A study in production functions from farm record data

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A STUDY IN PRODUCTION FUNCTIONS
FROM FARM RECORD DATA

by

Clifford George Hildreth

A Thesis Submitted to the Graduate Faculty
for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Agricultural Economics

Approved:

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1947
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INTRODUCTION

In this study, an attempt was made to estimate parameters of assumed production functions using data on the operation of 75 Iowa crop farms and 281 hog farms during the calendar year, 1944. These data were compiled from records kept by the farm operators who were members of the Iowa Farm Business Associations.

Two theoretical models were employed. In one case unique estimates could not be obtained, but a system of estimation equations was derived; in the other case, unique estimates were obtained. These estimates and estimation equations were compared with certain ranges derived from theoretical considerations and from general knowledge of farm operations. Some discrepancies were noted and these led to a re-examination of the estimation procedures used. Possible explanations for the discrepancies were examined and some suggestions for improvement of future investigations resulted.

The plan of presentation is that a theoretical background of the production determination process and the possibility of empirically estimating production functions is presented in Chapter I. Chapters II and III contain descriptions of two theoretical models used as bases for making estimates and the resulting estimates and estimation equations. A critical discussion of the models and procedures is presented in
Chapter IV along with suggestions for future study. The Appendix contains a mathematical demonstration of use in the discussion of Chapter II.
CHAPTER I.

THEORETICAL BACKGROUND

The Production Determination Process

As a point of reference for later discussion, it may be useful to present a brief outline of the production determination process. A comprehensive analysis of this process has been worked out by Hart and his analysis includes more extensive discussion of most of the points mentioned in the present outline. Here it is desired to inquire briefly what determines the particular quantity of product that emerges from a given firm's operations over a particular time interval. It will be considered that the interval is sufficiently short that the distribution of production within the interval is not of sufficient significance to be of interest. The production of many agricultural products is planned with reference to a natural season making this an appropriate interval to use in analysis.

The chain of causal events could in principle be traced to any stage of remoteness. For present purposes, it seemed that to consider the process as starting with the entrepreneur's observation of data relevant to the firm's operations would be the most useful point of

The entrepreneur has available for organization and operation

the present asset and hardware stock, the opinions of experts, and
and produces the sales of the firm's production and turnover. Then, the
history of the economy, particularly of the prices of raw materials, the
range of resource data, which influences among other things, the
recent developments. The entrepreneur has available for organization a

---

**Notes:**

- The text appears to be a section discussing the entrepreneur's role in production and the factors influencing it. It mentions the availability of resources, the entrepreneur's decision-making process, and the importance of historical data.

---

**References:**

- The text seems to be a continuation of a larger discussion, possibly from a business or economics publication, but specific references are not provided in the visible part of the image.
The entrepreneur has certain tastes, desires, or preferences which determine his objectives in the conduct of the firm's business and the relative subjective weights of these objectives. These desires and preferences are to be called "decision relations" in future references in this paper and they, in conjunction with his expectations, determine his business decisions. Essentially, his expectations define the probable outcomes of various decisions and his decision relations define which set of probable outcomes he prefers. The desire for profit is the most universally recognized component of the decision relations and is undoubtedly of first importance. Where expectations take the form of probabilities or probability distributions, the risk preference has been proposed. Other tastes and preferences may have some importance in various instances. A farmer, for example, might have a preference for working with new machinery, for a large barn, for raising cattle rather than hogs, or a prejudice against borrowing money.

The decisions in which we are particularly interested are those that determine the "inputs" of his firm where inputs are defined as services of the factors of production. The entrepreneur also makes other decisions, his financing plan, for example, but if these are to affect his production, they must also affect his inputs. Therefore, we shall not try to take account of these other decisions in the present outline.

---

The chosen inputs interact in the production process according to certain physical and psychological principles. These are usually quite complex and, together with the collection of inputs chosen, determine the quantities of "outputs" that result. These principles which relate inputs and outputs are called "production relations" in the remainder of this section.

The process that is outlined above might be diagrammed as follows:

```
   a. Environmental Data
    A. Expectation Relations
    b. Expectations
       B. Decision Relations
    c. Inputs
       C. Production Relations
    d. Outputs
```

The diagram is intended to represent a chronological sequence in the sense that the third step cannot occur unless the first and second have taken place and the second step cannot occur unless the first has taken place. However, the first two steps may be repeated or revised a number of times and to try to trace the actual time sequence leading to a particular output might be quite complex.
Production Functions for Groups of Firms

In this study, as in most cross-section investigations, the available data consisted of material from which the variables \( c \) and the variables \( d \) in the diagram page 6 could be approximated for a number of firms. In economic theory it is frequently considered that the production relations, \( C \), can be expressed as a single functional relationship between inputs and outputs. This is the familiar production or transformation function.

In empirical work, rather simple algebraic equations are generally used under the assumption that they will represent the underlying production relations with sufficient accuracy that estimates of parameters in the simple relationship will give useful information about the actual relationship.

Since the various firms that have been observed are not identical, it is to be expected that each firm has a unique production function associated with it. However, since the firms are quite similar in other significant respects, it seems reasonable to believe that there is a basic similarity in the individual production functions and that inputs and outputs of the observed firms can be used to estimate parameters of some ideal or average production functions.

The methods used in Chapters II and III are based on the further assumption that individual production functions differ in that each contains a particular value of a certain random parameter and that
this parameter enters the individual equations additively. This random parameter is often called the "disturbance" of the production equation.\(^1\)

It is obviously more realistic to postulate that the same basic production function applies to a group of firms if the group is fairly homogeneous. It was largely for this reason that crop farms and hog farms were treated separately in the present study.

Homogeneity and Fixed Proportions of Inputs and Outputs

Another problem of homogeneity arises in classifying and defining the inputs and outputs to be used in a particular study. The difficulties of computing estimates for equations involving a large number of

\(^1\)This assumption is made by Jacob Marschak and William H. Andrews in their model described in their monograph, "Random simultaneous equations and the theory of production", Econometrica, Vol. 12, July-October, 1944. pp. 143-205. This procedure neglects errors of observation of the variables, which is reasonable if the errors of observation are relatively small in comparison to the disturbances. Tinmner has developed a procedure for taking these errors into account but it involves neglecting the disturbance and to use his method in a cross-section study would require a priori knowledge of the variances of the errors. Tinmner's procedure is outlined in, "Multiple regression for systems of equations", Econometrica, v. 14, January, 1946. pp. 5-35. The problem of estimation with both errors of observation and disturbances in equations has been discussed by Leonid Hurwicz and T. H. Anderson, Jr. in a mimeographed report not yet published.
variables and the fact that clerical and computational resources available for most studies are not infinite frequently means that, if computations are to be made at all, the number of variables used must be kept small.

This involves certain dangers. If we consider two inputs \( x \) and \( y \) and suppose that it is necessary to treat them as a single input, say \( Z = x + y \), then there are two circumstances under which this involves little or no loss of information. If \( x \) and \( y \) are used in fixed proportions, or nearly so, by the observed firms, then the data contains little or no information on the relations of \( x \) or \( y \) alone to other inputs and to output and little or nothing is lost by treating them as a single input.

Similarly, if \( x \) and \( y \) are known to have similar effects in the production process, they can safely be treated together although a weighting problem arises in this case. If \( x \) and \( y \) have different production effects then the effect of a given value of the variable \( Z = x + y \) will depend on the proportions of \( x \) and \( y \) in \( Z \). If \( x \) and \( y \) have the same production effects, then a given value of \( Z \), where \( Z = x + ky \) will enter the production relations in a consistent way if \( k \) is a constant such that \( k \) units of \( x \) have the same production effects as one unit of \( y \).

Similar considerations apply to outputs. Outputs that appear in fixed proportions may safely be combined and outputs that have similar relations to inputs may be safely combined if properly weighted.
Past studies have usually recognized several inputs but only one output. In many cases this is undoubtedly proper, but reflection on the heterogeneity of outputs of many agricultural firms and the wide variations in the proportions in which they appear leads the author to believe that proper classification of outputs may be as important as proper classification of inputs in seeking to obtain useful agricultural production functions.

For the Iowa farms included in the present study, outputs seemed to fall in two logical classes—crop outputs and livestock outputs. The particular outputs in each class are not strictly homogeneous, but the author is inclined to believe that a large fraction of the difficulty due to heterogeneity can be removed by recognizing these two categories of output. Recognizing two outputs in the present study led to a number of complications which will become apparent in the discussion of the procedures used in the present study.

**Systems of Equations**

Some very fundamental questions affecting proper estimation of production functions arise from the work of a number of writers\(^1\) who

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Koopmans, Tjalling, "Statistical estimation of simultaneous
have presented what is sometimes called the simultaneous equations approach. This approach grew out of a reformulation of the problem of estimating economic relationships when more than one relation exists among the observed variables. Fundamental relationships among the variables are called "structural" relationships and equations used to represent them are called structural equations. Supply functions,

(Footnote continued)


"Statistical confluence analysis by means of complete regression systems", Publication No. 5 from the Institute of Economics, Oslo, Norway, 1934.


Schultze, Henry, "The theory and measurement of demand", The University of Chicago Press, Chicago, Illinois, 1928, Ch. II.

demand functions, liquidity preference functions, and production functions are examples of structural relationships. It has been shown that, in general, estimates of parameters of a structural equation obtained by considering only that equation are biased. In special cases where the structural equation coincides with the regression equation, single equation estimates are unbiased. Where the structural equation does not coincide with the regression equation, the latter may still be of considerable interest and many be used for prediction when there is reason to believe that none of the structural relations will change. However, for many purposes, estimates of one or more of the structural equations are needed.

Single equation methods have been used in most production function studies in the past. In the application of these methods,

1 Hurwicz, Leonid, "Prediction and least squares", in "Statistical inference in dynamic economic systems", to be published as Cowles Commission Monograph No. 10.


(with Grace T. Gunn), "The production function for
parameters of the assumed production equations are ordinarily estimated by
least-squares methods treating output as a dependent variable and
various inputs as independent variables. This yields an estimate of
the regression of output on inputs. Is this also an estimate of the
production function? Referring to the diagram of page 6, we see that
inputs also enter into decision relations. If we think of certain
decision equations and the production equation as a set of simultaneous
equations, it can be shown that the regression equation and the structural
equation will be the same if output does not enter the decision equations
and if the disturbances of these equations are independent of the dis-
urbance of the production equation.

(Footnote continued)

American manufacturing in 1919", American Economic Review, Vol. 31,
March 1941, pp. 67-80.

Douglas, Paul H. (with Grace T. Gunn), "The production function
for Australian manufacturing", Quarterly Journal of Economics,

_________ (with Grace T. Gunn), "The production function
for American manufacturing for 1914", Journal of Political Economy,
Vol. 50, August, 1942, pp. 595-802.

_________ (with Patricia Daly and Ernest Olson), "The
production function for manufacturing in the United States, 1904",

_________ (with Patricia Daly), "The production function for
Canadian manufactures", Journal of American Statistical Association,

Tintner, Gerhard, "A note on the derivation of production functions

_________ and Brownlee, Q.s., "Production functions derived
from farm records", Journal of Farm Economics, Vol. 26, August 1944,
pp. 566-571.
Output does not exist at the time inputs are determined and hence can enter the decision relations only through anticipation. However, in some cases this anticipation may be a relatively minor factor and it may not be unrealistic in these cases to assume that output enters the decision relations. In other cases it may be more realistic to consider that only expectations and inputs enter the decision relations and that expectations are formed from historical data and subjective expectation relations. In the latter cases, output would enter only the production equation and this would coincide with the regression equation if, in addition, the disturbance of the production equation were independent of the disturbance in the other equations. It is the current judgment of the author that, while the matter cannot be considered to be conclusively settled, cases in which the regression function does coincide with the production function are probably a fairly small minority.

The simultaneous equations system used for illustration by Marschak and Andrews involves a production equation and a decision equation for each input. The decision equations are derived from profit maximization assumptions. The author is inclined to believe that there may be a danger, particularly in a cross-section study, in assuming that all inputs have been chosen so as to maximize profits for the interval for which observations are available. The following reasons may be cited for considering that such an assumption might be unrealistic in certain instances:
1. The period of time over which the entrepreneur tries to maximize his economic position may be considerably longer than the interval of observation. Thus it may not be rational for him to try to attain maximum profits for that particular interval.

2. The entrepreneur can only regulate expected gain at the time of decision making.

3. The decision relations may involve important desires and preferences other than the desire for profit such as risk preference and non-economic tastes and desires.

4. The entrepreneur may not be free to vary certain inputs during the interval of observation. Technical, contractual, and financial barriers may exist.

These considerations seem to apply with more force to some inputs than others. For example, to assume that a particular farmer’s input of land in a given year was such as to maximize his profit for that year might be completely unrealistic. Capital-rationing is undoubtedly a more potent factor in determining the land input of many farmers than profit maximization. Even the farmer who has ample funds does not find it practical to alter this input very frequently. It would also appear that non-economic considerations also enter quite heavily in decisions with respect to land. We hear farmers described as "land hungry" or "land crazy" and sentiment against "breaking up the homestead" is quite strong among many cultural groups.

On the other hand, the assumption that a particular farmer’s live-
stock feed in a given year was such as to maximize his profit for that year is likely to be a close approximation to reality. The choice of this input is a day to day decision. If the farmer has had much experience with animals, he probably knows the results of various rations fairly well. Funds needed for feeding can usually be obtained on short-term loans more easily than other types of credit. It is also difficult to imagine a situation in which the goal of maximum profits from current feeding operations would conflict with longer-run economic objectives.

It would seem then, that essentially static profit maximization assumptions might reasonably be made with respect to some inputs but not with respect to others. Considerations of this sort have been used in setting up the model of Chapter III. This appears to be a rather crude method of specifying the theoretical model. The author feels that a still more general and more realistic set of equations should be set up describing the whole production determination process. If the more general set must involve variables not ordinarily observed, it should still be possible to reveal the nature of estimates obtained by using single-equation models and other simplified sets. The particular simplified models used as a basis for computations in this study are explained in the next two chapters.
CHAPTER II.
THE PRODUCTION EQUATIONS MODEL

Description of the Model

In the early stages of the present study an attempt was made to define fairly homogeneous classes of inputs and outputs in accordance with considerations mentioned in Chapter I. This resulted in the specification of a large number of classes. It was decided that outputs could reasonably be classed into livestock output and crop output, but the resulting equations were still unwieldy and it was decided to sacrifice homogeneity of inputs to obtain manageable equations. Accordingly inputs were divided into three broad and generally non-homogeneous categories. From the standpoint of obtaining useful and reliable results, this is probably as serious an error as recognizing only one output. It did, however, enable the author to investigate some problems of handling production equations involving two separate outputs. Since this has not been done in previous studies and since the author believes it to be of importance in many agricultural studies, this seemed a useful procedure.

The two classes of output used in this study were livestock output represented by $y_1$ and crop output represented by $y_2$. The three classes of input used were called $x_1$, $x_2$, and $x_3$. $x_1$ is defined to include
where capital letters are used to represent the logarithms of the

\[ \log Z = \log Z + \log Z \]  

\[ \log Z = \log Z + \log Z + \log Z \]  

\[ \log Z = \log Z + \log Z + \log Z \]  

It is assumed that logarithms are valid in the

production of other than \( \log Z \) and that the relationship between

the data \( \log Z \) represents a series of readings in the

production of other than \( \log Z \) and that the relationship between

the cost of the series of data used during the years 1 to

such as land, farm improvements and equipment, in addition to the

measured \( \log Z \) and output in dollars, and for fixed assets,

some of these were very needed. The \( \log Z \) was decided to

such as labor, improvements, measurements, and farm improvements

introduced into use in both the \( \log Z \) and \( \log Z \) to

introduce into use in both the \( \log Z \) and \( \log Z \) to
respective variables. (1) represents the production function for livestock, (2) represents the crop production function and (3) connects the unobserved variables with the observed variable, \( Z_3 \). \( U_1 \) and \( U_2 \) are random disturbances and the \( \gamma \)'s are structural parameters to be estimated. It is assumed further that the random disturbances, \( U_1 \) and \( U_2 \), are normally distributed and are uncorrelated with the inputs. This assumes ten relations of the type,

\[
\gamma_{21} U_j = 0 \quad 1 = 1, 2, 3, 4, 5; j = 1, 2
\]

A number of questions can properly be raised about the choice of this particular set of equations and this matter will be discussed in Chapter IV.

Investigation of Identification

From a statistical point of view, this set of equations is of interest in that it contains two variables which cannot be observed, but their sum can be observed. The first question to be answered is whether or not the parameters of this system can be estimated. The system can be simplified slightly by using equation (3) to eliminate one of the unknown variables changing equation (2) to

\[
(2') \quad Y_2 + \gamma_{22} Z_2 + \gamma_{25} Z_3 = \gamma_{25} Z_4^* = U_2
\]

Using moment methods on equations (1) and (2'), the author concluded that the system was unidentified—there seemed to be one degree of indeterminacy in the system. To check this conclusion, max-
imum likelihood considerations were employed. The system was rewritten in the following form:

\[(4) \quad x_1 + \gamma_{11} x_3 = v_1\]

\[(5) \quad x_2 + \gamma_{22} x_4 + \gamma_{25} x_5 = v_2\]

\[(6) \quad x_3 = v_3\]

\[(7) \quad x_4 = v_4\]

\[(8) \quad x_5 = v_5\]

where the new variables have been defined from the old as follows:

\[y_1 = x_1\]

\[y_2 = x_2\]

\[z_1 = x_3\]

\[z_2 = x_4\]

\[z_3 = x_5\]

\[u_1 = \gamma_{14} z_4^* = v_1\]

\[u_2 + \gamma_{25} z_4^* = v_2\]

The \(X\)'s are now observed variables. The assumptions of the form \(\sigma_{z_1 u_j} = 0\) are retained and can be used to obtain the following restrictions on the variance-covariance matrix of the disturbances of
equations (4) to (8):

\[ \frac{\gamma_{23}}{\gamma_{13}} = \frac{\gamma_{24}}{\gamma_{14}} = \frac{\gamma_{25}}{\gamma_{15}} = -\frac{\gamma_{25}}{\gamma_{14}} \]

where

\(\gamma_{ij}\) is used to represent \(\gamma_{i\gamma_j}\).

The problem is to derive maximum likelihood estimation equations for the fifteen variances and covariances of the \(v_i\)'s and the three \(\gamma_i\)'s in the system (4) to (8), subject to restrictions (9), and to see if the parameters \(\gamma_{11}, \gamma_{22}, \text{ and } \gamma_{25}\) and the fifteen variances and covariances of the \(v_i\)'s can be estimated from these equations. This process was carried out and the opinion was confirmed that there is lack of identification in the system (4) to (9). The demonstration is given in Appendix I where the likelihood function is presented, estimation equations are derived, and it is shown that one of the estimation equations is redundant.

This means that the original three-equation system as specified on page 18 is indeterminate or unidentified. It is not possible to derive a set of estimation equations that uniquely determines the unknown parameters without making use of additional information or an additional assumption.

One possibility is to modify the system by adding an assumption regarding profit maximisation. Such a modification is made in Chapter III. Before this is done, it is possible to make certain interesting
computation based on the present system.

Presumptions About the Unknowns

Referring again to the set of equations:

(1) \( \gamma_1 z_1 + \gamma_{14} Z_4^* = \mu_1 \)

(2) \( \gamma_2 z_2 + \gamma_{12} Z_2 + \gamma_{25} Z_3 - \gamma_{25} Z_4^* = \mu_2 \)

it is possible to derive the following equations involving moments:

(i) \( \gamma_1 z_1 + \gamma_{11} \sigma_{11} z_1 + \gamma_{14} \sigma_{12} Z_4 = 0 \)

(ii) \( \gamma_2 z_2 + \gamma_{11} \sigma_{12} z_2 + \gamma_{14} \sigma_{13} Z_4 = 0 \)

(iii) \( \gamma_1 z_3 + \gamma_{11} \sigma_{13} z_3 + \gamma_{14} \sigma_{14} Z_4 = 0 \)

(iv) \( \gamma_2 z_3 + \gamma_{12} \sigma_{21} z_2 + \gamma_{25} \sigma_{25} z_5 - \gamma_{25} \sigma_{25} Z_4 = 0 \)

(v) \( \gamma_2 z_3 + \gamma_{12} \sigma_{22} z_2 + \gamma_{25} \sigma_{23} z_3 - \gamma_{25} \sigma_{25} Z_4 = 0 \)

(vi) \( \gamma_2 z_3 + \gamma_{12} \sigma_{23} z_2 + \gamma_{25} \sigma_{24} z_4 - \gamma_{25} \sigma_{25} Z_4 = 0 \)

The symbol \( \hat{\gamma} \) indicates that the expression over which it is placed can be estimated directly from the data and the symbol \( \tilde{\gamma} \) indicates an unknown quantity for which an estimate is desired. Since there are seven such quantities, four \( \gamma \)'s three covariances \( \sigma_{11} Z_4 \), the six equations do not determine the estimates uniquely. They do, however, restrict them in such a way that if one of the unknown quantities
could be estimated independently of the system, the system would yield corresponding estimates of the other six.

There are also certain presumptions about the unknowns that can be drawn from our general knowledge of agricultural production and from previous studies. It is possible to make computations from the system (1) to (vi) which enable us to see whether or not this system is consistent with our other presumptions. If their signs are changed, the \( \gamma \)'s represent "average" output elasticities. A negative elasticity would mean that to add the collection of inputs for which the elasticity were negative would decrease output. An elasticity greater than one would mean that a small increase in the input in question would increase output more than proportionally. The author considers either virtually inconceivable on a collection of operating farms of the sort included in the present study. It can also be argued that when inputs are defined in fairly broad aggregates, inputs are almost certain to be complementary and we would expect inputs to be positively correlated. Results of previous studies verify that this is the general rule in agriculture. We thus expect, with a very high probability, that, say, \( Z_1 \) and \( Z_4 \) are positively correlated and, therefore, that \( \sigma_{Z_1Z_4} > 0 \).

Similarly, we could argue \( \sigma_{Z_1Z_5} > 0 \), and since \( \sigma_{Z_1Z_4} + \sigma_{Z_1Z_5} = \sigma_{Z_1Z_3} \) \(^2\)

\(^1\)Hart, op. cit., p. 38.

Hicks, J.R., "Value and capital", Oxford Univ. Press., 1939, p. 95.

\(^2\)This equality is derived by multiplying equation (3), page 18 by \( Z_1 \), assuming normalized variables, and taking expected values of the products.
we have the inequality

\[(\text{vii}) \quad 0 < \sigma_{14}^{Z} \sigma_{34}^{Z} < \sigma_{13}^{Z} \]

By analogous arguments:

\[(\text{viii}) \quad 0 < \sigma_{24}^{Z} \sigma_{34}^{Z} < \sigma_{23}^{Z} \quad \text{and} \quad \sigma_{23}^{Z} \]

\[(\text{ix}) \quad 0 < \sigma_{34}^{Z} \sigma_{34}^{Z} < \sigma_{33}^{Z} \]

The presumptions concerning the \( \gamma \)'s can be stated:

\[(\text{x}) \quad 1 < \gamma_{11} < 0 \]

\[(\text{xi}) \quad 1 < \gamma_{14} < 0 \]

\[(\text{xii}) \quad 1 < \gamma_{22} < 0 \]

\[(\text{xiii}) \quad 1 < \gamma_{25} < 0 \]

The quantities \( \sigma_{13}^{Z} \), \( \sigma_{23}^{Z} \), and \( \sigma_{33}^{Z} \) can be estimated from the data. The author believes that there are additional safe presumptions that could be used to narrow the range of these inequalities and to derive others. This possibility will be foregone since the set (vii) to (xiii) turns out to be sufficient for present purposes.

Restrictions Computed from the Data

To compare these restrictions with restrictions (i) to (vi), let us rewrite the set, (i) to (vi):
(i') \( \frac{1}{r_{14}} \hat{\sigma}_{y_1 z_1} + \frac{\tilde{r}_{11}}{r_{14}} \hat{\sigma}_{z_1 z_1} + \tilde{\sigma}_{z_1 z_4} = 0 \)

(ii') \( \frac{1}{r_{14}} \hat{\sigma}_{y_1 z_2} + \frac{\tilde{r}_{11}}{r_{14}} \hat{\sigma}_{z_1 z_2} + \tilde{\sigma}_{z_1 z_4} = 0 \)

(iii') \( \frac{1}{r_{14}} \hat{\sigma}_{y_1 z_3} + \frac{\tilde{r}_{11}}{r_{14}} \hat{\sigma}_{z_1 z_3} + \tilde{\sigma}_{z_1 z_4} = 0 \)

(iv') \( \frac{1}{r_{25}} \hat{\sigma}_{y_2 z_1} + \frac{\tilde{r}_{22}}{r_{25}} \hat{\sigma}_{z_2 z_2} + \tilde{\sigma}_{z_2 z_4} + \hat{\sigma}_{z_2 z_3} = 0 \)

(v') \( \frac{1}{r_{25}} \hat{\sigma}_{y_2 z_2} + \frac{\tilde{r}_{22}}{r_{25}} \hat{\sigma}_{z_2 z_2} + \tilde{\sigma}_{z_2 z_4} + \hat{\sigma}_{z_2 z_3} = 0 \)

(vi') \( \frac{1}{r_{25}} \hat{\sigma}_{y_2 z_3} + \frac{\tilde{r}_{22}}{r_{25}} \hat{\sigma}_{z_2 z_3} - \tilde{\sigma}_{z_2 z_4} + \hat{\sigma}_{z_2 z_3} = 0 \)

By treating \( \tilde{\sigma}_{z_1 z_4} \) as though it were a known quantity, it is possible to solve for the other six unknowns in terms of \( \tilde{\sigma}_{z_1 z_4} \). The moments estimated directly from the crop farm data and needed in this process are:

\( \hat{\sigma}_{y_1 z_1} = .05477 \)

\( \hat{\sigma}_{z_1 z_1} = .05075 \)

\( \hat{\sigma}_{y_1 z_2} = .03715 \)

\( \hat{\sigma}_{y_2 z_1} = .03387 \)

\( \hat{\sigma}_{y_1 z_3} = .03266 \)

\( \hat{\sigma}_{z_1 z_3} = .02533 \)
The solutions obtained from the set (1, 1, 0) are:

- $2 \theta^{12}$
- $8 \theta^{299}$
- $4 \theta^{494}$
- $2 \theta^{639}$
- $1 \theta^{916}$
- $0 \theta^{252}$
These solutions are shown in Figure 1 for the relevant range of $\tilde{\sigma}_1 Z$. The values of $\tilde{\sigma}_2 Z$ and $\tilde{\sigma}_3 Z$ have been multiplied by ten before plotting to make them distinguishable from the horizontal axis.

It can readily be seen that the restrictions imposed by the estimation equations (i') to (vi') are inconsistent with the presumptions (vii) to (xiii). As was stated above, the author considers that these presumptions are quite conservative and that the arguments for them are quite strong. It is, of course, possible that peculiarities in the sample farms or errors of observation are responsible for this inconsistency. The author believes that this is unlikely and that these results do constitute a fairly strong argument that there is some inadequacy in the system examined here.

Turning to the hog farm data, the moments estimated directly from the data are:

\[
\hat{\sigma}_{11} = .04506 \\
\hat{\sigma}_{11} = .05070 \\
\hat{\sigma}_{12} = .02324 \\
\hat{\sigma}_{12} = .03149 \\
\hat{\sigma}_{13} = .02314 \\
\hat{\sigma}_{13} = .02510 \\
\hat{\sigma}_{33} = .02309
\]
Figure 1. Production Equation Solutions with Crop Farm Data
\[ \hat{\sigma}_{21}^y = .03556 \]
\[ \hat{\sigma}_{22}^y = .03344 \]
\[ \hat{\sigma}_{22}^z = .03785 \]
\[ \hat{\sigma}_{23}^z = .02271 \]
\[ \hat{\sigma}_{23}^y = .02417 \]

The solutions of estimation equations, (i') to (vi') are:

\[ \tilde{\gamma}_{11} = -0.771 \frac{\hat{\sigma}_{14}^2 + .0535}{\hat{\sigma}_{14}^2 + .0464} \]

\[ \tilde{\gamma}_{14} = -0.00599 \frac{\hat{\sigma}_{14}^2 + .0464}{\hat{\sigma}_{14}^2 + .0464} \]

\[ \tilde{\gamma}_{22} = .579 \frac{\hat{\sigma}_{14}^2 + .0103}{\hat{\sigma}_{14}^2 + .0175} \]

\[ \tilde{\gamma}_{25} = .01755 \frac{\hat{\sigma}_{14}^2 + .01753}{\hat{\sigma}_{14}^2 + .01753} \]

\[ \tilde{\sigma}_{24}^2 = .0020 + .6639 \sigma_{14}^2 \]
\[ \tilde{\sigma}_{25}^z = .0065 + .6351 \sigma_{14}^2 \]

These are shown in Figure 2. In this case there is a small range, 0 < \( \hat{\sigma}_{14}^2 < .00018 \), in which the estimates have values that do not
Figure 2. Production Equation Solutions with Hog Farm Data
Theorem will be discussed further in Chapter 12. This theorem represents the production function relationship representing the model does not depend on the author's interpretation. Two sets of assumptions were made. These assumptions were derived from general knowledge of farm operation and from research. These were in form in which these could be compared with other researches on the values of the unknowns. And it was possible to integrate these into a model and to estimate parameters and represent relationships. The assumption that the model could be estimated from the data examination of the model. It was hoped that these assumptions were not possible to make the model meaningful. The relations were not specified, but the assumptions were introduced to make the model meaningful. Certain assumptions were introduced to make the assumptions meaningful into farm production. An effort was made to make the theoretical model used for prediction represent the variations. It was difficult to summarize the data and the presentation used as a test. The data also showed some evidence of inoculation research. These figures have been collected relatively recently. Since the values that order the limits of the precision remain the same, the range of these estimates in the theoretical presentation that have been made. However, in
CHAPTER III.

THE PRODUCTION AND PROFIT EQUATIONS MODEL

Description of the Model

Using the notation of page 20, the livestock production equation is:

\[ (1) \quad x_1 + \gamma_1 x_2 = v_1 \]

where \( x_1 \) represents the logarithm of livestock output and \( x_2 \) represents the logarithm of inputs used only in livestock production. \( x_3 \) consists principally of livestock feed. It was argued in Chapter I that to assume that the quantity of feed fed in a given year was approximately the amount that would maximize profit subject to the previously chosen levels of other inputs would not be unreasonable. This amount can be expressed as the amount that equates marginal revenue and marginal cost of the input. Since, in this study, both inputs and outputs are measured in dollars' worth, the marginal revenue of feed input is the partial derivative of livestock output with respect to feed input, \( \frac{\partial x_1}{\partial x_2} \); and the marginal cost is unity.

Converting the production equation, (1), to input and output rather than their logs and differentiating, we have
33.

\[ x_1 = e^{v_1} x_3 - 11 \]

\[ \frac{x_1}{x_3} = -\gamma_{11} e^{v_1} x_3 - \gamma_{11}^{-1} \]

\[ z = -\gamma_{11} x_1 x_3^{-1} \]

The profit maximization condition is that this derivative equal unity; this would make

\[ -\gamma_{11} x_1 x_3^{-1} = 1 \]

\[ x_1 x_3^{-1} = \frac{1}{-\gamma_{11}} \]

Taking logs of both sides yields

\[ x_1 - x_3 = \log \frac{1}{-\gamma_{11}} \]

The right hand side is a constant. Since we do not expect profit to be exactly maximized, a random component is added to the right side of the equation and this can be combined with the constant. The sum of the constant and the random component is represented by \( \gamma_3 \) and the resulting equation is substituted for the third equation of the previous system (page 20) giving the following as the set of equations to be used in this chapter.

(1) \( x_1 + \gamma_{11} x_3 \) = \( v_1 \)
(2) \( x_2 + \gamma_{22} x_4 + \gamma_{25} x_5 \) = \( v_2 \)
(3) \( x_1 - x_3 \) = \( v_3 \)
(4) \( x_4 \) = \( v_4 \)
(5) \( x_5 \) = \( v_5 \)
As before, prior considerations impose the restrictions,

\[ \frac{\sigma_{x_3 y_2}}{\sigma_{x_3 y_1}} = \frac{\sigma_{x_4 y_2}}{\sigma_{x_4 y_1}} = \frac{\sigma_{x_5 y_2}}{\sigma_{x_5 y_1}}. \]

In terms of moments of the \( v \)'s, \( \frac{\sigma_{22}}{\sigma_{11}} = \frac{\sigma_{23}}{\sigma_{13}} = \frac{\sigma_{24}}{\sigma_{14}} = \frac{\sigma_{25}}{\sigma_{15}}. \) It is further assumed that \( \sigma_{23} = 0, \)

i.e. the disturbance in the crop production equation is uncorrelated with the disturbance in the profit maximization equation. Since the latter relates only to livestock output and to inputs used only in livestock production, the author considers this a reasonable assumption.

Estimates and Comparisons

Maximum likelihood estimates could be obtained for the parameters of this system, estimation equations being derived by the procedure illustrated in the Appendix. Solution of the estimation equations would, however, be a long and complex process as the equations in the unknown parameters would be non-linear.

The author has chosen instead to use less efficient but computationally simpler estimation equations derived by moment methods. These estimation equations are:

(i) \( \hat{\sigma}_{x_1 x_3} + \hat{\gamma}_{11} \hat{\sigma}_{x_2 x_3} = \hat{\sigma}_{x_3 y_1} \)

(ii) \( \hat{\sigma}_{x_1 x_4} + \hat{\gamma}_{11} \hat{\sigma}_{x_2 x_4} = \hat{\sigma}_{x_4 y_1} \)

(iii) \( \hat{\sigma}_{x_1 x_5} + \hat{\gamma}_{11} \hat{\sigma}_{x_2 x_5} = \hat{\sigma}_{x_5 y_1} \)
(iv) \( \hat{\Sigma}_{1}^{X} + \hat{\gamma}_{22} \hat{\Sigma}_{2}^{X} + \hat{\gamma}_{25} \hat{\Sigma}_{3}^{X} = \hat{K} \hat{X}_{1} \)

(v) \( \hat{\Sigma}_{2}^{X} + \hat{\gamma}_{22} \hat{\Sigma}_{4}^{X} + \hat{\gamma}_{25} \hat{\Sigma}_{5}^{X} = \hat{X}_{4} \)

(vi) \( \hat{\Sigma}_{2}^{X} + \hat{\gamma}_{22} \hat{\Sigma}_{4}^{X} + \hat{\gamma}_{25} \hat{\Sigma}_{5}^{X} = \hat{X}_{4} \)

(vii) \( (\hat{\Sigma}_{1}^{X} - (\hat{\Sigma}_{2}^{X})) + 22 (\hat{\Sigma}_{1}^{X} - (\hat{\Sigma}_{3}^{X}))) + 25 (\hat{\Sigma}_{1}^{X} - (\hat{\Sigma}_{5}^{X})) = 0 \)

The quantities under the symbol \( \hat{\ldots} \) are estimated directly from the data. These include \( \hat{\Sigma}_{1}^{X} \) in addition to the moments used in the last chapter. For crop farms

\( \hat{\Sigma}_{1}^{X} = .0562 \)

and the estimates obtained from the system are:

\( \hat{\gamma}_{11} = -.77 \)

\( \hat{\gamma}_{14} = \frac{25}{k} = -2.11 \)

\( \hat{\gamma}_{22} = -.42 \)

\( \hat{\gamma}_{25} = -.71 \)

\( \hat{\Sigma}_{3}^{X} = .0081 \)

\( \hat{\Sigma}_{4}^{X} = .0111 \)

\( \hat{\Sigma}_{5}^{X} = .0132 \)

Again, we have for comparison strong presumptions from general knowledge that
\[ -1 < \gamma_{11} < 0 \]
\[ -1 < \gamma_{14} < 0 \]
\[ -1 < \gamma_{22} < 0 \]
\[ -1 < \gamma_{25} < 0 \]

\[ 0 < \hat{\sigma}_{3} \hat{V}_{1} < \hat{\sigma}_{3} \hat{X}_{5} \quad \hat{\sigma}_{3} \hat{X}_{5} = .0253 \]
\[ 0 < \hat{\sigma}_{4} \hat{V}_{1} < \hat{\sigma}_{4} \hat{X}_{5} \quad \hat{\sigma}_{4} \hat{X}_{5} = .0237 \]
\[ 0 < \hat{\sigma}_{5} \hat{V}_{1} < \hat{\sigma}_{5} \hat{X}_{5} \quad \hat{\sigma}_{5} \hat{X}_{5} = .0233 \]

The only estimate that falls outside this range is \( \hat{\gamma}_{14} = -2.11 \).

For hog farms

\[ \hat{\sigma}_{1} \hat{X}_{2} = .0513 \]

and the estimates derived from equations (i) to (vii) are

\[ \hat{\gamma}_{11} = -.870 \]
\[ \hat{\gamma}_{14} = -.108 \]
\[ \hat{\gamma}_{22} = -.065 \]
\[ \hat{\gamma}_{25} = -2.052 \]
\[ \hat{\sigma}_{3} \hat{V}_{1} = .000874 \]
\[ \hat{\sigma}_{4} \hat{V}_{1} = .000862 \]
\[ \hat{\sigma}_{5} \hat{V}_{1} = .001320 \]

The presumptions are almost the same in this case, the estimates of upper limits of \( \hat{\sigma}_{3} \hat{V}_{1} \), \( \hat{\sigma}_{4} \hat{V}_{1} \), and \( \hat{\sigma}_{5} \hat{V}_{1} \) being \( \hat{\sigma}_{3} \hat{X}_{5} = .0251 \), \( \hat{\sigma}_{4} \hat{X}_{5} = .0227 \), and \( \hat{\sigma}_{5} \hat{X}_{5} = .0231 \) respectively; and the \( \gamma \)'s are again presumed to lie between -1 and 0. The estimate, \( \hat{\gamma}_{25} \), falls outside the presumed range in this case and the estimates, \( \hat{\gamma}_{22} \), \( \hat{\sigma}_{3} \hat{V}_{1} \), \( \hat{\sigma}_{4} \hat{V}_{1} \), \( \hat{\sigma}_{5} \hat{V}_{1} \) are suffi-
ciently close to the lower limit, zero, that a somewhat more narrow set of presumptions might easily have excluded them. Thus, the results of computations with this model are similar to those obtained with the model of Chapter II. The sets of estimates obtained using crop farm data and hog farm data both contain at least one estimate inconsistent with a set of fairly conservative presumptions about the true values of the quantities being estimated. Again, sampling peculiarities and errors of observations must be admitted as possible explanations though the author does not consider it likely that they are the sole explanations.
CHAPTER IV.

FURTHER CONSIDERATION OF THE ESTIMATES

Alternative Explanations

In general, it seems to the author that unreasonable looking estimates of the sort derived in Chapters II and III might be explained on one or several of the following bases:

a. The grounds on which the estimates are described as unreasonable may be inconclusive.

b. The unreasonableness may be attributed to sampling error.

c. There may be biases in the observation of the data.

d. Biased estimates may be caused by unreal assumptions in the specification of the population.

The last-mentioned possibility is the subject of most of the chapter because it is felt that (a) is not a likely explanation in this case and that (b) and (c) can be adequately considered only after the investigator is reasonably sure of the specification employed. However, brief comments on the first three possibilities might be in order before (d) is discussed.
Examination of the Presumptions

The two a priori presumptions against which some of the computed estimates have been checked are:

(1) The inputs are positively correlated in the population.

(2) The elasticity of output with respect to a single input is greater than zero and less than one.

(1) rests on the general presumption of complementarity among inputs, and the fact that inputs have been treated in large aggregates in this study. Exceptions to this general presumption of complementarity have been noted by Hicks\(^1\) to be largely confined to inputs having the same "generic name". Thus No. 2 corn is quite an effective substitute for No. 3 as a feed and perhaps oats and barley may also be in this category. But when the classification of inputs is so broad that all livestock feeds and several additional items are treated as a single input, we would expect this input to be the technical complement of other inputs.

The other presumption, that elasticity of output with respect to one of the inputs lies between zero and unity, implies two claims. To claim that elasticity is positive is equivalent to saying that a small increase in the application of the input in question will increase rather than decrease output. Theoretical discussions commonly presuppose that output increases monotonically as a particular input is increased. There is no guarantee that this is always true. However,

\(^1\) Op. cit., Chapter VII.
even if we assume that output is a decreasing function of the input
in question over some intervals of the input, for the entrepreneur to
operate within one of these intervals would represent a serious in-
efficiency. While this might be true for an occasional entrepreneur,
we should hardly expect it to be typical of a large group. If, in
addition, we consider the general pressure of productive activities
against resources available on farms, the almost universal situation is
that a farmer could usefully employ more of the services of any of his
productive resources and for a small increment of some of these services
to affect product adversely would be quite surprising.

The contention that elasticity of output with respect to a single
input is less than one also seems to the author to depend for a large
part of its force on our general knowledge of how farms operate. While
all theoretical discussions of which the author is aware are consistent
with this presumption, hypothetical cases to the contrary can rather
easily be imagined. Those that the author has been able to construct
or has heard of involve either extreme misuse of resources or an impor-
tant change in technology in the process of increasing an input and seem
quite remote from conditions we would expect to find on the farms being
studied. If all inputs were homogeneous and freely divisible, we would
expect an increase of, say, 1 per cent in the application of all inputs
to result in a 1 per cent increase in product.\footnote{This situation is described as constant returns to scale and was assumed by Professor Douglas and his associates in their earlier production function studies. In later studies, this assumption was dropped but the results indicated approximately constant returns to scale. In a study by Tintner and one by Tintner and Brownlee results showing slightly decreasing returns to scale were obtained. References to these studies are included in Footnote 2 on page 12.} If some important inputs were held constant and others were increased 1 per cent, we would expect product to increase less than 1 per cent which would mean an elasticity of less than one with respect to the collection of inputs increased. If the collection of inputs held constant is not completely divisible, then there may be a range of small applications of variable inputs over which elasticity is greater than one. According to the traditional principle of diminishing returns, elasticity would be less than one for larger applications. Under perfect competition, average variable costs are decreasing over the range for which elasticity is greater than one and increasing over the range for which elasticity is less than one. Production in the range of decreasing average variable costs involves an extreme misuse of resources and, while isolated examples might be found, it would hardly be the typical case.

**Sampling Error**

With regard to the possibility of unreasonable results being due to sampling error, this possibility is nearly always present in statistical investigations. Standard statistical procedures include tests of significance which give the investigator an idea of the chance that this is the true explanation. The author has not worked out tests of significance for
the procedures used in this study. To do this would go beyond the author's present knowledge of mathematical statistics. The author is also inclined to believe that, at the present stage of inquiry, additional research effort might better be directed towards developing methods from somewhat broader theoretical models than those employed here rather than developing tests of significance for the particular methods employed here.

In the absence of tests of significance, several relevant features of the estimates might be casually noted. While the sample size is fairly large, inefficient methods of estimation were used. The author's guess would be that sampling fluctuations might be fairly large, but that it would be unwise to attribute the whole difficulty to sampling fluctuations. The fact that unreasonable looking estimates were obtained in four separate trials using two different theoretical models suggests that one might well examine some of the characteristics common to the two models. This will be done after a few observations on the data used are noted.

Errors of Observation

The data for the study were obtained from rather carefully kept and handled records and such errors as may have entered could hardly be assigned primary responsibility for some of the unlikely looking estimates obtained. There are, however, certain difficulties in estimating inputs and outputs from these data and it may be worth while to briefly mention them. While they were not considered to be of crucial importance in the present study, they can be expected to be of more significance as more powerful estimation procedures are developed and other errors of estimation
are decreased.

In determining inputs and outputs for a specific interval goods in process at the beginning and end of the interval are always troublesome. Besides being difficult to evaluate, they cause certain conceptual difficulties. They are really neither outputs nor inputs but a sort of hybrid. There are several ways of treating them which would seem to satisfy accounting requirements, but any of these treatments could result in a distortion of the true relations among inputs and outputs which the production function is intended to represent. Three such alternatives are (a) to treat opening inventory of goods in process as an addition to input and closing inventory as an addition to output, (b) to treat opening inventory as an addition to input and closing inventory as a subtraction from input, (c) to treat opening inventory as a reduction in output and closing inventory as an addition to output. In the present study were handled according to (c). Either (b) or (c) would work quite well if the inventories were of about the same size and composition, but might cause some distortion if the size or composition changed greatly.

One way to minimize this difficulty would be to choose the interval of observation in such a way that goods in process inventories are at a

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1The desire for formal completeness might suggest the additional possibility of treating opening inventory as a reduction in output and closing inventory as a reduction in input. However, if the two inventories are at all similar in composition, it would seem unreasonable to associate the one which enters into production processes of the period with output and the one which emerges from these processes with input.
minimum at the time chosen for the boundary between intervals. In the case of Iowa farm records this might mean defining a fiscal year beginning and ending on March 1 or March 15 and using this as the record-keeping interval rather than the calendar year.

Another difficulty occurs in estimating labor input from the records. Operator and family labor are listed and evaluated according to conventions which may obscure farm to farm differences in this part of the labor input. If farm records are to serve very extensively as data for this sort of study, it may be worth while to make a further effort to detect and measure these differences in operator and family labor somewhat better.

Problems of Specification

Some of the shortcomings of the estimation procedures and the underlying theory from which they were developed have already been discussed and perhaps require only brief mention here. Removal of difficulties such as the use of heterogeneous input classes and inefficient estimation equations is largely a matter of employing more time and more computational resources. The desirability of constructing a broader theoretical model explaining more completely the production determination process and including more general decision relations and perhaps expectations and/or expectation relations was discussed in Chapter I.

There remain two problems generally encountered in formulating theoretical models which the author feels might be profitably discussed.
The author is willing to argue that these assumptions about the

\[ \frac{\partial^2 \phi}{\partial z^2} + \lambda (1) \]

independence testable and observed notation we would have

production function is linear in the log and that the inputs are not

subsequence that is independent of the input, consider that the time-

case in which there are in principle a single output and a random da-

assumptions from a theoretical standpoint. Consider a

In the absence of such a test, the author has re-examined the set of

of the assumptions and then to test the assumptions from the data.

If it would have been possible to make assumptions independent

had not been essential to the assumptions presented in Chapter II

assumed to be independent of the disturbances. If these assumptions

In the two production functions of this study the inputs were

*apparent

the problems posed can readily be seen to have a general

apply these are discussed in the present study.

of the algebraic equations used to represent production functions

between purely of variables and the other is the problem of the form

in this study. One is the problem of the assumption of independence
not regulate these assumptions of independence.

There would still be a premium on the development of methods with which

there is no guarantee that these are not other underlying influences.

Would make independent assumption somewhat more plausible though.

Introduction to the left hand side of the production equation.

This in some cases to the extent that possible to estimate the management input and

assumptions of independent between observed input and the disturbances.

Arbitrary assumptions might not be necessary in such a circumstance. It would be necessary to

In a number of empirical studies management have been recognized as

* sort 0 would be justified.

If were such an input, we would have $\Pi$ and assumptions of the

* form, we would be justified in making assumptions in the production equation because

are not justified. One set of exceptions to conditions (2) would

If conditions (2) hold, assumptions of the sort

$0 = \Delta \Pi$ could be written without apparent that

$\Delta = \frac{\Pi_2 - \Pi_1}{t_2 - t_1}$

where the new disturbances

$\Delta = \frac{\Pi_2}{t_2 - t_1} + \lambda$ (11)

would be worked out with the disturbance

the production equation with which the disturbance to

side of equation one, but has become one of the factors contributing to

this is unobservable and not been taken into account on the left.

Moreover consider now that in an actual investigation, one implicates...
The fact that it is usually necessary to use fairly simple algebraic expressions to represent the relationships involved in an empirical study was mentioned in Chapter I. It would be surprising if the relationships exactly coincided with the equations used to represent them in very many cases. It is usually hoped that the equations will approximate the true relationships in the range covered by the observation and the range to be used in applying the results of the investigation. Frequently the latter is entirely or almost entirely included in the former. When this is not true, serious problems of extrapolation arise.

The chances of a good approximation are enhanced if the form of the equation permits it to correspond to a priori knowledge of the relationship. The form of the production functions used in this study is linear in the logarithms and is an example of the Cobb-Douglas type. This form has been used in production function studies because of two generally recognized advantages; it is relatively simple to handle, it has a small number of parameters and can be stated in a linear form, and it allows for the possibility of diminishing returns. However, there is no guarantee that it actually represents the true production relationship any better than other functions that might be suggested.

The author considered that it would be useful to consider alternatives, though computational considerations limited the choice to simple alternatives. Accordingly, the procedures of Chapter II were applied to the crop farm data using equations linear in the actual values of inputs and outputs and to equations which were linear in the actual values of outputs and the logs of inputs. The results using these forms of the equations were similar to the results using the Cobb-Douglas form in that the estimation equations derived could be shown to be inconsistent with the set of presumptions applied in Chapter II. For reasons that will be explained below, no attempt was made to apply the procedures of Chapter III using these other forms for the equations.

Using notation similar to that of Chapter II, we let $y$ represent output and $z_1$ and $z_2$ represent inputs. Capital letters represent logarithms of the corresponding quantities. The three forms of equations suggested can then be written:

(i) $y = y_0 + y_1z_1 + y_2z_2$, logarithms or Cobb-Douglas form

(ii) $y = y_0 + y_1z_1 + y_2z_2$, linear form

(iii) $y = y_0 + y_1z_1 + y_2z_2$, mixed form

The following characteristics of these equations are readily noted. (i) implies constant elasticities; (ii) implies constant marginal productivities; (i) and (iii) allow for the possibility of diminishing returns, (ii) does not. On the basis of just these observations there would seem to be a slight reason for preferring (i) or (iii) to (ii), though it is not at all impossible that the tendency towards diminishing
returns might be very slight in the ranges of the variables in which the investigator is interested and (ii) might then represent the true relationship in this range better than (i) or (iii).

If we assume pure competition and consider that profits are approximately maximized with respect to one input, say $z_1$, then additional considerations appear. If we consider that $z_1$ represents variable factors and $z_2$ represents fixed factors, the above assumption might be reasonable. Profit maximization for equation (i) would then require that $z_1 = \frac{\gamma_1 \cdot y \cdot (\text{price of } y)}{(\text{price of } z_1)}$. To simplify later discussion, let us consider that all prices are unity or that all quantities are measured in dollars' worth as has been done in some of the earlier discussion. Then we could write $z_1 = \gamma_1' y = (\gamma_1' y_0 z_2) \left(\frac{1}{1 \text{ unit}}\right)$ for the profit maximization condition using equation (i). For (ii) we would have, $z_1 \rightarrow \infty$ for $\gamma_1 > 1$ and $z_1 = 0$ for $\gamma_1 < 1$; and for (iii) the condition would be $z_1 = \gamma_1'$. Of these, the first is the only one that seems to make much sense. The condition that $z_1$ be either 0 or infinity seems completely unacceptable and the condition for (iii), that $z_1$ equal a constant also seems unreal. Our general notions of how inputs interact would require that the optimal application of $z_1$ should depend upon the amount of $z_2$ available. The general presumption of complementarity among inputs

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1 The only difference this makes in succeeding equations is that certain price ratios that would be clumsy to write do not appear.
would lead us to expect that the optimal amount of \( z_1 \) should increase as \( z_2 \) increases and this is the case using equation (i). Because the profit maximization conditions derived from (ii) and (iii) looked unreal, they were omitted from consideration in framing the production and profit equations model of the previous chapter.

While equations of the Cobb-Douglas type thus conform more closely to economic principles than the other forms of equal simplicity examined above, a new difficulty arises if the investigator wishes to consider more than one output. Consider an equation of the form,

\[
(iv) \quad y_1 + \beta y_2 = \gamma_0 + \gamma_1 z_1 + \gamma_2 z_2
\]

This equation could be written

\[
y_1 y_2^\beta = \gamma_0 z_1 z_2 \quad \text{and for given values of} \quad z_1 \text{ and } z_2 \text{ we would have } y_1 y_2^\beta = K. \quad \text{If } \beta \text{ were negative, it would be possible to increase } y_1 \text{ and } y_2 \text{ simultaneously without altering inputs. Hence we would expect } \beta \text{ to be positive making the curve showing the relationship between } y_1 \text{ and } y_2 \text{ concave from above and asymptotic to the axes. Thus an infinite quantity of either output could be obtained by reducing the other output to zero. Furthermore, if we consider alternative combinations of output that can be produced with given inputs, the usual first order condition of maximum profits, that the marginal rate of transformation between } y_1 \text{ and } y_2 \text{ should equal the ratio of their prices, would actually give a minimum in this case. These consider-

\[1\] The two production equations of Chapter III could be reduced to this form with one more input appearing on the right hand side. Thus the characteristics noted for (iv) also apply to this pair.
would seem to be worth a rather large effort.

However, it is likely that the successful development of such functions

- to see them as a basis for establishing a number of criteria to estimate the results of production function studies

- difficulties to be overcome before we will be justifiably in having

The author is inclined to the belief that there are a number of

- parameter estimation

- forms for the functional relationship or to develop methods of non-

outputs are recognized, it is likely that the production functions in which two

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It is desired to show that the maximum likelihood method does not yield a solution for the system of equations:

\[ x_1 + \gamma_{11} x_3 = v_1 \]
\[ x_2 + \gamma_{22} x_4 + \gamma_{25} x_5 = v_2 \]
\[ x_3 = v_3 \]
\[ x_4 = v_4 \]
\[ x_5 = v_5 \]

when the only additional information consists of these restrictions on the covariances of the \( v \)'s:

\[ \frac{\sigma_{23}}{\sigma_{13}} = \frac{\sigma_{24}}{\sigma_{14}} = \frac{\sigma_{25}}{\sigma_{15}} = k \]

where \( k \) is arbitrarily defined to be equal to the constant ratio. These restrictions make it possible to write the determinant of the covariance matrix of the system in the following form:

\[
\begin{vmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\
\sigma_{12} & \sigma_{22} & k & \sigma_{13} & \sigma_{14} \\
\sigma_{13} & k & \sigma_{33} & \sigma_{14} & \sigma_{15} \\
\sigma_{14} & k & \sigma_{14} & \sigma_{44} & \sigma_{45} \\
\sigma_{15} & k & \sigma_{15} & \sigma_{45} & \sigma_{55}
\end{vmatrix}
\]

It is useful to state the restrictions in terms of equivalent restrictions on the elements of the inverse of the covariance matrix of the \( v \)'s. Consider the quantities
\[
\sigma^{-13} = \frac{\text{Cofactor } \sigma_{13}}{\sigma_{13}} \text{ in } \sigma_{13} = (k \sigma_{12} - \sigma_{22})
\]

\[
\sigma^{-23} = - (k \sigma_{11} - \sigma_{12}) \begin{vmatrix} \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \frac{13}{13} & \frac{14}{14} & \frac{15}{15} \end{vmatrix}
\]

\[
\sigma^{-14} = - (k \sigma_{12} - \sigma_{22}) \begin{vmatrix} \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \frac{13}{15} & \frac{14}{15} & \frac{15}{15} \end{vmatrix}
\]

\[
\sigma^{-24} = - (k \sigma_{11} - \sigma_{12}) \begin{vmatrix} \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \frac{15}{15} & \frac{14}{15} & \frac{15}{15} \end{vmatrix}
\]

\[
\sigma^{-15} = (k \sigma_{12} - \sigma_{22}) \begin{vmatrix} \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \frac{15}{15} & \frac{14}{15} & \frac{15}{15} \end{vmatrix}
\]

\[
\sigma^{-25} = - (k \sigma_{11} - \sigma_{12}) \begin{vmatrix} \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \frac{15}{15} & \frac{14}{15} & \frac{15}{15} \end{vmatrix}
\]
It is evident that

$$
\frac{\sigma_{23}}{\sigma_{15}} = \frac{\sigma_{24}}{\sigma_{14}} = \frac{\sigma_{25}}{\sigma_{15}} = \frac{(k \sigma_{11} - \sigma_{12})}{(k \sigma_{12} - \sigma_{22})} = K
$$

where $K$ is arbitrarily defined equal to the constant ratio. These restrictions can be stated

1. $\mathcal{I}_1^1 = \mathcal{I}_1^{13} \sigma_{24} \sigma_{14} \sigma_{23} = 0$
2. $\mathcal{I}_2^1 = \mathcal{I}_2^{14} \sigma_{25} \sigma_{15} \sigma_{24} = 0$

The likelihood function for this system is

$$
\mathcal{L} = \left( \frac{\lambda}{2 \pi} \right)^{n/2} \exp \left( -\frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} x_{ij}^2 \right)
$$

where $\sigma_{ij}$ is the determinant of the inverse of the covariance matrix. $F$ is defined as the log of the likelihood function plus $\mathcal{I}_1$ and $\mathcal{I}_2$, each with a Lagrange multiplier.

$$
F = \frac{5n}{2} \log 2\pi + \frac{n}{2} \log \sigma + \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} x_{ij}^2 + \sum_{i=1}^{n} \lambda_i \mathcal{I}_i
$$

If estimates of the three $\mathcal{I}$'s and the variances and covariances of the $x$'s can be obtained, the estimates which maximize $F$ will also maximize $\mathcal{L}$ subject to restrictions (1) and (2). Estimation equations are obtained by differentiating $F$ with respect to each of the unknown quantities and equating the derivatives to zero. The Jacobian of transformation equals unity in this case and can be ignored.
(3) \( \frac{dF}{d\sigma_{ij}} = n\sigma_{ij} - \sum_{i,j} v_{ij} + \lambda_1 \frac{dH_1}{d\sigma_{ij}} + \lambda_2 \frac{dH_2}{d\sigma_{ij}} \)

for \( i, j = 1, 2, 3, 4, 5; \ i \neq j \)

\[ \sum_{i,j} v_{ij} \] is written for \( \sum_{i,j} v_{ij} \). If \( i = j \) we may write

(4) \( \frac{dF}{d\sigma_{ii}} = \frac{n}{2} \sigma_{ii} - \frac{1}{2} \sum_{i,j} v_{ij}^2 \) \( i = 1, 2, 3, 4, 5 \)

Differentiation with respect to the \( \gamma^2 \)'s yields

(5) \( \frac{dF}{d\gamma_{11}} = \sigma_{11} ^{12} \sum_{v_{1v_3}} + \sigma_{12} ^{12} \sum_{v_{2v_3}} + \sigma_{13} ^{12} \sum_{v_{3v_3}} + \sigma_{14} ^{12} \sum_{v_{4v_3}} + \sigma_{15} ^{12} \sum_{v_{5v_3}} \)

(6) \( \frac{dF}{d\gamma_{22}} = \sigma_{12} ^{12} \sum_{v_{1v_4}} + \sigma_{22} ^{12} \sum_{v_{2v_4}} + \sigma_{23} ^{12} \sum_{v_{3v_4}} + \sigma_{24} ^{12} \sum_{v_{4v_4}} + \sigma_{25} ^{12} \sum_{v_{5v_4}} \)

(7) \( \frac{dF}{d\gamma_{25}} = \sigma_{12} ^{12} \sum_{v_{1v_5}} + \sigma_{22} ^{12} \sum_{v_{2v_5}} + \sigma_{23} ^{12} \sum_{v_{3v_5}} + \sigma_{24} ^{12} \sum_{v_{4v_5}} + \sigma_{25} ^{12} \sum_{v_{5v_5}} \)

Setting (3) equal to zero and dividing by \( n \) yields estimation equations for the ten covariances.

(3') \( \sigma_{ij} \sigma_{ij} = \frac{\sum_{v_{ij}}}{n} + \lambda_1 \frac{dH_1}{n} \sigma_{ij} + \lambda_2 \frac{dH_2}{n} \sigma_{ij} = 0 \) \( i, j = 1, 2, 3, 4, 5; \ i \neq j \)

Similarly, dividing (4) by \( \frac{n}{2} \) and equating to zero yields estimation equations for the five variances.

(4') \( \sigma_{ii} ^2 \sigma_{ii} = \frac{\sum_{v_{ii}}}{n} = 0 \) \( i = 1, 2, 3, 4, 5 \)

Dividing (5) by \( n \) and equating to zero yields

(5') \( \sigma_{11} ^{12} \sum_{v_{1v_3}} \sigma_{12} ^{12} \sum_{v_{2v_3}} + \sigma_{13} ^{12} \sum_{v_{3v_3}} + \sigma_{14} ^{12} \sum_{v_{4v_3}} + \sigma_{15} ^{12} \sum_{v_{5v_3}} = 0 \)
Substituting from (3') and (4') for the included expressions

\[ \sum_{i,j} v_i v_j \]  

gives

\[ (5^\prime) \quad \hat{\epsilon}_{11} (\hat{\gamma}_{13} + \lambda_1 \hat{\gamma}_{14}^2) + \hat{\epsilon}_{12} (\hat{\gamma}_{23} - \lambda_1 \hat{\gamma}_{15}^2) + \hat{\epsilon}_{13} \hat{\gamma}_{33} + \hat{\epsilon}_{14} \hat{\gamma}_{34} + \hat{\epsilon}_{15} \hat{\gamma}_{35} = 0 \]

(6) and (7) may be handled similarly.

\[ (6^\prime) \quad \hat{\epsilon}_{12} (\hat{\gamma}_{14} - \lambda_2 \hat{\gamma}_{23} + \lambda_1 \hat{\gamma}_{25}^2) + \hat{\epsilon}_{22} (\hat{\gamma}_{24} + \lambda_1 \hat{\gamma}_{13} - \lambda_2 \hat{\gamma}_{15}^2) + \hat{\epsilon}_{23} \hat{\gamma}_{34} + \hat{\epsilon}_{24} \hat{\gamma}_{44} + \hat{\epsilon}_{25} \hat{\gamma}_{45} = 0 \]

\[ (7^\prime) \quad \hat{\epsilon}_{12} (\hat{\gamma}_{15} - \lambda_2 \hat{\gamma}_{14}^2) + \hat{\epsilon}_{22} (\hat{\gamma}_{15} + \lambda_1 \hat{\gamma}_{13}^2) + \hat{\epsilon}_{23} \hat{\gamma}_{35} + \hat{\epsilon}_{24} \hat{\gamma}_{45} + \hat{\epsilon}_{25} \hat{\gamma}_{55} = 0 \]

The ten equations (3') and the five equations (4'), together with (1), (2), (5'), (6'), and (7') are twenty equations in twenty unknowns.

As indicated in Chapter II, previous work using moment methods leads to the belief that the system is not identified. If this is true, it should be possible to show that one of the twenty equations is redundant. In (5'), the terms \( \hat{\epsilon}_{11} \hat{\gamma}_{13} + \hat{\epsilon}_{12} \hat{\gamma}_{23} + \hat{\epsilon}_{13} \hat{\gamma}_{33} + \hat{\epsilon}_{14} \hat{\gamma}_{34} + \hat{\epsilon}_{15} \hat{\gamma}_{35} \) constitute an estimate of a quantity known to be zero and hence can be eliminated leaving

\[ (5''\prime) \quad \lambda_1 \hat{\epsilon}_{11} \hat{\gamma}_{14}^2 - \lambda_1 \hat{\epsilon}_{12} \hat{\gamma}_{14}^2 = \lambda_1 \hat{\epsilon}_{14} (k \hat{\gamma}_{11} - \hat{\gamma}_{12}) = 0 \]

Expressions of this sort can also be eliminated from (6') and (7'), leaving

\[ (6''\prime) \quad - \lambda_1 \hat{\epsilon}_{13} (k \hat{\gamma}_{12} - \hat{\gamma}_{22}) + \lambda_2 \hat{\epsilon}_{15} (k \hat{\gamma}_{12} - \hat{\gamma}_{22}) = 0 \]

\[ (7''\prime) \quad - \lambda_2 \hat{\epsilon}_{14} (k \hat{\gamma}_{12} - \hat{\gamma}_{22}) = 0 \]
In general, neither $\lambda_2$ nor $\hat{J}_{14}$ is zero and $(7''')$ requires that 

$$(K\hat{J}_{12} = \hat{J}_{22}) = 0,$$

making $(6'''')$ redundant and the system indeterminate. In the trivial case in which $\hat{J}_{14} = 0$, neither $(5'''')$ nor $(7'''')$ adds any information and the system is still not identified.