamma_1 t \) and \( x = \gamma_2 t \), when \( \sqrt{2D_1 \beta^2} < v < \sqrt{2D_1 (\beta^2 + \beta^4 + k_1) } \), were not studied.

The velocity of the load could be expressed as a variable instead of a constant, possibly a function that would decrease with time. There are possibilities that the plate could be changed into a semi-infinite plate and not just a strip of
finite width, that is, expand the plate in the y-direction also. The applied load could be changed from a constant load to one that was a variable load. The $s$-integration could be investigated by some process other than the "saddlepoint" method. It would be useful to have expressions for the deflection at any time $t$.

No attempt was made to determine explicit values of the deflection caused by the block load, the line load, and the point load moving along the finite plate. If the $s$-integration can be performed, these deflection functions would be expressed as double infinite series.
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APPENDICES

APPENDIX A

Lemma: If $f(x)$ satisfies a Hölder condition for $0 \leq x \leq \infty$, that is if $|f(x+\Delta x) - f(x)| \leq H|\Delta x|^\gamma$ for $0 \leq \gamma \leq 1$, and if $|f(x)| \leq \phi(x)$ where $\phi(x)$ is a monotone decreasing function of $x$, and $\int_0^\infty \phi(x) \, dx$ exists, then for $h > 0$, $\sum_{m} hf(mh)$ converges and

$$\lim_{h \to 0} \sum_{m} hf(mh) = \int_0^\infty f(x) \, dx.$$

Further if $f(x)$ depends on a parameter $t$, and $H$ and $\phi(x)$ are independent of $t$, then the limit exists uniformly in $t$.

Proof:

$$\sum_{m=1,3,5,\ldots}^{\infty} 2h|f(mh)|$$

converges since

$$2h|f(mh)| \leq \int_{mh}^{(m+2)h} \phi(x) \, dx \quad \text{and} \quad \sum_{m=1,3,5,\ldots}^{\infty} \int_{mh}^{(m+2)h} \phi(x) \, dx = \int_0^\infty \phi(x) \, dx$$

which converges. Let

*This Lemma and its proof was copied from a thesis written by Lawrence E. Payne (9).*
\[
N = (2h)^{-1} - \frac{\gamma}{2} = \frac{1}{2h} \left( 1 + \frac{\gamma}{2} \right)
\]

Then
\[
\sum_{m=1}^{\infty} 2hf(mh) - \int_{mh}^{\infty} f(x)\,dx = \sum_{m=1}^{N} 2hf(mh) - \int_{mh}^{(m+2)h} f(x)\,dx + \sum_{N+2}^{\infty} 2hf(mh) - \int_{(N+2)h}^{\infty} f(x)\,dx.
\]

Now consider the absolute value of the right members.
\[
\left| \sum_{m=1}^{N} 2hf(mh) - \int_{mh}^{(m+2)h} f(x)\,dx \right| \leq \sum_{m=1}^{N} \int_{mh}^{(m+2)h} |f(mh) - f(x)|\,dx \\
\leq \sum_{m=1}^{N} H \int_{mh}^{(m+2)h} (x - mh) \,dx \\
\leq \frac{(2h)^{\gamma/2}}{1 + \gamma}.
\]

Also
\[
\left| \sum_{m=1}^{\infty} 2hf(mh) \right| \leq \sum_{m=1}^{\infty} 2hf(mh) \leq \sum_{m=1}^{\infty} \phi(x)\,dx \leq \int_{\mathbb{R}^+} \phi(x)\,dx
\]

and
\[
\left| \int_{(N+2)h}^{\infty} f(x)\,dx \right| \leq \int_{(N+2)h}^{\infty} \phi(x)\,dx \leq \int_{\mathbb{R}^+} \phi(x)\,dx.
\]
Therefore, given an $\varepsilon > 0$, one can choose $h$ so small that each of the above terms is less than $\varepsilon/3$. Hence

$$\lim_{h \to 0} \sum_{m}^{\infty} 2 hf(mh) = \int_{0}^{\infty} f(x)dx.$$ 

It can easily be shown by the same procedure

$$\lim_{h \to 0} \sum_{m}^{\infty} hf(mh) = \int_{0}^{\infty} f(x)dx.$$ 

In most plate problems one has $\sum_{m}^{\infty} \cos (m \pi x/a) \text{ or } \sum_{m}^{\infty} \sin (m \pi x/a)$. So one sets $h = \pi/a$. Then $mn/a = mh$. Thus $h$ goes to zero as the a dimension goes to infinity. The $H$ in the above lemma is a positive constant. The $h$ is a variable that approaches zero as a limit.
APPENDIX B

The relation, \( [i \lambda^2] = (\frac{s^2_{1}}{v^2} - \beta^2) \), will be established here.

\[
s_1 = \left( \frac{\sqrt{\frac{v}{2} (\frac{\nu^2}{D} - 2\beta^2)} \left[ 1 + \sqrt{1 - \frac{4(\beta^4 + \frac{k}{D})}{(\frac{\nu^2}{D} - 2\beta^2)^2}} \right]}{2} \right) - \beta^2
\]

\[
s_3 = -s_1.
\]

\[
\frac{s_1^2 - \beta^2}{i[\lambda^2]} = \frac{s_1^2}{s^2_{s_{s}} - \beta^2} = \frac{s_1^2}{s^2_{s_{s}} + k}
\]

\[
\sqrt{\frac{\nu^2 (\frac{\nu^2}{D} - 2\beta^2)}{2D}} \left[ 1 + \sqrt{1 - \frac{4(\beta^4 + \frac{k}{D})}{(\frac{\nu^2}{D} - 2\beta^2)^2}} \right] - \frac{k}{D}
\]

\[
\frac{\nu^2 (\frac{\nu^2}{D} - 2\beta^2)}{2D} \left[ 1 + \sqrt{1 - \frac{4(\beta^4 + \frac{k}{D})}{(\frac{\nu^2}{D} - 2\beta^2)^2}} \right] - \beta^2
\]

\[
\sqrt{\frac{\nu^2 (\frac{\nu^2}{D} - 2\beta^2)}{2D}} \left[ 1 + \frac{1}{(\frac{\nu^2}{D} - 2\beta^2)} \sqrt{\frac{\nu^2}{D} - \frac{4\beta^2 v^2}{D} - \frac{4k}{D}} \right] - \beta^2
\]

\[
\sqrt{\frac{\nu^2 (\frac{\nu^2}{D} - 2\beta^2)}{2D}} \left[ 1 + \frac{1}{(\frac{\nu^2}{D} - 2\beta^2)} \sqrt{\frac{\nu^2}{D} - \frac{4\beta^2 v^2}{D} - \frac{4k}{D}} \right] - \frac{k}{D}
\]
\[
\begin{align*}
\sqrt{\frac{\rho^2 v^4}{2D^2} \frac{\rho v^2 \beta^2}{D} - \frac{k}{D} + \frac{\rho v^2}{2D} \frac{\rho v^2}{D^2} - \frac{4 \rho \beta^2 v^2}{D^2} - \frac{4k}{D}}
= -\frac{\rho v^2}{2D} \frac{1}{2} \sqrt{\frac{\rho^2 v^4}{D^2} - \frac{4 \rho \beta^2 v^2}{D^2} - \frac{4k}{D}}
\end{align*}
\]

= ± 1.

The plus sign is used for the expression involving \(s_1\) and the minus sign is used with \(s_2\).
APPENDIX C

EVALUATION OF THE $N_a$ INTEGRAL

This is a summary of work done by Dörr (3) on pages 176-179. The symbols of this thesis are used. Dörr had only one integral in his paper whereas in this thesis there is an infinite series of integrals. Each of these integrals is of the same form as the integral of Dörr's paper. If each integral is equal to zero, then the infinite sum will equal zero.

In order to evaluate the integral

$$N_a = \int_{C_1} \sum_{n} \left( s - \sqrt{s^2 - i \lambda^2} \right) e^{-z\sqrt{s^2 - i \lambda^2}} G(s) ds,$$

where

$$K = \frac{i v^3 \tilde{P} \sin \beta y \sin \beta y_e}{2 \pi n^2 D},$$

and

$$G(s) = \frac{s^2 - \beta}{(s^2 - v^2 \beta_e)^2 + v^4 \lambda^4},$$

let

$$s = s_s + q,$$

where $s_s$ is the saddlepoint.
If this substitution is made, then

\[(s - \delta \sqrt{\beta^2 - 1 \lambda^2})t = (s - \delta \sqrt{\beta^2 - 1 \left(\frac{s_s + q}{D}\right)^2 + k})t\]

\[= (s_s + q - \delta \sqrt{\beta^2 - 1 \left(\frac{s_s + q}{D}\right)^2 + k})t\]

\[= \delta \left(s_s - \sqrt{\beta - 1 \left(\frac{s_s q}{D}\right)^2 + k}\right)\]

\[+ \delta \left[\frac{q}{\delta} - \sqrt{\beta - 1 \left(\frac{s_s q}{D}\right)^2 + k}\right] \left(\sqrt{\beta - 1 \left(\frac{s_s q}{D}\right)^2 + k} - 1\right)\]

\[= \delta p_s(\delta) t + \delta p(q)t. \quad (2)\]

Also

\[-z\sqrt{\beta^2 - 1 \lambda^2} = z(p_e + p - \frac{s_s q}{\delta}). \quad (3)\]

Using these expressions, \(N_2\) can be written as

\[N_2 = \sum_{n}^{\infty} \frac{p_e \delta t + (p_e - \frac{s_s}{\delta})z}{Ke} \int_{e}^{p} f t (p - \frac{q(p)}{\delta})z \]

\[\times G(s_s + q(s)) \frac{dq(p)}{dp} dp. \quad (4)\]
The function \( q(p) \) can be expressed as an infinite series of increasing powers of \( \sqrt{p} \), because \( p \) can be expressed as a series of increasing powers of \( q \),

\[
p = a_2 q^2 + a_3 q^3 + a_4 q^4 + \ldots \tag{5}
\]

This series has a radius of convergence differing from zero, when \( \delta > 0 \). The contour can then be made to go around the saddlepoint on a circle of radius differing from zero. Within this circle \( p(q) \) is always regular.

When the coefficient \( a_2 \) is different from zero, then from equation (5),

\[
q = b_1 p^{1/2} + b_2 p^{2/2} + b_3 p^{3/2} + \ldots = \sum_{k} b_k p^{k/2} \tag{6}
\]

The coefficient \( a_2 \) is different from zero when at \( q = 0 \) or \( s = s_s \),

\[
\frac{d^2}{ds^2} (s - \delta \sqrt{\beta^2 - i \alpha^2}) \neq 0. \tag{7}
\]

The value of \( s \) that makes this second derivative vanish has its real part not equal to zero, therefore it will not coincide with the saddlepoint, \( s_s \).
The expression $G(s_s + q) \frac{dq}{dp}$ can be expressed as

$$G(s_s + q) \frac{dq}{dp} = \sum_{k=0}^{\infty} C_k p^{(k-1)/2}.$$  \hspace{1cm} (8)

It can be shown that this series has a radius of convergence different from zero when $\delta > 0$.

Also

$$e^{(p - \frac{q(p)}{\delta})z} = \sum_{m=0}^{\infty} \frac{z^m}{m!} (p - \frac{q(p)}{\delta})^m$$ \hspace{1cm} (9)

is convergent over the entire $q$-plane.

Upon inserting the expression for $q$ from equation (6) into equation (9),

$$e^{(p - \frac{q(p)}{\delta})z} = \sum_{m=0}^{\infty} \frac{z^m}{m!} (p - \frac{1}{\delta} \sum_{k=0}^{\infty} b_k p^{k/2})^m.$$ \hspace{1cm} (10)

This series has the same radius of convergence as equation (6).

If equation (8) is multiplied by equation (10),

$$e^{(p - \frac{q(p)}{\delta})z} G(s_s + q) \frac{dq}{dp} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} d_{mk} z^m p^{(k-1)/2} dp.$$ \hspace{1cm} (11)

This equation has the same radius of convergence as equation (8). Denote this radius of convergence by $\theta(\delta)$. When $\delta > 0$, $\theta(\delta) > 0$. 
If the expression in equation (11) is inserted in equation (4), then

\[
N_2 \sim \sum_{n=1}^{\infty} K e^{-p_0 \delta t} + (p_0 - \frac{s_0}{\delta}) z
\]

\[
\chi \int e^{-p_0 \delta t} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} d_{mk} \frac{m}{p} \left( \frac{k-1}{2} \right) \frac{1}{p} dp.
\]

(12)

The contours of integration are shown in the figure below.

The integration of equation (12) is performed by stopping the summation at the Mth term and extending the contours in the p-plane toward the left to infinity. The integration will then to to a Gamma function,

\[
N_2 \sim \sum_{n=1}^{\infty} 2 K e^{-p_0 \delta t} + (p_0 - \frac{s_0}{\delta}) z \chi
\]
where \( k \) is whole and even. The remainder, \( R_M \), is

\[
|R_M| < 2K(z) \left| \frac{\Gamma\left(\frac{M+1}{2}\right)}{(\delta t)^{\left(M+1\right)/2}} \right|
\]

(14)

This can be shown as follows: a positive real number \( K = K(z) \) can always be found so that on the contours between \(-\infty \) and \(-\theta\),

\[
K(z) \left| p^{(M-1)/2} \right| > \left| \sum_{m}^{M} \sum_{k}^{M} d_{mk} \frac{m}{p^{(k-1)/2}} \right|
\]

(15)

and from \(-\theta\), through the origin, and back to \(-\infty\),

\[
K(z) \left| p^{(M-1)/2} \right| > \left| \sum_{m}^{\infty} \sum_{k}^{\infty} d_{mk} \frac{m}{p^{(k-1)/2}} \right|
\]

\[
- \sum_{m}^{M} \sum_{k}^{M} d_{mk} \frac{m}{p^{(k-1)/2}} \right|
\]  

(16)

Then

\[
|R_M| < 2K(z) \left| e^{p_0 \delta t} + (p_0 - \frac{\delta s}{\delta})z \right| \int e^{p_0 \delta t} \left| \frac{p}{p^{(M-1)/2}} \right| ds.
\]

(17)
If $p_0 \delta t + \left(p_0 - \frac{s_0}{\delta}\right)z$ is pure imaginary, then equation (14) is valid.

It follows that

$$\lim_{t \to \infty} N_2 = 0 \text{ for } x = \delta t + z, \delta > 0.$$
APPENDIX D

The expression for \( s_1 \) is

\[
\begin{align*}
\frac{4S}{R^2} & \left( 1 + \sqrt{1 - \frac{4(\beta^4 + \frac{k}{D})}{(\frac{D}{D} - 2\beta^2)^2}} \right) \\
& = i \sqrt{\frac{2R}{v}} \left( 1 + \sqrt{1 - \frac{4S}{R^2}} \right),
\end{align*}
\]

where \( R = \frac{D}{D} - 2\beta^2 \),

and \( S = (\beta^4 + \frac{k}{D}) \).

Using these symbols,

\[
\begin{align*}
\frac{s_1}{v} & = i \sqrt{\frac{2R}{v}} \left( 1 + \sqrt{1 - \frac{4S}{R^2}} \right) = i \sqrt{\frac{R}{v} + \sqrt{\frac{R^2}{4S} - 1}}
\end{align*}
\]

\[
= i \sqrt{\frac{R}{2} + \sqrt{\frac{R^2}{4S} - 1}}
\]

\[
= i \sqrt{\frac{R}{2} + \sqrt{\frac{R^2}{4S} - 1}} \left[ \sqrt{\frac{R^2}{4S}} + 1 \right]
\]

\[
= i \sqrt{\frac{R}{2} + 2 \left[ \sqrt{\frac{R^2}{4S} - 1} \right] \left[ \sqrt{\frac{R^2}{4S}} + 1 \right]}
\]

\[
= i \sqrt{\frac{R}{2} + 2 \sqrt{\frac{1}{2} \left( \frac{R^2}{4S} - 1 \right) \left[ \frac{1}{2} \left( \frac{R^2}{4S} + 1 \right) \right]}}
\]
\[
\sqrt{-\frac{R^2}{4\pi^2} - 21 \left( \frac{v}{2\pi} \sqrt{1 - \frac{R^2}{4\pi^2}} \right) \left( \frac{v}{2\pi} \sqrt{1 + \frac{R^2}{4\pi^2}} \right)}
\]

= \sqrt{(f - ig)^2} = f - ig,

where

\[
f = \sqrt{\frac{v}{2\pi}} \sqrt{1 - \frac{R^2}{4\pi^2}}
\]

\[
= \sqrt{\beta^4 + \frac{k_D}{D}} \sqrt{\frac{1}{2} \left[ 1 - \sqrt{\frac{\left( \frac{v^2}{D} - 2\beta^2 \right) \left( \frac{v^2}{D} - 2\beta^2 \right)}{4(\beta^4 + \frac{k_D}{D})}} \right]}
\]

and

\[
g = \sqrt{\frac{v}{2\pi}} \sqrt{1 + \frac{R^2}{4\pi^2}}
\]

\[
= \sqrt{\beta^4 + \frac{k_D}{D}} \sqrt{\frac{1}{2} \left[ 1 + \sqrt{\frac{\left( \frac{v^2}{D} - 2\beta^2 \right) \left( \frac{v^2}{D} - 2\beta^2 \right)}{4(\beta^4 + \frac{k_D}{D})}} \right]}
\]

It can be shown in a similar manner that

\[
\frac{s_2}{v} = \sqrt{\frac{\beta^2}{2}} \left[ 1 - \sqrt{1 - \frac{4(\beta^4 + \frac{k_D}{D})}{(\frac{v^2}{D} - 2\beta^2) \left( \frac{v^2}{D} - 2\beta^2 \right)}} \right]
\]

= -(f + ig).
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