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Capillary fringe and water flow in soil

Dale Swartzendruber
Iowa State College

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CAPILLARY FRINGE AND WATER FLOW IN SOIL

by

Dale Swartzendruber

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Soil Physics

Approved:

Signature was redacted for privacy.

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1954
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INTRODUCTION

Movement of water in soil saturated under positive water pressure is fairly well understood and can be well described in its fundamental nature by Darcy's law. While the pressure-saturated case (meaning that the pressure in the soil water is greater than atmospheric) undoubtedly is of considerable importance in soil, it is really just a beginning. Agriculturally it is required that soils shall be drained to some point where adequate aeration is possible. This will not occur in a pressure-saturated soil.

Consider the removal of ponded water by some drainage system, such as open ditches or tile drains. As the water seeps downward through the soil, the problem is one of pressure-saturated flow as long as free water remains on the surface of the region being considered. However, as the free water surface subsides beneath the soil surface, a whole new series of considerations is introduced.

First, from the physical standpoint, the moisture content of the upper layers is reduced as the larger pores yield their water. As the free water surface (or water table) recedes, progressively smaller pores are drained. As the pores lose water, air is drawn into the soil. In addition, the further removal of water from the unsaturated region follows a pattern different from the one which existed formerly, since drainage can only proceed through pore channels which contain water. This region of soil above the water table may be termed the capillary region since surface tension forces are of considerable consequence.
Secondly, the mathematical treatment of the more complex physical picture also becomes more involved. Equations and formulas become more complicated and restricted in application. In fact, the total complexity of the mathematical and physical processes has proved so far, in the solution of the problem, to be an insurmountable barrier as far as the general case of unsaturated moisture conditions is concerned.

It should be mentioned that at this complex stage practical interest in the problem is greater than ever. Excess water has been withdrawn, and in its place is found the air which is so necessary for plant growth. The formerly flooded region, other essential factors permitting, has become a better medium for plants of agronomic value.

The purpose of this study was to investigate the contribution of the capillary region above the water table to the movement of soil water. In the past, solutions to ground water flow problems, where a subsoil water table exists, have in many cases been based on the assumption that water movement in the capillary region above the water table is negligible compared to the flow beneath. This assumption is hardly reasonable since it is known by experiment that many soils are essentially saturated for a finite distance above the water table; hence, the hydraulic conductivity in this near-saturated capillary region should be practically the same as in the pressure-saturated region below the water table. Thus, for the same geometry of flow the contribution of the saturated capillary layer—the so-called capillary fringe—should be of the same order of magnitude as that of the zone beneath.
The specific objectives as originally envisioned were three, namely (1) to ascertain how the capillary fringe has been handled in the past by the few investigators who have attempted to take it into account, (2) to study the effect of the fringe for several simple flow cases and hence to demonstrate the order of magnitude of its contribution, and (3) to present, if possible, a general method for taking the fringe into account in flow problems other than the ones studied specifically. As the problem developed it was attacked both theoretically and experimentally.

In addition, a more or less visionary objective was kept in mind. Science seeks to solve complex phenomena in terms of the less complex. Since pressure-saturated flow is less complex than the unsaturated case, it seems correct to attack the latter from the vantage point of the former. This approach would seem even more plausible if a region of transition could be found between the two extremes. Now the capillary fringe is a transitional region. From the standpoint of moisture saturation it resembles the pressure-saturated region below it. By virtue of its position above the water table its moisture is under tension; thus, on this score the capillary fringe resembles the tension-unsaturated condition of the even higher regions of soil. The term tension-unsaturated means that the soil moisture is under tension, and that the soil is unsaturated. Hence, one might expect the flow phenomena in the capillary fringe to be intermediate between the pressure-saturated region below and the tension-unsaturated region above. So, an elucidation of the principles governing flow in the capillary fringe conceivably might lead to a better understanding of flow phenomena in the unsaturated region above it.
REVIEW OF LITERATURE

Literature relating to the capillary fringe may be found in abstracting journals or indexes under the general heading of "capillarity in soils". However, the amount of published material on this latter subject is voluminous, and the present literature investigation does not purport to be exhaustive. Rather, an attempt has been made to gather together pertinent references which deal more specifically with the problems concerning the capillary fringe itself.

The Capillary Fringe

Definition and specification

According to Tolman (41)*, the capillary fringe is "the zone immediately above the water table in which water is held above the water table by capillarity." This definition is not restricted to a saturated zone and hence differs from the implied definition given previously in the Introduction of this thesis.

Information on the definition of the fringe seems meager, but Luthin and Miller (26) imply that the one given by Tolman is the one generally accepted by research workers, especially in dry regions where the region above the water table is wetted from below. They attribute to Childs (5, p. 318) the restriction of the term so as to refer only to the saturated region.

*Figures in parentheses refer to Literature Cited.
However, it appears that the distinction, at least in principle, was recognized earlier by Walter (49). In a proposed classification of subsoil waters he suggested that three zones be recognized, (a) the zone of aeration, (b) the zone of complete capillary saturation, and (c) the zone of positive hydrostatic pressure. It is easily seen that (b) corresponds to Childs' capillary fringe, and that (a) and (c) are respectively equivalent to the tension-unsaturated and pressure-saturated regions described in the Introduction.

Walter also suggested names for the lines (or surfaces) of demarcation between the three zones. For the boundary between zones (a) and (b) he proposed "saturation line", and for the zero-pressure line (actually atmospheric-pressure line) between zones (b) and (c) he suggested "phreatic line". It should be pointed out that the phreatic line (or surface) is equivalent to the water table as defined earlier in the Introduction.

It seems to the present author that Walter's suggestions are good, particularly with regard to his saturation line as designating the upper boundary of the saturated capillary region. The existence of such a line has not been emphasized in the past, but it is an important concept, particularly in drainage problems where the level of moisture must be reduced below saturation to provide proper environment for the plant root. In a soil which has the capacity to develop a rather thick capillary layer, it might be relatively simple to reduce the height of the water table, but perhaps more difficult to lower the line of saturation. In a coarse-textured soil the saturation line might essentially correspond to the phreatic line, but in fine textured soils (which usually are more in need
of drainage) the two lines might be relatively widely separated. Hence, in the remainder of this dissertation the capillary fringe will designate the saturated capillary region, while phreatic line and saturation line will be used as Walter has suggested.

If a sample of soil is saturated with water under pressure and the pressure then released until there is actually a pressure deficiency (tension), the moisture status of the soil will depend in some manner upon the tension applied. The particular pattern of dependence will depend upon the pore-size distribution, but by applying enough tension it is always possible to reach some point (as A in Figure 1) beyond which the moisture content noticeably decreases with the applied tension.

The determination of such curves as in Figure 1 has been of common occurrence in soils work; hence Childs (3) suggested that they be called soil moisture characteristic curves, with abbreviation to soil moisture characteristic. Common usage has abbreviated the term still further to simply moisture characteristic, which will be used here in subsequent discussion.

The moisture characteristic is useful in specifying the capillary fringe. By the very nature of things the tension of the soil moisture increases with increasing height above the phreatic surface, since the tension, in length of water column, is simply the height above the phreatic surface. Hence, a plot of soil moisture versus height above the phreatic surface is simply the moisture characteristic.

The height of the capillary fringe, following Childs (5), is specified by the point at which the moisture characteristic breaks noticeably from a previous course at or near saturation. Referring to the
drying curve of Figure 1, the height of capillary fringe for this material, which is sand, would be a height of water column equal to a pressure of $10 \, (T/r)$ units, where $T$ is the surface tension and $r$ is the radius of curvature of the air-water interface. A more convenient procedure would be the plotting of moisture values against tension expressed in length of water column so that the height of the fringe could be read directly from the abscissa. The curves in Figure 1 were taken from Haines (17) and the tensions in terms of water column heights were not given.

Figure 1, however, illustrates an additional complication in specification of the capillary fringe, and that is brought out by the curve labelled "wetting". In this second curve a plateau occurs at about 80 to 85 percent saturation, but at tensions greater than about $5 \, (T/r)$ units the moisture content falls markedly. In no case is there 100 percent saturation, not even at zero tension. Both curves, however, were determined on the same sample of sand. It is obvious that one's decision regarding the capillary fringe would be vastly different depending upon the curve used. In the strict sense, the sand would not possess a capillary fringe on the basis of the wetting curve.

The type of behavior noted here is the hysteresis effect and was first reported by Haines (17), from whose work these curves are taken. The drying curve was obtained by first wetting the sand under hydrostatic pressure and then applying tension so that water would be removed. After a certain amount of tension had been applied (more than shown on the graph),
Figure 1. Moisture characteristic curves for sand, data of Haines (17). For the drying curve, the capillary fringe region extends from the left-hand ordinate to the point A.
the soil was allowed to re-absorb water at progressively smaller tensions, thus resulting in the wetting curve. It is obvious that the wetting and drying are not reversible; that is, the moisture status is not a single-valued function at a given pressure value, but depends upon the manner in which the pressure value is approached.

Even if the plateau of the wetting curve were displaced upwards to coincide with the drying curve, the fringe, while existing in this event, would still not be the same as the fringe determined from the drying curve. Hence, to specify the capillary fringe status of a soil one must also state the manner in which moisture changes are being effected; that is, whether the soil is drying (or draining), or whether it is wetting. In setting up the definition of the fringe, Childs (5), while not saying so specifically, was undoubtedly referring to the drying part of the hysteresis loop. In drainage work this would be the proper choice of curve.

The effect of hysteresis on the capillary fringe has also been noted by Tolman (41, p. 157). He illustrates, with diagrams, the difference in the fringe for a rising water table compared to one that is falling. From his treatment it is also possible to infer that the height of the fringe for the wetting case (rising water table) will be smaller than for the drying case (falling water table).

Figure 2 is a schematic diagram of a capillary fringe under static conditions. The shaded upper boundary of the fringe corresponds to point A in Figure 1.
Figure 2. Representation of a static capillary fringe as it might occur in the field.
1. Soil is unsaturated
2. $P \ll P_a$

Water table previously in this region

1. Soil is essentially saturated
2. $P \ll P_a$

Water Table
$P = P_a$

1. Soil is saturated
2. $P > P_a$

Impermeable layer somewhere
It should be pointed out that not all porous materials will exhibit a sharp break in the moisture characteristic as at point A in Figure 1. As the pore-size range increases, the curve tends to break less sharply because the drainage process occurs more evenly. This makes it somewhat difficult to determine the saturation line at the top of the fringe; hence, the upper boundary in Figure 2 is shown as diffuse.

It may be well to illustrate the locations of Walter's (49) various zones in Figures 1 and 2. The zone of capillary saturation has already been indicated. The zone of hydrostatic pressure would appear to the left of the left-hand ordinate in Figure 1, and below the water table in Figure 2. The zone of aeration occurs to the right of point A in Figure 1 for the drying curve, and to the right of the left-hand ordinate for the wetting curve. In Figure 2 the zone of aeration is located above the diffuse "line" of saturation.

The concept of a saturated capillary zone has also appeared in the writings of other soils workers. Zunker's (51) classification of soil water, which is widely quoted, contains a "Grundwasser zone", a "Kapillarwasser zone", and a "lufthaltige zone". From his diagram and a somewhat free translation of German, it appears that these are equivalent respectively to Walter's zones of hydrostatic pressure, capillary saturation, and aeration.

Terzaghi (40) in studies of capillary rise into dry sands from a phreatic surface has designated the lower part of the wetted sand column as being "capillary saturated." He does not give this region a name,
but notes that this height of capillary saturation is less than the maximum height to which the wetted front will rise. This capillary saturated section may correspond to the plateau of the wetting curve of Figure 1.

**Pressure characteristics**

Walter (49) has pointed out that the most important difference between the zone of capillary saturation and the zone of hydrostatic pressure is in the type of pressure considerations. Below the phreatic surface the pressure increases above atmospheric linearly with depth. Above the phreatic surface the pressure decreases below atmospheric linearly with height. Note that the latter part of the preceding sentence is equivalent to saying that tension increases linearly with height.

Sitz (38) has made similar observations about the pressure considerations. Even though he confines himself to capillary tubes, he demonstrates rather clearly that not only is there the linear pressure dependence, as stated above, but also that this linear dependence in the capillary fringe is merely a continuation of the same linear dependence in the region below the phreatic line. He also shows that one may think of the water in the capillary fringe as being under a vacuum, and he illustrates this nicely with diagrams. In this connection it is well to note that Vedernikov (45) has used the concept of a vacuum at the fringe boundary to solve a problem in fringe flow. This will be referred to later.
Childs (5), whose work will also be referred to later, has also used the concept of vacuum, or negative pressure, in his studies of the fringe. He criticizes Wyckoff, Botset, and Muskat (50) for having asserted that the pressure at the upper fringe boundary is atmospheric. In this connection Sitz's ideas may be useful. He points out that in passing from a region below the phreatic surface, up through the capillary tube, and out through the meniscus, the pressure varies in the following manner: Below the phreatic surface the pressure is greater than atmospheric and decreases linearly to atmospheric at the phreatic surface. Continuing up through the capillary tube, the pressure decreases below atmospheric on the same straight line as formerly until the meniscus is reached. In passing through the meniscus the pressure jumps, discontinuously, back to atmospheric. Hence, both Childs and Wyckoff et al. are right provided that the former was thinking of pressure underneath the meniscus, and that the latter were thinking of the pressure outside the meniscus.

Sitz further points out that the capillary fringe will quickly adjust itself to dynamic conditions and will continue to be present for cases in which the phreatic surface is curved as a result of flow. Thus, he concludes that one should expect an effect on the flow if a fringe is present, and he cites Wyckoff et al. (50) as evidence that it does happen in practice. He does not mention changes in the fringe itself as a result of viscous flow through it.

The fact that the pressure of the water in the capillary fringe is less than atmospheric has an important consequence for the fringe in the dynamic state. This is embodied in the outflow law stated by Richards (36)
that: "Outflow of free water from soil occurs only if the pressure in
the soil water exceeds atmospheric." Hence, there is placed a certain
restriction upon flow in the capillary fringe. Water can move into the
fringe from the phreatic surface, but the fringe likewise must discharge
its flow into a phreatic surface, since water cannot flow directly across
a fringe boundary into the atmosphere (except as a vapor). To do so
would violate the outflow law since by definition the fringe is under
tension. Thus, if the fringe is to make an addition to the flow which
occurs beneath the phreatic surface, it will have to perform similarly
to a siphon. A siphon can transfer water only from one phreatic surface
to another. Even then flow will not occur unless there is a difference
of potential between the two phreatic surfaces.

Effect of the Capillary Region on Water Movement

Since the term capillary fringe has been restricted to apply to the
saturated zone, another term must be employed to designate both the
saturated and unsaturated capillary zones. Hence, the term capillary
region will be used in the sense of including both the saturated and
unsaturated zones.

Capillary siphon

Versluys (48) constructed a U-shaped pipe 5 cm. in diameter and
filled it with dune sand. The pipe was inverted to form an arch, the
one leg of the U being placed in contact with a free water surface.
Provision was made to raise and lower the free water surface in the one
leg, but was always maintained above the end of the other leg of the U which extended into the atmosphere. The top of the U-shaped arch was open to the atmosphere. Measurements of flow were made after the sand had been completely wetted by capillarity and water was being discharged at the lower end of the outer leg. The height of complete capillary saturation for the sand was 29.3 cm.

The initial measurements were made with the water level near the arch of the siphon. Succeeding measurements were taken with the water level progressively lower. The flow rate was reduced as the distance from the water level to the crest of the siphon increased, but it did not stop until this distance was 154 cm., or 5.3 times the height of complete capillary saturation. Thus, even with a substantial amount of unsaturation of the soil in the siphon, a measurable amount of water was conducted through it.

McLaughlin (28) performed a slightly different experiment. He built a flume about 10 ft. long with a short 45° hook at one end. The flume was filled with soil and the hook-shaped end placed in contact with a free water surface. Water had to move upwards 4 in. by capillarity before it could move laterally, and eventually downward, in the soil of the flume. The main body of the flume was projected downward at an angle of 45°, with the lower end well below the free water surface. Four days after the whole column was wet by capillarity, water was dripping from the lower end of the flume. Moisture samples revealed that nowhere in
the whole column, except possibly at the lower end, was the percentage of moisture as great as that of capillary saturation.

The results of his study led McLaughlin to suggest that swamping of lands might in some cases be caused by natural capillary siphons. As another example of such a siphon he suggested earthen dams whose core walls do not extend very far above the level of water which they must contain.

Actually this second case had occurred in Germany several years before McLaughlin's experiments. Zunker (51) reports that a 20 km. section of the Berlin-Stettin Canal contained clay core walls (in the banks) which extended only 30 cm. above the level of the water in the canal. The seepage loss was 28 l. per sec. When another 40 cm. of height was added to the core wall, the loss was reduced to 6 l. per sec. The difference in seepage loss presumably represented the effect of the capillary siphon which existed when the core wall was too low.

Radial flow of water

As mentioned in the Introduction, various flow problems involving water under hydrostatic pressure have been solved using Darcy's law. The mathematical form for this law is

\[ Q = KA \Delta H/L, \]  

where

- \( Q \) = quantity of water per unit time, or flow
- \( K \) = hydraulic conductivity
- \( A \) = cross sectional area of soil column
\[ \Delta H = \text{driving head, or potential difference} \]

\[ L = \text{length of soil column.} \]

The differential form for the law is \[ q = -K \frac{\partial \phi}{\partial s}, \]
where \( q = \text{flow per unit area} \), \( \phi = \text{the potential function} \), and \( s = \text{the displacement in any arbitrary direction} \). The minus sign indicates that the velocity of flow is in the direction of potential decrease. By using the differential form along with the equation of continuity for an incompressible liquid such as water, there results the familiar Laplace equation \[ \nabla^2 \phi = 0. \]

The process is outlined in detail by Muskat (31, p. 132).

The Laplace equation is a common one in physics. It possesses an infinite number of solutions. Using it, along with the proper boundary conditions, it is possible, at least in principle, to obtain a solution for any flow problem.

Starting with the rigorous approach just outlined, Wyckoff, Botset, and Muskat (50) attempted to solve the problem of radial flow into a vertical well. This problem can be solved easily for certain types of geometry, but they were interested in the case where a phreatic line is present. They built a sand model which was a radial wedge of 15°, with a radius \( r_1 = 152.4 \text{ cm} \). A sectional view perpendicular to an axial plane is given in Figure 3. The inlet level of water is maintained at a height \( Z_1 \), and the outlet level, which corresponds to the level of water in the well of radius \( r_w \), is at a height \( Z_w \). The phreatic line through the sand is indicated by the variable \( Z_s \).
Figure 3. Schematic view of the capillary fringe in the radial flow problem of Wyckoff et al. (50). The view is sectional, and perpendicular to an axial plane. The indicated streamline is schematic.
It can be seen that the phreatic line is not continuous between the inlet and outlet levels. The distance $Z_{sw} - Z_w$ represents the height of what is termed a surface of seepage. The capillary fringe is indicated by the shaded portion. Both the surface of seepage and the fringe complicate the analytical solution, so certain empirical simplifications were deduced. Only the treatment of the fringe will be considered here.

As mentioned earlier, Wyckoff et al. considered the pressure to be atmospheric, both at the top and bottom of the fringe. Childs (5) has criticized this, but Wyckoff et al. did not actually use these pressure considerations in handling the fringe. Instead, they merely considered it to be a radial conducting layer, of thickness $Z_c$, subject to the same potential difference as the region beneath the phreatic line. This led to the expression

$$Q_{\text{total}} = \frac{\pi k \gamma g (Z_1 - Z_w) (Z_1 + Z_w + Z_c)}{\ln \frac{r_1}{r_w}}$$

where $Q_{\text{total}}$ is the total flow, $\pi = 3.1416$, and $k \gamma g$ corresponds to the hydraulic conductivity. The other quantities in Equation 2 are shown in Figure 3.

For the flow $Q_g$ in the region below the phreatic surface, Wyckoff et al. gave the following expression

$$Q_g = \frac{\pi k \gamma g (Z_1^2 - Z_w^2)}{\ln \frac{r_1}{r_w}}$$

Clearly, the flow $Q_{\text{total}}$ of Equation 2 reduces to $Q_g$ of Equation 3 when
there is no fringe; that is, when \( z_c = 0 \). It is seen, then that the correction for the fringe involves an additional height \( z_c \), the height of the fringe.

Table 1 shows some comparisons between theory and experiment. In all cases \( z_1 \) was enough smaller than \( h_s \) to permit the full value of \( z_c \) (9.5 cm. for the sand which was used) to be developed. Note the good agreement between \( Q_{\text{total}} \) and the observed value of flow \( Q \). The difference between \( Q_{\text{total}} \) and \( Q \) represents the flow in the capillary fringe.

Table 1. Observed flow and calculated flow for radial water movement (50), in cubic centimeters per second

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<thead>
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<th>Run No.</th>
<th>( Q_g ), Equation 2</th>
<th>( Q_{\text{total}} ), Equation 1</th>
<th>( Q ) observed</th>
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<tr>
<td>1</td>
<td>3.06</td>
<td>3.76</td>
<td>3.90</td>
</tr>
<tr>
<td>2</td>
<td>3.09</td>
<td>4.98</td>
<td>4.95</td>
</tr>
<tr>
<td>3</td>
<td>2.70</td>
<td>4.47</td>
<td>4.55</td>
</tr>
<tr>
<td>4</td>
<td>1.18</td>
<td>2.37</td>
<td>2.50</td>
</tr>
</tbody>
</table>

That flow actually did occur in the capillary fringe was demonstrated by placing small quantities of black ink at various vertical positions along the inlet surface. The resulting streamers in the upper part of the sand crossed the phreatic surface into the fringe, but dropped back down beneath the phreatic surface near the outlet. This general trend is indicated by the schematic line marked "streamline" in Figure 3.
Childs, Cole, and Edwards (10) have also applied corrections for the capillary fringe in radial flow. Their main subject of interest was in regard to the measurement of hydraulic conductivity by boring two vertical holes (wells) into the soil beneath the water table and pumping water from one hole to the other until a steady flow is achieved. The phreatic surface at the well with lower water level is similar in shape to that indicated in Figure 3.

The fringe correction is made as follows. Assume that the flow Q between the wells for a potential difference $\Delta \phi$ is given by

$$\frac{Q}{\Delta \phi} = k h + F$$

(4)

where $k$ is the hydraulic conductivity per unit length of well below the phreatic surface, $h$ is the height of water in the wells, and $F$ is the correction for the capillary fringe. Assuming now that $F = kh_f$, where $h_f$ is a height correction on $h$ necessary because of the fringe, Equation 4 becomes

$$\frac{Q}{\Delta \phi} = k (h + h_f).$$

(5)

Experimental measurements were made of all the quantities in Equation 5 except $h_f$. A plot of the data indicated a linear relation between $Q/\Delta \phi$ and $h$. From this plot an intercept value of $h_f$ was determined. It turned out to be about half the height of the capillary fringe as determined from the moisture characteristic. Thus, to correct for fringe flow in the two-hole method of determining hydraulic
conductivity, it is necessary to add the value $h_f$ to the value of $h$ wherever the latter appears in a formula.

The ellipse equation for the water table

Aronovici and Donnan (2) have proposed an equation to permit the calculation of tile spacings for artificially drained land. The derivation of this equation depends upon two assumptions, (a) that only horizontal flow of water occurs between tile lines, and (b) that the hydraulic gradient at any horizontal distance from the tile line is equal to the slope of the water table at that point. Their analysis leads to the following expression.

$$S = \frac{4P(b^2 - a^2)}{Q} \quad (6)$$

where $S$ is the tile spacing, $P$ is the hydraulic conductivity, $a$ is the height of the drain tile above the impermeable layer, $b$ is the height, above the impermeable layer, of the water table midway between the tile lines, and $Q$ is the flow per unit length of tile line. A diagram of the arrangement is given in (A) of Figure 4. The curved phreatic line is an ellipse.

Van Schilfgaarde (43) has discussed Equation 6 at some length. He points out that Aronovici and Donnan were not first in presenting it. Hooghoudt (18), and apparently others even before him, have given similar equations on the basis of similar assumptions. These assumptions in principle are embodied in the approximate Dupuit-Forchheimer theory which both Muskat (31) and Van Schilfgaarde (43) have analyzed rather carefully. Both investigators find that the Dupuit-Forchheimer theory
Figure 4. Diagrammatic illustrations of elliptically shaped water tables and the associated capillary fringes. (A) is adapted from Aronovici and Donnan (2); (B) is adapted from Hooghoudt (18).
leads to a mathematical absurdity. Muskat concludes that any agreement of Dupuit-Forchheimer theory with observed facts is wholly fortuitous. Van Schilfgaarde, while fully cognizant of the approximation, points out that in cases where the region of flow is of large horizontal extent relative to the depth, the streamlines (for either an open ditch or tile) will be approximately horizontal through a large part of the medium; hence, vertical velocities can be neglected and a useful result may be obtained in fairly simple form.

Returning now to the ellipse equation, Donnan (15) tested it experimentally in a sand tank. He found that theory and experiment agreed fairly well when the water table was near the soil surface. Discrepancies became fairly large as the water table fell farther below the soil surface. At this stage he discovered the work of Hooghoudt (18) and with it the suggestion of horizontal flow in the capillary fringe.

Hooghoudt's case is shown in (B) of Figure 4. His equation is

$$Q = \frac{k (H_0^2 - h_0^2)}{e}$$  (7)

where $Q$ as usual represents the flow, $k$ is the hydraulic conductivity, and the other symbols are as shown in (B) of Figure 4. The similarity between Equations 6 and 7 is evident. Both predict that the shape of the water table will be elliptical.

Since a basic premise in the derivation of Equation 7 is the assumption of horizontal flow (from Dupuit-Forchheimer theory), it seemed reasonable to Hooghoudt simply to add to $H_0$ and $h_0$ the height
of the capillary fringe to correct for the horizontal flow occurring in it. By making this correction, his experimental values checked closely with the theoretical, thus indicating an appreciable flow in the fringe.

Since he also had assumed only horizontal flow, Donnan corrected the a and b in his equation by adding to each of them the height of the capillary fringe. This brought his calculated values very nicely in line with the observed values, within an error range of 3 to 5 percent.

Van Schilfgaarde (43) has given the mathematical justification for Donnan's and Hooghoudt's procedure. It consists of incorporating the effect of an added conducting layer into the differential equation which comes from the Dupuit-Forchheimer theory. When the analysis is completed there emerges a modified form of Equation 7 in which the height of the fringe is added to both $H_0$ and $h_0$.

The fringe in electric analogue studies

Childs (4,5,6,7,8) has investigated the problem of drainage by the use of electric analogues. Reference to Equation 1 shows that if $Q$ is interpreted as electric current, $\Delta H$ as difference in electric potential (voltage), and $K$ as electrical conductivity, then Ohm's law results where the resistance $R = L/KA$. Thus, if one can construct an electric flow model completely analogous to the desired water flow problem, the electric model will give results which are as applicable to the flow problem as to the electric model itself.

Childs (4) begins his study with a complete rejection of the ellipse equation discussed in the previous section. He says that the
simplifying assumptions are so invalid that to make them is equivalent to assuming a solution.

In his second paper (5) he considers the equilibrium water table above a tile drain for the case of constant rainfall. As his basic working unit he chooses a section consisting of a half-spacing as shown in Figure 5. The straight line A'AC' is located at the midpoint between the tile lines. Line B'B'D' passes through the tile drain located at B. In general two cases were considered for the location of the impermeable layer, at A' B' and at AB. All of the results quoted in this discussion will be for the impermeable layer located at AB.

For the case of steady rainfall, the broken line CD was found to represent the position of the water table. When a fringe (50 cm., a large but reasonable value) was imposed on the water table, the position of the water table was shifted to the solid line CD, which is but little different from the broken line CD. Thus, the effect of the fringe on the water table was practically negligible.

The measured value of potential (in this case pressure) along the fringe boundary C'D' indicated a value of -44.3 cm. of water instead of -50 cm. as would be true for the static case. The difference can be accounted for as being the potential required to maintain viscous flow through the fringe.

Childs points out that his result for the fringe contribution differs from Hooghoudt's finding mentioned earlier. However, Childs notes that in his own case the fringe adds a layer of conducting medium,
but since the flow is appreciably vertical, the addition of the fringe increases the resistance. The increase in potential \((-44.3 - (-50) = 5.7\text{ cm. of water})\) largely offsets the additional resistance and hence the water table is only slightly affected. In Hooghoudt's case the fringe is added in a manner that causes the effective cross section open to flow to be increased.

In a later paper Childs (8) investigates the effect of the fringe on the falling water table. This was done by establishing a steady water table at the surface of the soil and then stopping the "rain". The shape of the water table was then determined at different times.

A set of three curves, all corresponding to the same intermediate time value, is shown in Figure 5. The curve FG represents the water table without a capillary fringe, BG represents the water table with a fringe, while E'G' represents the upper boundary of the fringe corresponding to EG.

The vertical positions of these curves with respect to each other seem to hold fairly well throughout the whole drainage period; that is, E'G' > FG > EG. The lag of E'G' behind FG is credited by Childs to the increased resistance to vertical flow caused by the fringe, without an addition of potential as was the case for the steady water table.

However, the complete set of curves shows that the point E' does catch up with F before F reaches its final equilibrium point at the top of the tile line. However, E' never goes lower than the height specified by the height of the tile line plus EE' (or GG'), whereas F will drop to
Figure 5. Diagrammatic illustration of water tables and capillary fringes adapted from Childs (5,8). CD and C'D' are for a steady water table, FG, EG, and E'G' for the falling water table. The fringes shown are for the impermeable barrier A'B' at AB.
a point which is level with the top of the tile line. The increase in velocity of \( E' \) relative to \( F \) is explained by Childs as being the result of increased horizontal movement in the fringe as the levels drop and the water table curves become flattened.

With regard to Childs' work the distinction between the phreatic line and the saturation line seems especially relevant. For his work discussed here, one's evaluation of the effect of the capillary fringe on drainage will depend upon whether the phreatic line or the saturation line is observed. From the standpoint of the phreatic line the capillary fringe is either of negligible effect, or actually aids drainage. In terms of the saturation line the fringe is in all cases a detriment to drainage.

One more result of Childs' (7) will be discussed briefly. This relates to downhill seepage of water in an inclined, permeable bed resting upon impermeable material. A drain tile is placed perpendicular to the flow net to intercept the seepage.

If the drain is placed in the middle of the bed, so-called perfect control will result if there is no capillary fringe. That is, 50 percent of the flow will be diverted. In the presence of a capillary fringe of one-tenth the thickness of the bed, only 40 percent of the flow is diverted. Thus, this represents another case in which the presence of a fringe reduces the efficiency of drainage.
Seepage from a canal

Vedernikov (45) has considered the seepage from a canal rectangular in cross section as shown in Figure 6. The sides of the canal AA' and DD' are impermeable walls. Water seeps out of the bottom of the canal into an infinite medium of equigranular soil (as, for instance, uniform glass beads), which possesses a saturated capillary rise of \( h_v \) units of water column. Since this is an ideal soil the saturation line is sharp and distinct; the moisture content is zero at heights above the capillary fringe.

The water level in the canal is assumed to be constant with a depth \( H \). Steady state flow into the infinite medium is assumed, meaning that a drainage layer occurs at some great depth and remains constant; that is, the water seeping from the canal does not build up a mound. It is necessary to assume this to ensure that the flow net at great depth will be as indicated in the lower part of Figure 6.

To take the capillary fringe into account Vedernikov considers the pressure along the line of saturation to be negative, or equal to \(-h_v\). From this he proceeds to solve the problem by a conformal mapping procedure used by himself in an earlier paper (44). His final result is

\[
Q = k_f (b + 2H_v K/K_1),
\]

where \( Q \) is the flow seepage per unit length of canal, \( k_f \) is the hydraulic conductivity, \( b \) is the width of canal, \( H_v = H + h_v \), \( K \) is the elliptic integral of argument \( k \), and \( K_1 \) is the elliptic integral of argument \((1 - k^2)^{\frac{1}{2}}\). For this particular case \( k \) is determined by \( b \),
Figure 6. Flow net for seepage from a canal, taken from Vedernikov (45). Streamlines are indicated by arrows, equipotentials by broken lines.
H_v, and h (see Figure 6 for the significance of h).

If there were no capillary fringe present, \( h_v = 0 \) and \( H_v = H \). It is clear that the effect of the fringe in this problem is to add a correction to the hydraulic head. Thus the presence of the capillary fringe, or soil vacuum as Vedernikov states it, increases the flow in the same manner as an increase in the level of water in the canal.

This effect is further illustrated by Vedernikov by using the numerical values of \( h \), \( h_v \) and \( H \) shown in Figure 6. The numerical values are expressed in meters. For the case of a capillary rise of 0.8 m., with a water level in the canal of 1.2 m., the flow is 21.2 percent larger than it would be if there were no capillary rise.

If the capillary rise were increased to 2 m., and the height of water in the canal reduced essentially to zero, then the flow is increased 120 percent over what it would be without capillary rise. It is important to note in this second case that the height of water in the canal is limited to being a very thin film over the bottom.

This particular example of Vedernikov's presents a difficulty in fitting the definition of capillary fringe. The only region in Figure 6 in which the pressure in the soil water is greater than atmospheric, is the half-moon area indicated by ABCD. The rest of the region included in the flow net is all under tension even though still saturated. Thus, a very considerable portion, in fact most, of the flow region is saturated and under tension, but located under the water table. Hence it would seem that the definition of the fringe should imply the following three things: (a) saturation, or near-saturation, (b) pressure
less than atmospheric, and (c) a location in contact with the water table, rather than specifying a location in any particular direction.

This particular problem of seepage from a canal also appears in a much shorter article by Vedernikov (47) in German. However, the original Russian publication as used here gives much important detail which is omitted in the German article.

The water table and flow in the capillary region

In 1934, Van Mourik Broekman and Buisman (42) were studying the effect of the capillary layer on the stability of dams. Their primary concern was to demonstrate that the capillary region above the water table did contribute to instability as the result of differential tensions caused by flow of water through the region. By means of dye streamers they showed that the streamlines going into the capillary region crossed the phreatic surface, and hence the phreatic line could not represent the limiting line of flow. Furthermore, they concluded that the streamline system in the capillary region formed a more or less continuous extension of the system existing below the water table.

These two conclusions were stated in almost identical terms by Wyckoff et al. (50) two years earlier. Also, two years later in 1936, Walter (49) and Vedernikov (46) emphasized them again. Vedernikov especially was very critical of many previous studies on seepage since in most cases the phreatic line had been taken as the limiting streamline, thus ignoring the capillary region. He even suggests that neglecting the flow in the capillary region is responsible for the
apparent breakdown of the Dupuit assumptions when applied to seepage of water through dams.

It is surprising, in view of the rather early recognition of the capillary contribution, that so few investigators have attempted to take it into account. Muskat (30) in 1935 published an elaborate paper on seepage through dams, but he continued to assume that the phreatic line was the limiting streamline. The capillary effect is not even mentioned. This in spite of the fact that he was an author with Wyckoff and Botset (50) in 1932.

The contribution of the capillary region when the conductivity is decreased

Most attempts to handle the capillary region have been restricted to the case of the hydraulic conductivity remaining essentially constant throughout both the pressure-saturated and capillary regions. Vedernikov (46), however, suggested that an exact solution could be obtained for the whole capillary region provided that the vertical position coordinate be expressed as \( h = t + a. h_K \), where \( t \) is the height of the pressure saturated region, \( h_K \) is the total height of capillary rise and \( a \) is a function which registers any change in hydraulic conductivity with height. As a first approximation, to be used with the Dupuit assumptions, he suggested that \( a \) be set equal to unity and that \( h_K \) be set equal to \( h_{KH} \), where \( h_{KH} \) is the height of capillary saturation. Interestingly enough this is the exact procedure followed by Van Schilfgaarde (43) as mentioned earlier.
Luthin and Miller (26) have pointed out that the starting point for an exact and rigorous solution of the complete capillary region is supplied by a modified form of Laplace's equation which is

\[ \frac{\partial}{\partial x} (k \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y} (k \frac{\partial \phi}{\partial y}) = 0 \]  

(9)

where the hydraulic conductivity \( K = K(x,y) \).

Irmay (20) proposes that \( K \) is a universal function of the degree of liquid saturation, the form of which is a cubic parabola. His experimental work, however, is confined to sand. Richards and Moore (37), Childs and Collis-George (9), and Collis-George (13,14) have also studied the problem and have proposed methods for measuring the conductivity as a function either of tension or moisture saturation.

Summary of Literature Review

The basic approach used by Wyckoff et al. (50), Hooghoudt (18), Donnan (15), and Childs (5,7,8,10) was to consider the capillary fringe as an added conducting layer in which the conductivity was equal to the conductivity of the pressure-saturated region. The correction was effected by increasing the appropriate position or length variable by a constant amount. In all cases, except one of Childs' (10), the added amount was made equal to the height of the capillary fringe. In the single exception (Childs') the amount of length correction was determined empirically, and turned out to be about half of the fringe height.
Vedernikov (45) corrected for the fringe in terms of a pressure adjustment. The conductivity was assumed to remain constant. Again the correction was the full height of the fringe added to the hydrostatic head. It must be pointed out that Vedernikov's contribution was a theoretical treatment of a theoretical fringe, but it well illustrates certain interesting aspects.

All of the investigators except Childs have assumed that the fringe remains unchanged with respect to thickness when water moves through it. His empirical result mentioned in the preceding paragraph automatically included any effect of dynamic changes.

In most of the cases considered, it was found that the presence of a conducting capillary region increased the flow. Childs (5, 7, 8), however, has shown cases in which the effect on flow is negligible, or even becomes negative; that is, the flow rate is decreased because of the capillary region.

When a capillary region is present above a curved phreatic surface, it is fallacious to assume that the phreatic line is the limiting streamline. This conclusion is supported by several investigators (42, 46, 49, 50).
THE CAPILLARY FRINGE AS AN ARRAY OF CAPILLARY TUBES

Keen (22) has emphasized that the water in the soil does not behave identically the same as does the water in a single capillary tube, even though there are points of similarity. The analysis which follows is made with that limitation clearly in mind. It is felt that a treatment on the basis of capillary tubes is useful in at least indicating general trends.

Theoretical Models

In Figure 7 are shown cross sectional side views of three theoretical models involving capillary tubes. Water flow through the systems is indicated by arrows. In (A) and (B) the main conducting tube IO is a capillary of the same diameter as the side tubes shown branching from it. The side tubes in (A) are continuous; those in (B) are not. In (C), water is simply flowing downward, and out of the system.

In (A), if the capillary fringe is to contribute to the flow, water must move as indicated by the tortuous path which enters and leaves each individual capillary composing the fringe. The driving gradient along this path will be small compared to the gradient along the horizontal paths IO.

In (B) is shown the effect of interconnecting capillaries composing a fringe. It is fairly obvious that the increase in an effective cross section for flow is much greater for the interconnected system than for
Figure 7. Cross sectional side views of theoretical flow models formed from capillary tubes.
the system shown in (A).

A porous medium consists of many interconnected capillary channels, and so is more like (B) than (A). Nevertheless, one should use considerable caution in applying conclusions from (B) to porous media. The capillaries in porous media are not only interconnected but are of many different sizes and shapes as well.

In (C), water is draining out of the system, and the boundary of menisci (saturation line) is moving downward with time. If the tubes are long enough, there will be a phreatic line at some specified distance below them. Hence, (C) represents a simple ideal case of a falling water table.

It should be noted in (C) that not only does the water encounter resistance from the walls of the tube, but that surface tension forces exert a certain drag at the curved air-water interfaces. Furthermore, the basic nature of the flow is perfectly represented in one tube. Even if there were interconnections the basic behavior would be little changed, since the flow net for the system is rectilinear.

Both (B) and (C) form the basis for extended investigation. A model somewhat like (B) but involving porous media will be investigated later on in this thesis. The case for (C), using a single capillary, will be treated in the following section.
The Falling Water Table, Simple Case

Mathematical treatment

Consider the single capillary tube shown in A of Figure 8. The equilibrium height of capillary rise is equal to $y_{\infty}$. The initial height of the meniscus is $y_0$ at time $t = 0$. At time $t$ the height of the meniscus is $y$. The position coordinate $z$ can be varied independently of $y$, but can never exceed $y$. The radius of the capillary is $r$. Other symbols which will be used are $T$ for surface tension, $\rho$ for the density of the liquid (water), $g$ for the gravitational constant, $\eta$ for viscosity, and $\pi = 3.1416$.

Poiseuille's law for the viscous flow of a fluid in a capillary tube states that

$$Q = \frac{\pi r^4 (P_i - P_o)}{8\eta L},$$

where

- $Q$ = flow
- $P_i$ = pressure at inlet end of tube
- $P_o$ = pressure at outlet end of tube
- $L$ = length of tube.

The other symbols are used as defined above.

Equation 10 applies only to horizontal tubes. If gravity enters the problem, a corrected form of Equation 10 must be used which is

$$Q = \frac{\pi r^4}{8\eta} \left[ \frac{P_i - P_o}{L} \pm \rho g \right],$$

(11)
Figure 8. A. Single capillary tube model of the falling water table, simple case. B. Theoretical curves of height y and potential $\phi$ as a function of time. C. Theoretical curves of flow ratio $(Q/Q_0)$ or potential ratio $(\phi/\phi_0)$ as functions of height or time.
where the sign of $\varphi g$ is positive if flow is in the direction of gravity and negative if it is opposed to gravity. For the justification of this see Muskat (31, p. 287).

Returning now to $A$ in Figure 8, the pressure at the bottom (outlet) of the capillary is atmospheric, or $P_a$. At the meniscus ("inlet," height $y$) the pressure is a negative $2T/r$. Hence,

$$P_i = P_a - 2T/r$$

$$P_o = P_a,$$

and Equation 11 becomes, taking $\varphi g$ positive,

$$Q = \frac{\pi r^4 \varphi g}{8\eta} \left[ 1 - \frac{2T}{y \varphi g r} \right]. \quad (12)$$

Now $2T/\varphi gr$ is the equilibrium height of rise $y_\infty$, $\pi r^2$ is the area $A$ of the capillary, and $r^2 \varphi g/8\eta$ can be set equal to the hydraulic conductivity $K$. Hence,

$$Q = KA \left( 1 - \frac{y_\infty}{y} \right) = KA \left( y-y_\infty \right)/y. \quad (13)$$

If there were no curved meniscus present (i.e., wetting angle $= 90^\circ$), the height of capillary rise $y_\infty$ would be zero. Designating the flow for this case as $Q_0$, Equation 13 becomes

$$Q_0 = KA. \quad (14)$$

Now $Q_0$ in Equation 14 represents the flow in the absence of a capillary fringe, while the $Q$ in Equation 13 represents the flow with a fringe. Taking ratios between these two $Q$'s there results finally

$$Q/Q_0 = 1 - y_\infty /y. \quad (15)$$
For values of \( y \) much larger than \( y^* \), the flow ratio \( Q/Q_0 \) is essentially unity and \( Q \approx Q_0 \). However, as the meniscus approaches its equilibrium position, \( Q \) becomes negligible compared to \( Q_0 \). Thus, the effect of the fringe is negligible as flow begins, provided the height \( y \) is sufficiently large, but it retards the flow as time goes on. In no case is the flow increased over the non-fringe flow. This behavior is shown graphically by the broken line in C of Figure 8.

It is of further interest to note how the flow ratio and the height \( y \) vary with time. Define \( q \) as the total quantity of water discharged in time \( t \); then \( Q = dq/dt \). Also, a slight increment of water \( dq \) corresponds to a slight decrement \(-dy\) of the height of water in the capillary. Thus,

\[
dq = -\pi r^2 dy.
\]

By combining this with Equation 12, and remembering that \( Q = dq/dt \) and \( y_\infty = 2I/\varphi gr \), one finds

\[
\frac{dy}{dt} = -K \left[ \frac{y - y_\infty}{y} \right]. \tag{16}
\]

Separating variables, integrating, and supplying the initial condition that \( y = y_0 \) when time \( t = 0 \), there results

\[
y_0 - y + y_\infty \ln \frac{y_0 - y_\infty}{y - y_\infty} = Kt. \tag{17}
\]

The time dependence of the flow ratio \( Q/Q_0 \), calculated from Equations 15 and 17, is shown by the solid line in C of Figure 8. The time dependence of the height \( y \) is shown by the solid curved line in B of
Figure 8. It is clear that complete drainage cannot occur if there is a capillary fringe.

Zunker (51) has derived a number of formulas for water movement in capillary tubes, but does not give the development outlined here. Lambe (25) has made a similar analysis to the one made here, especially for the case of interconnected capillaries. However, he gives few analytical results.

Horton (19) has derived the equivalent of Equation 17 for the case of a capillary tube inclined at an angle \( \alpha \) from the horizontal, but with a different position coordinate. The basic difference between the two results is that the right-hand side of Equation 17 must be multiplied by \( \sin \alpha \) to obtain his result. Experimental data of both Lambe and Horton will be presented later.

An examination of Equation 13 reveals that the potential gradient is the expression which multiplies \( KA \). This is true because at any time \( t \) the difference in head (or potential) across the length \( y \) is equal to \( y - y_{\infty} \). Hence, at any time \( t \), one can obtain the head distribution through the height \( y \) by setting

\[
\frac{d\varphi}{dz} = \frac{y - y_{\infty}}{y} z.
\]

For the integration with respect to \( z \), \( y \) is constant and hence

\[
\varphi = \frac{y - y_{\infty}}{y} z + C.
\]

By choosing the potential to be zero when \( z \) is zero, it is found that the
constant $C = 0$. Thus,

$$\phi = \frac{y - y_\infty}{y} z.$$  \hspace{1cm} (18)

Equation 18 shows that the potential varies linearly with height throughout the water column at any particular time. However, the slope of the line, which is given by the coefficient of $z$, varies with $y$ and hence with time. Initially, if $y$ is very large, the slope will be almost unity, but decreases with $y$ (and hence with time) until $y = y_\infty$, at which point the potential (or head) is zero. These results are shown graphically by the solid lines of Figure 9-1. The dotted line represents the distribution for $y_\infty = 0$ (the case of no fringe). For this latter case Equation 18 simplifies to $\phi = z$, where $\phi_0$ is the potential when the fringe is absent. By forming the ratio $\phi/\phi_0$ it is seen that

$$\phi/\phi_0 = \frac{y - y_\infty}{y},$$

which is the same as for $Q/Q_0$. Hence, the ordinate in C of Figure 8 may be considered to be either the flow ratio or the potential ratio.

As was pointed out in deriving Equation 18, the total potential at any time is $y - y_\infty$. This can be verified very simply from Equation 18. To obtain the total potential at any time, it is necessary to set $z = y$; hence, $\phi = y - y_\infty$. A plot of $\phi$ against time would simply be a plot of $y$ against time minus the constant value $y_\infty$. Thus, the $\phi$-curve is merely displaced downward from the $y$-curve by one $y_\infty$ unit, and is shown as the dotted curved line in B of Figure 8.
Figure 9. I. Theoretical curves of potential distribution through the capillary tube of Figure 8A at different times, dashed line is for case of no fringe. II. Theoretical (solid lines) and experimental (dashed lines) potential distributions for sand columns. Circles are experimental points, taken from Lambe (25). Dashed and solid lines are superposed on the top curve.
The curve-plotting scheme for Figures 8 and 9 has been based upon units of \( y_\infty \) and \( y_\infty / K \) in order to avoid assuming specific numerical cases. As will be shown in the next section, experimental results can be compared to the theoretical simply by using a reduced unit of length based upon \( y_\infty \), and a reduced unit of time based upon \( y_\infty / K \). That the quotient \( y_\infty / K \) is really a time unit can be shown very simply by dimensional analysis. The height \( y_\infty \) is expressed in length, while \( K \) is in length per unit time. Hence, the reduced time unit is length divided by length/time, resulting in a time unit the size of which depends upon the numerical values of \( y_\infty \) and \( K \).

**Experimental verification**

Horton (19), as mentioned earlier, investigated the fall of the meniscus in a sloping capillary tube both theoretically and experimentally. His analytical result is essentially Equation 17, the term \( Kt \) being multiplied by the sine of the angle of slope. For the data reported in Table 2, \( y_0 = 35.38 \) in., \( y_\infty = 8.38 \) in., and \( K = 1.294 \) in./sec. Hence, for the comparison to theoretical values, the new time unit is \( 8.38/1.294 = 6.46 \) sec. The new length unit is \( y_\infty = 8.38 \) in. Thus, Horton's figures for length must be divided by 8.38 and his time values by 6.46.

The results in Table 2 verify the theory rather well. Even the largest error is probably no more than could be accounted for by the variation in bore of the capillary tube.
Table 2. Theoretical and observed values of time t required for meniscus in a capillary tube to attain a height y.
Adapted from data of Horton (19)

<table>
<thead>
<tr>
<th>Height, in.</th>
<th>Time, sec.</th>
<th>Height y, units</th>
<th>Time t, y°°K units</th>
<th>Percent error</th>
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<tbody>
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<td>3.63</td>
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<td>25.6</td>
<td>1.84</td>
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</tr>
<tr>
<td>10.38</td>
<td>36.8</td>
<td>1.24</td>
<td>5.54</td>
<td>5.70</td>
</tr>
</tbody>
</table>

Lambe (25) studied the drainage from a vertical column of sand, giving the following experimental results:

(a) A plot of the cumulative quantity of drainage water versus time.
(b) A plot of the visual line of saturation versus time.
(c) Numerical values for hydraulic conductivity (0.0266 cm./min.) and porosity (0.375), but both were measured on a different column of sand. However, both columns had been filled from a common source of fairly uniform sand.
(d) Curves of the head distribution pattern through the column at various time intervals.

By graphical differentiation of the curve in (a), values of the flow Q were obtained, including the value of Q = Q₀ at time zero. Using Q₀, and a value of cross sectional area A determined from the porosity and the curves in (a) and (b), a value of K for the column was calculated.
from the relation \( K = \frac{Q_0}{A} = 0.204 \text{ cm./min.} \). This value differed from the one in (c) by a factor of 7.66. Values of \( y_0 \) and \( y_\infty \) were inferred from the curve in (b).

The comparison between theory and experiment is shown in Figure 9-II. The theoretical curves are represented by solid lines, the series \( A_1, A_2, A_3 \) corresponding to reduced time units of 0.0165, 0.986, and 1.664. The series \( B_1, B_2, B_3 \) corresponds to reduced time units of 0.00021, 0.1289, and 0.214. The corresponding actual time units in minutes are 5, 300, and 500. The \( A \)-series is based upon \( y_\infty/k = 62.0/0.204 = 304 \text{ min.} \), whereas the \( B \)-series is based upon \( y_\infty/k = 62.0/0.0266 = 2330 \text{ min.} \).

The experimental curves are shown by dashed lines and circles and are numbered to correspond to the subscripts of the \( A \)'s and \( B \)'s; however, the curve which would be No. 1 in the series has the experimental points so closely superimposed upon \( A_1, B_1, B_2 \) and \( B_3 \) that no dashed line can be shown.

It is seen from Figure 9-II that only at the beginning of drainage does the experimental head distribution agree with the theoretical. At later times the experimental curves (2 and 3) are much flatter than the corresponding curves of either theoretical series (\( A_2, A_3 \) or \( B_2, B_3 \)). This means that the potential gradient (the slope of the line) decreases even more rapidly with time in the sand than it does in a single capillary tube. It is also seen that the agreement between the experimental curves and the \( B \)-series (based upon the reported conductivity of 0.0266 cm./min.) is poorer than the comparison of the experimental curves to the \( A \)-series.
(based upon the calculated conductivity of 0.204 cm./min.).

In Figure 10 is shown a set of flow ratio curves, theoretical and experimental. Both experimental curves verify the relatively rapid reduction of the potential gradient as compared to the theoretical. Again the departure from the theoretical curve is much greater for the reported hydraulic conductivity (0.0266 cm./min.) than for the calculated value (0.204 cm./min.).

It may seem that the agreement between theory and experiment in Figures 9-11 and 10 is very poor. No doubt a substantial part of the discrepancy is the result of having assumed a theoretical model which is much too simple. However, the results do show that:

1. The potential gradient in the sand decreases even more rapidly than in the simple theoretical model.
2. As a result of (1), the flow ratio \( Q/Q_0 \) also falls off more rapidly.
3. The flow ratio \( Q/Q_0 \), instead of being constant, as it would be if the fringe had no effect, decreases asymptotically with time even if the curve of its time dependence does not show an inflection point.

Point No. 3 is also borne out by the work of Luthin and Miller (26). In an experiment similar to Lambe's except that soil was used instead of sand, a similar asymptotic decrease with time is noted for the flow. None of their experimental results are shown here, since the graphs are
Figure 10. Experimental flow ratio ($Q/Q_0$) curves for sand compared to theoretical flow ratio curves for a single capillary tube, all as a function of time. Experimental points are taken from Lambe (25).
so small that transfer of the necessary information would have been subject to considerable error.

At this point it is of relevance to reconsider very briefly the results of Childs (8) for the moving water table, discussed in the Review of Literature. It was pointed out there that the downward movement of the line of saturation for the case of the capillary fringe lagged the downward movement of the phreatic line for the case of no fringe. This is exactly the type of result predicted by the capillary tube analysis of the simple case of the falling water table.
INVESTIGATION I. SEMICIRCULAR MODEL

In the Review of Literature some mention was made concerning the hydraulic conductivity of the capillary fringe. Most measurements of conductivity as a function of tension seem to indicate a fairly rapid initial decline; see, for instance, the data of Richards and Moore (37). Yet, in investigations dealing with the capillary fringe, it was seen in a number of cases that a good correction for the fringe could be made by assuming the existence of an added layer of medium which had the same conductivity as the pressure-saturated region in contact with it.

In view of this, it seemed desirable to perform a preliminary experiment designed to give information about the hydraulic conductivity of the capillary fringe. Particularly it was desired to have a conductivity value determined on the basis of the flow lines that enter the fringe through one part of the phreatic surface and bend downward finally in discharging their flow into a phreatic surface at lower potential.

Choice and Analysis of Model

To investigate the hydraulic conductivity of the capillary fringe a semicircular flow model (Figure 11) was chosen. The two reasons for the choice were:

1. A semicircular path for flow permits the determination of an average value of hydraulic conductivity, on the basis
Figure 11. Schematic side view of semicircular flow model.
of flow lines that bend upward into the fringe and then downward again as indicated above.

2. The calculation of the average conductivity can be made on the basis of a relatively simple mathematical solution.

For the mathematical analysis of Figure 11 consider the inlet level of water to be at $L_1'$ and the outlet level at $L_o'$, so that the soil will be saturated under hydrostatic pressure. For this case the flow net is similar to that of a line source or sink Muskat (31, p. 151) except that the positions of the streamlines and equipotential lines are reversed. For the semicircle in Figure 11 the stream function $\psi$ is given by

$$\psi = A \ln r + B,$$

and the potential function $\phi$ by

$$\phi = C \theta + D,$$

where $r$ is the variable radius, $\theta$ is the variable angle, and $A$, $B$, $C$, and $D$ are constants.

To evaluate $C$ and $D$, note from Figure 11 that $\phi = L_1'$ when $\theta = 0$. When $\theta = \pi$, then $\phi = L_o'$. Thus,

$$L_1' = D, \text{ and } L_o' = \pi C + L_o'.$$

Solving for $C$,

$$C = \frac{L_o' - L_1'}{\pi}.$$

The potential function then becomes
where \( \phi = \frac{-\Delta \phi'}{\pi} \theta + L_1' \), \( \pi \)

Now the total flow \( Q \) for the model is given by

\[
Q = -\int_{\text{area}} \frac{\partial \phi}{\partial \theta} \frac{dS}{K}
\]

where \( K \) is the hydraulic conductivity and \( dS \) is an element of area perpendicular to the streamlines. In this case \( dS = t \, dr \) where \( t \) is the thickness of the conducting column. The partial derivative, using Equation 19 is

\[
\frac{\partial \phi}{\partial \theta} = -\frac{\Delta \phi'}{\pi}
\]

and, substituting for \( \phi' / \partial \theta \) and \( dS \) in the integral and carrying out the integration there results finally, for the total flow \( Q \),

\[
Q = \frac{K \Delta \phi'}{\pi} t \ln \frac{r_2}{r_1} \quad .
\]

Solving for \( K \) yields

\[
K = \frac{Q \pi}{\Delta \phi' t \ln r_2/r_1} \quad .
\]

Equation 21 does not correct the hydraulic conductivity \( K \) for viscosity effects. To do this it is convenient to calculate a \( K \) for some standard viscosity of water at a reference temperature, say 20°C. If the viscosity of water is \( \eta \) at 20°C, and \( \eta \) at other temperatures,
then

\[ K = k/r^2, \text{ and } K_{20} = k/r_{20}^2, \]

where \( k \) is a constant dependent only upon the internal geometry of the porous medium. Solving these two equations simultaneously by eliminating \( k \) yields

\[ K = K_{20} r_{20}/r. \]

Combining this with Equation 21 gives the final result that

\[ K_{20} = \frac{n \pi Q}{\Delta \phi \ln r_2/r_1}. \] (22)

Experimental Methods

The model

Return now to Figure 11, the diagrammatic sketch of the flow model. It was built of plexiglass, with the exception of the outer periphery which was sheet brass. The model had an inner radius of 15 cm, and an outer radius of 20 cm. The inside thickness was 5.5 cm. At the positions marked I, II, ...., VI, short brass tubes for stoppers were soldered into the brass boundary with screens across their ends to keep the soil in place. These were to function as air inlets, but initially they were closed with rubber stoppers. At positions A, B, ...., G, tensiometers (sintered glass discs at the ends of glass tubes) were sealed into the brass boundary with a mixture of 15 percent beeswax and 85 percent resin. At positions 1, 2, 3, ...., 18, circular holes
were cut in the side panels, the solid-line circles representing holes in one panel and the dotted-line circles representing holes in the other panel. These holes were also closed with closely fitting rubber stoppers which were flush with the inner sides of the panels. These positions served as moisture sampling stations. A sharpened brass cylinder was inserted at each hole and a sample of moist soil withdrawn. A photograph of the model and related apparatus is shown in Figure 12.

The operation for a typical run is as follows. The model is carefully filled with sand or soil, care being taken to tamp in the dry material as well as can be achieved without special equipment. The brass screens at the ends of the semicircular soil column are then fastened in place. The air in the column is flushed out with carbon dioxide. A head of water, $\Delta \phi$ in the diagram, is applied and flow is continued until the rate is constant, or at least decreases very slowly with time. Under these conditions the soil column is at every point saturated. Water piezometers connected to the tensiometers can be read to determine the dissipation of head across the column.

Following this, the inlet and outlet water levels are lowered to $L_1$ and $L_0$ respectively. Data are taken for the calculation of the saturated hydraulic conductivity. Head dissipation patterns are obtained from the water piezometers.

Next, the rubber stoppers at stations I to VI are removed and replaced with vented stoppers which are stoppers with small holes for introducing atmospheric pressure at positions I, II, III,.... in
Figure 12. Photograph of experimental setup containing the semi-circular flow model.
Figure 11. The holes are made very small to minimize evaporation loss. The non-fringe water is permitted to drain, and the remaining water is held in the column only by capillarity. If the outer radius of the model is equal to or less than the height of the capillary fringe, all of the soil in the model should be essentially saturated. After steady state conditions have obtained, conductivity and head dissipation data are again taken. The final step in the procedure is the sampling of the column for moisture content, to check as to whether capillary fringe conditions have been maintained. It should be pointed out that in operation this model is nothing more than a type of capillary siphon.

The procedure is designed to first provide values of the saturated hydraulic conductivity. Then the solid stoppers are removed, the vented stoppers inserted, and the hydraulic conductivity is again measured, this time for the capillary fringe.

The geometrical constants for the model are as follows:

- thickness $t = 5.50$ cm.
- inner radius $r_1 = 14.98$ cm.
- outer radius $r_2 = 20.1$ cm.

Combining these with the value of $\pi \approx 3.1416$, Equation 22 becomes

$$K_{20} = 1.92 \eta Q / \Delta \theta'$$  \hspace{1cm} (23)

In actual use, $\Delta \theta'$ is replaced by $\Delta \theta$. 
Flow liquids

Since the object of interest is the conductivity of the capillary fringe for water, the flow liquid obviously must be water. In the soil, however, the water usually carries some salts in solution. Using pure distilled water as the flow fluid would be unrealistic, but there is another even more serious objection. The passing through the soil of pure water would constitute a vigorous leaching process which would effectively remove soluble salts from the soil. The flocculation state of the clay would almost certainly be affected, very likely adversely. A deflocculated clay would have a smaller water conductivity than it would have if it were flocculated.

To standardize the flocculation status of the clay, a water solution of 200 ppm. of calcium chloride was used. This admittedly would result in a calcium-saturated clay, but the calcium ion usually promotes flocculation. Thus, the water conductivity should not be reduced because of dispersed clay.

Another phenomenon which frequently affects conductivity measurements is microbial growth during the run (1). The microbes clog the soil pores and reduce the conductivity gradually but definitely with time. To counteract this effect, 100 ppm. of mercuric chloride were added, but the resulting solution attacked the brass screens and fittings in the soil model. The mercuric chloride solution was kept only for the first run on sand and was thereafter discarded.
In subsequent runs a near-saturated water solution of toluene was employed as an anti-bactericidal agent. The actual flow solution consisted of water containing 200 ppm. of calcium chloride and 400 ppm. of toluene.

The use of this solution, however, raised two other questions; namely, what about viscosity and surface tension effects? One would expect the viscosity change, if any, to be small. However, to make sure, the viscosity of the toluene-calcium chloride solution was compared to that of distilled water, using an Ostwald viscosimeter. A comparison of outflow times showed the value for distilled water to be 85.28 sec.; for the solution 85.26 sec. Both figures were the means of five determinations. It is clear that the solutes had a negligible effect on the viscosity.

One would expect a possible effect on the surface tension of the water when toluene and calcium chloride are added, especially a lowering due to the toluene. A lowering of the surface tension would have a direct effect on the capillary fringe, since a given pore size would support a shorter column of water. The surface tension of the solution was compared to that of pure water and again no appreciable differences were found. The measurement was made by the maximum bubble pressure method. The pressure in centimeters of water required to just force off a bubble from a given capillary orifice was 9.74 for pure water and 9.72 for the solution. Again the effect of the solutes is unimportant.
Figure 13. The moisture status of a column of quartz sand as a function of height. The lower curve shows the moisture content on the dry-weight basis, the upper curve is in terms of moisture saturation.
Another factor influencing the water conductivity is the presence of trapped air in the porous medium. Christiansen (11) has shown that it is difficult to remove all of the trapped air without resorting to the procedure of wetting the porous medium under vacuum. It would not, however, have been very practicable to apply a vacuum to the model. Instead, the original air in the porous medium was displaced with carbon dioxide, a procedure which has been reported by Christiansen, Fireman, and Allison (12). Hence, the trapping of this gas in the soil pores would be of relatively short duration, since the bubble would dissolve rather quickly in the percolating water.

Porous materials used

The graph in Figure 13 shows two types of moisture characteristic curves for quartz sand. The lower curve is on the basis of actual moisture content, the upper curve is in terms of moisture saturation. To calculate the upper curve from the lower, it is necessary to have values of volume weight (bulk density) or porosity corresponding to each moisture content. Hence, the resulting moisture saturation values are based upon two experimental values instead of one and thus exhibit much more variation.

In spite of the greater variation, it was intended to express the moisture status on the basis of moisture saturation, following the suggestion of Irmay (20). However, experimental difficulties prevented this. In sampling the model for moisture it was found almost impossible to remove the sample in such a way that a dependable volume weight
measurement could be obtained. Hence, all moisture curves determined from the model were expressed on the basis of the oven-dry moisture percentage.

Referring once again to Figure 13 it is seen that the moisture characteristic makes its most pronounced departure from saturation at points higher than 22 cm. above the water level. Hence, the height of the capillary fringe for the quartz sand is on the order of 20 cm. This was the basis for choosing the outer radius of the model equal to 20 cm.

A size analysis of the quartz sand appears in Table 3. The percentages in the various size ranges are averages of duplicate determinations. It may seem surprising that this sand can support such a high fringe when 36 percent of its particles are larger than 0.5 mm. However, a sizable portion of the sand is also in the relatively fine size range below 0.25 mm. Apparently the smaller particles tend to

Table 3. Size distribution of quartz sand

<table>
<thead>
<tr>
<th>Size range</th>
<th>Percentage</th>
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<tbody>
<tr>
<td>larger than 1.00 mm.</td>
<td>0.04</td>
</tr>
<tr>
<td>1.00 - 0.50 mm.</td>
<td>37.92</td>
</tr>
<tr>
<td>0.50 - 0.25 mm.</td>
<td>26.87</td>
</tr>
<tr>
<td>0.25 - 0.10 mm.</td>
<td>29.75</td>
</tr>
<tr>
<td>smaller than 0.10 mm.</td>
<td>5.42</td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
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</tbody>
</table>
close up the voids which normally exist between particles of larger size. Furthermore, the moisture characteristic was determined on the drying portion of the hysteresis loop.

In addition to the sand, two runs were made with soil, one a Marshall silty clay surface soil, and the other an Edina clay taken at a depth of 2 feet. Both soils were air-dried and then screened. The part used for filling the model was that which passed a 1 mm. sieve and was retained on a 0.25 mm. sieve. The mechanical analysis of these soils, by the hydrometer method, appears in Table 4.

Table 4. Mechanical analysis of soils used for flow measurements

<table>
<thead>
<tr>
<th>Soil</th>
<th>Percent sand</th>
<th>Percent silt</th>
<th>Percent clay</th>
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</thead>
<tbody>
<tr>
<td>Marshall</td>
<td>15.4</td>
<td>49.6</td>
<td>35.0</td>
</tr>
<tr>
<td>Edina</td>
<td>13.4</td>
<td>31.6</td>
<td>55.0</td>
</tr>
</tbody>
</table>

Results and Discussion

The data on hydraulic conductivity are plotted against time in Figure 14. The time origin was purposely omitted since such an origin has no particular significance. In each case, the important time point is that represented by the vertical broken line. It indicates the time of the last measurement of pressure-saturated conductivity. After this measurement was taken the vented stoppers were put into the model.
Figure 14. Water conductivity values, from the semicircular flow model, plotted against time. Numerical values under each curve designate the time in hours that the model was pressure-saturated. Curve A is for Edina clay, B and C are for quartz sand, and D is for Marshall silty clay. Numbers 1 and 2 on Curve D are time points at which potential distributions (Figure 16) were measured. Note the break in the scale of the ordinates.
Thus, up to and including the time represented by the vertical broken line, the model was operating as a pressure-saturated system. Use of the vented stoppers made it a capillary system. At times beyond the broken line a capillary fringe was present in the model. The numerical values underneath each curve and located on the vertical broken line, refer to the time (in hours) after the initial wetting of the material. The dotted line sections of each curve indicate the lapse between the last pressure-saturated conductivity measurement and the first measurement of conductivity in the fringe.

Curve B represents the first run with the quartz sand described earlier, while Curve C is a duplicate run on the same sand. In neither case is there a suggestion of extreme decrease in conductivity when the curves are viewed as a whole. Over the pressure-saturated range Curves B and C agree within about 5 percent. Both show a break, but in C it is much less pronounced than in B.

The moisture determinations, dry-weight basis, are shown in Figure 15. The curves in this figure are not moisture characteristics in the strict sense since the abscissa is in terms of angular displacement around the model. It is seen, though, that only at the top of the arch (halfway around the model) is there a tendency for definite unsaturation. In view of this, it is somewhat surprising that the conductivity curves indicate so little decrease, since the moisture determinations of necessity represent the moisture status after the last conductivity measurements have been taken.
Figure 15. Moisture percentage (dry-weight basis) plotted against the percentage of angular displacement around the semicircular model. As in Figure 14, A is for Edina clay, B and C are for quartz sand, and D is for Marshall silty clay. Note the break in the scale of the ordinates.
Return now to Figure 14 and Curve D for Marshall soil. Here there is a more definite decrease, but still only on the order of 20 percent. However, a glance at Curve D in Figure 15 indicates that the reduction in moisture content was more general around the model than was the case in the sand. It appears, then, that fringe conditions (water saturation under tension) were very poorly approximated in the Marshall soil; hence, one might have expected a greater decrease in conductivity than actually did occur.

For the quartz sand and the Marshall soil the general relationship between capillary fringe conductivity and moisture content seems to be as follows:

1. The conductivity in the fringe is always lower than in the same region when pressure-saturated.
2. The decrease, however, may not be more than 5 to 10 percent unless definite unsaturation occurs.

The Edina soil exhibits a different type of behavior. Referring to Curve A of Figure 14 it is seen that a definite break appears in the conductivity curve as the change to fringe conditions is made. The decrease in conductivity is only about 8 percent, but appears to be real in view of the small variation of the points on both sides of the vertical broken line.

However, reference to Curve A in Figure 15 does not reveal any particular trend toward reduced moisture content. Even at the top of the arch, where such a trend would be most expected and was found in
the other cases, there is no suggestion of a general decrease.

It thus appears that essential saturation is not in itself sufficient to ensure against a drop in conductivity. It may be that only a few large pores are drained as a result of the tension in the fringe. Yet, in view of the disproportionate contribution of the large pores to the flow, the overall conductivity may be somewhat decreased.

It should be pointed out that while the apparatus and procedure for the Edina soil were basically the same as for the Marshall soil and quartz sand, there were some slight modifications. Referring back to Figure 11 note the small brass venting tubes numbered I, II, III, etc. These were 9/16 in. in diameter, with screens across their inner ends to keep them from filling with soil. Before filling the model with Edina soil, the two uppermost venting tubes (III and IV in the diagram) were replaced with very large brass tubes the diameters of which were almost equal to the width of the soil model itself. This was done to ensure that a very substantial part of the uppermost surface of the arch of soil would be opened to the atmosphere. Tubes I, II, V, and VI were left unchanged.

In shifting from pressure-saturated to capillary fringe conditions a slight modification of procedure was also made. As before, the soil column was saturated under pressure by maintaining the inlet and outlet head levels at \( L_1 \) and \( L_0 \) respectively (see Figure 11). Then the head levels were changed to \( L'_1 \) and \( L'_0 \). After four saturated conductivity
readings were taken, the levels \( L_0 \) and \( L_1 \) (Figure 11) were shifted to a level \( L \) halfway between the original positions. The vented stoppers were next inserted, and any free water standing in the peripheral brass venting tubes drained away, raising somewhat the level \( L \) to the level \( L' \). The level was restored to \( L \) and allowed to remain there for several minutes to ensure that all free water had drained. Then the levels were again set at \( L_0 \) and \( L_1 \), and conductivity measurements were begun for the fringe condition. The total elapsed time between the last pressure-saturated measurement and the first fringe measurement was 50 minutes. The measurements of conductivity in the fringe were continued for about 8 hours, after which moisture sampling was performed to determine the moisture distribution around the semicircular column.

The final item of interest in this investigation was the distribution of potential (or head) around the periphery of the model. This was measured, as mentioned previously, by the tensiometers (A, B, ..., G in Figure 11) connected to water piezometers. The heights of the water levels in the piezometers were read with a cathetometer.

A great many head distribution patterns were taken, for both fringe and pressure-saturated cases, but the differences between the patterns were very small; so small that it was impossible to detect any trends. To illustrate the magnitude of the differences involved, two such patterns are shown in Figure 16.

The plot in Figure 16 represents the unused head (or potential) as a function of angular displacement. Over the inlet boundary all of the head is as yet unused (or remains); over the outlet boundary it is
Figure 16. Distribution of potential (or head) around the periphery of the semicircular model. Curves 1 and 2 correspond respectively to time points No. 1 and 2 of Curve D in Figure 14.
all used up. Hence, the total head difference $\Delta \phi$ corresponds to 100 percent.

Curve 1 in Figure 16 was taken at the time point No. 1 of Curve D in Figure 14, and hence represents a distribution pattern for the pressure-saturated case. Thus, Curve 1 in Figure 16 should be linear from the upper left-hand corner of the graph to the lower right-hand corner. It is so, except for small variations no doubt due to differences in packing.

Curve 2 of Figure 16 corresponds in time to No. 2 on Curve D of Figure 14. Curve 2 could be considered as a head distribution pattern for the capillary fringe, except that the moisture content was too greatly reduced. However, this particular pattern was chosen deliberately so as to represent the greatest change in conductivity from the pressure-saturated condition. Clearly it differs but little from Curve 1.

On the basis of these experiments the following conclusions may be stated:

1. The hydraulic conductivity of the capillary fringe as here determined is equal to or slightly less than (by perhaps 10 percent) the hydraulic conductivity of the pressure-saturated region.

2. The existence of moisture saturation as determined by the moisture percentage is not sufficient to guarantee that a slight decrease in hydraulic conductivity will not occur.
3. The head dissipation pattern of the capillary fringe is essentially equivalent to that of the same material when pressure-saturated.
INVESTIGATION II. RECTANGULAR MODEL

In an earlier section, reference was made to (B) of Figure 7 as forming the basis for extended investigation. In this connection it was also stated that the interconnected capillaries of Figure 7(B) provide an increased cross section for the flow IO. In the present investigation this increase in cross section will be studied both theoretically and experimentally, not for idealized capillary tubes but for the interconnected capillaries in porous media.

Choice and Analysis of Model

In Figure 17 are shown side views of three flow models for porous media, all with unit thickness. In (A) is the counterpart of Figure 7(B). The head difference ΔH is relatively small compared to the length of the model L; hence, in the lower parts of the model, the flow lines will be essentially horizontal, and parallel to the dimension L.

The initial measurement of flow q₀ is made with the height of the porous medium h equal to h₀, the height of the inlet level of water. Thus, q₀ represents primarily the pressure-saturated flow. For heights h greater than h₀, corresponding flow values q can be determined. Thus, the contribution of the fringe can be evaluated by comparing q to q₀.

Mathematical solution of McNown and Hsu

Along with the experimental analysis, it was desired to have a suitable mathematical analysis also. In this connection the work of McNown and Hsu (29) proved to be valuable. A diagram of their problem
Figure 17. (A) Experimental model for studying the effect of the capillary fringe in the case of flow which is nearly horizontal ($\Delta H$ small). Initial flow measurement is made with the height of the porous medium $h$ equal to $h_0$.

(B) Case of McNown and Hsu (29).

(C) Experimental model in (B) cut through the line of symmetry $MM'$ and rearranged to superimpose line $II'$ upon $OO'$. 

In all three cases the broken vertical lines represent screens.
appears in (B) of Figure 17.

McNown and Hsu were interested also in a flow problem of rectangular geometry but for a different reason. They studied the case in Figure 17(B), where the porous medium is confined at the top and bottom by impermeable barriers, to determine the effectiveness of a cutoff plate $M'M''$ (of length $d$) in reducing the pressure-saturated flow through a block of porous medium of height $h$ and length $L$. They solved the problem analytically by the method of conformal mapping.

The geometry of McNown and Hsu's problem can be rearranged in the following manner. First of all, since the cutoff $M'M''$ is located in the middle of the length $L$, the flow net will be symmetrical on each side of the line $MM'M$. Thus, the potential will be constant on a line drawn as a continuation of $M'M$. Since the potential is constant over $II'$ and $OO'$, the geometry of flow can be rearranged as in (C) of Figure 17 without changing the solution. In this rearrangement, the lines $II'$ and $OO'$ are superimposed and an impermeable plate $M'M''$ (of length $d$) now appears at each end of the flow medium. So far it has been assumed that the inlet and outlet water levels have remained at $J$ and $N$ respectively.

If, now, the water levels in Figure 17(C) are lowered to $J'$ and $N'$, the resulting arrangement begins to resemble the arrangement in (A) of the same figure. In fact, the plate $M'M''$ at the outlet end could be removed without changing the solution, since water could not flow out of the fringe boundary $M'M''$ (except for a negligibly small surface of
seepage as will be described later) without violating the outflow law of Richards (36).

At the inlet end, however, the circumstances are different. If there the plates M'M'' were removed, the distance from M' to the water surface J' would no longer be equal to d. Hence, the McNown and Hsu solution would no longer hold.

For this reason, the problem represented in Figure 17(A) was attacked directly, using, as did McNown and Hsu, the method of conformal mapping. First, however, the results of McNown and Hsu will be stated but not derived. Secondly, the conformal mapping procedure will be applied directly to the geometry of Figure 17(C) to obtain results equivalent to but more convenient than those of McNown and Hsu; and thirdly, conformal mapping will be applied directly to the geometry of Figure 17(A). In all of the solutions it will be assumed that the hydraulic conductivity is constant throughout the medium.

In using the method of conformal mapping it should be noted that this method can be applied to two-dimensional flow problems provided that Laplace's equation is true for the potential; that is, \( \nabla^2 \phi = 0 \) throughout the medium where \( \phi \) is the potential function. In flow problems this amounts to assuming Darcy's law and the equation of continuity as discussed in the Review of Literature.

The diagrams of the McNown and Hsu solution are shown in Figure 18. The flow pattern in (a) is inverted from its position in Figure 17(B) for mathematical convenience. Following the procedure of Streeter (39),
Figure 18. Conformal transformation planes, Mcnown and Hsu (29); (a) flow pattern; (b) complex potential; (c) auxiliary plane. Dashed lines indicate middle streamlines, schematic in (a) and (c).
the Schwarz–Christoffel theorem is used to transform the interior of a polygon (i.e., the flow pattern) in the $z$-plane into the upper half of the $t$-plane; and likewise for the polygon of complex potential in the $w$-plane.

The Schwarz–Christoffel theorem can be written for the $z:t$ transformation as

$$\frac{dz}{dt} = \frac{M}{(t - a_1)^{\alpha_1/\pi} (t - a_2)^{\alpha_2/\pi} \cdots} \quad (23)$$

where $M$ is a constant of proportionality, $\alpha_1$ is the exterior angle at one of the vertices of the polygon in the $z$-plane, and $a_1$ is the corresponding point on the real axis of the $t$-plane. The formula can be written in similar fashion for the $w:t$ transformation with $w$ replacing $z$. However, a different constant of proportionality must be used, and the $\alpha$'s must be correctly interpreted in the $w$-plane.

Omitting the derivations, the results of McNown and Hsu are

$$L \frac{h}{n} = \frac{2K[\sqrt{(m^2 - 1)/(n^2 - 1)}]}{K[\sqrt{(n^2 - m^2)/(n^2 - 1)}]} \quad (24)$$

$$d \frac{h}{n} = \frac{F[\sin^{-1} \frac{1}{m}, \sqrt{(n^2 - m^2)/(n^2 - 1)}]}{K[\sqrt{(n^2 - m^2)/(n^2 - 1)}]} \quad (25)$$

$$q \frac{q_0}{2K(m/n)} = \frac{L}{h} \quad (26)$$
where \( q \) is the flow for a cutoff of length \( d \), and \( q_0^\prime \) is the flow for no cutoff. Elliptic integrals of the first kind, complete and incomplete, are designated respectively by \( K \) and \( F \). The length of the soil column is \( L \) and its height is \( h \). The hydraulic conductivity \( k \) cancels out and does not appear explicitly.

The quantities \( m \) and \( n \) are real numbers derived from the \( t \)-plane, which satisfy \( n > m > 1 \), and serve as parameters in the implicit relation between \( q/q_0^\prime \), \( L/h \), and \( d/h \) as specified by Equations 24, 25, and 26. McNown and Hsu have plotted \( q/q_0^\prime \) as functions of \( L/h \) and \( d/h \) for values in the practical range.

The methods of conformal mapping next will be applied directly to the flow problem illustrated by Figure 17(C), with the water levels at \( J \) and \( N \). The transformation diagram appears in Figure 19, since the impermeable boundaries \( FG \) and \( DC \) in (a) of Figure 19 are both of length \( d \), this problem will hereinafter be referred to as the balanced case.

**Mathematical solution, balanced case**

Applying Equation 23 to (a) and (c) of Figure 19 and integrating from \( z_1 \) to \( z_2 \), and \( t_1 \) to \( t_2 \) there results

\[
\begin{align*}
  z_2 - z_1 &= \int_{t_1}^{t_2} \frac{dt}{\sqrt{(t^2 - 1)(t^2 - n^2)}}.
\end{align*}
\]

(27)
Figure 19. Conformal transformation planes, balanced case; (a) flow pattern; (b) complex potential; (c) auxiliary plane. Dashed lines indicate middle streamlines, schematic in (a) and (c). To avoid confusion with elliptic integral K, the hydraulic conductivity is indicated by K.
(a) $z$-plane

$\psi = q$

$\psi = q/2$

$\phi = \frac{K \Delta H}{2}$

$\phi = 0$

$\psi = 0$

(b) $w$-plane

(c) $t$-plane
For a similar procedure applied to (b) and (c) of Figure 19, using a constant N, it is found that

\[ w_2 - w_1 = N \int_{t_1}^{t_2} \frac{dt}{\sqrt{(t^2 - 1)(t^2 - m^2)}} \]  

(28)

Applying Equation 27 to the line segment OB in Figure 19 (a) and (c) there results

\[ L = 2M \int_{0}^{1} \frac{dt}{\sqrt{(t^2 - 1)(t^2 - n^2)}} \]  

(29)

A similar application to the segment BD yields, since \( i = \sqrt{-1} \),

\[ h = M \int_{1}^{n} \frac{dt}{\sqrt{(n^2 - t^2)(t^2 - 1)}} \]  

(30)

and similarly for CD it is found that, with \( i = \sqrt{-1} \),

\[ d = M \int_{m}^{n} \frac{dt}{\sqrt{(1 - t^2)(t^2 - n^2)}} \]  

(31)

Applying Equation 28 to the line segment OB in Figure 19 (b) and (c), it is found that
\[
\bar{K} \Delta H = 2N \int_0^1 \frac{dt}{\sqrt{(t^2 - 1)(t^2 - m^2)}} ,
\]  

(32)

where \( \bar{K} \) = hydraulic conductivity and \( \Delta H \) = total head difference, the symbol \( \bar{K} \) being used to avoid confusion with the elliptic integral \( K \).

Similarly, for the line segment BC, with \( i = \sqrt{-1} \), there results

\[
q = N \int_1^m \frac{dt}{\sqrt{(t^2 - 1)(m^2 - t^2)}} ,
\]  

(33)

where \( q \) is the flow.

All of the integrals in Equations 29 to 33 can be evaluated from Peirce's Tables (35). Two useful identities in this connection are

\[
\text{sn}^{-1}(x,k) = F(\text{sn}^{-1} x,k) ,
\]  

(34)

and

\[
\text{sn}^{-1}(1,k) = F(\pi/2,k) = K(k) .
\]  

(35)

Specifically, the integrals in Equation 29 and 32 can be evaluated by No. 536 in Peirce, and Equations 30, 31, and 33, by No. 540. Using Equations 34 and 35 where needed, and taking ratios to eliminate the \( M \)'s, there results for the balanced case

\[
\frac{L}{h} = \frac{2K(1/n)}{K(\sqrt{n^2 - 1}/n^2)} ,
\]  

(36)
\[
\frac{d}{h} = \frac{F(\sin^{-1} \sqrt{(n^2 - m^2)/(n^2 - 1)}, \sqrt{(n^2 - 1)/n^2})}{K(\sqrt{n^2 - 1}/n^2)}. \tag{37}
\]

Referring to Figure 17(C), define \( h_0 \) to be the length \( n^m \). It is seen that \( h \) is always greater than \( h_0 \), and when \( d = 0 \), then \( h = h_0 \). Further, define \( q_0 \) to be the flow when \( h = h_0 \) and \( d = 0 \). From Darcy's law (Equation 1), the flow for this case is

\[
q_0 = \frac{\bar{K} \Delta H h_0}{L}, \tag{38}
\]

since the model is of unit thickness. After the integrals in Equations 32 and 33 are evaluated, and \( N \) is removed by taking ratios, the result can be combined with Equation 38 to yield

\[
\frac{q}{q_0} = \frac{K(\sqrt{(m^2 - 1)/m^2}) L}{2K(1/m) h_0}. \tag{39}
\]

As before, \( K \) and \( F \) respectively are complete and incomplete elliptic integrals of the first kind. Again \( n \) and \( m \) are the parameters in Equations 36, 37, and 39, and satisfy \( n > m > 1 \). It also should be emphasized that they are different from the parameters in the original McNown and Hsu solution. Another point of difference between the solutions is the choice of reference flow, since \( q_0 \) and \( q_0' \) are not the same. However, when the difference in reference flow is taken into account, the solution for the balanced case (Equations 36, 37, and 39)
is identical to that of McNown and Hsu (Equations 24, 25, and 26); that is, for the same numerical values of $L/h$ and $d/h$, the values of $q/q_0$ are identical when calculated numerically. This, of course, is as it should be and merely verifies the correctness of the rearrangement by symmetry demonstrated in (B) and (C) of Figure 17.

There is an advantage in using the solution for the balanced case as compared to the one of McNown and Hsu. In the McNown and Hsu solution, each of the variables $L/h$, $d/h$, and $q/q_0$ are functions of both parameters $m$ and $n$. In the solution of the balanced case only $d/h$ is a function of both parameters, while $L/h$ and $q/q_0$ respectively are functions only of $n$ and $m$. This makes it easier to calculate numerical values for certain specific cases of the general solution.

The next analysis, which will correspond to (A) of Figure 17, is based upon the diagrams of Figure 20. In the strict sense the solution which will be obtained applies to (C) of Figure 17 (water levels at J and N) where the length of the plate at the left of the diagram is no longer equal to $d$, but a distance $d_1$ as shown in (a) of Figure 20. The length of the plate at the outlet end is $d_2$, and because $d_1$ is less than $d_2$, the flow pattern is no longer symmetrical. It therefore will be referred to as the unbalanced case.

Mathematical solution, unbalanced case

The inequality $d_2 > d_1$ introduces several basic changes into the diagrams of Figure 20 as compared to Figure 19. First, the mid-point of length, $E$ in (a), no longer corresponds to the mid-point on the
Figure 20. Conformal transformation planes, unbalanced case; (a) flow pattern; (b) complex potential; (c) auxiliary plane. Dashed lines indicate middle streamlines, schematic in (a) and (c).
(a) z-plane

(b) w-plane

(c) t-plane
complex potential diagram of (b). Also, instead of two parameters m and n, another one, j, is required. The parameter p is introduced at D' and F' merely to permit calculation of the distribution of potential along the line FF'ED'D in (a). This is also the reason for the introduction of $\tilde{R} \triangle u_1$, $\tilde{R} \triangle u_2$, $v_1$, and $v_2$, symbols shown in the figure. The parameters must always satisfy $p > n > m > j > 1$.

The Schwarz-Christoffel transformation corresponding to Equation 27 remains unchanged; the one corresponding to Equation 28 becomes

$$w_2 - w_1 = N \int_{t_1}^{t_2} \frac{dt}{\sqrt{(t^2 - 1)(t + m)(t - j)}}.$$  \hspace{1cm} (40)

Applying Equation 27 to the line segment OB in (a) and (c) of Figure 20, it is found that

$$L = 2M \int_0^1 \frac{dt}{\sqrt{(t^2 - 1)(t^2 - n^2)}}.$$  \hspace{1cm} (41)

For the same procedure on line segment BD, with $i = \sqrt{-1}$, there results

$$h = M \int_1^n \frac{dt}{\sqrt{(n^2 - t^2)(t^2 - 1)}}.$$  \hspace{1cm} (42)
For segment GF, with \( i = \sqrt{-1} \), there results

\[
d_1 = M \int_m^n \frac{dt}{\sqrt{(1 - t^2)(t^2 - n^2)}} ,
\]

(43)

and, finally, for segment CD, with \( i = \sqrt{-1} \),

\[
d_2 = M \int_j^n \frac{dt}{\sqrt{(1 - t^2)(t^2 - n^2)}}.
\]

(44)

Using Equation 40 on line segment AB in (b) and (c) of Figure 20 yields

\[
\bar{K} \Delta H = N \int_{-1}^1 \frac{dt}{\sqrt{(t^2 - 1)(t + m)(t - j)}} ,
\]

(45)

and for segment AG, with \( i = \sqrt{-1} \), the result is

\[
q = N \int_1^m \frac{dt}{\sqrt{(t^2 - 1)(t - m)(t + j)}} .
\]

(46)

The integral in Equation 41 is evaluated by No. 536 in Peirce, while those in Equations 42, 43, and 44 are solved by No. 540. The one in Equation 45 is evaluated by No. 553, and the one in Equation 46 by
No. 551. With the help of the identities (Equations 34 and 35) there then results

\[ L = \frac{2M}{n} K(1/n), \]  

(47)

\[ h = \frac{M}{n} K \left( \sqrt{\frac{n^2 - 1}{n^2}} \right), \]  

(48)

\[ d_1 = \frac{M}{n} F \left( \sin^{-1} \sqrt{\frac{n^2 - m^2}{n^2 - 1}}, \sqrt{\frac{n^2 - 1}{n^2}} \right), \]  

(49)

\[ d_2 = \frac{M}{n} F \left( \sin^{-1} \sqrt{\frac{n^2 - j^2}{n^2 - 1}}, \sqrt{\frac{n^2 - 1}{n^2}} \right), \]  

(50)

\[ K \Delta H = \frac{2N}{\sqrt{(m + 1)(j + 1)}} K \left( \sqrt{\frac{2(m + j)}{(m + 1)(j + 1)}} \right), \]  

(51)

and

\[ q = \frac{2N}{\sqrt{(m + 1)(j + 1)}} K \left( \sqrt{\frac{m - 1}{(m + 1)(j + 1)}} \right). \]  

(52)

Forming the ratios \( L/h, d_1/h, \) and \( d_2/h \) there results, using Equations 47, 48, 49, and 50,

\[ \frac{L}{h} = \frac{2K(1/n)}{K(\sqrt{(n^2 - 1)/n^2})}, \]  

(53)
\[ d_1 = \frac{F(\sin^{-1}\sqrt{(n^2 - m^2)/(n^2 - 1)}, \sqrt{(n^2 - 1)/n^2})}{K(\sqrt{(n^2 - 1)/n^2})}, \]  
(54)

and

\[ d_2 = \frac{F(\sin^{-1}\sqrt{(n^2 - j^2)/(n^2 - 1)}, \sqrt{(n^2 - 1)/n^2})}{K(\sqrt{(n^2 - 1)/n^2})}. \]  
(55)

By defining the reference flow \( q_o = \frac{K \Delta H h_o}{L} \), and taking ratios between Equations 51 and 52, there results

\[ \frac{q}{q_o} = \frac{K\sqrt{(m-1)(j-1)/(m+1)(j+1))}}{K(\sqrt{2(m+j)/(m+1)(j+1))}} \frac{L}{h_o}. \]  
(56)

It is thus seen that the solution for the unbalanced case involves one more equation than does the solution for the balanced case. It is also seen that the expressions for \( L/h \) and \( d_1/h \) are very similar to those obtained for \( L/h \) and \( d/h \) in the solution of the balanced case. Also, if \( j \) is set equal to \( m \), the expression for \( d_2 \) becomes equal to \( d_1 \) as it should. However, the equivalence of Equations 56 and 39 is not obvious analytically for \( j = m \), but for the same values of \( L/h \) and \( d/h \) both formulas for \( q/q_o \) (\( j = m \) in Equation 56) will give the same numerical answer. Thus, the two solutions do merge for \( j = m \), but the numerical values of \( m \) and \( n \) for the unbalanced case will in general
be different from the values of $m$ and $n$ for the balanced case.

The calculation of the potential distribution along the boundary $FF'DD'D$ is also of interest. To do this it is necessary to choose some value of $p > n$. Equation 40 is then applied to the line segment $GF'$ in (b) and (c) of Figure 20. The result is

$$ K\Delta u_1 = N \int_m^P \frac{dt}{\sqrt{(t^2 - 1)(t - m)(t + j)}}. $$

(57)

Using No. 550 in Peirce's table, and combining the result with Equation 51, one finds

$$ \frac{\Delta u_1}{\Delta H} = \frac{F(\sin^{-1}\sqrt{\frac{(j + 1)(p - m)}{(m + j)(p - 1)},\sqrt{\frac{2(m + j)}{2(m + j)/(m + 1)(j + 1)}})}}{K(\sqrt{2(m + j)/(m + 1)(j + 1)})}. $$

(58)

Using Equation 27 on the line segment $FF'$ in (a) and (c) of Figure 20 there results

$$ v_1 = M \int_n^P \frac{dt}{\sqrt{(t^2 - 1)(t^2 - n^2)}}. $$

(59)

This integral also can be evaluated using No. 550 in Peirce. Combining the evaluated result with Equation 47 leads to
The quotient $\Delta u_1/\Delta H$ represents the fraction of the head $\Delta H$ which is used up at the point $F^*$. The distance of $F^*$ from $F$ is given in terms of the fraction $v_1/L$. Hence, for a given geometrical form specified by $n$, $m$, and $j$, it is possible to specify the fraction of head dissipated ($\Delta u_1/\Delta H$) for any point $F^*$ located anywhere along the boundary from $F$ to $E$. The fraction of the head undissipated (or the fraction of head remaining) is given by $1-\Delta u_1/\Delta H$.

A similar analysis can be performed for the boundary $DE$ using $D'$ in the same capacity as $F^*$. The result for $\Delta u_2/\Delta H$, which now is the fraction of head undissipated, is identical to Equation 58 except that the positions of $m$ and $j$ are reversed. The expression for $v_2/L$ is found to be identical to that for $v_1/L$, indeed a reasonable result.

A final aspect involves the form of the solution when the line $FF'ED'D$ is moved to infinity; that is, $h/L$ becomes very large. It is reasonable to assume that for a constant head difference $\Delta H$, some limiting value of the flow ratio $q/q_0$ could be found. However, the consequences of allowing $h/L$ to become infinite in Equations 53 to 56 are not readily apparent. Hence it is desirable to return to the Schwarz-Christoffel theorem and consider the effect of having line $FF'$ $ED'D$ at infinity. For this case the integrated form of the theorem for the zit transformation is

$$v_1 = \frac{n}{L} \frac{F(\sin^{-1}\left(\frac{n+1}{2n(p-1)}\right)}{\sqrt{\frac{2n(p-1)}{n+1}}}, \quad K(l/n) \right) \tag{60}$$
By applying Equation 61 to the proper line segments of Figure 20, evaluations of \( h_0, h_1, \) and \( L \) can be obtained. The final results are

\[
h_0/L = (1/\pi) \cosh^{-1} m_1 ,
\]

and

\[
h_1/L = (1/\pi) \cosh^{-1} j_1 ,
\]

where \( m_1 \) and \( j_1 \) are the limiting values of \( m \) and \( j \) for the case of \( h/L \) very large.

Even though \( h/L \) becomes very large, the complex potential diagram of Figure 20(b) remains unchanged. Hence, to calculate the limiting value of the flow ratio \( q/q_0 \), it is necessary only to substitute the values of \( m_1 \) and \( j_1 \) from Equations 62 and 63 into Equation 56.

**Graphic illustration of the solution for the unbalanced case**

The mathematical solution represented by Equations 53, 54, 55, and 56 is not explicit; that is, the dependent variable \( q/q_0 \) is related to the independent variables \( L/h \), \( d_1/h \), and \( d_2/h \) implicitly through the parameters \( j, m, \) and \( n \). Hence, the behavior of \( q/q_0 \) as a function of the three independent variables can be visualized most easily by a properly constructed graph.
The elliptic integrals $K(k)$ and $F(\phi, k)$ of Equations 53 to 56 are tabulated in Pearson's table (33) which is a reproduction of an earlier table prepared by Legendre. Dwight (16) has prepared a useful table for $K$ when $k$ is near to unity. Only recently Kennedy (23) has presented some new approximation formulas for $F$ and $K$ which may be useful for certain critical values of the arguments $\phi$ and $k$. Note that $\phi$ and $k$ are not to be confused with the symbols for potential and hydraulic conductivity used earlier.

In calculating the numerical results it is useful to note a certain characteristic form in Equations 53 and 56. Now $K'(k')$ is defined to be $K(\sqrt{1-k^2})$, see Peirce (35). If in Equation 53, $k$ is set equal to the argument $1/n$ of the numerator $K$, then the argument of the denominator turns out to be $k'$. A similar relation holds in Equation 56.

Since the ultimate objective is the analysis of the model in (a) of Figure 17, the variation of $q/q_0$ will be brought about in the following manner. The initial height of the conducting medium shall be $h = h_0$ (Figure 20(a)). The length $L$ remains constant, and the difference in head $\Delta H$ remains constant. However, $h$ increases through values greater than $h_0$, and as it does so, $d_1$ becomes greater than zero and $d_2$ also increases, but $\Delta d = d_2 - d_1$ remains constant. Hence, at the initial height when $h = h_0$, $d_2 = \Delta d$ since $d_1$ is zero when $h = h_0$. The resulting curves of flow ratio $q/q_0$ versus $h/h_0$ are shown in Figure 21 for different combinations of $h_0$ and $\Delta d$. The limiting values of the flow ratios are calculated by the aid of Equations 61 and 62. The calculations for
Curve A of Figure 21 are shown in Table 9 of the Appendix.

In Figure 21, the straight line D represents the relationship that would hold if the flow increased proportionately to the height. However, reference to (a) of Figure 20 shows that as \( h, d_1, \) and \( d_2 \) increase in the manner here specified, the flow lines through the upper part of the model will be bent very sharply. Thus, the leveling off of \( q/q_0 \) as shown by curves A, B, B', C, and C' of Figure 21 is caused by the constricting effect of the geometry. When \( h_0 \) is small relative to the length, the added height of conducting medium is almost as effective in providing a flow path as is the original column of medium. As the height increases the flow is "pinched off" by geometrical limitations.

The efficiency of the upper layers in contributing to the flow is also reduced as the imbalance, indicated by \( \Delta d \), is increased. In the sequence of curves A, B, C the imbalance is always equal to \( h_0/2 \), so as \( h_0 \) increases, the imbalance increases. The A, B', C' sequence of curves possesses a constant imbalance. By comparing these two sequences of curves it is seen that the contribution of the upper layers, relative to the initial layer, is much reduced as the imbalance increases.

In studying Figure 21 it should be remembered that \( q/q_0 \) is a relative measure of the flow increase. If \( h_0 \) is large, then the reference flow will be large. If \( h_0 \) is small, the reverse is true. Hence, in such a series as A, B', C' much of the spread between the curves is due to the fact that the reference flow is not the same for all curves.
Figure 21. Theoretical curves of flow ratio against height ratio for the unbalanced case. Numerical values above the curves along the right-hand ordinate are limiting values as $h/h_0$ increases without bound. Symbols are as on Figure 20(a), $\Delta d$ being $d_2 - d_1$. Values of $h/h_0$ less than unity have no meaning.
The curves of Figure 21 have important implications for the capillary fringe. If, as in Figure 17(A), the water levels are lowered below the top of the medium, the part of the medium above the phreatic line (which connects the water levels) will be in the capillary fringe if the pore sizes are small enough. The driving head $\Delta H$ will equal $\Delta d$. If, as indicated in Investigation I, the hydraulic conductivity of the fringe is almost equal to that of the pressure-saturated region, then the curves of Figure 21 will be very nearly correct for the capillary fringe, provided that $h$ does not exceed the height of the fringe. For Curve A the maximum increase due to the fringe is approximately 170 percent, a contribution which cannot be ignored. However, in curve C the relative increase is a maximum of 13 percent, and illustrates rather well that as the flow region below the phreatic surface is increased relative to the fringe, the fringe contribution becomes less important.

Application of Curve C in Figure 21 to Figure 17(A) may, however, be inaccurate from another standpoint. Since $\Delta H \approx \Delta d$ by the very nature of things, the potential difference across the section is increased as the imbalance increases. This increases the amount of flow, and if $\Delta H$ increases sufficiently, a surface of seepage will form at the outlet boundary above the outlet water level. That is, the phreatic line no longer intersects the outlet boundary of the medium at the outlet water level but at some point above it, as indicated in Figure 3. Existence of a surface of seepage will, of course, invalidate the
theoretical solutions represented in Figure 21, and the use of this mathematical solution must be confined to cases in which the ratio \( \Delta H/L \) is small enough that an appreciable surface of seepage cannot form.

Muskat (30) has analyzed the effect of surfaces of seepage on the seepage of water through rectangular dams which resemble the model in Figure 17(A). However, while considering the surface of seepage, his analysis was carried through mainly for the case in which the inlet water level is at or near the top of the porous medium. At any rate, the solution was carried through on the assumption that the fringe did not contribute to the flow, that is, it was assumed that the limiting streamline coincided with the phreatic line. Therefore, his solution is not applicable here.
Experimental Methods

The model and associated devices

To investigate the capillary fringe experimentally, a plexiglass model was constructed along the general lines of Figure 17(A). A semi-detailed drawing of the model (not to scale) is shown in Figure 22. Only recently Luthin and Day (27) have constructed a model similar in principle to the one described here, but their technique of operation differs in several important respects.

The model was constructed to have a water reservoir at each end, in which water could be maintained at any desired level. In every instance except one, the porous medium was separated from the reservoirs by screens as shown in the figure. In the case of the single exception (to be discussed later), an impermeable plate replaced the screen at the outlet end of the model. At the bottom of this plate was a rectangular opening, covered with screen, and with a height equal to 1/20 the length of the model. All of the flow would in this case be forced to move through the rectangular opening; no surface of seepage could occur.

Perforated tubes, about 1/4 in. in diameter and made of 100-mesh brass screen, were placed vertically in the model at equally spaced horizontal intervals. The upper ends of these tubes were open to the atmosphere, while the lower ends were imbedded in the base of the model. The lower end of each tube was connected to an individual outlet
Figure 22. Detailed drawing (not to scale) of rectangular flow model. At the far left of the diagram is shown an end view. The top view, upper center, does not show the angle braces and tube support, for reasons of simplicity. The sectional views particularly illustrate the detail at the bottoms of the perforated tubes used for measuring the position of the phreatic surface.
INLET RESERVOIR

C.D.AND E, NOT SHOWN ON THIS VIEW

BRASS TUBE

3 3/4 IN.

TO PIEZOMETERS

SECTION A-A'

TO OUTLET PIEZOMETER SECTION A-A'

TO PIEZOMETERS

SECTION B-B'

FLOW INLET

FLOW OUTLET

PERFORATED TUBES

SOIL CAVITY

C-ANGLE BRACE, FRONT

D-ANGLE BRACE, REAR

E-TUBE SUPPORT

STOPPERS A'-SCREEN

TENSIOMETERS

D—TENSIOMETERS

TOPPERS7 A'
tube through the base of the model, and from thence to a glass tube hereinafter referred to as a piezometer. The detail of the outlets in the base of the model is shown in sections BB' and AA' in Figure 22. By measuring the water levels in the piezometers, it was possible to record the position of the phreatic surface through the model.

The tops of the perforated tubes were held in place by the tube support E shown in Figure 22. On one side of this support, provision was made to fasten tensiometers alongside the perforated tubes as shown in the figure. Each tensiometer consisted of a sintered glass disc fused across the end of a glass tube. When the model was completely filled with soil or other porous medium, the tensiometers could be mounted on the tube support and placed in contact with the upper boundary of soil. Each tensiometer was connected to a piezometer, so by measuring the height of water in the piezometer, the head at a point on the upper boundary could be determined.

Along the middle of the front panel of the model, a staggered row of holes was located. The holes were evenly spaced in the vertical direction, and in such a manner that the shortest vertical spacing between subsequent holes of the staggered pattern was 3 cm., center to center. The diameters of the holes also were of the order of 3 cm., and this was the reason for staggering the pattern. If the holes had been placed one directly above the other, a definite line of weakness would have been introduced into the front panel.
These holes were closed with stoppers during the time that the model was being used for flow measurements. Then, after all the flow data were taken, the stoppers were removed and samples of the porous medium were removed with sharpened brass sampling cylinders. Because of the difficulties encountered in a similar sampling scheme for the semicircular model, a small hole was drilled into the back panel opposite each large hole. Each small hole was threaded to take a screw with a rubber gasket. During the sampling of the porous material, these screws were removed, thus permitting the small holes to release any vacuum formed in withdrawing the sampling tube and sample from the model. This technique worked well, and with a little care it was usually possible to remove a sampling tube with the contents intact. Thus, it was possible to determine volume weight and moisture percentage on each sample.

An important part of the model which is not shown in Figure 22 is a special two-piece cover made of sheet plexiglass to fit over the top of the model. Without a cover, excessive evaporation of moisture might have occurred from both the soil and the free water surfaces in the reservoirs. However, the cover was not completely air-tight.

The system of external bracing also is not shown in Figure 22. A triangular system of angle-iron braces was constructed both on the back side and front side of the model. The front braces are visible in the photographs of Figure 24.
At the bottom of each reservoir are two external water connections. The larger of these connections serves as either an inlet or an outlet for the flow; the smaller serves either the inlet or outlet piezometer. It is from the inlet and outlet piezometers that the total head loss is determined; hence, the dual arrangement for the outlets is necessary to prevent error in the piezometer readings due to dynamic loss through the inlet and outlet openings.

The water levels in the reservoirs of the model are maintained by apparatus shown in Figure 23. The water supply reservoirs, which can be raised and lowered, are located at the left of the diagram and operate as Mariotte bottles. Normally only the larger water supply reservoir is used, but when its supply of water is nearly exhausted, the smaller supply reservoir is thrown into the line by means of the two-way stopcock (see figure), while the larger vessel is refilled. Hence, the water levels in the flow model are not disturbed. If it is desired to measure the amount of water entering the model, the smaller supply reservoir may be replaced by the modified buret at the extreme left of the diagram. Normally, however, the discharge from the model is collected by the buret at the far right of the diagram. By recording the time required to fill the buret a certain amount, the flow rate can be determined.

One difficulty that had been encountered in the experiments with the semicircular model was the accurate measurement of the head difference across the model. The apparatus used for maintaining the water
Figure 23. Diagrammatic representation of the devices used to maintain water levels in the model. In the diagram, H represents the time-average head difference.
A, A', A" - SCREW CLAMPS
B, B' - CAPILLARY TUBES
C, C', C" - AIR INLET TUBES
D, D' - PINCH CLAMPS
E, E' - OUTLET AND INLET PIEZOMETERS
H - HEAD DIFFERENCE
levels was very similar to that in Figure 23 except for the absence of the screw clamps A, A', and A" as well as the capillaries B and B'.

The water level in the piezometer E would fluctuate as bubbles were pulled from the air inlet tube of the supply reservoir. The water level in the piezometer E' would fluctuate as droplets were forced from the end of the discharge tube. Hence, in the measurement of H in Figure 23 there was considerable error. A similar difficulty had been encountered previously by Kirkham (24, p. 97).

The fluctuation was very materially decreased by placing capillary tubes B and B' into the piezometer lines. Due to the flow resistance of the capillary tubes, fluctuations in the piezometers were essentially choked out and the measurement of H, a time-average head difference, could be made accurately.

Even so, for very slow movement of water, fluctuation again became troublesome, particularly in the outlet piezometer. This time the fluctuation was effectively removed by placing a screw clamp A" on the outlet line and adjusting it until the water droplets fell steadily. The resulting constriction of the outlet line apparently caused enough flow resistance to counteract any backlash from surface tension forces at the outlet nozzle. A similar technique, using clamps A and A', worked very well in the inlet lines. These constriction techniques, however, are applicable only to steady state flow.

Ivie and Richards (21) have developed a flow meter for very slow rates of flow. However, measurement of the head difference in their equipment is still only approximate because of surface tension effects.
Figure 24. Two photographic views of the rectangular model in its experimental arrangement. Note the pattern of head dissipation in the battery of piezometers located at the extreme right of the upper photograph. In both photographs the flow in the model is from right to left.
Photographs of the experimental setup are shown in Figure 24. The lower photograph shows a side view of the discharge buret, model, water supply reservoirs, piezometers, and two-way stopcock. Note the white markings on the side of the model. These were white gummed labels placed to mark off equal increments of height. The vertical distance between the tops of these labels was equal to one-tenth the length of the soil column in the model. At the time the picture was taken the level of soil in the model was at the top of the 7th increment of height. Note also the capillary tube in the outlet piezometer line near the lower right-hand corner of the model.

The upper photograph shows an end view. The water levels in the battery of piezometer tubes are visible at the extreme right in the picture. The inlet and outlet piezometers are located at the immediate left of the battery of piezometers. The water levels in all of the piezometers are read with a cathetometer not shown in the photographs.

Operation of the model

The basic principles in operating the model can best be understood by referring to Figure 17(A). First of all the model is filled with porous medium to the height \( h_0 \). Water is introduced into the end reservoirs of the model slowly so that the porous medium can wet by capillarity, since it is not feasible to displace the air with carbon dioxide. After the water levels in both end reservoirs have risen to a height slightly greater than \( h_0 \), so that the porous medium is everywhere saturated under pressure, the inlet level is restored to \( h_0 \).
(or very slightly below) and the outlet level is reduced so that a head difference $\triangle H$ is maintained. The experimental evaluation of $\triangle H$ is the time-average head difference $H$ shown in Figure 23. When the flow rate has become constant, the values of the time-average head and the flow are measured. A pattern of head dissipation (phreatic line) through the model is also taken, as well as the temperature of the water to correct for viscosity. It is from this set of measurements that a corrected reference flow $q_o$ is eventually calculated.

After the steady state flow for the column of height $h_o$ has been measured, more medium is added, wetted by capillarity, saturated under pressure, and then the levels are restored to their former positions, or very close thereto. The flow, the time-average head difference, the temperature, and the phreatic line are again measured for the steady state. The resulting flow now includes flow in the fringe. The process is repeated until the top of the model is reached. If desired, the tensiometers can then be stationed along the upper surface of the medium and the head dissipation pattern determined for the upper boundary. After the last flow measurement is taken, the model is sampled for moisture as described earlier. A complete set of data for flow, phreatic line, and moisture, will be hereinafter referred to as a run. Only one set of tensiometer readings on the upper boundary was taken during the whole course of experimentation.
For the first run, the outlet boundary of the porous medium was separated from the outlet reservoir by a brass plate with only a small rectangular screened opening at the bottom, the height of which was equal to $L/20$. This was done to make sure that there would be no surface of seepage on the outlet boundary above the outlet water level. The plate was not used in subsequent experiments for reasons which will now be explained.

If only an initial measurement can be taken to specify $q_0$, it becomes questionable whether this is a valid reference flow since a complete run may require a week's time. In spite of one's best efforts to prevent it, the hydraulic conductivity may change somewhat with time, especially if more material is piled upon the wetted medium in the bottom of the model, possibly resulting in compaction. Hence, it was deemed desirable to check on the pressure-saturated flow at regular intervals. This could be done as follows. When the medium of height $h$ has been pressure-saturated by raising the water levels slightly above $h$, (Figure 17(A)) one then restores the inlet level to $h$ and the outlet level to a distance $\Delta H$ below $h$, where $\Delta H$ is of the same order of magnitude as previously, but not at the same location as before. The resulting flow can be corrected to the flow through the column of height $h_0$ by multiplying by $h_0/h$. However, this latter procedure is correct only if there is no impermeable plate over the outlet boundary.

Now the proper geometrical correction for the plate could be made, but there is still another objection to it. The presence of the plate with only a small opening at the bottom will give rise to appreciable
vertical flow when \( h \) approaches \( L \) in magnitude. Thus, the water would tend to flow down through the perforated tubes, and this would impose an unknown amount of error on the flow. So, the plate was replaced by a screen in all following experiments and \( \Delta H \) never exceeded \( L/10 \); hence, the surface of seepage was assumed to be negligible. This was borne out later by the shape of the phreatic line. In no case was there clearcut evidence for the existence of an appreciable surface of seepage.

For all runs after the first, complete screens 2 ft. long were used to contain the ends of the soil column as shown in Figure 22. This made it possible to determine a reference value of flow based upon the whole soil mass, rather than upon only the lower part of it. Thus, the difference in time between a determination of the reference flow and a flow value with the water levels properly lowered, could be greatly reduced.

The model and the mathematical solution

In deriving the mathematical solution represented by Equations 53, 54, 55, and 56, it has been assumed that \( \bar{K} \) and \( \Delta H \) remain constant. Experimentally, however, it is easier to measure \( \Delta H \) (or its equivalent, the time-average head \( H \)) and correct the flow accordingly. Furthermore, it is advisable also to correct the flow for viscosity. All experiments were conducted in a constant-temperature room kept at 25° C.; however, since the temperature did fluctuate about 1° C., viscosity corrections were applied to all flow measurements.
Following the procedure used in developing Equation 22, \( \bar{K} = \bar{K}_{20} \frac{\eta_{20}}{\eta} \), where \( \bar{K}_{20} \) is independent of viscosity, \( \eta_{20} \) is the viscosity of water at 20° C, and \( \eta \) is the viscosity at the experimentally recorded temperature. Using this relationship to rewrite the equation for the reference flow \( q_0 \) it is found that

\[
q_0 = \bar{K}_{20} \frac{\eta_{20}}{\eta} \frac{\Delta H_0 h_0}{L},
\]

where \( \eta_0 \) is the viscosity of the water when \( q_0 \) is determined, and \( \Delta H_0 \) is the head difference. Substituting \( \bar{K} = K_{20} \frac{\eta_{20}}{\eta} \) into Equation 51, and then combining Equations 51, 52, and 64 there results finally

\[
\left( \frac{q}{q_0} \right)_c = \frac{K \left( \frac{m - 1}{j - 1} / (m + 1)(j + 1) \right) L}{h_0}
\]

where \( \left( \frac{q}{q_0} \right)_c \) is the corrected flow ratio and is defined by

\[
\left( \frac{q}{q_0} \right)_c = \frac{q \eta_0 / \eta_{20} \Delta H}{q_0 \eta_0 / \eta_{20} \Delta H_0}
\]

All measurements of flow were corrected as indicated by Equation 66.

Finally, a special geometrical correction for \( q_0 \), as determined from a column of height \( h_0 \) or \( h \), needs to be applied. As an example, refer to Figure 17(A) and assume that the height of medium is equal to
The flow $q_0$ is defined to be the pressure-saturated flow through the column of height $h_0$. However, to measure the flow experimentally it is necessary to maintain a head difference $\Delta H$, and this of necessity means that the level of water in the outlet reservoir must be lower than $h_0$ by the amount of $\Delta H$. Hence, there exists the equivalent of an impermeable plate of length $\Delta H$ at the outlet boundary. Thus, the flow at the outlet boundary is forced to move through a height $h_0 - \Delta H = h_1$ (see Figure 20) instead of a height $h_0$.

If the McNown and Hsu model of Figure 17(B) is cut on the line $M'M'M$ and only the left half of it considered, it is seen that this is the case of the previous paragraph. Hence the McNown and Hsu solution can be used to correct the measured value of flow to obtain a more correct value for $q_0$. This correction was always applied whether $q_0$ was being determined from the flow through a column of height $h_0$ or $h$. In no case did the correction amount to more than 2 percent.

When $q_0$ was determined from a column of height $h$, the complete method of obtaining $q_0$ was:

(a) Measure the flow and correct for viscosity and head.

(b) Correct the result of (a) by the solution of McNown and Hsu.

(c) Multiply the result of (b) by $h_0/h$ to obtain the value of the experimental $q_0$.

When $q_0$ was determined from a height $h_0$, step (c) above was omitted. It should also be emphasized that the modifications above apply only to $q_0$ and not to the $q$ which contains substantial fringe flow. In any event
the flow in the fringe can only be measured with both the inlet and outlet levels substantially lowered as in Figure 17(A).

It is also of interest to know how the theoretical flow ratio curve is affected by slight variations of the lengths \( h_0 \) and \( h_1 \) (Figure 20) as would occur experimentally. To check this, the case of \( h_0 = L/10 \) (Curve A, Figure 21) was chosen. It was felt that the maximum possible error would be 2 mm. in \( h_0 \) and 4 mm. in \( h_1 \). For the actual dimensions of the model these would correspond to percentage errors of 3.3 and 13.1 respectively. Choosing these errors to induce the maximum variation in \( q/q_0 \), it was found that the error in \( q/q_0 \) amounted to 3.2 percent when \( h/h_0 \) was near unity, and 1.3 percent when \( h/h_0 \) was very large. Hence, the flow ratio curve is relatively insensitive to minor variations in \( h_0 \) and \( h_1 \).

**Flow liquids**

This subject has been adequately discussed in Investigation I. The flow liquid used for the rectangular model was the same as used for the semicircular model except for the type of calcium compound in solution. Since calcium chloride solution is somewhat difficult to prepare, gypsum (\( \text{CaSO}_4 \cdot 2\text{H}_2\text{O} \)) was used instead. To provide approximately the same amount of calcium ion in solution as a 200 ppm. solution of calcium chloride, it is necessary to use a 250 ppm. solution of calcium sulfate (\( \text{CaSO}_4 \)), or 316 ppm. of gypsum (\( \text{CaSO}_4 \cdot 2\text{H}_2\text{O} \)). Thus, the flow liquid for all experimentation with the rectangular model was a water solution of 250 ppm. of calcium sulfate and 400 ppm. of toluene.
Porous materials used

Sand and Marshall silty clay were used in the rectangular model.

The sand was taken from a natural deposit in a sand pit operated by the Iowa Highway Commission south of Belle Plaine, Iowa. The Marshall silty clay was the same as that used in the semicircular model, except that three different size ranges of dry aggregates were used. These were the ranges from 1.17 - 0.42 mm, 0.42 - 0.15 mm, and the material less than 0.15 mm, in size. The numerical values refer to the sizes of the screen openings.

Table 5 shows a size analysis of the sand and Marshall soil, determined on air-dry samples. No size analysis is shown for the Marshall soil in the size range below 0.15 mm, since the smallest screen used for the size analysis was 0.10 mm.

Table 5, Air-dry size analysis of sand and Marshall silty clay

<table>
<thead>
<tr>
<th>Size range, mm.</th>
<th>Percentage of sample</th>
<th>Sand</th>
<th>Marshall, 1.17-0.42 mm.</th>
<th>Marshall, 0.42-0.15 mm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00 &lt;</td>
<td>0.4</td>
<td>16.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.00 - 0.50</td>
<td>7.8</td>
<td>68.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.50 - 0.25</td>
<td>47.7</td>
<td>13.6</td>
<td>53.6</td>
<td></td>
</tr>
<tr>
<td>0.25 - 0.10</td>
<td>41.5</td>
<td>0.7</td>
<td>43.1</td>
<td></td>
</tr>
<tr>
<td>0.10 &gt;</td>
<td>2.6</td>
<td>1.4</td>
<td>3.3</td>
<td></td>
</tr>
</tbody>
</table>
It may seem odd that the Marshall contains particles smaller than the supposed lower size limit. This merely is a measure of the difficulty of removing all of the fine particles in a given sample of material. Also, the sieving process itself may exert a degrading effect on the particles.

**Description of runs**

As mentioned before, a "run" is used to refer to the collection of data for flow, phreatic line, and moisture on a given material filled to a given height in the model. The term "increment" as used below refers to a vertical distance of one-tenth the length of the model as discussed in connection with Figure 24. In all cases $\Delta d = \Delta H$.

**Run 1:** The height $h_0$ was equal to $L/10$; $\Delta H = L/20$ for all determinations of flow. The material used was sand, the final height of which was equal to $L$. The outlet boundary of the sand was separated from the outlet reservoir by a brass plate which contained a rectangular opening at the bottom, of height $L/20$ and width equal to the width of the model. This was the only run for which the plate was used. For the remaining runs, the plate was replaced by a screen as indicated in Figure 22.

**Run 2:** The material used was sand. Data were obtained for the case of $h_0 = L/5$, $\Delta H = L/10$; and $h_0 = L/10$, $\Delta H = L/20$. The head distribution along the upper sand boundary was measured for $h = L$, both for the case of the inlet water level at $L$ and at $L/2$. Finally,
with the inlet water level at L/2 and the outlet water level at 0.45 L (H = L/20), two brass plates, of width equal to the width of the model were inserted to a depth of L/2. These plates were located at a distance of L/3 from each end of the column. The flow was determined both with and without the plates to provide an evaluation of the effect of the plates. The reason for this latter experimentation will be explained in the Results and Discussion Section.

Run 3: For the first part of the experiment, the sand was allowed to wet only by capillarity. To ensure capillary wetting, the inlet and outlet water levels never exceeded a height L/10 above the bottom of the model. The maximum height of material was 0.7L. After the last flow measurement for the wetting capillary fringe was taken, the medium was pressure-saturated by raising the water levels. The final flow measurement was taken for a drying capillary fringe. The moisture distribution for the wetting fringe was determined on a separate sample. For all flow measurements, h_o = L/10, ΔH = L/20.

Run 4: The material used was Marshall silty clay, 1.17 - 0.42 mm, with height h_o = L/10, ΔH = L/20. Final height of soil was L.

Run 5: Again the material used was Marshall silty clay, 1.17 - 0.42 mm, h_o = L/10, ΔH = L/20. The final height of soil was slightly above the top of the second increment from the bottom of
the model.

Run 6: The material used was Marshall silty clay, 0.42 - 0.15 mm., \( h_0 = L/10 \), \( \Delta H = L/20 \). The final height of soil was 0.7L, or the top of the 7th increment from the bottom of the model.

Run 7: The material used was Marshall silty clay, 0.15 mm., \( h_0 = L/10 \), \( \Delta H = L/20 \). The final height of soil was located about mid-way in the 3rd increment from the bottom of the model.

In all cases a determination of the shape of the phreatic line was taken for every flow measurement in every run. In all cases the moisture distribution, in a vertical direction, was determined after the last flow measurement was taken. Notice that in all cases except one (Run 3) the capillary fringe was that specified by the drying part of the hysteresis loop as discussed in the Review of Literature.
Results and Discussion

Flow ratios, sand

The corrected experimental flow ratios (see Equation 66) were calculated for Runs 1 and 2 and plotted against the height ratios as shown in Figure 25-I. Curves $A_1$ and $A_2$ are to be compared with the theoretical Curve $A$, while $B_1$ is to be compared with $B$. It is seen that the experimental curves are similar in shape to the theoretical curves, but disagree quantitatively especially at the larger values of $h/h_0$. The experimental values of $q/q_0$ are higher than the theoretical ones. Hence, there is no indication that the fringe is less efficient in contributing to the flow than is expected theoretically with the assumption of the same hydraulic conductivity in the fringe as in the pressure-saturated region.

In Figure 26-I are shown the moisture data corresponding to the curves in Figure 25-I. The sand does not appear to have been completely saturated at any point, even at values of $h/h_0 < 1$ which are below the phreatic surface. However, determinations of moisture and volume weight below the phreatic surface were difficult to make, and in many cases could not be obtained for the sand. Hence, the saturation status beneath the water table is not known with certainty, but one would expect the percentage saturation beneath the water table to be as high as, or higher than, the percentage saturation in the fringe.
Figure 25. Experimental and theoretical flow ratio curves. Curves A, B, and D are reproduced from Figure 21. The subscripts of the letters which label the experimental curves refer to the numbers of the runs described earlier in the text. The solid circle in II represents the flow ratio as measured after the sand column, previously wetted only by capillarity, had been wetted under hydrostatic pressure to create a drying capillary fringe. Note the close agreement of this check point with Curve A.
A, B - THEORETICAL CURVES
A, A - EXPERIMENTAL CURVES CORRESPONDING TO A
B, B - EXPERIMENTAL CURVE CORRESPONDING TO B

D - THEORETICAL CURVE, A
=D - EXPERIMENTAL CURVE A (DRYING), AVERAGE OF A, AND A
=G - EXPERIMENTAL CURVE A (WETTING)
=D - DRYING CHECK POINT OF WETTING CURVE A

FLOW RATIO, \( q/q_0 \)

HEIGHT RATIO, \( h/h_0 \)
Figure 26. Percentage moisture saturation of sand as a function of height ratio. The curves are to be compared with those of Figure 25 as indicated at the bottoms of the diagrams below. For all of the percentage moisture saturation curves shown, the water table is located approximately at a height $h_0$ or $h/h_0 = 1$. Experimentally, Curve 1 of II below corresponds to the check point (solid circle) of Figure 25-II.
1 - CORRESPONDS TO $\frac{q}{q_0}$ - CURVE $A_1$
2 - CORRESPONDS TO $\frac{q}{q_0}$ - CURVE $A_2$
3 - CORRESPONDS TO $\frac{q}{q_0}$ - CURVE $B_1$

1 - CORRESPONDS TO $\frac{q}{q_0}$ - CURVE $A$ (DRYING)
2 - CORRESPONDS TO $\frac{q}{q_0}$ - CURVE $A_3$ (WETTING)
Curve 1 of Figure 26-I appears to break at a height ratio of about 5, and reference to $A_1$ of Figure 25-I indicates very little increase in the flow ratio beyond the same point. A similar analysis on Curve 2 of Figure 26-I and Curve $A_2$ of Figure 25-I does not reveal such a specific behavior. Curve $A_2$ seems to continue increasing even though the moisture saturation decreases sharply. The same behavior is noted in $B_1$ when compared to its corresponding moisture curve, Curve 3 in Figure 26-I. However, the similar behavior of Curves $A_1$ and $B_1$ is not surprising since both were determined on the same column of sand. This behavior of Curves $A_1$ and $B_1$ suggests that there may be cases in which the zone above the capillary fringe makes a measurable contribution to the total flow.

The flow ratio results of the wetting capillary fringe of Run 3 are shown as Curve $A_3$ of Figure 25-II. It is seen that the curve breaks sharply at a value of $h/h_0 = 4$. In Figure 26-II a pronounced separation also occurs between Curves 1 and 2, again at a value of $h/h_0$ near 4. Thus, the consistent behavior of the flow ratio curves and the moisture curves illustrates clearly the hysteresis effect on the fringe. In the particular case illustrated here, the wetting fringe is very similar to the drying fringe except that its height is less. This is reflected in both the flow ratio curves and the moisture curves.
It should be pointed out that Curve 1 of Figure 26-II could not actually be determined for Curve A in Figure 25-II. However, since the check point (solid circle in Figure 25-II) falls so closely to Curve A, it is felt that one is justified in using Curve 1 of Figure 26-II for comparison purposes. Experimentally, Curve 1 does correspond to the check point.

**Head distribution, sand**

The dissipation through the sand column of the total head difference $\Delta H$ was measured by determining the heights of the water levels in the perforated tubes positioned vertically in the model. In a given tube, the amount of head undissipated, which is the same as the amount of head remaining, is simply the difference between the level of water in the tube and the level of water in the outlet reservoir. Thus, a plot of the head remaining, as a function of distance along the length of the model, would also represent the position and shape of the phreatic line as referred to the outlet water level.

To reduce all graphs to a common unit, the head remaining was expressed as the percentage of the total head difference $\Delta H$. The distance of each perforated tube from the inlet end of the column was expressed as the percentage of the total distance $L$, the length of the model.

**Head distribution along the water table.** One of the first objectives of the studies of head distribution was the evaluation of the effect of the capillary fringe upon the shape of the phreatic line.
This can be done by comparing the pattern of the phreatic line when 
$h = h_0$ to the patterns obtained when $h$ is greater than $h_0$. Strictly 
speaking, the pattern obtained when $h = h_0$ does not represent the 
case of zero fringe. In any experimental determination of reference 
flow $q_0$ (whether on the basis of $h$ or $h_0$, see page 118), there will 
always be a small wedge-shaped capillary fringe, since of necessity 
the outlet water level must be lower (by $\Delta H$) than the inlet water 
level. Although this fringe is small, rather than terming it "no fringe" 
it will be referred to as "minimal fringe".

In Figure 27 are shown patterns of the phreatic line as determined 
from Run 2, where $\Delta H = L/10$ and $h_0 = L/5$. For Figure 27, and also for 
the four more similar figures which follow later, the numerical values 
of $\Delta H$ for each curve are given in the Appendix, Table 10.

Curve 1 in Figure 27 shows the case of minimal fringe. The curve 
is essentially linear. As more and more fringe is added, the curves 
bulge upwards over about 60 percent of the length of the model nearest 
the outlet end. The distortion is greatest for the first increment 
of fringe (see Curve 2), and becomes progressively smaller thereafter. 
Reference to Curve $B_1$ of Figure 25-I, which is the flow ratio curve 
corresponding to Figure 27, indicates that the flow ratio was increasing 
as $h/L$ changed from 0.60 to 1.00 (or $h/h_0$ changed from 3 to 5 in Figure 
25-I). Hence, it appears that the phreatic line will be distorted as 
long as there is an appreciable contribution of the fringe to the flow.
Figure 27. Patterns of phreatic lines through sand (Run 2) plotted as percentage of head remaining (referred to outlet level) against distance percentage from inlet. Curves are shown as broken lines near the outlet to indicate possible uncertainty due to a surface of seepage indicated by the arrow. The flow ratio curve corresponding to this figure is Curve B₁ of Figure 25-I.
NOMINAL HEAD DIFFERENCE
$\Delta H = L/10$ FOR ALL CURVES.
This latter point is brought out more clearly by Figure 28, which is a set of phreatic curves corresponding to the wetting fringe flow ratio curve, \( A_3 \), of Figure 25-II. Again Curve 1 of Figure 28 shows the case of minimal fringe. As the height increases to where \( h/L = .4, \ (h/h_o = 4) \), the phreatic line bulges as shown by Curves 2 and 3.

From Figure 25-II it is also seen that the flow ratio (Curve \( A_3 \)) increases only very slowly as \( h/h_o \) changes from 4 to 7. Returning to Figure 28 it is seen that Curve 4 is only slightly different from Curve 3 over the right half of the diagram. However, as the change is made from the wetting fringe to the drying fringe, the flow ratio increases sharply as indicated by the check point in Figure 25-II. Again, a corresponding bulge is found in Curve 5 of Figure 28 as compared to Curve 4.

In both Figures 27 and 28 there seems to occur a depression of the phreatic line over the initial distance from the inlet end of the model. As more fringe is added, the trend seems to reverse. However, the behavior near the inlet seems less regular and certain than the behavior over the part of the model near the outlet end.

Near the lower right-hand corners of Figures 27 and 28 appear small arrows drawn as linear extrapolations of the last two points of the uppermost curves (Curve 4 in Figure 27 and Curve 5 in Figure 28). These are used to establish a maximum limit for a possible surface of seepage. In Figure 27 this would be about 4 percent of the head difference and
Figure 28. Patterns of phreatic lines through sand. All curves are for the wetting capillary fringe except Curve 5 which is for the drying check point. For Curves 1 to 4 the corresponding flow ratio curve is $A_2$ of Figure 25-II. Broken-line sections in the curves indicate possible uncertainty from the surface of seepage indicated by the arrow in the lower right-hand corner.
NOMINAL HEAD DIFFERENCE
$\Delta H = L / 20$ FOR ALL CURVES.
in Figure 28 about 8 percent. Since the head difference is only half
as great in Figure 28 as in Figure 27, both estimations yield about
the same result, or about 2.4 mm, for the head differences involved.
When it is remembered that a 4 mm, variation in the outlet height h₁
(see previous section, Experimental Methods) affected the flow ratio
curve by a maximum of 3.2 percent, it is seen that the surface of
seepage would be of the same order of magnitude as the experimental
error. Furthermore if one determined the surface of seepage from
Figures 27 and 28 by a curvilinear extrapolation, as would be more
realistic, the surface of seepage would be still smaller. Hence, neglect
of the surface of seepage seems fully justified for this model as long
as the head difference ΔH does not exceed L/10.

Head distribution along fringe boundary. Only in Run 2 were
measurements made of the head distribution pattern along the upper
boundary of porous medium, in this case sand. Two cases were investi­
gated, (a) the inlet water level at the top of the porous medium
(of height L) with the outlet water level lower by ΔH = L/10; and (b)
the inlet water level at L/2 with the outlet water level again lower
by ΔH = L/10. In both (a) and (b) measurements were taken of the
phreatic line as well as of the distribution pattern afforded by the
tensiometers. Theoretical values for the distribution patterns along
the upper boundary were calculated from Equations 58 and 60. All of the
results, both experimental and theoretical, are shown in Figure 29.
Figure 29. Head distribution patterns along the upper sand boundary determined by tensiometers imbedded slightly beneath the surface. For A the phreatic line was near the surface; for B it was positioned approximately halfway down in the model.
PHREATIC SURFACE, MEASURED
Tensiometers: MEASURED - - - - A AND B
THEORETICAL

PERCENTAGE HEAD REMAINING

PERCENTAGE OF DISTANCE

PERCENTAGE OF DISTANCE
Figure 29A gives the results when the water levels were near the upper boundary. It is seen that all three curves agree closely except near the outlet. This discrepancy between the theoretical curve and the phreatic line is not surprising. Near the outlet, the phreatic line will need to be lower than the head at some point on the upper boundary in order to provide a difference in potential which will produce a downward flow component. The tensiometer discrepancy will be discussed in the next paragraph.

When the water levels are lowered to the middle of the column, Figure 29B, the result for the head distribution along the upper boundary changes markedly, the curve becoming more flat and somewhat sigmoidal. The agreement between theory and experiment is satisfactory, but the experimental curve is slightly rotated toward the phreatic line. Some of this may be accounted for by the fact that the tensiometer discs were not located right on the surface of the sand but were imbedded about 1-2 cm. below it; hence, they were nearer to the phreatic surface than they should have been. This also would explain some of the deviation of the last tensiometer point in Graph A.

Figure 29B also illustrates the potential mechanism operating to produce flow in the fringe. Near the inlet side of the model, the head along the upper boundary is lower than the phreatic line, this being necessary to drive water upward into the fringe. Near the outlet side the head along the upper boundary is higher than the phreatic line. This is necessary to drive water out of the fringe toward the
outlet level. Along the upper boundary itself the head at the outlet end is lower than the head at the inlet end. This is necessary to drive water along the upper boundary.

Reference to 26-I indicates that for the case just considered the upper boundary of the capillary fringe should not have been below the upper boundary of the model. To supply additional information on this point, the measured flow ratio was compared to the theoretical flow ratio for this particular geometry. The theoretical value was 0.659; the experimental value 0.726; or about 10 percent greater than the theoretical. Thus, it appears that fringe conditions were maintained.

Results with Marshall silty clay

Flow results, coarse aggregates. In Figure 30 are shown the theoretical flow curve A and the experimental flow curves for Marshall silty clay. Curve $A_4$ is for the 1.17-0.42 mm. fraction, Run 4. In shape, $A_4$, the experimental curve, is similar to the theoretical, but it does not rise as high.

In Figure 31-I is shown the moisture saturation curve. Obviously there is a relatively small capillary fringe present since the moisture curve breaks very sharply at a height ratio of about 3. Thus, in Figure 30 the failure of Curve $A_4$ to rise higher may be attributed to a small height of capillary fringe.

This conclusion is further supported by Figure 32. Curve 1 of this figure represents the phreatic line for the case of the minimal fringe, and Curve 2 shows the beginning fringe contribution. However,
Figure 30. Flow ratio curves for Marshall silty clay. Curves A, B, and D are reproduced from Figure 21. As in Figure 25 the subscripts refer to run numbers.
FLOW RATIO, \( q/q_0 \)

HEIGHT RATIO, \( h/h_0 \)

THEORETICAL CURVE, A

EXPERIMENTAL CURVE, A4

EXPERIMENTAL CURVE, A5

EXPERIMENTAL CURVE, A6

EXPERIMENTAL CURVE, A7

SEE TEXT
Figure 31. Percentage moisture saturation of Marshall silty clay as a function of height ratio. The curves are to be compared with those of Figure 30 as indicated at the bottoms of the diagrams below. For all of the percentage moisture saturation curves shown, the water table is located approximately at a height $h_0$, or $h/h_0 = 1$. 
1 - CORRESPONDS TO $q/q_0$ - CURVE A4
2 - CORRESPONDS TO $q/q_0$ - CURVE A5

1 - CORRESPONDS TO $q/q_0$ - CURVE A6
2 - CORRESPONDS TO $q/q_0$ - CURVE A7
Figure 32. Patterns of phreatic lines through Marshall silty clay of size 1.17-0.42 mm. Curves are shown as broken lines near the outlet where it is thought that a surface of seepage, however small, might exist. The corresponding flow ratio curve for this set of phreatic line patterns is $A_4$ of Figure 30.
NOMINAL HEAD DIFFERENCE $\Delta H = L/20$ FOR ALL CURVES.

<table>
<thead>
<tr>
<th>CURVE</th>
<th>SYMBOL</th>
<th>$h/L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>●</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>○</td>
<td>0.20</td>
</tr>
<tr>
<td>3</td>
<td>○</td>
<td>0.40</td>
</tr>
<tr>
<td>4</td>
<td>△</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Curves 3 and 4 coincide over the half-length of model nearest the outlet. In Figure 30 this corresponds to the flat portion of Curve $A_4$ from $h/h_0 = 4$ to $h/h_0 = 10$. Hence, this is supporting evidence that there was relatively little flow in the added Marshall soil after the height ratio reached a value of the order of 3 or 4.

However, on the basis of Curve 1 of Figure 31-I, one would expect a full theoretical contribution up to a value of $h/h_0 = 3$. Yet it is clear from Curve $A_4$ in Figure 30 that at no measured point was the full theoretical contribution realized.

Now the moisture determinations, of necessity, must be performed after the last flow measurements have been taken. This means that the moisture curve may reflect any effect of compaction caused by the added soil. This might mean that the pore sizes in the lower parts of the model would be reduced. Thus, they would support a higher capillary fringe than they did when the actual flow measurement was taken.

Run 5 was initiated to check this possibility. Only enough soil was put into the model to permit the taking of four moisture samples. Only one flow ratio value could be determined.

These results appear as Curve $A_5$ in Figure 30 and as Curve 2 in Figure 31-I. They do not point out any behavior basically different from the previous run, No. 4, with Marshall. However, one is forced to think of a possible parallel to the Edina soil of Investigation I. Perhaps the hydraulic conductivity is lowered because only a few
large pores are drained, but there are not enough of them to greatly affect the moisture content.

Aggregate stability. Since the Marshall silty clay of size range 1.17-0.42 mm. did not possess a very large capillary fringe, it was decided to use a finer size range of dry aggregates. The coarser range had been chosen originally since it was feared that finer particles might slake and seal when wetted. That such a breakdown is not excessive in the Marshall silty clay, especially in the coarser aggregates, is shown in Table 6. The dry size analysis of Table 5 is shown along with a wet-sieve analysis using the same size screens. Results for two air-dry size ranges of Marshall are shown.

Table 6. Water-stability of Marshall silty clay

<table>
<thead>
<tr>
<th>Size range, mm.</th>
<th>Marshall, 1.17-0.42 mm.</th>
<th>Marshall, 0.42-0.15 mm.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Dry-sieved</td>
<td>Wet-sieved</td>
</tr>
<tr>
<td>&gt;1.00</td>
<td>16.0</td>
<td>7.9</td>
</tr>
<tr>
<td>1.00-0.50</td>
<td>68.3</td>
<td>41.5</td>
</tr>
<tr>
<td>0.50-0.25</td>
<td>13.6</td>
<td>16.6</td>
</tr>
<tr>
<td>0.25-0.10</td>
<td>0.7</td>
<td>13.6</td>
</tr>
<tr>
<td>&lt;0.10</td>
<td>1.4</td>
<td>20.4</td>
</tr>
</tbody>
</table>
It is seen in Table 6 that when wet-sieved, over 60 percent of the coarser Marshall (1.17-0.42 mm.) remains in water-stable aggregates larger than 0.25 mm. This is the reason for the rather small capillary fringe. On the other hand, the finer Marshall (0.42-0.15 mm.) has over 60 percent of its water-stable particles smaller than 0.10 mm. It should support a rather high capillary fringe.

Flow results, fine aggregates. The Marshall silty clay with air-dry size range 0.42-0.15 mm., was used in Run 6. The flow ratio results appear as Curve C in Figure 30. Again it is noted that nowhere is the full theoretical contribution realized. Yet, reference to Curve 1 of Figure 31-II shows that the capillary fringe extends to a height corresponding to an h/h₀ ratio of 6.

Returning to Figure 30, note the three heavy circles located under Curve C and corresponding to height ratio values of 5, 6, and 7. The heavy circle farthest to the left was a flow ratio point determined according to the previously explained standard procedure. After it was determined, the next increment (6th) of soil was added, wetted by capillarity, pressure-saturated, and the value of q₀ determined. The water levels were then lowered in preparation for the fringe measurements, but while the latter were being taken, it was seen that the surface of the soil had noticeably subsided, apparently because of the tension in the capillary fringe. When the corresponding flow ratio was calculated, it was the middle one of the three heavy circles. So, instead of adding more soil, the water levels were once again raised, the already wet
medium was pressure-saturated a second time, and a new value of $q_0$ was obtained. The levels were once again lowered and another capillary flow measurement taken. When the flow ratio was calculated according to the second set of measurements, it was considerably larger than before. In the figure it is the point on $A_6$ for $h/h_0 = 6$.

A similar procedure was repeated on the next increment (7th) and again the two calculated values of the flow ratio were different, to about the same extent as for the 6th increment. Hence, the three heavy circles were omitted in drawing Curve $A_6$.

On the basis of the behavior just described, one might postulate that Curve $A_6$ fails to coincide with the theoretical curve because of cumulative compaction in the lower part of the model. This would seem plausible since the reference flow $q_0$ is determined on the basis of the whole soil column. A decrease in conductivity in the lower layers might be counterbalanced by the fairly large conductivity in the relatively uncompacted upper layers; hence, $q_0$ might not decrease markedly because of the compaction. However, when the water levels are lowered for the measurement of the fringe flow, the compaction in the lower layers could result in pronounced changes in the flow. Hence, because of the compaction, a false lowering of the flow ratio curve could result.

An independent means of detecting flow in the capillary fringe is needed. This, fortunately, is provided by the set of phreatic line patterns shown in Figure 33. From the figure it is seen that the phreatic
Figure 33. Phreatic line patterns through Marshall silty clay of size 0.42-0.15 mm. Curves are shown as broken lines near the outlet where it is thought that a surface of seepage, however small, might exist. The corresponding flow ratio curve for this set of phreatic line patterns is $A_6$ of Figure 30.
NOMINAL HEAD DIFFERENCE
$\Delta H = L/20$ FOR ALL CURVES.

<table>
<thead>
<tr>
<th>CURVE</th>
<th>SYMBOL</th>
<th>h/L</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>·</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>○</td>
<td>0.20</td>
</tr>
<tr>
<td>3</td>
<td>·</td>
<td>0.30</td>
</tr>
<tr>
<td>4</td>
<td>○</td>
<td>0.50</td>
</tr>
<tr>
<td>5</td>
<td>○</td>
<td>0.70</td>
</tr>
</tbody>
</table>
line bulges definitely upward for every added increment of soil. The comparison between Curves 3 and 4 is especially illuminating. These curves indicate that an appreciable flow contribution does exist for the 5th increment; however, the flow ratio value (left heavy circle under Curve A<sub>5</sub> in Figure 30) as originally determined implies a decrease. Therefore, it appears that one is justified in discarding this erratic value of the flow ratio. In general, Figure 33 rather strongly implies that the extreme lowering of Curve A<sub>5</sub> in Figure 30 is not real.

The results of Run 7, involving Marshall silty clay smaller than 0.15 mm., are shown in A<sub>7</sub> of Figure 30 and Curve 2 of Figure 31-II. Nothing particularly new is illustrated; however, the single experimental point for the flow ratio is not far below the theoretical value.

Additional compaction study. Since volume weight data were taken in connection with the moisture data, it was possible to study the matter of compaction somewhat further. This was done by plotting the volume weights against height above the bottom of the model. The results are shown in Figure 34. Since the points exhibit considerable scatter, linear regression lines were fitted to each set of data by the method of least squares (32).

Curves 1 and 2 are for the sand of Runs 1 and 2 respectively. Curves 3 and 4 are for the respective size ranges of Marshall soil in Runs 4 and 6. In general it is noted that the lines have very little slope.
Figure 34. Scatter diagram of volume weight plotted against height.
1.72
1.68
1.64

n—I—I—I—I—r

O ®

8

® ® ® ® _

® o ^ ;7 —

o o ^

1-1.60

° 8 ®

O

1-SAND

2-SAND

3-MARSHALL

SOIL, 1.17-0.42MM

4-MARSHALL

SOIL, 0.42-0.15MM

15 20 25 30 35 40 45 50

DISTANCE ABOVE BOTTOM OF MODEL, CM
Making the usual statistical assumptions of normally distributed
independent errors, the significance levels of the slopes (regression
coefficients) of the regression lines were calculated using the procedure
outlined by Ostle (32). The significance levels were calculated by
transforming the computed F-values and using the incomplete beta-
function as tabulated by Pearson (34). The significance levels, along
with the regression coefficients, are shown in Table 7.

A negative regression coefficient is synonymous with a negative
slope of the regression line, which in turn implies compaction in the
lower parts of the model. However, from Table 7, only Curve 3 of
Figure 34 appears to represent evidence of real compaction. Yet it
was the soil column of Curve 4 (Run 6) that presented the strong
evidence for compaction on the basis of flow and phreatic line measure-
ments.

Table 7. Significance levels of the regression coefficients
for the regression lines in Figure 34

<table>
<thead>
<tr>
<th>Curve no. (Fig. 34)</th>
<th>Regression coefficient</th>
<th>Significance level, percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+0.000410</td>
<td>52</td>
</tr>
<tr>
<td>2</td>
<td>-0.000621</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>-0.001547</td>
<td>0.05</td>
</tr>
<tr>
<td>4</td>
<td>-0.000648</td>
<td>29</td>
</tr>
</tbody>
</table>
The failure of Curve 4 in Figure 34 to have a regression coefficient of greater significance, may have been due to a compacting effect induced by the additional cycles of water level raising and lowering as mentioned earlier. The Marshall soil corresponding to Curve 3 was not subjected to such additional cycles.

It should be pointed out that the volume weight study of compaction is not very conclusive. However, on an overall basis, it is implied that compaction is unimportant in the sand; but that it may occur in the soil. Hence, any future studies regarding the capillary fringe should be conducted according to a procedure which eliminates compaction, or at least minimizes its effect, especially if the studies involve soil.

From the foregoing analysis of compaction, one might suppose that the experimental control of this factor was in all cases very poor. Actually the difficulty occurred only in soil, and then only when dry soil was put into the model on top of wet soil. For the first layer of dry soil placed in the model, there is no suggestion that variations in compaction were sufficiently large to cause marked variations of the hydraulic conductivity. This is illustrated by the phreatic line patterns of Figure 35. For each curve, the soil column had been placed in the previously empty model at one time; no dry soil had been placed upon wet soil. In each case the height of the soil column was approximately 0.2L. Marshall silty clay of different size ranges was the soil used. Curve 1 is for the 1.17-0.42 mm. range, Curve 2 is for the 0.42-0.15 mm.
Figure 35. Phreatic line patterns (minimal fringe—see text) for different air-dry size ranges of Marshall silty clay. Curves 1, 2, and 3 correspond to respective air-dry size ranges in millimeters of 1.17–0.42, 0.42–0.15, and less than 0.15. The units for the hydraulic conductivity values in the box are centimeters per hour.
range, and Curve 3 is for the range less than 0.15 mm.

All of the curves are for a minimal fringe and in each case the total head difference is small, in a range from about L/20 to L/30. Now if the packing of the soil is such that the conductivity is constant throughout, a smooth, nearly linear phreatic line should result. In this regard the curves of Figure 35 are indeed satisfactory. Furthermore, the pattern is little changed even when the saturated hydraulic conductivity is widely different from one curve to the next.

Effect of obstructions in the fringe

Since the capillary fringe is a region of tension, any holes, crevices, or cracks in the medium can function as impermeable barriers. To study the effect of barriers experimentally, obstructions were formed by inserting two brass plates into the fringe through the upper boundary when the model had been filled with sand (Run 2). The inlet water level was maintained at a height L/2 and the outlet level slightly lower, by the amount ΔH. Each plate, of width equal to the width of the model, was inserted to a depth of L/2. The horizontal distance between the plates was L/3. This was also the distance from each plate to the nearer end of the model.

A consideration of this geometry shows it to be merely a combination of three unit cells. Two of these are balanced (Figure 19), with \( d = L/2, \quad h = L, \quad \text{and length } = L/3 \). The other cell is unbalanced (Figure 20), with \( d_1 = L/2, \quad d_2 = L/2 - ΔH, \quad h = L, \quad \text{and length } = L/3 \). However, since ΔH is small, \( d_2 \) is set equal to \( d_1 \) and the balanced case applies to this
unit cell also. Since the McNown and Hsu solution is equivalent to the solution for the balanced case, the theoretical value for \( q/q_0 \) can be taken from the graph of McNown and Hsu (29). In using their solution, however, one must properly adjust for the different choice of reference flow as mentioned previously in the discussion of their solution.

In Table 8 are shown the experimental values of flow ratio, with and without plates. Appropriate theoretical values are shown for comparison. It is seen that the plates do not reduce the flow as much as they should if the percentage error is to be kept the same. This could be accounted for by leakage around the edges of the plates, since they could not be placed against the walls of the model tightly enough to completely prevent water loss around the edges.

Table 8. Effect of obstructions (plates) in the capillary fringe

<table>
<thead>
<tr>
<th>Condition of the capillary fringe</th>
<th>Flow ratio, ( q/q_0 )</th>
<th>Percent error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theoretical</td>
<td>Experimental</td>
</tr>
<tr>
<td>Fringe only</td>
<td>1.42</td>
<td>1.50</td>
</tr>
<tr>
<td>Fringe with two plates</td>
<td>1.15</td>
<td>1.31</td>
</tr>
</tbody>
</table>
The case represented by Table 8 was made the subject of extended theoretical analysis. How does the flow ratio decrease as more plates are added? Again the analysis was first made on the basis of balanced unit cells in which $d/h = 1/2$. The $L/h$ variable for the unit cell decreases as more plates are added. The geometry is illustrated by Case I in Figure 36.

A second analysis was made by assuming every other plate to be only half as long as the others. Thus, the unit cell is unbalanced; for it $d_1/h = 1/4$, and $d_2/h = 1/2$. Again the $L/h$ variable for the unit cell decreases as more plates are added. The unbalanced cells are illustrated by Case 2 in Figure 36.

The variable plotted on the abscissa in Figure 36 is the total number of plates per unit length of model. Strictly speaking it is not a continuous variable, but it does approach continuity as the number of plates increases. At any rate it serves as a useful indicator of the number of plates added.

It is seen from Curves 1 and 2 that the Case 1 procedure is the more effective method for reducing the flow in the fringe. Even so, the Case 1 procedure requires about 1 plate in every foot to reduce the fringe flow to about 10 percent of the flow beneath the phreatric line. Hence it appears that the effect of the fringe is not easily suppressed unless the obstructions are exceedingly numerous.
Figure 36. Theoretical curves showing the effect of plates in reducing flow in the capillary fringe.
2.00
T
L = 50 FT.
h = 5 FT.
t = hh/2
CASE 1

CASE 2

CURVE | CASE
-----|-----
1    | 1   
2    | 2   
3    | 3   

K_H / K_V = 4

NUMBER OF PLATES PER UNIT LENGTH, PLATES / FT.
Anisotropy

All analyses so far have involved isotropic media. In natural soils, however, the hydraulic conductivity is not always the same in all directions; that is, natural soils are anisotropic in many instances. Frequently this means that the horizontal hydraulic conductivity $K_h$ is greater than the vertical hydraulic conductivity $K_v$.

According to McNown and Hsu (29), the case of anisotropy can be analyzed for their rectangular geometry by calculating a new geometrical equivalent length $L'$ from the relation

$$L' = L\sqrt{K_v/K_h}.$$  \hfill (67)

The value $L'$ is substituted into the previous solution in place of $L$; otherwise the solution remains unchanged if kept in the form of $q/q_0$.

This procedure was applied to the unit cell of Case 1, Figure 36, assuming that $K_h/K_v = 4$. The results are shown in Curve 3. With respect to reducing fringe flow, it is obvious that the plates of Case 1 are more effective in an anisotropic soil (of the type considered here) than they are in an isotropic soil. From another point of view this implies, for the case here considered, that anisotropy reduces the relative fringe contribution to flow.

This result is not surprising if one considers what $K_h/K_v = 4$ implies for the fringe. As far as horizontal conductivity is concerned, the fringe is not handicapped as compared with the region below. However, vertical flow into the fringe must occur before the fringe can contribute
to the total flow. Hence, since the vertical conductivity is reduced, relatively speaking, the fringe is handicapped since its contribution depends upon vertical components of flow. If, on the other hand, $K_v$ were greater than $K_h$, the fringe contribution would be enhanced. The reasoning for this case follows the same lines as previously.

It is also of interest to apply anisotropy considerations to Curve A of Figure 21. Again assume that $K_h/K_v = 4$. This means, applying Equation 67, that

$$L' = L\sqrt{1/4} = L/2.$$  

Also, since $h_0 = L/10$ and $d = L/20$, the new quantities, indicated by primes, become:

$$h_0' = 2L'/10 = L'/5$$
$$d' = 2L'/20 = L'/10.$$  

However, if the primes are omitted, it is seen that the flow ratio curve for this case would be identical with Curve B. Thus it is shown once more that for the anisotropic case of the horizontal conductivity greater than the vertical, the relative contribution of the fringe is reduced.
CONCLUSIONS

In the following paragraphs it is to be remembered that the capillary fringe is defined as an essentially saturated zone, under tension, and in contact with a saturated zone in which the hydrostatic pressure in the water is greater than atmospheric.

1. The contribution of the capillary fringe to the flow of water in porous media has been accounted for by most investigators simply by assuming the existence of an added conducting layer. The layer is assumed to have a height equal to the capillary rise of the material and a conductivity equal to that of the medium containing water under pressure.

2. The hydraulic conductivity of the capillary fringe is nearly equal to that of the same porous medium when saturated under hydrostatic pressure, the difference being on the order of 10 percent. Usually the conductivity is lower in the fringe. A small decrease may occur even if the fringe appears to be saturated as determined from moisture measurements.

3. On the basis of head distribution along the boundaries of the flow systems investigated, potential considerations for the capillary fringe appear to be the same as if the region were under hydrostatic pressure.
4. If a capillary fringe is so located that the flow lines are lengthened, without an increase in potential along these lines, the fringe will retard the flow as compared to that which would occur without a fringe. This occurs theoretically for the simple case of the falling water table. Data taken from the literature, for both the simple falling water table and the falling water table above tile lines, verify the retarding effect of the fringe.

5. If, as in horizontal flow, a capillary fringe is added so as to increase the flow region and at the same time increase the effective cross section for flow, the flow contribution of the capillary fringe is limited only by geometrical considerations. The theoretical solution for this geometry, worked out for the first time in this thesis, predicts a maximum fringe contribution of 170 percent when the height of the region under hydrostatic pressure is one-tenth of the length. The theory is adequately verified for sand and also to a satisfactory degree for soil. The results for soil are somewhat masked by experimental difficulties with compaction.

6. For horizontal flow in an anisotropic medium, where the horizontal conductivity is greater than the vertical conductivity, the fringe contribution is lower than for the same geometry in an isotropic medium. If the horizontal conductivity is less than the vertical, the fringe contribution is increased as compared to the isotropic case.
7. In horizontal water movement, the shape of the phreatic line is altered by the presence of a capillary fringe. The most prominent feature is an upward bulge near the outlet boundary of the flow region, apparently necessary to drive the fringe flow through the outlet boundary beneath the outlet water level.

8. Hysteresis affects the flow contribution of the capillary fringe by influencing the height of the fringe. The fringe formed by allowing the medium to wet only by capillarity from the water table is always smaller in height than the fringe formed by first raising the water table and then lowering it. Experimentally, the hysteresis effect on the fringe contribution to flow is nicely verified for sand.

9. It is difficult to reduce the flow in the capillary fringe by inserting plates into the fringe perpendicular to the direction of flow. A large number of sizable plates is needed to reduce the fringe flow to a negligible fraction of the flow beneath the water table. This implies that natural barriers in the fringe, such as cracks, crevices, or worm holes, have little effect in reducing the fringe flow unless they are extremely numerous.

10. There is some evidence to indicate that even the unsaturated capillary zone above the fringe may make a measurable contribution to the flow.
No general solution for the capillary fringe flow is presented. Each case must be considered on its own merits. The general approach mentioned in Conclusion 1 seems fairly adequate and was used to good advantage in this study. However, over and above the general approach, one must specify the problem before the fringe effect can be analyzed.
SUMMARY

The capillary fringe as defined in this study is a zone of porous medium which is (a) essentially saturated with water, (b) under tension, and (c) in immediate contact with a saturated zone in which the pressure in the water is greater than atmospheric. In the usual cases, the capillary fringe is located directly above the subsoil water table.

For a horizontal falling water table, the capillary fringe was analyzed theoretically on the basis of a falling meniscus in a capillary tube. It was found that the presence of the fringe retarded the flow as compared to that occurring without a fringe. Experimental data from the literature supported this conclusion.

The fringe was further investigated with laboratory flow models filled with sand or soil. The first model consisted of a semicircular arch constructed as a capillary siphon. It could be operated to work as either a system everywhere under hydrostatic pressure, or as a tension system simulating the capillary fringe. The mathematical solution for the flow was easily obtained using methods of potential theory.

The results from this model indicated that the head distribution along the outer boundary was practically unaffected when the operation of the model was shifted from hydrostatic pressure conditions to fringe conditions. Likewise, the hydraulic conductivity decreased only a small amount provided that essential saturation was maintained throughout the model.
The second model was rectangular in shape and contained a water reservoir at each end, with porous medium between the reservoirs. By ponding water in the reservoirs and maintaining a small head difference between them, the effect of the fringe for horizontal flow could be determined. In addition, the problem was solved mathematically, assuming the hydraulic conductivity to be the same both above and below the water table. An exact solution was obtained in elliptic integrals.

Experimental results with sand verified the theoretical solution for the rectangular model. As an example, consider the case in which the part of the soil column beneath the water table is one-tenth as high as it is long. It was found that the theory predicts a maximum flow contribution from the fringe of 170 percent when compared to the flow occurring beneath the water table. This was verified experimentally. Furthermore, with respect to hydraulic conductivity and the distribution of head along the fringe boundary, the experimental results for the rectangular model were in agreement with those found for the semicircular model.

For soil the fringe contribution to the flow seemed to be masked by variations in compaction. However, supplementary studies indicated that flow in the fringe for the case of soil would be correctly described by the theoretical solution if the compaction error did not exist.

The mathematical solution was derived for isotropic flow, but the theory can be extended to include the anisotropic case wherein the hydraulic conductivity of the medium is not independent of the direction
of flow. For anisotropy in which the horizontal conductivity is greater than the vertical, which is the more common case in soil, the flow contribution of the fringe is reduced as compared with the isotropic case. When the vertical conductivity is greater than the horizontal, the fringe contribution is increased as compared to the isotropic case. Both results are easily explained qualitatively by considering the flow resistance encountered along a streamline which passes through the fringe.

In horizontal flow, the shape of the water table is altered by the presence of a capillary fringe. The most prominent feature is an upward bulge near the outlet boundary of the flow region. Apparently the bulge is necessary to drive the fringe flow through the outlet boundary beneath the water table.

Hysteresis also affects the flow contribution of the capillary fringe. This is largely a matter of the height of the fringe. This height will be greater if the porous medium is first saturated under pressure and the water table subsequently lowered. A smaller height results if the porous medium wets by capillarity from a water table maintained at a constant level. The difference in fringe height and fringe contribution to the flow were demonstrated experimentally with sand. No measurements were made for soil.

Experimental and theoretical studies were conducted on reducing the flow in an existing capillary fringe by inserting plates into the medium perpendicular to the direction of flow. It was found
difficult to reduce the flow by such means; that is, a relatively large number of plates is required to reduce the flow in the capillary fringe to a negligible amount. Hence, it is concluded that natural barriers in the fringe, such as cracks, crevices, or worm holes, will not greatly reduce the flow contribution of the fringe.

Some of the experimental studies indicated that even the flow in the unsaturated zone above the fringe cannot always be neglected. However, the detailed study of this effect is left to future investigation.

No general solution for the flow contribution of the capillary fringe can be given. In all cases the capillary fringe increases the size of the conducting region. However, this size increase and the flow velocities through it will depend upon the boundary conditions of the specific problem.
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Table 9. Calculation of flow ratio curve A in Figure 21. Initial Height $h_0 = L/10$, $\Delta d = L/20$, $h_1 = L/20$

<table>
<thead>
<tr>
<th>$h/h_0$</th>
<th>$L/h$</th>
<th>$n$</th>
<th>$d_1/h$</th>
<th>$m$</th>
<th>$d_2/h$</th>
<th>$j$</th>
<th>$q/q_o$</th>
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<table>
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* Limiting value $m_1$ calculated from Equation 62.

** Limiting value $j_1$ calculated from Equation 63.
Table 10. Total head differences for the phreatic line patterns of Figures 27, 28, 32, 33, and 35. The head difference $\Delta H$ is expressed in centimeters of water.

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<th>Figure 27 Curve no.</th>
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<th>$\Delta H$</th>
<th>Figure 32 Curve no.</th>
<th>$\Delta H$</th>
<th>Figure 33 Curve no.</th>
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