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Relativistic effects in nucleon-nucleon scattering

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RELATIVISTIC EFFECTS
IN NUCLEON-NUCLEON SCATTERING

by
Spencer Macy

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Theoretical Physics

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1950
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I. INTRODUCTION

The recent availability of high energy nucleon sources has made it possible to obtain data on nucleon-nucleon scattering in the 100-400 Mev range. A great number of experimental papers have appeared giving the differential and total scattering cross sections for particular cases in which the scattered particle has an energy in this range. The most striking feature of these experiments is the apparent symmetry about 90° in the center of mass system of the angular distribution of the differential cross section for neutron-proton scattering. This has led Serber to suggest a type of exchange force which acts only in even states. This type of exchange force leads to considerable simplification of the present work and will be used in this paper.

Along with the experimental work have come theoretical papers attempting to explain and interpret the experimental results. Most of these papers use the Schrödinger equation

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with a potential function chosen to fit the experimental data. The ranges and well depths of the potential functions are derived from the low energy data. If tensor forces are assumed, they are chosen to give the correct electric quadrupole moment of the deuteron. One of the most recent attempts of this kind is that of Christian and Hart\textsuperscript{4}. They use the Serber potential with various combinations of central and tensor forces. Using a radial dependence of the Yukawa, exponential or square well type, they find a total cross section at least 10\% larger than the experimental cross section for an energy of about 90 Mev. This indicates that if the phenomenological description of the neutron-proton interaction in terms of a potential is to be applied to scattering in the region of 100 Mev, the potential should be chosen so that the relativistic corrections to be applied would decrease the cross section by about 10\%.

In order to deal with the relativistic corrections, there are two approaches that might be used. The first approach, the most satisfactory theoretically, is to use meson field theory. However, at the present time there is no theory which accounts for all the phenomena to be explained. The problem is somewhat the same as that of selecting a potential in the ordinary Schrodinger theory, in that there are

several types of meson fields to choose from. Marty\textsuperscript{5} chose a combination of pseudoscalar and vector fields. With this combination he found too large a total cross section, and the relativistic terms increased it. Alzofon\textsuperscript{6} took pseudoscalar coupling with exchange of both charged and neutral mesons to account for the symmetry about $90^\circ$. With this choice he found the relativistic terms decreased the cross section by 5–10%. Snyder and Marshak\textsuperscript{7} made a calculation using the Møller method\textsuperscript{8} on the scalar and vector theories. The correction to the scalar cross section was small, while the vector cross section was increased by about 10%. Thus it is seen that both the amount and direction of the correction depend upon the particular theory.

The second approach, the one used in this paper, is to apply the ordinary Dirac theory, choosing a potential function to express the interaction between the particles. Thus the nucleon is treated as a Dirac particle with spin $\frac{1}{2}$. The second order corrections to the Dirac equation describing the system can be found and the results used to find the

\footnotesize

\textsuperscript{5}C. Marty, Nature \textbf{165}, 361 (1950)\textsuperscript{L}.

\textsuperscript{6}F. E. Alzofon, Phys. Rev. \textbf{75}, 1773 (1949)\textsuperscript{L}.

\textsuperscript{7}H. Snyder and R. E. Marshak, Phys. Rev. \textbf{72}, 1253 (1947)\textsuperscript{L}.

\textsuperscript{8}C. Møller, Zeit. f. Physik \textbf{70}, 786 (1931).
change in the cross section. Since the only truly relativistic potential that can be chosen has the $\delta$-function as the radial dependence, a correction must be applied to the Hamiltonian when a finite range is used. This has been done for certain potentials by Breit. The analysis of this additional term will follow closely that given by Breit.

\footnotesize

\begin{itemize}
\end{itemize}
II. RELATIVISTIC CORRECTIONS TO THE WAVE EQUATION

A. Units and Notation

The units and notation in this paper are those used in Rosenfeld\(^1\). Mass and momentum are expressed in energy units so that \( M \) stands for \( Mc^2 \), and \( p \) stands for \( pc \). This means that Planck's constant becomes \( \hbar = \hbar c \), and the orbital angular momentum \( \mathbf{L} = \mathbf{p} \times \mathbf{r} \) will contain the velocity of light \( c \). These units are very convenient when we are dealing with the Dirac equation, because the velocity of light then does not explicitly appear.

Rosenfeld introduces the concept of a "dichotomic" variable to treat the case in which we must distinguish between two possible states of a system. Two well-known examples of such a variable are the ordinary spin \( s \), and the isotopic spin \( \tau \). If we let \( s = \frac{1}{2}h\sigma \), then the eigenvalues of \( \sigma \) are +1 and -1, corresponding to the two possible orientations of the spin. In the same way, the two eigenvalues of the isotopic spin correspond to the two states of different charge.

We can think of the spin \( \sigma \) as being the projection of a vector \( \mathbf{\sigma} \) on a unit vector \( \hat{n} \) in a cartesian coordinate system. The components of \( \mathbf{\sigma} \) can be taken as 2x2 matrices.

\(^1\)Rosenfeld, op. cit.
The $z$-component is usually taken to be diagonal. The representation of $\sigma$ is then

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

In the matrix notation, the components of all the dichotomic variables have formally the same representation, although, of course, they are in different vector spaces. These components are seen to satisfy the commutation relations

$$
\sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij} \quad (i, j = x, y, \text{or } z),
$$

as well as

$$
\sigma_x \sigma_y = i \sigma_z \quad (\text{cyclic}).
$$

In addition to the two variables already mentioned, we will use a third dichotomic variable $\varphi$. This variable serves to separate the four components of the wave function into two pairs. If we let the three components of $\varphi$ be $\varphi_1$, $\varphi_2$, and $\varphi_3$, then we have the choice of which component to make diagonal. If we choose $\varphi_3$ to be diagonal, as will be done in this paper, the two pairs become the large and small components of the wave equation. On the other hand, if $\varphi_1$ is diagonal, then the separation is into spinors. This latter representation is useful when dealing with the relativistic invariance of the wave.
equation, but is not convenient for our purpose.

We will need functions which are simultaneous eigenfunctions of the square and of the z-component of these variables. These functions are again formally the same for any of the variables, and will be illustrated by those for \( \sigma \). Let \( \chi_+ \) and \( \chi_- \) be eigenfunctions of \( \sigma_z \) such that \( \sigma_z \chi_+ = \chi_+ \) and \( \sigma_z \chi_- = -\chi_- \). Then we can form the following combinations of these functions for a system of two particles (1) and (2):

\[
\begin{align*}
1(\sigma)_0 &= (2)^{-\frac{3}{2}} [\chi_+ (1) \chi_- (2) - \chi_- (1) \chi_+ (2)] \\
3(\sigma)_0 &= (2)^{-\frac{3}{2}} [\chi_+ (1) \chi_- (2) + \chi_- (1) \chi_+ (2)] \\
3(\sigma)_1 &= \chi_+ (1) \chi_+(2) \\
3(\sigma)_{-1} &= \chi_- (1) \chi_- (2),
\end{align*}
\]

where the functions \( \chi_{\pm} \) have the matrix representation

\[
\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

The functions \( \sigma' (\sigma)_{M_z} \) are then such that they are eigenfunctions of \( S^2 = (\hbar^2/4)(\sigma^2 (1) + \sigma^2 (2)) \) with eigenvalue \( \hbar^2 \sigma' \), and of \( S_z = \frac{1}{2} \hbar (\sigma^z (1) + \sigma^z (2)) \) with eigenvalue \( \hbar M_z \). The singlet function with \( \sigma' = 1 \) corresponds to a total spin of zero, while the triplet functions with \( \sigma' = 3 \) correspond to a total spin 1, with three possible values of the
z-component.

For the isotopic spin, the values +1, 0, and -1 refer to the proton-proton, neutron-proton, and neutron-neutron states, respectively. The singlet and triplet eigenfunctions are antisymmetric and symmetric, respectively, to the interchange of (1) and (2). The particular eigenfunction used is fixed by the requirement that the total wave function be antisymmetric to interchange of the two particles. In the case of the functions for \( \mathbf{r} \), the values +1, 0, and -1 signify that the lowest order term of the accompanying function is of the zero order, first order, and second order, respectively, in \( v/c \).

For a system of two particles, the total angular momentum in the center of mass system is given by

\[
\mathbf{\hat{J}} = \mathbf{\hat{r}} \times \mathbf{\hat{p}} + \frac{\hbar}{2}(\mathbf{\hat{\sigma}}^{(1)} + \mathbf{\hat{\sigma}}^{(2)}),
\]

where \( \mathbf{\hat{r}} \) and \( \mathbf{\hat{p}} \) are the radius vector and linear momentum in relative coordinates. We will make use of eigenfunctions in the LSJm scheme, i.e., functions which are eigenfunctions of \( L^2, S^2, J^2, \) and \( J_z \). These eigenfunctions are denoted by \( \sigma_{LJ}^{(m)} \). The explicit representation of the \( Z \)'s is given in the appendix. The functions \( Y_{L}^{M} \) are the tesseral harmonics\(^2\), and are eigenfunctions of \( L^2 \) and \( L_z \).

\(^2\)Rosenfeld, op. cit., p. xii.
B. The Potential Function

The difficulty of applying the Dirac theory to the problem of nucleon-nucleon interaction lies in the requirement of the relativistic invariance of the interaction potential. The only completely relativistically invariant potentials are of the form

\[ J(\vec{r}_1 - \vec{r}_2) = C\omega_1 \delta(\vec{r}_1 - \vec{r}_2), \]

where \( \omega_1 \) has one of the five forms

\[
\begin{align*}
\omega_1 &= \rho_3^{(1)} \rho_3^{(2)} \\
\omega_2 &= 1 - \sigma^{(1)} \sigma^{(2)} \quad \rho_1^{(1)} \rho_1^{(2)} \\
\omega_3 &= \rho_3^{(1)} \rho_3^{(2)} \left[ 1 + \rho_1^{(1)} \rho_1^{(2)} \right] \sigma^{(1)} \sigma^{(2)} \\
\omega_4 &= \sigma^{(1)} \sigma^{(2)} - \rho_1^{(1)} \rho_1^{(2)} \\
\omega_5 &= \rho_3^{(1)} \rho_3^{(2)} \rho_1^{(1)} \rho_1^{(2)}.
\end{align*}
\]

Since we are to compare the results of a relativistic calculation with those of the Schrödinger theory, it is desirable to use a potential function with a finite range.

We are interested in the scattering of particles with a total energy of about 100 Mev, so that the value of \( (v/c)^2 \) is about 0.05. Therefore, if we keep terms only up to \( (v/c)^2 \), we will have a noticeable correction without making an appreciable error in neglecting higher terms.

With this in mind, we will choose our potential in such a
way that the expectation value of the potential, i. e.,
\[ \int \psi^* J \psi \, dx, \]
remains invariant to the second order in v/c under a Lorentz transformation. The value of this integral is invariant to all orders for the potentials above\(^4\), as it should be. However, an interaction potential with a finite range introduces a number of second order terms.

If we make a Lorentz transformation from a frame K to a frame K', then, as pointed out by Breit\(^5\), we must be sure that we are measuring the coordinates of particles (1) and (2) at the same time in K'. This is because the two points \((t, x_1, y_1, z_1)\) and \((t, x_2, y_2, z_2)\) which have the same time in frame K, will not have the same time coordinates in K'. Therefore, after the transformation we must obtain the value of the wave function for (2) at the time \(t'\) of (1). In the integral for the expectation value of J, we must consider the way in which the potential, the wave function, and the volume element transform under a Lorentz transformation. The details of the calculation are given by Breit. After a transformation in the x-direction to a frame moving with relative velocity \(u\), we find

\(^4\)Bethe, Rev. Mod. Phys. 9, 190 (1936).
\(^5\)Breit, op. cit., p. 257.
\[
\left[ \psi^* J \psi \, dx \right]' = \\
\int \psi^* \left[ (1-u^2/2c^2)J - (u/2c)(\alpha^{(1)}_x + \alpha^{(2)}_x)J - (u/2c)J(\alpha^{(1)}_x + \alpha^{(2)}_x) \right. \\
+ \left. (u^2/4c^2)(\alpha^{(1)}_x \alpha^{(2)}_x)J(\alpha^{(1)}_x + \alpha^{(2)}_x) - (u^2/2c^2)x \frac{dJ}{dx} \right. \\
\left. + (u^2/4c^2)(\alpha^{(1)}_x (2) + 1)J + (u^2/4c^2)J(\alpha^{(1)}_x \alpha^{(2)}_x + 1) \right] \psi \, dx \\
- \int (ux/c) \phi^* \left[ (v^{(2)}_x \alpha^{(2)}_x) J \chi \right. \\
\left. + \chi^* J(v^{(2)}_x \alpha^{(2)}_x) \right] \psi \, dx.
\]

In (2.2) we have set \( \vec{\alpha} = \vec{\rho} \), and the total wave function \( \psi = \phi(1) \chi(2) \). This separation corresponds to small accelerations and is valid in our case. Now let \( J(\vec{r}) = \omega_1 \psi(\vec{r}) \), where \( \omega_1 \) can be any combination of the \( \omega \)'s.

With this choice, (2.2) simplifies to

\[
- (u^2/2c^2) \int \psi^* \left[ \omega_1 \psi + (x^2/\vec{r}) \frac{d\psi}{d\vec{r}} \omega_1 \right] \psi \, dx \\
(2.3) \quad - (u/c) \int x \phi^* \left[ (v^{(2)}_x \alpha^{(2)}_x) \omega_1 \psi \chi \\
+ \chi^* \omega_1 \psi(v^{(2)}_x \alpha^{(2)}_x) \right] \phi \, dx.
\]

If we now take a form of \( \omega_1 \) which commutes with \( \alpha^{(2)}_x \), i.e., \( \omega_1 \) or \( \omega_2 \), we obtain after an integration by parts

\[
- (u^2/2c^2) \int \psi^* \left[ \omega_1 \psi + (x^2/\vec{r}) \frac{d\psi}{d\vec{r}} \omega_1 \right] \psi \, dx \\
(2.4) \quad + (u/c) \int \psi^* \left[ \alpha^{(2)}_x \omega_1 \psi + (x/r)(\alpha^{(2)}_x \psi) \frac{d\psi}{d\vec{r}} \omega_1 \right] \psi \, dx.
\]
We must now add additional terms to the potential which satisfy the conditions that they contribute nothing to the potential in the non-relativistic limit and they introduce terms in the integral of the potential which cancel the terms in (2.3). If \( \omega_1 \) is such that (2.4) holds, then, as Breit\(^6\) has shown, a correction term of the form

\[
(2.5) \quad \tilde{Q} = \frac{1}{2} \bar{\sigma}(1) \bar{\sigma}(2) \omega_1 \delta + \frac{1}{2} (\bar{\sigma}(1) \bar{F}) (\bar{\sigma}(2) \bar{F}) (1/r) \frac{d}{dr} \omega_1 \delta
\]

will satisfy these conditions. However, if \( \omega_1 \) is of the form \( \omega_3 \) or \( \omega_4 \), this term will not cancel the terms in the second integral of (2.3). For these two forms we can again integrate by parts, but there will be an additional term involving the gradient of the wave function which cannot be taken care of by any correction term satisfying the first condition above. Since we will be using a potential which does contain \( \omega_3 \) or \( \omega_4 \), we will take the function \( \tilde{Q} \) given above as the one most nearly satisfying the conditions that we have set forth.

If we attempt to include a tensor force, i. e., a term in the potential of the form

\[
(2.6) \quad L(12) = ( \bar{\sigma}(1) \bar{F}) (\bar{\sigma}(2) \bar{F}) -(1/3) \bar{\sigma}(1) \bar{\sigma}(2),
\]

\(^6\)Breit, op. cit., p. 257. The term \( \tilde{Q} \) given by Breit differs slightly from that here because he did not include the second term of \( \omega_2 \).
the situation is even less satisfactory. The terms in (2.2) arising from such a potential will depend on the direction of the relative velocity of the frames $K$ and $K'$, just as they do in (2.3), where the potential depends only on the distance $r$ between the particles. If we assume that the correction term that we add is to be symmetric in the indices (1) and (2) as well as to contribute nothing non-relativistically, then it is not difficult to show that this function must depend on the direction in which we make the Lorentz transformation. Since any term that we add should be spherically symmetric in order to have any physical significance, there is no function which can meet the demands that we have made. For this reason, as well as for the reason that the inclusion of tensor forces greatly increases the computational work, we will in this paper treat only the case of central forces.

G. The Wave Equation

As the wave equation for the two nucleon system in the laboratory system with total energy $E$, we will take

$$(2.7) \quad H\Psi = E\Psi$$

where

$$H = \rho^{(1)} \sigma^{(1)} \rho^{(1)} + \rho^{(2)} \sigma^{(2)} \rho^{(2)}$$

$$(2.8) \quad r (\rho^{(1)} + \rho^{(2)}) N + \omega v(r) + \tilde{q}.$$
Since there is no meaning to the center of mass system in relativity, we transform to a system in which the total momentum is zero. The transformation is the same as that to a center of mass system would be, namely,

\[ \tilde{x}_b = \frac{x_1 + x_2}{2}, \quad \tilde{p}_b = \frac{p_1 + p_2}{2}, \]

(2.9)

\[ \tilde{x} = x_1 - x_2, \quad \tilde{p} = \frac{p_1 - p_2}{2}. \]

We now make a Lorentz transformation from a system in which one of the particles is at rest and the other is moving with velocity \( v \) to one in which the total momentum is zero. A short calculation shows that the relative velocity of the two frames must be

\[ u = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \]

(2.10)

With this velocity it can be easily shown that the energy is equally divided between the two particles, and the relation between the relative momentum in the two frames is given by

\[ p_x^1 = (1 - u^2/c^2)^{\frac{1}{2}} p_x, \quad p_y^1 = p_y, \quad p_z^1 = p_z. \]

(2.11)

At this point we make our first approximation, namely,
we will keep only those terms of order \((v/c)^2\) which involve the potential. Correction terms of the kind introduced by the term \((1 - u^2/c^2)^{1/2}\) in (2.11) are essentially mass correction terms and will play no part in the calculation of the scattering cross section.

After these transformations, the Hamiltonian for the system becomes

\[
H = (\xi_1^{(1)} - \xi_1^{(2)}) \cdot \vec{p} + (\xi_3^{(1)} - \xi_3^{(2)}) \cdot \vec{k} + \omega_1 v + \vec{q}.
\]

With this Hamiltonian we can now calculate the corrections to be applied to the non-relativistic wave equation.

D. The Equation for the Large Component

We can write the total wave function in the form

\[
\psi = (0) + (1) + (2),
\]

with

\[
(0) = 3f_{1,1} \xi_1 \rho_1,
\]

\[
(1) = 3f_{0,0} \xi_0 \rho_0 + \frac{1}{3} f_{0,0} \xi_0 \rho_0,
\]

\[
(2) = 3f_{-1} \xi_{-1} \rho_{-1}.
\]

The (0), (1), and (2) signify that the functions are of
zero, first, and second order in \((v/c)\). We insert (2.13) into (2.7) with the Hamiltonian given by (2.12). Using the fact that eigenfunctions of \(\varphi\) in (2.14) are orthogonal, we set the coefficients of these eigenfunctions equal on the two sides of the equation. This leads to the set of four equations

\[
\begin{align*}
\vec{B} \cdot \vec{p}^3 f_0 - \vec{A} \cdot \vec{p}^1 f_0 + 2m^3 f_1 + \omega_1(a_1) v^3 f_1 &= \omega_1(a_2) v^3 f_{-1} + \omega_1(a_1) q^3 f_{-1} = E^3 f_1 \\
\omega_1(a_2) v^3 f_{-1} + \omega_1(a_1) q^3 f_{-1} &= E^1 f_0 \\
\vec{B} \cdot \vec{p}^3 f_{-1} - \vec{A} \cdot \vec{p}^3 f_1 + \omega_1(b) v^1 f_0 - \omega_1(a_1) q^1 f_0 &= E^3 f_0 \\
\omega_1(b) v^1 f_0 - \omega_1(a_1) q^1 f_0 &= E^1 f_0 \\
\vec{B} \cdot \vec{p}^3 f_0 + \vec{A} \cdot \vec{p}^1 f_0 - 2m^3 f_{-1} + \omega_1(d_1) v^3 f_{-1} &= \omega_1(d_2) v^3 f_1 + \omega_1(a_1) q^3 f_1 = E^3 f_{-1},
\end{align*}
\]

where we have set

\[
\begin{align*}
\vec{A} &= (2)^{-\frac{3}{2}} (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \\
\vec{B} &= (2)^{-\frac{3}{2}} (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}),
\end{align*}
\]

and where the function \(Q\) is now

\[
Q = \frac{1}{2} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} v + \frac{1}{2} (\vec{\sigma}^{(1)} \vec{r}) (\vec{\sigma}^{(2)} \vec{r}) (1/r) \frac{d}{dr} v.
\]
If we make the correspondence

\[
\omega_1 = \begin{pmatrix} 
\varepsilon_{10} \\
\varepsilon_{20} \\
\varepsilon_{30} \\
\varepsilon_{40}
\end{pmatrix},
\]

then the \(\omega_1\)'s in (2.15) become

\[
\begin{align*}
\omega_1^{(a_1)} &= \omega (d_1) \sim \begin{pmatrix} 
1 \\
1 - \sigma(1) \sigma(2) \\
\sigma(1) \sigma(2) \\
0
\end{pmatrix}, \\
\omega_1^{(b)} &= \begin{pmatrix} 
-1 \\
1 + \sigma(1) \sigma(2) \\
0 \\
1
\end{pmatrix}, \\
\omega_1^{(a_2)} &= \omega (d_2) \sim \begin{pmatrix} 
0 \\
-\sigma(1) \sigma(2) \\
\sigma(1) \sigma(2) \\
-1
\end{pmatrix}, \\
\omega_1^{(c)} &= \begin{pmatrix} 
-1 \\
1 - \sigma(1) \sigma(2) \\
-2 \sigma(1) \sigma(2) \\
-(1 - \sigma(1) \sigma(2))
\end{pmatrix},
\end{align*}
\]

(2.16)

If we wish to approximate the equation for the large component \(3f_1\) to order \(v/c\), then the last equation of (2.15) does not appear, and the equation becomes the same as the ordinary Schrödinger equation. We note that if the potential is of the first order in \(v/c\), then there are no correction terms of the second order.

We can derive the equation for \(3f_1\) to the second order by successively substituting the last three equations
of (2.16) into the first and keeping only terms of this order. We will make the approximation that the potential is small compared with the total (non-relativistic) energy. Thus we drop terms which contain powers of $V(r)$ higher than the first. With this approximation we obtain

\[
\left\{ \frac{d^2}{dt^2} - \omega_1^{(a_1)} V + \frac{1}{4M^2} \left[ - (\vec{B} \cdot \vec{p} \omega_1^{(a_1)})(\vec{B} \cdot \vec{p}) - (\vec{A} \cdot \vec{p} \omega_1^{(a_1)})(\vec{A} \cdot \vec{p}) \\
- V(\vec{B} \cdot \vec{p} \omega_1^{(a_1)})(\vec{B} \cdot \vec{p}) - V(\vec{A} \cdot \vec{p} \omega_1^{(a_1)})(\vec{A} \cdot \vec{p}) \\
+ (\vec{B} \cdot \vec{p} \omega_1^{(a_1)})^{(\omega_1^{(a_1)})}(\vec{B} \cdot \vec{p}) - (\vec{A} \cdot \vec{p} \omega_1^{(a_1)})^{(\omega_1^{(a_1)})}(\vec{A} \cdot \vec{p}) \\
+ V(\vec{B} \cdot \vec{p} \omega_1^{(a_1)})^{(\omega_1^{(a_1)})}(\vec{B} \cdot \vec{p}) + V(\vec{A} \cdot \vec{p} \omega_1^{(a_1)})^{(\omega_1^{(a_1)})}(\vec{A} \cdot \vec{p}) \\
- (\vec{r}(1) \cdot \vec{p})(\vec{r}(2) \cdot \vec{p})^{(\omega_1^{(a_1)})} \right) \right. \\
- (\vec{r}(1) \cdot \vec{p})(\vec{r}(2) \cdot \vec{p})^{(\omega_1^{(a_1)})} - \omega_1^{(a_2)} V(\vec{r}(1) \cdot \vec{p})(\vec{r}(2) \cdot \vec{p}) \\
+ (\vec{B} \cdot \vec{p} \omega_1^{(a_1)})^{(\omega_1^{(a_1)})} - (\vec{A} \cdot \vec{p} \omega_1^{(a_1)})^{(\omega_1^{(a_1)})} \\
+ Q(\vec{B} \cdot \vec{p} \omega_1^{(a_1)})^{(\omega_1^{(a_1)})} - (\vec{A} \cdot \vec{p} \omega_1^{(a_1)})^{(\omega_1^{(a_1)})} \\
- (\vec{r}(1) \cdot \vec{p})(\vec{r}(2) \cdot \vec{p})^{(\omega_1^{(a_1)})} - \omega_1^{(a_1)} Q(\vec{r}(1) \cdot \vec{p})(\vec{r}(2) \cdot \vec{p}) \right)^2 f_1 \\
= (E - 2M)^2 f_1.
\]

It was stated in the introduction that the experimental data indicated that an appropriate potential would be one which acts only in even states. This means that if we let
then the operation of the potential function on \( \frac{3}{2} f_1 \) will give zero if \( L \), the eigenvalue of the orbital angular momentum, is odd. In order to form such a potential, we must first define the isotopic function that we use in a neutron-proton system. Thus we let

\[
(2.19) \quad \tau' = (2)^{-\frac{3}{2}} \{ u_+(1) - u_-(2) \} = (-1)^L u_+(1) u_-(2) \]

where the - or + sign is used for the triplet or singlet spin state, respectively. With this choice our wave function will be totally antisymmetric for an exchange of the two particles. We can now write the desired potential as

\[
(2.20) \quad V(r) = -\frac{i}{2}(1 - \frac{1}{2} \sigma_1 \sigma_2 \cdot L) \cdot \frac{1}{2} \sigma_1 \sigma_2 \cdot \frac{1}{2} \sigma_1 \sigma_2) V(r).
\]

It is easy to show that this term gives \(-\frac{1}{2}(1 + (-1)^L)\)
when operating on \( \frac{3}{2} f_1 \) as given by (2.18).

The \( \omega_1 \)'s in (2.15) are therefore replaced by a linear combination which gives (2.20) for \( \omega_1^{(a1)} \). It is seen from (2.16) that we have a choice of four combinations of the \( \omega_1^{(a1)} \)'s, each of which gives (2.20). When we insert this potential into (2.17), we find that the result depends on whether (2.18) is a singlet or triplet state, and whether \( L \) is even or odd. The terms involving \( Q \) will be
the same for any combination of the $\omega_i$'s, while the rest of the terms will differ. Thus for example, the correction terms for the potential

$$- \mathcal{H}(r) = \frac{\omega_1}{2} \left( 1 + \frac{\tau(1)\tau(2)}{2} \right) V(r)$$

will be as follows:

**Singlet, $L$ even,**

$$\left(\frac{1}{4M^2}\right) \left[ - (\mathbf{P} \cdot \mathbf{V})(\mathbf{P} \cdot \mathbf{P}) - \frac{3}{2} \mathbf{P}^2 + \frac{7}{4} V(\mathbf{\sigma}(1) \cdot \mathbf{P})(\mathbf{\sigma}(2) \cdot \mathbf{P}) \right]$$

(2.21a)

$$\frac{9}{2} h^2 \frac{dV}{R dr} + (\mathbf{\sigma}(1) \cdot \mathbf{P})(\mathbf{\sigma}(2) \cdot \mathbf{P}) \mathbf{Q}$$

**Singlet, $L$ odd,**

$$\left(\frac{1}{4M^2}\right) \left[ 3 (\mathbf{P} \cdot \mathbf{V})(\mathbf{P} \cdot \mathbf{P}) + \frac{9}{4} \mathbf{P}^2 - \frac{15}{4} V(\mathbf{\sigma}(1) \cdot \mathbf{P})(\mathbf{\sigma}(2) \cdot \mathbf{P}) \right]$$

(2.21b)

$$- \frac{9}{4} h^2 \frac{dV}{R dr} + \mathbf{Q}(\mathbf{\sigma}(1) \cdot \mathbf{P})(\mathbf{\sigma}(2) \cdot \mathbf{P}) - \mathbf{P}^2$$

**Triplet, $L$ even,**

$$\left(\frac{1}{4M^2}\right) \left[ 2 (\mathbf{A} \cdot \mathbf{P} V(\mathbf{A} \cdot \mathbf{P}) - (\mathbf{P} \cdot \mathbf{V})(\mathbf{P} \cdot \mathbf{P}) + \mathbf{P}^2 - \frac{3}{2} \mathbf{P}^2 \right]$$

(2.21c)

$$+ 3 V(\mathbf{\sigma}(1) \cdot \mathbf{P})(\mathbf{\sigma}(2) \cdot \mathbf{P}) + (\mathbf{\sigma}(1) \cdot \mathbf{P})(\mathbf{\sigma}(2) \cdot \mathbf{P}) \mathbf{Q}$$

$$+ 2 \mathbf{Q}(\mathbf{\sigma}(1) \cdot \mathbf{P})(\mathbf{\sigma}(2) \cdot \mathbf{P}) + (\mathbf{A} \cdot \mathbf{P} \mathbf{Q})(\mathbf{A} \cdot \mathbf{P}) + \mathbf{Q}^2$$
triplet, \( L \) odd,

\[
\frac{1}{4m^2} \left[ 3(\vec{E} \cdot \vec{P}) \vec{E} \cdot \vec{P} + 3\nabla p^2 - 3\nabla(\vec{E} \cdot \vec{P})(\vec{E} \cdot \vec{P}) \right] \\
(2.21d) + \frac{i\hbar^2}{rdr} + Q(\vec{E} \cdot \vec{P})(\vec{E} \cdot \vec{P}) + Qp^2 \\
-(\vec{E} \cdot \vec{P})(\vec{E} \cdot \vec{P})
\]

the other possible combinations give corrections with these same terms, but with different coefficients. These are given in the appendix. A valuable check on the computations is provided by the fact that if we take the combinations \((\omega_1, \omega_3), (\omega_2, \omega_4), (\omega_1, \omega_4), \) and \((\omega_2, \omega_3)\), then the sum of the terms from the first two is equal to the sum of terms from the last two.
III. THE SCATTERING CROSS SECTION

A. Derivation of the Formulas

In order to find the cross section for neutron-proton scattering, we need to determine the asymptotic form of the scattered wave. From (2.13) and (2.14) we can see

\[ |\psi|^2 = |3f_1|^2 + |3f_0|^2 + |1f_0|^2 + |3f_{-1}|^2, \]

using the orthogonality of the \( \varphi \) eigenfunctions. The differential cross section for scattering into the solid angle \( d\Omega \) is given by

\[ d\sigma = |\psi_{sc}|^2 r^2 d\Omega. \]

It is apparent that we need to consider only the large and first order components when computing the cross section to the second order in \( v/c \). Further, only the lowest order terms have to be retained in \( 3f_0 \) and \( 1f_0 \). In the first approximation \( E = 2M \), and by (2.15) we can write

\[ 3f_0 = (1/2M) B \cdot p \ 3f_1 \]
\[ 1f_0 = -(1/2M) A \cdot p \ 3f_1 \]

In the approximation used, the total wave function may be written

$$\psi_t = 3_{f_1} 3(\rho)_1 + 3_{f_0} 3(\rho)_0 + 1_{f_0} 1(\rho)_0$$

(3.4) $$= 3_{f_1} 3(\rho)_1 + \frac{1}{2M} B \cdot p 3_{f_1} 3(\rho)_0 - \frac{1}{2M} A \cdot p 3_{f_1} 1(\rho)_0,$$

with an analogous representation of the incident wave.

The scattered wave is then

$$\psi_{sc} = (3_{f_1} - 3_{f_{11}}) 3(\rho)_1 + \frac{1}{2M} B \cdot p (3_{f_1} - 3_{f_{11}}) 3(\rho)_0$$

(3.5) $$- \frac{1}{2M} A \cdot p (3_{f_1} - 3_{f_{11}}) 1(\rho)_0,$$

where $3_{f_{11}}$ is the large component of the incident wave.

When we separate the wave equation (2.17) into partial waves according to the LSJM scheme, the correction terms mix states of equal parity according to equations (A.7) to (A.10) in the appendix. Thus the situation is similar to the case of tensor forces. Rohrlich and Eisenstein\textsuperscript{2} have given the analysis of the scattering by a tensor force using the method of partial waves and the following derivation will be similar to theirs.

\textsuperscript{2}F. Rohrlich and J. Eisenstein, Phys. Rev. 75, 705 (1949).
We must first express the incident wave in the LSJm notation. For the singlet state there is no problem, since \( L = J \). The incident wave can then be taken in the usual \( L_S M_L M_S \) scheme\(^3\), and the asymptotic form of the scattered wave is

\[
\psi_{sc}^0 \approx \frac{e^{ikr}}{r} \left( \sum_{J} \frac{(2J+1)}{k} \right) \sin \delta^J_J e^{i\delta^J_J} \sin \left( \frac{1}{2} \right) \frac{1}{2} \left( J \right)^m \]

In (3.6), \( k = (Mc)^{1/2} / \hbar \) with \( \epsilon = E - 2N \), and \( \delta^J_J \) is the phase shift for the scattered wave. We write \( \psi_{sc}^m \) to differentiate between states of different spin multiplicity and between states with different \( m \). Since \( m = M_L + M_S \) is a constant of the motion, and since for a plane wave \( M_L = 0 \), then we have \( m = M_S \) in both the singlet and the triplet states.

For the triplet states we must find the matrix elements \( (LSJm/ L_S M_L M_S) \) for the transformation from the \( L_S M_L M_S \) to the LSJm representation. From the above considerations, these may be written as \( (LlJm/ Ll0m) \). Because of the mixing of states with different \( L \) but the same parity, \( L \) is no longer a good quantum number, but \((-1)^L\) is. However, the representation of the incident wave (i.e., a plane wave) is the same for either \((-1)^LsJm\)

\(^3\)Rosenfeld, op. cit., p. 99.
or LSJm, since there is no potential.

In the $LSM, MS$ case the incident wave is

$\begin{align*}
(3.7) \quad 3\psi^m_1 &= \sum_L (4\pi)^{\frac{3}{2}} (2L + 1)^{\frac{3}{2}} l^L j_L(kr) \ U_L^0 \ 3(\sigma)_{MS},
\end{align*}$

where

$\begin{align*}
(3.8) \quad j_L(kr) &= (\pi/2kr)^{\frac{3}{2}} J_{L+\frac{3}{2}}(kr)
\end{align*}$

is the spherical Bessel function. In the LSJm notation,

$\begin{align*}
(3.9) \quad 3\psi^m_1 &= \sum_J \sum_L (4\pi)^{\frac{3}{2}} (2L + 1)^{\frac{3}{2}} l^L j_L(kr) \ U^0_L \ 3^2 J^m_J(LlJm/Ll0m),
\end{align*}$

where $L$ takes on the values $J - 1$, $J$, and $J + 1$. The matrix elements for the transformation are given in Table 1.

<table>
<thead>
<tr>
<th>$L \setminus m$</th>
<th>$J - 1$</th>
<th>$J$</th>
<th>$J + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J - 1$</td>
<td>$-(J + 1)/2$</td>
<td>$J$</td>
<td>$(J/2)^{\frac{3}{2}}$</td>
</tr>
<tr>
<td>$J$</td>
<td>$-(2J - 1)/2$</td>
<td>$0$</td>
<td>$(J - 1)^{\frac{3}{2}}$</td>
</tr>
<tr>
<td>$J + 1$</td>
<td>$J$</td>
<td>$0$</td>
<td>$(J + 1)^{\frac{3}{2}}$</td>
</tr>
</tbody>
</table>

The derivation of these quantities may be found in Condon

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and Shortley\textsuperscript{5}. The incident wave has the asymptotic form

\[(3.10) \quad 3\psi_{1}^{m} \simeq \frac{(4\pi)^{\frac{1}{2}}}{kr} \sum_{L} \sum_{J} (2L + 1)^{\frac{1}{2}} \sin(kr - \pi L/2) \cdot 3Z_{J}^{(L)m} (l_{1}J_{m}/l_{1}0m).\]

Since \(L\) can take on the above values, the eigenfunctions in the \((-1)^{L}\) scheme will either be the same as those in the LSJm scheme, namely, for the value \(L = J\), or they will be linear combinations of the eigenfunctions for \(L = J - 1\) and \(L = J + 1\). Therefore

\[\psi_{J,m}^{\alpha} = \frac{1}{r} u_{J}(r) 3Z_{J}^{(J-1)m} - \frac{1}{r} w_{J}(r) 3Z_{J}^{(J+1)m}\]

\[\psi_{J,m}^{\beta} = \frac{1}{r} v_{J}(r) 3Z_{J}^{(J)m},\]

with the asymptotic forms

\[\psi_{J,m}^{\alpha} \simeq \frac{1}{r} \sin(kr - (J-1)\pi/2 + \delta_{J}) \cdot (3Z_{J}^{(J-1)m} + \eta_{J}^{\alpha} 3Z_{J}^{(J+1)m})\]

\[\psi_{J,m}^{\beta} \simeq \frac{1}{r} \sin(kr - (J+1)\pi/2 + \delta_{J}) \cdot (3Z_{J}^{(J-1)m} + \eta_{J}^{\beta} 3Z_{J}^{(J+1)m})\]

\[\psi_{J,m}^{\beta} \simeq \frac{1}{r} \sin(kr - J\pi/2 + \delta_{J}) 3Z_{J}^{(J)m} .\]

The asymptotic form of the total triplet wave will then consist of a linear combination of the functions in (3.12) summed over \( J \). After subtracting off the incident wave we find

\[
\mathcal{S} \psi_{sc} = \frac{\hbar k}{r} \frac{4m^3}{r} \sum_{J} \left\{ \alpha_{J} e^{i \delta_{J}} \sin \delta_{J} \left( \frac{3Z}{J} - \frac{\eta_{J}^{\alpha}}{J} \right) + \beta_{J} e^{i \delta_{J}} \sin \delta_{J} \left( \frac{3Z}{J} \right) + \gamma_{J} e^{i \delta_{J}} \sin \delta_{J} \left( \frac{3Z}{J} - \frac{\eta_{J}^{\gamma}}{J} \right) \right\},
\]

where

\[
\alpha_{J} = - \frac{(J+1)^{\frac{3}{2}} - \frac{3Z}{J}}{(2)^{\frac{3}{2}} (1 + (\eta_{J}^{\alpha})^2)} \quad \alpha_{J}^{\circ} = \frac{J^\frac{3}{2} + \frac{\eta_{J}^{\alpha}}{J}}{1 + (\eta_{J}^{\alpha})^2} \quad\quad\quad (3.14)
\]

\[
\beta_{J} = m \left( \frac{2J + 1}{2} \right)^{\frac{3}{2}} \quad \beta_{J}^{\circ} = \frac{J^\frac{3}{2} + \eta_{J}^{\beta}(J+1)^{\frac{3}{2}}}{1 + (\eta_{J}^{\beta})^2} \quad\quad\quad (3.14)
\]

\[
\gamma_{J} = - \frac{(J+1)^{\frac{3}{2}} - \frac{3Z}{J}}{(2)^{\frac{3}{2}} (1 + (\eta_{J}^{\gamma})^2)} \quad \gamma_{J}^{\circ} = \frac{J^\frac{3}{2} + \eta_{J}^{\gamma}(J+1)^{\frac{3}{2}}}{1 + (\eta_{J}^{\gamma})^2} \quad\quad\quad (3.15)
\]

The quantities \( \eta_{J}^{\alpha} \) and \( \eta_{J}^{\gamma} \) are called the "amounts of admixture", and can be shown to satisfy the identity

\[
(3.15) \quad \eta_{J}^{\alpha} \eta_{J}^{\gamma} = -1.
\]

\[\text{\bibitem{Rohrlich and Eisenstein, op. cit., p. 717.}}\]
The quantities $\alpha_j$, $\beta_j$, and $\gamma_j$ are found by writing the asymptotic form of the total triplet wave and comparing coefficients with (3.9) when the phase shifts are set equal to zero.

The total incident wave, singlet plus triplet, may be written

\begin{equation}
\psi_1 = \sum_{\sigma^i=1,3} \sum_{N_S} \sigma'_{J_{\rho}} \sigma_1^{N_S},
\end{equation}

where the quantities $\sigma'_{J_{\rho}}$ are in general arbitrary. The incident wave is usually normalized to unity. We do this in the following way. Because of the orthogonality of the $\sigma$ eigenfunctions, we can write

\begin{equation}
|\psi_1|^2 = |1_0^J \psi_0|^2 + |3_{j1}^3 \psi_{11}|^2 + |3_{y_0}^y \psi_{01}|^2
\end{equation}

\begin{equation}
+ |3_{j-1}^y \psi_{-11}|^2.
\end{equation}

In the approximation that we are using, we have

\begin{equation}
|\sigma'_{J_{\rho}}|^2 = |3_{f_{11}}|^2 + (1/4\pi^2) |B \cdot p \cdot 3_{f_{11}}|^2
\end{equation}

\begin{equation}
+ (1/4\pi^2) |A \cdot p \cdot 3_{f_{11}}|^2.
\end{equation}

Thus we set

\begin{equation}
3_{f_{11}} = e^{ikz} \sigma_1^{(\sigma)_{N_S}}.
\end{equation}
and (3.18) becomes

\[ |\psi' M_3|^2 = 1 + \left( \frac{\hbar^2 k^2}{2M^2} \right). \]

If the incident particles are unpolarized, as we shall assume, then the squares of the \( J \)'s in (3.16) are all equal. Thus, if \( |\psi_1|^2 \) is to be unity, then from (3.17) and (3.19)

\[ G = |J_{M_3}|^2 = \frac{1}{4} \left( 1 + \left( \frac{\hbar^2 k^2}{2M^2} \right)^{-1} \right). \]

In the same way the total scattered wave must be written

\[ \psi_{sc} = \sum_{\sigma' = 1, 3} \sum_{M_3} \sigma' J_{M_3} \sigma' \psi_{sc}^M. \]

Therefore

\[ |\psi_{sc}|^2 = \sum_{\sigma' = 1, 3} \sum_{M_3} |\sigma' J_{M_3} \sigma' \psi_{sc}^M|^2, \]

where

\[ |\sigma' \psi_{sc}^M|^2 = |3f_{lsc}|^2 + \left( \frac{1}{4M^2} \right) |B \cdot p \cdot 3f_{lsc}|^2 \]

\[ + \left( \frac{1}{4M^2} \right) |A \cdot p \cdot 3f_{lsc}|^2. \]

The contribution of the first term on the right in (3.23) to the total cross section, i. e., the differential cross section (3.2) integrated over all angles, will be
given by

\[ S_{np} = \frac{4\pi C_e}{k^2} \sum_J (2J + 1) (\sin^2 \delta_J^1 + \sin^2 \delta_J^\alpha + \sin^2 \delta_J^\beta + \sin^2 \delta_J^r), \]

as in the case of tensor forces. For the case in which there are only central forces, this equation becomes the usual equation

\[ S_{np} = \frac{4\pi C_e}{k^2} \sum_L \sum_{\sigma' = 1, 3} (2L + 1) \sigma' \sin^2 \delta_L^{\sigma'}, \]

since \( \delta_{J+1}^\alpha = \delta_J^\beta = \delta_J^r \). When we include the relativistic terms, the phase shifts are slightly altered and this equality is no longer true.

To find the contribution of the last two terms of (3.23), we must use the following relations:

\[
\overline{A} \cdot \overline{p} \, \overline{\psi}^m_{\text{sc}} \propto (2J + 1) \frac{e^{i k r}}{r} (4\pi)^{\frac{5}{2}} \sum_j \left[ \left( \frac{m_j}{J} \right)^{\frac{1}{2}} e^{i \delta_j^\alpha} \sin \delta_j^\alpha \right.
\]

\[ \times \left[ - \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} + \eta_J^\alpha \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} \right] \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} \]

\[ + \gamma_J \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \sin \delta_J^r \left[ - \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} + \eta_J^r \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} \right] \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} \]

\[ + \beta_J \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \sin \delta_J^\beta \left[ - \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} + \eta_J^\beta \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} \right] \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} \]

\[ + \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} \]

(3.26a)

\[ \overline{A} \cdot \overline{p} \, l \psi^0_{\text{sc}} = 0 \]
\[ \mathbf{b} \cdot \mathbf{p} \Psi_{sc}^{m} = (2)^{\frac{3}{2}} \hbar \frac{e}{r} \left( \frac{4\pi}{3} \right)^{\frac{1}{2}} \sum_{J} \left\{ s_{J} \, e^{i \delta_{J}^{m}} \sin \delta_{J}^{m} \right\} \]

(3.26c)

\[ + \left[ \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} + \eta_{J} \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \right] \rho^{m} e^{i \delta_{J}^{m'}} \sin \delta_{J}^{m'} \left[ \left( \frac{J}{2J+1} \right)^{\frac{1}{2}} + \eta_{J} \left( \frac{J+1}{2J+1} \right)^{\frac{1}{2}} \right] \] \[ \times \beta^{J} J_{A}^{J} \mathbf{r}_{J} \]

\[ \mathbf{b} \cdot \mathbf{p} \Psi_{sc}^{0} = (2)^{\frac{3}{2}} \hbar \frac{e}{r} \left( \frac{4\pi}{3} \right)^{\frac{1}{2}} \sum_{J} e^{i \delta_{J}^{1}} \sin \delta_{J}^{1} \]

(3.26d)

\[ \left( J^{\frac{1}{2}} \beta^{J} J_{A}^{J} \mathbf{r}_{J}^{J} \right) \]

The contribution to the singlet cross section is given by (except for the multiplying factor \( |J_{A}|^{2} \))

(3.27) \[ \frac{1}{4M^{2}} \int \mathbf{b} \cdot \mathbf{p} \Psi_{sc}^{0} |^{2} \, d\mathbf{n} = \]

\[ \left( \frac{\hbar^{2} k^{2}}{2M^{2}} \right) \frac{4\pi}{k^{2}} \sum_{J} (2J + 1) \sin^{2} \delta_{J}^{1} \]

In the same way, using (3.15) and the orthonormality of the \( Z \)'s, a term

(3.28) \[ \frac{\hbar^{2} k^{2}}{2M^{2}} \frac{4\pi}{k^{2}} \sum_{J} (2J + 1)(\sin^{2} \delta_{J}^{m} + \sin^{2} \delta_{J}^{m'}) \]

\[ \times \sin^{2} \delta_{J}^{m'} \]

is added to the triplet cross section. If we now add this to (3.24) after using (3.20) and (3.22), we obtain an equation for the total cross section which has the same form as in the non-relativistic case, namely,
Thus the only change brought about by the inclusion of the relativistic terms is to change the values of the phase shifts. After the phase shifts have been determined, the problem is the same as in the non-relativistic calculation. The method of finding the phase shifts is given in the next section.
B. The Variational Principle for the Phase Shifts

If in (2.17) we set

\[(3.30) \quad S_{J-1} = (p_J/r) Z_J^{(J)} m + (u_J/r) Z_J^{(J-1)m} \]

\[+ (v_J/r) Z_J^{(J)} m + (w_J/r) Z_J^{(J+1)m}, \]

and equate coefficients of the \(Z\)'s on the two sides of the equation, we obtain the radial equations

\[(3.31a) \quad \left( \frac{d^2}{dx^2} + \kappa^2 - \frac{J(J-1)}{x^2} \right) u_J(x) = (f_J(x)u_J(x) + g_J^{(1)}(x)w_J(x)) \]

\[(3.31b) \quad \left( \frac{d^2}{dx^2} + \kappa^2 - \frac{(J+1)(J+2)}{x^2} \right) w_J(x) = (g_J^{(2)}u_J + h_Jw_J) \]

\[(3.31c) \quad \left( \frac{d^2}{dx^2} + \kappa^2 - \frac{J(J+1)}{x^2} \right) v_J(x) = l(x)v_J(x) \]

\[(3.31d) \quad \left( \frac{d^2}{dx^2} + \kappa^2 - \frac{J(J+1)}{x^2} \right) p_J(x) = m(x)p_J(x) . \]

If we let \(r_0\) be the range of the potential, then in the above equations \(\kappa = kr_0\), and \(x = r/r_0\).

The set of Green's functions for the problem will be

\[G^u_J(x,x') = -\frac{1}{\kappa} kx_J J_{J-1}(\kappa x \cdot \kappa x_\ell) \eta_{J-1}(\kappa x) \]

\[G^w_J(x,x') = -\frac{1}{\kappa} kx_J J_{J+1}(\kappa x \cdot \kappa x_\ell) \eta_{J+1}(\kappa x) \]

\[G^v_J(x,x') = -\frac{1}{\kappa} kx_J J_J(\kappa x \cdot \kappa x) \eta_J(\kappa x_\ell) = G^p_J(x,x') \]
which satisfy
\[
\left( \frac{d^2}{dx^2} + \kappa^2 - \frac{J(J-1)}{x^2} \right) \Phi_{J}(x) = -\delta(x - x'), \text{ etc.}
\]
and vanish at \( x_* = 0 \). The notation \( x_* \) and \( x_\tau \) designates the smaller and the larger, respectively, of \( x \) and \( x' \). The function \( \eta_{J}(\kappa x) \) is the spherical Neumann function.

The detailed analysis leading to the variational principle is given by Rohrlich and Eisenstein\(^7\), and will not be reproduced here. The phase shift for \( \delta_{J}^{\beta} \) satisfies the equation
\[
-\cot \delta_{J}^{\beta} = \frac{\int [1(x)\Phi_{J}(x)]\Phi_{J}(x)dx}{\kappa \left( \int x_{J-\frac{1}{2}}(\kappa x)1(x)\Phi_{J}(x)dx \right)^2}
\]
An analogous equation holds for \( \delta_{J}^{1/2} \). This equation can easily be shown to be stationary with respect to small variations in the radial function \( \Phi_{J}(x) \). Thus if the error in the wave function is of the first order, then the error in the phase shift is of the second order. In order to compute the phase shift, a trial function is used which is taken from a closely related or simpler

\(^7\)Rohrlich and Eisenstein, op. cit., p. 719.
problem. The simplest trial function to use is the solution
for zero potential. From (3.31) this will be

\[ \bar{v}_j(x) = \kappa x j_{-1}(\kappa x) \]
\[ \bar{w}_j(x) = \kappa x j_{+1}(\kappa x) \]
\[ \bar{u}_j(x) = \kappa x j_j(\kappa x) = p_j(x). \]

This corresponds to the Born approximation, but gives
much better results than the ordinary Born approximation\(^8\).
The values for the phase shifts are better the higher the
energy and the higher the value of \( J \).

The phase shifts arising from the coupled equations in
(3.31) satisfy a more complicated equation. These phase
shifts are given by the two solutions of the equation

\[ (\alpha J - \beta J^2) \cot^2 \delta J + (\alpha J + \alpha J a_J - 2b J b_J) \cot \delta J \]
\[ + (a J a_J - b J^2) \] \[ = 0, \]

where

\[ a_J = \int (h_J w_J) w_J dx + \int \varepsilon_J^{(1)} w_J dx \varepsilon_J^{u} dx \varepsilon_J^{(2)} u_J \]
\[ + \int h_J w_J dx \varepsilon_J^{W} dx \varepsilon_J^{W} w_J. \]

\(^8\)M. Hamermesh, *Nuclear Physics* (New York University,
New York, 1947), p. 90. These mimeographed notes contain
a detailed analysis of the variational principle used in
this paper, most of which is taken from the work of J.
Schwinger.
(3.36b) \[ h_j = \frac{1}{2} \int (g_j^{(1)} w_j) u_j \, dx + \frac{1}{2} \int (g_j^{(2)} u_j) w_j \, dx \]

\[ + \int f_j u_j \, dx \int g_j^u \, dx' g_j^{(1)} w_j \]

\[ + \int h_j w_j \, dx \int g_j^w \, dx' g_j^{(2)} u_j \]

(3.36c) \[ c_j = \int (f_j u_j) u_j \, dx + \int g_j^{(2)} u_j \, dx \int g_j^u \, dx' g_j^{(1)} u_j \]

\[ + \int f_j u_j \, dx \int g_j^u \, dx' f_j u_j \]

(3.36d) \[ d_j = \kappa \left( \int x_{j-1}(\kappa x) g_j^{(1)} w_j \, dx \right)^2 \]

\[ + \kappa \left( \int x_{j+1}(\kappa x) h_j w_j \, dx \right)^2 \]

(3.36e) \[ \beta_j = \kappa \left( \int x_{j-1}(\kappa x) f_j u_j \, dx \right) \left( \int x_{j-1} g_j^{(1)} w_j \, dx \right) \]

\[ + \kappa \left( \int x_{j+1}(\kappa x) g_j^{(2)} u_j \, dx \right) \left( \int x_{j+1} h_j w_j \, dx \right) \]

(3.36f) \[ \gamma_j = \kappa \left( \int x_{j-1}(\kappa x) g_j^{(2)} u_j \, dx \right)^2 \]

\[ + \kappa \left( \int x_{j-1}(\kappa x) f_j u_j \, dx \right)^2 \]

It will be noticed that if \( g^{(1)} = 0 \) and \( g^{(2)} = 0 \), as would be the case for central forces, then each of the two solutions of (3.35) takes the form of (3.35), as it should.

These are the equations that we will use in the next section to compute the cross section for a specific problem. Although they appear somewhat complicated, they are quite convenient to apply. By using (3.34), most of
the integrals can be evaluated analytically. Inspection shows further that a specific integral may appear several times. This is especially true of (3.36), where a certain integral may appear in the equations for several different values of J.
IV. APPLICATION TO A SCATTERING PROBLEM

We will now apply the theory that has been developed to a specific scattering problem. We will assume a Serber potential, and let the radial dependence be that of a square well. Christian and Hart\(^1\), as well as other authors\(^2\), have found that there is strong evidence that there is considerable difference in the ranges in the triplet and singlet states. In particular, the triplet range appears to be much shorter than hitherto believed. In accordance with this we will use the ranges \(r_s = 2.8 \times 10^{-13}\) cm., and \(r_t = 1.5 \times 10^{-13}\) cm. The corresponding well depths are \(V_s = 11.9\) Mev and \(V_t = 59.1\) Mev. We will assume an energy of 100 Mev for the incident particle.

The choice of a square well simplifies the problem somewhat. The integrals in (3.33) and (3.35) become simple in form. Because of the finite discontinuity of the potential at \(x = 1\), we can write

\[
(4.1) \quad \frac{d}{dx} V(x) = -V_0 \delta(x - 1).
\]

We also drop derivatives higher than the first.

We must now compute the correction terms for such

\(^1\)Christian and Hart, op. cit., p. 452.
\(^2\)J. M. Blatt, Phys. Rev. 74, 92 (1948).
a potential from (2.21) and those in the appendix. As mentioned before, the terms in $Q$ are the same for any of the possible combinations of the $\omega_1$'s. Hence we can write these down separately. We need to determine the results of the operation of the $Q$ terms on $\frac{1}{2}z(2J+1)$, $\frac{3}{2}z(2J)$, $\frac{3}{2}z(2J-1)$, and $\frac{3}{2}z(2J+1)$, where the bar indicates that these are the radial functions of (3.34). We must multiply the terms of (2.21), etc. by $\frac{M_{rr}}{h^2}$ to put these corrections in the dimensionless units used in (3.31). The results of these calculations are given in Table 2a. The functions $l(x)$, $m(x)$, etc. in (3.31) will in general contain a zero order term plus a correction term brought in by the relativistic term in the wave equation. The latter are the terms given in Table 2a. When used in (3.31), all terms in the table must be multiplied by $(1/4M)$. Likewise, the values for $h_J$ and $f_J$ must be multiplied by $1/(2J+1)$, and those for $g_J^{(1)}$ and $g_J^{(2)}$ are to be multiplied by $(J(J+1))^{3/2}/(2J+1)$. We observe that the terms for $L = J$ are independent of $J$, while the terms for $L = J - 1$ and $L = J + 1$ vary with $J$.

The remainder of the relativistic correction terms will depend on the particular choice of $\omega_1$'s used. It is therefore from these terms that we would expect to be able to choose a suitable potential which would decrease the cross section enough to give agreement with the experimental results. Let us use the following notation:
Table 2a

Correction Terms for $Q$

<table>
<thead>
<tr>
<th>$L$ even</th>
<th>$m(x)$</th>
<th>3</th>
<th>-1</th>
<th>-5</th>
<th>3</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l(x)$</td>
<td>2</td>
<td>-2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_J$</td>
<td>$J-1$</td>
<td>$-\frac{6J^2+7J-1}{2J+1}$</td>
<td>$-2(J^2+5J-4)$</td>
<td>$\frac{9(J-1)}{2(2J+1)}$</td>
<td>$J(J+1)(6J+1) - \frac{3}{2}(J-1)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$g_J$</td>
<td>$J-1$</td>
<td>$-J^4$</td>
<td>$2J^2-J-1$</td>
<td>$\frac{3(J+2)}{2(2J+1)} + J(J+1)(6J+5)$</td>
<td>$\frac{3}{2}(J+2)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$h_J$</td>
<td>$J-1$</td>
<td>$-J^2$</td>
<td>$2J^2-J-1$</td>
<td>$\frac{3(J+2)}{2(2J+1)} + J(J+1)(6J+5)$</td>
<td>$\frac{3}{2}(J+2)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$g_J^{(1)}$</td>
<td>$J-1$</td>
<td>$\frac{4J}{2J+1}$</td>
<td>$0$</td>
<td>$\frac{2(J+2)^2}{2J+1} + (J+1)$</td>
<td>$6J^2+9J+3$</td>
<td>$3(2J+1)$</td>
<td></td>
</tr>
<tr>
<td>$L$ odd</td>
<td>$m(x)$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$l(x)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_J$</td>
<td>$(J+1)$</td>
<td>$-\frac{2J^2+J+1}{2J+1}$</td>
<td>$-J(2J-3)$</td>
<td>$\frac{J-1}{2(2J+1)} + J^2(2J-3)$</td>
<td>$-\frac{1}{2}(J-1)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$g_J$</td>
<td>$(J+1)$</td>
<td>$-J^4$</td>
<td>$2(J+2)$</td>
<td>$4$</td>
<td>$\frac{(J-1)}{2J+1} - 4J$</td>
<td>$J(2J+1)$</td>
<td>$-2J+1$</td>
</tr>
<tr>
<td>$h_J$</td>
<td>$(J+1)$</td>
<td>$(J+1)$</td>
<td>$2J^2-3J-5$</td>
<td>$\frac{J+2}{2(2J+1)} + (J+1)(2J+5)$</td>
<td>$\frac{3}{2}(J+2)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$g_J^{(1)}$</td>
<td>$(J+1)$</td>
<td>$-1$</td>
<td>$-6$</td>
<td>$\frac{J+2}{2J+1} + 4(J+1)$</td>
<td>$2J^2+3J+1$</td>
<td>$2J+1$</td>
<td></td>
</tr>
</tbody>
</table>
The results of these calculations are given in Table 2b.

The choice of potential that we have made leads to a considerable simplification in the use of the variational principle. In the first place, when L is odd, there are no zero order contributions and the phase shifts are determined solely by the relativistic terms. Inspection shows that the phase shift so determined gives a contribution to the scattering cross section which is proportional to the square of the phase shift, i.e., \( \sin^2 \delta \), while the contribution of the terms for L even is of the first order in the correction to the phase shift. That is, for the latter we have

\[
\sin^2(\delta + \delta') = \sin^2 \delta + \delta' \sin 2\delta.
\]

This means that we can neglect the terms for L odd, and the relativistic terms do not change the symmetry about 90°.

Since we only keep the first order terms in (3.36), we can drop the second term in (3.36a) and (3.36c), as well as the first term in (3.36d) and (3.36f). For L even, (3.36b) and (3.36e) have no zero order terms. There-

\[
A \sim (\omega_1, \omega_3)
\]
\[
B \sim (\omega_2, \omega_4)
\]
\[
C \sim (\omega_1, \omega_4)
\]
\[
D \sim (\omega_2, \omega_3)
\]
Table 2b

Relativistic Corrections to the Wave Equation

<table>
<thead>
<tr>
<th>State</th>
<th>( \delta(x-l) V_0 \frac{d}{dx} )</th>
<th>( \frac{1}{x} \delta(x-l) V_0 )</th>
<th>( V_0 x^2 )</th>
<th>( \frac{V_0}{x^2} )</th>
<th>( \frac{1}{x} V_0 \frac{d}{dx} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singlet</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L even</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A ( m(x) )</td>
<td>-2</td>
<td>-17/4</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>-2</td>
<td>-5</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>-3/4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>-4</td>
<td>-17/2</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L odd</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>6</td>
<td>33/4</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>-2</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
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<td>19/4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>L = J</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L even</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A ( l(x) )</td>
<td>4</td>
<td>1/4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>-4</td>
<td>-1</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>-1/4</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>-2</td>
<td>-1/2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L odd</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>-1/4</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>1/4</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
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<td>-1/2</td>
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</table>
Table 2b (Continued)

<table>
<thead>
<tr>
<th>State</th>
<th>$\delta(x-1)\frac{dV_0}{dx}$</th>
<th>$\frac{1}{x}\delta(x-1)V_0$</th>
<th>$V_0k^2$</th>
<th>$V_0/x^2$</th>
<th>$\frac{1}{x}V_0\frac{d}{dx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Triplet</strong>&lt;sup&gt;a&lt;/sup&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L = J-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>L even</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A $f_{J(n)}$</td>
<td>(2(J+2)) (-2J^2-(7/2)J - \frac{1}{2})</td>
<td>(-2(J-1)) (-3(J-1))</td>
<td>(0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s_{J}^\prime)</td>
<td>(-6) (6J)</td>
<td>(-6) (6J(2J+1))</td>
<td>(-6) (6J(2J+1))</td>
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<td></td>
</tr>
<tr>
<td>B</td>
<td>(-2(3J+2)) (6J^2+2J - 1)</td>
<td>(-2(3J+1)) ((J-1))</td>
<td>(0)</td>
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<td></td>
</tr>
<tr>
<td>(2)</td>
<td>(-2J)</td>
<td>(2) (-2J(2J+1))</td>
<td>(2) ((2J+1))</td>
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<td></td>
</tr>
<tr>
<td>C</td>
<td>(2(3J+1)) (-6J^2-(5/2)J - \frac{1}{2})</td>
<td>(2(3J+2)) ((J-1))</td>
<td>(0)</td>
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<tr>
<td>(2)</td>
<td>(-2J)</td>
<td>(2) (-2J(2J+1))</td>
<td>(2) ((2J+1))</td>
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<td></td>
</tr>
<tr>
<td>D</td>
<td>(-2(5J+1)) (10J^2+J - \frac{1}{2})</td>
<td>(-2(5J+4)) (-3(J-1))</td>
<td>(0)</td>
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<td></td>
</tr>
<tr>
<td>(-6)</td>
<td>(6J)</td>
<td>(-6) (6J(2J+1))</td>
<td>(-6) ((2J+1))</td>
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<td></td>
</tr>
<tr>
<td><strong>L odd</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>(6J) (-6J^2-(1/2)J - \frac{1}{2})</td>
<td>(6(J+1)) (3(J-1))</td>
<td>(0)</td>
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<td></td>
</tr>
<tr>
<td>(6)</td>
<td>(-6J)</td>
<td>(6) (-6J(2J+1))</td>
<td>(6) ((2J+1))</td>
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<td></td>
</tr>
<tr>
<td>B</td>
<td>(-2J) (2J^2)</td>
<td>(-2(J+1)) (-3(J-1))</td>
<td>(0)</td>
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<tr>
<td>(-2)</td>
<td>(2J)</td>
<td>(-2) (2J(2J+1))</td>
<td>(-2) ((2J+1))</td>
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<td></td>
</tr>
<tr>
<td>C</td>
<td>(2(J+1)) (-2J^2-(3/2)J + \frac{1}{2})</td>
<td>(2J) (-3(J-1))</td>
<td>(0)</td>
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<tr>
<td>(-2)</td>
<td>(2J)</td>
<td>(-2) (2J(2J+1))</td>
<td>(-2) ((2J+1))</td>
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<td></td>
</tr>
<tr>
<td>D</td>
<td>(2(J-1)) (-2J^2+J - \frac{1}{2})</td>
<td>(2(J+2)) (3(J-1))</td>
<td>(0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6)</td>
<td>(-6J)</td>
<td>(6) (-6J(2J+1))</td>
<td>(6) ((2J+1))</td>
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</table>
### Table 2b (Continued)

<table>
<thead>
<tr>
<th>State</th>
<th>$f(x-1) V \frac{d}{dx}$</th>
<th>$\frac{1}{x} f(x-1) V_o$</th>
<th>$V_o k^2$</th>
<th>$V_o / x^2$</th>
<th>$\frac{1}{x} V_o \frac{d}{dx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Triplet</strong>&lt;br/&gt;$L = J+1$&lt;br/&gt;L even</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>$2(J-1)$</td>
<td>$2J^2 + (1/2) J - 7/4$</td>
<td>$2(J-1)$</td>
<td>$3(J+2)$</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$-2(3J+1)$</td>
<td>$-6J^2 - 10J - 3$</td>
<td>$-2(3J+1)$</td>
<td>$-(J+2)$</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>$2(3J+2)$</td>
<td>$6J^2 + 19/2 J + 15/2$</td>
<td>$2(3J+2)$</td>
<td>$-(J+2)$</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>$-2(5J+4)$</td>
<td>$-10J^2 - 19J - \frac{1}{3}$</td>
<td>$-2(5J+4)$</td>
<td>$3(J+2)$</td>
<td>0</td>
</tr>
<tr>
<td><strong>L odd</strong>&lt;br/&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>$6(J+1)$</td>
<td>$6J^2 + 23/2 J + 23/4$</td>
<td>$6(J+1)$</td>
<td>$-3(J+1)$</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$-2(J+1)$</td>
<td>$-2(J+1)^2$</td>
<td>$-2(J+1)$</td>
<td>$(J+2)$</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>$2J$</td>
<td>$2J^2 + (5/2) J + \frac{1}{4}$</td>
<td>$2J$</td>
<td>$(J+2)$</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>$2(J+2)$</td>
<td>$2J^2 + 5J - \frac{1}{3}$</td>
<td>$2(J+2)$</td>
<td>$-3(J+2)$</td>
<td>0</td>
</tr>
</tbody>
</table>

*For $L = J - 1$ and $L = J + 1$, the first entry for a given potential is to be multiplied by $1/(2J + 1)$, while the second entry is to be multiplied by $(J(J+1))^{3/2}/(2J + 1)$.*
fore in (3.35) we may neglect the terms which include these quantities. The two solutions of (3.35) then take exactly the same form as (3.33). This aids materially in the computation.

The results of the calculation using $p_0$, $p_2$, and $v_1$ are given in Table 3. The non-relativistic values (N.R.) are included for comparison.

Table 3
Phase shifts at 100 Mev.

<table>
<thead>
<tr>
<th></th>
<th>$\delta_0^1$</th>
<th>$\delta_2^1$</th>
<th>$\delta_1^\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N.R.</td>
<td>0.162</td>
<td>0.0635</td>
<td>0.616</td>
</tr>
<tr>
<td>A</td>
<td>0.159</td>
<td>0.0716</td>
<td>0.616</td>
</tr>
<tr>
<td>B</td>
<td>0.160</td>
<td>0.0729</td>
<td>0.570</td>
</tr>
<tr>
<td>C</td>
<td>0.147</td>
<td>0.0582</td>
<td>0.625</td>
</tr>
<tr>
<td>D</td>
<td>0.176</td>
<td>0.0817</td>
<td>0.629</td>
</tr>
</tbody>
</table>

The non-relativistic phase shifts are calculated from the variational principle, although for a square well they can be calculated exactly. To illustrate that the variational values are better the larger the value of $J$, we find the exact value of $\delta_0^1$ to be 0.196, compared with the variational value 0.162, while the exact value of $\delta_2^1$ is 0.0646, compared with 0.0635. Thus the value for
$J = 0$ is in error by more than 16%, while the value for
$J = 2$ is only about 1% off. It seems reasonable to assume
that the relativistic corrections would be of the same
order of magnitude as in Table 3, if all the phase shifts
were calculated exactly.

The phase shifts for $L = 0$ can be expected to give the
main contribution to the cross section. There is no
triplet term for $L = J = 0$, since if this equality holds,
the total spin must also be zero. Thus $\delta_{1}^{q}$ is the only
triplet phase shift for $L = 0$. We notice the large
increase in the phase shift as we go from the singlet
to the triplet states, i. e., as we decrease the range
and consequently increase the well depth. Also, we observe
that both the direction and amount of the relativistic
corrections depend on the particular theory. Thus it
appears that A affects the total cross section very little,
while B decreases it the most and $D$ produces the largest
increase. The corrections could easily increase or
decrease the total cross section by 5-10%.

Because of the depth of the potential well in the
triplet state, the assumption that this depth is small
compared with the non-relativistic energy of the particles,
i. e., 100 Mev, is not too good. With a well depth of
59.1 Mev, we are ignoring terms which may be comparable
to the terms that we keep. This is especially evident
when we attempt to compute the corrections to the phase shifts for values of $J$ greater than 1. In these cases the correction becomes as large as the zero order term. However, from the behavior of the phase shifts for the singlet state, one feels that the inclusion of the terms that we have omitted would change the details, but no the general conclusions.
V. SUMMARY

We are now in a position to compare the results of this investigation with those of others. It was found that on the meson theory, the corrections to the total cross section could be about 10%, and that the direction of the correction depended on which of the available theories one used. Similar results have been found in this paper.

We have found that the form of the equation for the scattering cross section is the same as in the non-relativistic case. However, the inclusion of the relativistic terms in the wave equation changes the phase shifts, and also mixes states of equal parity. We can choose the potential in four different ways, each of which gives the same result in the non-relativistic limit. Depending on the particular potential chosen, we can either increase or decrease the total cross section by 5-10%.

It is reasonable to conclude, therefore, that the description of the interaction by means of a potential is still satisfactory at an energy as high as 100 Mev. Most of the details of the differential cross section can be accounted for by the range, well depth, amount of tensor force, etc., leaving only the fact that the cross section is too large in the non-relativistic calculation.
to be explained. In order to choose a particular potential, a much more exact calculation would have to be done. Until the accuracy of the experimental results becomes better than it is now, such a calculation would have little meaning. At present the neutron-proton cross section at 90 Mev is found to be $7.9 \pm 1.0 \times 10^{-26}$ cm$^2$. With this accuracy it would be possible only to discard those potentials which increase the cross section.
VI. ACKNOWLEDGEMENTS

The author would like to thank Dr. J. F. Carlson for his interest and advice during the investigation. The author also wishes to acknowledge the support received during the first stages of the work from the National Research Council under a Pre-Doctoral Fellowship. The prompt and willing services of the typist, Miss Jean Wygle, is greatly appreciated.
VII. APPENDIX

The normalized eigenfunctions of $J^2$, $L^2$, $S^2$, and $J_z$ can be shown to be

\[(A.1) \quad l_Z(L)_m = l(\sigma)_m \gamma_L^m \]

for the singlet state, and

\[(A.2) \quad S_Z(L)_m = (2L(L+1))^{-\frac{1}{2}} \left[ (2 \frac{3}{2})_m \gamma(L)_o \left\{ \gamma_L^m \right\} 
\quad + [(L+m)(L-m+1)]^\frac{1}{2} 3(\sigma)_1 \gamma_L^{m-1} 
\quad - [(L-m)(L+m+1)]^\frac{1}{2} 3(\sigma)_{-1} \gamma_L^{m+1} \right] \]

\[(A.3) \quad S_Z(L)_m = (2L(2L+1))^{-\frac{1}{2}} \left[ (2 \frac{3}{2})_m \gamma(L)_o \left\{ \gamma_L^m \right\} 
\quad + [(L+m)(L-m+1)]^\frac{1}{2} 3(\sigma)_1 \gamma_L^{m-1} 
\quad - [(L-m)(L+m+1)]^\frac{1}{2} 3(\sigma)_{-1} \gamma_L^{m+1} \right] \]

for the triplet state. These functions are orthogonal with respect to integration over the angles.

In order to obtain the results given in the text, it
is necessary to know the effect of a number of operators involving the spin and momentum on these eigenfunctions. In particular, there are the operators analogous to (2.21) for the other combinations of the $\omega_1$'s. The terms in $Q$ will be the same for $B$, $C$, and $D$, as for $A$, i.e., (2.21). The remaining terms contain only five different operators.

Table 4

Coefficients of Correction Term Operators

<table>
<thead>
<tr>
<th>State</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singlet</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L even</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>1/2</td>
<td>3/4</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>-2</td>
<td>-1/2</td>
<td>7/2</td>
<td>9/2</td>
</tr>
<tr>
<td>L odd</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>2</td>
<td>7/4</td>
<td>-9/4</td>
<td>-3/4</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>3/2</td>
<td>9/2</td>
</tr>
<tr>
<td>Triplet</td>
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</tr>
<tr>
<td>L even</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>-2</td>
<td>-1</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>2</td>
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<td>1/2</td>
</tr>
<tr>
<td>D</td>
<td>-1</td>
<td>-4</td>
<td>-5</td>
<td>3</td>
<td>1/2</td>
</tr>
<tr>
<td>L odd</td>
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<td></td>
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<tr>
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<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>D</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>-3</td>
<td>1/2</td>
</tr>
</tbody>
</table>
In Table 4 we have used the abbreviations

\[ a = (\hat{\mathbf{A}} \cdot \hat{\mathbf{p}})(\hat{\mathbf{A}} \cdot \hat{\mathbf{p}}), \quad b = (\hat{\mathbf{B}} \cdot \hat{\mathbf{p}})(\hat{\mathbf{B}} \cdot \hat{\mathbf{p}}), \quad c = Vp^2, \]
\[ (A.3) \]
\[ d = V(\hat{\sigma}(1) \cdot \hat{\mathbf{p}})(\hat{\sigma}(2) \cdot \hat{\mathbf{p}}), \quad e = \hbar^2 \frac{1}{r} \frac{dV}{dr}. \]

Also, all terms must be multiplied by \(1/4M^2\) to correspond with (2.21).

Let us use the following notation:

\[ M = \hat{\mathbf{r}} \cdot \hat{\mathbf{p}}, \quad N = (\hat{\sigma}(1) \cdot \hat{\mathbf{p}})(\hat{\sigma}(2) \cdot \hat{\mathbf{p}}), \]

\[ O = (\hat{\sigma}(1) \cdot \hat{\mathbf{r}})(\hat{\sigma}(2) \cdot \hat{\mathbf{p}}) + (\hat{\sigma}(2) \cdot \hat{\mathbf{r}})(\hat{\sigma}(1) \cdot \hat{\mathbf{p}}), \]

\[ P = (\hat{\sigma}(1) + \hat{\sigma}(2)) \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{p}}). \]

We then find the following expansions

\[ (A.4) \quad N \cdot V = \frac{\hbar}{i} \frac{1}{r} \frac{dV}{dr} (0 - i\hbar \hat{\sigma}(1) \hat{\sigma}(2)) + V \cdot N \]

\[ (A.5) \quad (\hat{\mathbf{A}} \cdot \hat{\mathbf{p}})(\hat{\mathbf{A}} \cdot \hat{\mathbf{p}}) = \frac{\hbar}{i} \frac{1}{2r} \frac{dV}{dr} (2M + \text{Im} + O) \]

\[ (A.6) \quad (\hat{\mathbf{B}} \cdot \hat{\mathbf{p}})(\hat{\mathbf{B}} \cdot \hat{\mathbf{p}}) = \frac{\hbar}{i} \frac{1}{2r} \frac{dV}{dr} (2M + \text{Im} - O) \]

Therefore we must find the result of operating on (A.1) and (A.2) by the four operators \(M, N, O, \text{ and } P\). This is given in (A.7) to (A.10) below.

\[ (A.7) \quad \frac{M \cdot R(r)}{r} \sigma'_{\frac{Z}{J}}(L)^m = -i\hbar \left( \frac{dR}{dr} - \frac{R}{r} \right) \sigma'_{\frac{Z}{J}}(L)^m \]
\[ P \cdot 1_{z(J)}^m = 0 \]
\[ P \cdot 3_{z(J)}^m = -2\hbar 3_{z(J)}^m \]
\[ (A.10) \]
\[ P \cdot 3_{z(J-1)}^m = 2(J-1)\hbar 3_{z(J+1)}^m \]
\[ P \cdot 3_{z(J+1)}^m = -2(J+2)\hbar 3_{z(J+1)}^m \]